REAL SOLUTIONS OF POLYNOMIAL SYSTEMS

JORDY LOPEZ GARCIA, KELLY MALUCCIO, FRANK SOTTILE, AND THOMAS YAHL

Abstract. The *Macaulay2* package `RealRoots` contains methods to symbolically explore real roots of univariate polynomials and real solutions of multivariate systems. It updates and expands the package under the same name, developed by Frank Sottile and Dan Grayson. For univariate polynomials, the package provides tools to count and isolate real roots, as well as to determine when a polynomial is *Hurwitz stable*. For multivariate polynomial systems, it provides general methods of elimination to solve zero-dimensional systems, which include the *rational univariate representation* of a zero-dimensional ideal. In particular, for zero-dimensional real multivariate systems, it contains the *trace form* method, which counts the number of real solutions without multiplicity. We provide the algebraic geometry background behind the methods, illustrating them with examples and computations. Among the theorems implemented, we prove a general version of *Sylvester's Theorem*.

Introduction

Since the 19th century, understanding the number of real solutions of polynomial systems is ubiquitious in the study of real algebraic geometry. Grayson and Sottile [3] developed the *Macaulay2* package `RealRoots` to study enumerative aspects of real algebraic geometry. We build upon their work and implement tools to symbolically explore real solutions of polynomial systems.

The paper is divided into three sections. Section 1 is devoted to algorithms on univariate polynomials. Among its methods are `BudanFourierBound` that gives a bound to the number of real roots of a univariate polynomial in an interval $(a, b]$, and `realRootIsolation` which isolates its real roots. The original package contained methods to count real roots of univariate polynomials by means of *Sturm's Theorem*. We implement a general version of Sturm's theorem, called *Sylvester's Theorem*, and provide a proof that, to the best of our knowledge, did not previously exist in literature. We also define *Hurwitz stability*, and present algorithms to determine the Hurwitz stability of a univariate polynomial.

Section 2 covers elimination methods to solve zero-dimensional systems. In particular, we expand the `eliminant` method by dividing it into the methods `univariateEliminant` and `minimalPolynomial`. We also implement algorithms to compute the *rational univariate representation* of a zero-dimensional ideal [2].

Section 3 focuses on methods to count real solutions of zero-dimensional, real multivariate systems. In particular, we implement a multivariate version of Sylvester's theorem, called `traceSignature`.

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1. Real Univariate Polynomials

Our story begins with a real univariate polynomial f of the form

$$c_0x^{a_0} + c_1x^{a_1} + \cdots + c_mx^{a_m},$$

where $i = 0, 1, \ldots, m$, where $c_i \neq 0$ are real numbers, and where $a_0 < a_1 < \cdots < a_m$ are nonnegative integers. Descartes' Rule of Signs [4] gives us an upper bound for the number of positive real roots of f.

Theorem 1 (Descartes' Rule of Signs)**.** *Let r be the number of positive real roots of f, counting multiplicity, and let v be the cardinality of the set $\{i \mid 1 \leq i \leq m, c_{i-1}c_i < 0\}$. Then $r \leq v$ and $v - r \equiv 0 \mod 2$.*

Equivalently, we find v by counting the number of sign changes between consecutive elements of the sequence $(c_0, c_1, \ldots, c_m)$. The *variation* of a finite sequence of numbers c, denoted by $var(c)$, is the number of times its consecutive elements have opposite sign, after removing any zero terms. Hence $v = var((c_0, c_1, \ldots, c_m))$.

If $F = (f_0, f_1, \ldots, f_k)$ is a finite sequence of real univariate polynomials, and $a \in \mathbb{R}$, we define $var(F, a)$ to be the variation of the sequence $(f_0(a), f_1(a), \ldots, f_k(a))$. If $a = \pm\infty$, then, for $j = 0, \ldots, k$, $var(F, \infty)$ is the variation of the leading coefficients of the polynomials $f_j(t)$, and $var(F, -\infty)$ is the variation of the leading coefficients of the polynomials $f_j(-t)$.

If g is a univariate polynomial of degree l, we denote by δg its sequence of derivatives:

$$\delta g = \left(g(x), g'(x), g''(x), \ldots, g^{(l)}(x)\right).$$

Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ such that $a < b$, and denote by $r(f, a, b)$ the number of real roots of f in the interval $(a, b]$. Budan and Fourier [4] generalized Descartes' Rule of Signs.

