Relative Anosov representations and convex projective structures

Teddy Weisman
University of Texas at Austin

The big idea:

deformations of geometric structures on manifolds

$\leftrightsquigarrow$

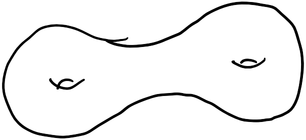
Stable dynamics of discrete subgroups of Lie groups

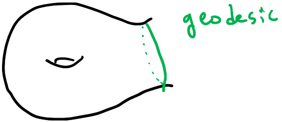
Convex cocompactness in hyperbolic space

$g \in PGL(d+1)$ preserving bilinear form w/ signature $(d,1)$.

geometric structure

M compact hyperbolic manifold with convex boundary





$PO(2,1) \cong PSL(2,\mathbb{R})$

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \Big\} d$$

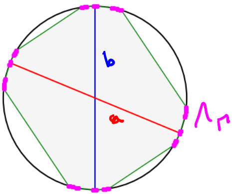
discrete subgroups of Lie groups

$\Gamma \subset PO(d,1)$ discrete

$PO(d,1)$ is $\mathrm{Isom}(\mathbb{H}^d)$

Λ_Γ = limit set = accumulation pts. of $\Gamma \cdot x$ in $\partial\mathbb{H}^d$, $x \in \mathbb{H}^d$

Γ acts convex cocompactly if it acts cocompactly on $\mathrm{Conv\,Hull}(\Lambda_\Gamma)$.



a component of

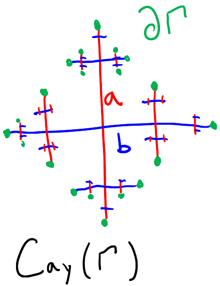
$$\{ v \in \mathbb{R}^{2,1} : \langle v, v \rangle_{2,1} = -1 \} \cong \mathbb{H}^2$$

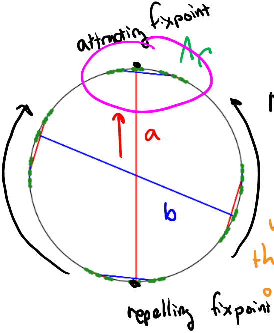
2-sheeted hyperboloid in $\mathbb{R}^{2,1}$ preserved by $PO(2,1)$

Dynamics of convex cocompact groups

$\Gamma \subset PO(d,1)$ convex cocompact.

$\Longrightarrow$ Γ is word-hyperbolic, and Gromov boundary $\partial\Gamma$ embeds equivariantly into $\partial \mathbb{H}^d$.





Γ acts with North-south dynamics on $\Lambda_\Gamma \subset \partial \mathbb{H}^2$.

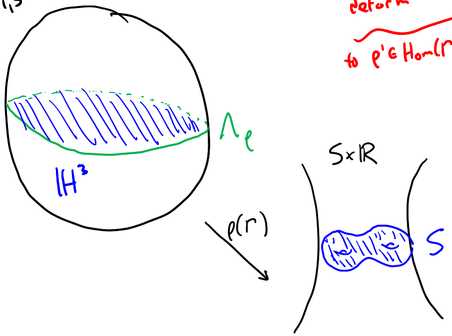
Can play ping-pong with Λ_Γ to see that small deformations of a and b in $PO(2,1)$ still generate a discrete free group.

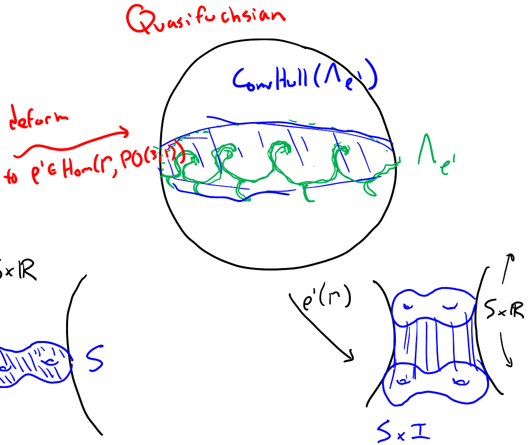
Stability:

Thm (Sullivan):

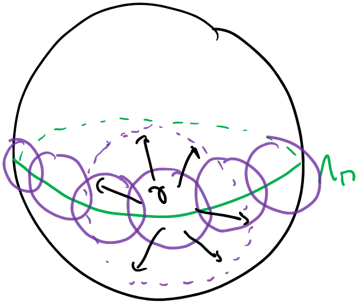
Let $\rho : \Gamma \to PO(d,1)$ be convex cocompact. Then an open neighborhood of ρ in $\mathrm{Hom}(\Gamma, PO(d,1))$ consists of convex cocompact representations.

$$\rho : \underset{\pi_1 S}{\underset{\cong}{\Gamma}} \xrightarrow{\text{Fuchsian}} PO(2,1) \hookrightarrow PO(3,1)$$





Stability from dynamics



Cover Λ_Γ with finitely many open nbhds so some $\gamma \in \Gamma$ is expanding on each neighborhood.

