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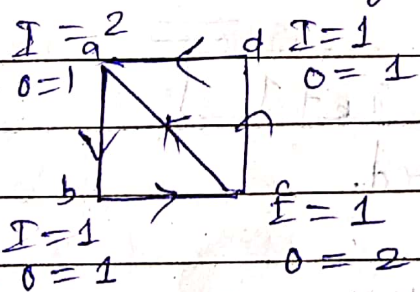
Some Imp Results:-

1. An undirected contains Eulerian circuit iff it is connected & all its vertices are of even degree.
2. An undirected graph contain Eulerian path if and only if it is connected and there are 0 or 2 vertices of odd degree.
3. A directed graph contains a Eulerian circuit if & only if it is connected & incoming degree of each vertex is equal to outgoing degree of that vertex.
4. A directed graph contains a Eulerian path iff it is connected & the incoming degree of one vertex is one larger than its outgoing degree and the incoming degree of the other vertex is one less than its outgoing degree.



There should be one change in either indegree / outdegree or change in none of them.

→ Square, triangle are some regular graphs in which all paths, i.e. Hamiltonian path, circuit; Eulerian path, circuit will exist

→ By default, complete ~~dired~~ graph is undirected graph.

Planar Graph (G = V, E):-

Necessary condition to check whether the graph is

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planar or not for vertices greater than or equal to 3 ($v \geq 3$) :-

This is necessary condition
(1) $e \leq 3v-6$
(2) $e \leq 2v-4$ (if there is no cycle of length 3).
(3) $r \leq 2v-4$

not sufficient condition

therefore we can say maybe planar graph if any one of them is not satisfied, then it is sure it is not planar graph.

• (1) Prove that :- $e \leq 3v-6$ for a connected planar graph having vertices ≥ 3.
(every region is bounded by atleast 3 edges, each edge contributes / joins 2 edges)

Proof:- For a connected planar graph have vertices ≥ 3, each region (except the outside / infinite region), bounded by atleast 3 edges and each edge joins atmost 2 regions, in simpler terms when we have more than one edge $2e \geq 3r$ will be always true.

$$2e \geq 3r$$

$$\frac{2e}{3} \geq r$$

$$v - e + r = 2 \quad \text{(Euler's Formula)}$$

$$v - e + \frac{2e}{3} \geq 2$$

$$v - \frac{e}{3} \geq 2$$

$$3v - e \geq 6 \implies \boxed{e \leq 3v-6}$$

$O(n)$ ← no. of vertices (less complexity, faster than Kuratowski's condition)

(2) Proof: Each region will be bounded by atleast 4 edges

$\therefore 2e \geq 4r$

$\frac{2e}{4} \geq r$

$\frac{e}{2} \geq r$

~~v+e~~ $v - e + \frac{e}{2} \geq 2$

$v - \frac{e}{2} \geq 2$

$2v - e \geq 4$

$\Rightarrow \boxed{e \leq 2v - 4}$

define & draw the graph

(3) From first & result it is proved that $r \leq 2v - 4$.

(*) Sufficient conditions for existence of planar graph:-

(Imp) Kuratowski's Condition:-

A graph is planar if and only if it does not contain any subgraph that is two-degree isomorphic to within the vertices of degree-two to either K_5 or $K_{3,3}$.

(Exponential complexity)

$K_{3,3}$, ~~vertices = 3+3 = 6; edges = m×n = 3×3 = 9~~ $v=6$, $e=9$

Q.1) check whether K_4 and $K_{2,2}$ are planar or not.

K_4, vertices = 4, edges = 6

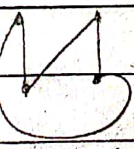
$e \leq 3v - 6$

$6 \leq 3 \times 4 - 6$

$6 \leq 12 - 6 = 6$

$e \leq 2v - 4$, 3

Satisfied



$K_{2,2}$, $v = 4$, $e = 4$

$v - e + r = 2$

$4 - 4 + r = 2$

$r = 2$

$e \leq 3v - 6$

$4 \leq 12 - 6$

$4 \leq 6$

Satisfied

$e \leq 8 - 4$

$4 \leq 4$

$r \leq 8 - 4$

$2 \leq 4$

$v - e + r = 2$

$6 - 4 + r = 2$

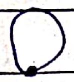
all condition satisfied $\Rightarrow$ Planar Graph.

