A unified model of demographic time

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Abstract

We describe a three-dimensional model that relates six different aspects of lifespans and time. The six aspects of demographic time considered are chronological age, thanatological age, lifespan, year of birth, year of death, and period. Two versions of the model are described: a relatively intuitive extension of the right-angled Lexis diagram, and an isotropic extension based on the regular tetrahedron.

The so-called Lexis diagram relates the chronological age (A), period (P), and birth cohort (C) indices of demographic time, APC, but it does not account for remaining years of life (thanatological age), and other related time indices. The thanatological counterpart to APC is an identity between thanatological age (T), period (P), and death cohort (D), TPD. A third identity exists between chronological age (A), thanatological age (T), and lifespan (L), ATL, and a fourth between year of birth (C), year of death (D) and lifespan (L), CDL. Each of these four triad identities may be sufficiently described by any two of its consituent indices, making the third index redundant. Each of these four identities also lacks a major dimension of time. The ATL identity lacks calendar time, the CDL identity is ageless, APC lacks years left, and TPD lacks years lived.

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To our knowledge, the only tripartite identity that has received serious treatment at the time of this writing is the APC identity. Different aspects of the APC identity have been discussed since at least 1868 (Knapp 1868), and discussion remains lively today. Here it is our objective to relate the six major indices of time in a geometric identity, in much the same spirit as the work on APC done between the late 1860s and mid 1880s.[1]

Each of the four above-mentioned triad identities may be thought of as a two-dimensional plane fully defined by any two of its three constituent time indices. In this case, we may imagine the third "lacking" dimension as providing depth, for the sake of a mental image. Having a third dimension implies a multitude of parallel planes for the given identity, each plane belonging to a unique value of the third time dimension. Any of the identities can be extended in this way to fill a space. A space derived by extending any of the triad identies into its lacking dimension implies each of the other triad identities, making a total of six time indices. In essence, the four triad identities may be thought of as parallel to the four faces of a tetrahedron. In this case, the four "lacking" dimensions may be assigned to the four vertices of the tetrahedron, and the six demographic time indices match to the six edges of the tetrahedron. This three-dimensional construct unifies the six indices of demographic time, and is the subject of this paper.

Let us first more rigorously define the previously-mentioned tetrahedron. Luckily, the edges and vertices of a tetrahedron are easily rendered in a two-dimensional graph, as seen in Figure 1, with vertices labeled in red and the six time indices labeled as blue edges.[2] The reader may also imagine this graph as a transparent 3d object, in which case the four faces become aparent. There are two intuitive ways to imagine the graph as 3d, either the vertex **4** is on top, and we gaze from a bird's-eye-view, or the vertex **4** is in the back, behind the other three vertices. Assume we gaze from the top, for the sake of a consistent description.

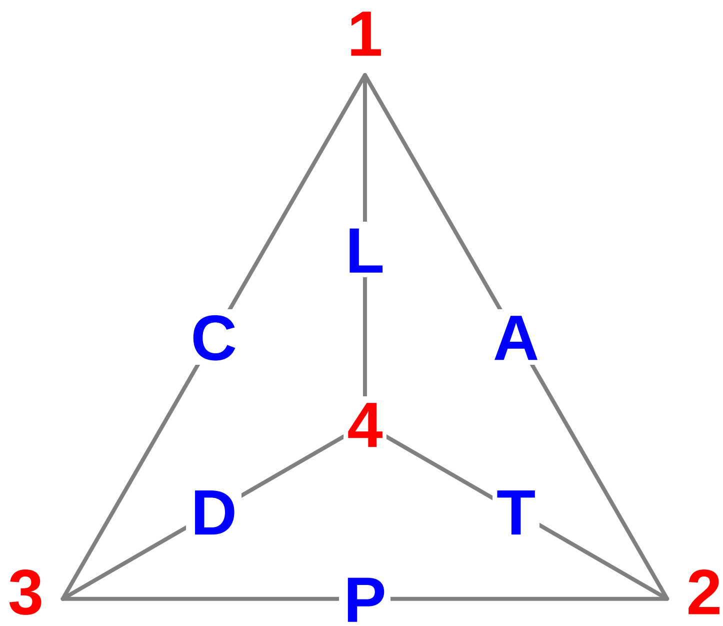
The edges **APC** at the base define the much-studied APC plane. If the only information we have is chronological age, period, and birth cohort (or just two of these), then we have no access to the vertex **4**. Each of the faces of the tetrahedron have this quality. The South face **TDP** has no access to **1**. The Northeast face, **ATL** has no connection to **3**, and the Northwest face **CDL** lacks a connection to **2**. The triads that make up the faces of the tetrahedron are stuck in "flatland". However, there are twenty ways to

