

zkSNARKs Cryptography Protocol Survey

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1 zkSNARKs Cryptography Protocol Survey

1.1 Basic information

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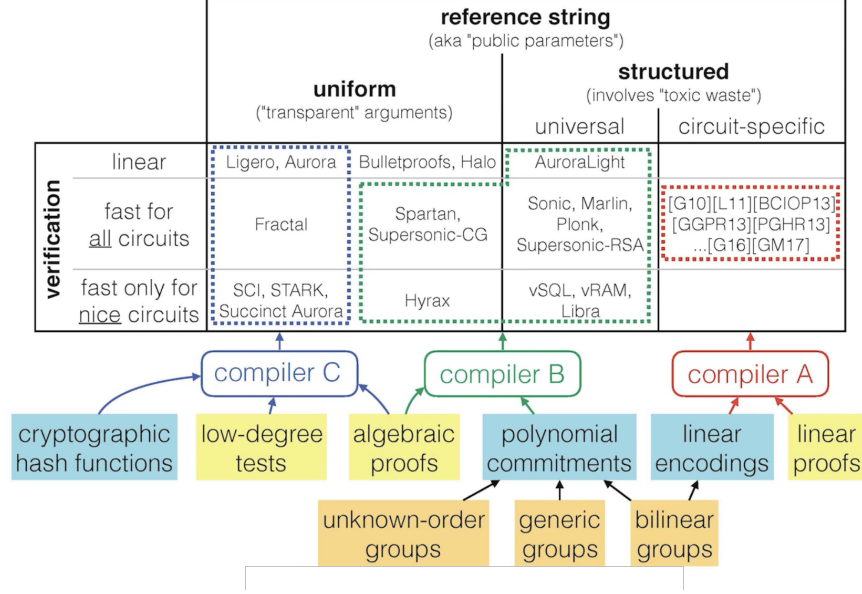
1.2 Research scope

We use the term SNARK for what is sometimes called a "SNARK with pre-processing" (see e.g. [Gen+13]) where one allows a one-time verifier computation that is polynomial rather than polylogarithmic (or linear) to the circuit size. This limit our discussion to SNARKs with universal or circuit-specific SRSs, so the proof is fully succinct¹ or equivalent. In return, the SNARK is expected to work for all non-uniform circuits, rather than only statements about uniform computation.

As the following diagram (from Alessandro Chiesa's [State of the SNARG-scape speech](#)) illustrated:

¹A zk-SNARK for circuit satisfiability is fully succinct if: 1) The pre-processing phase/SRS generation run time is quasilinear in circuit size. 2) The prover run time is quasilinear in circuit size. 3) The proof length is logarithmic in circuit size. 4) The verifier run time is polylogarithmic in circuit size.

A Slice of The SNARG-scape



the research scope of this survey will be focusing on the SNARKs in the right two columns (under structured reference string, i.e. universal and circuit-specific). The constructions on the left side, such as STARK, Fractal, Bulletproofs etc. will be left for future work.

The only exception is Halo. Due to the special (growing) importance of recursive SNARKs, Halo will be included in the scope.

1.3 Overview

zkSNARK is one of the most important Zero-Knowledge Proof (ZKP) methods used in privacy computing and blockchain technology. zkSNARK stands for: zero-knowledge Succinct Non-Interactive ARgument of Knowledge, which is a protocol through which a prover can quickly convince a verifier on knowledge of a secret without revealing anything about it. Nir Bitansky, Ran Canetti, Alessandro Chiesa, and Eran Tromer introduced the term zk-SNARKs in 2012 [Bit+12]. The core theoretical components of zkSNARK are summarized below:

- zero-knowledge (zk) According to Goldwasser, Micali, and Rackoff [GMR85], who first introduced the "zero-knowledge" concept as "*Zero-knowledge proofs are defined as those proofs that convey no additional*

knowledge other than the correctness of the proposition in question".

There are three essential attributes of zk featuring protocols:

- Completeness: If the statement is true, then the verifier always accepts the proof π .
 - Soundness: If the statement is false, then the prover can only convince the verifier with a negligible probability.
 - Zero-knowledge: a correct proof π does not leak any information about the secret.
- Succinctness (S) The first constructions of SNARK protocols were inspired by the PCP theorem [AS98] which shows that NP statements have "short" Probabilistic Checkable Proofs (PCP). New instantiations were found which allow faster and shorter proofs, when a pre-processing state is permitted. With proper pre-processing and arithmetization, the proof becomes realistically feasible w.r.t proof size, running time, verification time etc. The formal definition is below [Bit+12]:

The length of the proof Π that $\mathcal{P}(y, w, vgrs)$ outputs, as well as the running time of $\text{nu}(\text{priv}, y, \Pi)$, is bounded by:

$$p(k + |y|) = p(k + |M| + |x| + \log t)$$

where p is a universal polynomial that does not depend on $\mathcal{R} \subseteq \mathcal{R}_{\mathcal{U}}$. $\mathcal{R}_{\mathcal{U}}$ is the universal relation [BG02], a.k.a the set $\mathcal{R}_{\mathcal{U}}$ of instance-witness pairs (y, w) , where $y = (M, x, t)$, $|w| \leq t$ and M is a Turing machine such that M accepts (x, w) after at most t steps.

- "Non-interactive" (N) means the prover can publish a single one-way message to the verifiers, with no need of further back-and-forth interactions.
- "ARguments" (AR): An argument system for an NP relationship \mathcal{R} is a protocol between a computationally-bounded prover P and a verifier V . AR in SNARK means V is only protected against computationally-bounded P . Provers with computationally unlimited power can fake proofs/arguments on wrong statements. This requirement is also called "computational soundness", as opposed to "perfect soundness".²

²Christian Reitwiessner's blog: [zkSNARKs in a nutshell](#)

- "of Knowledge":

SNARKs are SNARGs where computational soundness is replaced by knowledge soundness, which roughly means the prover is not able to construct a proof without knowing the witness w .²

1.4 Big idea

As the name hints, zkSNARKs have four main components²:

- Encoding the problem as a polynomial

The problems are compiled into polynomials, e.g. $t(x)h(x) = w(x)v(x)$, the Quadratic Span Programs (QSP) as in [Gen+13], where the equality holds *iff* the program is computed correctly. The prover convinces the verifier that the above equality holds. The formal definition in [Bit+12] is:

... membership of an instance y in an NP language L can be verified in time that is bounded by $p(k, |y|, \log t)$, where t is the time to evaluate the NP verification relation for L on input y , p is a fixed polynomial independent of L , and k is a security parameter that determines the soundness error.

In general, zkSNARKs can be constructed from four categories [Nit20]:

- PCP

The oldest SNARK construction methodology is explained:

...is based on PCP characterization of NP, and it is first achieved in the random oracle model (ROM), which gave only heuristical security. The idea is to apply the random oracle-based Fiat-Shamir transform to Kilian's succinct PCP-based proof system [Kil92], achieving logarithmic proof size and verification time. Later, the construction is improved by removing the use of the random oracles and replacing them with extractable collision-resistant hash functions (ECRH).

