Extended geometrically finite representations

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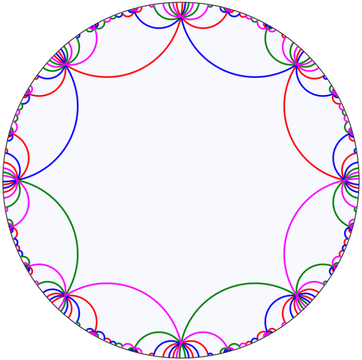
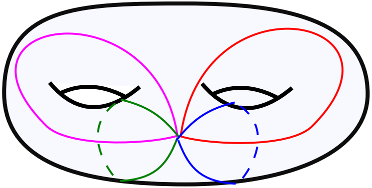
Example: $\Gamma \simeq \pi_1 M$ for M a closed hyperbolic d-manifold.





Proposition (Gromov, Coornaert, Bourdon)

A discrete group $\Gamma \subset \mathrm{SO}(d,1)$ is convex cocompact if and only if Γ is (abstractly) word-hyperbolic, and its Gromov boundary $\partial\Gamma$ embeds equivariantly into $\partial\mathbb{H}^d$.

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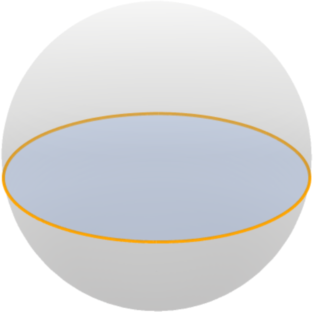
$\pi_1 S \to \mathrm{SO}(2,1) \hookrightarrow \mathrm{SO}(3,1)$



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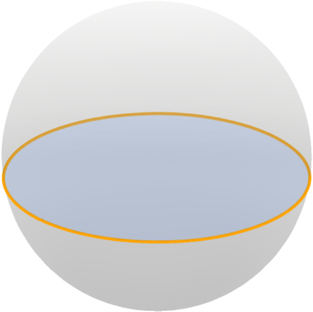

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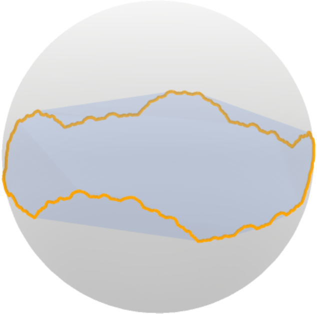

Convex cocompactness in higher rank

Definition

Let $\rho : \Gamma \to \mathrm{SL}(d, R)$ be a representation of a word-hyperbolic group. We say ρ is P_1-*Anosov* if there are ρ-equivariant embeddings

$$\xi : \partial\Gamma \to \mathbb{R}P^{d-1}, \quad \xi^* : \partial\Gamma \to (\mathbb{R}P^{d-1})^*$$

which are *transverse* and *dynamics-preserving*.

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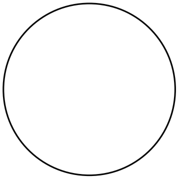

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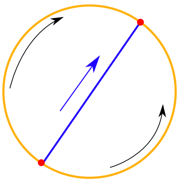

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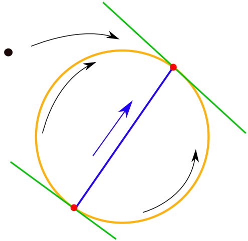
ξ maps attracting fixed points to attracting fixed points



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Theorem (Labourie, Guichard-Wienhard)

Let $\rho : \Gamma \to \mathrm{SL}_d \mathbb{R}$ *be a* P_1*-Anosov representation. Then an open neighborhood of* ρ *in* $\mathrm{Hom}(\Gamma, \mathrm{SL}_d(\mathbb{R}))$ *consists of* P_1*-Anosov representations.*

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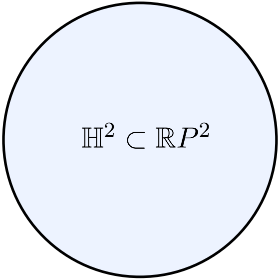