[1]See e.g., Keiding (2011) for an overview of that literature.

[2]The same graph could be composed in four basic ways, depending on which vertex is in the middle. These are given in an appendix.

choose three time indices from our total of six, and the four above-name triads are the only four of these that will not yield the full 3d space and imply the other three. The sixteen other combinations of three indices will recreate the full tetradhedron (hexad identity).

Figure 1: Graph of tetrahedron, with edges labeled by the six demographic time indices.



For example, say we are at vertex **1**, and we therefore have information on year of birth **C**, completed lifespan **L**, and chronological age **A**. Clearly, **A** and **C** imply **P** ($C + A = P$). **A** and **L** imply **T** ($L - A = T$). Finally, **C** and **L** imply **D** ($C + L = D$), and we have the full hexad identity. Each vertex has this property, as is easily verified. However, there are twelve other sets of three that also have this property. A simple algorithm to produce a listing of these sets of three (with redundancy) goes as follows:

1. Start at a vertex, V^i
2. Select two of the three edges that touch V^i.

3. The third "eliminated" edge, E^-, links to some other vertex, V^-.
4. Two other edges also join to V^-.
5. Either of these vertices will complete the trio in place of E^-.
6. Repeat from (1) by eliminating in turn each of the other two edges touching V^i.
7. Repeat for each of the other four vertices.

This algorithm will produce an exhaustive set of all combinations of three indices that imply the full 3d space, but it will produce each "hidden" set twice, for 24 total. A complete and non-redundant set of such triads, including the original indices, is given in Table 1. Using intuitive logic, it can be verified that each of these triads implies the full hexad identity.

Table 1: All sets of three indices that imply the full six indices.

ACD	**ACL**	**ACT**	**ADL**
ADP	**ADT**	**ALP**	**APT**
CDP	**CDT**	**CLP**	**CLT**
CPT	**DLP**	**DLT**	**LPT**

This relationship is comparable to the reducibility characteristic of triad identities. For the APC identity, the list of dyads that give full information is easier to derive: **AC**, **AP**, **PC**. The triads in Table 1 give analogous information for the APCTDL identity. It may be further stated that no dyad of indices can give the hexad identity, and any quad (or greater) of indices will yield the hexad identity. Further, each index can be said to have an opposite index, with which it shares no information, and which is therefore maximally complimentary. These pairs are **A-D**, **L-P**, and **C-T**.

We have said that the four triad identities are parallel to the faces of the the above-described tetrahedron. In imagining this three-dimensional relationship, we are no longer confined to the extent of the tetrahedron, but instead each of its edges extends a certain distance in either direction. It may therefore help to first consider the extension of each axis (or index). Some indices have a lower bound of zero and an upper bound set by the maximum length of life, ω, while others are boundless. **A**, **T**, and **L** are clearly in the range $[0, \omega]$.[3] **P**, **C**, and **D** are bounded only by the inception and

[3] It's best to imagine some number like 122.45 years, for ω, rather than infinity. This is the longevity record at the time of this writing. Jeanne L. Calment would have had $T = 122.45$ at birth, $A = 122.45$ at death, and $L = 122.45$ for her entire life.

extinction of our species, but may be thought of as boundless for practicality, or benchmarked to our earliest and most recent observations for even more practicality.[4] As an abstraction, however, the dimension of calendar time in this model is infinite. Of the four triad identities, only one lacks an unbounded dimension, the **ATL** . Adding the absent dimension to **ATL** therefore makes it boundless. In this way, we may imagine a prism-like construct, where **A**, **T**, and **L**, compose the faces of a triangular cross-section of said prism, which extends infinitely "through" the triangle. We can think of the **ATL** triangle passing through time, extending the population forward to infinity. In this case, the **ATL** triangle may take either the period or cohort perspective, and this will be explained later.