GRAPH
(Continuation)

→ Prove : $v - e + r = 2$ [for edges]

(I) Basic of Induction:

For $n = 1$ (edge)



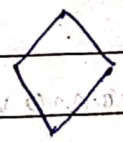
$2 - 1 + 1 = 2$ $\qquad$ $1 - 1 + 2 = 2$

(II) Induction Hypothesis:

Assume that result is true for $n = k$ (edges)

$V_k - e_k + r_k = 2 \Rightarrow r_k = e_k - V_k + 2$

G_k :



(III) Induction Step:

P.T: The result is true for $n = k+1$ (edges)

Case I:

$r_{k+1} = r_k + 1$

$V_{k+1} = V_k$

$e_{k+1} = e_k + 1$

edge added within region

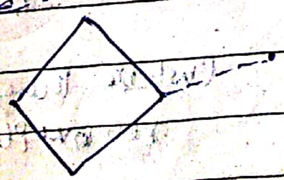
$r_{k+1} = e_{k+1} - V_{k+1} + 2$

$r_k + \cancel{1} = e_k + \cancel{1} - V_k + 2$

$\Rightarrow r_k = e_k - V_k + 2$

Case II:

edge added outside the region



$r_{k+1} = r_k$

$V_{k+1} = V_k + 1$

$e_{k+1} = e_k + 1$

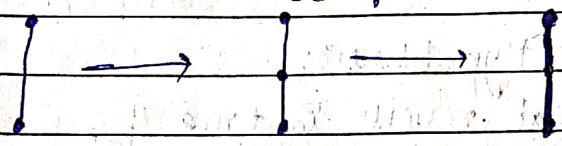
$r_k = e_k + 1 - (V_k + 1) + 2$

$\Rightarrow r_k = e_k - v_k + 2$

Hence proved.

(4) Isomorphic to within vertices of degree - 2

adding   removing

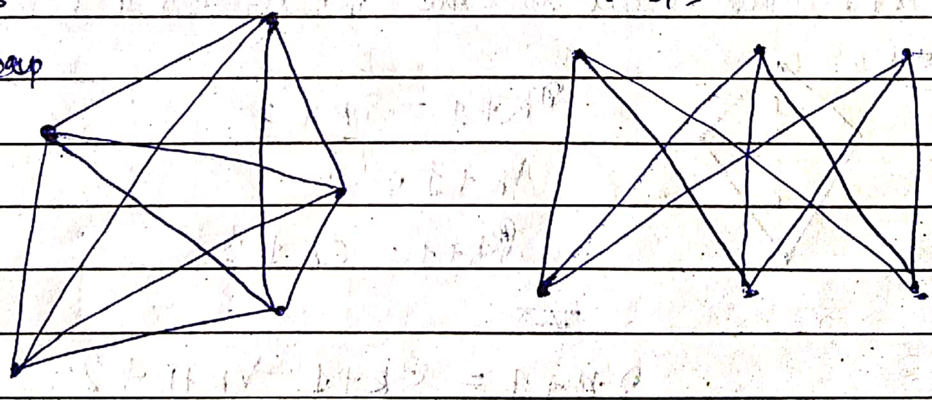


$v - e + r = 2$

$3 - 2 + 1 = 2$

$\Rightarrow$ by Inserting a vertex b/w or removing the vertex, the planarity of the graph remains unaffected.

K_5 $K_{3,3}$



$e = 10$

$V = 5$

$e = 9$

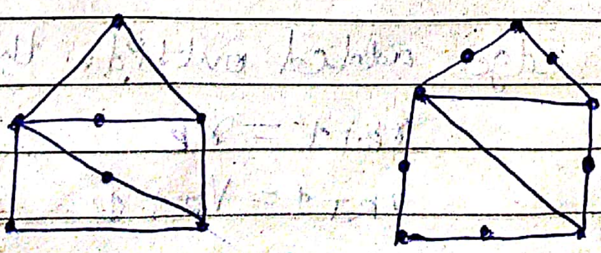
$v = 6$

$\underbrace{{}^{10}C_5 = X \qquad {}^{10}C_6 = X_2}$

vertices is the criteria.