The work of [CL08] proposed the PCP + Merkle Tree (MT)+ Private Information Retrieval (PIR) approach to “squash” Kilian’s four-message protocol into a two-message protocol.

In both cases, we do not obtain knowledge soundness, but only plain adaptive soundness.

– QAP

The most popular and widely implemented SNARK construction shares a central starting point on quadratic programs introduced by [Gen+13]. The proof/verification framework build SNARKs for programs encoded as boolean or arithmetic circuits.

/Arithmetic Circuits/: Informally, an arithmetic circuit consists of wires that carry values from a field F and connect to addition and multiplication gates.

/Boolean Circuits/: A boolean circuit consists of logical gates and of a set of wires between the gates. The wires carry values over 0, 1.

This approach has led to fast progress towards practical verifiable computations. The first implementation is Pinocchio [Par+13].

A SNARK scheme for a circuit has to enable verification of proofs for (Arithmetic or Boolean) Circ-SAT problem, i.e., a prover, given a circuit (\mathcal{C}) has to convince the verifier that it knows an assignment of its inputs that makes the output true.

Circ-SAT problem is the NP-complete decision problem of determining whether a given circuit has an assignment of its inputs that makes the output true. A very important line of works focuses on building SNARKs for circuit satisfiability and have as a central starting point the framework based on quadratic span programs (QSP) [Gen+13]. QSP consists of multiple polynomials $\nu_0, \dots, \nu_i; \omega_0, \dots, \omega_i$ over a Galois field F and a target polynomial t . The QSP is accepted *iff* t divides $\nu_a \times \omega_b$ where ν_a and ω_b is constructed from the witness w and the original polynomials $\nu_0, \dots, \nu_i; \omega_0, \dots, \omega_i$.

Further more, QSP can be improved by QAP. [Nit20]

Parno et al. [Par+13] defined QAP, a similar notion for arithmetic circuits, namely Quadratic Arithmetic Programs (QAP). More recently, an improved version for boolean circuits, the Square Span Programs (SSP) was presented which consequentially has led to a simplified version for arithmetic circuits, Square Arithmetic Programs (SAP), proposed in [GM17].

- Linear Interactive Proof (LIP)

According to [Nit20],

The QAP approach was generalized under the concept of Linear Interactive Proof (LIP), a form of interactive ZK proofs where security holds under the assumption that the prover is restricted to compute only linear combinations of its inputs.

These proofs can then be turned into (designated-verifier) SNARKs by using an extractable linear-only encryption scheme.

- Polynomial Interactive Oracle Proof (PIOP)

According to [Nit20]:

As we saw previously, SNARKs are commonly built in a modular way. Recent works propose schemes that follow a new framework, enabling more possibilities such as: **transparent constructions (without a trusted setup), recursion, aggregation properties, post-quantum security**, etc.

Modular instantiations of recent SNARKs employ polynomial *interactive oracle proofs (IOP)* for the information theoretic step, and *polynomial commitments* for the cryptographic compilation. One advantage of using polynomial commitments is that **the setup is only needed for the commitment scheme and is thus independent of the statement being proven**. This allows these SNARKs

to be **universal** as opposed to the circuit-based SNARKs presented in the previous section. Another advantage is that the choice of polynomial commitment scheme facilitates finding the right trade-off between performance and security.

To build such SNARKs, one commonly follows 3 steps:

- * NP Characterization or Arithmetization for Circuits:

Start with a way to describe the computation to be proven as a system of constraints involving native operations over a finite field. Then, this system of constraints can be changed into a (low-degree) **univariate polynomial expression**.

- * Polynomial IOP or Algebraic Holographic Proofs: The proof consists in finding a way to show that such a polynomial equation holds. In a Polynomial Interactive Oracle Proof (PIOP), the prover sends low degree polynomials to the verifier, and rather than reading the entire list of coefficients, the verifier queries evaluations of these polynomials in a random point to an oracle interface.

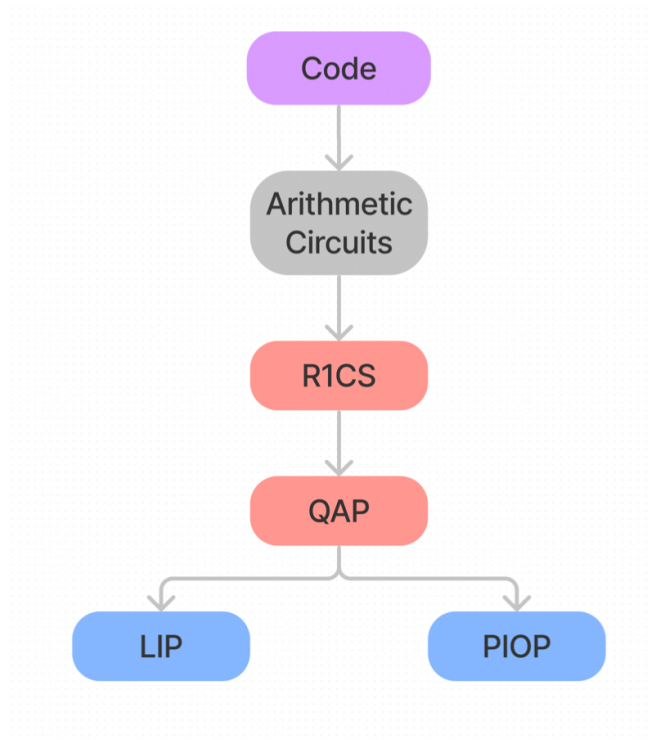
- * Cryptographic Compiler.

Using polynomial commitment schemes and Fiat-Shamir heuristic, the previous checks can be compiled into an efficient and non-interactive proof system. The resulting zk-SNARK has either universal updatable structured reference string (SRS), or transparent setup, depending only on the nature of the polynomial commitment scheme used in this step.

Remark that the only step where we employ cryptographic techniques is the final one. In this cryptographic compilation step the oracles from before are replaced by a suitable cryptographic realization: Polynomial Commitments (PC) in Sonic, Marlin and Plonk ([Mal+19], [Mal+19], [GWC19]).³ Polynomial commitments allow

to check efficiently a series of identities on low-degree polynomials resulting from the arithmetization step. This comes at the expense of introducing computational hardness assumptions for security and sometimes a trusted setup.

A typical modern zkSNARK process can be illustrated as follows:



Source of the above image is unknown, meanwhile it appears to be frequently credited to cryptographer Eran Tromer.

Polynomial commitments e.g. KZG, are a commitment scheme introduced by [KZG10] which are extremely powerful constructs that allow a prover to commit a polynomial and show evaluations of that polynomial on any point with a single group element-sized proof in constant time. As such, Kate commitments and its variants lie at the heart of many PIOP-style zkSNARK constructions (such as Marlin and PLONK).

