A unified framework of demographic time

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Abstract

Demographic thought and practice is largely conditioned by the Lexis diagram, a two-dimensional graphical representation of the identity between age, period, and birth cohort. This relationship does not account for remaining years of life or other related time measures, whose use in demographic research is both underrepresented and incompletely situated. We describe a three-dimensional relationship between six different measures of demographic time: chronological age, time to death, lifespan, time of birth, time of death, and period. We describe four identities among subsets of these six measures, and a full identity that relates the six of them. One of these identities is the age-period-cohort identity, while the others are relatively novel. We provide a topological overview of the diagrams that pertain to these identities. The 3-d geometric representation of the full six-way identity is proposed as a coordinate system that fully describes temporal variation in demographic data. We offer this framework as an instrument to enable the discovery of yet-undescribed relationships and patterns in formal and empirical demography.

Keywords. Age structure, formal demography, data visualization, age period cohort.

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Introduction

In the course of training, all demographers are introduced to the Lexis diagram, a convenient graphical identity between the three main time measures used to structure demographic stocks and flows: Age, period, and birth cohort. This popular representation does not account for remaining years of life and other related time indices that may be of interest to researchers and policy makers.

We wish to draw attention to three time indices that are complementary to age (A), period (P) and birth cohort (C). The first such index is time-to-death, which we refer to as "thanatological age" (T) in contrast to "chronological age" (A). The second index is death cohort (D), which groups all individuals (of different ages) dying in the same time period. Finally, lifespan (L) or age-at-death itself is an index by which data may be structured. We therefore have six time measures in total to relate. We call these measures of demographic time because each, except period, depends on the timing of birth, death, or both.

Just as the Lexis diagram has been a fundamental instrument to teach demography for decades, we hope that the demographic time measures and their graphical depictions presented here will be helpful to teachers and young demographers. The temporal relationships we describe will also be useful for researchers to better understand the temporal structure of data, and for methodologists to better account for the temporal structure of data in demographic methods.

The Lexis diagram can be understood as an APC plane that relates age (A), period (P), and birth cohort (C). Other such planes are also identifiable. The "thanatological" counterpart to APC is an identity between thanatological age (T), period (P), and death cohort (D), TPD. A third identity relates thanatological age (T), chronological age (A), and lifespan (L), TAL. Finally, a potentially less intuitive graphical identity relates lifespan (L), birth cohort (C), and death cohort (D), LCD. We call three-way identities of this sort "triad identities".

Each of these four triad identities (APC, TPD, TAL, and LCD) is sufficiently described by any two of its constituent indices, making the third index redundant. For instance, if the exact age (A) of an individual at a particular time (P) is known, the birth cohort (C) to which he or she belongs can be immediately derived. Each of these four identities also lacks a major dimension of time. The TAL identity lacks calendar time, the LCD identity is ageless, APC lacks an endpoint in time, and TPD lacks a starting point in time. To our knowledge, the only triad identity that has received serious treatment at the time of this writing is the APC identity. Different aspects of the APC identity have been discussed since at least 1868 (Knapp 1868), and discussion remains lively today. Here we relate the six major indices of time in a geometric identity, in much the same spirit as the work on APC relationships done between the late 1860s and mid 1880s.[1]

Our goal is to describe the geometric identity between all six measures of demographic time, a hexad identity, that may be useful or an intuitive referent for demographers in the same way as the Lexis diagram. At the same time, this identity relates the four triad identities we have mentioned. We give a bottom-up description of how the six dimensions of time relate in a single framework, building from familiar components to the full relationship.

[1]See e.g., Keiding (2011) for an overview of that literature.

We begin by defining some terms used throughout the manuscript. We then explore all combinations of two time measures, the dyadic relationships, followed by the four triad identities, and finally the hexad identity. We give a systematic topological overview of the different elements of demographic time.

Definitions

Technical terminology

In describing this relationship we attempt to adhere to a rigorous terminology. The following list describes some of the more important terms we use.

Time measures are any of the six time indices discussed to describe demographic time: chronological age (A), period (P), birth cohort (C), thanatological age (T), lifespan (L), and death cohort (D).

Dyads, triads, and hexads are any set of two, three, or six unique time measures, respectively.

An informative dyad is any pair of two time measures from which a third time measure can be derived. For example, A and P form an informative dyad from which C can be derived.

A triad identity is the union of an informative dyad and its one derived time measure. Analogously, a triad identity is the union of three different time measures, with the property that any of them can be derived from the other two with no additional information. There are four triad identities: APC, TPD, TAL, and LCD.

A temporal plane is any (x, y)-mapping of a dyad of time measures.

Using this terminology, we say that the "Lexis" measures constitute a triad identity between chronological age, period, and birth cohort. Each dyad combination of elements in this identity can be mapped to a temporal plane, the Lexis diagram. If we know that Mindel turned 50 on the 21st of May, 1963, then we also also can derive that she was born on the 21st of May, 1913. Hence, any two pieces of information in this case will give the third, which means that any dyad from this set is informative. The same holds for the other triad identities.

Time measures

Our model description is conceived in absolute, linear, Newtonian time. We describe time in terms of years, the dominant time scale for human demography, although all relationships are scalable to any time unit. We therefore speak of calendar time, imagining the modern Gregorian calendar. We also describe the framework in terms of human lifespans, although it applies in a more general sense to any durations observed over time. This is to say, birth may be translated to entry, and death to exit, or any other terminal state. The six measures

Table 1: Definitions of the six time measures.

Time measure	**Short**	**Demographic def.**	**Event history def.**
chronological age	A	Time since birth	Time since start of exposure
period	P	calendar time	calendar time
birth cohort	C	calendar time of birth	calendar time of exposure start
thanatological age	T	time until death	time until event
death cohort	D	calendar time of death	calendar time of event
lifespan	L	duration of life	duration of exposure

of time we consider are defined in Table 1, both in the demographic sense we describe, as well as in a more general event history interpretation.

The concepts of thanatological age and death cohorts are likely less famialar to readers than the other measures we consider. Thanatological age is sometimes referred to in the literature as remaining years of life, time-to-death, prospective age (Sanderson and Scherbov 2007), or residual lifespan, but we prefer the term thanatological age (Riffe 2015). Age in this general sense marks a position on a lifeline with respect to one of its endpoints. Chronological age and thanatological age are in this way complementary. Thanatological age is meaningful without much justification: It is the measure we all want to know, the thing we approximate with remaining life expectancy.

Cohorts in general associate individuals that share a characteristic. In demography the grouping characteristic is often a combination of place and time, such as a cohort of young demographers passing through a particular graduate program. In this instance already, we accommodate the notion of a cohort for both the start and endpoints of the program, saying for example, “the class of 2015” instead of the “graduating cohort of 2015”, in contrast to “cohort 37”, the 37th class of entering students since the start of the program. These concepts are analogous to the ideas of birth and death cohorts we use here, but we do not often refer to the deaths of a given year as a death cohort. In the time preceding death, the members of a given death cohort have much in common, despite heterogeneity with respect to time of birth.[2] If the reader accepts this premise, then the abstract construct of a death cohort is also meaningful in the way that other cohort measures are.

Much of the work of demography is directed at the study of lifespan. Lifespan is synonymous both with longevity, chronological age at death, and thanatological age at birth. One’s ultimate completed lifespan is constant throughout life, though we have no knowledge of it until death: It is assigned retrospectively. Demographers have more often used lifespan or age-at-death as a measure of mortality, or similar, than as a measure on which to compare individuals or structure data.

Treating lifespan, death cohorts, and thanatological age as temporal structuring variables enables new classes of comparisons, models of understanding, and discovery, akin to those unlocked by breaking down demographic phenomena by chronological age, period, and birth cohort. The following sections, in this sense, provide an exhaustive classification of the ways in which these six measures of time can be juxtaposed to such ends.

[2]Death cohorts lack a shared identity, so any kind of emergent homogeneity in a death cohort probably has a physiological basis.

