A unified model of demographic time

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Abstract

We describe a three-dimensional model that relates six different aspects of lifespans and time. The six aspects of demographic time considered are chronological age, thanatological age, lifespan, year of birth, year of death, and period. Two versions of the model are described: a relatively intuitive extension of the right-angled Lexis diagram, and an isotropic extension based on the regular tetrahedron.

The so-called Lexis diagram relates the chronological age (A), period (P), and birth cohort (C) indices of demographic time, APC, but it does not account for remaining years of life (thanatological age), and other related time indices. The thanatological counterpart to APC is an identity between thanatological age (T), period (P), and death cohort (D), TPD. A third identity exists between chronological age (A), thanatological age (T), and lifespan (L), ATL, and a fourth between year of birth (C), year of death (D) and lifespan (L), CDL. Each of these four triad identities may be sufficiently described by any two of its consituent indices, making the third index redundant. Each of these four identities also lacks a major dimension of time. The ATL identity lacks calendar time, the CDL identity is ageless,

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APC lacks an endpoint in time, and TPD lacks a starting point in time. We refer to these four identities as triad identities.

To our knowledge, the only triad identity that has received serious treatment at the time of this writing is the APC identity. Different aspects of the APC identity have been discussed since at least 1868 (Knapp 1868), and discussion remains lively today. Here it is our objective to relate the six major indices of time in a geometric identity, in much the same spirit as the work on APC relationships done between the late 1860s and mid 1880s.[1]

A tetrahedron relates the six time indices.

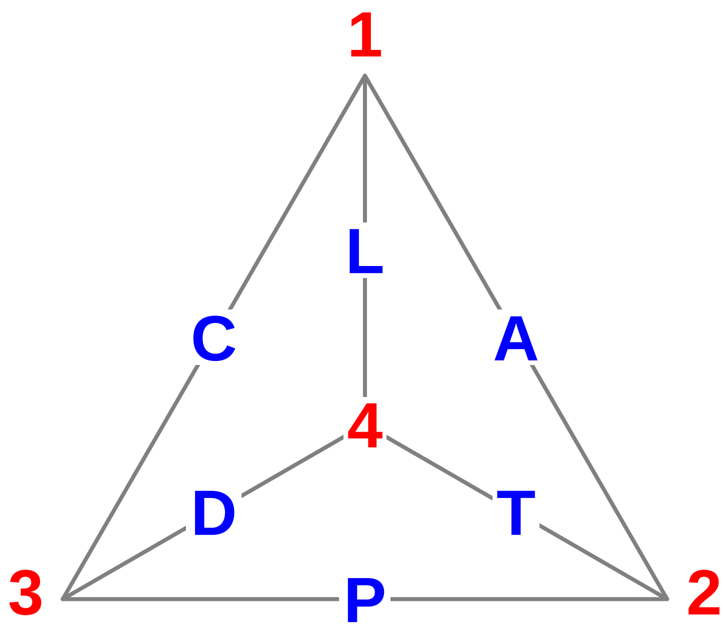
Each of the four above-mentioned triad identities may be thought of as a two-dimensional plane fully defined by any two of its three constituent time indices. In this case, we may imagine the third "lacking" dimension as providing depth, for the sake of a mental image. Having a non-redundant third dimension implies a multitude of parallel planes for the given identity, each plane belonging to a unique value of the third time dimension. Any of the identities can be extended in this way to fill a space. A space derived by extending any of the triad identies into its lacking dimension implies each of the other triad identities, making a total of six time indices. In essence, the four triad identities may be thought of as parallel to the four faces of a tetrahedron. In this case, the four "lacking" dimensions may be assigned to the four vertices of the tetrahedron, and the six demographic time indices match to the six edges of the tetrahedron. This three-dimensional construct unifies the six indices of demographic time, and is the subject of this paper.

