A unified model of demographic time

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Abstract

We describe a three-dimensional model that relates six different measures of lifespans and time. The six measures of demographic time considered are chronological age, thanatological age, lifespan, year of birth, year of death, and period. Two versions of the model are described: a relatively intuitive extension of the right-angled Lexis diagram, and an isotropic extension based on the regular tetrahedron.

The Lexis diagram relates the chronological age (A), period (P), and birth cohort (C) measures of demographic time, APC, but it does not account for remaining years of life (thanatological age), and other related time indices. The thanatological counterpart to APC is an identity between thanatological age (T), period (P), and death cohort (D), TPD. A third identity exists between thanatological age (T), chronological age (A), and lifespan (L), TAL, and a fourth between year of birth (C), year of death (D) and lifespan (L), CDL. Each of these four identities may be sufficiently described by any two of its consituent indices, making the third index redundant. Each of these four identities also lacks a major dimension of time. The TAL identity lacks calendar time, the CDL identity is ageless, APC lacks an endpoint in time, and TPD lacks a starting point in time. We refer to these four identities as the triad identities.

To our knowledge, the only triad identity that has received serious treatment at the time of this writing is the APC identity. Different aspects of

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the APC identity have been discussed since at least 1868 (Knapp 1868), and discussion remains lively today. Here it is our objective to relate the six major indices of time in a geometric identity, in much the same spirit as the work on APC relationships done between the late 1860s and mid 1880s.[1]

This paper provides a bottom-up description of the model, building from familar components to the full relationship. We begin by defining some terms used throughout the manuscript. We then explore all combinations of two time measures, the dyadic relationships, followed by the four triad identities, and finally the hexad identity. At the price of some redundancy, we give a systematic topological overview of the different elements of demographic time. This paper contains no data visualizations, but it may serve as a basis for designing some.

Definitions

0.1 Technical terminology

We attempt to adhere to a rigorous terminology in this paper. The following list describes some of the more important terms we use.

Time measures are any of the six demographic perspectives discussed that can be used to index time: chronological age (A), period (P), birth cohort (C), thanatological age (T, time-to-death), lifespan (L), and death cohort (D).

Dyads, triads, and hexads are any set of two, three, or six unique time measures, respectively.

A temporal plane is any (x, y)-mapping of a dyad of time measures.

A temporal space is any (x, y, z)-mapping of a triad of time measures that generates a 3-dimensional space.

A triad identity is a triad that is the union of an informative dyad and its one derived time measure. For example, A and P form an informative dyad from which C can be derived.

A hexad identity is a unique combination of the six time measures.

Using this terminology, for example, we say that the "Lexis" measures constitute a triad identity between chronological age, period, and birth cohort. Each dyad combination of elements in this identity is informative, and can be mapped to a temporal plane, the Lexis diagram. If we know that Mindel turned 50 on the 21st of May, 1963, then we also also can derive that she was born on the 21st of May, 1913. Any two pieces of information in this case

[1]See e.g., Keiding (2011) for an overview of that literature.

will give the third, which means that any dyad from this set is informative. Three other such triad identities are also to found within the six measures of time we discuss.

Time measures

This model description is conceived in absolute, linear, Newtonian time, and we do not consider situating the model in any other perspective or model of time itself. This relationship is scalable to any time unit, but we decribe it in terms of years, the dominant human time scale. We therefore speak of calendar time, imagining the modern Gregorian calendar, though this is not necessary. The six measures of time we consider are defined in Table 1, both in the demographic sense we describe, as well as in a more general event history interpretation.

The concepts of thanatological age and death cohorts are likely less familar to readers than the other measures we consider. Thanatological age is sometimes referred to in the literature as remaining years of life, or time-to-death (TTD), but we prefer the term thanatological age, and to think of the concept of age in general as marking a position on a lifeline with respect to one of its endpoints. Chronological age and thanatological age are in this way complementary. Thanatological age is different from the notion of prospective age, used by Sanderson and Scherbov (2007), since prospective age is a relative term that reflects a comparison of expectancies. Prospective age scales chronological age by comparing mortality schedules, but it is neither an expectancy nor a statement of remaining years of life. Thanatological age is meaningful without much justification; it is the measure we all want to know, the thing we approximate with remaining life expectancy.

Cohorts in general associate individuals that share a characteristic. In demography the grouping characteristic is often a combination of place and time, such as the cohorts of young demographers passing through a particular graduate program. In this instance already, we accomodate the notion of a cohort for both the start and endpoints of the program, but we say e.g., "the class of 2015" instead of the "graduating cohort of 2015", in contrast

Table 1: Definitions of the six time measures.