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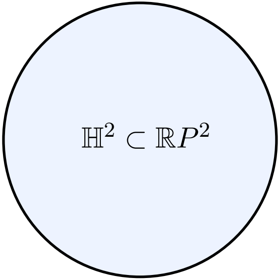
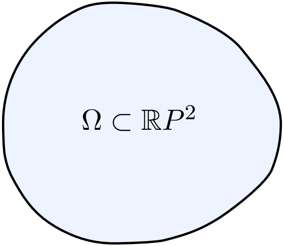


Deform in $\mathrm{Hom}(\Gamma, \mathrm{SL}_3(\mathbb{R}))$:



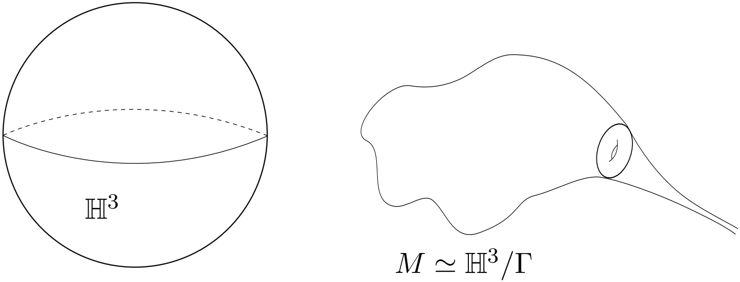
Invariant under deformed action, quotient is *convex projective surface*

What about geometrically finite groups?

Definition

Let $\Gamma \subset \mathrm{SO}(d,1)$ be a *finitely generated* discrete group. We say Γ is *geometrically finite* if it acts with finite covolume on a convex Γ-invariant subset of $\mathbb{H}^d$ *with nonempty interior*.

Example: M = complete finite-volume noncompact hyperbolic 3-manifold, $\Gamma = \pi_1 M \subset \mathrm{SO}(3,1)$.



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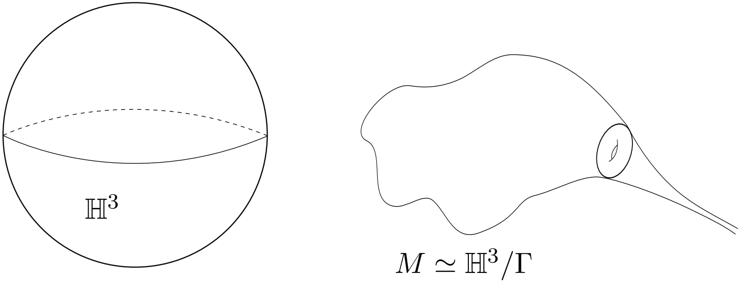
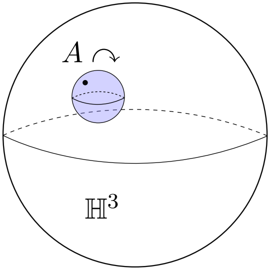
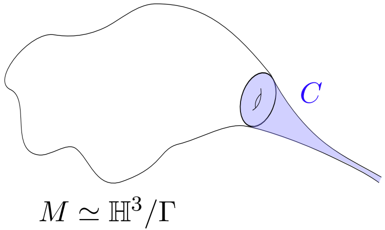


Γ is *not* a word-hyperbolic group.

Any geometrically finite group Γ is *relatively hyperbolic*, relative to its *cusp subgroups* $\mathcal{P} = \{\pi_1 C : C \text{ a cusp of } \mathbb{H}^d/\Gamma\}$.





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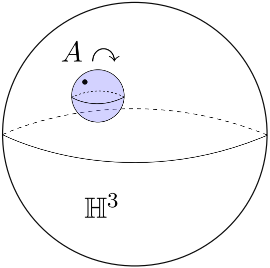

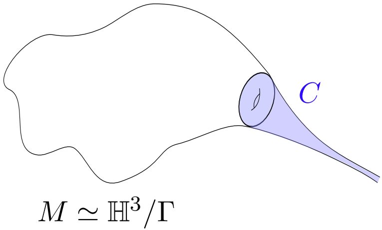

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A is the stabilizer of a point in $\partial\mathbb{H}^3 = \partial(\Gamma, \mathcal{P})$, the *Bowditch boundary* of the pair $(\Gamma, \mathcal{P})$