There are also numerous ways that this three dimensional construct can be proportioned, of which I present two in this paper. The first stems from the respect given to right angles in the most common representation of the Lexis diagram. For this reason, it will likely be the most intuitive rendition of the model, and it will be presented first. The second version presented is isotropic with respect to time units in each of the six temporal indices. In this case, the four tripartite identities are based on equilateral triangles between their three constituent indices, and the four planes are joined together such that each is parallel to a face from the regular tetrahedron, a construct known in geometry as an octahedral-tetrahedral honeycomb.

In this paper we first state the relationships between all six dimensions of demographic time. The APC, TDL, and ATL planes are introduced as degenerate cases of the unified model. The case of the ATLC cross-sectional plane is introduced, followed by the full three-dimensional unified model, the APCTDL model. Finally, we demonstrate the use of this coordinate system for the case of end-of-life trajectories of some characteristics of morbidity. I explain the utility of this model by demonstrating a case where heterogeneity with respect to unaccounted-for time dimensions has caused serious misunderstandings in the scientific literature and in public policy.

[4] We explain the choice of the word "benchmarked". Say we have a data series that runs from 1751 to 2011, and an upper age interval of 110+. Then we could say that **P** is in the range $[1751, 2011]$, but by another reading, **P** must range from at least as early as the earliest **C** and until at least as late as the latest **D**. Someone dying at 110 in 1751 had a **C** of 1640, and an infant born in 2011 that is destined to live to 110 will die in 2121. In this case a **P** that *contains* the observed population will extend well before and after the observed data series, even moreso if we take into account that $\omega > 110$.

Intersecting planes

The model can be introduced in terms of four planes intersecting in Euclidean 3d space. Two of the planes motivate the model, and the other two are in this case artifactual, although they may have demographic meaning. The first plane is the widely used APC plane. This plane omits remaining years of life (T).

APC

The Lexis diagram has long been used in demography to relate chronological age (A) with birth cohorts (C) and calendar years (P). Since the so-called Lexis diagram could have been named for others (Vandeschrick 2001, Keiding 2011), and since we compare with other temporal configurations, let us refer to it as the APC diagram, as seen in Figure 2 When a value (data) is structured by APC coordinates, we refer to it as an APC surface.

The APC diagram in Figure 2 represents years lived on the y axis, calendar years on the x axis, and birth cohorts as the right-ascending diagonals.

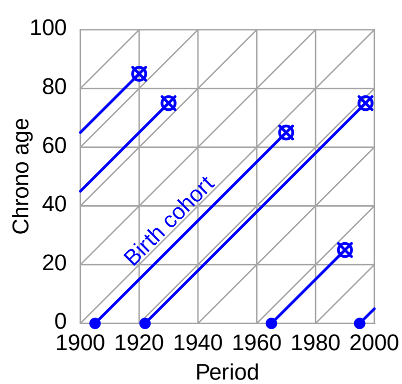


Figure 2: Lifelines in the APC diagram

This is the most common of several possible configurations of the APC dimensions. Individual lifelines are aligned in the cohort direction, starting with birth (filled circle) at chronological age zero, and death ()

Any APC surface can be interpreted along each of these three dimensions of temporal structure. Such interpretation is a descriptive task, and it does not succumb to problems of overidentification. Variation along these three dimensions can not be parsimonsiously separated into the three effects of A, P, and C. This is the so-called APC problem, and it is not the concern of the present work.

