# A unified model of demographic time

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**Abstract**

We describe a three-dimensional model that relates six different measures of lifespans and time. The six measures of demographic time considered are chronological age, thanatological age, lifespan, year of birth, year of death, and period. We show three identities similar to the well-known age-period-cohort identity, and we then present an identity consisting of all six time measures. Two versions of the model are described: a relatively intuitive extension of the right-angled Lexis diagram, and an isotropic extension based on the regular tetrahedron.

The Lexis diagram relates the chronological age (A), period (P), and birth cohort (C) measures of demographic time, APC, but it does not account for remaining years of life (thanatological age), and other related time indices. The thanatological counterpart to APC is an identity between thanatological age (T), period (P), and death cohort (D), TPD. A third identity exists between thanatological age (T), chronological age (A), and lifespan (L), TAL, and a fourth between year of birth (C), year of death (D) and lifespan (L), CDL. Each of these four identities may be sufficiently described by any two of its constituent indices, making the third index redundant. Each of these four identities also lacks a major dimension of time. The TAL identity lacks calendar time, the CDL identity is ageless, APC lacks an endpoint in time, and TPD lacks a starting point in time. We refer to these four identities as the triad identities.

To our knowledge, the only triad identity that has received serious treatment at the time of this writing is the APC identity. Different aspects of the APC identity have been discussed since at least 1868 (Knapp 1868), and discussion remains lively today. Here it is our objective to relate the six major indices of time in a geometric identity, in much the same spirit as the work on APC relationships done between the late 1860s and mid 1880s.[1]

This paper provides a bottom-up description of the model, building from familiar components to the full relationship. We begin by defining some terms used throughout the manuscript. We then explore all combinations of two time measures, the dyadic relationships, followed by the four triad identities, and finally the hexad identity. At the price of some redundancy, we give a systematic topological overview of the different elements of demographic time. This paper contains no data visualizations, but it may serve as a basis for designing some.

# Definitions

## Technical terminology

We attempt to adhere to a rigorous terminology in this paper. The following list describes some of the more important terms we use.

**Time measures** are any of the six demographic perspectives discussed that can be used to index time: chronological age ($A$), period ($P$), birth cohort ($C$), thanatological age ($T$, time-to-death), lifespan ($L$), and death cohort ($D$).

**Dyads, triads, and hexads** are any set of two, three, or six unique time measures, respectively.

**A temporal plane** is any $(x, y)$-mapping of a dyad of time measures.

**A temporal space** is any $(x, y, z)$-mapping of a triad of time measures that generates a 3-dimensional space.

**A triad identity** is a triad that is the union of an informative dyad and its one derived time measure. For example, $A$ and $P$ form an informative dyad from which $C$ can be derived.

**A hexad identity** is a unique combination of the six time measures.

Using this terminology, for example, we say that the "Lexis" measures constitute a triad identity between chronological age, period, and birth cohort. Each dyad combination of elements in this identity is informative, and can be mapped to a temporal plane, the Lexis diagram. If we know that Mindel

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[1]See e.g., Keiding (2011) for an overview of that literature.

turned 50 on the 21st of May, 1963, then we also also can derive that she was born on the 21st of May, 1913. Any two pieces of information in this case will give the third, which means that any dyad from this set is informative. Three other such triad identities are also to found within the six measures of time we discuss.

## Time measures

This model description is conceived in absolute, linear, Newtonian time, and we do not consider situating the model in any other perspective or model of time itself. This relationship is scalable to any time unit, but we describe it in terms of years, the dominant human time scale. We therefore speak of calendar time, imagining the modern Gregorian calendar, though this is not necessary. The six measures of time we consider are defined in Table 1, both in the demographic sense we describe, as well as in a more general event history interpretation.

The concepts of thanatological age and death cohorts are likely less familiar to readers than the other measures we consider. Thanatological age is sometimes referred to in the literature as remaining years of life, or time-to-death (TTD), but we prefer the term thanatological age, and to think of the concept of age in general as marking a position on a lifeline with respect to one of its endpoints. Chronological age and thanatological age are in this way complementary. Thanatological age is different from the notion of prospective age, used by Sanderson and Scherbov (2007), since prospective age is a relative term that reflects a comparison of expectancies. Prospective age scales chronological age by comparing mortality schedules, but it is neither an expectancy nor a statement of remaining years of life. Thanatological age is meaningful without much justification; it is the measure we all want to know, the thing we approximate with remaining life expectancy.

Cohorts in general associate individuals that share a characteristic. In demography the grouping characteristic is often a combination of place and time, such as the cohorts of young demographers passing through a particular graduate program. In this instance already, we accommodate the

Table 1: Definitions of the six time measures.

| Time measure | Short | Demographic def. | Event history def. |
|---|---|---|---|
| chronological age | A | Time since birth | Time since study entry |
| period | P | calendar time | calendar time |
| birth cohort | C | calendar time of birth | calendar time of study entry |
| thanatological age | T | time until death | time until event |
| death cohort | D | calendar time of death | calendar time of event |
| lifespan | L | duration of life | duration of exposure |

notion of a cohort for both the start and endpoints of the program, but we say e.g., "the class of 2015" instead of the "graduating cohort of 2015", in contrast to "cohort 37", the 37th class of entering students since the start of the program. These concepts are analogous to the ideas of birth and death cohorts we use here, though we do not often refer to the deaths of a given year as a death cohort. In the time preceding death, the members of a given death cohort have much in common, despite heterogeneity with respect to time of birth. If the reader accepts this premise, then the abstract construct of a death cohort is also meaningful in the way that the other measures are.