Expansion $\Leftrightarrow$ convex cocompactness.

$\Uparrow$

stable under deformation

deformations of geom. structures $\longleftrightarrow$ expansion dynamics of $\Gamma \subset PO(d,1)$

Want: to understand in higher rank groups

Definition

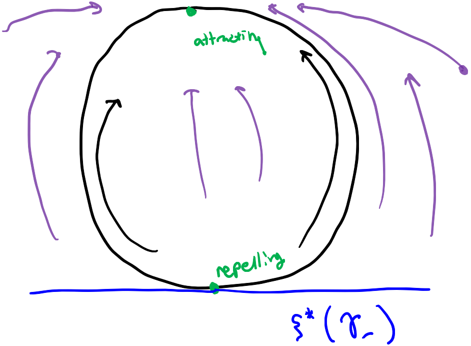
Let Γ be a hyperbolic group. A representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is *P_1-Anosov* if there exist equivariant embeddings

$$\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d), \qquad \xi^* : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)^*$$

hyperplanes in $\mathbb{R}^d$

which are *transverse*, and *preserve the dynamics of* Γ.

$\mathbb{H}^2 \subset \mathbb{R}P^2$ round ball



$\Gamma \cong \pi_1 S$ surface group

$\rho : \Gamma \to PO(2,1) \hookrightarrow PGL(3, \mathbb{R})$

$$\rho(\gamma) = \begin{bmatrix} \lambda & & \\ & \begin{bmatrix} 1 & \\ & \lambda^{-1} \end{bmatrix} \end{bmatrix}$$

Thm (Labourie, Guichard-Wienhard):

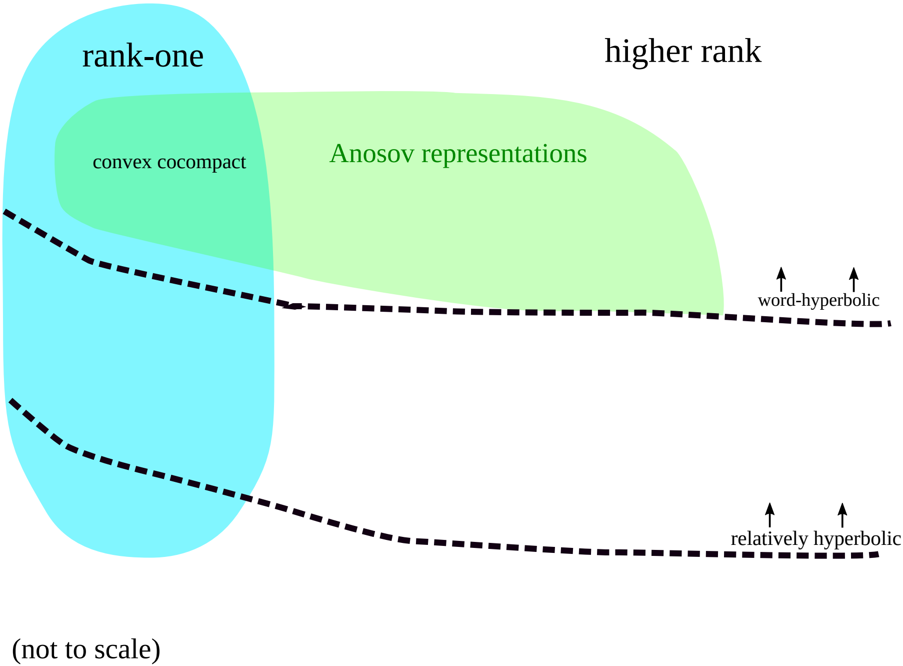
Γ word-hyperbolic group. $\rho : \Gamma \to G$ Anosov representation.