- Succinctness by random sampling

The verifier randomly samples a secret input s for evaluation with the polynomials, thus reducing the problem from polynomials operations to simple evaluation on numbers, e.g. $t(s)h(s) = w(s)v(s)$

- Homomorphic encoding/encryption

An encoding/encryption function e is used that has some homomorphic properties. Combining the research on pairing-based double homomorphic encryption scheme [BGN05] and NIZK leads to the solution on unconditional zero-knowledge:

ZK proof for NP-complete languages	Computational zero knowledge	Unconditional zero knowledge
Interactive	Goldreich-Wigderson-Micali 1986	Brassard-Cr�peau 1986
Non-interactive	Blum-Feldman-Micali 1988	Groth-Ostrovsky-Sahai 2006

– Core idea ([GOS06]) to prove circuit satisfiability with zk⁴:

- * Commit to wire values in circuit as $g^a h^r, g^b h^s, g^c h^t, \dots$
- * Verifier can easily add committed elements: $g^a h^r \cdot g^b h^s = g^{a+b} h^{r+s}$
- * Prover can demonstrate multiplicative relation: $e(g^a h^r, g^b h^s) = e(g^c h^t, g) e(\pi, h)$

- Obfuscation

The prover obfuscates the values $e(g^a h^r, g^b h^s), e(g^c h^t, g), e(\pi, h)$ by multiplying with a non-zero secret point k from the contextual elliptic curve (EC) on both sides, thus the verifier can still verify the correctness on hidden values.

1.5 Theoretical development

1.5.1 Pre-Pinocchio

Based on the retrospective summary by [Par+13],

⁴Jens Groth's lecture [Bilinear Pairing-based Zero-Knowledge Proofs](#)

Considerable systems and theory research has looked at the problem of verifying computation. However, most of this work has either been function specific, relied on assumptions we prefer to avoid, or simply failed to pass basic practicality requirements. Function specific solutions are often efficient, but only for a narrow class of computations. More general solutions often rely on assumptions that may not apply. For example, systems based on replication assume uncorrelated failures, while those based on Trusted Computing or other secure hardware assume that physical protections cannot be defeated. Finally, the theory community has produced a number of beautiful, general purpose protocols ([GMR85], [Kil92], [Mic00], [GKR15], [Gro10], [GGP10], [CKV10], [Gen09]), that offer compelling asymptotics. In practice however, because they rely on complex Probabilistically Checkable Proofs (PCPs) [AS98] or fully-homomorphic encryption (FHE) [Gen09], the performance is unacceptable – verifying small instances would take hundreds to trillions of years. . .

1.5.2 Pinocchio

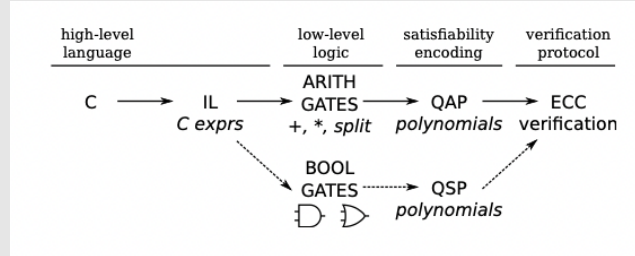
Pinocchio [Par+13] was the first **practical** implementation of a zk proving system; for instance, before adoption of [Gro], earliest version of zCash implemented Pinocchio to deliver their original shielded transaction protocol [BCT20]. (*Heavily quoted from [Par+13] below*).

- Major features
 - Supports public verifiable computation, which allows an untrusted worker to produce signatures of computation
 - Supports zero-knowledge verifiable computation, in which the worker convinces the client that it knows an input with a particular property, without revealing any information about the input.
 - Verification time is typically 10ms: 5-7 orders of magnitude less than previous work.

- The first general-purpose system to demonstrate verification cheaper than native execution (for some apps).
- Reduces the worker’s proof effort by an additional $19\text{-}60\times$
- Generalizes to zero-knowledge proofs at a negligible cost over the base protocol.
- Provides an end-to-end toolchain that compiles a subset of C into programs that implement the verifiable computation protocol.
- Design ideas

To achieve efficient verifiable computation, Pinocchio combines quadratic programs (QSP), a computational model introduced by Gennaro et al. [Gen+13], with a series of theoretical refinements and systems engineering to produce an end-to-end toolchain for verifying computations. Specifically, via an improved protocol and proof technique, we slash the cost of key generation by 61

From a developer’s perspective, Pinocchio provides a *circuit compiler* that transforms C code into a circuit representation (both Boolean and arithmetic supported), thus converts the circuit into a quadratic program, and then generates programs to execute the cryptographic protocol.



- Contributions
 1. An end-to-end system for efficiently verifying computation performed by one or more untrusted workers. This includes a compiler that converts C code into a format suitable for verification, as well as a suite of tools for running the actual protocol.

2. Theoretical and systems-level improvements that bring performance down by 5-7 orders of magnitude, and hence into the realm of plausibility.
3. An evaluation on seven real C applications, showing verification faster than 32-bit native integer execution for some apps.

1.5.3 Groth16

In 2016, Jens Groth formalized a proving system that significantly improved performance [Gro] compared to Pinocchio.

Despite the required trusted setup, Groth16 currently remains the fastest known zk-SNARK with the smallest proof size. Groth16's trusted setup is per circuit thus non-universal.⁵ New SNARKs often use Groth16 as the performance benchmark.

Below are heavily quoted from [Gro] and [Nit20]:

- Major features
 - For arithmetic circuits, Groth16 proofs consist of only 2 elements in G1 and 1 in G2, compared to Pinocchio's 7 elements in G1, and 1 in G2.
 - Verification is faster. Verifiers need to compute a number of exponentiations proportional to the statement size and check a single pairing product equation with just 3 pairings instead of 12.
 - Shorter CRS.
 - The construction can be instantiated with any type of pairings including Type III pairings [Nad], which are the most efficient pairings.

Based on the observation of "arithmetic circuit satisfiability in the Table, of the size of the common reference string (CRS), the size of the proof, the prover's computation, the verifier's computation, and the number of pairing product equations used to verify a proof", Groth16

⁵Comparing General Purpose zk-SNARKs

performs better than the state of the art on all efficiency parameters when published.

	CRS size	Proof size	Prover comp.	Verifier comp.	PPE
[PHGR13]	$7m + n - 2\ell \mathbb{G}$	$8 \mathbb{G}$	$7m + n - 2\ell E$	$\ell E, 11 P$	5
This work	$m + 2n \mathbb{G}$	$3 \mathbb{G}$	$m + 3n - \ell E$	$\ell E, 3 P$	1
[BCTV14a]	$6m + n + \ell \mathbb{G}_1, m \mathbb{G}_2$	$7 \mathbb{G}_1, 1 \mathbb{G}_2$	$6m + n - \ell E_1, m E_2$	$\ell E_1, 12 P$	5
This work	$m + 2n \mathbb{G}_1, n \mathbb{G}_2$	$2 \mathbb{G}_1, 1 \mathbb{G}_2$	$m + 3n - \ell E_1, n E_2$	$\ell E_1, 3 P$	1

Figure 1: Comparison for arithmetic circuit satisfiability with ℓ -element statement, m wires, n multiplication gates.

- Design ideas.

Groth16’s construction is simplified compared to the one in [Par+13]. Popular research [Nit20] have observed that the proof elements are not duplicated with respect to some random factor and this is not needed for the extraction, since the security relies on a stronger model, the Generic Group Model(GGM) and not on q-PKE assumption.