Much of the work of demography is directed at the study of lifespan. Lifespan is synonymous both with longevity, chronological age at death, and thanatological age at birth. One's ultimate completed lifespan is constant throughout life, though we have no knowledge of it until death: It is assigned retrospectively. Demographers have more often used lifespan or age-at-death as a measure of mortality, or similar, than as a measure on which to compare individuals.

Treating lifespan, death cohorts, and thanatological age as temporal structuring variables enables new classes of comparisons, models of understanding, and discovery, akin to those unlocked by breaking down demographic phenomena by chronological age, period, and birth cohort. The following sections will, in this sense, provide an exhaustive classification of the ways in which these six measures of time can be juxtaposed to such ends. We begin with all sets of two time measures, the informative and uninformative temporal dyads.

## Informative and uninformative dyads

Any mapping of two time measures to a $(x, y)$ coordinate system constitutes a temporal plane. If the two given time measures are members of the same triad identity, the third member is called a derived measure. For instance, if we take the dyad $AP$, $C$ is called the derived measure, and we make this dyadic relationship explicit by writing $AP(C)$. The temporal plane that corresponds to this informative dyad is the contemporary representation of the Lexis diagram. The dyads $AC(P)$ and $CP(A)$ also belong to the Lexis identity, but imply different less-common rotations and projections of the Lexis diagram.

For each dyad there is a fundamental question of how to map the constituent coordinates to a cartesian temporal plane. Typically one forces parity between time units within a specified dyad, and maps one element directly to $x$ and the second element directly to $y$, resulting in a 90° angle between the $x$ axis and $y$ axis. In this case the convention is to force a unity aspect ratio between the $x$ and $y$ axes, such that the derived measure, if any, is then *accidentally* present in a 45° ascending or descending angle,

depending on the dyad and axis orientation.

It has long been noted (Zeuner 1869, Perozzo 1880) that the derived time measure, when plotted at $45°$, is relatively longer than either the age or years axes. If a right angle and unity aspect ratio is forced between the dyad, the derived measure is always stretched by $\sqrt{2}$, or 41%. In the case of dyads that imply a derived measure, another logical mapping is to translate to an $(x, y)$ coordinate that forces $60°$ angles between the three measures. Such a mapping ensures that the spatial units are equal for the three measures, and we therefore refer to it as the isotropic mapping. The isotropic mapping is comparable to using a ternary coordinate system, which we do not discuss. It does not make sense to provide an isotropic mapping of an uninformative dyad. The primary justification for isotropic temporal planes comes from a data visualization perspective, where it may be hypothesized that the viewer's ability to compare slopes is hindered if the axes are not on the same scale.
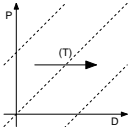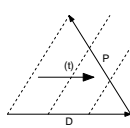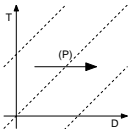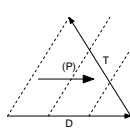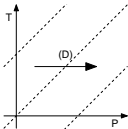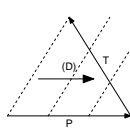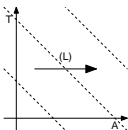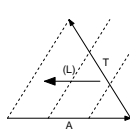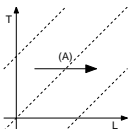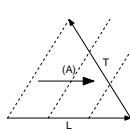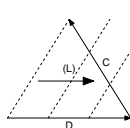
15 dyads are possible from our set of six time measures, 12 of which are informative, and three of which have no derived time measure, and are therefore called uninformative. These dyads, an explanation or simple example, and the corresponding graphical representations are summarized in Table 2.

Table 2: The 15 temporal dyads, with corresponding translations to Euclidean and isotropic temporal planes.[2]

VARIANTS OF APC

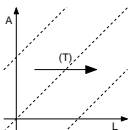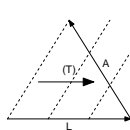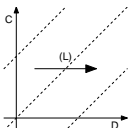| | | | |
|---|---|---|---|
| $AP(C)$ <br> $C = P - A$ | The $AP(C)$ temporal plane constitutes the classical Lexis diagram. |  |  |
| $AC(P)$ <br> $P = C + A$ | The $AC(P)$ temporal plane is equivalent to the Lexis diagram except that birth cohort is given and period is derived. |  |  |
| $CP(A)$ <br> $A = P - C$ | The $CP(A)$ temporal plane is equivalent to the Lexis diagram except that birth cohort is given and age is derived. |  |  |

VARIANTS OF TPD

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[2]Contrary to mathematical convention we name the ordinate scale first and the abscissa scale second. This is to be consistent with the established $APC$ and $ACP$ terms.

| | | |
|---|---|---|
| $PD(T)$<br>$T = D - P$ | Mindel died in 1973 (D). In 1953 (P) she had 20 (T) years left to live. | |
| $TD(P)$<br>$P = D - T$ | Irene died in 1974 (D). When she had 30 (T) remaining years of life the year must have been 1944 (P). | |
| $TP(D)$<br>$D = P + T$ | Helen had 30 yeasrs of life left (T) in 1971 (P) and therefore belonged to the 2001 death cohort (D). | |

<center>VARIANTS OF TAL</center>

| | | |
|---|---|---|
| $TA(L)$<br>$L = T + A$ | The time already lived and the time still left sum up to the total lifespan. | |
| $TL(A)$<br>$A = L - T$ | Helen lived to the age of 86. When she had 20 years left she must have been 66. | |
| $AL(T)$<br>$T = A - L$ | Tim is 34 years old (A) and will live to the age of 96 (L), leaving him 62 years (T) to settle affairs. | |

