Graduation Thesis

Run length limited de Bruijn sequences for quantum communication

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Satellite Quantum Key Distribution (QKD)

Quantum Key Distribution.

- Cryptographic protocol involving components of quantum mechanics.
- Enables two parties to produce a shared random secret key known only to them.

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Quantum Key Distribution.
Satellite QKD: share random key between satellite and ground station.

[1] Peide Zhang et al. "Timing and synchronisation for high-loss free-space quantum communication with Hybrid de Bruijn Codes". In: *IET Quantum Communication* 2.3 (2021), pp. 80–89.
[2] Isaac Khader et al. "Time synchronization over a free-space optical communication channel". In: *Optica* 5.12 (2018), pp. 1542–1548.

Satellite Quantum Key Distribution (QKD)

Quantum Key Distribution.
Satellite QKD: share random key between satellite and ground station.
A classical channel is used along to synchronise quantum channel[1,2].

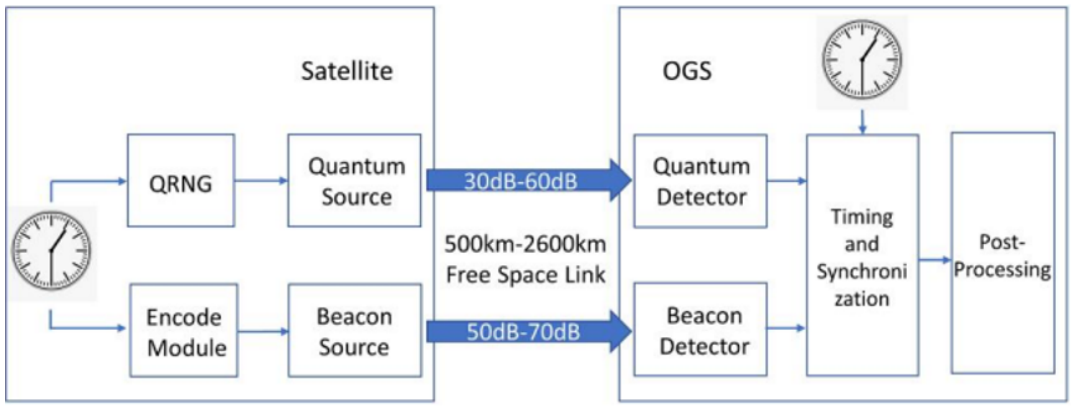

Figure 1: High-level satellite Quantum Key Distribution schematic[1].

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Zhang et al.[1]: de Bruijn based timing-synchronization system (dBTS).

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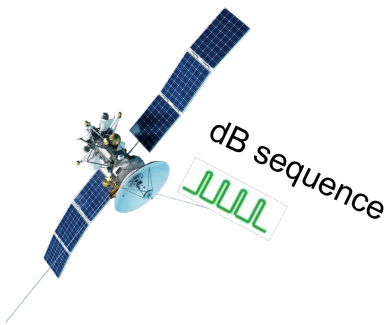

- Linear feedback shift register (LFSR): generate an order k de Bruijn sequence.

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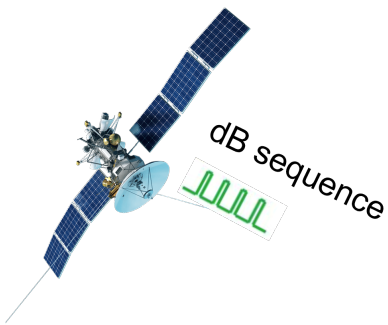

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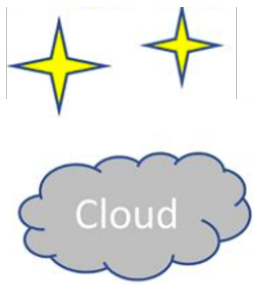
Noisy channel



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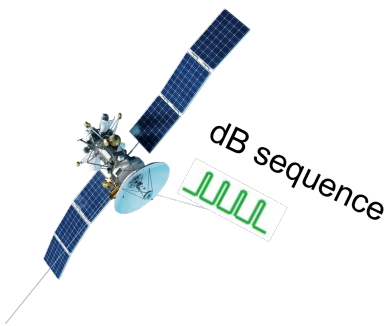

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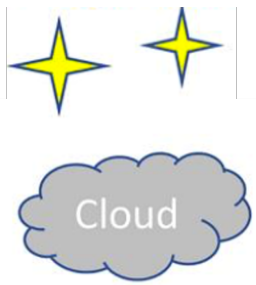

Decode

- Look-up table: locate the position of a length k subsequence in the whole sequence.