Informative and uninformative dyads

Any mapping of two time measures to an (x, y) coordinate system constitutes a temporal plane. If the two given time measures are members of the same triad identity, the third member is a derived measure. For instance, if we take the dyad AP, C is the derived measure, making AP an informative dyad. We make this dyadic relationship explicit by writing AP(C). The temporal plane that corresponds to this informative dyad is the contemporary representation of the Lexis diagram (Lexis 1875). The informative dyads AC(P) and CP(A) also belong to the Lexis identity, but imply different less-common rotations and projections of the Lexis diagram.

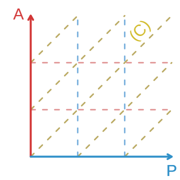
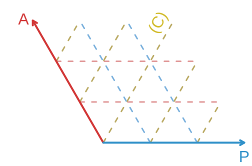
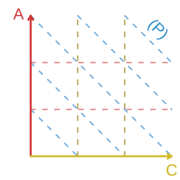
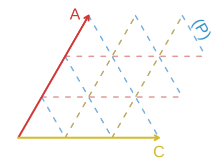
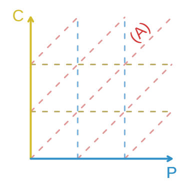
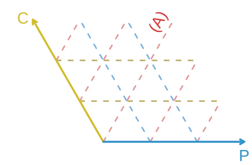
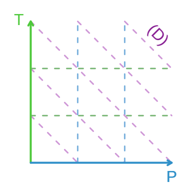
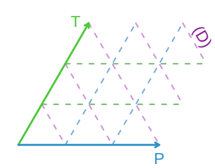
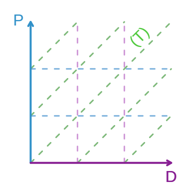
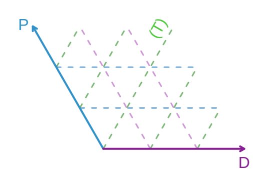
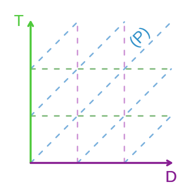
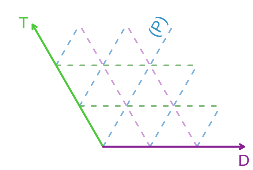
For each dyad there is a fundamental question of how to map the constituent coordinates to a Cartesian temporal plane. Typically one forces parity between time units within a specified dyad, and maps one element directly to x and the second element directly to y, resulting in a 90° angle between the x and y axes. In this case the convention is to force a unity aspect ratio between the x and y axes, such that the derived measure, if any, is then *accidentally* present in a 45° ascending or descending angle, depending on the dyad and axis orientation.

It has long been noted (Zeuner 1869, Perozzo 1880) that the derived time measure (usually birth cohort) is longer than either the age or period axes when plotted at 45°. If a right angle and unity aspect ratio is forced between the dyad, the derived measure is always stretched by $\sqrt{2}$, or 41%. In the case of dyads that imply a derived measure, another logical mapping is to translate to (x, y) coordinates that forces 60° angles between the three measures. Such a mapping ensures that the spatial units are equal for the three measures, and we therefore refer to it as the isotropic mapping. The isotropic mapping is comparable to using ternary or barycentric coordinate systems. Under this representation, the three variants of each triad identity are simple rotations of one another, and they imply no rescaling.

There are $15 = \binom{6}{2}$ possible ways to form a dyad from our set of six time measures, 12 of which are informative, and three of which have no derived time measure, and are therefore called uninformative. These dyads, an explanation or simple example, and the corresponding graphical representations are summarized in Table 2. Each informative dyad is a subset consisting of two elements from one of the four triad identities (APC, TPD, TAL, LCD), which we analyze in detail in further sections. The uninformative dyads are simply pairs of time measures that are not contained in any of these four triad identities.

Table 2: All dyadic juxtapositions of the six measures of demographic time.

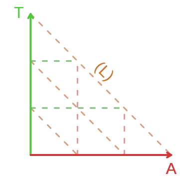
Note: The temporal planes are named after the two given time scales. The derived scale is appended in parentheses. Contrary to mathematical convention we name the ordinate scale first and the abscissa scale second. This is to be consistent with the established *APC* and *ACP* terms.

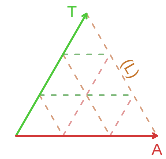
<table>
<tr><th>Relationship</th><th>Description</th><th>Cartesian</th><th>Isotropic</th></tr>
<tr><td colspan="4">VARIANTS OF APC</td></tr>
<tr><td>$AP(C)$
$C = P - A$</td><td>The $AP(C)$ temporal plane constitutes the classical Lexis diagram.</td><td>
</td><td>
</td></tr>
<tr><td>$AC(P)$
$P = C + A$</td><td>The $AC(P)$ temporal plane is equivalent to the Lexis diagram except birth cohort is given and period is derived rather than the other way around.</td><td>
</td><td>
</td></tr>
<tr><td>$CP(A)$
$A = P - C$</td><td>The $CP(A)$ temporal plane is equivalent to the Lexis diagram except birth cohorts are given and age is derived rather than the other way around.</td><td>
</td><td>
</td></tr>
<tr><td colspan="4">VARIANTS OF TPD</td></tr>
<tr><td>$TP(D)$
$D = P + T$</td><td>Helen had 30 years of life left (T) in 1971 (P) and therefore belonged to the 2001 death cohort (D)</td><td>
</td><td>
</td></tr>
<tr><td>$PD(T)$
$T = D - P$</td><td>Mindel died in 1973 (D). In 1953 (P) she had 20 years left to live (T).</td><td>
</td><td>
</td></tr>
<tr><td>$TD(P)$
$P = D - T$</td><td>Irene died in 1974 (D). When she had 30 remaining years of life (T) the year must have been 1944 (P).</td><td>
</td><td>
</td></tr>
<tr><td colspan="4">VARIANTS OF TAL</td></tr>
</table>

$TA(L)$

$L = T + A$

The time already lived and the time still left sum up to the total lifespan.

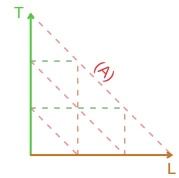


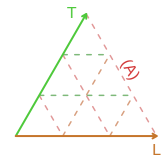


$TL(A)$

$A = L - T$

Helen lived to the age of 86 (L). When she had 20 years left (T) she must have been 66 (A).

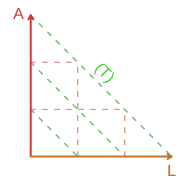


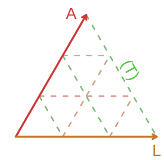


$AL(T)$

$T = A - L$

Tim is 34 years old (A) and will live to the age of 96 (L), leaving him 62 years (T) to settle affairs.



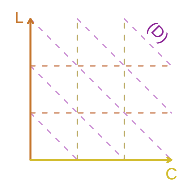


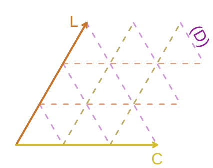
Variants of LCD

$LC(D)$

$D = C + L$

ngels was born in 1940 (C) and she lived to be 64 (L), implying an untimely death in 2004 (D)

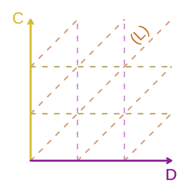


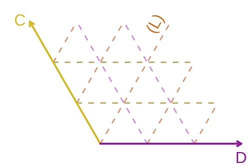


$CD(L)$

$L = D - C$

Pascal was born in 1893 (C) and died in 1964 (D), implying a lifespan of 71 (L), or so.

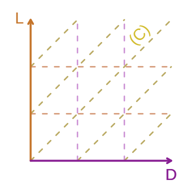


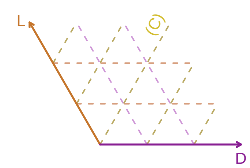


$LD(C)$

$C = D - L$

Margaret died in Dec., 1995 (D) with a completed lifespan of 96 (L), putting her birth year in 1900 (C).