It has long been noted (Zeuner 1869, Perozzo 1880) that the birth cohort dimension, as represented in Figure 2, is relatively longer than either the age or years axes. If a right angle and unity aspect ratio is forced between any two of the APC dimensions, the third dimension is always be stretched by $\sqrt{2}$. Another long-standing, but less common variant, is to represent

TPD

Thanatological age (T), period (P) and death cohort (D) form a coordinate system best imagined as the opposite of APC. One may take the same lifelines from Figure 2 and realign them in descending fashion to create the diagram in Figure 3

ATL

The second plane is ATL, a valid coordinate system for processes that vary over the lifecourse, but not over time (P). Since the lifecourse belongs to the cohort perspective, it is best to think of the ATL plane as belonging to some particular birth cohort. Alternatively, an ATL triangle may be taken as a cross-section along through the period dimension, a sort of synthetic ATL plane.

APCT

I propose a geometric identity that unifies all such temporal notions into a single (simple) spatial relationship that serves as an omnibus conceptual aid to demographers, much as the Lexis diagram does for APC relationships. The full result is a three dimensional space that can be disected by any of four different planes, each of which is parallel to the faces of a regular tetrahedron (see Figure 4 for a first mock-up of the model). Each dissecting plane relates three indices of demographic time in proportion to one another

(1:1:1 ternary aspect ratio). The complete space can be described in geometry nomenclature as the tetrahedral-octahedral honeycomb, which is a kind of space-filling tessellation.[5] One of these planes is the familar Lexis plane (horizontal planes in Figure 4, and the other three will be new surprises for demographers. This three dimensional space is not only useful for the sake of formalizing observed temporal relationships, but also for encolosing demographic time in the past and future (e.g., before the first census and after the most recent census).

A property of the geometry that I propose is that the time units in every direction (with respect to each index) are proportional. The Lexis diagram based on right angles and 45° birth cohort lines does not have this property, whereas Lexis diagrams and surfaces based on equilateral triangles, such as some early proposals (inter alia, Lexis 1875, Lewin 1876), the masterful stereogram of Perozzo (1880), or the more recent APC diagram of Ryder

[5]Constructs following this geometry exist both in nature and in man-made structures.

[6]This and other figures to be replaced with vector graphics, although I may bring this model to the presentation, since it helps explain concepts.

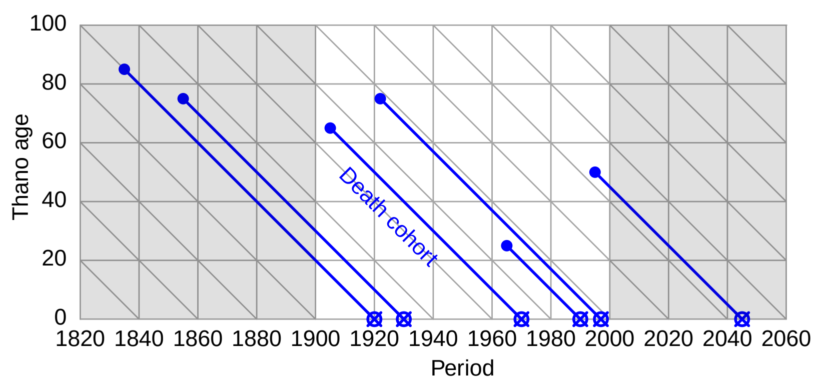
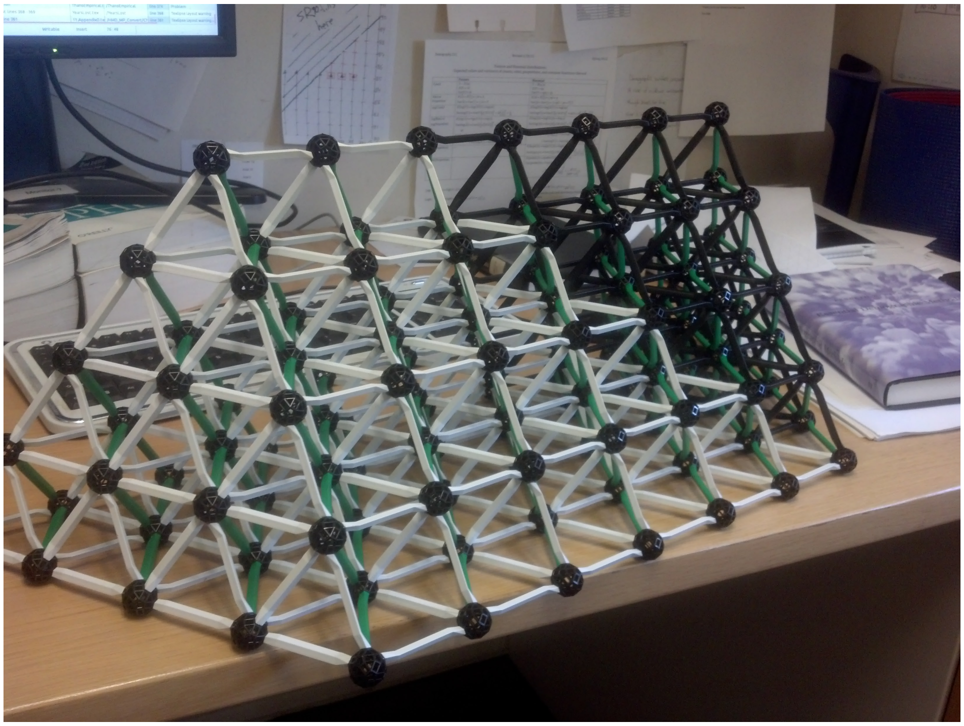