<center>VARIANTS OF CDL</center>

| | | |
|---|---|---|
| $CD(L)$<br>$L = D - C$ | Pascal was born in 1893 (C) and died in 1964 (D), implying a lifespan of 71 (L), or so. | |

<center>6</center>

| | | |
|---|---|---|
| $LD(C)$<br>$C = D - L$ | Margaret died in Dec., 1995 (D) with a completed lifespan of 96 (L), putting her birth year in 1900 (C). |  |
| $LC(D)$<br>$D = C + L$ | Àngels was born in 1940 (C) and she lived to be 64 (L), implying an untimely death in 2004 (D). |  |

<div align="center">THE UNINFORMATIVE DYADS</div>

| | | |
|---|---|---|
| $LP(-)$ | The $LP$ plane is *non-informative*. No additional dimensions can be derived knowing just lifespan and period. |  |
| $CT(-)$ | The $CT$ plane is *non-informative*. No additional dimensions can be derived from birth cohort and thanatological age. |  |
| $AD(-)$ | The $AD$ plane is *non-informative*. No additional dimensions can be derived from death cohort and chronological age. |  |

Most of what we know about how rates change over age and time comes from the very first juxtaposition in Table 2, $AP(C)$. While $CP(A)$ and $AC(P)$ are statistically redundant, they are not fully redundant in terms of perception and data visualization if using Euclidean coordinates, as demographers typically do. The other dyadic juxtapositions can be considered as either rare or novel ways of structuring or viewing temporal variation in demography.

## The triad identities

Visualizations of data structured by any dyad belonging to a triad identity are inherently richer in information than juxtapositions of uninformative dyads. There is no reason not to explore all possible dyadic juxtapositions, but the triad identities have more apparent meaning, even in the absence of

data, due to the underlying relationship between measures. Each of the triad identities can represent some version of a lifeline. In the present section, we therefore lay out the four primary diagrams that belong to the triad identities.

## APC

The so-called Lexis diagram has long been used in demography as a conceptual tool for structuring data, observations, and rate estimation, as inspiration for work on statistical identification, and as the coordinate basis of contemporary Lexis-surfaces. Since the Lexis diagram could have been named for others (Vandeschrick 2001, Keiding 2011), and since we compare with other temporal configurations, let us refer to it as the APC diagram, as seen in Figures 1a and 1b.

The APC diagram in Figure 1a represents years lived on the y axis, calendar years on the x axis, and birth cohorts as the right-ascending diagonals. This is the most common of several possible configurations of the APC dimensions. Individual lifelines are aligned in the birth cohort direction, starting with birth (filled circle) at chronological age zero, and death (circled x). Any APC surface can be interpreted along each of these three dimensions of temporal structure.

Figure 1: An APC diagram in two projections.

(a) Right angles



(b) Isotropic
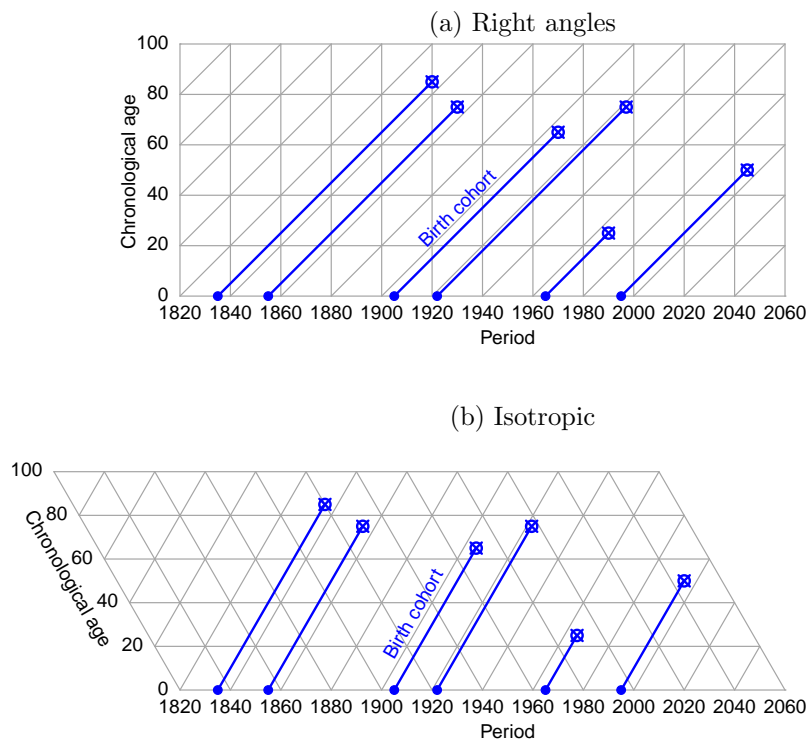
## TPD

Thanatological age (T), period (P) and death cohort (D) form a coordinate system best imagined as the inverse of APC. One may take the same lifelines from Figure 1 and realign them in descending fashion to create the diagram in Figure 2. Such diagrams have to our knowledge only appeared once in the literature, as a visual aid to a discrete mathematical proof of the Brouard-Carey equality (Villavicencio and Riffe 2015). TPD coordinates may in general be used to arrange events or durations that are logically aligned (or may only be aligned) by time of termination, and in general any situation in which a terminal event predicts preceding patterns of variation. Examples may include lifelines preceding deaths from infectious or acquired conditions where the time of infection or acquisition is unknown.