(it can be done acc. to edges also)

eg:- ${}^{10}C_9$



(where there is vertex remove it & where there is not add vertex).

2 degree isomorphic

Tree

(continuation)

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→ class of graph, tree is object of class.

→ Connected (undirected) graph with no simple circuit (cycle) is called tree.

→ connected ⟹ there is no isolated vertex.

→ A tree must satisfy following 3 properties:-

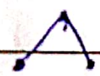
Properties:-

(1) There is always a unique path between every 2 vertices in a tree.

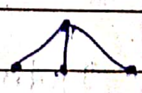
(2) The no. of vertices in a tree is always one more than the no. of edges ($e = v - 1$).

(3) A tree with 2 or more than 2 vertices contains atleast 2 leaves.

2 vertices, 2 leaves



3 vertices, 2 leaves



4 vertices, 3 leaves

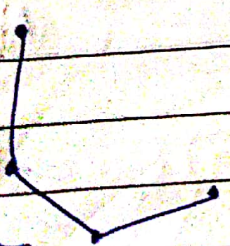
leaf - any vertex with degree 1

→ collectively 3 properties are called Invariants of a tree.

meaning:- does not vary, constant

Property 1:-

Proof:-



connected ⟹ there is atleast one vertex

∴ from definition we got to know graph is connected ⟹ there exist one path.

→ now if we add edge, it will form cycle and by

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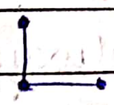
definition there should not exist circuit.
∴ The path is unique.

Property 2:
Proof: PMI on vertices not edge, b/c if there is no vertex, graph/tree will not exist.

* Basis of Induction :-
$e = v - 1$ } To proof

(i) $v = 1$
$e = 1 - 1 = 0$ //

(ii) $v = 2$ —— edges $= 1$
$e = 2 - 1$
$= 1$

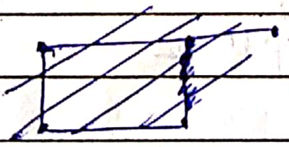


$v = 3$
$e = 3 - 1$
$= 2$
$e = 2$

Optimal solution is obtained in DP's (dynamic problem solution)

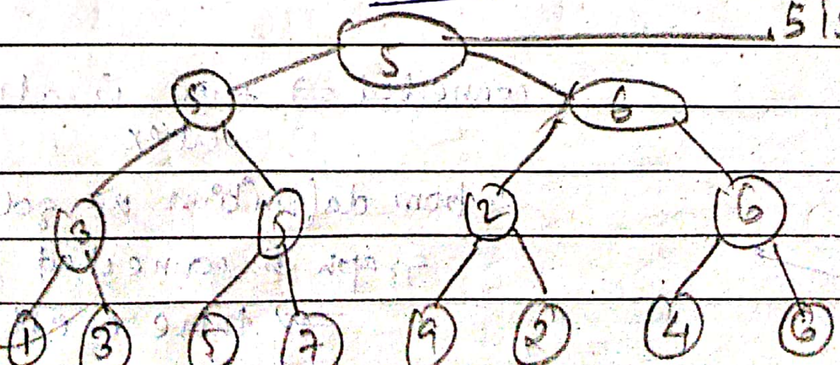
* Induction Hypothesis: proved
~~assume that the~~
assume result is true for $v = k$. $e_k = v_k - 1$

* Induction step:
now for $v = k + 1$



$v_{k+1} = k + 1$
$v_{k+1} = e_k + 1$
$e_{k+1} = v_{k+1} - 1$
$e_{k+1} = v_k + 1 - 1$
$e_{k+1} = v_k - 1$

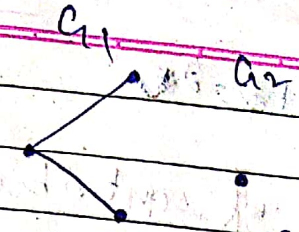
~~Hence proved~~.



