Representations of Coxeter groups via linear reflections and the canonical representation

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Recall that a Coxeter group is a group with a presentation of the following form:

- The generating set is $S = \{s_1, \ldots, s_n\}$.
- Each generator s_i satisfies $s_i^2 = 1$
- For each pair of distinct generators s_i, s_j, there is an integer $m_{ij} \in \{2, 3, \ldots\} \cup \{\infty\}$ so that $(s_i s_j)^{m_{ij}} = 1$. If $m_{ij} = \infty$ then $s_i s_j$ has infinite order.

We can think of the integers m_{ij} as specifying a matrix, called the *Coxeter matrix* for the Coxeter group. (We take $m_{ii} = 1$ for every i, so the matrix is also defined along the diagonal). A given Coxeter matrix also determines a Coxeter group. If W is a Coxeter group, and S is the generating set for W given by the presentation above, then the pair (W, S) is called a *Coxeter system*; this pair determines the Coxeter matrix.

1 The canonical representation

Given a Coxeter system (W, S) (equivalently, a Coxeter matrix), it is always possible to find a special representation of W, called the *canonical representation.* Here is the procedure.

First, write down the *cosine matrix*: the $|S| \times |S|$ matrix C whose entries are

$$c_{ij} = \begin{cases} -\cos \pi/m_{ij}, & m_{ij} \neq \infty \\ -1, & m_{ij} = \infty \end{cases}.$$

The cosine matrix is a symmetric matrix, so it determines a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{|S|}$: for vectors v, w in $\mathbb{R}^{|S|}$, we have

$$\langle v, w \rangle = v^T C w.$$

Since $m_{ii} = 1$, we have $c_{ii} = -\cos \pi = 1$. This means that the standard basis vectors in $\mathbb{R}^{|S|}$ are unit vectors, i.e. $\langle e_i, e_i \rangle = 1$ for every i.

We can now define a representation of our Coxeter group by reflections.

Definition 1. For every generator s_i, we define the reflection r_i by

$$r_i(x) = x - 2\langle x, e_i \rangle e_i.$$

You can check that this reflection preserves the bilinear form $\langle \cdot, \cdot \rangle$.

We need to check that this is actually a representation of the Coxeter group, i.e. that it is a group homomorphism from W to $\mathrm{GL}(\mathbb{R}^{|S|})$. This means we need to check that all of the relations specified by the Coxeter matrix are satisfied, i.e. that for any i, j, we have

$$(r_i r_j)^{m_{ij}} = 1.$$

The key here is the bilinear form:

Proposition 2. *For any i, j with $m_{ij} \neq \infty$, the bilinear form $\langle \cdot, \cdot \rangle$ restricts to a positive definite inner product on the vector space spanned by e_i, e_j.*

Proof. The restriction of $\langle \cdot, \cdot \rangle$ to the vector space spanned by e_i, e_j is described by the 2×2 matrix

$$M = \begin{pmatrix} 1 & -\cos \pi/m_{ij} \\ -\cos \pi/m_{ij} & 1 \end{pmatrix}.$$

A matrix is positive definite exactly when, for any nonzero vector $v \in \mathbb{R}^2$, we have $v^T M v > 0$, so we need to show this holds for M.

Since M is symmetric, it is diagonalizable by an orthogonal matrix we call A, i.e. $AMA^{-1} = AMA^T = D$ for a diagonal matrix D. We can think of A as a change of basis matrix: we're writing down the same bilinear form, with respect to new coordinates which are a transformation of the old coordinates by the matrix A. In symbols, this means:

$$v^T M v = v^T A^{-1} AMA^{-1} Av = v^T A^T DAv = (Av)^T D(Av).$$

Since A is invertible, M is positive definite if and only if D is positive definite, so we really just need to find out if D is positive definite.

The key fact now is that *an n-dimensional diagonal matrix specifices a positive definite bilinear form if and only if all of its diagonal entries are positive.* This is easy enough to check directly: if the diagonal entries of D are $d_1, \ldots, d_n$, then for any column vector $v = (x_1, \ldots x_n)$ in $\mathbb{R}^n$, we have

$$v^T D v = \sum_{i=1}^{n} d_i x_i^2.$$

Since x_i^2 is always positive, this sum is positive for every nonzero vector v if and only if every d_i is positive.

Returning back to the 2×2 matrix M we were thinking about originally: we now just need to show that the diagonalization of M has all positive entries, or equivalently, that *every eigenvalue of M is positive.*

Since M is just a 2×2 matrix, this is easy enough to check: we want to know if the characteristic polynomial

$$(1 - \lambda)^2 - \cos^2 \pi/m_{ij}$$

has positive roots. But, if λ is nonpositive, then $(1 - \lambda)^2 \geq 1$. Since $\cos^2 \pi/m_{ij} < 1$ always, this polynomial can never vanish for nonpositive λ. $\square$

So now we know that the bilinear form $\langle \cdot, \cdot \rangle$ is positive definite on the span of e_i and e_j whenever m_{ij} is finite. Here is another key fact:

Proposition 3. *Let $\langle \cdot, \cdot \rangle$ be any positive definite symmetric bilinear form on $\mathbb{R}^n$. There is an invertible matrix A so that for any vectors $v, w \in \mathbb{R}^n$, we have*

$$\langle v, w \rangle = Av \cdot Aw.$$

In other words, up to a change of coordinates, *any* positive symmetric bilinear form is essentially the same as the standard dot product on $\mathbb{R}^n$. The proof of this proposition is below, but **feel free to skip it on a first read-through**—it is not that important (but also not that hard).