Let us first more rigorously define the previously-mentioned tetrahedron. Luckily, the edges and vertices of a tetrahedron are easily rendered in a two-dimensional graph, as seen in Figure 1, with vertices labeled in red and the six time indices labeled as blue edges.[2] The reader may also imagine this graph as a transparent 3d object, in which case the four faces become aparent. There are two intuitive ways to imagine the graph as 3d, either the vertex 4 is on top, and we gaze from a bird's-eye-view, or the vertex 4 is in the back, behind the other three vertices. Assume we gaze from the top, for the sake of description.

[1]See e.g., Keiding (2011) for an overview of that literature.

[2]The same graph could be composed in four basic ways, depending on which vertex is in the middle. These are given in an appendix.

Figure 1: Graph of tetrahedron, with edges labeled by the six demographic time indices.

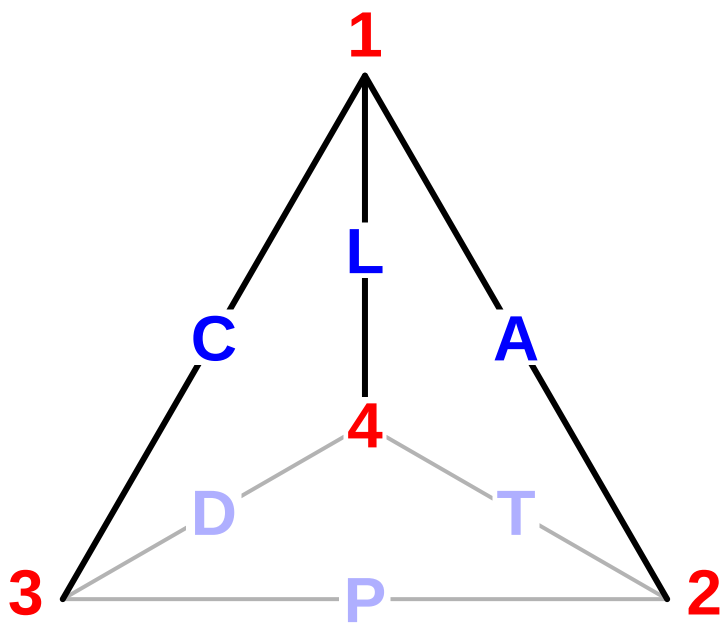


Information criteria to derive the tetrahedron.

The edges APC at the base define the much-studied APC plane. If the only information we have is chronological age, period, and birth cohort (or just two of these), then we have no access to the vertex 4. Each of the faces of the tetrahedron has this quality. The South face TDP has no access to 1. The Northeast face, ATL has no connection to 3, and the Northwest face CDL lacks a connection to 2. The triads that make up the faces of the tetrahedron are stuck in "flatland". However, there are twenty ways in total to choose three time indices from our total of six, and the four above-named triads are the only four of these that will not yield the full 3d space and imply the other three. The sixteen other combinations of three indices will recreate the full tetradhedron (hexad identity).

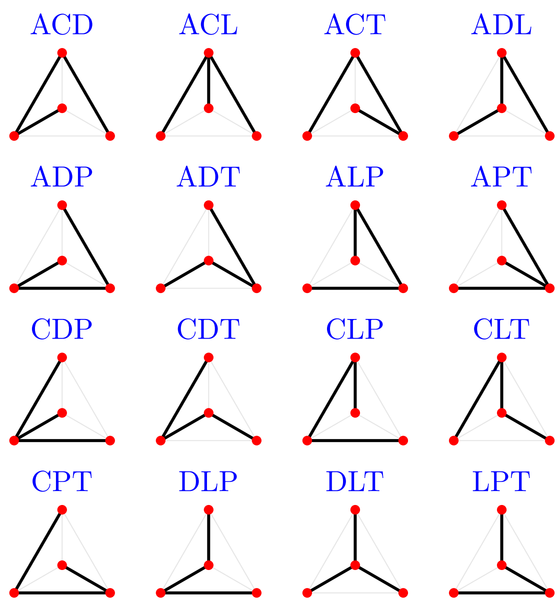
For example, say we are at vertex 1, and we therefore have information on year of birth C, completed lifespan L, and chronological age A. Clearly, A and C imply P ($C + A = P$). A and L imply T ($L - A = T$). Finally, C and L imply D ($C + L = D$), and we have the full hexad identity. In the tetrahedron graph, we have three edges that connect to the four vertices. This is the essential property of a fully informed triad.

