Chapter 2. Distance Measures for Shapes and Patterns

Lecturer: Prof. Dr. H. Alt
Notes prepared by: Heinrich Kruger (3085066)

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When defining distance measures for shapes and patterns, the idea is to obtain a function $d(A, B) \in \mathbb{R}$ that takes two shapes or patterns, and gives a measure of how similar the patterns are to each other.

1 Points

Some of the simplest distance measures, are those that measure the distance between points in $\mathbb{R}^2$. Let $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$. The following are some examples of distance measures between points:

a. The Euclidean distance $d_2(p_1, p_2) = \sqrt{(x_1 - x_2)2 + (y_1 - y_2)2}$. This is simply the length of a straight line segment between the two points.

b. $d_1(p_1, p_2) = |x_1 - x_2| + |y_1 - y_2|$. This is sometimes referred to as the "Manhattan distance".

c. $d_\infty(p_1, p_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$. This is sometimes called the "maximum distance".

   It is easy to see that the following inequalities hold:

$$d_\infty \le d_1 \le 2d_\infty$$

   and

$$d_2 \le d_1 \le \sqrt{2}d_2$$

d. The distance measures above, are all special cases of one general distance measure d_p for points. For $1 \le p < \infty$ $d_p(p_1, p_2) = \sqrt[p]{|x1 - x2|^p + |y_1 - y_2|^p}$. It can be shown that as $p \to \infty$, $d_p \to d_\infty$. This is also called the L_p distance.

Any L_p distance $d = d_p$ satisfies the following properties:

a. $d(p, q) = d(q, p)$. i.e. d is symmetric.

b. $d(p, q) = 0 \Leftrightarrow p = q$

c. $d(p, q) \le d(p, r) + d(r, q)$ $\forall p, q, r$. This is known as the triangle inequality, since it states that the length of one side of a triangle is always less or equal to the sum of the lengths of the other two sides.

A distance function d which satisfies the above properties, is called a *metric*. The underlying space (in this case $\mathbb{R}^2$) and the metric d together, are called a metric space. For example $(\mathbb{R}^2, d_p)$ is a *metric space* $\forall 1 \le p < \infty$

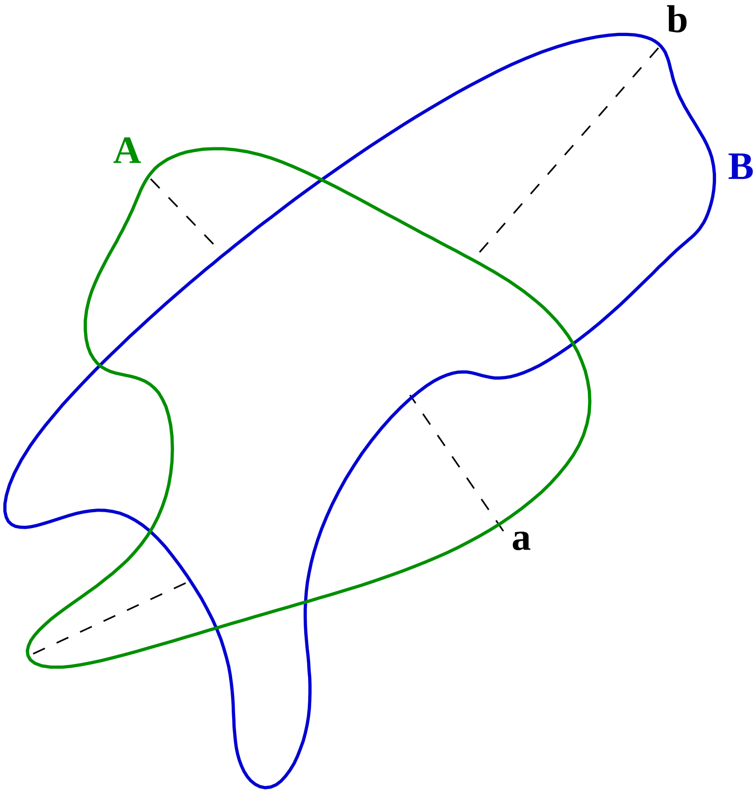

Figure 1: The distance between sets

2 The Hausdorff distance

2.1 Definition and properties of d_H

We need to define distance functions for sets. Using the minimum distance between the sets would clearly not work, since the distance would be 0 if the sets intersect. Thus it would make sense to use the maximum distance between the two sets. However, it would not do to simply take the greatest distance between two points (one from each set), since the distance would then be greater than 0 even if the sets are identical. Thus, given sets A and B, we define the distance between them as

$$\overrightarrow{d_H}(A,B) = \max_{a \in A} \min_{b \in B} d_2(a,b)$$

This is called the directed Hausdorff distance from A to B.

$\overrightarrow{d_H}$ is not symmetric, for example, in Figure 1 $\overrightarrow{d_H}(A,B)$ occurs at point a in A, while $\overrightarrow{d_H}(B,A)$ (at b) is clearly larger. Also $\overrightarrow{d_H}(A,B) = 0 \Leftrightarrow A \subseteq B$. Thus the directed Hausdorff distance is not a metric.