Proof. The main idea is to use the spectral theorem. For any symmetric bilinear form on $\mathbb{R}^n$, we can find a matrix M so that

$$\langle v, w \rangle = v^T M w$$

for every pair of vectors $v, w \in \mathbb{R}^n$. (This is not hard: the i, j entry of the matrix M is just $\langle e_i, e_j \rangle$). Since the form is symmetric, so is the matrix M. So there's an orthogonal matrix (let's call it Q) diagonalizing M; we use the same trick we did before, and write

$$\langle v, w \rangle = (Qv)^T D (Qw),$$

where D is a diagonal matrix. As we saw before, the diagonal entries of D all have to be positive (because the form is positive definite). Let's write it out:

$$D = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}.$$

Positive definite diagonal matrices are special: they have square roots. We can define the matrix $\sqrt{D}$ by

$$\sqrt{D} = \begin{pmatrix} \sqrt{d_1} & & & \\ & \sqrt{d_2} & & \\ & & \ddots & \\ & & & \sqrt{d_n} \end{pmatrix}.$$

You can see immediately that the matrix $\sqrt{D}^{-1} D \sqrt{D}^{-1}$ is the identity, so we have

$$(Qv)^T D(Qv) = v^T Q^T D Q v = v^T Q^T \sqrt{D}\sqrt{D}^{-1} D \sqrt{D}^{-1} \sqrt{D} Q w = (\sqrt{D} Q v)^T (\sqrt{D} Q w).$$

Then setting $A = \sqrt{D} Q$ we get the formula $\langle v, w \rangle = (Av)^T (Aw) = Av \cdot Aw$. $\square$

Armed with this proposition, we can *finally* go back and check that the product of the reflections r_i and r_j has the correct order!

Proposition 4. *For $m_{ij} \neq \infty$, the reflections r_i, r_j satisfy $(r_i r_j)^{m_{ij}} = 1$.*

Proof. Let V_{ij} be the 2-dimensional subspace spanned by e_i and e_j. We observe that both reflections r_i and r_j act by the identity on the $(|S| - 2)$-dimensional subspace

$$V_{ij}^\perp = \{v \in \mathbb{R}^{|S|} : \langle e_i, v \rangle = \langle e_j, v \rangle = 0\}.$$

Also, both reflections preserve the decomposition of $\mathbb{R}^{|S|}$ into $V_{ij}^{\perp} \times V_{ij}$—so we just need to check that $r_i r_j$ has the correct order when restricted to V_{ij}.

Let L_i be the one-dimensional subspace of V_{ij} given by $e_i^{\perp} = \{v \in V_{ij} : \langle e_i, v \rangle = 0\}$, and similarly let $L_j = e_j^{\perp}$.

Using the previous proposition, we can find a 2×2 matrix A so that for any pair of vectors $v, w \in V_{ij}$, we have $\langle v, w \rangle = (Av) \cdot (Aw)$.

Now, we know that $\langle e_i, e_j \rangle = -\cos \pi/m_{ij}$, so $(Ae_i) \cdot (Ae_j) = -\cos \pi/m_{ij}$, and the angle between the vectors Ae_i and Ae_j with respect to the standard Euclidean inner product is $\cos^{-1}(-\cos \pi/m_{ij}) = \pi/2 + \pi/m_{ij}$.

We also know that $\langle e_i, L_i \rangle = 0$, so $Ae_i \cdot AL_i = 0$, and Ae_i and AL_i are orthogonal (again with respect to the standard Euclidean inner product). Similarly, the vector Ae_j and the line AL_j are orthogonal. If we combine this with the fact that the angle between Ae_i and Ae_j is $\pi/2 + \pi/m_{ij}$, we see that the angle between the lines AL_i and AL_j is π/m_{ij}.

Now, let's think about the conjugates Ar_iA^{-1} and Ar_jA^{-1}. Remember that, if v is an eigenvector of a matrix M, then Av is an eigenvector of the conjugate matrix AMA^{-1}, with the same eigenvalue. This means that the conjugate Ar_iA^{-1} is also a linear reflection, with mirror AL_i and -1-eigenvector Ae_i. In fact (and you can check this directly), the formula for the conjugate reflection is given by

$$Ar_iA^{-1}x = x - (x \cdot Ae_i)Ae_i.$$

This means that the conjugate reflection *preserves the standard Euclidean inner product.* Similarly, the conjugate reflection Ar_jA^{-1} also preserves the standard Euclidean inner product. So the product of the conjugates is just a Euclidean rotation by the angle between the reflection walls... which we have already seen is π/m_{ij}. So we have proved that the product of the *conjugate reflections*

$$(Ar_iA^{-1})(Ar_jA^{-1})$$

has order m_{ij}. But this is good enough! The product of a conjugate is the conjugate of a product, and conjugating a matrix preserves its order, so this also tells us that r_ir_j has order m_{ij} as well. □