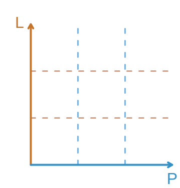




The Uninformative Dyads

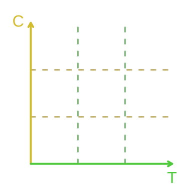
LP(-)

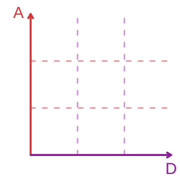
The LP plane is *non-informative*. No additional measures can be derived knowing just lifespan and period.



CT(-)

The CT plane is *non-informative*. No additional measures can be derived knowing just cohort and thanatological age.



AD(-)	The AD plane is *non-informative*. No additional measures can be derived knowing just death cohort and age.	

Most of what we know about how rates change over age and time comes from the very first juxtaposition in Table 2, AP(C). While CP(A) and AC(P) are statistically redundant, they are not fully redundant in terms of geometric mappings if using Cartesian coordinates, as demographers typically do. The other dyadic juxtapositions can be considered as either rare or novel ways of structuring or viewing temporal variation in demography.

The triad identities

There are $20 = \binom{6}{3}$ ways to choose three time indices out of six, of which four form a triad identity: APC, TPD, TAL, and LCD. Given the three time measures from any of the triad identities, one can derive no further time measures. If one selects three random time indices that do not form any of these four triad identities ($20 - 4 = 16$ possibilities), this property does not hold. For instance, in the triad APT, age and period are not sufficient to determine thanatological age. Given the triad APT one can however derive the remaining three time measures.[3]

While there is no reason not to visualize all possible dyadic juxtapositions, triad identities have more apparent meaning, even in the absence of data, due to the underlying relationship between measures. Each of the triad identities can accommodate some version of a lifeline, for instance. In the following, we therefore lay out the four primary diagrams that belong to the triad identities.

APC

The so-called Lexis diagram has long been used in demography as a conceptual tool for structuring data, observations, and rate estimation, as inspiration for work on statistical identification, and as the coordinate basis of contemporary Lexis-surfaces [4]. Since the Lexis diagram could have been named for others (Keiding 2011, Vandeschrick 2001), and since we compare with other temporal configurations, let us refer to it as the APC diagram, as seen in Figs. 1a and 1b.

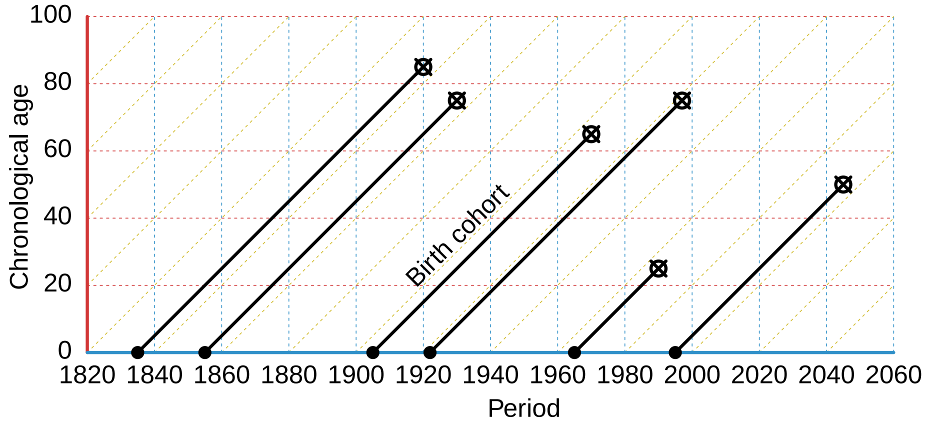
The APC diagram in Figure 1a represents years lived on the y axis, calendar years on the x axis, and birth cohorts as the right-ascending diagonals. This is the most common of several possible configurations of the APC dimensions. Individual lifelines (black) are aligned in the birth cohort direction, starting with birth (filled circle) at chronological age zero, and death (circled x). Any APC surface can be interpreted along each of these three dimensions of temporal structure.

[3]We return to the case of APT and similar constructs in later sections.

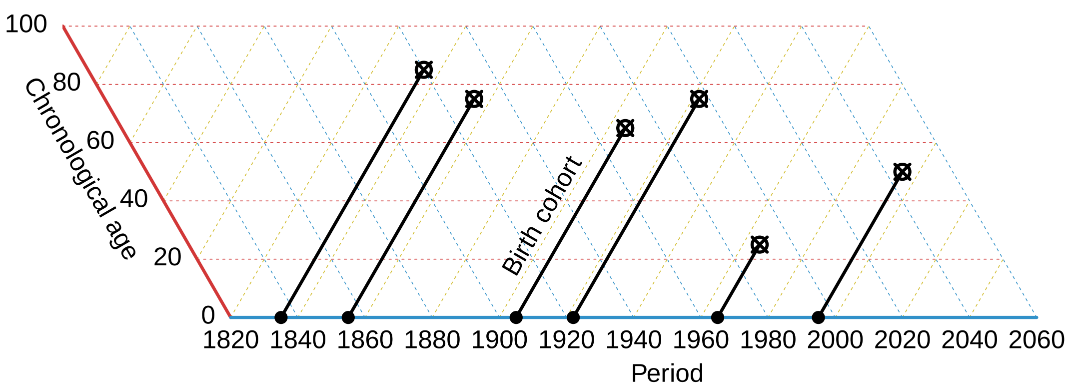
[4]Some prefer the term Lexis *surface*, while others prefer to call them contour maps, heatmaps, or stereograms

Figure 1: An APC diagram in two projections.

(a) Right angles



(b) Isotropic

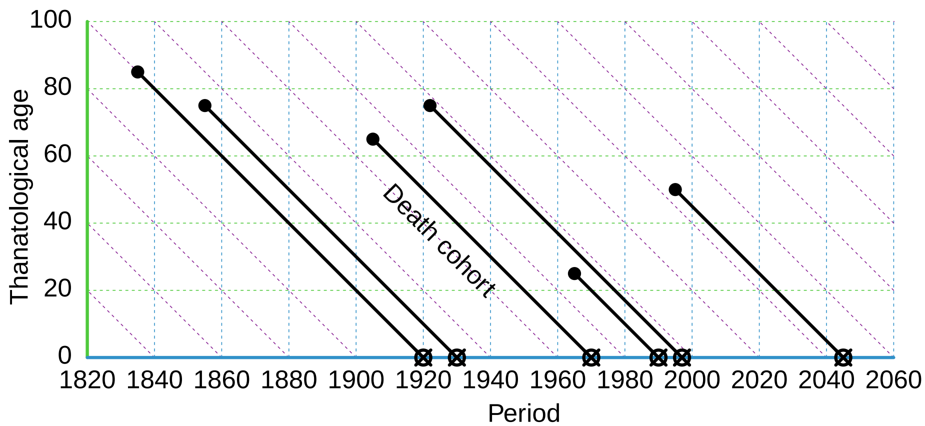


TPD

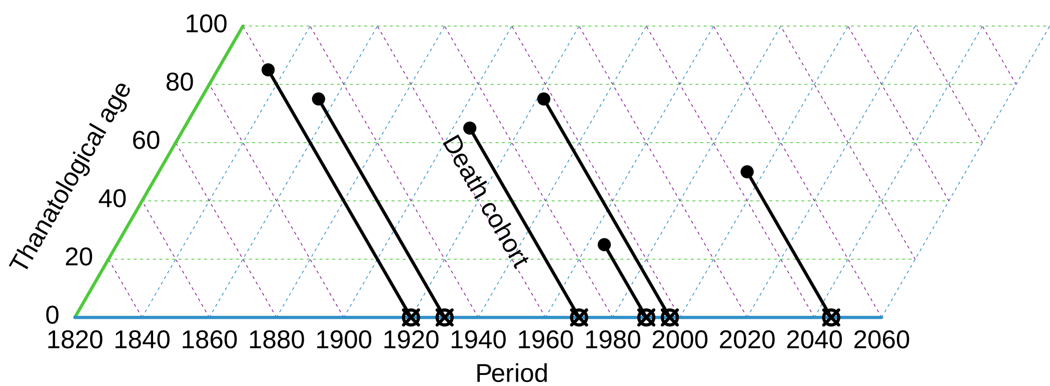
Thanatological age (T), period (P) and death cohort (D) form a coordinate system best imagined as the inverse of APC. One may take the same lifelines from Figure 1 and realign them in descending fashion such that all endpoints align to thanatological age 0, creating the diagram in Figure 2. Such diagrams have to our knowledge only appeared once in the literature, as a visual aid to a formal proof of the symmetries between life lived and left in finite stationary populations (Villavicencio and Riffe 2015). TPD coordinates may in general be used to arrange events or durations that are logically aligned (or may only be aligned) by time of termination, and in general any situation in which a terminal event aligns preceding patterns of variation. Examples may include lifelines preceding deaths from infectious or acquired conditions where the time of infection or acquisition is unknown. The main justification for visualizing data in such coordinates is when birth cohort and chronological age do not display regular empirical variation, but thanatological age or death cohort do.