Related Work

Zhang et al.[1]: de Bruijn based timing-synchronization system (dBTS).

Transmit a modulated sequence

- Constraint: avoid long period of no pulse .
- Requirement: positioning sequence.
- Method: de Bruijn sequence, pulse modulation:
  $1 \rightarrow \mathrm{on}-\mathrm{on},\ 0 \rightarrow \mathrm{on}-\mathrm{off}$, called Hybrid de Bruijn (HdB) sequence.

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Drawback

- $\mathrm{Rate} = 0.5$.
- Encode: LFSR (prerequisite: suitable primitive polynomial).
- Decode: use look-up table (exponential complexity).

Contributions

Propose a new combinatorial object RdB

- Can be encoded and decoded efficiently.
- Can replace HdB sequence: higher rate and maximal asymptotic rate, more general and adaptive.

Determine the maximal length of RdB

- Explicit formula.

Encoding and Decoding Algorithm

- Encoder: Constant amortized time per symbol.
- Decoder: Sub-linear time with respect to the length of the sequence.

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De Bruijn Sequence (Positioning Sequence)

1. De Bruijn sequence:
   - A cyclic binary de Bruijn of order k is a length 2^k sequence such that each length k string appears exactly once.
   - Example: A de Bruijn sequence of order 4.
     Cyclic : 0000100110101111.
     Acyclic : 00001001101011110000.
   - De Bruijn sequence $\equiv$ longest simple path in de Bruijn graph (Eulerian cycle).

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2. De Bruijn graph of order k, G_k:
   - Each vertex is labeled by a sequence of length $k-1$.
   - A directed edge from $\boldsymbol{x} = [x_0 x_1 \dots x_{k-2}]$ to $\boldsymbol{y} = [y_0 y_1 \dots y_{k-2}]$
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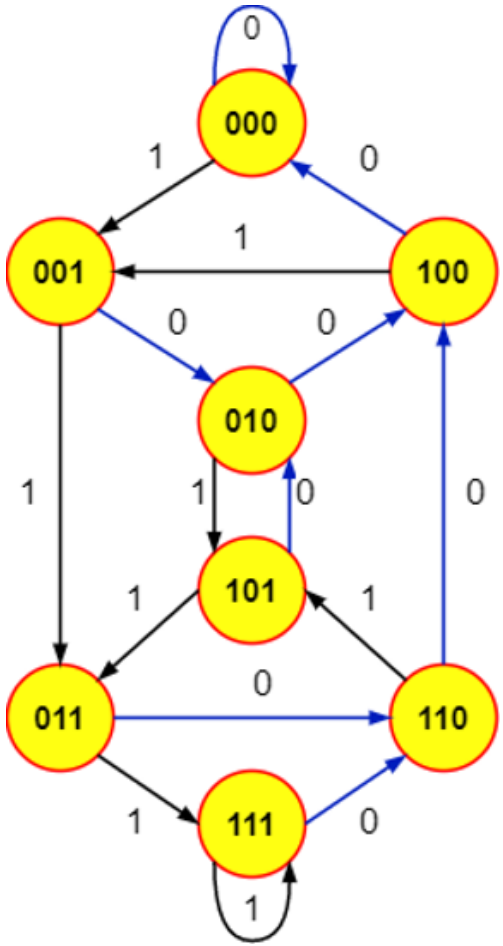

Figure 2: de Bruijn graph of order 4.

Run Length Limited de Bruijn (RdB) sequences

Definition 1

A (k, s)-RdB sequence: a de Bruijn sequence of order k containing at most s consecutive bit 0's.