This proof was a little laborious, but pay attention to the key role played by conjugation! We started with a positive definite inner product, but (via conjugation) we were able to basically ignore the fact that this product was not exactly the standard Euclidean product. This is an extremely common trick in linear algebra: you have some object (maybe a bilinear form, maybe a linear map, maybe something more complicated) which can look very complicated. But by applying a change-of-basis, you can pretend that your complicated object was actually in some standard form all along—because ultimately you might not have cared too much about your original coordinates anyway.

Altogether we have now proved the following fact:

Proposition 5. *For a Coxeter group W, the reflections r_i defined above determine a homomorphism $\rho : W \to \mathrm{GL}(\mathbb{R}^{|S|})$ (the canonical representation).*

Part of the reason the canonical representation is important is the following (hard!) theorem, due to Jacques Tits:

Theorem 6. *The representation $\rho : W \to \mathrm{GL}(\mathbb{R}^{|S|})$ is faithful (i.e. injective).*

The proof is given in Davis, spread out across several chapters (and appendices).

2 Representations by linear reflections

The canonical representation is one good source of reflection groups, but, as we have seen, there are other representations of Coxeter groups by reflections—for instance, the hexagonal tiling of Euclidean space does *not* come from a canonical representation. (You can tell, because the canonical representation of a Coxeter group with n generators always lands in $\mathrm{GL}(\mathbb{R}^n)$, but there are six reflections in the sides of a hexagon, which land in the affine group, a subgroup of $\mathrm{GL}(\mathbb{R}^3)$).

Below we'll give a theorem which allows us to verify when a collection of (arbitrary!) linear reflections gives a faithful representation of a Coxeter group.

Definition 7. Let V be a (finite dimensional, real) vector space. The dual space V^* is the space of linear maps $V \to \mathbb{R}$; it is a vector space with the same dimension as V.

There is a natural identification from V^{**} to V. (Actually, because V and V^* have the same dimension, they are already isomorphic, but the isomorphism is not "natural" in a precise sense.)

Given a vector $v \in V$ and a dual vector $\alpha \in V^*$ such that $\alpha(v) = 1$, we can always define a *linear reflection* $V \to V$ by

$$x \mapsto x - 2\alpha(x)v.$$

A couple of things to notice here:

- The reflection is determined by α and v, but if someone gives you a linear map of this form, then α and v are not determined: if you replace α with $\lambda\alpha$ and v with v/λ for some $\lambda \in \mathbb{R}$, you'll get the same reflection.

- If A is a linear map on the n-dimensional space V, and it has an $(n-1)$-dimensional eigenspace for the eigenvalue 1, and a 1-dimensional eigenspace for the eigenvalue -1, then A is a reflection, and can be written in the above form. To see this, you can recognize that any $(n-1)$-dimensional subspace of V is precisely the kernel of some linear map $\alpha : V \to \mathbb{R}$. So let v be an eigenvector with eigenvalue -1, let α be a dual vector whose kernel is the 1-eigenspace, and rescale v and/or α so $\alpha(v) = 1$.

We now want to know:

Question 8. When does a collection of linear reflections generate a Coxeter group? How can we tell which Coxeter group they generate?

As for the canonical representation, the first thing we need to do is figure out: *what is the order of a product of linear reflections?*

2.1 Finite-order products of reflections

We let r_1, r_2 be linear reflections, determined by vectors v_1, v_2 and dual vectors α_1, α_2.

Here's our claim:

Proposition 9. *If $\alpha_1(v_2)$ and $\alpha_2(v_1)$ are both negative, and $\alpha_1(v_2)\alpha_2(v_1) = \cos^2 \pi/m$, then the order of $r_1 r_2$ is m.*

In addition, if $\alpha_1(v_2) = \alpha_2(v_1) = 0$, then $r_1 r_2$ has order 2.

Proof. The reflection r_1 acts by the identity on $\ker(\alpha_1)$ (you can just check this with the formula), and similarly r_2 acts by the identity on $\ker(\alpha_2)$. So they both act by the identity on the subspace $V_{12}^* = \ker(\alpha_1) \cap \ker(\alpha_2)$, which has dimension $\dim V - 2$. If V_{12} is the subspace spanned by v_1 and v_2, both reflections also preserve the decomposition of V as $V_{12}^* \times V_{12}$. So once again we just need to know what the order of $r_1 r_2$ is on V_{12}.