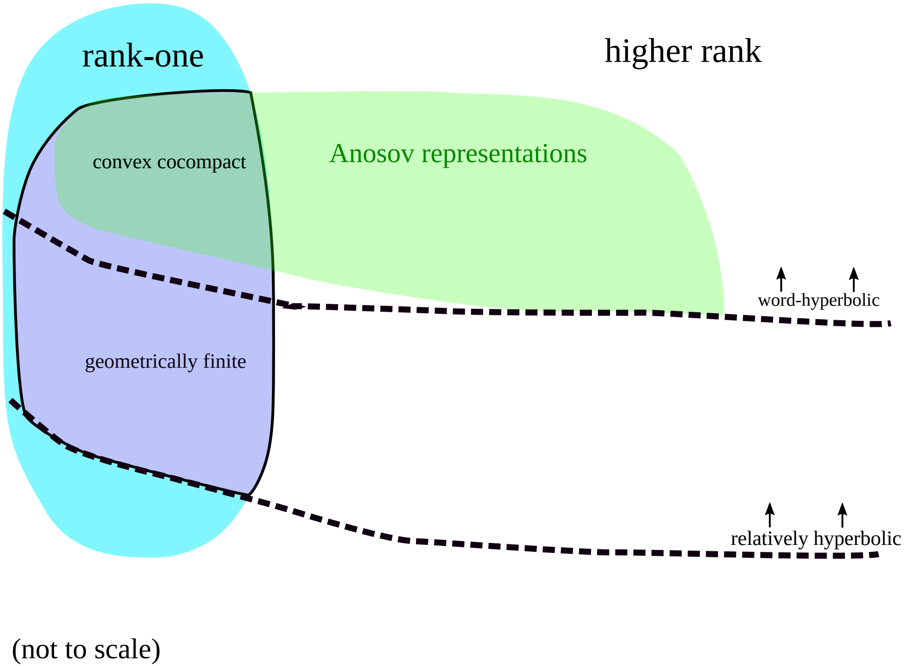
An open neighborhood of ρ in $\mathrm{Hom}(\Gamma, G)$ consists of Anosov representations.

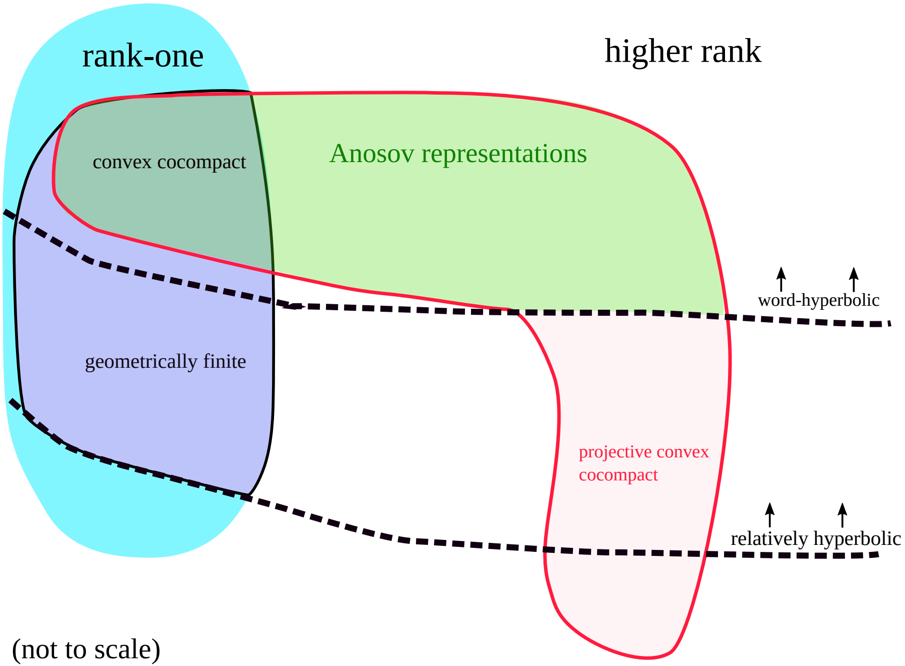
Geometric structures associated to Anosov reps. can have deformations.