5 is winner
After 7 match, winner declares
Branches $= 8$
$8 - 1 = 7$ //

this is called single elimination tournament, if you lose single match out ok tournament

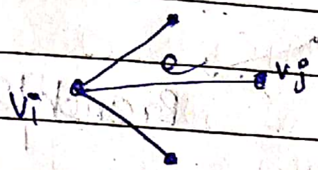
Knockout



add one more vertex.
assume that there is edge e which connects v_i & v_j.
Since G is a tree it implies that there is a unique path from v_i to v_j. If we delete edge e, then G will be decomposed into 2 distinct components.

m_2 vertices $\Rightarrow e = m_2 - 1$

m_1 vertices $\Rightarrow e = m_1 - 1$



$$e = (m_1 - 1) + (m_2 - 1) + 1$$
$$= m_1 + m_2 - 1$$
$$= k + 1 - 1$$
$$= k.$$

2 components are G1 & G2 respectively.
It is clear that both G1 & G2 will contain less than k+1 no. of vertices, hence we can write,
Total no. of edges $e = (m_1 - 1) + (m_2 - 1) + 1$

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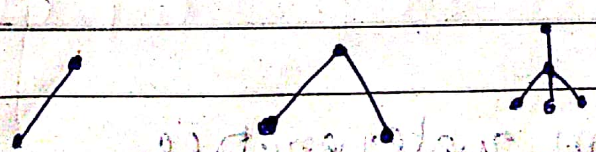
Property 3:-
Proof:
By Hand shaking th^m, we know sum of degree of vertices is twice the no. of edges.

$$\sum_{i=1}^{n} degree(v_i) = 2e$$
$$= 2(v-1)$$
$$= 2v - 2$$

If we are having more than one vertex then it will contain atleast 2 leaves.



2 leaves 2 leaves 4 leaves

$v = 1+1+4+1+1 = 8$,
$2e = 2\times5 - 2 = 10 - 2 = 8$//

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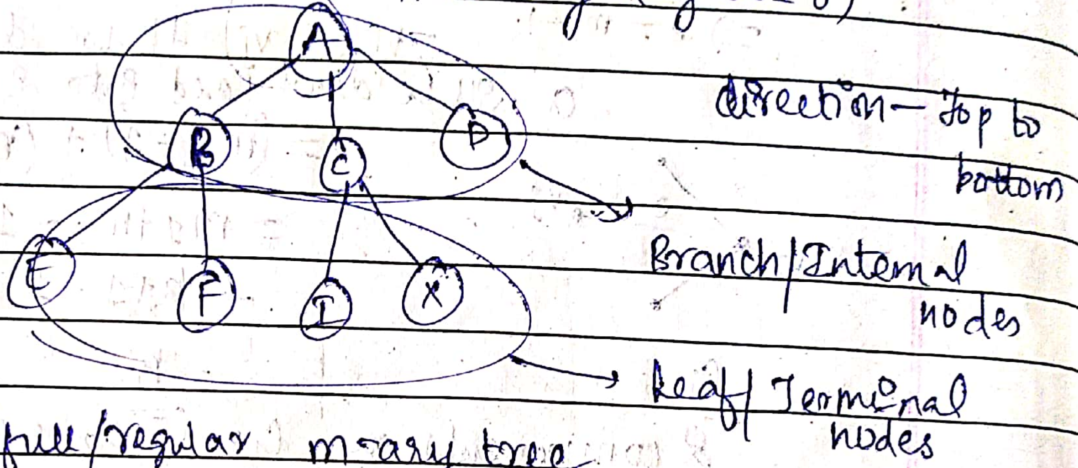
$$\sum_{i=1}^{n} \text{degree}(v_i) + 2 = 2e$$

Presence of 2 is because of contribution of 2 leaves.