Trivial cases:

- $s \geq k$: original de Bruijn sequence.
- $s = k - 1$: remove 0^{k-1} in the de Bruijn graph.

$\Rightarrow$ Interested in:

- $s < k - 1$: eliminate vertices with more than s consecutive bit 0's.

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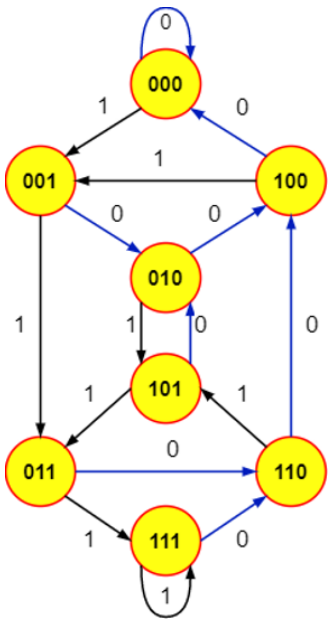

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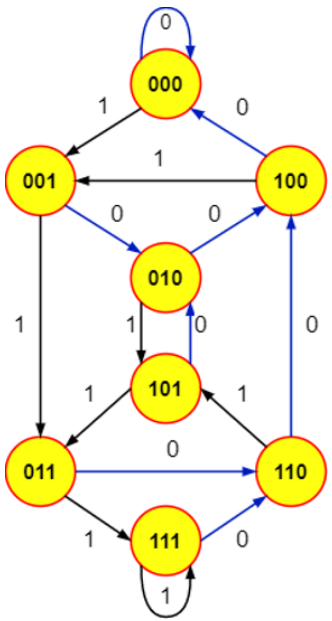

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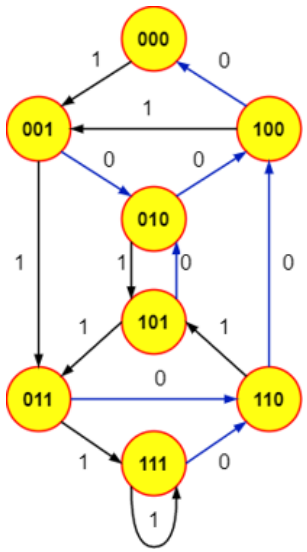

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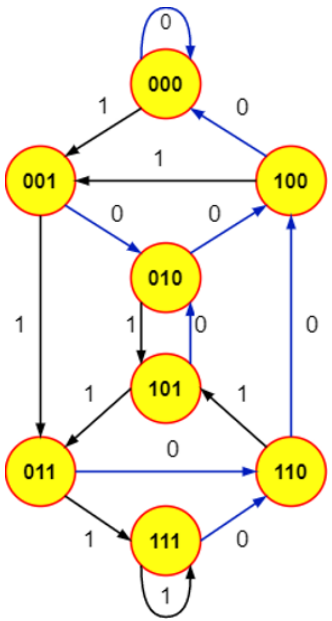

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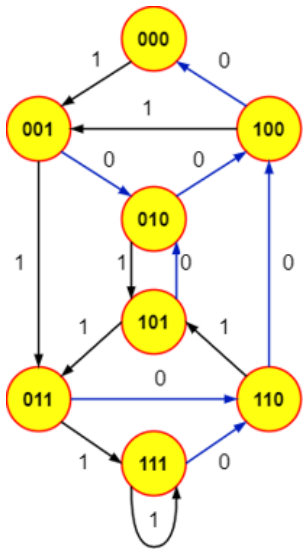

Figure 4: $(4, 3)$-RdB graph.

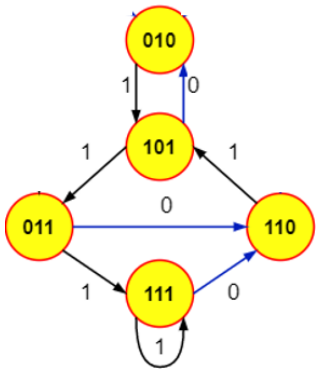

Figure 5: $(4, 1)$-RdB graph.