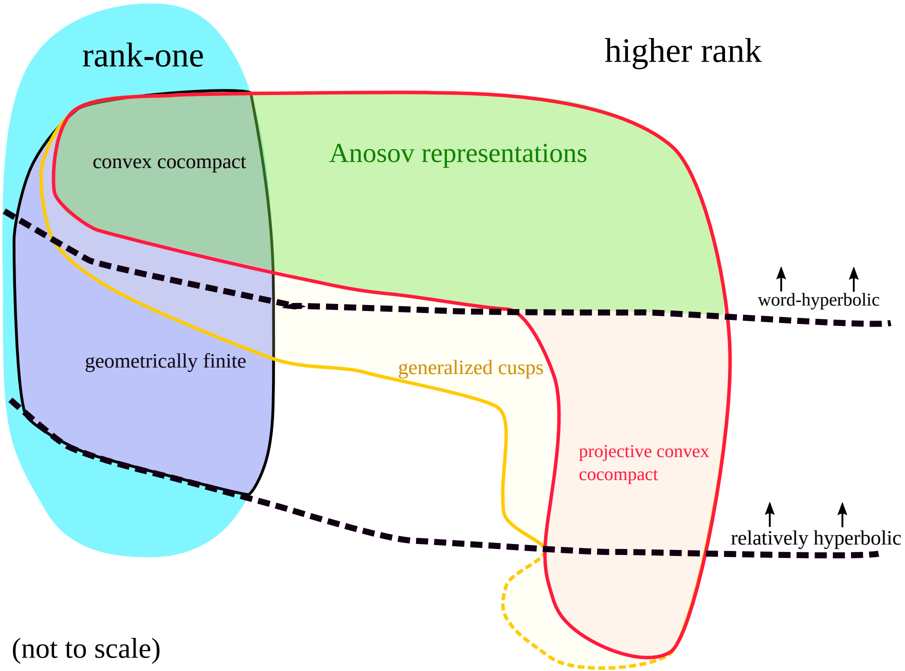
Q: What if Γ isn't word-hyperbolic? e.g. relatively hyperbolic?

- What kinds of geometric structures?
- What kinds of dynamics?

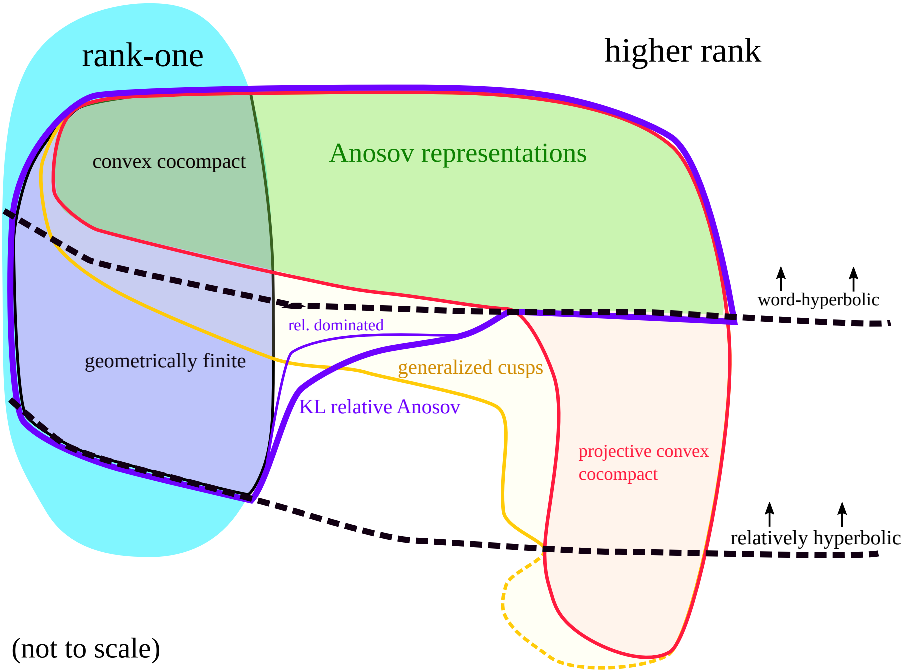

rank-one
higher rank
convex cocompact
Anosov representations
word-hyperbolic
relatively hyperbolic
(not to scale)


rank-one
higher rank
convex cocompact
Anosov representations
word-hyperbolic
geometrically finite
relatively hyperbolic
(not to scale)