Time measure	**Short**	**Demographic def.**	**Event history def.**
chronological age	A	Time since birth	Time since study entry
period	P	calendar time	calendar time
birth cohort	C	calendar time of birth	calendar time of study entry
thanatological age	T	time until death	time until event
death cohort	D	calendar time of death	calendar time of event
lifespan	L	duration of life	duration of exposure

to "cohort 37", the 37th class of entering students since the start of the program. These concepts are analagous to the ideas of birth and death cohorts we use here, though we do not often refer to the deaths of a given year as a death cohort. In the time preceding death, the members of a given death cohort have much in common, despite heterogeneity with respect to time of birth. If the reader accepts this premise, then the abstract construct of a death cohort is also meaningful in the way that the other measures are.

Much of the work of demography is directed at the study of lifespan. Lifespan is synonymous both with longevity, chronological age at death, and thanatological age at birth. One's ultimate completed lifespan is constant throughout life, though we have no knowledge of it until death: It is assigned retrospectively. Demographers have more often used lifespan or age-at-death as a measure of mortality, or similar, than as a measure on which to compare individuals.

Treating lifespan, death cohorts, and thanatological age as temporal structuring variables enables new classes of comparisons, models of understanding, and discovery, akin to those unlocked by breaking down demographic phenomena by chronological age, period, and birth cohort. The following sections will, in this sense, provide an exhaustive classification of the ways in which these six measures of time can be juxtaposed to such ends. We begin with all sets of two time measures, the informative and uninformative temporal dyads.

Informative and uninformative dyads

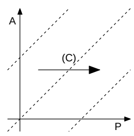
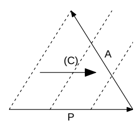
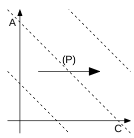
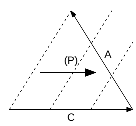
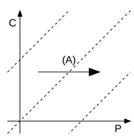
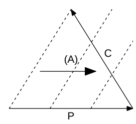
Any mapping of two time measures to a (x, y) coordinate system constitutes a temporal plane. If the two given time measures are members of the same triad identity, the third member is called a derived measure. For instance, if we take the dyad AP, C is called the derived measure, and we make this dyadic relationship explicit by writing $AP(C)$. The temporal plane that corresponds to this informative dydad is the contemporary representation of the Lexis diagram. The dyads $AC(P)$ and $CP(A)$ also belong to the Lexis identity, but imply different less-common rotations and projections of the Lexis diagram.

For each dyad there is a fundamental question of how to map the constituent coordinates to a cartesian temporal plane. Typically one forces parity between time units within a specified dyad, and maps one element directly to x and the second element directly to y, resulting in a $90°$ angle between the x axis and y axis. In this case the convention is to force a unity aspect ratio between the x and y axes, such that the derived measure, if any, is then *accidentally* present in a $45°$ ascending or descending angle, depending on the dyad and axis orientation.

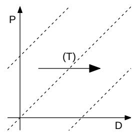
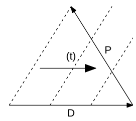
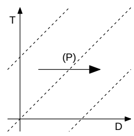
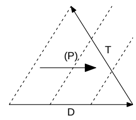
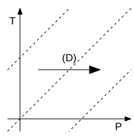
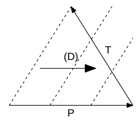
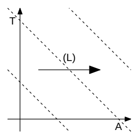
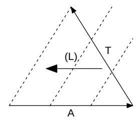
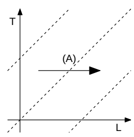
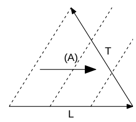
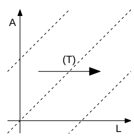
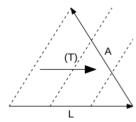
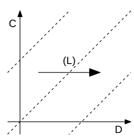
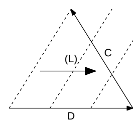
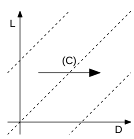
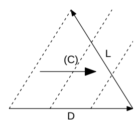
It has long been noted (Zeuner 1869, Perozzo 1880) that the derived time measure, when plotted at 45°, is relatively longer than either the age or years axes. If a right angle and unity aspect ratio is forced between the dyad, the derived measure is always stretched by $\sqrt{2}$, or 41%. In the case of dyads that imply a derived measure, another logical mapping is to translate to an (x, y) coordinate that forces 60° angles between the three measures. Such a mapping ensures that the spatial units are equal for the three measures, and we therefore refer to it as the isotropic mapping. The isotropic mapping is comparable to using a ternary coordinate system, which we do not discuss. It does not make sense to provide an isotropic mapping of an uninformative dyad. The primary justification for isotropic temporal planes comes from a data visualization perspective, where it may be hypothesized that the viewer's ability to compare slopes is hindered if the axes are not on the same scale.