Figure 2: A TPD diagram in two projections.

(a) Right angles



(b) Isotropic

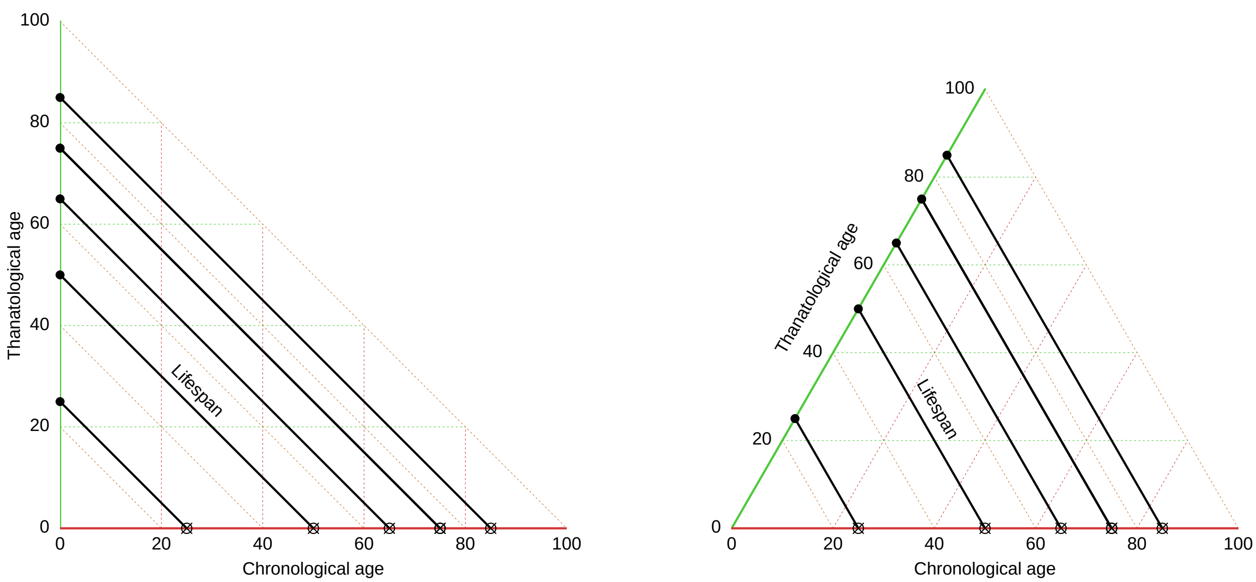


TAL

TAL is an appropriate coordinate system to examine variation in processes over the lifecourse. Since the lifecourse belongs to the cohort perspective, it is best to think of the TAL plane as belonging to some particular birth cohort. Alternatively, a TAL triangle may be taken as a cross-section through the period dimension, a sort of synthetic TAL plane. To our knowledge, the TAL diagram has only appeared once in the literature, in an exploration and classification of late-life health conditions (Riffe, T. et al. 2015). The TAL diagram in Fig. 3 contains no such indication of period or cohorts. The lifelines depicted are identical to those shown in APC Fig. 1 and TPD Fig. 2.[5] In this view, one can juxtapose some intensities that vary over the lifecourse with respect to years lived and years left, and separated by lifespan, but time trends are blended out, and cohorts overlapped.

Figure 3: A TAL diagram in two projections.

(a) Right angles (b) Isotropic



LCD

The LCD diagram completes our set of identities. It consists in an identity between lifespans, birth cohorts, and death cohorts. This identity bears resemblance to an APC surface of mortality, but we suggest some differences due to blending out age. In Fig. 4a, lifespans are indexed by the y-axis, while birth cohorts are indexed by the x-axis. Lives are lived within birth cohorts, for the length of the lifelines in the figure. However, the death cohort diagonals of the diagram are only valid with reference to the endpoints of the lifelines. To

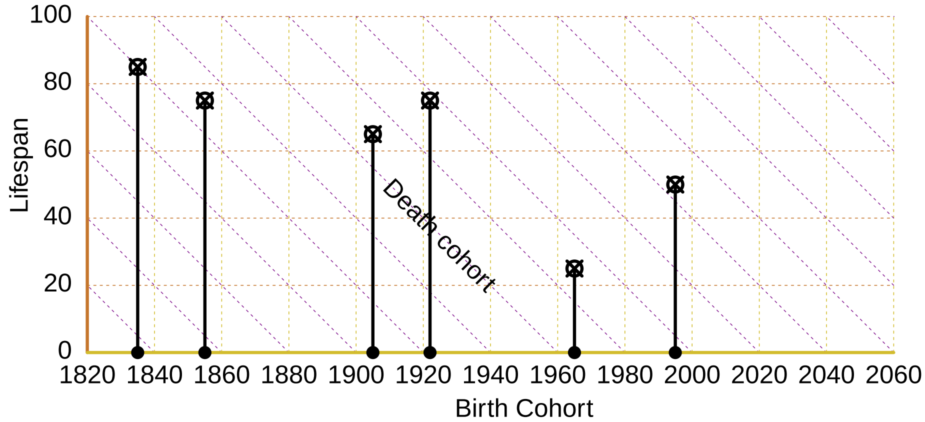
[5]The prior figures contained six lifelines each, but since two of them were of equal length (75), they are overlapped in Fig. 3 and appear to be five.

structure data on these three time measures implies ignoring variation within the lifespan. Following this, lifelines are better represented with points.

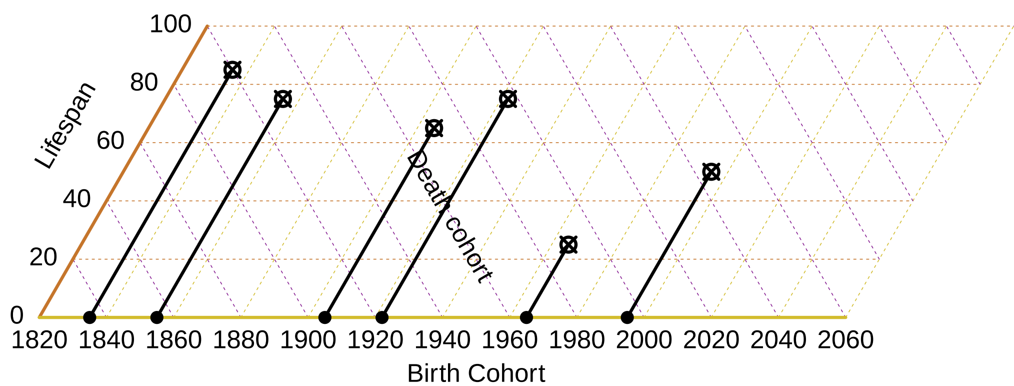
We recommend this mapping for plotting surfaces of values that vary over time by year of birth or death and that vary by lifespan, e.g., that are cumulative or static over the lifecourse. Imagine an LCD surface of cumulative lifecourse consumptive surplus or deficit, or anything else that might vary by lifespan and over time, such as children ever born, or years of retirement. In a sense, APC mortality surfaces already conform with this perspective, since the intensities visualized mark the endpoints of lifelines. However a mortality surface tells us only the relative intensity of death, which translates to densities of life lines, but does not identify other kinds of values. Fertility surfaces are in this way very different from mortality surfaces, as they plot event intensities over the ages in which they occur. A health surface is more like a fertility surface than a mortality surface, but in this case proximity to death is an important time dimension.