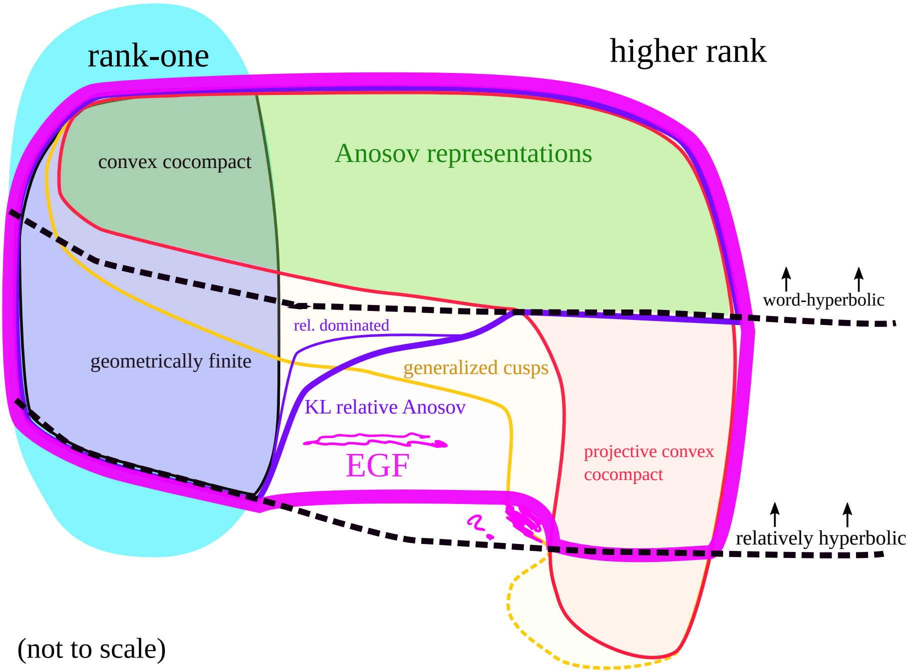

rank-one
higher rank
convex cocompact
Anosov representations
word-hyperbolic
geometrically finite
projective convex
cocompact
relatively hyperbolic
(not to scale)

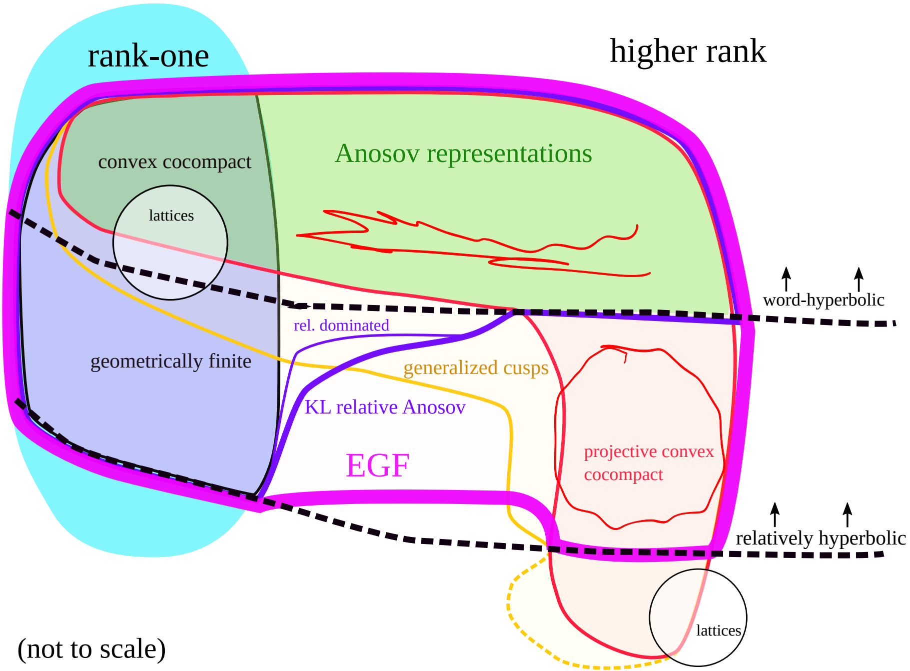


rank-one
higher rank
convex cocompact
Anosov representations
word-hyperbolic
geometrically finite
generalized cusps
projective convex cocompact
relatively hyperbolic
(not to scale)



rank-one
higher rank
convex cocompact
Anosov representations
word-hyperbolic
rel. dominated
geometrically finite
generalized cusps
KL relative Anosov
projective convex
cocompact
relatively hyperbolic
(not to scale)


rank-one
higher rank
convex cocompact
Anosov representations
word-hyperbolic
rel. dominated
geometrically finite
generalized cusps
KL relative Anosov
EGF
projective convex
cocompact
?
relatively hyperbolic
(not to scale)