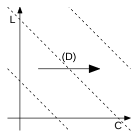
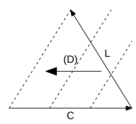
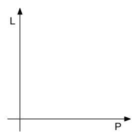
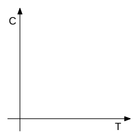
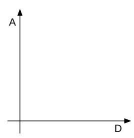
15 dyads are possible from our set of six time measures, 12 of which are informative, and three of which have no derived time measure, and are therefore called uninformative. These dyads, an explanation or simple example, and the corresponding graphical representations are summarized in Table 2.

Table 2: The 15 temporal dyads, with corresponding translations to Euclidean and isotropic temporal planes.[2]

Variants of APC			
$AP(C)$ $C = P - A$	The $AP(C)$ temporal plane constitutes the classical Lexis diagram.		
$AC(P)$ $P = C + A$	The $AC(P)$ temporal plane is equivalent to the Lexis diagram except that birth cohort is given and period is derived.		
$CP(A)$ $A = P - C$	The $CP(A)$ temporal plane is equivalent to the Lexis diagram except that birth cohort is given and age is derived.		
Variants of TPD			

[2] Contrary to mathematical convention we name the ordinate scale first and the abscissa scale second. This is to be consistent with the established APC and ACP terms.

$PD(T)$ $T = D - P$	Mindel died in 1973 (D). In 1953 (P) she had 20 (T) years left to live.		
$TD(P)$ $P = D - T$	Irene died in 1974 (D). When she had 30 (T) remaining years of life the year must have been 1944 (P).		
$TP(D)$ $D = P + T$	Helen had 30 yeasrs of life left (T) in 1971 (P) and therefore belonged to the 2001 death cohort (D).		
	Variants of TAL		
$TA(L)$ $L = T + A$	The time already lived and the time still left sum up to the total lifespan.		
$TL(A)$ $A = L - T$	Helen lived to the age of 86. When she had 20 years left she must have been 66.		
$AL(T)$ $T = A - L$	Tim is 34 years old (A) and will live to the age of 96 (L), leaving him 62 years (T) to settle affairs.		
	Variants of CDL		
$CD(L)$ $L = D - C$	Pascal was born in 1893 (C) and died in 1964 (D), implying a lifespan of 71 (L), or so.		
$LD(C)$ $C = D - L$	Margaret died in Dec., 1995 (D) with a completed lifespan of 96 (L), putting her birth year in 1900 (C).		

$LC(D)$ $D = C + L$	Àngels was born in 1940 (C) and she lived to be 64 (L), implying an untimely death in 2004 (D).		
	The uninformative dyads		
$LP(-)$	The LP plane is *non-informative*. No additional dimensions can be derived knowing just lifespan and period.		
$CT(-)$	The CT plane is *non-informative*. No additional dimensions can be derived from birth cohort and thanatological age.		
$AD(-)$	The AD plane is *non-informative*. No additional dimensions can be derived from death cohort and chronological age.		

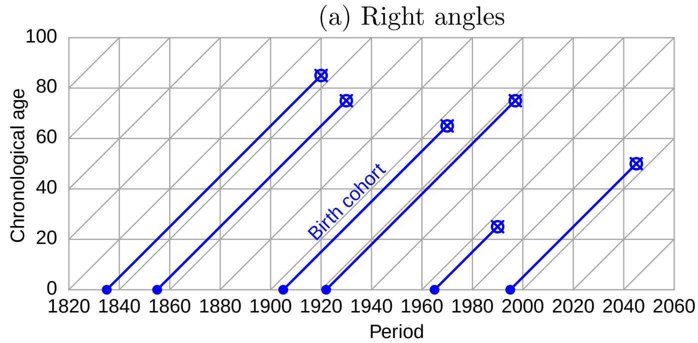
Most of what we know about how rates change over age and time comes from the very first juxtaposition in Table 2, $AP(C)$. While $CP(A)$ and $AC(P)$ are statistically redundant, they are not fully redundant in terms of perception and data visualization if using Euclidean coordinates, as demographers typically do. The other dyadic juxtapositions can be considered as either rare or novel ways of structuring or viewing temporal variation in demography.