Figure 4: An LCD diagram in two projections.

(a) Right angles



(b) Isotropic

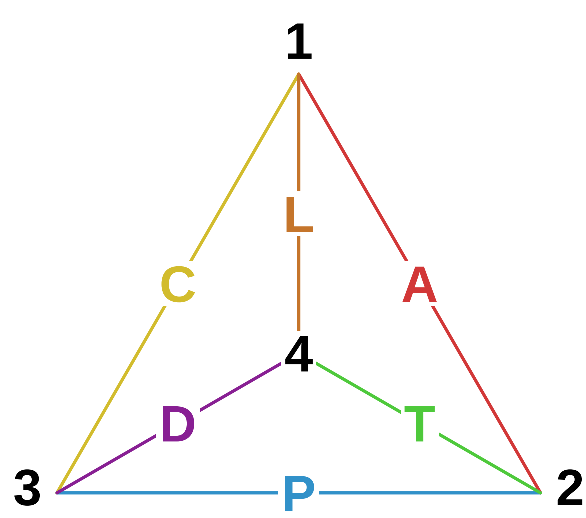


A tetrahedron relates the six time indices.

Each of the four above-mentioned triad identities may be thought of as a two-dimensional plane fully defined by any two of its three constituent time indices. In this case, we may imagine any of the excluded time measures as capable of providing depth, a potential z-coordinate, for the sake of a mental image. Having a non-redundant third dimension implies a multitude of parallel planes for the given triad identity, each plane belonging to a unique value of the third time dimension. Any of the identities can be extended in this way to fill a space. A space derived by extending any of the triad identities into its lacking dimension implies each of the other triad identities, making a total of six time indices. In essence, the four triad identities may be thought of as the four faces of a tetrahedron. If an additional time measure is added to any face (triad identity), the six demographic time indices can be derived, matching the six edges of the tetrahedron. This three-dimensional construct unifies the six indices of demographic time, and is the subject of this paper.

Let us first more rigorously define the previously-mentioned tetrahedron. Luckily, the edges and vertices of a tetrahedron are easily rendered in a two-dimensional graph, as seen in Fig. 5, with vertices labeled in black and the six time indices colored following the pattern from Table 2. The tetrahedron is composed with the APC plane at the base and vertex 4 on top. The same graph could be composed in four basic ways, depending on which identity forms the base.

Figure 5: Graph of tetrahedron, with edges labeled by the six demographic time indices. The APC plane is at the base, and vertex four on top.



Information criteria to derive the tetrahedron.

The edges APC at the base define the much-studied APC plane. If the only information we have is chronological age, period, and birth cohort (or just two of these), then we have no access to the vertex 4. Each of the faces of the tetrahedron has this quality. The South face TDP has no access to 1. The Northeast face, ATL has no connection to 3, and the Northwest face CDL lacks a connection to 2. The four triad identities that make up the faces of the tetrahedron are stuck in "flatland" and do not yield the full 3d space. However, the 16 other possible combinations of three time indices will recreate the full tetrahedron (hexad identity).

A geometrical analogy is pertinent at this point. Any pair of intersecting edges of the tetrahedron may be interpreted as two vectors $\vec{u}$ and $\vec{v}$ that determine a 2-dimensional plane in a Cartesian 3-dimensional space ($\mathbb{R}^3$).[6] Therefore, any third vector $\vec{w}$ of that plane can be expressed as a linear combination of $\vec{u}$ and $\vec{v}$ (formally, $\vec{w} = \alpha\vec{u} + \beta\vec{v}$ for some $\alpha, \beta \in \mathbb{R}$), which is usually described by saying that $\vec{w}$ is linearly dependent on $\vec{u}$ and $\vec{v}$. A similar property can be derived from the information contained in the tetrahedron: $A = P - C$ is a linear combination of C and P and it "depends" on them because they all belong to the same APC plane. Analogously, $P = C + A$ "depends" on C and A, and $C = P - A$ "depends" on P and A. Given that the faces of the tetrahedron represent the triad identities, any pair of intersecting edges has the same property: The third edge located in the same face of the tetrahedron can be determined by the first two by a simple linear relationship.

Once a 2d-plane is defined in $\mathbb{R}^3$, an additional vector may be sufficient to cover a 3d-space. Nonetheless, this third vector needs to be linearly independent of any pair of vectors of the 2d plane—that is, it cannot be expressed as a linear combination of any two vectors on that plane. Again, an analogous property can be observed in the tetrahedron: Say we only have information about the indices of the APC plane; A, C and P are not sufficient to determine a thanatological age T, death cohort D, or lifespan L (the three indices that do not belong to the APC plane). So, T, D and L are "independent" of the overall information that can be extracted from the APC plane. However, if two of the three constituent time indices of the APC plane are known (the third one would be unnecessary as it could be derived from the other two), the additional information provided by any of the three "independent" indices T, D or L would be sufficient to cover the whole tetrahedron. For example, suppose we have information about thanatological age T in addition to C and P, then $A = P - C$, $D = P + T$ and $L = T + A = T + P - C = D - C$.

Hence, as with vectors in a 3d space, any triad of indices that are independent of each other—that is, none of them can be expressed as the sum or the difference of the other two—generates a full hexad identity or, using an analogous terminology, covers the whole "space" of demographic indices presented here. Graphically, this is equivalent to choosing any combination of three indices that do not belong to the same face of the tetrahedron.

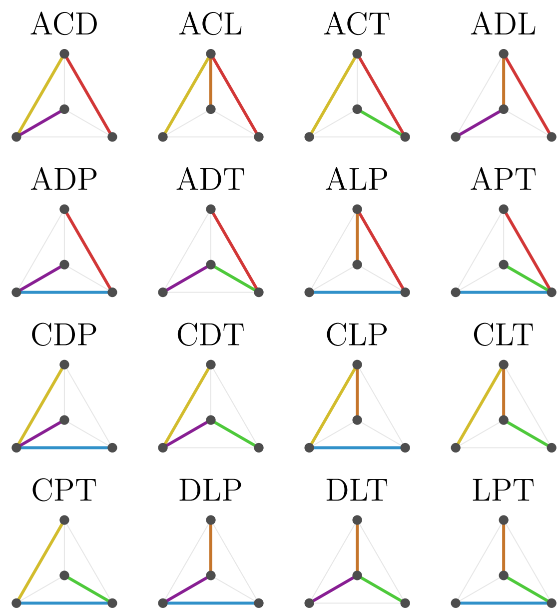
Table 3 gives the full set of 16 index-triads that are informative in the sense that each of them implies the full hexad identity. Practically, this means that if a given dataset contains variables in one of the combinations displayed in Table 3 that the entire temporal relationship is available to the researcher.

Note that the 12 possible pairs of intersecting edges on the tetrahedron are, in fact, the informative dyads described in Table 2, whereas the uninformative dyads LP, CT, and AD are the pairs of opposite edges of the tetrahedron. As discussed, all 12 informative dyads

[6] A 2d plane in a 3d space is determined by two linearly independent vectors (with different direction) and a point, but the inclusion of a point is not necessary for the intuitive analogy that we describe here.

generate one of the four triad identities, but no dyad will generate the hexad identity since a third "independent" time dimension is necessary. Similarly, any quad of indices is sufficient to complete the hexad identity, as at least one of them will not belong to the same face of the tetrahedron, but a triad may be sufficient if its edges do not all belong to the same plane forming a triad identity.

Table 3: All triads from which the full tetrahedron is derivable (same orientation).



The extension of time axes.

