Graduation Thesis

Run length limited de Bruijn sequences for quantum communication

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Coding Theory

- The study of the properties of codes and their respective fitness for specific applications.

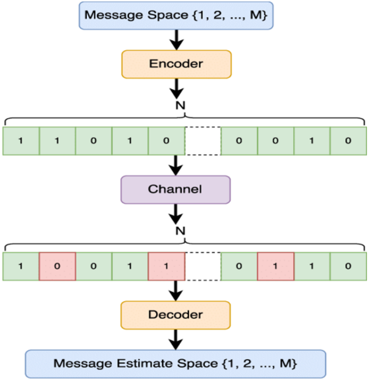


Figure 1: Coding Process.

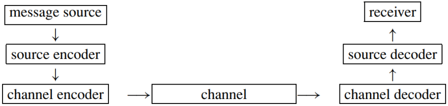
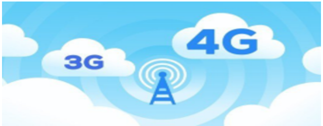


Figure 2: Source and Channel Coding.

Coding Theory - Applications

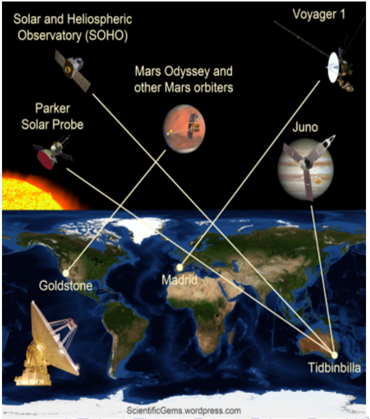
- Run length limited code is used in CD, DVD.

- 3G,4G networks use Reed-Solomon code.



- 5G networks use Turbo code.

- LDPC code is used in Data modems, telephone transmissions, and the NASA Deep Space Network.



Constrained Code

- Code: set of (binary) sequences, strings.
  $\rightarrow$ Constrained code $\mathbb{C}$: set of sequences satisfying given constraints.
- Presentation: a directed graph $G = (V, E)$.
- Path: sequence of edges $(e_1, e_2, \ldots, e_n)$ such that the end vertex of e_i is the start vertex of e_{i+1} $e_i \in E$.
- Simple path: path with no repeated edges.

De Bruijn Code (Positioning Code)

1. De Bruijn graph of order k, G_k:
   - Each vertex is labeled by a sequence of length $k-1$.
   - A directed edge from $\boldsymbol{x} = [x_0 x_1 \ldots x_{k-2}]$ to $\boldsymbol{y} = [y_0 y_1 \ldots y_{k-2}]$ $\Leftrightarrow$ $x_1 x_2 \ldots x_{k-2} = y_0 y_1 \ldots y_{k-3}$.
2. De Bruijn sequence:
   - A cyclic binary de Bruijn of order k is a length 2^k sequence such that each length k string appears exactly once.
   - Longest simple path in de Bruijn graph (Eulerian cycle) $\equiv$ de Bruijn sequence.
3. Applications:
   - Cryptography.
   - Interconnection networks.
   - VLSI testing.
   - Combine with biology: DNA storage.

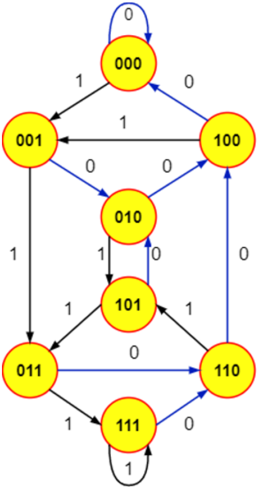


Figure 3: de Bruijn graph of order 4.

What do we concern about a code?

Rate

Rate and Maximal asymptotic rate tend to as high as possible.

Efficient encoding algorithm

Encoding algorithm: maps messages into codewords, generates sequences in the code, ...

Efficient decoding algorithm

Decoding algorithm: maps codewords into messages, locates the position of the sequences in the ordered code, ...

Et Cetera

Fault tolerant, robust positioning, ...