Figure 2: Graph of tetrahedron, edges eminating from vertice 1 highlighted.



It is easily verified that each vertex has this property. However, there are twelve other sets of three that also have this property. To locate these "hidden" triads, first note that each index has an opposite index, with which it shares no information. These pairs are A-D, L-P, and C-T, and can be found in Figure 1 as the three sets of perpendicular edges. Each of these pairs can be completed into a 'fully-informed" triad by the addition of any of the other four indices (thereby connecting the edges). Doing so for each of the opposite pairs will yield the remaining twelve triads.

Table 1: All sets of three indices that imply the full six indices, graphed given the previous orientation of the tetrahedron.



A like-organized table for the four triad identities simply highlights each of the four faces of the tetrahedron, as seen in Table 2. When graphed in this way, the vertex lacking a connection becomes clearer. We therefore say that each of the triad identities is incomplete.

Table 2: The four triad identities on the tetrahedron (same orientation)

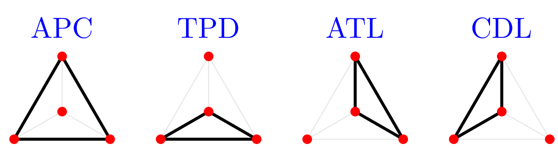


Table 1 gives the full set of sixteen index-triads that are complete in this sense. It can be verified that each of these triads implies the full hexad identity. This property is comparable to the reducibility characteristic of triad identities. For example, for the APC identity, the list of dyads that give full information is shorter: AC, AP, PC. The triads in Table 1 give analogous information for the APCTDL identity. It may be further stated that no dyad of indices will give the hexad identity, and any quad (or greater) of indices will yield the hexad identity. Any index complemented by any index other than its opposite will imply one of the triad identities.

The extension of time axes.

We have said that planes defined by the four triad identities are parallel to the faces of the the above-described tetrahedron. In imagining this three-dimensional relationship, we are no longer confined to the extent of the tetrahedron that we have used thus far for orientation. Instead each of its edges extends a certain distance in either direction. It may therefore help to first consider the extension of each axis (or index). Some indices have a lower bound of zero and an upper bound set by the maximum length of life, ω, while others are boundless. A, T, and L are clearly in the range $[0, \omega]$.[3] P, C, and D are bounded only by the inception and extinction of our species, but may be thought of as boundless for practicality, or benchmarked to our earliest and most recent observations for even more practicality.[4] As an abstraction, however, the dimension of calendar time in this model is infinite. Of the four triad identities, only one lacks an unbounded dimension, the ATL. Adding the absent dimension to ATL therefore makes its 3d extension boundless. In this way, we may imagine a prism-like construct, where A, T, and L, compose the faces of a triangular cross-section of said prism, which extends infinitely "through" the triangle. We can think of the ATL triangle passing through time, extending the population forward to infinity. In this case, the ATL triangle may take either the period or cohort perspective, and this will be explained later.

There are also numerous ways that this three dimensional construct can be proportioned, of which we present two in this paper. The first stems from the respect given to right angles in the most common representation of the Lexis diagram. For this reason, it will likely be the most intuitive rendition of the model, and it will be presented first. The second version presented is isotropic with respect to time units in each of the six temporal indices.

[3] It's best to imagine some number like 122.45 years, for ω, rather than infinity. This is the longevity record at the time of this writing. Jeanne L. Calment would have had $T = 122.45$ at birth, $A = 122.45$ at death, and $L = 122.45$ for her entire life.