We write down the following matrix:

$$\begin{pmatrix} \alpha_1(v_1) & \alpha_1(v_2) \\ \alpha_2(v_1) & \alpha_2(v_2) \end{pmatrix}.$$

This matrix is *not* symmetric... yet! But remember that for any fixed (positive) $\lambda \in \mathbb{R}$, if we replace α with α/λ and v with λv, we get the same reflection. Since we're assuming $\alpha_1(v_2)$ and $\alpha_2(v_1)$ are both negative, $\alpha_1(v_2)\alpha_2(v_1)$ is positive. So we replace v_1 with $v_1\sqrt{\alpha_1(v_2)\alpha_2(v_1)}$, and α_1 with $\alpha_1\sqrt{\alpha_1(v_2)\alpha_2(v_1)}^{-1}$, and now our matrix is symmetric. (If both $\alpha_1(v_2)$ and $\alpha_2(v_1)$ are zero, we don't need to do anything: the matrix is already symmetric.)

Notice that rescaling α_1, v_1 in this way does *not* change the product $\alpha_1(v_2)\alpha_2(v_1)$! So, since we now assume

$$\alpha_1(v_2) = \alpha_2(v_1),$$

and we assumed at the beginning of the proposition that

$$\alpha_1(v_2)\alpha_2(v_1) = \cos^2 \pi/m,$$

we can assume that both $\alpha_1(v_2)$ and $\alpha_2(v_1)$ are equal to $-\cos \pi/m$.

Now, we can define a symmetric bilinear form on V_{12}, by the formula:

$$\langle v_i, v_j \rangle = \alpha_i(v_j).$$

This only defines the bilinear form on the basis vectors v_1 and v_2, but this is good enough to determine the form on the whole vector space.

On the subspace V_{12}, we can rewrite the formula for the reflections r_1 and r_2 in terms of this bilinear form: we have

$$r_i = x - 2\langle x, v_i \rangle v_i.$$

So on V_{12}, the reflections actually preserve the bilinear form $\langle \cdot, \cdot \rangle$.

On the other hand, the symmetric matrix associated to this bilinear form is exactly the 2×2 matrix we wrote down above, which we have arranged to be

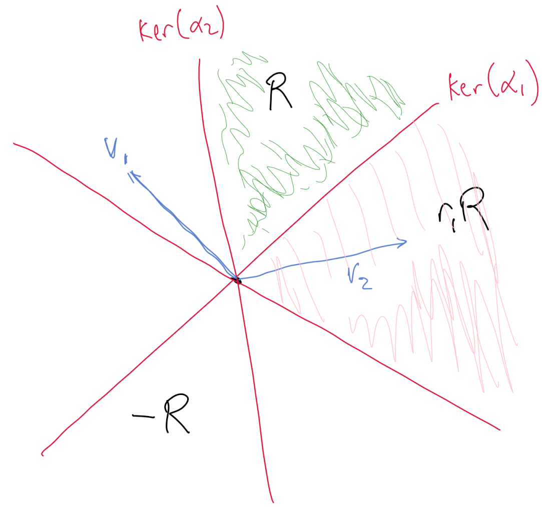
$$\begin{pmatrix} 1 & -\cos \pi/m \\ -\cos \pi/m & 1 \end{pmatrix}.$$

This means that (once again) we can identify V_{12} with 2-dimensional Euclidean space, and think of r_1 and r_2 acting by Euclidean reflections whose walls meet at an angle π/m. So, the order of $r_1 r_2$ is m. $\square$

One important fact about this argument: it doesn't *just* tell us that the product of reflections has finite order. Let R be the subset of V given by

$$R = \{v \in V : \alpha_1(v) < 0 \text{ and } \alpha_2(v) < 0\}.$$

If Γ is the group generated by r_1 and r_2, then the translates of R by elements of Γ give a tiling of V, with $2m$ tiles. These tiles all look like "cones" based at the origin in V.

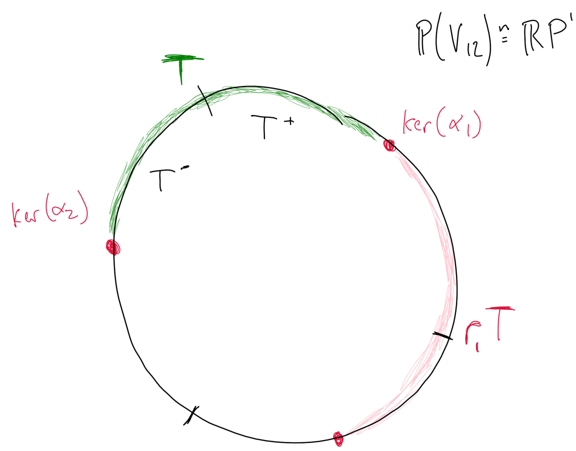


We get a corresponding tiling of projective space. Remember that we map $V - \{0\}$ to the projective space $\mathbb{P}(V)$ via the equivalence relation

$$v \sim w \iff v = \lambda w \text{ for some } \lambda \neq 0.$$

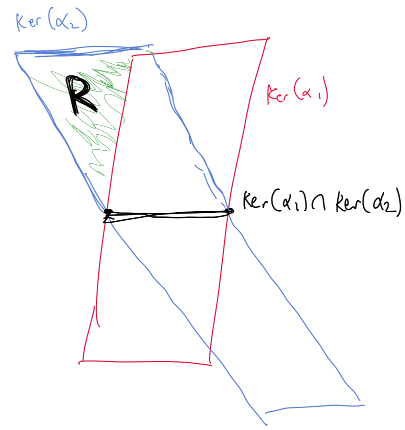
The projective space $\mathbb{P}(V_{12})$ is identified with $\mathbb{RP}^1$, which is a circle. Each of the wedge-shaped tiles from the earlier picture gets mapped to an interval in $\mathbb{RP}^1$. The region R gets mapped to an interval T.