The Generic Group Model [Sho97], [Mau05] is an idealized cryptographic model, where algorithms do not exploit any special structure of the representation of the group elements and can thus be applied in any cyclic group. In this model, the adversary is only given access to a randomly chosen encoding of a group, instead of efficient encodings, such as those used by the finite field or elliptic curve groups used in practice. The model includes an oracle that executes the (additive) group operation. Therefore, one can efficiently extract the coefficients used to express an output of the oracle as a linear combination of initial group elements.

To describe Groth’s construction, we consider the relation \mathcal{R} implemented as an arithmetic circuit for a total number of wires m . We denote by $\mathcal{T}_{io} = 1, 2, \dots, \ell$, the indices corresponding to the public input and public output values of the circuit wires (corresponding to the statement u) and by $\mathcal{T}_{mid} = \ell + 1, \dots, m$, the wire indices corresponding to private input and non-input, non-output

intermediate values (for the witness w).

Below is the birdview of Groth16's algorithm.

<p>Groth.Gen($1^\lambda, \mathbb{C}$)</p> <hr/> <p> $\alpha, \beta, \gamma, \delta \leftarrow \mathbb{Z}_p^*, \quad s \leftarrow \mathbb{Z}_p^*,$ $\text{crs} = \left(QAP, g^\alpha, g^\beta, g^\delta, \{g^{s^i}\}_{i=0}^{d-1}, \left\{ g^{\frac{\beta v_k(s) + \alpha w_k(s) + y_k(s)}{\gamma}} \right\}_{k=0}^\ell, \left\{ g^{\frac{\beta v_k(s) + \alpha w_k(s) + y_k(s)}{\delta}} \right\}_{k>\ell}, \right.$ $\left. \left\{ g^{\frac{s^i t(s)}{\delta}} \right\}_{i=0}^{d-2}, h^\beta, h^\gamma, h^\delta, \{h^{s^i}\}_{i=0}^{d-1} \right)$ $\text{vk} := (P = g^\alpha, Q = h^\beta, \{S_k = g^{\frac{\beta v_k(s) + \alpha w_k(s) + y_k(s)}{\gamma}}\}_{k=0}^\ell, H = h^\gamma, D = h^\delta)$ $\text{td} = (s, \alpha, \beta, \gamma, \delta)$ return (crs, td) </p>	<p>Groth.Ver(vk, u, π)</p> <hr/> <p> $\pi = (A, B, C)$ $v_{io}(x) = \sum_{i=0}^\ell a_i v_i(x)$ $w_{io}(x) = \sum_{i=0}^\ell a_i w_i(x)$ $y_{io}(x) = \sum_{i=0}^\ell a_i y_i(x)$ $f_{io} = \frac{\beta v_{io}(s) + \alpha w_{io}(s) + y_{io}(s)}{\gamma}$ Check $e(A, B) = e(g^\alpha, h^\beta) e(g^{f_{io}}, h^\gamma) e(C, h^\delta)$ </p>
<p>Groth.Prove(crs, u, w)</p> <hr/> <p> $u = (a_1, \dots, a_\ell), a_0 = 1$ $w = (a_{\ell+1}, \dots, a_m)$ $v(x) = \sum_{k=0}^m a_k v_k(x)$ $v_{mid}(x) = \sum_{k \in I_{mid}} a_k v_k(x)$ $w(x) = \sum_{k=0}^m a_k w_k(x)$ $w_{mid}(x) = \sum_{k \in I_{mid}} a_k w_k(x)$ $y(x) = \sum_{k=0}^m a_k y_k(x)$ $y_{mid}(x) = \sum_{k \in I_{mid}} a_k y_k(x)$ $h(x) = \frac{(v(x)w(x) - y(x))}{t(x)}$ $f_{mid} = \frac{\beta v_{mid}(s) + \alpha w_{mid}(s) + y_{mid}(s)}{\delta}$ $r, r' \leftarrow \mathbb{Z}_p^*$ $a = \alpha + v(s) + r\delta,$ $b = \beta + w(s) + r'\delta$ $c = f_{mid} + \frac{t(s)h(s)}{\delta} + r'a + rb - r'r\delta$ return $\pi : (A = g^a, B = h^b, C = g^c)$ </p>	<p>Groth.Sim(td, u)</p> <hr/> <p> $a, b \leftarrow \mathbb{Z}_p^*$ $c = \frac{ab - \alpha\beta - \beta v_{io}(s) + \alpha w_{io}(s) + y_{io}(s)}{\delta}$ return $\pi : (A = g^a, B = h^b, C = g^c)$ </p>

Figure 2: Groth16 Construction from QAP.

- Contributions
 - Our first contribution is a pairing-based (pre-processing) SNARK for arithmetic circuit satisfiability, which is an NP-complete language. In our SNARK we work with:

1. *asymmetric pairings* for higher efficiency,
 2. a proof is only 3 group elements
 3. verification consists of checking *a single pairing product equations using 3 pairings in total*.
- As our second contribution we answer an open question of Bitansky, Chiesa, Ishai, Ostrovsky and Paneth (TCC 2013) by showing that *2-move linear interactive proofs cannot have a linear decision procedure*. It follows from this that *SNARGs where the prover and verifier use generic asymmetric bilinear group operations cannot consist of a single group element*. This gives the *first lower bound for pairing-based SNARGs*. It remains an intriguing open problem whether this lower bound can be extended to rule out 2 group element SNARGs, which would prove optimality of our 3 element construction.

1.5.4 Sonic

Sonic [Mal+19] is an early general purpose zk-SNARK protocol. The paper was published in January 2019. Sonic supports a universal and updatable SRS. Sonic proofs are constant size, but verification is expensive. In theory, multiple proofs can be verified in batches to achieve better performance. Many modern zk-SNARKs are based on Sonic. Sonic is the first fully succinct NIZK with SRS.¹

What follows are heavily quoted from [Mal+19] and [Nit20]:

- Major features:
 - Universal and continuously updatable SRS that scales linearly in size.
 - Arithmetization for circuits

Sonic represents the circuit as three vectors of left, right, and output wires. Then, the consistency of both the multiplication gates and the linear constraints are reduced to one large polynomial equation in a variable Y . The equation is then embedded into

the X^0 terms of a bivariate polynomial $t(X, Y)$. The verifier will check that the prover computed the bivariate polynomials from witness and circuit specific bivariate polynomials such that the X^0 terms cancel out. Using a PIOP, these bivariate polynomials can be simulated with univariate ones under some conditions.

– Polynomial commitment

As a pairing-based SNARKs and rely on trusted setup, Sonic using a variant of KZG [KZG10]. KZG is a pairing-based construction that allows for openings of evaluations on a point with a single group element-sized proof of correct opening and a constant time verifier for this check. More precisely, $\text{KZG.PC} = (\text{KZG.ComGen}, \text{KZG.Com}, \text{KZG.Open}, \text{KZG.Check})$ is defined over bilinear groups $gk = (p, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T)$ with $\mathbb{G}_1 = \langle g \rangle, \mathbb{G}_2 = \langle h \rangle$ as follows:

$\text{KZG.ComGen}(1^\lambda, n) \rightarrow (\text{ck}_{\text{kzg}}, \text{vk}_{\text{kzg}})$: Set keys $\text{ck}_{\text{kzg}} = \{g^{\alpha^i}\}_{i=0}^{n-1}, \text{vk}_{\text{kzg}} = h^\alpha$.
 $\text{KZG.Com}(\text{ck}_{\text{kzg}}; f(X)) \rightarrow C_f$: For $f(X) = \sum_{i=0}^{n-1} f_i X^i$, computes $C_f = \prod_{i=0}^{n-1} g^{f_i \alpha^i} = g^{f(\alpha)}$.
 $\text{KZG.Open}(\text{ck}_{\text{kzg}}; C_f, x, y; f(X)) \rightarrow \pi$: For an evaluation point x , a value y , compute the quotient polynomial $q(X) = \frac{f(X) - y}{X - x}$ and output a proof $\pi = C_q = \text{KZG.Com}(\text{ck}_{\text{kzg}}; q(X))$.
 $\text{KZG.Check}(\text{vk}_{\text{kzg}} = h^\alpha, C_f, x, y, \pi) \rightarrow 1/0$: Check if $e(C_f \cdot g^{-y}, h) = e(C_q, h^\alpha \cdot h^{-x})$.

– Batched verifications and potential of improved efficiency

Sonic presented a generally useful technique in which untrusted "helpers" can compute advice that allows batches of proofs to be verified more efficiently. Sonic proofs are constant size, and in the "helped" batch verification context the marginal cost of verification is comparable with the most efficient SNARKs in the literature.

• Design ideas:

– The best of both worlds:

At the time Sonic was invented, the most efficient scheme described in the literature is Groth16 which contains only three group elements. Typically, such

zkSNARKs require a trusted setup, a pairing-friendly elliptic curve, and rely on strong assumptions.

In contrast, proving systems such as Bulletproofs [Bün+18] do not require a trusted setup and depend on weaker assumptions. Unfortunately, although its proof sizes scale logarithmically with the relation size, Bulletproof verification time scales linearly, even when applying batching techniques. As a result, Bulletproofs are good for simple relations but with limited practicality in real-world applications.

The trusted setup has emerged as a barrier for deployment and wide adoption.

The research in [Gro+18], which proposed the first universal SRS that allows a single setup to support all circuits of some bounded size, which is also updatable, meaning that an open and dynamic set of participants can contribute secret randomness to it indefinitely. The drawback is that it requires an SRS that is quadratic with respect to the number of multiplication gates in the supported arithmetic circuits. In a concrete setting such as ZCash, which has a circuit with 217 multiplication gates, the SRS would be on the order of terabytes and is thus prohibitively expensive. Moreover, updating the SRS requires a quadratic number of group exponentiations, and verifying the updates requires a linear number of pairings. Expensive Gaussian elimination process is also involved in computing SRS.

– Core technique

* Two-variate polynomial equation

... Sonic defines its constraint system with respect to the two-variate polynomial equation used in Bulletproofs [Bün+18].

In the Bulletproof’s polynomial equation, there is one polynomial that is determined by the instance of the language and a second that is determined by the constraints.

The polynomial determined by the instance \mathbf{a} is

given by:

$$\sum_{i,j} a_{i,j} X^i Y^j$$

Each element of the instance is used to scale a monomial in the overall polynomial. For this reason, an SRS that contains only hidden monomial evaluations suffices for committing to the instance. Groth et al. [Gro+18] showed that an SRS that contains monomials is updatable; the second polynomial that is determined by the constraints is known to the verifier. Sonic uses this knowledge to allow the verifier to obtain evaluations of the polynomial while avoiding putting constraint-specific secrets in the SRS.

* Polynomial Commitment

Sonic use KZG [KZG10] for polynomial commitment. Sonic proved the commitment scheme secure in the Algebraic Group Model [FKL18], which is a model that lies somewhere between the standard model and the generic group model. KZG scheme has constant size and verification time, but is designed for single-variate polynomials, whereas Sonic use two-variate. To account for this, Sonic hid only one evaluation point in the reference string.

• Main contributions:

- Updatable universal SRS: Sonic requires a trusted setup, but unlike conventional SNARKs the structured reference string supports all circuits (up to a given size bound) and is also updatable, so that it can be continually strengthened. This addresses many of the practical challenges and risks surrounding such setups; Sonic’s SRS is linear in size with respect to the size of supported circuits. The SRS does not need to be specialized or pre-processed for a given circuit. This makes a large, distributed and never-ending setup process a practical reality.

- Batch proof verification: Sonic consists of a constant number of pairing checks. Unlike other zk-SNARKs, all proof elements are in the same source group, which has several advantages. Most significantly, when verifying many proofs at the same time, the pairing operations need to be computed only once. Sonic also removed the requirement for operations in the second source group, which are typically more expensive.
- Succinct Evaluation: Sonic’s verification includes checking the evaluation of a sparse bivariate polynomial in the scalar field. Sonic introduced a method to check this evaluation succinctly (given a circuit-dependent pre-computation) and thus maintain our zk-SNARK properties. Our proof of correct evaluation introduces a new permutation argument and a grand-product argument.
- Proof aggregation: Sonic can achieve better concrete efficiency if an untrusted “helper” party aggregates a batch of proofs. This batching operation computes advice to speed up the verifier. In a blockchain application, this helper could be a miner-type client that already processes and verifies transactions for inclusion in the next block.

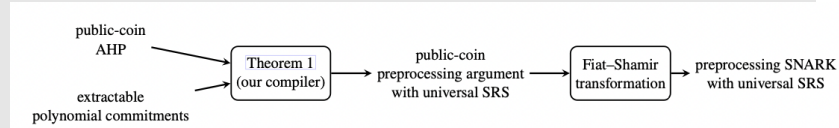
1.5.5 Marlin

Marlin is an improvement on Sonic with 10x better prover time and 4x better verification times.

The following are heavily quoted from [Chi+19].

- Major features
 - Fast verification: Marlin exploited a novel use of holography [Bab+91], where fast verification is achieved provided the statement being checked is given in an encoded form.

- SRS and argument size: We use our methodology to obtain a pre-processing zkSNARK where the SRS has linear size and arguments have constant size.
- Improvements on Sonic [Mal+19] in all efficiency parameters: proving is an order of magnitude faster and verification is thrice as fast, even with smaller SRS size and argument size.
- Algebraic Group Model: Our construction is most efficient when instantiated in the algebraic group model (also used by Sonic).
- Design ideas



- Main contributions
 - A new methodology: The methodology in fact produces succinct interactive arguments that can be made non-interactive via the Fiat–Shamir transformation. [FS87]. Marlin’s key observation is that the ability to pre-process a circuit in an offline phase is closely related to constructing “holographic proofs” [Bab+91], which means that the verifier does not receive the circuit description as an input but, rather, makes a small number of queries to an encoding of it. These queries are in addition to queries that the verifier makes to proofs sent by the prover. Moreover, Marlin focused on the setting where the encoding of the circuit description consists of low-degree polynomials and also where proofs are themselves low-degree polynomials — this can be viewed as a requirement that honest and malicious provers are “algebraic”, known as Algebraic Holographic Proofs (AHPs).
 - An efficient AHP for R1CS: Marlin designed an algebraic holographic proof (AHP) that achieves linear proof length and constant query complexity, among

other useful efficiency features. The protocol is for rank-1 constraint satisfiability (R1CS), a well-known generalization of arithmetic circuits where the “circuit description” is given by coefficient matrices (see definition below).

- Extractable polynomial commitments: Marlin proposed a definition for polynomial commitment schemes that incorporates the functionality and security that suffice for standalone use. Moreover, Marlin shows how to extend the construction of [KZG10] to fulfill this definition in the plain model under non-falsifiable knowledge assumptions, or via a more efficient construction in the algebraic group model under falsifiable assumptions.

- Darlin: Merlin’s evolution with recursive SNARK features:

...a succinct zero-knowledge argument of knowledge based on the Marlin SNARK [Chi+19] and the ‘dlog’ polynomial commitment scheme from [Boo+16], [Bün+18]. Darlin addresses recursive proofs by integrating the amortization technique from Halo [BGH20] for the non-succinct parts of the dlog verifier, and we adapt their strategy for bivariate circuit encoding polynomials to aggregate Marlin’s inner sumchecks across the nodes the recursive scheme...

1.5.6 Plonk

- Major features
 - Plonk is an improvement on Sonic with a $5\times$ better prover time. [GWC19]. Sonic still requires relatively high proof construction overheads. Plonk presented a universal SNARK construction with fully succinct verification, and significantly lower prover running time (roughly 7.5-20 times fewer group exponentiations than [Mal+19] in the fully succinct verifier mode depending on circuit structure).

- At a high level the improvements stem from a more direct arithmetization of a circuit. Plonk represents the vector of wire values as well as the different gates selectors as polynomials using interpolation.
- A polynomial division check ensures that the prover knows satisfying inputs and outputs for each gate, but does not ensure a correct wiring, e.g. that the output of one gate is the input to another. Vitalik⁶ described the solution as "copy constraints" encoded with permutation of polynomials. A permutation argument establishes the consistency of the assignment of wires to gates.

- Design ideas

A full formal description can be find in [GWC19] §8.4. The core idea is that Plonk relied on a permutation argument over univariate evaluations on a multiplicative subgroup based on [BG12]. Plonk focused on evaluation on a subgroup rather than over coefficients of bivariate polynomial as in [Mal+19].

A more plain rephrase would be: the prover first constructs a series of commitments to various polynomials that represent the constraints represented by circuits. After committing to these values, the prover constructs the opening proofs for these polynomials, which the verifier can verify using elliptic curve pairings.

From the prover's perspective, Plonk has five rounds in total, each of which computes a new Fiat-Shamir challenge⁷. The following are all of the public and private inputs for the PlonK protocol:

⁶[Blog post](#)

⁷A short [description](#)

Common preprocessed input:

$$\begin{aligned}
& n, (x \cdot [1]_1, \dots, x^{n+2} \cdot [1]_1), (q_{Mi}, q_{Li}, q_{Ri}, q_{Oi}, q_{Ci})_{i=1}^n, \sigma(X), \\
& q_M(X) = \sum_{i=1}^n q_{Mi} L_i(X), \\
& q_L(X) = \sum_{i=1}^n q_{Li} L_i(X), \\
& q_R(X) = \sum_{i=1}^n q_{Ri} L_i(X), \\
& q_O(X) = \sum_{i=1}^n q_{Oi} L_i(X), \\
& q_C(X) = \sum_{i=1}^n q_{Ci} L_i(X), \\
& S_{\sigma_1}(X) = \sum_{i=1}^n \sigma(i) L_i(X), \\
& S_{\sigma_2}(X) = \sum_{i=1}^n \sigma(n+i) L_i(X), \\
& S_{\sigma_3}(X) = \sum_{i=1}^n \sigma(2n+i) L_i(X)
\end{aligned}$$

Public input: $\ell, (w_i)_{i \in [\ell]}$

Prover algorithm:

Prover input: $(w_i)_{i \in [3n]}$

The common pre-processed input contains the values from the trusted setup ceremony (the $x \cdot [1]_1, \dots, x^{n+2} \cdot [1]_1$ values, where $x \cdot [1]_1 = g_1^x$ and g_1 is the generator of an elliptic curve group); these are shared by both the prover and verifier.

The big idea of PLONK is to prove that some polynomial $f(x)$ vanishes on some domain $H \subset \mathbb{F}$ (treating permutation argument as just another proof). To prove it, we reduce the problem to some other problem. Incrementally, it looks like this ⁸:

- Proving the previous statement is equivalent to proving that the polynomial is divisible by $Z_H(x)$, the polynomial that has all the elements of H as roots (also called vanishing polynomial). [RZ21] has more discussions.
- Which is equivalent to proving the following identity (for some quotient polynomial t):

$$f(x) = t(x) \cdot Z_H(x) \quad \forall x \in \mathbb{F}$$

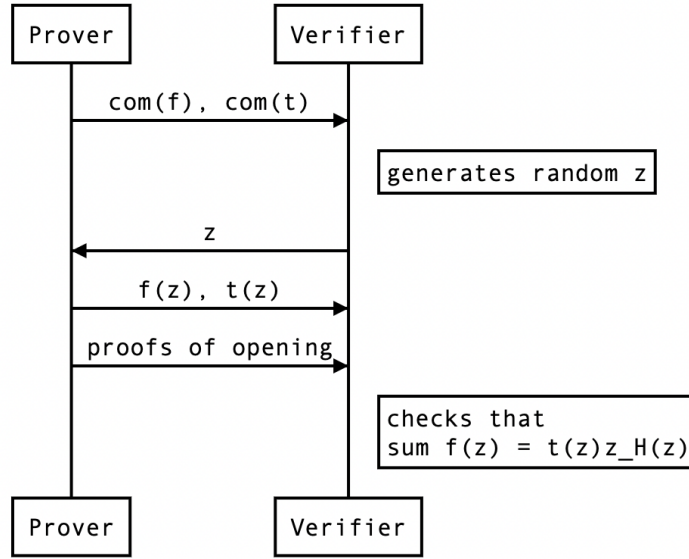
- Which is equivalent to proving the identity on some random point z (from the Schwartz-Zippel lemma):

$$f(z) = t(z) \cdot Z_H(z)$$

To prove the last statement, the prover uses of polynomial commitment scheme (KZG) to commit to the polynomial f and t .

⁸Blog post

The prover then sends the commitments to the verifier. At that point, the verifier has to check that for some random point z : $f(z) = t(z) \cdot Z_H(z)$. This is done by sending a random point z to the prover and doing an "opening" of the commitments at this point: the prover sends the values $f(z)$ and $t(z)$ as well as a proof that these are the correct evaluations.



- Main contributions

- Plonk’s universal SNARK requires the prover to compute 5 polynomial commitments, combined with two opening proofs to evaluate the polynomial commitments at a random challenge point. There are two flavors of Plonk: by increasing the proof size by two group elements, the total prover computations can be reduced by $\approx 10\%$. The combined degree of the polynomials is either $9(n+a)$ (larger proofs) or $11(n+a)$ (smaller proofs, reduced verifier work), where n is the number of multiplication gates and a is the number of addition gates.
- Kate commitment [KZG10] was improved in PlonK as a batched form, making parallel processing (opening) of commitments for each evaluation point possible. As quoted from [GWC19], "Ultimately, in the “grand product argument”, this reduces to check-

ing relations between coefficients of polynomials at “neighboring monomials”.

- Plonk defined a low-degree polynomial, which is a slightly simplified version from [KZG10] and [Mal+19], in order to achieve the desired properties for polynomial commitments:
 - * Plonk proved that the simplified low-degree polynomial commitment achieved knowledge soundness against algebraic adversaries under the AGM model [FKL18].
 - * Plonk described the pre-processing (trusted setup) of ranged polynomials. Converting a ranged protocol to a polynomial protocol only incurs one additional prover polynomial.

- Improvements

- SHPLONK: multiple commits
- TurboPLONK: Custom gates (e.g. EC point addition gates, greatly reducing pairing’s computation cost)
- Redshift
- Kimchi
- PLOOKUP: SNARK-unfriendly functions
- Zexe PLONK: PLONK single-layer rollup with BLS12-377 + BW6-761
- FFlonk
- Plonky/Plonky2
- UltraPlonk

1.5.7 Halo

- Major features Halo is the first zk-SNARK that supports recursive proof composition without a trusted setup, using the discrete log assumption over normal cycles of elliptic curves, such that proofs constructed with one curve can efficiently verify proofs constructed over the other. Recursion works using “nested amortization”: repeatedly collapsing multiple proofs together over cycles of elliptic curves. Unlike the other new constructs, Halo’s verification time is linear, making it the only new construct that isn’t succinct. However, proof size and verification time in our protocol does not increase with the depth of recursion.

- Design ideas The summary of the big idea of Halo is quoted from Vitalik's recent [blog post](#):
 - We don't know of a way to make a commitment to a size- n polynomial where evaluations of the polynomial can be verified in $< O(n)$ time directly. The best result so far is to make a $\log(n)$ sized proof, where all of the work to verify it is logarithmic except for one final $O(n)$ -time piece.
 - But what we can do is merge multiple proofs together. Given m proofs of evaluations of size- n polynomials, you can make a proof that covers all of these evaluations, that takes logarithmic work plus a single size- n polynomial proof to verify.
 - With some clever trickery, separating out the logarithmic parts from the linear parts of proof verification, we can leverage this to make recursive SNARKs.
 - These recursive SNARKs are actually more efficient than doing recursive SNARKs "directly". In fact, even in contexts where direct recursive SNARKs are possible (eg. proofs with KZG commitments), Halo-style techniques are typically used instead because they are more efficient.
 - It's not just about polynomials; other games used in SNARKs like R1CS can also be aggregated in similar clever ways.
 - No pairings or trusted setups required.
- Main contributions
 - Polynomial Commitments: with Amortized Succinctness Halo's main protocol is a variant of Sonic (Plonk for Halo2) [Mal+19] that is adapted to a new polynomial commitment scheme based on the inner product argument (IPA) of [Boo+16] and [Bün+18] inspired by Hyrax from [Wah+17] (which is itself a variant of the inner product argument first presented in [Boo+16], with adaptations from [Bün+18]). By exploiting the smooth structure of vectors that the verifier must work with, Halo can amortize away (across many proofs) the linear-time verification operation for commitment openings with the assistance of an untrusted third

party "helper". Thus the verifier perform the same linear-time operation, but this time only once for the entire batch of proofs. This is similar to the helped variant of the [Mal+19], except that Halo’s approach avoids the need for a trusted setup. Halo achieved smaller proof size than Hyrax from [Wah+17] and construct U using verifier challenge inspired by [Bün+18].

- Nested Amortization: Halo had a novel approach for reducing the verification circuit size by exploiting the amortization strategy (with "helper"). In short, the verification circuit never performs linear-time operations itself, but rather takes the input and (claimed) output of the linear-time operation to be public inputs to the circuit, i.e. they are encoded in the statement being proven. This effectively defers the full verification of the “inner” proof to the verifier, who must also perform a similar linear-time operation to check the “outer” proof. The verifiers can collapse computations on their side together into one with the assistance of a helper. In the recursive context Halo can simply embed the verifier of this amortization argument at each depth of the recursion so as to continually collapse the cost of verifying arguments. The linear-time verification operation is thus only performed once at the end of the recursive chain, and never as part of the verification circuit itself.
- 2-cycle Curves: Halo performed a search for the 2-cycle curves that had highly 2-adic scalar fields. Both fields are expected to have large 2k primitive roots of unity for applying radix-2 FFTs to accelerate polynomial multiplication. Both fields are expected to have elements of multiplicative order 3 so that we can apply curve endomorphisms to optimize circuits, and that $\gcd(p-1, 5) = \gcd(q-1, 5) = 1$ in order to allow instantiating the Rescue hash function with $\alpha = 5$. Halo also affectionately referred to their found curves as Tweedledum and Tweedledee. A recent upgrade Halo2 applies Pasta curves: Pallas and Vesta.

$$\begin{aligned} E_p/\mathbb{F}_p : y^2 &= x^3 + 5 \text{ of order } q \text{ is called Tweedledum;} \\ E_q/\mathbb{F}_q : y^2 &= x^3 + 5 \text{ of order } p \text{ is called Tweedledee;} \end{aligned}$$

where p and q are 255-bit primes:

$$\begin{aligned} p &= 2^{254} + 4707489545178046908921067385359695873; \\ q &= 2^{254} + 4707489544292117082687961190295928833. \end{aligned}$$

1.5.8 Comparative analysis

A brief performance benchmark from [GWC19] can shed some lights on the complexity and efficiency of the above algorithms (*this work* means Plonk):

	size $\leq d$ SRS	size = n CRS/SRS	prover work	proof length	succinct	universal
Groth'16	-	$3n + m$ \mathbb{G}_1	$3n + m - \ell$ \mathbb{G}_1 exp, n \mathbb{G}_2 exp	$2 \mathbb{G}_1, 1 \mathbb{G}_2$	✓	✗
Sonic (helped)	$12d \mathbb{G}_1, 12d \mathbb{G}_2$	$12n \mathbb{G}_1$	$18n \mathbb{G}_1$ exp	$4 \mathbb{G}_1, 2 \mathbb{F}$	✗	✓
Sonic (succinct)	$4d \mathbb{G}_1, 4d \mathbb{G}_2$	$36n \mathbb{G}_1$	$273n \mathbb{G}_1$ exp	$20 \mathbb{G}_1, 16 \mathbb{F}$	✓	✓
Auroralight	$2d \mathbb{G}_1, 2d \mathbb{G}_2$	$2n \mathbb{G}_1$	$8n \mathbb{G}_1$ exp	$6 \mathbb{G}_1, 4 \mathbb{F}$	✗	✓
This work (small)	$3d \mathbb{G}_1, 1 \mathbb{G}_2$	$3n + 3a \mathbb{G}_1, 1 \mathbb{G}_2$	$11n + 11a \mathbb{G}_1$ exp , $\approx 54(n+a)\log(n+a) \mathbb{F}$ mul	$7 \mathbb{G}_1, 6 \mathbb{F}$	✓	✓
This work (fast prover)	$d \mathbb{G}_1, 1 \mathbb{G}_2$	$n + a \mathbb{G}_1, 1 \mathbb{G}_2$	$9n + 9a \mathbb{G}_1$ exp , $\approx 54(n+a)\log(n+a) \mathbb{F}$ mul	$9 \mathbb{G}_1, 6 \mathbb{F}$	✓	✓