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Satellite Quantum Key Distribution (QKD)

Commercial QKD: deployed over optical fibre.
But:

$$\text{range} < 1000\text{km}$$

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But:

$$\text{range} < 1000\text{km}$$

$\rightarrow$ Satellite QKD: alternative method to establishing intercontinental secure communication links.
Challenges:

- High channel losses.
- Rapid relative motion between the transmitter and receiver.

Satellite Quantum Key Distribution (QKD)

Commercial QKD: deployed over optical fibre.
→ Satellite QKD: alternative method to establishing intercontinental secure communication links.
→ A classical channel is used along to synchronise quantum channel[1,2].

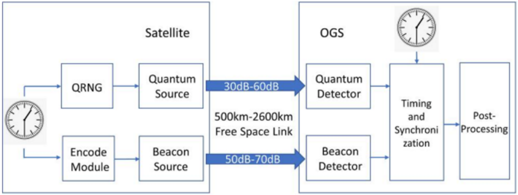

Figure 4: High-level satellite Quantum Key Distribution schematic.

[1] Shiqi Duan, Shuang Cong, and Yuanyuan Song. "A survey on quantum positioning system". In: *International Journal of Modelling and Simulation* 41.4 (2021), pp. 265–283.
[2] Isaac Khader et al. "Time synchronization over a free-space optical communication channel". In: *Optica* 5.12 (2018), pp. 1542–1548.

Related Work

Zhang et al.[3]: de Bruijn based timing-synchronization system (dBTS).

[3] Peide Zhang et al. "Timing and synchronisation for high-loss free-space quantum communication with Hybrid de Bruijn Codes". In: *IET Quantum Communication* 2.3 (2021), pp. 80–89.

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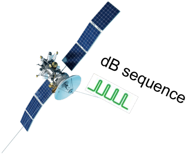
Encode



- Linear feedback shift register (LFSR): generate an order k de Bruijn sequence.

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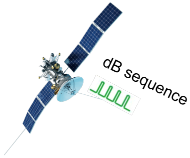

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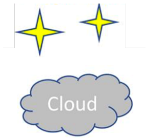
Noisy channel



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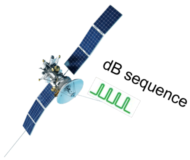

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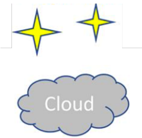

Decode

- Look-up table: locate the position of a length k subsequence in the whole sequence.

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Transmit a modulated sequence

- Constraint: avoid long period of no pulse .
- Requirement: positioning sequence.
- Method: de Bruijn sequence, pulse modulation:
  $1 \to \mathrm{on} - \mathrm{on},\ 0 \to \mathrm{on} - \mathrm{off}$, called Hybrid de Bruijn (HdB) sequence.

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Drawback

- $\mathrm{Rate} = 0.5$.
- Encode: LSFR (prerequisite: suitable primitive polynomial).
- Decode: use look-up table (exponential complexity).

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Contributions

Propose a new combinatorial object RdB

- Can be encoded and decoded efficiently.
- Can replace HdB sequence: higher rate and maximal asymptotic rate, more general and adaptive.
- Potential application: DNA storage.

Determine the maixmal length of RdB

- Explicit formula.

Encoding Algorithm

- Constant amortized time per symbol.

Decoding Algorithm

- Sub-linear time with respect to the length of the sequence.

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Run Length Limited de Bruijn (RdB) code

Definition 1

A (k, s)-RdB sequence: a de Bruijn sequence of order k containing at most s consecutive bit 0's.

Trivial cases:

- $s \geq k$: original de Bruijn sequence.
- $s = k - 1$: remove $0^k - 1$ in the de Bruijn graph.

$\Rightarrow$ Interested in:

- $s < k - 1$: eliminate vertices with more than s consecutive bit 0's.

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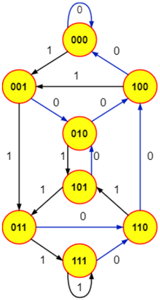

Figure 5: de Bruijn graph of order 4.