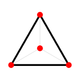
[4] We explain the choice of the word "benchmarked". Say we have a data series that runs from 1751 to 2011, and an upper age interval of 110+. Then we could say that P is in the range $[1751, 2011]$, but by another reading, P must range from at least as early as the earliest C and until at least as late as the latest D. Someone dying at 110 in 1751 had a C of 1640, and an infant born in 2011 that is destined to live to 110 will die in 2121. In this case a P that *contains* the observed population will extend well before and after the observed data series, even moreso if we take into account that $\omega > 110$.

In this case, the four tripartite identities are based on equilateral triangles between their three constituent indices, and the four planes are joined together such that each is parallel to a face from the regular tetrahedron, a construct known in geometry as an octahedral-tetrahedral honeycomb.

Intersecting planes

The APC, TPD, ATL, and CDL planes can be conceived of as *compressions* of this 3d space, or as cross-sections of the 3d space. To compress the space in this sense is to ignore the missing dimension, whereas a cross-section sets a given triad identity against a particular position of the missing dimension. APC has typically been treated as a compression, and myriad such examples are familiar to demographers. A compressed TPD diagram has thus far only appeared in Villavicencio and Riffe (2015) as an aid in explaining a mathematical proof. A cross-sectional ATL diagram and surfaces have thus far only appeared in Riffe, T. et al. (2015). This ATL usage was selected for the 1915-1919 birth cohort, and therefore belongs to the 3d space.

APC



The Lexis diagram has long been used in demography, both as a conceptual tool for structuring data and observations, as inspiration for work on statistical identification, and as the coordinate basis of contemporary Lexis-surfaces. Since the so-called Lexis diagram could have been named for others (Vandeschrick 2001, Keiding 2011), and since we compare with other temporal configurations, let us refer to it as the APC diagram, as seen in Figure 3 When a value (data) is structured by APC coordinates, we refer to it as an APC surface.

The APC diagram in Figure 3 represents years lived on the y axis, calendar years on the x axis, and birth cohorts as the right-ascending diagonals. This is the most common of several possible configurations of the APC dimensions. Individual lifelines are aligned in the cohort direction, starting with birth (filled circle) at chronological age zero, and death ()

Any APC surface can be interpreted along each of these three dimensions of temporal structure. Such interpretation is a descriptive task, and it does not succumb to problems of overidentification. Variation along these three dimensions can not be parsimoniously separated into the three effects of A, P, and C. This is the so-called APC problem, and it is not the concern of the present work.

It has long been noted (Zeuner 1869, Perozzo 1880) that the birth cohort dimension, as represented in Figure 3, is relatively longer than either the age or years axes. If a right angle and unity aspect ratio is forced between any two of the APC dimensions, the third dimension is always be stretched by $\sqrt{2}$. Another long-standing, but less common variant, is to represent

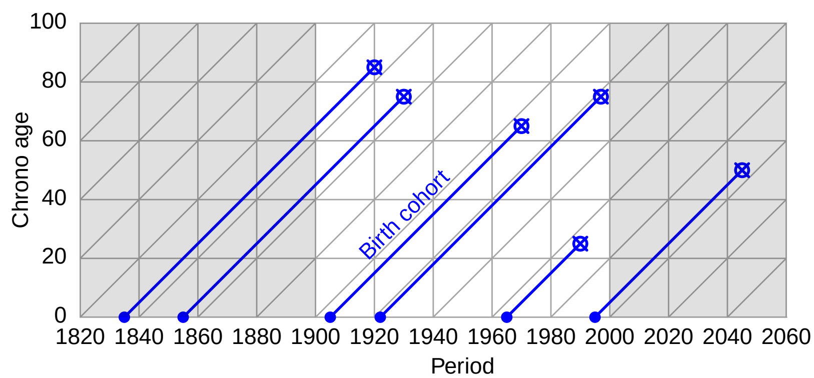
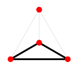
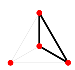


Figure 3: Lifelines in the APC diagram

TPD 

Thanatological age (T), period (P) and death cohort (D) form a coordinate system best imagined as the opposite of APC. One may take the same lifelines from Figure 3 and realign them in descending fashion to create the diagram in Figure 4

ATL 

The second plane is ATL, a valid coordinate system for processes that vary over the lifecourse, but not over time (P). Since the lifecourse belongs to the cohort perspective, it is best to think of the ATL plane as belonging to some particular birth cohort. Alternatively, an ATL triangle may be taken as a cross-section along through the period dimension, a sort of synthetic ATL plane.