The region $-R$ also gets mapped to T, since $-R$ and R are equivalent in projective space. However, we still have $2m$ tiles: the group element taking R to $-R$ in V_{12} is a reflection across the tile T. So if we split T into two smaller intervals T_+ and T_-, the circle $\mathbb{RP}^1$ is tiled by $2m$ copies of T_+.



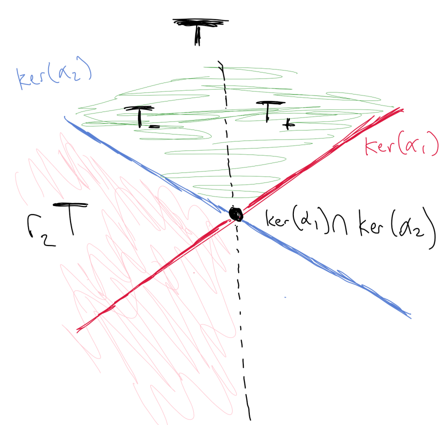
When $\dim V$ is larger, the pictures don't change too much: remember that $\ker(\alpha_1)$ and $\ker(\alpha_2)$ meet in an invariant subspace V_{12}^*. The pictures in V look like products of the previous pictures with this subspace.

For instance, if V is 3-dimensional, then the reflection walls $\ker(\alpha_1)$ and $\ker(\alpha_2)$ are 2-dimensional subspaces in V. Once again they cut out $2m$ wedges; each wedge is a region R that looks like this:



There's no easy way to draw all of $\mathbb{RP}^2$ at once, so there's no version of the "circle" picture above. But we can still draw what these "wedges" look like inside of an affine chart in $\mathbb{RP}^2$, if we pick an affine chart where the intersection $\ker(\alpha_1) \cap \ker(\alpha_2)$ is *not* at infinity.

The $2m$ regions in V again correspond to m regions in the projective space $\mathbb{P}(V)$: the region R maps to a region T in projective space. The group element taking R to $-R$ is a reflection preserving T, so we again get a tiling of projective space but cutting T into two pieces, T_+ and T_-:



2.2 Infinite-order products of reflections

We'd like to get similar pictures when we have products of reflections with *infinite* order: we still want the reflection walls to give a tiling of projective space, or at least of some region in projective space. Here's the description of when this happens.

Proposition 10. *Suppose r_1 and r_2 are reflections coming from vectors $v_1, v_2 \in V$ and dual vectors $\alpha_1, \alpha_2 \in V^*$.*

If $\alpha_1(v_2)$ and $\alpha_2(v_1)$ are both negative, and

$$\alpha_1(v_2)\alpha_2(v_1) > 1,$$

then r_1 and r_2 generate a discrete infinite dihedral group in $\mathrm{GL}(V)$, and their reflection walls give a tiling of a region in $\mathbb{P}(V)$.

Proof. Once again, we really only need to think about what's happening on the special subspace V_{12} spanned by v_1 and v_2. We also apply the same trick we did before, and rescale α_1 and v_1, which allows us to assume that $\alpha_1(v_2) = \alpha_2(v_1)$.

We also again define a symmetric bilinear form on V_{12} by

$$\langle v_i, v_j \rangle = \alpha_i(v_j),$$

which once again has matrix

$$\begin{pmatrix} 1 & \alpha_1(v_2) \\ \alpha_2(v_1) & 1 \end{pmatrix}.$$

The difference this time is that the bilinear form is *not* positive definite: it is instead *semidefinite.* However, we can still diagonalize it: using the same procedure as before, we can find an invertible matrix A so that

$$\langle v, w \rangle = (Av)^T \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} Aw.$$

That is, the bilinear form $\langle \cdot, \cdot \rangle$ is equivalent to the Minkowski inner product on $\mathbb{R}^2$.

This means that, after a change of basis, the product $r_1 r_2$ is a *hyperbolic rotation matrix*: a 2×2 matrix which preserves the "unit hyperbola"

$$x^2 - y^2 = 1$$

in $\mathbb{R}^2$. (If you want, you can think of this as the hyperboloid model for 1-dimensional hyperbolic space.)