Maximal length of (k, s)-RdB sequence

Given k, s. Notations:

- $\ell(k, s)$: maximal length of simple path (k, s)-RdB graph.
- $N(k, s)$: maximal length of (k, s)-RdB sequences $(= \ell(k, s) + k - 1)$.

Theorem 2

Given k, s. Then:

$$\ell(k, s) = |W(k, s)| - \left(\sum_{i=0}^{C} (s - i) \, |W(k - s - i - 3, s)| - s \right) \quad (1)$$

where :

- $C = \min(s - 1, k - s - 2)$.
- $W(n, s)$: set of length n sequences containing at most s consecutive bit 0's.

Maximal length of (k, s)-RdB sequence

Lemma 3

$$\ell(k,s) \leq |W(k,s)| - \left(\sum_{i=0}^{C}(s-i)\,|W(k-s-i-3,s)| - s\right)$$

Observe:

- Vertices: balanced, right-unbalanced, left-unbalanced.
- Number of left(right)-unbalanced vertices of form $0^s 1 \ldots 10^i$ is $|W(k-i-s-3,s)|$ with $i \leq C = \min(s-1, k-s-2)$.

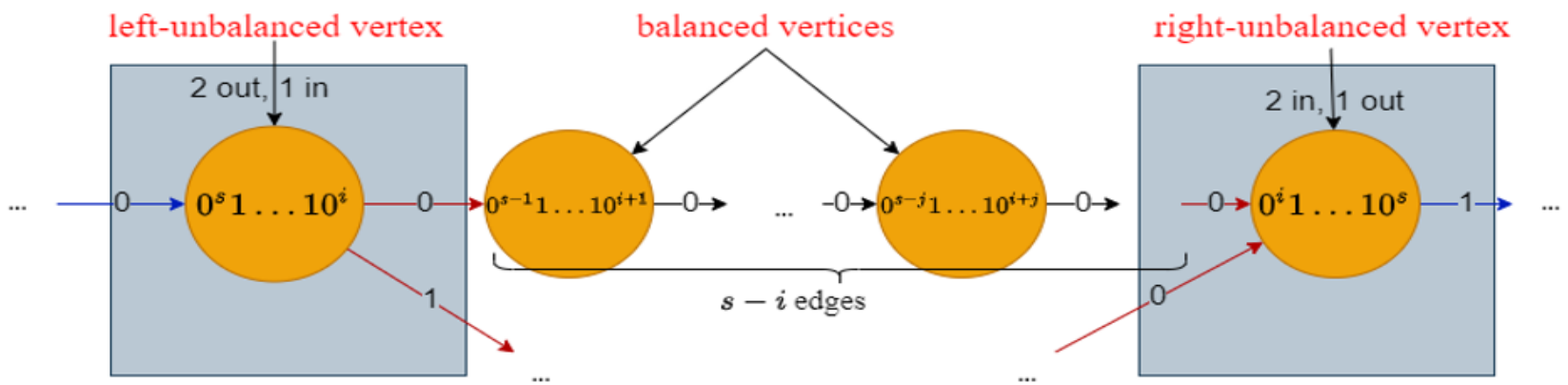

Figure 6: Vertices in RdB graphs.

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Definition of rate

Recall $N(k,s) = \ell(k,s) + k - 1$. Given k, s:

- Let $\boldsymbol{x}_{k,s}$: a (k,s)-RdB sequence. The (information) rate of $\boldsymbol{x}_{k,s}$:

$$R_{\boldsymbol{x}_{k,s}} = \frac{\log\left(|\boldsymbol{x}_{k,s}|\right)}{k} \tag{2}$$

- Maximal rate of (k,s)-RdB sequences:

$$R_{k,s} = \frac{\log(N(k,s))}{k} \tag{3}$$

- Maximal asymptotic rate of (k,s)-RdB sequences:

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Maximal Asymptotic Rate

Theorem 4

$$R_1 = \log\left(\frac{1+\sqrt{5}}{2}\right) = 0.6942$$

which is larger than 0.5, rate of HdB sequences.