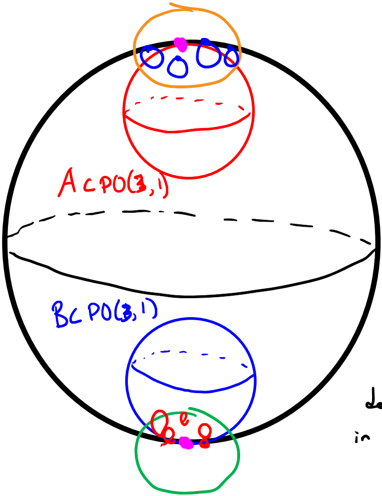
rank-one
higher rank
convex cocompact
Anosov representations
lattices
word-hyperbolic
rel. dominated
generalized cusps
geometrically finite
KL relative Anosov
projective convex
cocompact
EGF
relatively hyperbolic
lattices
(not to scale)

Thm (W.):

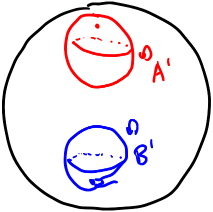
Let $\rho : \Gamma \to G$ be an EGF representation, and let $W \subseteq \mathrm{Hom}(\Gamma, G)$ be a peripherally stable subspace. Then an open neighborhood of ρ in W consists of EGF representations.

Peripheral Stability:

$\mathbb{H}^3 \subset \mathbb{RP}^3$:



deform in $PO(3,1)$



NOT discrete

$A * B$ is discrete

set up ping-pong

deform in $PGL(4)$

A' & B' preserves $\Omega \subset \mathbb{RP}^3$

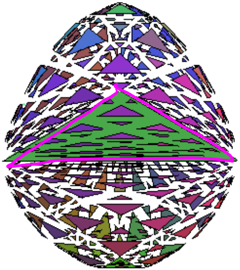
Def: A convex projective structure on a manifold M is a diffeomorphism $M \to \Omega/\Gamma$ for $\Omega \subset \mathbb{RP}^d$ properly convex and $\Gamma < PGL(d+1, \mathbb{R})$ discrete group preserving Ω.

Thm (Danciger-Guéritaud-Kassel, Zimmer): If a compact convex proj. manifold $M = \Omega/\Gamma$ has strictly convex bdry and $\pi_1 M$ is word-hyperbolic, then
$$\Gamma \hookrightarrow PGL(d+1, \mathbb{R}) \text{ is Anosov.}$$

Conversely: every Anosov rep arises in this way (kind of).

Convex projective manifolds with relatively hyperbolic fundamental group:

Benoist: $\Gamma \subset PGL(4,\mathbb{R})$



$\subset \mathbb{RP}^3$

$\curvearrowleft \mathbb{Z}^2 \subset \Gamma$

$\Gamma \cong \pi_1 M$ for 3-mfd M whose JSJ decomposition has hyperbolic pieces.

Other examples

- Ballas - Danciger - Lee
- Choi - Lee - Marquis
- Danciger - Guéritaud - Kassel - Lee - Marquis

Examples with cusps

- Ballas
- Ballas - Marquis
- Bobb

Definition

Let Γ be a hyperbolic group. A representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is *P_1-Anosov* if there exist equivariant embeddings

$$\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d), \qquad \xi^* : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)^*$$

which are *transverse*, and *preserve the dynamics of* Γ.

Definition

Let Γ be a hyperbolic group. A representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is *P_1-Anosov* if there exist equivariant embeddings

$$\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d), \qquad \xi^* : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)^*$$

which are *transverse*, and *preserve the dynamics of* Γ.

Definition (Kapovich-Leeb)

Let Γ be a relatively hyperbolic group. A representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is *relatively asymptotically embedded* if there exist equivariant embeddings

$$\xi : \partial(\Gamma, \mathcal{P}) \to \mathbb{P}(\mathbb{R}^d), \quad \xi^* : \partial(\Gamma, \mathcal{P}) \to \mathbb{P}(\mathbb{R}^d)^*$$

which are *transverse*, and *preserve the dynamics of* Γ.