Figure 2: A TPD diagram in two projections.

(a) Right angles

(b) Isotropic
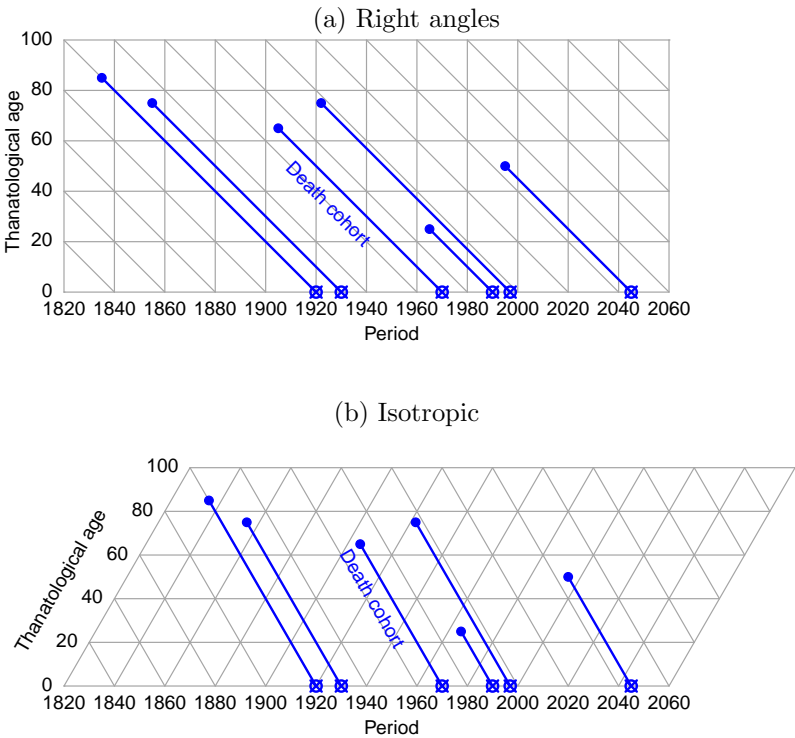
## TAL

TAL is an appropriate coordinate system to examine variation in processes over the lifecourse. Since the lifecourse belongs to the cohort perspective, it is best to think of the TAL plane as belonging to some particular birth cohort. Alternatively, an TAL triangle may be taken as a cross-section along through the period dimension, a sort of synthetic TAL plane. To our knowledge, the TAL diagram has only appeared once in the literature, in an exploration and classification of late-life health conditions (Riffe, T. et al. 2015). The TAL diagram in Figure 3 contains no such indication of period or cohorts. The lifelines depicted are identical those shown in APC Figure 1 and TPD Figure 2.[3] In this view, the user can juxtapose some intensities that vary over the lifecourse with respect to years lived and years left, and separated by lifespan, but time trends are blended out, and cohorts overlapped.

Figure 3: A TAL diagram in two projections.

(a) Right angles              (b) Isotropic
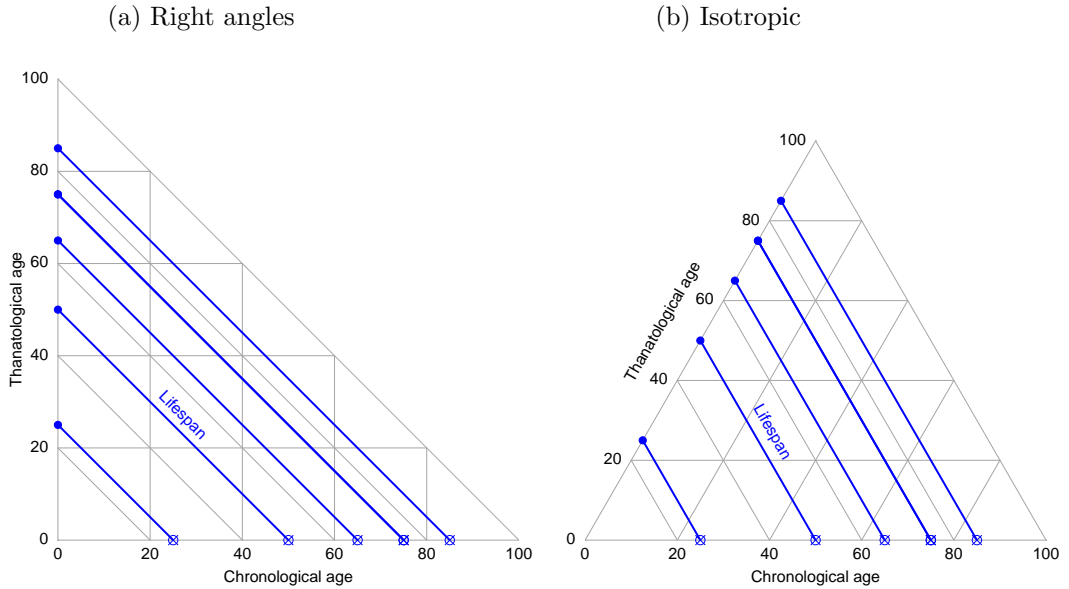


## LCD

*section in progress, will contain 2 figures, like the above* This diagram completes the tetrahedron. Here is an example figure, followed by some speculation that will be replaced in the final paper version. An LCD diagram

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[3]The prior figures contained six lifelines each, but since two of them were of equal length (75), they are overlapped in Figure 3 and appear to be five.