(*) Rooted Tree:-

(exactly one vertex with incoming degree = 0)



complete/full/regular m-ary tree

no. of terminal nodes = $(m-1)i + 1$ ↳ no. of internal nodes

↳ Mathematically the tree must be full as it is complete.

but for programming point of view we consider complete binary tree, filled to the second last (compulsory). m-ary tree

Q.1) Consider a problem of connecting 19 electric bulbs through extension cords. Each extension cord contains exactly 4 outlets. Find out no. of extension cords required.

→ $t = 19$

It will form quaternary tree

$t = (m-1)i + 1$

$19 = (4-1)i + 1$

$3i = 18 \qquad i = 6$

no. of extension cords required are 6.

Path – no. of edges you take from root to leaf node.

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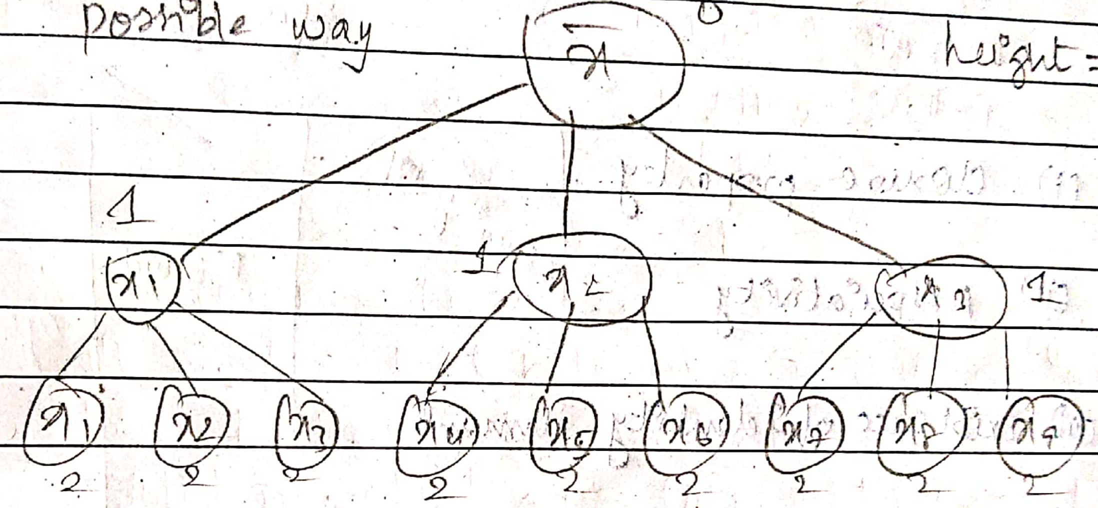
Path length in rooted trees:-

Consider a comp. that performs addition of no's. It has an instruction to perform addition of 3 no's at a time $(x_1+x_2+x_3)$. Suppose we want to perform addition of 9 no's. Draw the possible ways of doing the task