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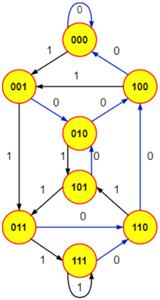

Figure 5: de Bruijn graph of order 4.

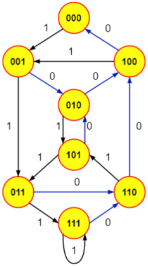

Figure 6: $(4, 3)$-RdB graph.

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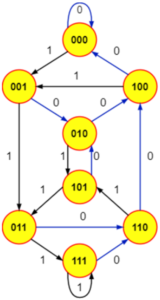

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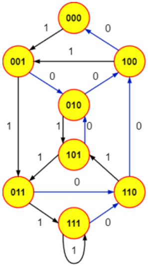

Figure 6: (4, 3)-RdB graph.

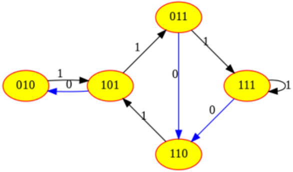

Figure 7: (4, 1)-RdB graph.

Maximal length of (k, s)-RdB sequence

Given k, s. Notations:

- $\ell(k, s)$: maximal length of simple path (k, s)-RdB graph.
- $N(k, s)$: maximal length of (k, s)-RdB sequences $(= \ell(k, s) + k - 1)$.

Theorem 2

Given k, s. Then:

$$\ell(k, s) = |W(k, s)| - \left(\sum_{i=0}^{C} (s - i) \, |W(k - s - i - 3, s)| - s \right) \tag{1}$$

where :

- $C = \min(s - 1, k - s - 2)$.
- $W(n, s)$: set of length n sequences containing at most s consecutive bit 0's.

Maximal length of (k, s)-RdB sequence

Lemma 3

$$\ell(k,s) \le |W(k,s)| - \left(\sum_{i=0}^{C}(s-i)\,|W(k-s-i-3,s)| - s\right)$$

Observe:

- Vertices: balance, right-unbalanced, left-unbalanced.
- Number of left(right)-unbalanced vertices of form $0^s1\dots10^i$ is $|W|\,(k-i-s-3,s)$ with $i \le C = \min(s-1, k-s-2)$.

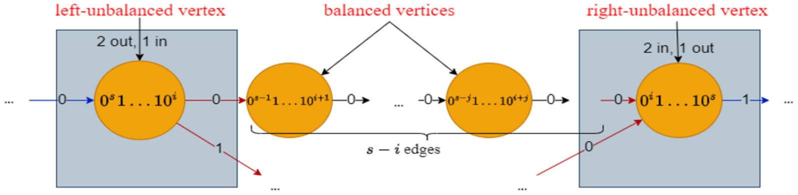

Figure 8: Vertices in RdB graphs.

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Definition of rate

Recall $N(k,s) = \ell(k,s) + k - 1$. Given k, s:

- Let $\boldsymbol{x}_{k,s}$: a (k,s)-RdB sequence. The (information) rate of $\boldsymbol{x}_{k,s}$:

$$R_{\boldsymbol{x}_{k,s}} = \frac{\log\left(|\boldsymbol{x}_{k,s}|\right)}{k} \tag{2}$$

- Maximal rate of (k,s)-RdB sequences:

$$R_{k,s} = \frac{\log(N(k,s))}{k} \tag{3}$$

- Maximal asymptotic rate of (k,s)-RdB sequences:

$$R_s = \lim_{n\to\infty} \frac{\log(N(k,s))}{k} \tag{4}$$

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Maximal Asymptotic Rate

Theorem 4

$$R_1 = 0.6942$$

which is larger than 0.5, rate of HdB sequences.

Theorem 5

$$R_s = \log(|\omega|)$$

where ω is the root of equation: $x^{s+1} - x^s - \ldots - x - 1 = 0$ such that $|\omega|$ is the largest.