Relative hyperbolicity in higher rank

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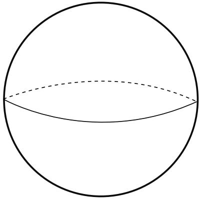

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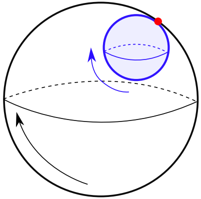

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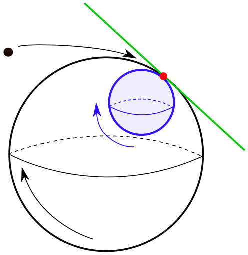

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Deforming relative Anosov representations in $\mathrm{SL}_d(\mathbb{R})$

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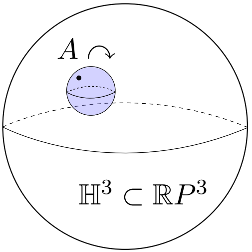
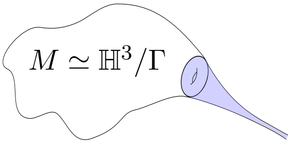


$A \simeq \mathbb{Z}^2 \subset \{\text{upper triangular}\}$



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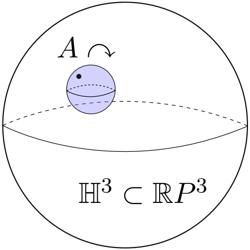

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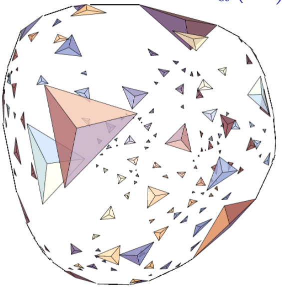

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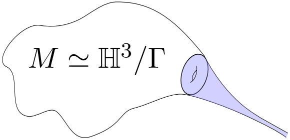

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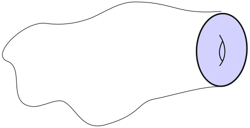

Get *convex projective 3-manifd.*

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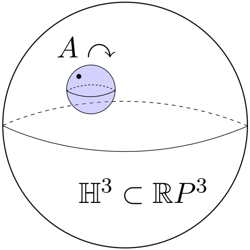

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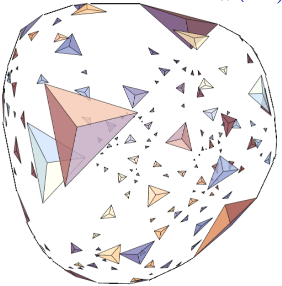

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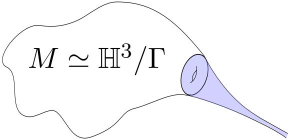

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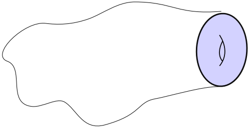

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Bowditch boundary $\partial\mathbb{H}^3$ is *not* equivariantly embedded into $\mathbb{R}P^3$!

Definition (W.)

Let $\rho : \Gamma \to \mathrm{SL}(d, \mathbb{R})$ be a representation of a relatively hyperbolic group. We say that ρ is *extended geometrically finite* if there are Γ-invariant subsets $\Lambda \subset \mathbb{R}P^{d-1}$, $\Lambda^* \subset (\mathbb{R}P^{d-1})^*$ and surjective transverse maps

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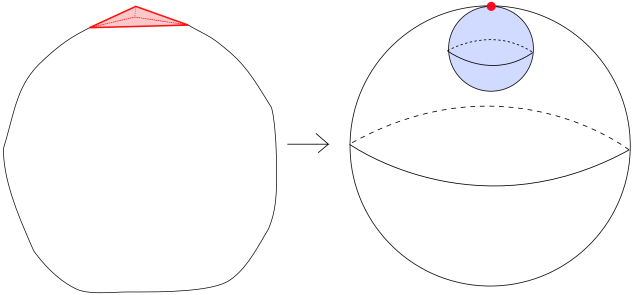