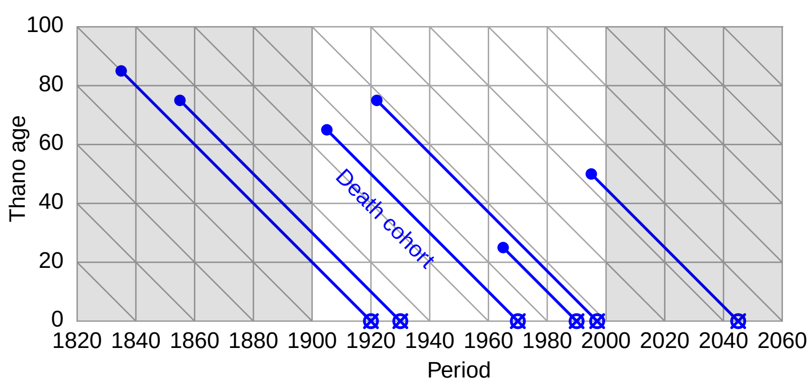


Figure 4: Lifelines in the TPD diagram

APCT

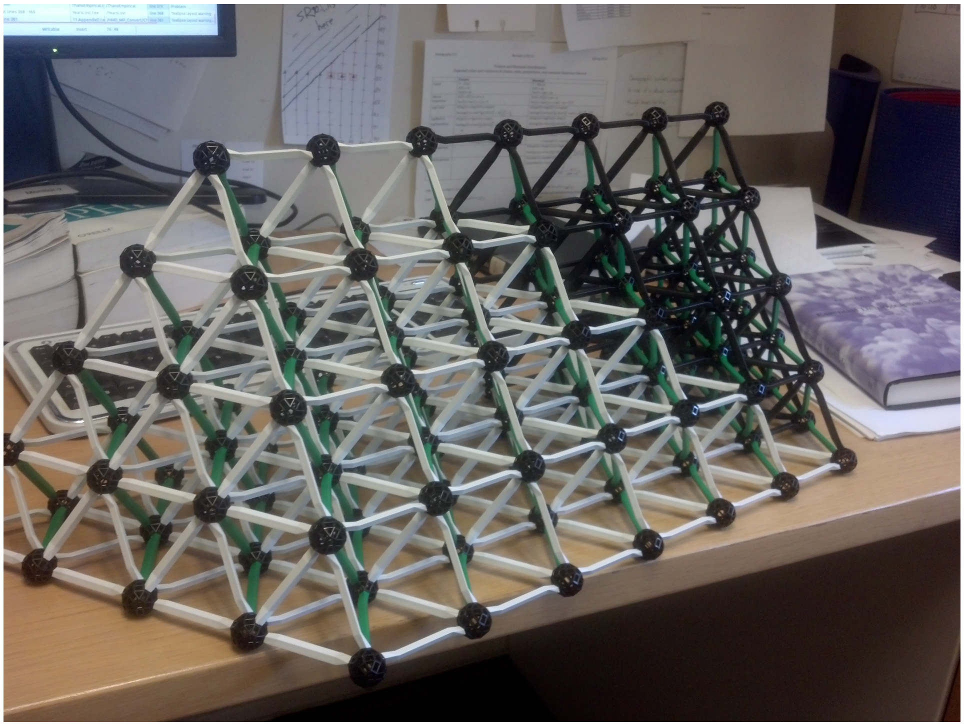
I propose a geometric identity that unifies all such temporal notions into a single (simple) spatial relationship that serves as an omnibus conceptual aid to demographers, much as the Lexis diagram does for APC relationships. The full result is a three dimensional space that can be disected by any of four different planes, each of which is parallel to the faces of a regular tetrahedron (see Figure 5 for a first mock-up of the model). Each dissecting plane relates three indices of demographic time in proportion to one another (1:1:1 ternary aspect ratio). The complete space can be described in geometry nomenclature as the tetrahedral-octahedral honeycomb, which is a kind of space-filling tessellation.[5] One of these planes is the familar Lexis plane (horizontal planes in Figure 5, and the other three will be new surprises for demographers. This three dimensional space is not only useful for the sake of formalizing observed temporal relationships, but also for encolosing demographic time in the past and future (e.g., before the first census and after the most recent census).