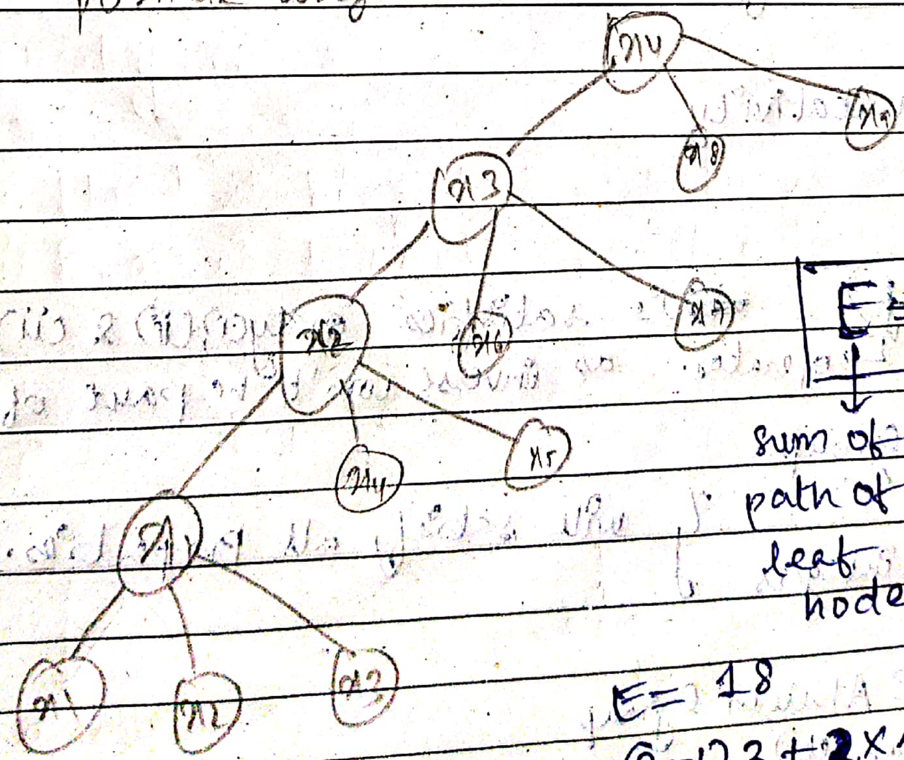
1st possible way

height = 2



2nd possible way

height = 4



no. of internal nodes

$$\boxed{E = (m-1)I + mi}$$

E – sum of path of leaf node

i – sum of path of internal nodes

$E = 18$

$(3-1)3 + 3\times4$

$= 6 + 12 = 18$ //

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(Self study)

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Group Theory

* **Algebraic structure:**

A non empty function on G $(G \neq \emptyset)$, together with a binary operator (*) is defined as Algebraic structure if it follows:

Dad \ Wight	Father	Mother
Mother		
Father		

(i) closure property
(ii) Associativity — (semi-group)
(iii) existence of Identity element — Monoid
(iv) Existence of inverse element — Group
(v) commutativity — Abelian group (satisfy all properties)

Monoid

eg:- $(Z^+, +)$ — this satisfies only (i), (ii) & (iii) property as inverse won't be part of Z^+.
(Z^+ → set, $+$ → operator)

Abelian group:

eg:- $(Z, +)$ will satisfy all properties.
(Z → all Integers)

Group ⊇ Abelian group
(subset)

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Kenneth Rosen (7th edition) ⟸ predicate logic

Group theory ⟹ gate example (cse)

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(multiplication)

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω $(\omega^3 \cdot \omega = 1 \cdot \omega)$

$\because \omega^3 = 1$

Abelian grp as it satisfy all conditions.

⟹ Algebraic structure

$\times$	1	i	$i^2(-1)$
1	1	i	
i	i	-1	
$i^2(-1)$	-1	$\boxed{-i}$	

$\therefore$ Not an Algebraic structure.

$-i$ does not belong to $1, i, i^2(-1)$

(Closure property not satisfied)

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Recurrence
~~Recursive~~ Relation:-

An equation of the form,

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \dots + C_k a_{r-k} = f(r)$$

is called linear ~~recursive~~ recurrence relation of order k.

We must first convert this recurrence relation into a polynomial form & then find out its homogeneous solution.

$$C_0 \alpha^k + C_1 \alpha^{k-1} + C_2 \alpha^{k-2} + \dots + C_k \alpha^0 = 0.$$

eg:- $a_r + 5a_{r-1} = 6$

$$1\alpha^1 - 5\alpha^0 = 0$$

$$\alpha - 5 = 0 \longrightarrow \text{here we ignore } f(r)$$

$$\underline{\underline{\alpha = 5}}$$

Homogeneous solution $= a_r^{(h)} = A_1 (5)^r$

General formula, $\big(\underbrace{A_1 r^{m-1}}_{m \to \text{ multiplicity of the root}} + A_2 r^{m-2} + A_3 r^{m-3}\big)(\text{root})^r$

eg:- If $\alpha = 2, 2, 2$

$$\underline{m = 3}$$

$$a_r^{(h)} = (A_1 r^2 + A_2 r + A_3)(2)^r$$

eg:- $T(n) = 2T(n-1) + 1$

$t_n = 2t_{n-1} + 1$, $\alpha^1 - 2\alpha_0 = 0 \Rightarrow \boxed{\alpha = 2}$

$$\boxed{a_r^{(h)} = A_1 (2)^n}$$

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \dots C_k a_{r-k} = f(r)$$

$f(r) = 0$ (homogeneous)

$f(r) \neq 0$ homogeneous solution (2 marks) + Particular solution (4 marks) (6 marks).

