We are trying to find the maximum value of c such that the equation

$$538x + 19y = c \tag{1}$$

has no non-negative integral solutions for x and y. The above equation is called a linear Diophantine equation, which takes the form $ax + by = c$. In general, there exist infinitely many integral solutions (that is, integers x and y that could be negative, zero, or positive) to the equation whenever $gcd(a, b)$ evenly divides c. Notice that $gcd(538, 19) = 1$; thus, if the bank were to offer arbitrarily large loans of either denomination, there would always be a way to convert dollars to Dios and Phanti.

However, we are interested in the case where the bank does not offer any loans. First, notice that a base solution for the Diophantine equation can be found via the extended Euclidean algorithm, which finds integers n and m such that

$$gcd(a, b) = an + bm. \tag{2}$$

In the case where $a = 538$ and $b = 19$, the algorithm finds the base solution $n = -3$ and $m = 85$. All other solutions will be of the form

$$x = \frac{nc}{gcd(a, b)} + \frac{bk}{gcd(a, b)} \tag{3}$$

$$y = \frac{mc}{gcd(a, b)} - \frac{ak}{gcd(a, b)} \tag{4}$$

for an integer k. Since we are restricting ourselves to $x, y \geq 0$, the following inequalities must be true:

$$nc + bk \geq 0 \tag{5}$$

$$mc - ak \geq 0. \tag{6}$$

Since a and b are positive in our example, these inequalities reduce to

$$-\frac{nc}{b} \leq k \leq \frac{mc}{a}. \tag{7}$$

Thus, there does not exist a positive solution to the Diophantine equation if and only if there does not exist an integer between $-nc/b$ and mc/a. Notice that this implies that if $mc/a - (-nc/b) \geq 1$, then there will necessarily be a positive solution. This inequality reduces to

$$c \geq \frac{1}{\frac{m}{a} + \frac{n}{b}} \tag{8}$$

so long as $m/a + n/b$ is positive.

Returning to our specific example, recall that $a = 538$, $b = 19$, $n = -3$, and $m = 85$. Thus, $m/a + n/b$ is positive, with

$$\frac{1}{\frac{m}{a} + \frac{n}{b}} = \frac{1}{\frac{85}{538} - \frac{3}{19}} = 10222. \tag{9}$$

Thus, for all dollar amounts $c \geq 10222$, there is a way to convert to non-negative amounts of Dios and Phanti. So, if we find the largest dollar amount less than \$10222 that cannot be converted, we will have found the largest dollar amount that cannot be converted in general. Iterating through values of c less than 10222 and using inequality (7) to determine if a positive integral solution exists, the largest amount that cannot be converted is \$9665.