Theorem 5

$$R_s = \log(|\omega|)$$

where ω is the root of equation: $x^{s+1} - x^s - \ldots - x - 1 = 0$ such that $|\omega|$ is the largest.

Proof: Approximation: $|W(k,s)| \approx a\,|\omega|^n$. Hence:

$$N(k,s) \approx |\omega|^{k-s-2}\left(\sum_{i=0}^{s} a(i+1)\,|w|^{s-i} + \frac{s+k-1}{|w|^{k-s-2}}\right)$$

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$$\Rightarrow R_s = \lim_{n\to\infty} \frac{(k-s-2)\log(|\omega|)}{k} = \log(|\omega|).$$

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Encoder

- Based on lexicographic minimal de Bruijn sequence (granddaddy by Knuth):
  Example with order 5: 0 00001 00011 00101 00111 01011 01111 1
  $\rightarrow$ Cut the sequence at the "right place".
- Complexity = Complexity to generate granddaddy sequence.
  = Constant amortized time per symbol[3].
- The correctness can be proved easily.

Cut at $\boldsymbol{u} = 0^{s+1}1^{k-s-1}$.
Example $k = 5, s = 2$:

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Encoder

Algorithm 1: Encode (k,s)-RdB

Input : k, and descending ordered set $Ł^{(n)}$.
Output: (k, s)-RLL dBs

w $\leftarrow$emptystring
for $\lambda \in Ł^{(n)}$ **do**
    w.*prepend*(λ)
    if $\lambda == 0^{s+1}1^{k-s-1}$ **then**
        $\mathbf{w} = \mathbf{w}[2, \ell]0^s$
        break
return w

$Ł^{(n)}$: generated by FKM algorithm[4,5].

[4] Harold Fredricksen and James Maiorana. "Necklaces of beads in k colors and k-ary de Bruijn sequences". In: *Discrete Mathematics* 23.3 (1978), pp. 207–210.
[5] Harold Fredricksen and Irving J Kessler. "An algorithm for generating necklaces of beads in two colors". In: *Discrete mathematics* 61.2-3 (1986), pp. 181–188.

The Optimality of The Encoder

- We're proving that our encoder generate the longest (k, s)-RdB sequences.

Example 6 ($k = 5, s = 2$)

$$0\ 00001\ 0|\underbrace{0011\ 00101\ 00111\ 01011\ 01111\ 100}_{\text{this length}=N(k,s)\ ?}$$

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Equivalent to:

$$0\ 00001\ 0|\underbrace{0011\ 00101\ 00111\ 01011\ 01111\ 1}_{\text{How long is this sequence? (=X)}}00$$

We'll prove that $X = N(k, s) - s$

The Optimality of The Encoder

$\langle \boldsymbol{v} \rangle$ minimal rotation of $\boldsymbol{v}$ (exp: $\langle 110 \rangle = 011$, $\langle 1001 \rangle = 0011$).
$S(\boldsymbol{v}) = \left\{ \boldsymbol{x} \in \Sigma^{|\boldsymbol{v}|} : \langle \boldsymbol{x} \rangle < \boldsymbol{v} \right\}$.
Ł$^{(n)}$: set of Lyndon words whose length is a divisor of n.

Lemma 7 (Lemma 29[6])

Let $\boldsymbol{v}$ be a Lyndon word. Define $Ł(\boldsymbol{v}) = \{\lambda \in Ł^k : \lambda^{\frac{k}{|\lambda|}} \leq \boldsymbol{v}\}$ to be the set of all Lyndon words smaller than $\boldsymbol{v}$ whose length is the divisor of k. Then: $\sum_{\lambda \in Ł(\boldsymbol{v})} |\lambda| = |S(\boldsymbol{v})|$.

$$\overbrace{0\ 00001\ 0}^{\text{length}=|S(\boldsymbol{u})|}\underbrace{0011\ 00101\ 00111\ 01011\ 01111\ 1}_{\text{length needs calculating}}$$

[6] Tomasz Kociumaka, Jakub Radoszewski, and Wojciech Rytter. "Efficient ranking of Lyndon words and decoding lexicographically minimal de Bruijn sequence". In: *SIAM Journal on Discrete Mathematics* 30.4 (2016), pp. 2027–2046.