Proof: Approximation: $|W(k,s)| \approx a\,|\omega|^n$. Hence:

$$N(k,s) \approx |\omega|^{k-s-2} \left(\sum_{i=0}^{s} a(i+1)\,|w|^{s-i} + \frac{s+k-1}{|w|^{k-s-2}} \right)$$

$$\Rightarrow R_s = \lim_{n\to\infty} \frac{(k-s-2)\log(|\omega|)}{k} = \log(|\omega|).$$

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Encoder

- Based on lexicographic minimal de Bruijn sequence (granddaddy by Knuth):
  Example with order 5: 0 00001 00011 00101 00111 01011 01111 1
  $\rightarrow$ Cut the sequence at the "right place".
- Complexity = Complexity to generate granddaddy sequence.
  = Constant amortized time per symbol[4]
- The correctness can be proved easily.

Cut at $\boldsymbol{u} = 0^{s+1}1^{k-s-1}$.
Example $k = 5, s = 2$:

[4] Frank Ruskey, Carla Savage, and Terry Min Yih Wang. "Generating necklaces". In: *Journal of Algorithms* 13.3 (1992), pp. 414–430.

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Encoder

Algorithm 1: Encode (k,s)-RdB

Input : k, and descending ordered set $Ł^{(n)}$.
Output: (k,s)-RLL dBs

```
w ← emptystring
for λ ∈ Ł^(n) do
    w.prepend(λ)
    if λ == 0^(s+1) 1^(k-s-1) then
        w = w[2, ℓ] 0^s
        break
return w
```

$Ł^{(n)}$: generated by FKM algorithm[5,6].

[5] Harold Fredricksen and James Maiorana. "Necklaces of beads in k colors and k-ary de Bruijn sequences". In: *Discrete Mathematics* 23.3 (1978), pp. 207–210.
[6] Harold Fredricksen and Irving J Kessler. "An algorithm for generating necklaces of beads in two colors". In: *Discrete mathematics* 61.2-3 (1986), pp. 181–188.

The Optimality of The Encoder

- We're proving that our encoder generate the longest (k, s)-RdB sequences.

Example 6 ($k = 5, s = 2$)

$$0\ 00001\ 0 \mid \underbrace{0011\ 00101\ 00111\ 01011\ 01111\ 100}_{\text{this length}=N(k,s)\ ?}$$

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$$0\ 00001\ 0 \mid \underbrace{0011\ 00101\ 00111\ 01011\ 01111\ 100}_{\text{this length}=N(k,s)\ ?}$$

Equivalent to:

$$0\ 00001\ 0 \mid \underbrace{0011\ 00101\ 00111\ 01011\ 01111\ 1}_{\text{How long is this sequence? } (=X)}\ 00$$

We'll prove that $X = N(k, s) - s$

The Optimality of The Encoder

$\langle \boldsymbol{v} \rangle$ minimal rotation of $\boldsymbol{v}$ (exp: $\langle 110 \rangle = 011$, $\langle 1001 \rangle = 0011$).
$S(\boldsymbol{v}) = \left\{ \boldsymbol{x} \in \Sigma^{|\boldsymbol{v}| : \langle \boldsymbol{x} \rangle < \boldsymbol{v}} \right\}$.
$Ł^{(n)}$: set of Lyndon words whose length is a divisor of n.

Lemma 7 (Lemma 29[7])

Let $\boldsymbol{v}$ be a Lyndon word. Define $Ł(\boldsymbol{v}) = \{\lambda \in Ł^k : \lambda^{\frac{k}{|\lambda|}} \leq \boldsymbol{v}\}$ to be the set of all Lyndon words smaller than $\boldsymbol{v}$ whose length is the divisor of k. Then: $\sum_{\lambda \in Ł(\boldsymbol{v})} |\lambda| = |S(\boldsymbol{v})|$.

$$\overbrace{0\ 00001\ 0}^{\text{length}=|S(\boldsymbol{u})|}\!\!\!\!\underbrace{|0011\ 00101\ 00111\ 01011\ 01111\ 1}_{\text{length needs calculating}}$$

[7] Tomasz Kociumaka, Jakub Radoszewski, and Wojciech Rytter. "Efficient ranking of Lyndon words and decoding lexicographically minimal de Bruijn sequence". In: *SIAM Journal on Discrete Mathematics* 30.4 (2016), pp. 2027–2046.