Figure 4: An LCD diagram in right angles
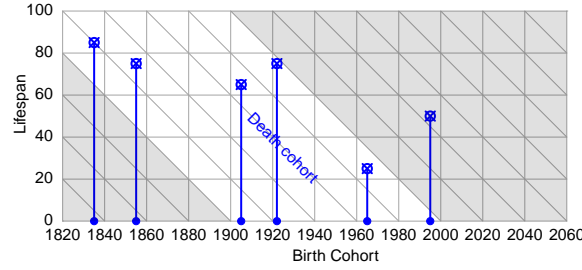


is probably best conceived without the lifelines. It is probably best to plot values that are cumulative over the lifecourse, assigned to the endpoint of the lifespan. Imagine an LCD surface of cumulative lifecourse deficit *a la* NTA, or anything that might vary by lifespan and over time, but where age is. In a sense, mortality Lexis surfaces already conform with this perspective, since the intensities that get plotted mark the endpoints of lifelines. Fertility surfaces are in this way very different from mortality surfaces, as they plot event intensities over the ages in which they occur. A health surface is more like a fertility surface than a mortality surface, but in this case proximity to death is an important time dimension.

## A tetrahedron relates the six time indices.

Each of the four above-mentioned triad identities may be thought of as a two-dimensional plane fully defined by any two of its three constituent time indices. In this case, we may imagine any of the excluded time measures as capable of providing depth, a potential $z-$coordinate, for the sake of a mental image. Having a non-redundant third dimension implies a multitude of parallel planes for the given triad identity, each plane belonging to a unique value of the third time dimension. Any of the identities can be extended in this way to fill a space. A space derived by extending any of the triad identities into its lacking dimension implies each of the other triad identities, making a total of six time indices. In essence, the four triad identities may be thought of as the four faces of a tetrahedron. If an additional time measure is added to any face (triad identity), the six demographic time indices can be derived, matching the six edges of the tetrahedron. This three-dimensional construct unifies the six indices of demographic time, and is the subject of

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this paper.

Let us first more rigorously define the previously-mentioned tetrahedron. Luckily, the edges and vertices of a tetrahedron are easily rendered in a two-dimensional graph, as seen in Figure 5, with vertices labeled in red and the six time indices labeled as blue edges. The reader may also imagine this graph as a transparent 3d object, in which case the four faces become apparent. There are two intuitive ways to imagine the graph as 3d, either the vertex 4 is on top, and we gaze from a bird's-eye-view, or the vertex 4 is in the back, behind the other three vertices. Assume we gaze from the top, for the sake of description.[4]

Figure 5: Graph of tetrahedron, with edges labeled by the six demographic time indices.
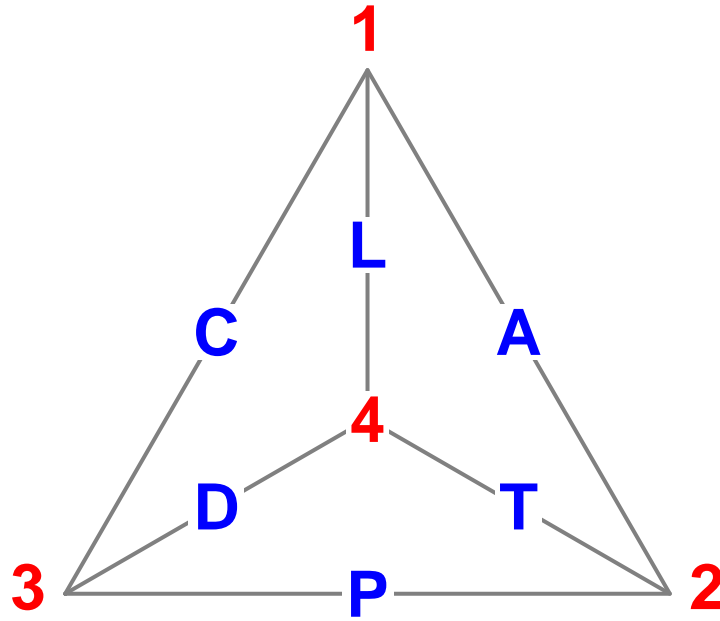


### Information criteria to derive the tetrahedron.

The edges APC at the base define the much-studied APC plane. If the only information we have is chronological age, period, and birth cohort (or just two of these), then we have no access to the vertex 4. Each of the faces of the

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[4]The same graph could be composed in four basic ways, depending on which triad-identity face forms the base. These are given in an appendix.

tetrahedron has this quality. The South face TDP has no access to 1. The Northeast face, ATL has no connection to 3, and the Northwest face CDL lacks a connection to 2. The triads that make up the faces of the tetrahedron are stuck in "flatland". However, there are $\binom{6}{3} = 20$ ways to choose three time indices out of six, and the four above-named triads are the only ones that will not yield the full 3d space. The sixteen other combinations of three indices will recreate the full tetradhedron (hexad identity).

A geometrical analogy is pertinent at this point. Any pair of intersecting edges of the tetrahedron may be interpreted as two vectors $\vec{u}$ and $\vec{v}$ that determine a 2-dimensional plane in a 3-dimensional space.[5] Therefore, any third vector $\vec{w}$ of that plane can be expressed as a linear combination of $\vec{u}$ and $\vec{v}$ (formally, $\vec{w} = \alpha\vec{u} + \beta\vec{v}$ for some $\alpha, \beta \in \mathbb{R}$), which is usually described by saying that $\vec{w}$ is linearly dependent on $\vec{u}$ and $\vec{v}$. A similar property can be derived from the information contained in the tetrahedron: Say we have information on year of birth $C$ and period $P$. Clearly, $C$ and $P$ are sufficient to determine the chronological age $A$, given that $A = P - C$. That is, $A$ is a linear combination of $C$ and $P$ and it "depends" on them because they all belong to the same $APC$ plane (see Figure 6). Analogously, $P = C + A$ "depends" on $C$ and $A$, and $C = P - A$ "depends" on $P$ and $A$. It can be easily verified that any pair of intersecting edges have the same property: The third edge located in the same face of the tetrahedron can be determined by the first two by a simple linear relationship.