The triad identities

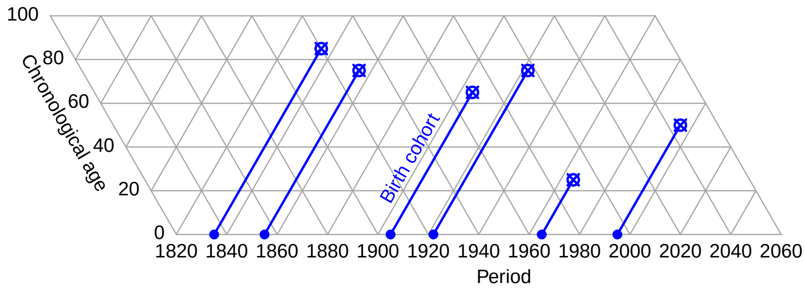
Visualizations of data structured by any dyad belonging to a triad identity are inherently richer in information than juxtapositions of uninformative dyads. There is no reason not to explore all possible dyadic juxtapositions, but the triad identities have more apparent meaning, even in the absence of data, due to the underlying relationship between measures. Each of the triad identities can accomodate some version of a lifeline. In the present section, we therefore lay out the four primary diagrams that belong to the triad identities.

Figure 1: An APC diagram in two projections.

(a) Right angles



(b) Isotropic



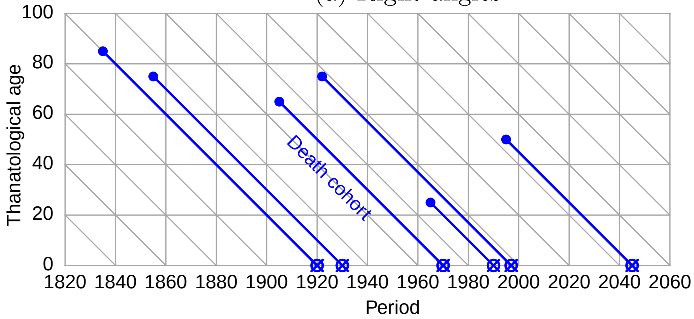
APC

The so-called Lexis diagram has long been used in demography as a conceptual tool for structuring data, observations, and rate estimation, as inspiration for work on statistical identification, and as the coordinate basis of contemporary Lexis-surfaces. Since the Lexis diagram could have been named for others (Vandeschrick 2001, Keiding 2011), and since we compare with other temporal configurations, let us refer to it as the APC diagram, as seen in Figures 1a and 1b.

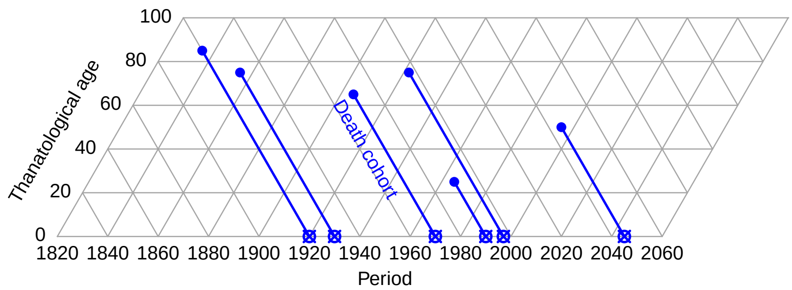
The APC diagram in Figure 1a represents years lived on the y axis, calendar years on the x axis, and birth cohorts as the right-ascending diagonals. This is the most common of several possible configurations of the APC dimensions. Individual lifelines are aligned in the birth cohort direction, starting with birth (filled circle) at chronological age zero, and death (circled x). Any APC surface can be interpreted along each of these three dimensions of temporal structure.

Figure 2: A TPD diagram in two projections.

(a) Right angles



(b) Isotropic



TPD

Thanatological age (T), period (P) and death cohort (D) form a coordinate system best imagined as the inverse of APC. One may take the same lifelines from Figure 1 and realign them in descending fashion to create the diagram in Figure 2. Such diagrams have to our knowledge only appeared once in the literature, as a visual aid to a discrete mathematical proof of the Brouard-Carey equality (Villavicencio and Riffe 2015). TPD coordinates may in general be used to arrange events or durations that are logically aligned (or may only be aligned) by time of termination, and in general any situation in which a terminal event predicts preceding patterns of variation. Examples may include lifelines preceding deaths from infectious or acquired conditions where the time of infection or acquisition is unknown.

TAL

TAL is an appropriate coordinate system for processes that vary over the lifecourse. Since the lifecourse belongs to the cohort perspective, it is best to think of the TAL plane as belonging to some particular birth cohort. Alternatively, an TAL triangle may be taken as a cross-section along through the period dimension, a sort of synthetic TAL plane. To our knowledge, the TAL diagram has only appeared once in the literature, in an exploration and classification of late-life health conditions (Riffe, T. et al. 2015).