Just as a (spherical) rotation matrix can always be written in the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta, \end{pmatrix}$$

a hyperbolic rotation matrix can always be written as

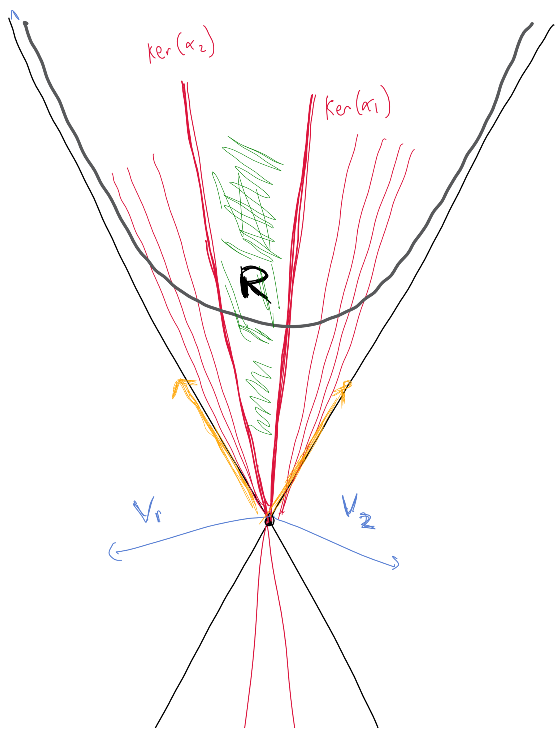
$$\begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}.$$

Here cosh and sinh are the hyperbolic trig functions, satisfying $\cosh^2(u) - \sinh^2(u) = 1$; the matrix above corresponds to translation along the unit hyperbola by "hyperbolic angle" u. This already tells us that the product $r_1 r_2$ has infinite order, so r_1 and r_2 generate an infinite dihedral group.

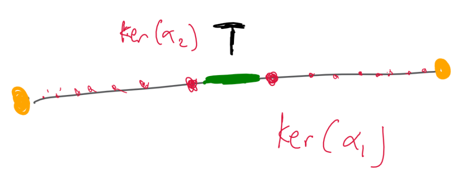
What about the tiling? The reflection walls of r_1 and r_2 are lines passing through the unit hyperbola, separated by a hyperbolic angle u. The region

$$R = \{v \in V_{12} : \alpha_1(v) < 0 \text{ and } \alpha_2(v) < 0\}$$

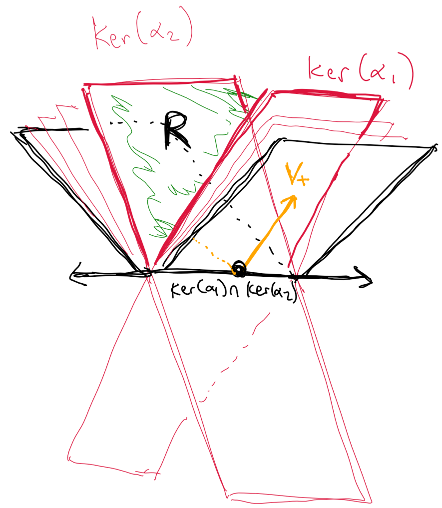
is once again a cone in V_{12}, lying inside the light cone for 2-dimensional Minkowski space. The reflections r_1 and r_2 take this to another cone inside of Minkowski space bordering the first one. We end up getting a tiling of the interior of the light cone by (increasingly distorted) copies of this cone:



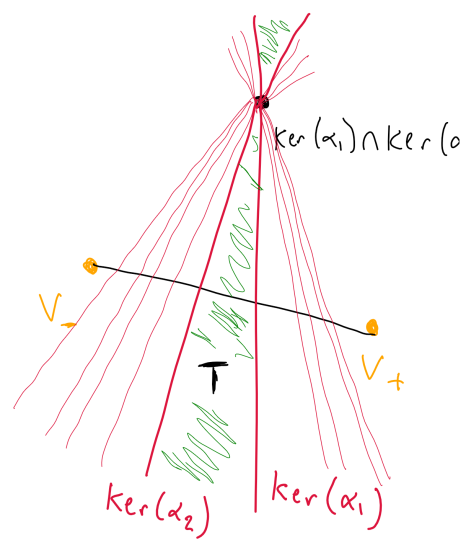
In projective space, the entire hyperbola is mapped to a single interval, and the region R is also mapped to a single interval T. So we get a tiling of an interval by smaller intervals:



We can also draw the picture one dimension up, in the vector space V (drawn here as 3-dimensional):



In an affine chart in the projective space $\mathbb{P}(V)$, the region R again gets mapped to a wedge-shaped tile T, which tiles a cone in projective space:



$\square$

There's one more important case to consider.

Proposition 11. *Suppose r_1 and r_2 are reflections coming from vectors $v_1, v_2 \in V$ and dual vectors $\alpha_1, \alpha_2 \in V^*$. Suppose further that $\ker(\alpha_1) \neq \ker(\alpha_2)$.*

If $\alpha_1(v_2)$ and $\alpha_2(v_1)$ are both negative, and

$$\alpha_1(v_2)\alpha_2(v_1) = 1,$$

then r_1 and r_2 generate a discrete infinite dihedral group in $\mathrm{GL}(V)$, and their reflection walls give a tiling of a region in $\mathbb{P}(V)$.

Proof. This case is a little different: we can still reduce everything down to a 2-dimensional picture, but we have to be a little bit more careful.