1) $a_r - 5a_{r-1} + 6a_{r-2} = 3r$

characteristic eqn is :

$$\alpha^2 - 5\alpha + 6 = 0$$

$$(\alpha - 2)(\alpha - 3) = 0$$

$$\boxed{\alpha = 2, 3}$$

$$\boxed{a_r^{(h)} = A_1 (2)^r + A_2 (3)^r} \text{ homogeneous solution}$$

Recurrence Relation (Next Chp)

Q. 1) $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 2r + 1$ $\longrightarrow$ 2nd order recurrence relation

$\alpha^2 + 5\alpha + 6 = 0$

$(\alpha+2)(\alpha+3) = 0$

$\alpha = -2, -3$

(1) $a_r^{(h)} = A_1(-2)^r + A_2(-3)^r$

(2) Particular solution

Let particular solution $a_r^{(P)} = P_1 r^2 + P_2 r + P_3$

$P_1 r^2 + P_2 r + P_3 + 5(P_1(r-1)^2 + P_2(r-1) + P_3) + 6(P_1(r-2)^2 - P_2(r-2) + P_3) = 3r^2 - 2r + 1$

$P_1 r^2 + P_2 r + P_3 + 5(P_1 r^2 - 2P_1 r + P_1 + P_2 r - P_2 + P_3) + 6(P_1 r^2 - 4P_1 r + 4P_1 + P_2 r - 2P_2 + P_2) = 3r^2 - 2r + 1$

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$(P_1 + 5P_1 + 6P_1)r^2 + (P_2 - 10P_1 + 5P_2 - 24P_1 + 6P_2)r + P_3 + 5P_1 - 5P_2 + 5P_3 + 24P_1 - 12P_2 + 6P_3 = 3r^2 - 2r + 1$

$12P_1 r^2 + (12P_2 - 34P_1)r + 29P_1 - 17P_2 + 12P_3 = 3r^2 - 2r + 1$

Comparing powers,

$12P_1 = 3$

$P_1 = 1/4$

$12P_2 - 34P_1 = -2$

$12P_2 - \frac{34}{4} = -2$

$P_2 = \frac{13}{24}$

$P_3 = \frac{71}{280}$

$$\boxed{a_r^{(P)} = \frac{1}{4}r^2 + \frac{13}{24}r + \frac{71}{280}} \quad a_r = (a_r^{(h)} + a_r^{(P)})$$

Homogeneous + Particular ⟸ final ans

Q.2) $a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$

$\alpha^2 + 5\alpha + 6 = 0$

$\alpha = -2, -3$

$a_r^{(h)} = A_1(-2)^r + A_2(-3)^r$

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Let assume, $a_r^{(P)} = P_1 r^2 + P_2 r + P_3$

Q.3) $a_r - 5a_{r-1} + 6a_{r-2} = 1$ (2nd order linear recurrence eqn)

$$\alpha^2 + (-5\alpha) + 6 = 0$$

$$\alpha = +2, +3$$

$$a_r^{(h)} = A_1(2)^r + A_2(3)^r$$

Let $a_r^{(P)} = P$

$$P - 5P + 6P = 1$$

$$2P = 1$$

$$P = 1/2$$

$$a_r^{(P)} = 1/2$$

$$a_r = a_r^{(h)} + a_r^{(P)}$$

$$= A_1(2)^r + A_2(3)^r + \frac{1}{2}$$

2nd case :- when exponential termed is involved in RHS. (Keep exponential term as it is as the part of particular solution)