Once a 2d-plane is defined, an additional vector may be sufficient to cover the whole 3d-space. Nonetheless, this third vector needs to be linearly independent of any pair of vectors of the 2d plane—that is, it cannot be expressed as a linear combination of any two vectors on that plane. Again, an analogous property can be observed in the tetrahedron: Say we only have information about the indices of the $APC$ plain; $A$, $C$ and $P$ are not sufficient to determine a thanatological age $T$, death cohort $D$, or lifespan $L$ (the three indices that do not belong to the $APC$ plane). So, $T$, $D$ and $L$ are "independent" of the overall information that can be extracted from the $APC$ plane. However, if two of the three constituent time indices of the $APC$ plane are known (the third one would be unnecessary as it could be derived from the other two), the additional information provided by any of the three "independent" indices $T$, $D$ or $L$ would be sufficient to cover the whole tetrahedron. For example, suppose we have information about thanatological age $T$ in addition to $C$ and $P$, then $A = P - C$, $D = P + T$ and $L = T + A = T + P - C$.

Hence, as with vectors in a 3d space, any triad of indices that are independent of each other—that is, none of them can be expressed as the

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[5]A 2d plane in a 3d space is determined by two linearly independent vectors (with different direction) and a point, but the inclusion of a point is not necessary for the intuitive analogy that we describe here.

Figure 6: Graph of tetrahedron, edges belomging to the APC face highlighted.
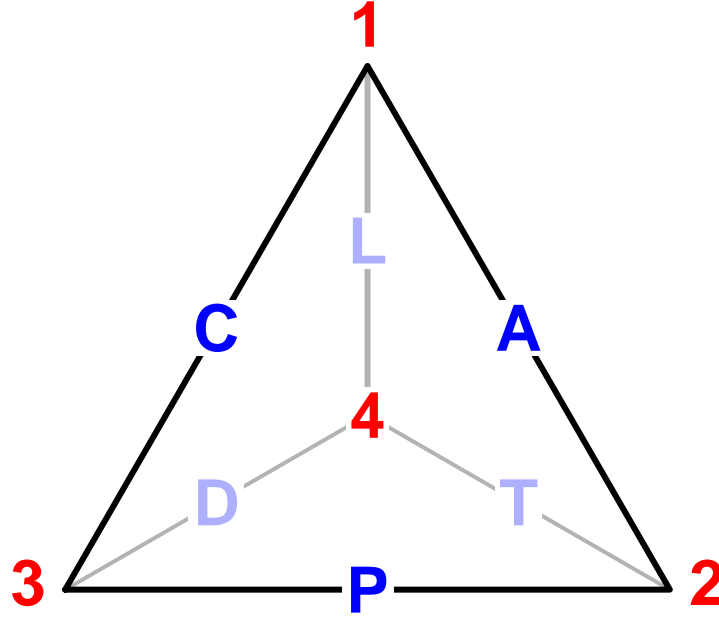


sum or the difference of the other two—generates a full hexad identity or, using an analogous terminology, covers the whole "space" of demographic indices presented in this paper. Graphically, this is equivalent to choosing any combination of three indices that do not belong to the same face of the tetrahedron, i.e., that do not form one of the four triad identities represented in Table 3.

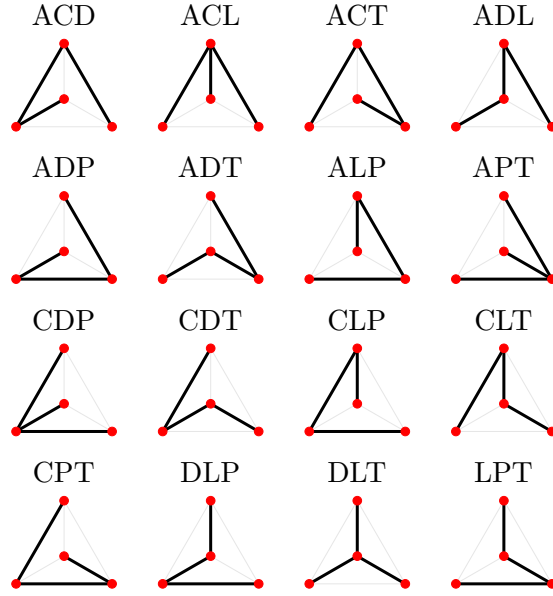Table 3: The four triad identities on the tetrahedron (same orientation)
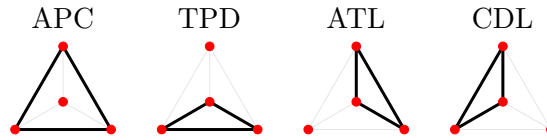


When graphed in this way, it is clearer that every triad identity surface lacks a connection to the opposite vertex of the tetrahedron. We therefore say that each of the triad identities is incomplete. Table 4 gives the full set of sixteen index-triads that are complete in this sense. It can be verified that each of these triads implies the full hexad identity.

12 different pairs of time indices generate one of the four triad identities in Table 3(and three do not), but no dyad will generate the hexad identity since a third "independent" time dimension is necessary. Similarly, any quad of indices is sufficient to complete the hexad identity, as at least one of them will not belong to the same face of the tetrahedron, but a triad may be sufficient if they do not all belong to the same plane forming a triad identity.