CDL

(section in progress, will contain 2 figures, like APC) * I've not thought much about this one, but it completes the tetrahedron :-P

A tetrahedron relates the six time indices.

Each of the four above-mentioned triad identities may be thought of as a two-dimensional plane fully defined by any two of its three constituent time indices. In this case, we may imagine any of the excluded time measures as capable of providing depth, a potential $z-$coordinate, for the sake of a mental image. Having a non-redundant third dimension implies a multitude of parallel planes for the given triad identity, each plane belonging to a unique value of the third time dimension. Any of the identities can be extended in this way to fill a space. A space derived by extending any of the triad identies into its lacking dimension implies each of the other triad identities, making a total of six time indices. In essence, the four triad identities may be thought of as the four faces of a tetrahedron. If an additional time measure is added to any face (triad identity), the six demographic time indices can be derived, matching the six edges of the tetrahedron. This three-dimensional construct unifies the six indices of demographic time, and is the subject of this paper.

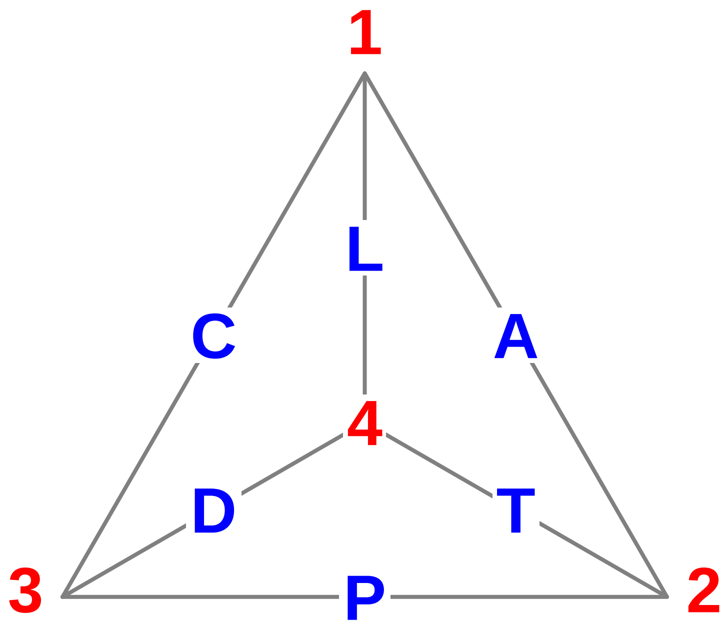
Let us first more rigorously define the previously-mentioned tetrahedron. Luckily, the edges and vertices of a tetrahedron are easily rendered in a two-dimensional graph, as seen in Figure 3, with vertices labeled in red and the six time indices labeled as blue edges. The reader may also imagine this graph as a transparent 3d object, in which case the four faces become apparent. There are two intuitive ways to imagine the graph as 3d, either the vertex 4 is on top, and we gaze from a bird's-eye-view, or the vertex 4 is in the back, behind the other three vertices. Assume we gaze from the top, for the sake of description.[3]

Information criteria to derive the tetrahedron.

The edges APC at the base define the much-studied APC plane. If the only information we have is chronological age, period, and birth cohort (or just

[3] The same graph could be composed in four basic ways, depending on which triad-identity face forms the base. These are given in an appendix.

Figure 3: Graph of tetrahedron, with edges labeled by the six demographic time indices.



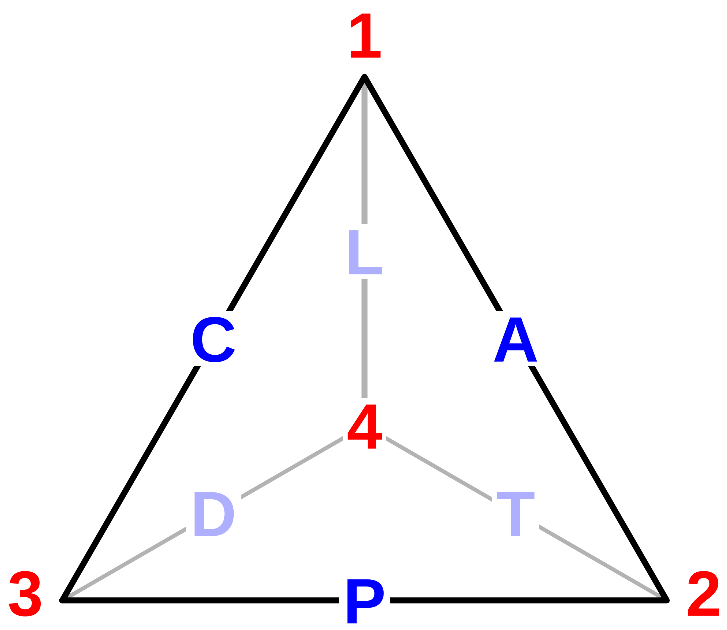
two of these), then we have no access to the vertex 4. Each of the faces of the tetrahedron has this quality. The South face TDP has no access to 1. The Northeast face, ATL has no connection to 3, and the Northwest face CDL lacks a connection to 2. The triads that make up the faces of the tetrahedron are stuck in "flatland". However, there are $\binom{6}{3} = 20$ ways to choose three time indices out of six, and the four above-named triads are the only ones that will not yield the full 3d space. The sixteen other combinations of three indices will recreate the full tetradhedron (hexad identity).