Q.4) $a_r + 5a_{r-1} + 6a_{r-2} = 42 \times 4^r$

2nd order linear recursive eqn,

$$\alpha^2 + 5\alpha + 6 = 0$$

$$\alpha = -2, -3$$

$$a_r^{(h)} = A_1(-2)^r + A_2(-2)^r$$

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Let $a_r^{(P)} = P.4^r$

$P.4^r + 5.P.4^{r-1} + 6P.4^{r-2} = 42.4^r$

$12P.4^r = 42.4^r$

$P = \frac{42.4^r}{12.4^r}$

$= \frac{\cancel{42}\ 21}{\cancel{12}\ 6}$

Case 3:-

$\underline{P = 16}$

When root is the part of exponential term (Imp)

Q.5) $\cancel{a_{r-2}}$ $a_r - 2a_{r-1} = 3.2^r$

characteristic eqn, (1st order)

$\alpha - 2 = 0$

$\alpha = 2$

$a_r^{(h)} = A_1(2)^r$

always multiply by r^m (m - multiplicity)

Let $a_r^{(P)} = P.2^r.r^1$ (most imp step)

$P.2^r.r^1 - 2[P.2^{(r-1)} \times (r-1)] = 3.2^r$

$r.P.2^r - \cancel{2}P.2^{r-1} \times (r-1) = 3.2^r$

$P.2^r.r - 2^r.P(r-1) = 3.2^r$

$P.2^r.r - 2^r P.r + 2^r P = 3.2^r$

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$$\underline{P=3}$$

Final ans, $a_r = \mathcal{F}.r^1.2^r + A_1(2)^r$

Q.5) $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)2^r$

$$\alpha^2 - 4\alpha + 4 = 0$$
$$\alpha^2 - 2\alpha - 2\alpha + 4 = 0$$
$$(\alpha-2)(\alpha-2) = 0$$
$$\alpha = 2, 2$$

$$a_r^{(h)} = A_1(2)^r . r + A_2(2)^r$$

$$a_r^{(P)} = (P_1 r + P_2).2^r.r^2$$

$$(P_1 r + P_2).2^r.r^2 - 4(P_1(r-1)+P_2).2^{r-1}.(r-1)^2 + 4(P_1(r-2)+P_2).2^{r-2}.(r-2)^2 = (r+1)2^r$$

$$2^r P_1 r^3 + P_2 2^r r^2 + (-4P_1 r + 4P_1 - 4P_2) 2^{r-1}.(r-1)^2 + 4.2^{r-2}(r^2 - 4r + 4)(P_1 r - 2P_1 + P_2) = (r+1)2^r$$

$$\cancel{2^r P_1 r^3 + P_2 2^r r^2 - 4 \times 2^{r-1} \times P_1 (r)(r-1)^2 + 2^{r-1}(}$$

$$2^r P_1 r^3 + P_2 2^r r^2 + 2^{r-1}(r^2 - 2r + 1)[-4P_1 r + 4P_1 - 4P_2] + 4.2^{r-2}[P_1 r^3 - 2P_1 r^2 + P_2 r^2 - 4P_1 r^2 + 8P_1 r - 24P_2 + 4P_1 r - 8P_1 + 4P_2] = (r+1)2^r$$

$$P_1 = 1/6, \quad P_2 = 1$$

Q. 6) $a_r = a_{r-1} + 7$

$a_r - a_{r-1} = 7$ — take 7 as $7.(1)^r$

$\alpha - 1 = 0$

$\alpha = 1$

and 1 is in exponential & root even ∴ Case 3

$a_r^{(h)} = A(1)^r$

Let assume $a_r = P \times 1^r \times r^1$

$= Pr$

~~P2 P=2P~~

∴ $Pr - P(r-1) = 7.1^r$

$P - Pr + P = 7.1^r$

$P = 7$

$a_r = A_1(1)^r + 7r$

~~Let~~ $a_r^{(P)}$

Q. 7) $a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r$

$a_r^{(h)} = A_1(2^r) + A_2(3)^r$

Let, $a_r^{(P)} = P_1 2^r r^2 + P_2 r + P_3$

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