We have said that planes defined by the four triad identities are parallel to the faces of the the above-described tetrahedron. In imagining this three-dimensional relationship, we are no longer confined to the extent of the tetrahedron used thus far for orientation. Instead each of its edges extends a certain distance in either direction. It may therefore help to first consider the extension of each axis (or index). Some indices have a lower bound of zero and an upper bound set by the maximum length of life, ω, while others are boundless. A, T, and L are clearly in the range $[0, \omega]$.[7] P, C, and D are bounded only by the inception and extinction of our species, but may be thought of as boundless for practicality, or benchmarked to our earliest and most recent observations for even more practicality.[8] As an abstraction, however,

[7]It's best to imagine some number like 122.45 years, for ω, rather than infinity. This is the longevity record at the time of this writing. Jeanne L. Calment would have had $T = 122.45$ at birth, $A = 122.45$ at death, and $L = 122.45$ for her entire life.

[8]We explain the choice of the word "benchmarked". Say we have a data series that runs from 1751 to 2011, and an upper age interval of 110+. Then we could say that P is in the range $[1751, 2011]$, but by another reading, P must range from at least as early as the earliest C and until at least as late as the latest D. Someone dying at 110 in 1751 had a C of 1640, and an infant born in 2011 that is destined to live to 110 will die in 2121. In this case a P that *contains* the observed population will extend well before and after the observed data series, even more so if we take into account that $\omega > 110$.

the dimension of calendar time in this model is infinite. Of the four triad identities, only one lacks an unbounded dimension, the TAL. Adding the absent dimension to TAL therefore makes its 3d extension boundless. In this way, we may imagine a prism-like construct, where T, A, and L, compose the faces of a triangular cross-section of the prism, which extends infinitely "through" the triangle. We can think of the TAL triangle passing through time, extending the population forward to infinity. In this case, the TAL triangle may take either the period or cohort perspective, which is illustrated later.

Intersecting planes

The APC, TPD, TAL, and CDL planes can be conceived of either as *compressions* of this 3d space, or as cross-sections of the 3d space. To compress space in this sense is to ignore the missing dimension, whereas a cross-section sets a given triad identity against a particular position of the absent dimension. This is a more general sense of the term *cross-section* than is often used in the demographic literature, where it typically implies a period cross-section. APC has thus far always been treated as a compression in this sense. As mentioned, a compressed TPD diagram has thus far only appeared in Villavicencio and Riffe (2015), and cross-sectional TAL diagrams and surfaces have thus far only appeared in Riffe, T. et al. (2015). We have been unable to locate an example in the literature of a compressed TAL or cross-sectional TPD diagrams, though it seems plausible that the former will have arisen.

We suppose that LCD diagrams of any kind are novel. A simultaneous juxtaposition of the six demographic time measures has also never appeared in the literature, either as a visual diagnostic or in any line of inquiry or argumentation.

Diagram of the hexad identity

There are different ways to proportion this three dimensional construct, of which we only present the isotropric mapping. In an isotropic projection, the tetrahedron is regular, such that all edges are of the same length, and the units of each of the six time measures are proportional. In this case, the four triad identities are based on equilateral triangles between their three constituent indices, and the four planes are joined together such that each is parallel to a face from the regular tetrahedron. If a plane parallel to each face is repeated in equal intervals, we create an isotropic 3d space.

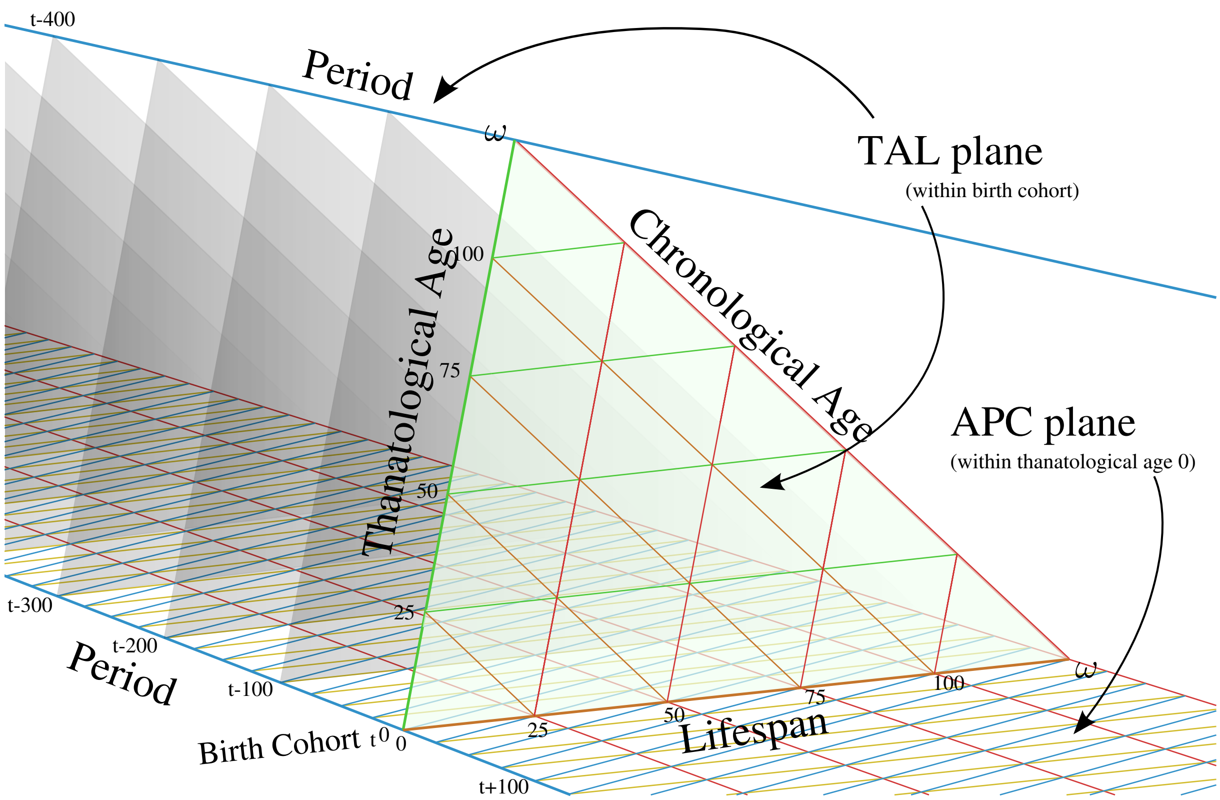
The isotropic space that results from this framework is known in other disciplines with different nomenclatures [9]. Constructs following this geometry exist in nature, in other theoretical settings, and in man-made structures. There are various ways in which one might diagram or visualize the 3d space of demographic time according to the hexad identity. Displaying all planes simultaneously creates a very dense and difficult-to-read diagram. We choose instead to emphasize particular planes and intersections.

In Fig. 6 we offer a view of the hexad identity, where birth-cohort TAL cross-sectional

[9] In geometry, this structure is called the tetrahedral-octahedral honeycomb, a variety of space-filling tessellation. In architecture, it is found in the octet truss system. In physics it is called the isotropic vector matrix.

planes are placed in sequence in a perspective drawing.[10] The most recent TAL plane, for year t, is placed in the front, whereas past TAL planes are stacked behind it, highlighted in century intervals. This view emphasizes how the juxtaposition of thanatological age, chronological age, and lifespan shifts over time. This is the view used in Riffe, T. et al. (2015) to describe late-life health outcomes, albeit for a single birth cohort. The base of this figure is the APC plane, drawn for thanatological age 0. Each of the TAL planes therefore sits atop a single birth cohort line from the APC plane that makes up the base of the figure.