Table 4: All complete sets of three indices, graphed given the previous orientation of the tetrahedron.



## The extension of time axes.

We have said that planes defined by the four triad identities are parallel to the faces of the the above-described tetrahedron. In imagining this three-dimensional relationship, we are no longer confined to the extent of the tetrahedron used thus far for orientation. Instead each of its edges extends a certain distance in either direction. It may therefore help to first consider the extension of each axis (or index). Some indices have a lower bound of zero and an upper bound set by the maximum length of life, $\omega$, while others are boundless. A, T, and L are clearly in the range $[0, \omega]$.[6] P, C, and D are bounded only by the inception and extinction of our species, but may be thought of as boundless for practicality, or benchmarked to our earliest

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[6]It's best to imagine some number like 122.45 years, for $\omega$, rather than infinity. This is the longevity record at the time of this writing. Jeanne L. Calment would have had $T = 122.45$ at birth, $A = 122.45$ at death, and $L = 122.45$ for her entire life.

and most recent observations for even more practicality.[7] As an abstraction, however, the dimension of calendar time in this model is infinite. Of the four triad identities, only one lacks an unbounded dimension, the TAL. Adding the absent dimension to TAL therefore makes its 3d extension boundless. In this way, we may imagine a prism-like construct, where T, A, and L, compose the faces of a triangular cross-section of the prism, which extends infinitely "through" the triangle. We can think of the TAL triangle passing through time, extending the population forward to infinity. In this case, the TAL triangle may take either the period or cohort perspective, and this will be explained later.

There are also numerous ways that this three dimensional construct can be proportioned, of which we present two in this paper. The first stems from the respect given to right angles in the most common representation of the Lexis diagram. For this reason, it will likely be the most intuitive rendition of the model, and it will be presented first. The second version presented is isotropic with respect to time units in each of the six temporal indices. In this case, the four tripartite identities are based on equilateral triangles between their three constituent indices, and the four planes are joined together such that each is parallel to a face from the regular tetrahedron, a construct known in geometry as an octahedral-tetrahedral honeycomb.

## Intersecting planes

The APC, TPD, TAL, and CDL planes can be conceived of either as *compressions* of this 3d space, or as cross-sections of the 3d space. To compress the space in this sense is to ignore the missing dimension, whereas a cross-section sets a given triad identity against a particular position of the absent dimension. APC has thus far always been treated as a compression in this sense. A compressed TPD diagram has thus far only appeared in Villavicencio and Riffe (2015) as an aid in proving the symmetries between chronological and thanatological age in stationary populations, and a cross-sectional one has never appeared. Cross-sectional TAL diagram and surfaces have thus far only appeared in Riffe, T. et al. (2015). This TAL usage was selected for the 1915-1919 birth cohort, and therefore belongs to the 3d space, ⟁. We have been unable to locate an example in the literature of a compressed TAL diagram, but it seems plausible one will have arisen in the field of biology, albeit with no relation to the present discourse. We suppose

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[7]We explain the choice of the word "benchmarked". Say we have a data series that runs from 1751 to 2011, and an upper age interval of 110+. Then we could say that P is in the range $[1751, 2011]$, but by another reading, P must range from at least as early as the earliest C and until at least as late as the latest D. Someone dying at 110 in 1751 had a C of 1640, and an infant born in 2011 that is destined to live to 110 will die in 2121. In this case a P that *contains* the observed population will extend well before and after the observed data series, even moreso if we take into account that $\omega > 110$.

that CDL diagrams of any kind are yet-unknown.

A simultaneous juxtaposition of the six demographic time measures has also never appeared in the literature, either as a visual diagnostic or in any line of inquiry or argumentation, although there are some novel temporal juxtapositions in the literature on event analysis in the analysis of big data (Watson 2015, see e.g.,).

## APCTDL

*section under construction* The final version of this section will contain different three-dimensional views of the tetrahedron with particular plan accentuated to match each of the four triad identities. These views will be in an expository table, similar to those above. The figures will probably by vector snapshots of an RGL model. Probably each will be centered on a single tetrahedron, with sides labeled, and one face of it (triad identity) will be highlighted and extended, and given depth). Each face will be seen from 2 angles, and also for the right and isotropic projections. use lighting, etc, to give depth.

A working mockup in WebGL can be seen here:[8] `http://www.demog.berkeley.edu/~triffe/RGL1` , though the snapshots in the final paper will be notably simpler.

The complete space can be named using geometry nomenclature as the tetrahedral-octahedral honeycomb, which is a kind of space-filling tessellation.[9] This three dimensional space is not only useful for the sake of formalizing observed temporal relationships, but also for enclosing demographic time in the past and future (e.g., before the first census and after the most recent census).

A property of the geometry that we propose is isotropy, the time units in every direction (with respect to each index) are proportional. In Lexis-nomenclature, the 3d projections of an AP square, and AC or PC parallelograms are all congruent shapes known as regular trigonal trapezohedra (RTT). The orientation of a given RTT uniquely defines the Lexis shape in question. Similar constructs exist in the other time dimensions.

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[8]This graphic is best viewed on a large screen, in a browser with javascript enabled, and manipulated with a mouse that includes a wheel.

[9]Constructs following this geometry exist both in nature and in man-made structures.