Theorem 2 (Budan-Fourier)**.** *Let v_a denote $var(\delta f, a)$ and v_b denote $var(\delta f, b)$. The real polynomial f satisfies*

$$v_a - v_b \geq r(f, a, b),$$

and $v_a - v_b - r(f, a, b) \equiv 0 \mod 2$.

The previous theorems bound the number of real roots of f. We now describe methods to find the *actual* number of its real roots. The Sylvester sequence of two real univariate polynomials f and g, denoted by $Syl(f, g)$, is the sequence $(f_0, f_1, \ldots, f_k)$, where $f_0 := f, f_1 := g, f_k := \gcd(f, g)$, and

$$-f_{i+1} = \text{remainder}(f_{i-1}, f_i).$$

Theorem 3 (Sylvester)**.** *The difference between the number of roots of f in $(a, b]$ where g is positive and the number of roots of f in $[a, b)$ where g is negative is counted by the difference in variation*

$$var(Syl(f, f'g), a) - var(Syl(f, f'g), b).$$

Proof. The proof when all the roots of f are in (a, b) is found in [1], where in their notation, a *Sturm sequence* refers to our definition of a *Sylvester sequence*. It is left to consider when a or b are roots of f. Set $g_i := f_i/f_k$, for $i = 0, \ldots, k$. By construction, $var((f_0(x), f_1(x)))$ (resp. $var((f_{i-1}(x), f_i(x), f_{i+1}(x))))$ and $var((g_0, g_1))$ (resp. $var((g_{i-1}(x), g_i(x), g_{i+1}(x))))$ are the same when x is not a root of f. The roots of $g_0 = f/f_k$ is the number of roots of f which are not roots of g. Moreover, g_i and g_{i+1} are relatively prime.

Let c be a root of g_i, and denote by c_- a number immediately to the left of c (avoiding any zeros), and c_+ a number immediately to its right. If c is a root of g_0, then c is not a root of g_1. We obtain the following cases.

	$c = a$	
f	0	$+$
$f'g$		$-$

$g(c) > 0, f'(c_-) < 0, f'(c_+) > 0$

		$c = b$
f	$-$	0
$f'g$	$+$	

$g(c) > 0, f'(c_-) > 0, f'(c_+) > 0$

	$c = a$	
f	0	$+$
$f'g$		$-$

$g(c) < 0, f'(c_-) < 0, f'(c_+) < 0$

		$c = b$
f	$-$	0
$f'g$	$+$	

$g(c) > 0, f'(c_-) < 0, f'(c_+) < 0$

As x leaves a, $var((f_0, f_1), x)$ increases by 1, and as x reaches b, $var((f_0, f_1), x)$ decreases by 1. If c is a root of g_i, for $i \geq 1$, then c is not a root of g_{i-1} nor g_{i+1}. Hence $g_{i-1}(c)g_{i+1}(c) < 0$. Hence, the variation of $(f_{i-1}(x), f_i(x), f_{i+1}(x))$, which is the same as $var((g_{i-1}(x), g_i(x), g_{i+1}(x)))$, remains unchanged. $\square$

Corollary 4 (Sturm's Theorem). *Let f be a univariate polynomial and $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$ and $f(a), f(b) \neq 0$. Then the number of zeros of f in the interval $(a, b]$ is the difference*

$$var(F, a) - var(F, b),$$

where F is the Sturm sequence of f.

Theorem 5. *Let $f(x) = \sum_{j=0}^{n} a_j x^j$ with $n \geq 1$ and $a_n > 0$. Then f is Hurwitz stable if and only if all the Hurwitz determinants $\delta_1, \ldots, \delta_n$ are all positive.*

2. Eliminations

3. Real Multivariate Systems

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J. Lopez Garcia, Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA
Email address: jordy.lopez@tamu.edu
URL: https://jordylopez27.github.io/

K. Maluccio, Department of Mathematics, Austin Community College, Austin, Texas 78752, USA
Email address: kmaluccio@gmail.com
URL: https://github.com/kmaluccio

F. Sottile, Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA
Email address: sottile@tamu.edu
URL: https://www.math.tamu.edu/~sottile

T. Yahl, Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA
Email address: thomasjyahl@tamu.edu
URL: https://tjyahl.github.io/