Figure 6: Birth cohort TAL planes over time

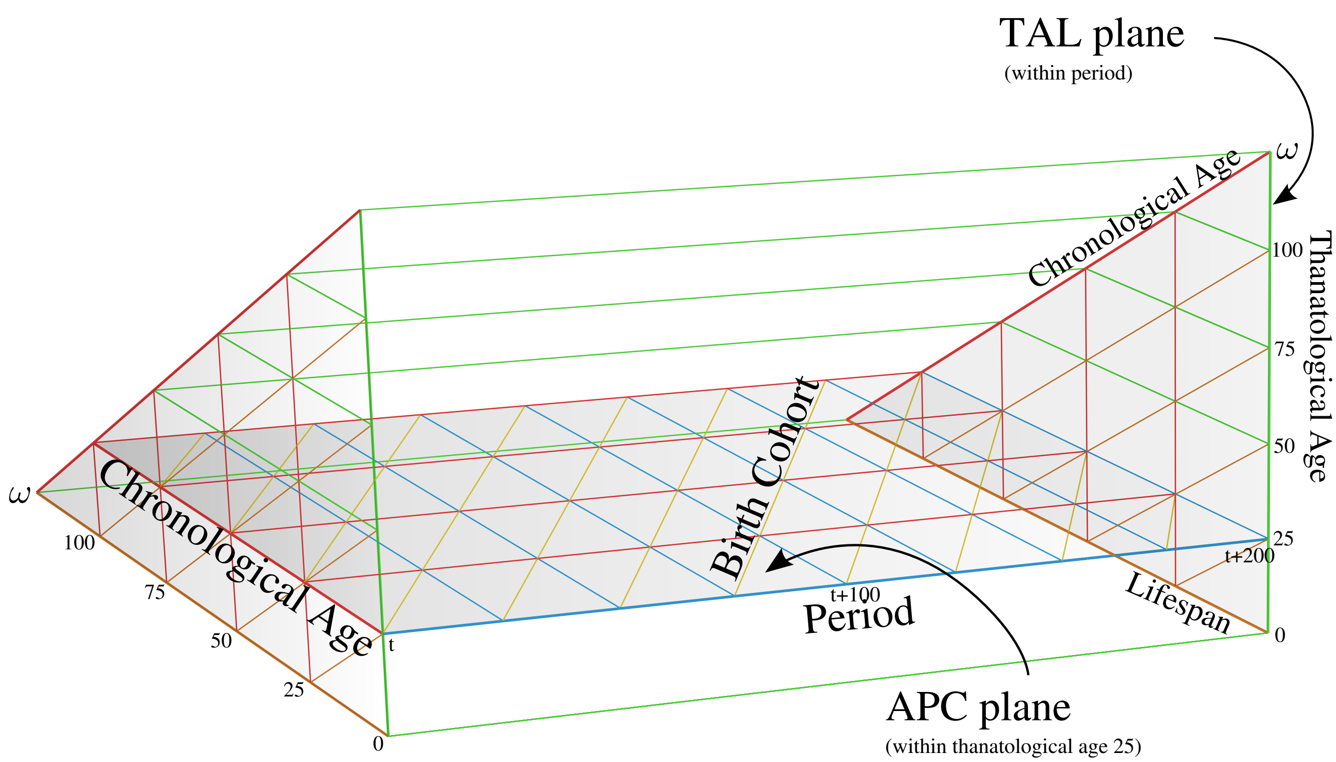


Since the APC plane at the base of Fig. 6 could have been drawn for any thanatological age, it is better to imagine the TAL plane slicing through the same birth cohort, t, of every possible thanatological APC plane. In Fig. 7 we gaze from a different angle and highlight different planes to emphasize how the APC planes stack by thanatological age. The space in this view is capped by period TAL planes on either side. Think of period TAL planes as population censuses that have been fully linked to mortality outcomes, such that each

[10]The coordinates used to render Figures 6 and 7 are isotropic. However, there are no 60° angles in this figure due to the use of parallax and an indirect viewing angle in this rendering for the sake of increased legibility.

person is categorized by thanatological age as well as chronological age (and each of the other indices). Period TAL planes shift over time, like the birth cohort TAL planes in Fig. 6, but the period and cohort TAL planes stand in intersection.

Figure 7: The APC plane of thanatological age 25, with period TAL planes.



The APC shown in Fig. 7 is drawn for thanatological age 25, but we can think of it as shifting up and down through the space. At the peak, near ω remaining years of life, the age axis of the plane is short, since only the lowest chronological ages may live so long. The APC base plane at thanatological age 0 is the largest because members of any chronological may die.

Application

The relationship between the six measures of demographic time is true in the same sense that mathematics is true: Under linear time it is an internally valid set of relationships, and this is self-evident. We have mentioned that the coordinates described here may be useful for the visualization of data, to enable discovery, and to better inform demographic methods. We have not yet mentioned how visualization can inspire such developments, or inform researchers of their necessity. We therefore give a shematic overview of our own process of scientific inquiry and reflection that was based this coordinate system, and that would not have arisen without it. This chain of inquiry is meant to demonstrate the usefulness of the present framework, but it is far from an exhaustive application of its potential for other substantive questions, nor is the case study described in complete rigor. Specifically, we reason that healthy life expectancies (HLE) for many health conditions cannot be projected, compared between groups, or compared over time without taking into account the time-to-death time measure and mortality differences.

There are three steps in our empirical inquiry. The first step is to visualize variables on health outcomes using our framework. The second step is to assess the primary time measures over which health outcomes vary. Under the assumption that these patterns of temporal variation are empirically regular, we procede to develop a method of standardizing health expectancy calculations for morbidity conditions whose prevalence is more closely related to time-to-death. Finally, one can reason that period estimates of health expecancies for certain health conditions are biased when mortality has been or will-be changing, and comparisons of HLE between populations with different mortality are also biased. We conclude that comparisons of health expecancies are biased in ways not previously documented.

Let us take the example of self-reported health (SRH). This variable is available from many different survey sources, and it is familiar to many researchers. There are many known pitfalls to this particular variable that already make it difficult to compare between sexes, over time, or between populations, and these do not concern us. We use this variable as an example because of the particular clear pattern of variation that it shows, which we use to make our more general point.

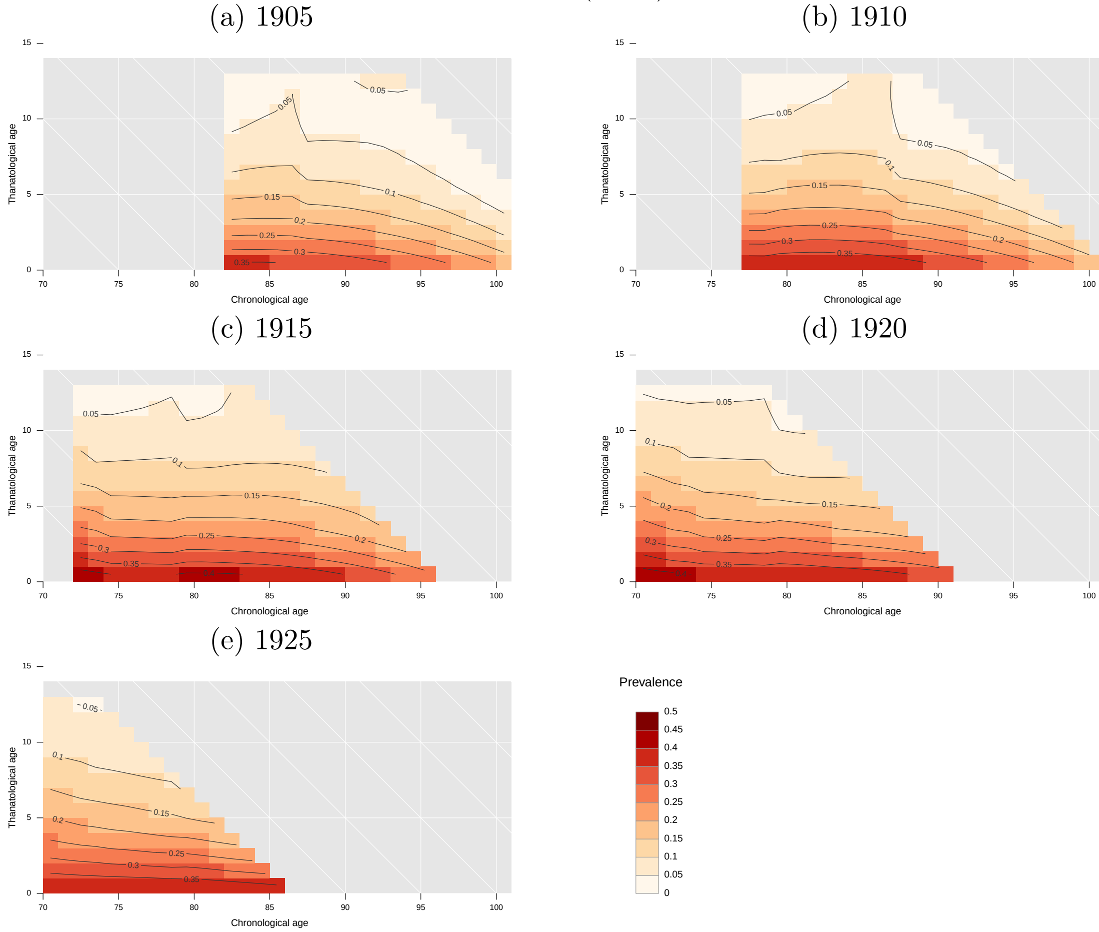
Figure 8 displays a series of TAL surface plots, each referring to a different quinquennial birth cohort (1905-1909, etc). The data come from the Rand version of the US Health and Retirement Study (HRS 2013). Since this survey has a mortality follow-up, we have each of the six time measures for each observation. Further methodological details are given by Riffe, T. et al. (2015). The TAL surfaces for each birth cohort are shifted by five years because the observation window available is from 1992 to 2011 for each cohort.