Definition

Let Γ be a relatively hyperbolic group. A representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is *relatively asymptotically embedded* if there exist equivariant embeddings

$$\xi : \partial(\Gamma, \mathcal{P}) \to \mathbb{P}(\mathbb{R}^d), \quad \xi^* : \partial(\Gamma, \mathcal{P}) \to \mathbb{P}(\mathbb{R}^d)^*$$

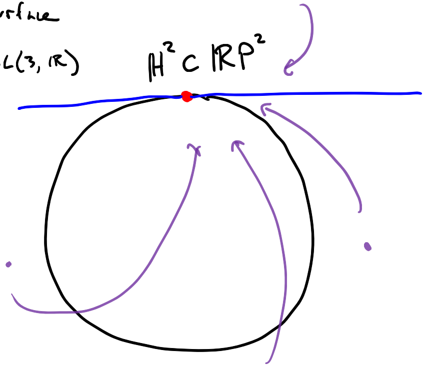
which are *transverse*, and *preserve the dynamics of* Γ.

$\Gamma \cong \pi_1 S$, S punctured surface

$\rho : \Gamma \to PO(2,1) \hookrightarrow PGL(3, \mathbb{R})$

$\mathbb{H}^2 \subset \mathbb{RP}^2$

$$\rho(\gamma) = \begin{bmatrix} 1 & \lambda & \lambda^2/2 \\ & 1 & \lambda \\ & & 1 \end{bmatrix}$$

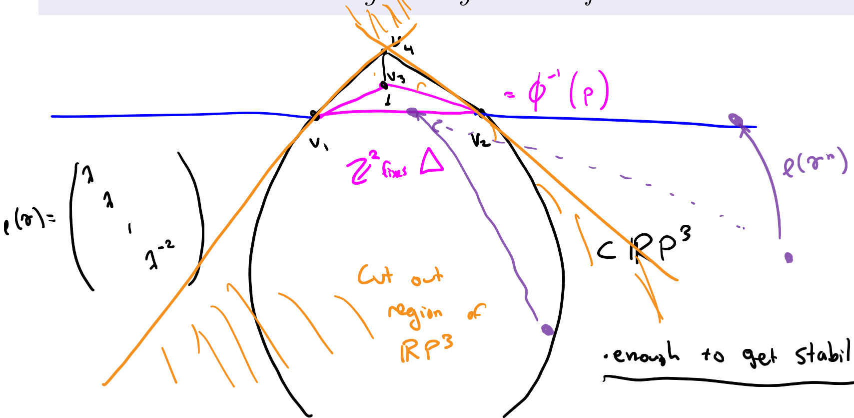


Definition (W.)

Let Γ be a relatively hyperbolic group. A representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is *extended geometrically finite* if there exists a closed set $\Lambda \subset \mathcal{F}_{1,d}$ and a transverse equivariant *extension*

$$\phi : \Lambda \to \partial(\Gamma, \mathcal{P})$$

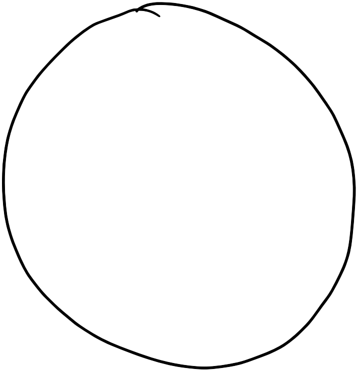
which *extends the convergence dynamics of* Γ.



· enough to get stability

M = finite-vol hyp. 3-mfd.

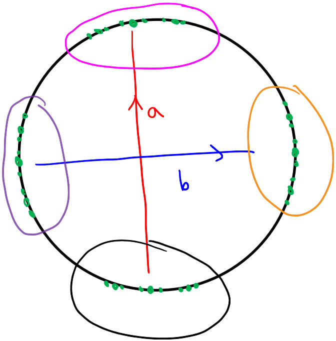
$\pi_1 M \curvearrowright \mathbb{H}^3$

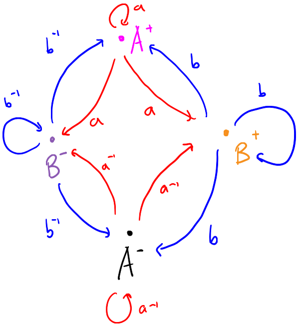


$\partial(\pi_1 M) \cong S^2$

Stability: proof idea

Cover limit set $\Lambda \subset F_{1,d}$ with finitely many open sets & construct finite directed graph with "ping pong" inclusions.





With parabolics: