Metric spaces

Foreword. This document contains notes from lecture by prof. P. Simon on metric spaces. All mistakes are on responsibility of authors.

Lecture 1

Metric space. Let M be a set and ρ a function $M^2 \to \mathbb{R}$. We call (M, ρ) metric space if ρ satisfies following conditions:

1) $\rho(x, y) = 0 \Leftrightarrow x = y$
2) $\rho(x, y) \geq 0$
3) $\rho(x, y) = \rho(y, x)$ *symmetry*
4) $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$ *triangle inequality*

Such a function is called *metric* (or a distance function). However, it shall be noted that conditions 1 and 4 imply the other two:

$$\rho(x, x) \leq \rho(x, z) + \rho(x, z) \Rightarrow 0 \leq 2\rho(x, z)$$

$$\rho(x, y) \leq \rho(x, x) + \rho(y, x) \Rightarrow \rho(x, y) \leq \rho(y, x);\ \rho(y, x) \leq \rho(x, y) \text{ analogically.}$$

We can make condition 4 even stronger: $\rho(x, y) \leq \max\{\rho(x, z), \rho(x, y)\}$, then ρ is called *ultrametric*.

Let us now introduce an example of a metric space. An *n-dimensional euclidean space* is $(\mathbb{R}^n, \rho)$, where ρ is defined as $\sqrt{\sum_{i=0}^{n}(x_i - y_i)^2}$. To be sure that it is really a metric space, we should check all conditions 1-4. The only nontrivial is 4, though, so we present only its proof: for $i = 1..n$ let a_i, b_i be arbitrary real numbers. Then, for $1 \leq i < j \leq n$ we have $0 \leq (a_ib_j - a_jb_i)^2 = a_i^2b_j^2 - 2a_ib_ja_jb_i + a_j^2b_i^2$, therefore $2a_ib_ja_jb_i \leq a_i^2b_j^2 + a_j^2b_i^2$. If we sum these inequalities for all i, j; we get:

$$2 \cdot \sum_{1 \leq i < j \leq n} a_ib_ja_jb_i \leq \sum_{1 \leq i < j \leq n} a_i^2b_j^2 + a_j^2b_i^2$$

$$\sum_{i=1..n, j=1..n, i \neq j} a_ib_ja_jb_i \leq \sum_{i=1..n, j=1..n, i \neq j} a_i^2b_j^2$$

$$\sum a_ib_ja_jb_i + \sum_{i=1..n} a_i^2b_i^2 \leq \sum a_i^2b_j^2 + \sum_{i=1..n} a_i^2b_i^2$$

$$\left(\sum_{i=1..n} a_ib_i\right)^2 \leq \sum_{i=1..n} a_i^2 \sum_{j=1..n} b_j^2.$$

This inequality is usually called *Cauchy-Schwarz*'s. After taking the square root of the inequality and multiplying it by two, we obtain $2\sum_{i=1..n} a_ib_i \leq 2|\sum_{i=1..n} a_ib_i| \leq 2\sqrt{\sum_{i=1..n} a_i^2}\sqrt{\sum_{i=1..n} b_i^2}$. But that can be rewritten as

$$\sum_{i=1..n} (a_i + b_i)^2 \leq \left(\sqrt{\sum_{i=1..n} a_i^2} + \sqrt{\sum_{i=1..s} b_i^2}\right)^2,$$

and taking the square root again and setting $a_i = x_i - z_i,\ b_i = z_i - y_i$ gives us desired triangle inequality.

Open and closed sets. In a metric space (M, ρ) we use call $\inf\{\rho(x, y) | y \in Y \subseteq M\}$ the *distance of x to the set Y* and write it as $\rho(x, Y)$. A subset X of M is a *closed set*, if $\{x | x \in M \wedge \rho(x, X) = 0\} \subseteq X$.

A subset X of M is an *open set*, if its complement $M\backslash X$ is a closed set. A *closure* of the set X is a set $\overline{X} = \{x \in M|\rho(x, X) = 0$. Clearly $X \subseteq \overline{X}$, and $X = \overline{X}$ iff X is closed.