A geometrical analogy is pertinent at this point. Any pair of intersecting edges of the tetrahedron may be interpreted as two vectors $\vec{u}$ and $\vec{v}$ that determine a 2-dimensional plane in a 3-dimensional space.[4] Therefore, any third vector $\vec{w}$ of that plane can be expressed as a linear combination of $\vec{u}$ and $\vec{v}$ (formally, $\vec{w} = \alpha\vec{u} + \beta\vec{v}$ for some $\alpha, \beta \in \mathbb{R}$), which is usually described by saying that $\vec{w}$ is linearly dependent on $\vec{u}$ and $\vec{v}$. A similar property can be derived from the information contained in the tetrahedron: Say we have

[4] A 2d plane in a 3d space is determined by two linearly independent vectors (with different direction) and a point, but the inclusion of a point is not necessary for the intuitive analogy that we describe here.

information on year of birth C and period P. Clearly, C and P are sufficient to determine the chronological age A, given that $A = P - C$. That is, A is a linear combination of C and P and it "depends" on them because they all belong to the same APC plane (see Figure 4). Analogously, $P = C + A$ "depends" on C and A, and $C = P - A$ "depends" on P and A. It can be easily verified that any pair of intersecting edges have the same property: The third edge located in the same face of the tetrahedron can be determined by the first two by a simple linear relationship.

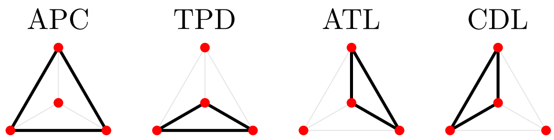
Figure 4: Graph of tetrahedron, edges belomging to the APC face highlighted.



Once a 2d-plane is defined, an additional vector may be sufficient to cover the whole 3d-space. Nonetheless, this third vector needs to be linearly independent of any pair of vectors of the 2d plane—that is, it cannot be expressed as a linear combination of any two vectors on that plane. Again, an analogous property can be observed in the tetrahedron: Say we only have information about the indices of the APC plain; A, C and P are not sufficient to determine a thanatological age T, death cohort D, or lifespan L (the three indices that do not belong to the APC plane). So, T, D and L are "independent" of the overall information that can be extracted from the APC plane. However, if two of the three constituent time indices of the APC plane are known (the third one would be unnecessary as it could be derived from the other two), the additional information provided by any of the three "independent" indices T, D or L would be sufficient to cover the whole tetrahedron. For example, suppose we have information about thanatological age T in addition to C and P, then $A = P - C$, $D = P + T$ and $L = T + A = T + P - C$.

Hence, as with vectors in a 3d space, any triad of indices that are independent of each other—that is, none of them can be expressed as the sum or the difference of the other two—generates a full hexad identity or, using an analogous terminology, covers the whole "space" of demographic indices presented in this paper. Graphically, this is equivalent to choosing any combination of three indices that do not belong to the same face of the tetrahedron, i.e., that do not form one of the four triad identities represented in Table 3.

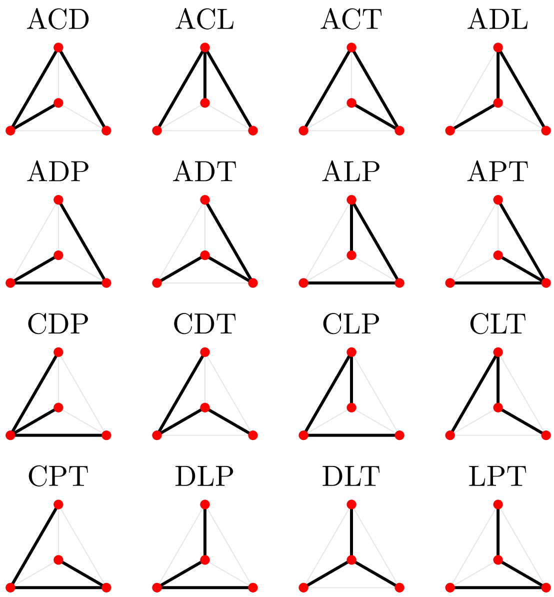
Table 3: The four triad identities on the tetrahedron (same orientation)



When graphed in this way, it is clearer that every triad identity surface lacks a connection to the opposite vertex of the tetrahedron. We therefore say that each of the triad identities is incomplete. Table 4 gives the full set of sixteen index-triads that are complete in this sense. It can be verified that each of these triads implies the full hexad identity.

12 different pairs of time indices generate one of the four triad identities in Table 3(and three do not), but no dyad will generate the hexad identity since a third "independent" time dimension is necessary. Similarly, any quad of indices is sufficient to complete the hexad identity, as at least one of them will not belong to the same face of the tetrahedron, but a triad may be sufficient if they do not all belong to the same plane forming a triad identity.

Table 4: All complete sets of three indices, graphed given the previous orientation of the tetrahedron.



The extension of time axes.

We have said that planes defined by the four triad identities are parallel to the faces of the the above-described tetrahedron. In imagining this three-dimensional relationship, we are no longer confined to the extent of the tetrahedron used thus far for orientation. Instead each of its edges extends a certain distance in either direction. It may therefore help to first consider the extension of each axis (or index). Some indices have a lower bound of zero and an upper bound set by the maximum length of life, ω, while others are boundless. A, T, and L are clearly in the range $[0, \omega]$.[5] P, C, and D are bounded only by the inception and extinction of our species, but may be thought of as boundless for practicality, or benchmarked to our earliest and most recent observations for even more practicality.[6] As an abstraction, however, the dimension of calendar time in this model is infinite. Of the four

[5]It's best to imagine some number like 122.45 years, for ω, rather than infinity. This is the longevity record at the time of this writing. Jeanne L. Calment would have had $T = 122.45$ at birth, $A = 122.45$ at death, and $L = 122.45$ for her entire life.

[6]We explain the choice of the word "benchmarked". Say we have a data series that runs from 1751 to 2011, and an upper age interval of 110+. Then we could say that P is in the range $[1751, 2011]$, but by another reading, P must range from at least as early as the earliest C and until at least as late as the latest D. Someone dying at 110 in 1751 had a C of 1640, and an infant born in 2011 that is destined to live to 110 will die in 2121. In this case a P that *contains* the observed population will extend well before and after the observed data series, even moreso if we take into account that $\omega > 110$.

triad identities, only one lacks an unbounded dimension, the TAL. Adding the absent dimension to TAL therefore makes its 3d extension boundless. In this way, we may imagine a prism-like construct, where T, A, and L, compose the faces of a triangular cross-section of the prism, which extends infinitely "through" the triangle. We can think of the TAL triangle passing through time, extending the population forward to infinity. In this case, the TAL triangle may take either the period or cohort perspective, and this will be explained later.

There are also numerous ways that this three dimensional construct can be proportioned, of which we present two in this paper. The first stems from the respect given to right angles in the most common representation of the Lexis diagram. For this reason, it will likely be the most intuitive rendition of the model, and it will be presented first. The second version presented is isotropic with respect to time units in each of the six temporal indices. In this case, the four tripartite identities are based on equilateral triangles between their three constituent indices, and the four planes are joined together such that each is parallel to a face from the regular tetrahedron, a construct known in geometry as an octahedral-tetrahedral honeycomb.

Intersecting planes

The APC, TPD, TAL, and CDL planes can be conceived of as *compressions* of this 3d space, or as cross-sections of the 3d space. To compress the space in this sense is to ignore the missing dimension, whereas a cross-section sets a given triad identity against a particular position of the absent dimension. APC has thus far always been treated as a compression, and myriad such uses and examples are familiar to demographers. A compressed TPD diagram has thus far only appeared in Villavicencio and Riffe (2015) as an aid in proving the symmetries between chronological and thanatological age in stationary populations, and a cross-sectional one has never appeared. Cross-sectional TAL diagram and surfaces have thus far only appeared in Riffe, T. et al. (2015). This TAL usage was selected for the 1915-1919 birth cohort, and therefore belongs to the 3d space, . We have been unable to locate an example in the literature of a compressed TAL diagram, but it seems likely one will have arisen in the field of biology, albeit with no relation to the present discourse. We suppose that CDL diagrams of any kind are yet-unknown.