Figure 7: A mock-up example of the unified model of demographic time.
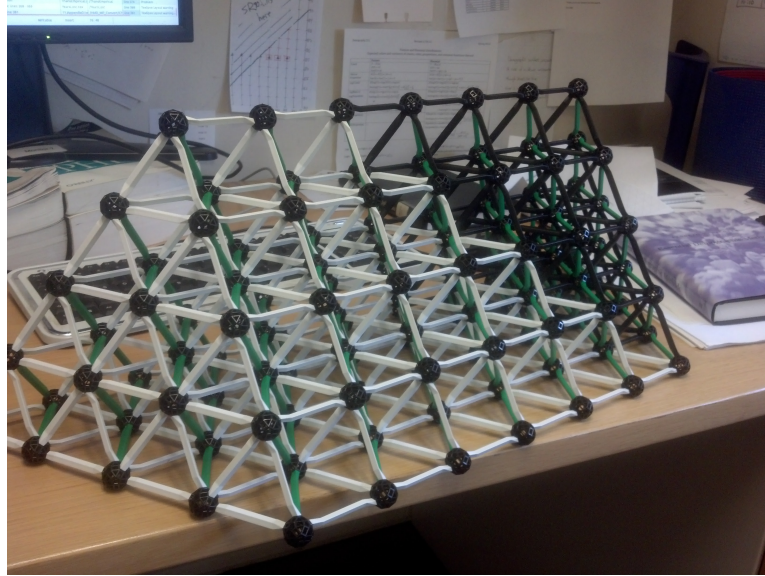


## Discussion

The contemporary practice of demography is based on the premise that vital rates, and other kinds of rates over the lifecourse, are the truest measure of demographic quantum. Rates are paramount because they tend to vary in empirically regular ways. The scalings and movements of primary vital rates fall within a limited range for humans. For this reason, many of the methods of demography are developed to estimate rates, independent of population composition, or to partition crude magnitudes into the effects of population age structure and pure vital rates. Controlling for age is in a more general sense controlling for temporal variation in stocks.

The best way to seek regular patterns in variation is via data visualization. The coordinate system proposed in this paper is conceived as one adequate to capture all such variation, and we suggest its use for visualizing data, probably via small multiples of successive time slices parallel to any of the four triad identities. Such strategies at this time are exploratory and speculative, but much may be learned from how quantities that vary through 3d volumes are visualized in other disciplines, such as neuroscience or geology.

The techniques used to age-standardize mortality and fertility estimation are at times applied to other kinds of transitions over the life course. For example, one may estimate an age pattern to some degenerative disease, or the ability to carry out some common activities of daily life. Much of the regular temporal variation for such conditions is by thanatological age or lifespan, rather than chronological age. Apparent chronological age patterns for such

conditions are artifactual and do not represent the same kind of intrinsic meaning as does the age pattern of mortality. Further kinds of temporal standardization must be developed in order to measure and understand the natural patterns of such conditions over the lifecourse. The measurement of such conditions may benefit from consideration of the complete model presented in this paper. To this end, Table 4 provides all combinations of information that are sufficient to derive the full space. Some panel surveys with mortality followups already provide the requisite information, as do linkable registers that include items such as health measures or proxies and relevant dates of birth, observation, and death.

Mortality determines three of the dimensions of demographic time, and it therefore makes little sense to model mortality using all six time measures. Any of the six measures may be pertinent in the case of conditions and states that vary over and within the lifecourse. Fertility is one such within-lifecourse event, although it may still turn out that most important variation in fertility is captured by APC. This remains to be seen until adequate data is brought to bear on the question. The most obvious application for the present model, given data commonly (and publicly) available at this time, are late-life health conditions, although there may be other substantive areas of application that have not occurred to us.

## A    Variants of tetrahedron graph

The graph depicted in Figure 5 could be drawn with any of the four vertices in the middle of the triangle (as well as other inversions and rotations). These would all serve equally well to present the same aspects of the model, and we have no insight as to whether one of these renditions is more or less intuitive. Figure 8 provides four perspectives on the tetrahedron, for the case that this aids in understanding. The reader may make a paper tetrahedron, with labeled edges and vertices to be convinced that these are identical graphs.
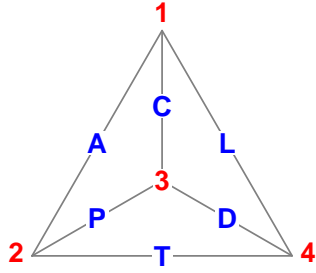
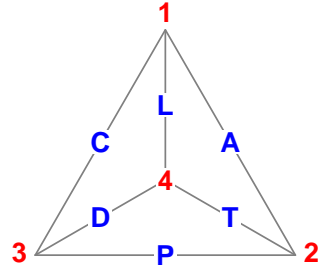Figure 8: Some variants of the graph of the APCTDL tetrahedron.
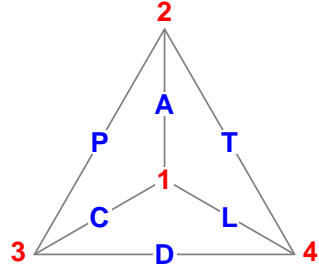


(a) Vertex 1 in middle. APC North-west.
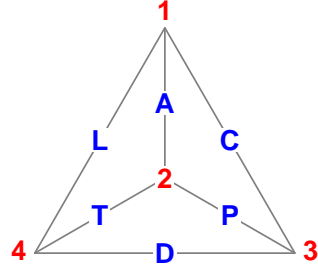
(b) Vertex 2 in middle. APC Northeast.

(c) Vertex 3 in middle. APC Northwest.

(d) Vertex 4 in middle, as in Figure 5. APC base.

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