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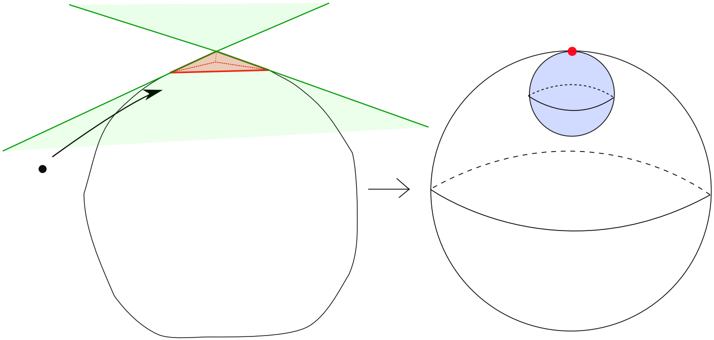

Extended geometrically finite representations are *relatively stable.*

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Let $\rho : \Gamma \to G$ be EGF, and let $W \subseteq \mathrm{Hom}(\Gamma, G)$ be a peripherally stable subspace at ρ. Then an open subset of W containing ρ consists of EGF representations.

In particular, the deformation of $\pi_1 M \to \mathrm{SO}(3,1) \hookrightarrow \mathrm{SL}_4 \mathbb{R}$ shown previously is peripherally stable.

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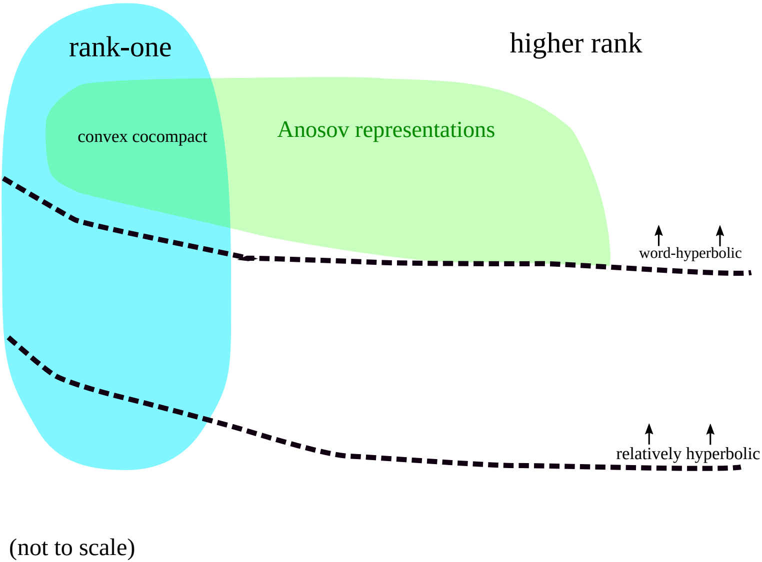
This works for any relatively hyperbolic group Γ and semisimple Lie group G.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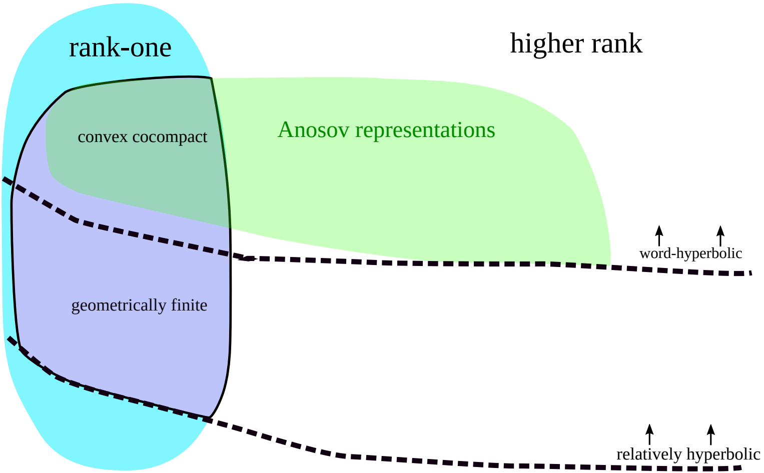