The Optimality of The Encoder

$$|S(\boldsymbol{u})| = 1 + \sum_{t=M}^{k} (k-t+1)\,|C(t-2,s)| + \sum_{t=1}^{k-s} (k-t+1) A_t \qquad (5)$$

where $|C(k,s))| = 2^k - |W(k,s)|,\ A_t = 2^{t-2},\ M = \max(s+2, k-s+1)$

The Optimality of The Encoder

For $s = 1$, $|S(\boldsymbol{u})| = 2^k - (|W(k,1)| - |W(k-4,1|)$

The Optimality of The Encoder

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$k = 5, s = 1$

$$\overbrace{0\ 00001\ 00011\ 00101\ \rlap{$\underbrace{\underbrace{}_{|W(k,1)|-|W(k-4,1)|+k}}_{=N(k,1)}$}0|0111}^{2^k-(|W(k,1)|-|W(k-4,1|)}\ 01011\ 01111\ 1$$

Decoder

1. Based on the decoder $\mathcal{D}_{KRR}$ proposed by Kociumaka, Radoszewski, and Rytter[7].
2. Based on the observation:
   - $i = \mathcal{D}_{KRR}(\boldsymbol{u} = 0^{s+1}1^{k-s-1})$: position of $\boldsymbol{u}$ in granddaddy sequence $\boldsymbol{x}$ of order k.
   - Location of $\boldsymbol{v}$ ($|\boldsymbol{v}| = k$) in encoded sequence $=$ Location of $\boldsymbol{v}$ in $\boldsymbol{x}$ - i
     $= \mathcal{D}_{KRR}(\boldsymbol{v}) - i$
3. Complexity $=$ Complexity of $\mathcal{D}_{KRR}$.

Decoder

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Decoder

Denote $c_{k,s}$ to be the encoded sequence.

Algorithm 2: Decode (k,s)-RdB $\boldsymbol{c}_{\mathbf{k,s}}$

Input : A word $\boldsymbol{v} = (v_1, \ldots, v_k)$ of length k

Output: a is the location of $\boldsymbol{v}$ in $\boldsymbol{c}_{\mathbf{k,s}}$

$i \leftarrow \mathcal{D}_{KRR}(\boldsymbol{u}_{k,s})$;

if $\boldsymbol{v} = 1^j 0^{k-j}$**, then**

  return $n - i + 1 - (k - j)$;

else

  return $\mathcal{D}_{KRR}(\boldsymbol{v}) - i$;

Table of Contents

Summary

A new combinatorial structure

- Run length limited de Bruijn sequences (RdB).
- Open new questions in research.
- Potential applications: Quantum communication, DNA storage, ...

Length and rate

- Explicit formula for maximal length and maximal asymptotic rate of RdB sequences.

Encoding and decoding algorithm

- Encode in constant amortized time per symbol.
- Decode in polynomial time (sub-linear with respect to the length of the whole sequence).

Future works

More Constraints

- Weight constraint.
- Locally constraint.
- ...

Alphabet

- Extend the size of alphabet.
- Alphabet $\{A, T, C, G\}$ (DNA).

Properties

- Lexicographic minimal RdB sequences ?
- How many sequences of the same size ?
- ...

Publications

My publications during the time doing this thesis:

- Yeow Meng Chee, Duc Tu Dao, **Tien Long Nguyen**, Duy Hoang Ta, Van Khu Vu. "Run Length Limited de Bruijn Sequences for Quantum Communications", The 2022 IEEE International Symposium on Information Theory.
- Tran Ba Trung, Lijun Chang, **Nguyen Tien Long**, Kai Yao, Huynh Thi Thanh Binh. "Verification-Free Approaches to Efficient Locally Densest Subgraph Discovery", The 39th IEEE International Conference on Data Engineering.

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