Figure 3: Lifelines in the TPD diagram

Figure 4: A mock-up example of the unified model of demographic time.[6]

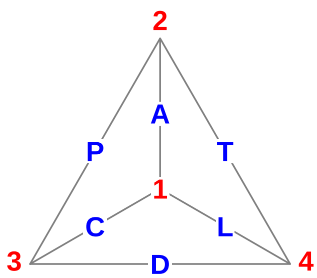


(1980), do have this property. The disecting planes of the model I propose are likewise composed of equilateral triangles. In Lexis nomenclature, the 3d projections of an AP square, and AC or PC parallelograms are all congruent shapes known as regular trigonal trapezohedra (RTT). The orientation of a given RTT uniquely defines the Lexis shape in question. Similar constructs exist in the other time dimensions, and these will also be described.

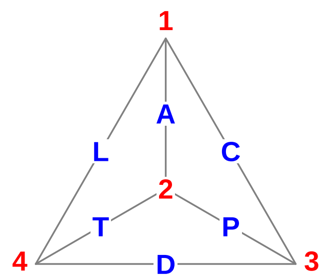
A Variants of tetrahedron graph

The graph depicted in Figure 1 could be drawn with any of the four vertices in the middle of the triangle (as well as other inversions and rotations). These would all serve equally well to present the same aspects of the model, and we have no insight as to whether one of these renditions is more or less intuitive. Figure 5 provides for perspectives on the tetrahedron, for the case that this aids in understanding. The reader may make a paper tetrahedron, with labeled edges and vertices to be convinced that these are

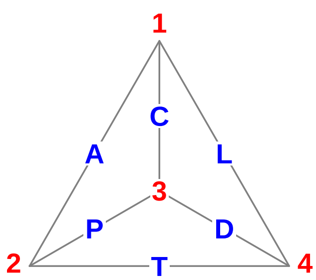
Figure 5: Some variants of the graph of the APCTDL tetrahedron.



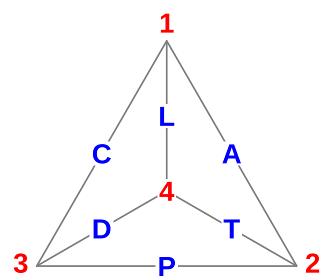
(a) Vertex **1** in middle. **APC** Northwest.



(b) Vertex **2** in middle. **APC** Northeast.



(c) Vertex **3** in middle. **APC** Northwest.



(d) Vertex **4** in middle, as in Figure 1. **APC** base.

identical graphs.

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