The x-axis of these plots is chronological age, while the y-axis is remaining years of life. Contour lines in the surfaces indicate the primary direction of variation, in this case over thanatological age. Downward diagonals indicate lifespans, which the reader may also think of as very specific birth-death cohorts. These are the diaongals along which lifelines may be imagined, as suggested in Figure 3. For each of these birth-death cohorts we have a prevalence trajectory— empirical examples of the lifeline morbidity trajectories often conceptually diagrammed in the literature on morbidity compression (e.g., Fries 2005). In each surface the primary axis of variation is along thanatological age, and not chronological age. The prevalence for those with t remaining years of life was similar, irrespective of chronological age, birth cohort, or ultimate lifespan.

When one looks at a chronological age pattern of SRH, as measured here (the Sullivan curve, (Sullivan 1971)), one sees an increasing tendency over age. However, such an increasing line is a marginal ruse, due to an interaction between the distribution of lifespans and the relatively fixed underlying pattern of morbidity seen in Figure 8. These surfaces can indeed be tidily summarized with a single line, but it is a line over the thanatological age margin rather than over chronological age.

Since the patterns for each of these cohorts can be presumed to be the same, any shifting in the distribution of lifetimes ought not produce a change in the expected years of poor health for a given lifespan. Further, the life years spent in poor health should also be

Figure 8: Proportion of males self-reporting poor health by chronological and thanatological age, by quinquennial birth cohorts, 1905-1925. (HRS)



approximately the same "on average", even if the underlying mortality patterns shift. If morbidity change is a pure function of time to death, an increase in life expectancy should increase healthy life expectancy by the same amount. This is not what we predict when we base analyses on the chronological age pattern of self-reported health. An underlying morbidity pattern this stable would predict improvements in the marginal chronological age pattern of self-reported health if the lifespan distribution were to shift right. This bias in the current status quo of morbidity measurement and prediction leads to pessimistic morbidity scenarios when mortality improvements are foreseen, and it undermines health expectancy comparisons between groups with different mortality.

Using the data from our example surfaces, we can calculate an average prevalence trajectory with the approach to death and calculate some basic results that support our case. Let us take the population of US males aged 60 and older, and assume that the trajectory derived from the Figure 8 surfaces is valid for them. If we apply this trajectory to a stationary population from 1980 and 2010 (Human Mortality Database), we can calculate the resulting healthy and unhealthy life expectancies, and compare these with the expectancies that we would have projected assuming the 1980 Sullivan curve. Total remaining life expectancy at age 60 increased 4.3 years from 17.4 to 21.7 years from 1980 to 2010. Assuming the time-to-death trajectory of morbidity, we calculate healthy life expectancies of 15.7 and 19.9, respectively, an increase of 4.2 years. Unhealthy life expectancy in this scenario increased just 0.1 years. Had we used the Sullivan curve from 1980 to calculate the 2010 values, we would have predicted an increase of 0.7 years in unhealthy life expectancy, or 39% versus the 4% "observed" in this exercise.

This is a large difference, and it is based on a relatively minor tweak to standard methodology, itself inspired by viewing data under the conditions enabled by this temporal framework and adjusting standard demographic methods to match the direction of temporal variation in data. There is a wide variety of prevalence patterns when viewed in this way (Riffe, T. et al. 2015, Wolf et al. 2015), and much empirical and methodological work is still required to verify these findings and understand the consequences for the standard ways of comparing and projecting HLE. Our objective in this application has been to demonstrate how viewing data under the rigorous conditions enabled by the time-framework we propose can lead to new scientific understandings of processes over the life course.

Discussion

It is straightforward to think of examples to derive unstated time measures based on other given time measures. In isolation, the various triad identities are also intuitive. The TPD diagram is similar to APC, but it is aligned to time of death rather than time of birth. The TAL diagram presents a clear way to classify events over the life course of a cohort. Finally, the LCD diagram can be used to structure quantities that vary over time and by length of life. Joined together, the relationship between all six time measures is more complex than any of the triad identities, but it condenses into a simple geometric representation that can be easily derived by ascribing the various time measures to the six edges of a tetrahedron. An understanding of how the six time measures relate is key to understanding the temporal structure of demographic processes, which itself underlies the comparability of demographic measures.

The contemporary practice of (macro) demography is based on the premise that vital rates, and other kinds of rates over the lifecourse, are the truest measure of demographic forces. Rates are paramount because they tend to vary in empirically regular ways over the life course. The scalings and movements of primary vital rates fall within a limited range for humans. For this reason, many of the methods of demography are developed to estimate rates, independent of population composition, or to partition crude magnitudes into the effects of population age structure and pure vital rates. Controlling for age is in a more general sense controlling for temporal variation in stocks. To the extent that regular temporal variation relates to the end of life, or the length of life, common age-standardization does not fully account for such structure.

The techniques used to age-standardize mortality and fertility estimation are at times applied to other kinds of transitions over the life course. For example, one may estimate an age pattern to some degenerative disease, or the ability to carry out some common activities of daily life. Much of the regular temporal variation for such conditions is by time-to-death or lifespan, rather than by chronological age. Apparent chronological age patterns for such conditions are artifactual and do not represent the same kind of intrinsic meaning as does the *age pattern* of mortality. Further kinds of temporal standardization must be developed in order to measure and understand the natural patterns of such conditions over the lifecourse. The measurement of such conditions may benefit from consideration of the framework presented in this paper. To this end, Table 3 provides all combinations of information that are sufficient to derive the full set of six time measures. Panel surveys with mortality followups already provide the requisite information, as do linkable registers that include items such as health measures or proxies and relevant dates of birth, observation, and death.

The best way to seek regular patterns in variation is via data visualization. The coordinate system proposed in this paper is conceived as one adequate to capture all such variation, and we suggest its use for visualizing data, probably via small multiples of successive time slices parallel to any of the four triad identities, similar to that shown in Figure 8. Such visualization strategies at this time are exploratory, and this is a technique that may benefit from further refinement.

Mortality determines three of the dimensions of demographic time, and it therefore makes little sense to model mortality using all six time measures. Any of the six measures may be pertinent in the case of conditions and states that vary over and within the lifecourse. The most obvious application for the present model, given data commonly (and publicly) available at this time, are late-life health conditions, although there may be other substantive areas of application.

Furthermore, we believe in the pedagogical value of the framework introduced in this paper. We hope that the present inquiry will be useful as a teaching instrument in the same way as APC diagrams have formed a part of basic demographic education. The relationship between the six dimensions of demographic time helps situate the APC paradigm in a broader framework. Just as scientific discovery in general depends partly on the development of finer optics and instrumentation, we hope that the framework we give will be an instrument to enable new discoveries in formal and empirical demography.

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