We can however define a new distance measure $d_H(A,B) = \max\{\overrightarrow{d_H}(A,B), \overrightarrow{d_H}(B,A)\}$, called the Hausdorff distance between A and B. If it is defined, the Hausdorff distance is a metric. However, the Hausdorff distance is not always defined. For example, Figure 2 shows two (unbounded) lines, clearly the Hausdorff distance is not defined, since there is no maximum distance between the lines.

Facts about d_H

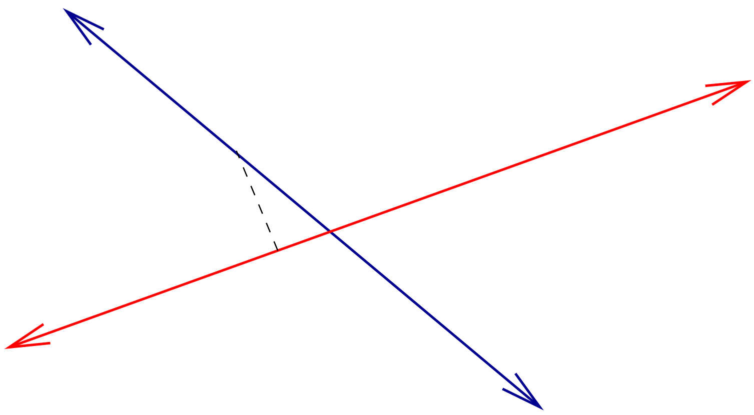

Figure 2: The Hausdorff distance is not defined for all sets

1. $d_H(A, B)$ is defined if A and B are both *compact* sets.

   A set is compact if it is bounded (there exists a circle containing the set completely) and closed (the set contains its boundary).

2. The set of all compact subsets of $\mathbb{R}^2$ with d_H form a metric space.

2.2 Computing d_H for finite sets (Point patterns)

Let A and B be finite sets in $\mathbb{R}^2$ such that $|A| = n$ and $|B| = m$. Naively, $\overrightarrow{d_H}(A, B)$ can be computed by scanning all points in B for every point in A to find the distance from each point in A, to the closest point to it in B, and then taking the maximum of all these values. Clearly this method takes $O(nm)$ time.

The following method is much more efficient:

1. Compute the Voronoi diagram of B and process it to obtain a point location data structure for planar subdivisions.

2. Locate each point in A in this structure, finding in this way the distance to its closest point in B.

3. Take maximum of distances found in 2.

The run time of this method is as follows: step 1 requires $O(m \log m)$ time, step 2 requires $O(n \log m)$ time, and step 3 requires $O(n)$ time. Thus the overall time complexity is $O((n + m) \log m)$ or $O(N \log N)$ where $N = n + m$.

To compute $d_H(A, B)$, use the method outlined above to compute $\overrightarrow{d_H}(A, B)$ and $\overrightarrow{d_H}(B, A)$ and take the maximum of these. This has overall time complexity $O((n + m) \log nm)$.

3 Computing Hausdorff distance for finite sets of line segments

Let A and B be the sets of points lying on the line segments. To calculate $\overrightarrow{d_H}(A, B)$ we use a similar approach to that described for finite sets above. However, since A is not a finite set, we cannot just compute the distance to the nearest point in B for each point in A. Thus we need to determine where in A, $\overrightarrow{d_H}(A, B)$ can occur.

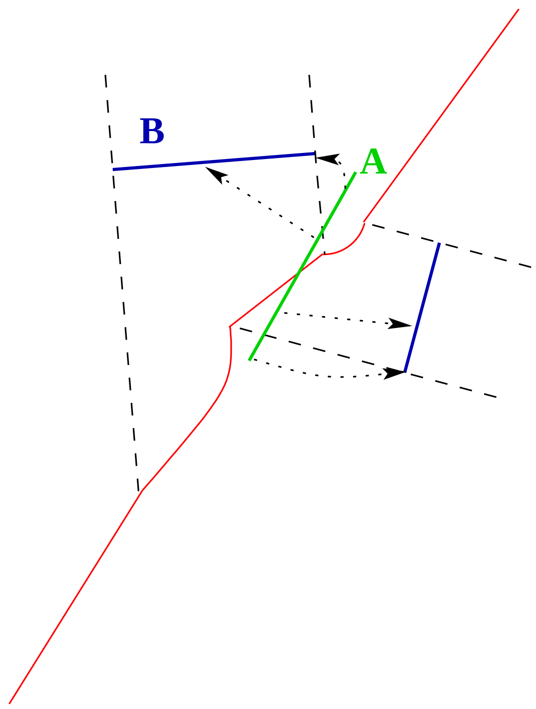

Figure 3: Using Voronoi diagram to compute directed Hausdorff distance for sets of line segments

Consider Figure 3. The dotted arrows in the figure indicate which part of B is closest to each part of the line segment from A. Now for each of the line segments formed by splitting the segment from A at its intersection points with $VD(B)$, the distance between the line segment and a point is maximum at one of the endpoints of the line segment. As one moves away from the endpoint, the distance to the point decreases and then (possibly) increases again.