Remember that (as always) we assume that $\alpha_1(v_2) = \alpha_2(v_1)$. So, we know that

$$\alpha_1(v_2) = \alpha_2(v_1) = -1.$$

But then $\alpha_1(v_1 + v_2) = \alpha_2(v_1 + v_2) = 0$—and now we have a problem, because $v_1 + v_2$ lies in $V_{12}^* = \ker(\alpha_1) \cap \ker(\alpha_2)$. So, it is no longer true that V splits as a direct product $V_{12} \times V_{12}^*$.

So instead, we will think about the action of r_1 and r_2 on the *quotient* vector space $\bar{V}_{12} = V/V_{12}^*$. This quotient is two-dimensional, and since r_1 and r_2 preserve V_{12}^*, they act on the quotient $\bar{V}_{12}$. Moreover, if we can prove the statement for the action on the quotient, the statement for the action on the larger vector space will follow. For any vector $v \in V$, we let $\bar{v}$ be the image of v in $\bar{V}_{12}$.

Notice also that α_1 and α_2 are also well-defined on $\bar{V}_{12}$, since we are modding out by the *kernels* of these forms. Here is a little more detail. An element of $\bar{V}_{12}$ is (by definition) a V_{12}^*-*equivalence class*: a set U of vectors, satisfying the property that the difference of any two elements in U is an element of V_{12}^*. So if w, w' are equivalent, then $w = w' + v$ for $v \in V_{12}^*$. Then

$$\alpha_1(w) = \alpha_1(w' + v) = \alpha_1(w') + \alpha_1(v) = \alpha_1(w').$$

So α_1 gives the same value when applied to *any* element of the equivalence class, meaning it gives a well-defined linear functional on the quotient vector space (and similarly for α_2).

Since $v_1 + v_2 \in V_{12}^*$, inside the quotient $\bar{V}_{12}$, we have

$$\bar{v}_1 + \bar{v}_2 = 0,$$

or in other words $\bar{v}_1 = -\bar{v}_2$. Since r_1 fixes v_1, and r_2 fixes v_2, this means that $\bar{v}_1$ is an eigenvector for the action of both r_1 and r_2 on $\bar{V}_{12}$. Now let $\bar{w} \in \bar{V}_{12}$ be any nonzero vector in $\ker(\alpha_1)$ (remember α_1 can be thought of as a dual vector in $\bar{V}_{12}$).

In the vector space $\bar{V}_{12}$, the intersection $\ker(\alpha_1) \cap \ker(\alpha_2)$ is trivial (remember we have taken the quotient by this intersection, so this is true by definition). So, we can rescale $\bar{w}$ so that $\alpha_2(\bar{w}) = 1$.

Then, we have $r_1\bar{w} = \bar{w}$, and $r_2\bar{w} = \bar{w} - 2\alpha_2(\bar{w})\bar{v}_1 = \bar{w} - 2\bar{v}_1$. Also, since $r_1\bar{v}_1 = -\bar{v}_1 = \bar{v}_2$, and $r_2\bar{v}_2 = -\bar{v}_2 = \bar{v}_1$, we have $r_2r_1\bar{v}_1 = \bar{v}_1$. So, in the basis $\{\bar{v}_1, \bar{w}\}$, the linear map r_2r_1 has matrix

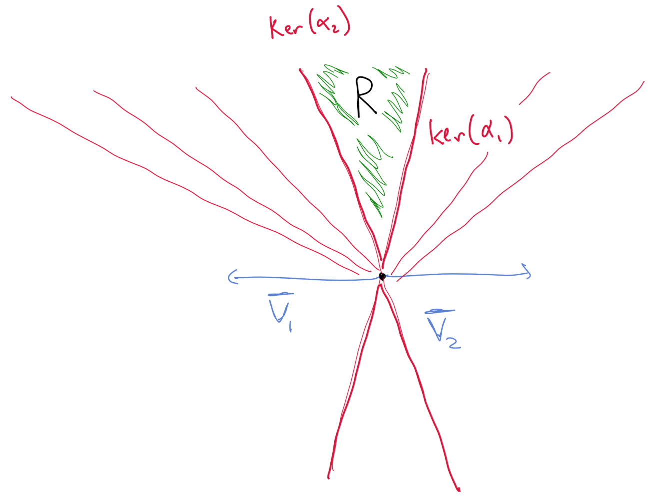
$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

You can check directly that this matrix has infinite order. In fact, this matrix is just a translation on an affine chart in $\mathbb{RP}^1$.