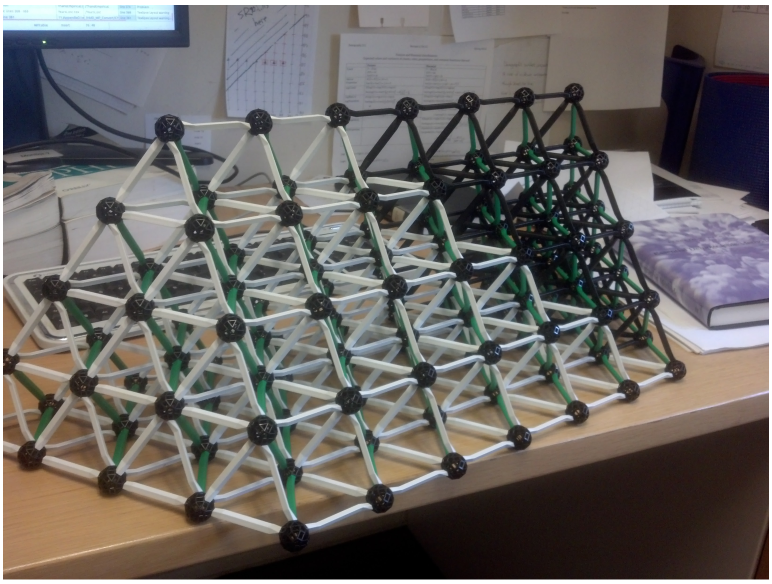
APCT

*section ignored for the present: plan for a table with different distilled 3d views of the contruct, probably vector snapshots of an RGL model. Probably each will be centered on a single tetrahedron, with sides labeled, and one face of it (triad identity) will be highlighted and extended, and given depth). Each face will be seen from 2 angles, and also for the right and isotropic projections. use lighting, etc, to give depth. Need to work hard on this, but no rush.

*Again, the following stuff is from the original proposal and will be completely superceded, though tidbits still hold.

The complete space can be described in geometry nomenclature as the tetrahedral-octahedral honeycomb, which is a kind of space-filling tessellation.[7] This three dimensional space is not only useful for the sake of formalizing observed temporal relationships, but also for encolosing demographic time in the past and future (e.g., before the first census and after the most recent census).

Figure 5: A mock-up example of the unified model of demographic time.[8]

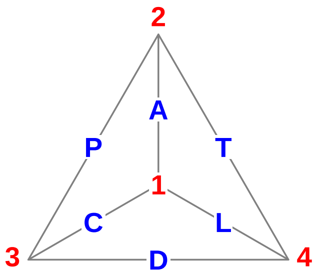


A property of the geometry that we propose is isotropy, the time units in every direction (with respect to each index) are proportional. In Lexis-nomenclature, the 3d projections of an AP square, and AC or PC parallelograms are all congruent shapes known as regular trigonal trapezohedra (RTT). The orientation of a given RTT uniquely defines the Lexis shape in question. Similar constructs exist in the other time dimensions, and these will also be described.

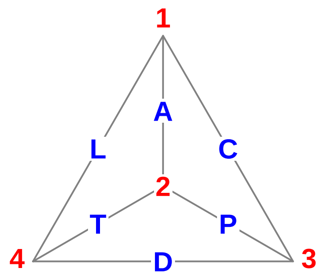
[7]Constructs following this geometry exist both in nature and in man-made structures.

[8]This and other figures to be replaced with vector graphics, although I may bring this model to the presentation, since it helps explain concepts.

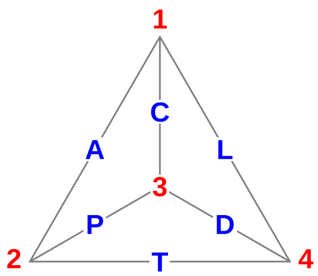
Figure 6: Some variants of the graph of the APCTDL tetrahedron.



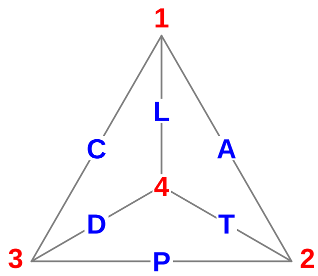
(a) Vertex 1 in middle. APC Northwest.



(b) Vertex 2 in middle. APC Northeast.



(c) Vertex 3 in middle. APC Northwest.



(d) Vertex 4 in middle, as in Figure 3. APC base.

A Variants of tetrahedron graph

The graph depicted in Figure 3 could be drawn with any of the four vertices in the middle of the triangle (as well as other inversions and rotations). These would all serve equally well to present the same aspects of the model, and we have no insight as to whether one of these renditions is more or less intuitive. Figure 6 provides four perspectives on the tetrahedron, for the case that this aids in understanding. The reader may make a paper tetrahedron, with labeled edges and vertices to be convinced that these are identical graphs.

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