The Optimality of The Encoder

$$|S(\boldsymbol{u})| = 1 + \sum_{t=M}^{k} (k - t + 1)\,|C(t-2, s)| + \sum_{t=1}^{k-s} (k - t + 1) A_t \qquad (5)$$

where $|C(k, s))| = 2^k - |W(k, s)|$, $A_t = 2^{t-2}$, $M = \max(s + 2, k - s + 1)$

The Optimality of The Encoder

For $s = 1$, $|S(\boldsymbol{u})| = 2^k - (|W(k,1)| - |W(k-4,1|)$

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$$\overbrace{0\ 00001\ 00011\ 00101}^{2^k-(|W(k,1)|-|W(k-4,1|)}\ \underbrace{0|0111\ 01011\ 01111\ 1}_{\underbrace{|W(k,1)|-|W(k-4,1)|+k}_{=N(k,1)}}$$

Decoder

1. Based on the decoder $\mathcal{D}_{KRR}$ proposed by Kociumaka, Radoszewski, and Rytter[6].
2. Based on the observation:
   - $i = \mathcal{D}_{KRR}(\boldsymbol{u} = 0^{s+1}1^{k-s-1})$: position of $\boldsymbol{u}$ in granddaddy sequence $\boldsymbol{x}$ of order k.
   - Location of $\boldsymbol{v}$ ($|\boldsymbol{v}| = k$) in encoded sequence = Location of $\boldsymbol{v}$ in $\boldsymbol{x}$ - i. $= \mathcal{D}_{KRR}(\boldsymbol{v}) - i$.
3. Complexity = Complexity of $\mathcal{D}_{KRR}$.

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Exp: $k = 5, s = 1$

then $\boldsymbol{u} = 00111$

say: $\boldsymbol{v} = 10111$

Decoder

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0 00001 00011 00101 $\underbrace{00111}_{i}$ 0101 $\underbrace{1\ 0111}_{j}$ 1 1 0

0111 0101 $\underbrace{1\ 0111}_{j-i}$ 1 1 0

Decoder

Denote $c_{k,s}$ to be the encoded sequence.

Algorithm 2: Decode (k,s)-RdB $\boldsymbol{c_{k,s}}$

Input : A word $\boldsymbol{v} = (v_1, \ldots, v_k)$ of length k
Output: a is the location of $\boldsymbol{v}$ in $\boldsymbol{c_{k,s}}$

$i \leftarrow \mathcal{D}_{KRR}(\boldsymbol{u}_{k,s})$;
if $\boldsymbol{v} = 1^j 0^{k-j}$, **then**
  return $n - i + 1 - (k - j)$;
else
  return $\mathcal{D}_{KRR}(\boldsymbol{v}) - i$;

Table of Contents

Summary

A new combinatorial structure

- Run length limited de Bruijn sequences (RdB).
- Encode and decode efficiently.

Length and rate

- Explicit formula for maximal length and maximal asymptotic rate of RdB sequences.

Encoding and decoding algorithm

- Encode in constant amortized time per symbol.
- Decode in polynomial time (sub-linear with respect to the length of the whole sequence).

Future works

More Constraints

- Weight constraint.
- Locally constraint.
- Extend the size of alphabet.

Properties

- Lexicographic minimal RdB sequences ?
- How many sequences of the same size ?

Publications

My publications during the time doing this thesis:

- Yeow Meng Chee, Duc Tu Dao, **Tien Long Nguyen**, Duy Hoang Ta, Van Khu Vu. "Run Length Limited de Bruijn Sequences for Quantum Communications". In proceeding of IEEE International Symposium on Information Theory, 2022.
- Tran Ba Trung, Lijun Chang, **Nguyen Tien Long**, Kai Yao, Huynh Thi Thanh Binh. "Verification-Free Approaches to Efficient Locally Densest Subgraph Discovery". Accepted as paper at The 39th IEEE International Conference on Data Engineering, 2023.

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