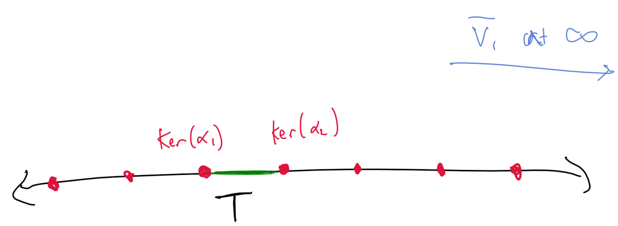
What about the tiling? Again we let R be the region

$$\{v \in V : \alpha_1(v) < 0 \text{ and } \alpha_2(v) < 0\}.$$

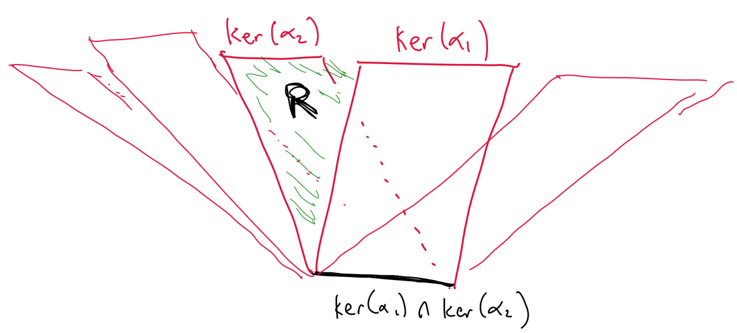
In $\bar{V}_{12}$, the reflections act to give us a tiling of the upper half-plane by copies of R:



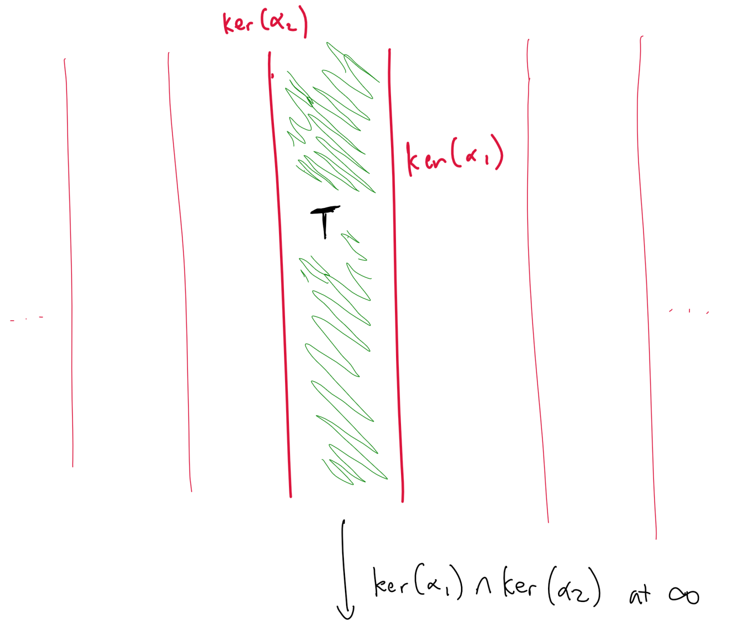
In the affine chart $y \neq 0$, R again maps to an interval T, so this just looks like a tiling of the line by intervals.



If $\dim V = 3$, we can see what the picture looks like one dimension up:



In the real projective plane, this gives a familiar tiling of affine space. Again the region R maps to a region T in $\mathbb{RP}^2$:



□

2.3 Putting it all together

Here's the big important theorem we'll really be focusing on:

Theorem 1 (Vinberg's theorem)**.** *Suppose that $r_1, r_2, \ldots, r_n$ are a collection of linear reflections in $\mathrm{GL}(\mathbb{R}^d)$, determined by vectors $v_1, v_2, \ldots, v_n$ and dual vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.*

Suppose that all of the following conditions hold:

1. $\alpha_i(v_i) = 1$ *for every* i.
2. $\alpha_i(v_j) \leq 0$ *for every* $i \neq j$*, and* $\alpha_i(v_j) = 0$ *if and only if* $\alpha_j(v_i) = 0$.
3. *If* $\alpha_i(v_j)\alpha_j(v_i) < 1$*, then* $\alpha_i(v_j)\alpha_j(v_i) = \cos^2 \pi/m_{ij}$ *for some integer* $m_{ij} \geq 2$.

Then, the reflections $r_i(x) = x - 2\alpha_i(x)v_i$ generate a reflection group which is isomorphic to the Coxeter group with matrix m_{ij} (where $m_{ij} = \infty$ if $\alpha_i(v_j)\alpha_j(v_i) \geq 1$). Moreover, the reflections tile a region Ω in projective space, and the fundamental tile is given by the projectivization of the region

$$R = \{v \in \mathbb{R}^d : \alpha_i(v) < 0 \text{ for all } 1 \leq i \leq n\}.$$

You can think of this theorem as a vast generalization of the Poincare polyhedron theorem. The proof is quite hard, and is beyond what we'll talk about this semester. However, we've already seen the proof of one part of it! The discussion above tells us that the collection of reflections r_i do actually give us a well-defined homomorphism from the abstract Coxeter group with Coxeter matrix m_{ij} into $\mathrm{GL}(\mathbb{R}^d)$.

Question 12. If W is the (a, b, c) triangle group for integers a, b, c, then the canonical representation of W should satisfy the hypotheses of Vinberg's theorem. What are the vectors v_i and dual vectors α_i for the canonical representation? What does the fundamental region R look like, in $\mathbb{R}^3$ and in $\mathbb{R}\mathrm{P}^2$?