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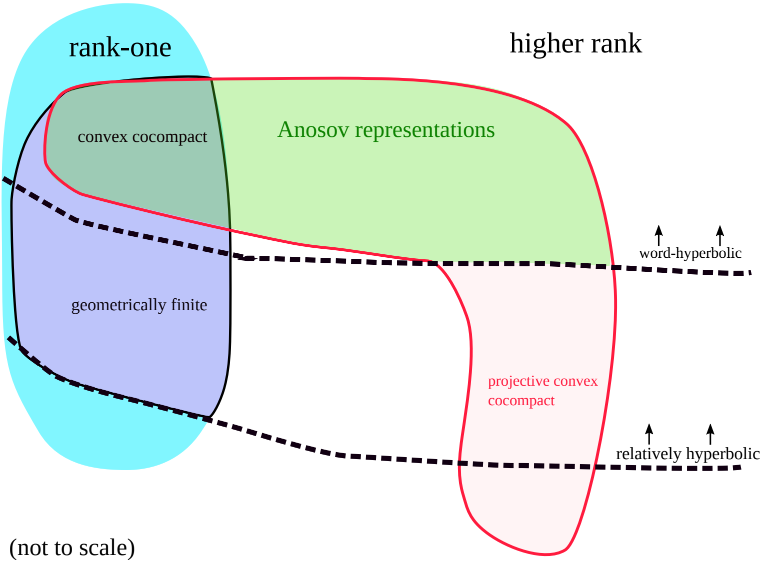
relatively hyperbolic

(not to scale)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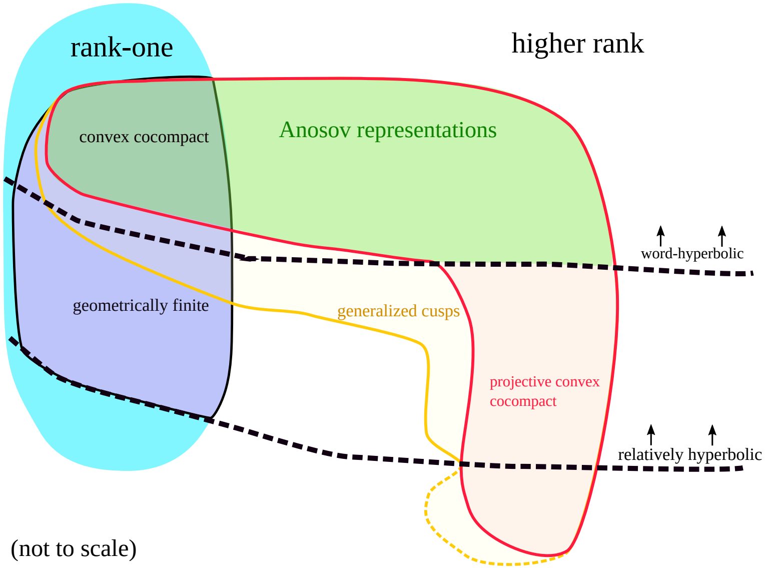
higher rank
convex cocompact
Anosov representations
word-hyperbolic
geometrically finite
projective convex
cocompact
relatively hyperbolic
(not to scale)