A property of the geometry that I propose is that the time units in every direction (with respect to each index) are proportional. The Lexis diagram based on right angles and 45° birth cohort lines does not have this property, whereas Lexis diagrams and surfaces based on equilateral triangles, such as some early proposals (inter alia, Lexis 1875, Lewin 1876), the masterful stereogram of Perozzo (1880), or the more recent APC diagram of Ryder (1980), do have this property. The disecting planes of the model I propose are likewise composed of equilateral triangles. In Lexis nomenclature, the 3d projections of an AP square, and AC or PC parallelograms are all congruent shapes known as regular trigonal trapezohedra (RTT). The orientation of a given RTT uniquely defines the Lexis shape in question. Similar constructs exist in the other time dimensions, and these will also be described.

[5] Constructs following this geometry exist both in nature and in man-made structures.

[6] This and other figures to be replaced with vector graphics, although I may bring this model to the presentation, since it helps explain concepts.

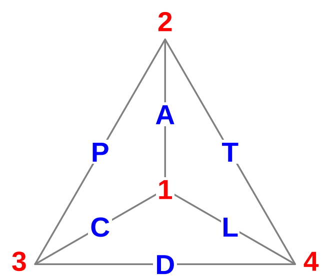
Figure 5: A mock-up example of the unified model of demographic time.[6]



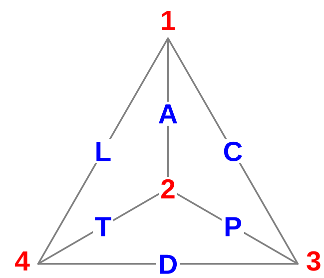
A Variants of tetrahedron graph

The graph depicted in Figure 1 could be drawn with any of the four vertices in the middle of the triangle (as well as other inversions and rotations). These would all serve equally well to present the same aspects of the model, and we have no insight as to whether one of these renditions is more or less intuitive. Figure 6 provides for perspectives on the tetrahedron, for the case that this aids in understanding. The reader may make a paper tetrahedron, with labeled edges and vertices to be convinced that these are identical graphs.

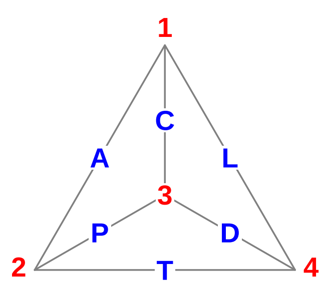
Figure 6: Some variants of the graph of the APCTDL tetrahedron.



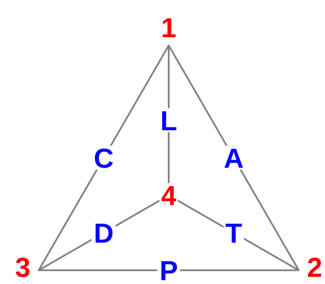
(a) Vertex 1 in middle. APC Northwest.



(b) Vertex 2 in middle. APC Northeast.



(c) Vertex 3 in middle. APC Northwest.



(d) Vertex 4 in middle, as in Figure 1. APC base.

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