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NOTES FOR MACAULAY2 REAL PROJECT

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Abstract. These are notes around the project on implementing rountines to study real roots of polynomial systems in Macaulay2.

1. Introduction

For a sequence $c = (c_0, \ldots, c_n)$ of real numbers, let $\mathrm{var}(c)$ be the variation (number of changes in sign) in the sequence c. This is the number of times consecutive elements of the sequence c have opposite signs, after removing any occurrences of 0 from c. Given a sequence $F = (f_0(t), \ldots, f_m(t))$ of real univariate polynomials and $a \in \mathbb{R}$, let $\mathrm{var}(F, a)$ be the variation in the sequence $(f_0(a), \ldots, f_m(a))$. We define $\mathrm{var}(F, \infty)$ to be the variation in the leading coefficients of the polynomials in F, which is $\mathrm{var}(F, a)$ for $a \gg 0$ sufficiently positive, and set $\mathrm{var}(F, -\infty)$ to be $\mathrm{var}(F, a)$ for $a \ll 0$ sufficiently negative.

For a univariate polynomial $f \in \mathbb{R}[t]$ of degree m, its derivative sequence δf is the sequence of its derivatives, $\delta f := (f(t), f'(t), f''(t), f^{(3)}(t), \ldots, f^{(m)}(t))$. For numbers $a < b$ in $\mathbb{R} \cup \{\pm\infty\}$, let $r(f, a, b)$ be the number of roots of f in the interval $(a, b]$.

Theorem 1.1 (Budan-Fourier Theorem). *Let $f \in \mathbb{R}[t]$ be a univariate polynomial and $a < b$ be numbers in $\mathbb{R} \cup \{\pm\infty\}$. Then $\mathrm{var}(\delta f, a) - \mathrm{var}(\delta f, b) \geq r(f, a, b)$, and the difference is even.*

(This implies Descartes' rule of signs for roots in $(0, \infty)$.)

Given univariate polynomials f, g, their *Sturm sequence* $\mathrm{St}(f, g)$ is the sequence

$$f_0 \ := \ f \,,\ f_1 \ := \ f'g \,,\ \ f_2 \,, f_3 \,,\ \ldots \,,\ f_m \,,$$

where f_m is a greatest common divisor of f and $f'g$, and for each $i \geq 1$, $-f_{i+1}$ is the negative of the remainder from the Euclidean algorithm. That is, f_{i+1} is the unique polynomial of degree less than the degree of f_i such that there is a polynomial q_i with $f_{i-1} = q_i f_i - f_{i+1}$.

Theorem 1.2 (Sylvester's Theorem). *Let $f, g \in \mathbb{R}[x]$ be univariate polynomials and suppose that $a < b$ are numbers in $\mathbb{R} \cup \{\pm\infty\}$ such that neither is a root of f. Then $\mathrm{var}(\mathrm{St}(f, g), a) - \mathrm{var}(\mathrm{St}(f, g), b)$ is the difference of the number of roots $x \in (a, b)$ of f with $g(x) > 0$ and those with $g(x) < 0$.*

Theorem 1.3 (Sturm's Theorem). *Let $f \in \mathbb{R}[x]$ be a univariate polynomial suppose that $a < b$ are two numbers in $\mathbb{R} \cup \{\pm\infty\}$ such that neither is a root of f. Then $\mathrm{var}(\mathrm{St}(f, 1), a) - \mathrm{var}(\mathrm{St}(f, 1), b)$ is the number of roots of f in the open interval (a, b).*

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Lemma 1.4. *Let $f = c_0 + c_1 t + \cdots + c_m t^m$ be a univariate polynomial and $M := |c_0/c_m| + \cdots + |c_{m-1}/c_m| + 1$. Then all roots of f lie in the interval $(-M, M)$.*

Proof. Let $|x| > M$. If $b_i := c_i/c_m$, then $f(x) = c_m x^m(b_0 x^{-m} + \cdots + b_{m-1}x^{-1} + 1)$. As

$$|b_0 x^{-m} + \cdots + b_{m-1}x^{-1}| \ < \ (|b_0| + \cdots + |b_{m-1}|)M^{-1} \ < \ 1\,,$$

we see that $f(x) \neq 0$. $\square$

This gives the start of a bisection algorithm for locating the roots of f.

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