rank-one
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convex cocompact
Anosov representations
word-hyperbolic
geometrically finite
generalized cusps
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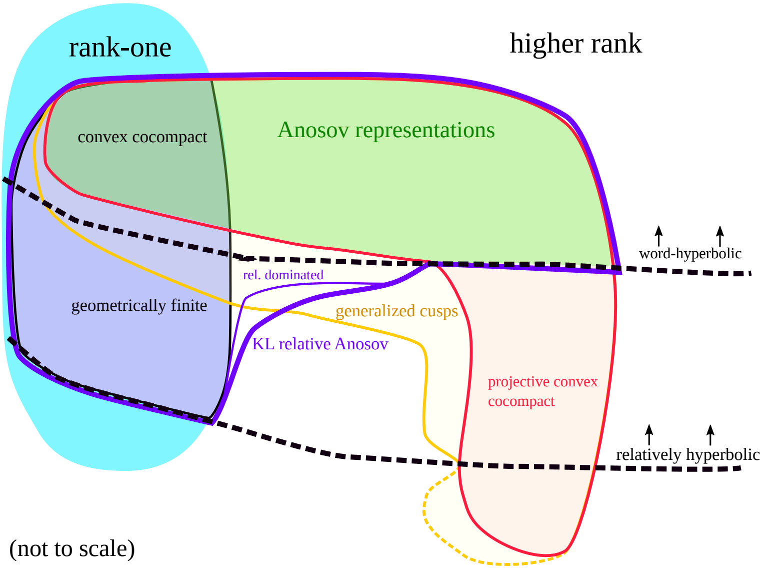
relatively hyperbolic
(not to scale)


rank-one
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convex cocompact
Anosov representations
word-hyperbolic
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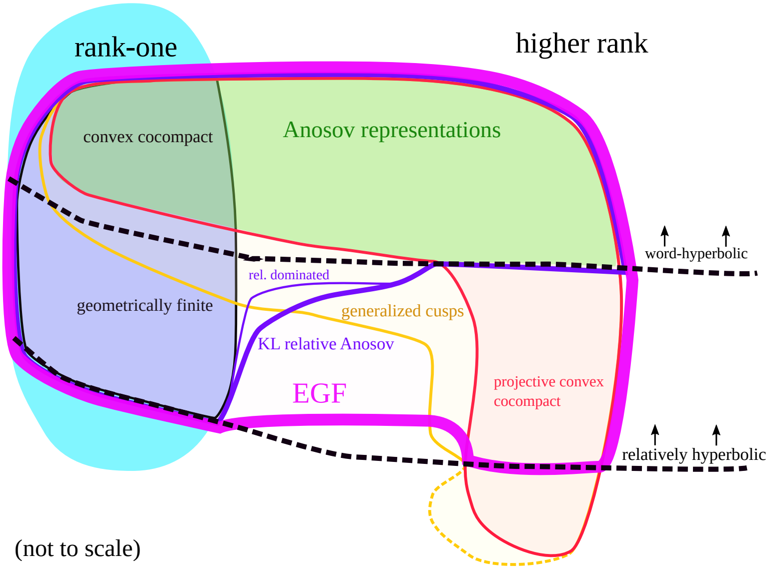
geometrically finite
generalized cusps
KL relative Anosov
projective convex
cocompact
relatively hyperbolic

(not to scale)


rank-one
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EGF
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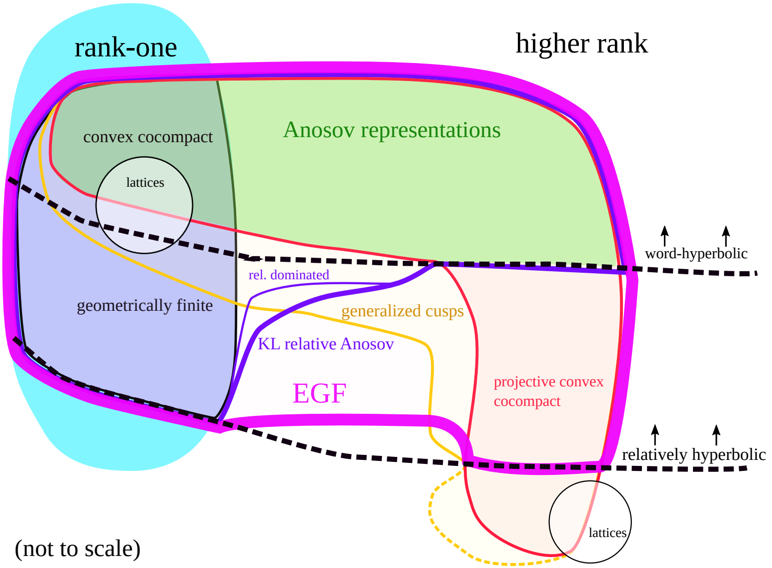
rank-one
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