Figure 3: Prover comparison: m = number of wires, n = number of multiplication gates, a = number of addition gates

	verifier work	elem. from helper	extra verifier work in helper mode
Groth'16	$3P, \ell \mathbb{G}_1$ exp	-	-
Sonic (helped)	$10P$	$3 \mathbb{G}_1, 2 \mathbb{F}$	$4P$
Sonic (succinct)	$13P$	-	-
Auroralight	$5P, 6 \mathbb{G}_1$ exp	$8 \mathbb{G}_1, 10 \mathbb{F}$	$12P$
This work (small)	$2P, 16 \mathbb{G}_1$ exp	-	-
This work (fast prover)	$2P, 18 \mathbb{G}_1$ exp	-	-

Figure 4: Verifier comparison per proof, P = pairing, l = number of public inputs.

1.5.9 Outlook

What follows are heavily quoted from [Nit20]:

1. Recursive ZK Proofs Recursive zero-knowledge SNARKs are one of the most interesting use cases for zero-knowledge technology that are almost ready for practical use. As the name suggests, recursive zk-SNARKs unlock the ability to take a proof and verify it inside another proof, allowing for a lot more composability without blowing up proving/verifying times. Besides the real-world composability advantages, recursive zero-knowledge SNARKs are also of great interest

because of their use case as a blockchain scaling solution, relying on the succinctness feature of SNARKs to compress secure verification of long blockchains. Mina Protocol is currently implementing such an approach.

It should, however, come as no surprise that efficient recursive ZK SNARK constructions are quite hard to pull off. With the typical method of SNARK constructions that uses elliptic curve fields, the initial, hairy problem to solve is to find a pairing-friendly curve. Attempting to solve for this yields many interesting trade-offs. Many of the options and possibilities have been explored in recent works such as Halo [BGH20], which uses elliptic curve cycles that do not require pairings and Fractal [COS19], that eliminates the use of elliptic curves altogether. Overall, the study of efficient recursive ZK proofs is of great importance, and ripe territory for further exploration.

2. **Post-Quantum Zero-Knowledge** Many of the techniques described in previous sections depend on the computational hardness of certain problems, such as computing the discrete logarithm, which have been shown solvable by quantum computers in polynomial time. Therefore, quantum computers may pose a significant threat to the security model and practicality of zkSNARKs that use these techniques. There has been some work on making quantum attack resistant zk-SNARKs. For instance, [Gen+18] recently proposed a lattice-based zk-SNARK starting with SSP — square span programs (an alternative to the QAP intermediate for boolean circuits). Lattice problems are known to hold against quantum attacks [Ajt96], so this is a very promising line of work.
3. **Fiat-Shamir-Compatible Hash Functions** Currently Fiat-Shamir heuristic’s security is only proven in the Random Oracle Model. It remains an open question whether there exist concrete hash functions that are compatible with the Fiat Shamir heuristic, i.e. if there exist hash functions that guarantee soundness for a transformed proof (and are also compatible with zero-knowledge proofs). There has been a lot of work on this topic, but there is still no known universal hash function that can be proven to be compatible without making strong, somewhat impractical, assumptions yet.

1.6 Application development

A brief overview of recent applications of zkSNARKs are illustrated [here](#):

Further work is required on elaborating the digram.

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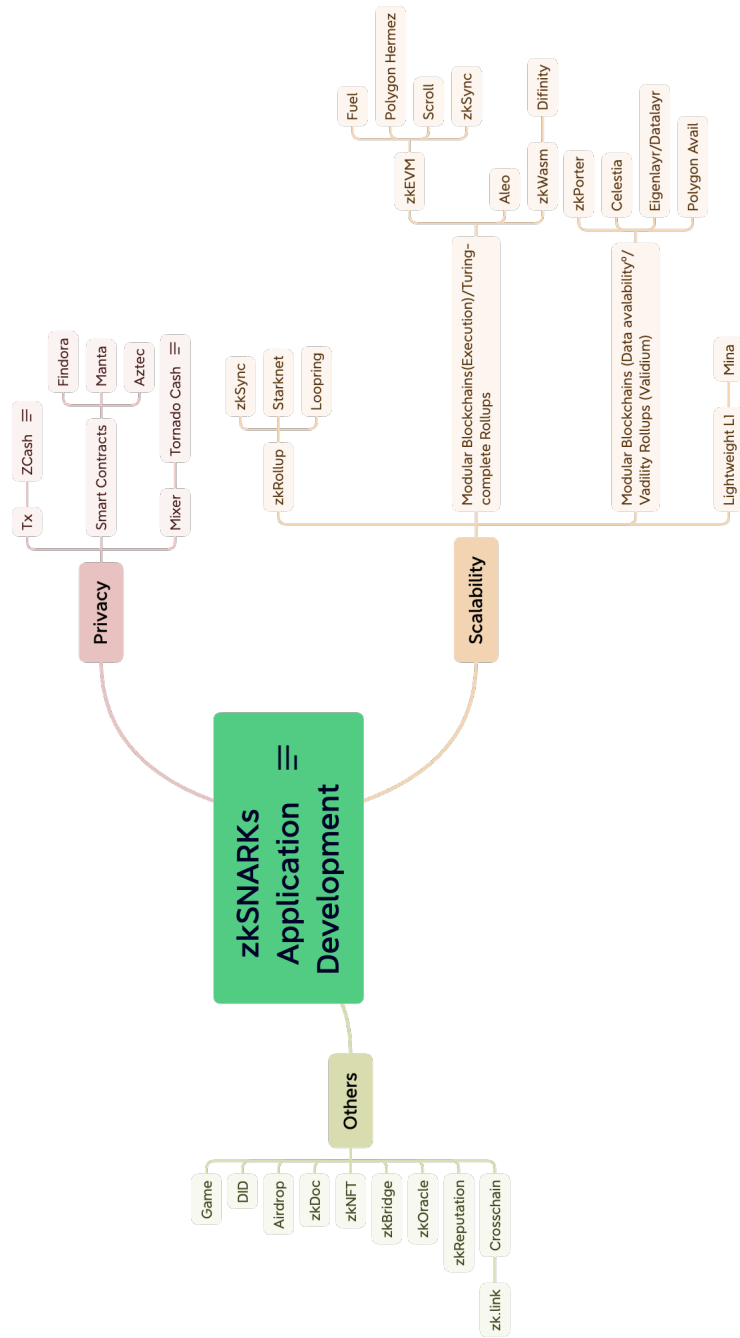


Figure 5: Recent development in zkSNARK applications