Closed set can be defined in an equivalent way: $\bigcap\{Y \subseteq M|Y \supseteq X \wedge Y \text{ is closed}\}$. Let us prove that this definition is equivalent to the previous one (for defintion's sake, $\bigcap\emptyset = M$). First, note that $\overline{X}$ is closed: if $\rho(x, \overline{X}) = 0$, then there is $y \in \overline{X}$ so that $\rho(x, y) = 0$; and $z \in X$ so that $\rho(y, z) = 0$, triangle inequality gives us $\rho(x, X) = 0$ and by definiton of closure is x in $\overline{X}$. Therefore, $\bigcap\{Y\} \subseteq \overline{X}$. On the other hand, if x is in $\overline{X}$, then by the definiton of closure $\rho(x, X) = 0$ which means that $\rho(x, Y) = 0$ for all $Y \in \bigcap\{Y\}$ (because Y is superset of X, and distance can get only smaller by taking a superset). But all these Y are closed, so $x \in Y$ and $\overline{X} \subseteq \bigcap\{Y\}$.

As one can easily see, M and $\emptyset$ are both closed and open. But we can tell more about closed and open sets. If S is a family of closed sets, then $\bigcap S$ is still a closed set: let $\rho(x, \bigcap S) = 0$. But then $\rho(x, X) = 0$ for all $X \in S$ (the superset argument again), and $x \in X$ which means $x \in \bigcap S$. If R is a finite union of closed sets X_i, S is still a closed set: let $\rho(x, R) = 0$. But $\rho(x, R) = \inf\{\rho(x, X_i)\}$, and if infimum of a finite set is zero, then one of its elements must be zero, i.e. $\rho(x, X_i) = 0$ for some i. That also means that $x \in X_i$ and $x \in R$. We can not extend the union to be infinte (a counterexample is the union of all closed subintervals of $(0, 1)$ in $\mathbb{R}$). One easily gets dual characterization of open sets using DeMorgan identities (union of a family of open sets is still open, intersection of a finite set of open sets is still open).

We can also describe open sets in another way. We call the set $B(x, \varepsilon) = \{y \in M|\rho(x, y) < \varepsilon\}$ an open ball centered at x with radius ε or just a ball (a closed ball is the same with $\leq$). Alternatively, X is open if $(\forall x \in X)(\exists\varepsilon > 0)(B(x, \varepsilon) \subseteq X)$. Let us prove that the two definitions are equivalent. Let $\rho(x, (M\backslash X)) = 0$, then $\forall\varepsilon > 0$ is $B(x, \varepsilon) \cap (M\backslash X) \neq \emptyset$ and $x \notin X$. On the other hand, let $M\backslash X$ be closed set. Then $\forall x \in X$ is $\rho(x, M\backslash X) = \varepsilon > 0$ which means that $B(x, \varepsilon) \cap (M\backslash X) = \emptyset$ and $B(x, \varepsilon) \subseteq X$.

Topological equivalency. Two metrics ρ_1, ρ_2 on the same set are called *topologically equivalent*, if $X \subseteq M$ is open in (M, ρ_1) iff X is open in (M, ρ_2). One can also say that these metrics induce the same topology on M.

Now we present some topologically equivalent metrics on $\mathbb{R}^2$.

- $\rho_1(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
- $\rho_2(x, y) = \max\{(x_1 - y_1), (x_2 - y_2)\}$
- $\rho_3(x, y) = |x_1 - y_1| + |x_2 - y_2|$

The equivalency is immediate because open ball in any of these metrics contains (properly) open ball of the other two metrics (*image coming soon*).

Lecture 2

Topological equivalency of metrics ρ_1, ρ_2 has also an alternative form:

$$\begin{aligned}
&(\forall x)(\forall\varepsilon > 0)(\exists\delta_1, \delta_2 > 0)(\forall y) \\
&\rho_1(x, y) < \delta_1 \Rightarrow \rho_2(x, y) < \varepsilon \\
&\rho_2(x, y) < \delta_2 \Rightarrow \rho_1(x, y) < \varepsilon,
\end{aligned}$$

meaning that for any ball in one metric we can find a sub-ball in the other one. To prove it formally, let ρ_1, ρ_2 be topologically equivalent. Then $B_{\rho_1}(x, \varepsilon)$ is open in both metrics, so there is an ρ_2-open ball centered at x with diameter δ_2; proof for δ_1 is analogous. Now, let ρ_1, ρ_2 suffice the condition. If X is open in ρ_1, then we can find a ε-open ball centered at every $x \in X$, and due to sufficiency a δ_2 sub-ball centered at x in ρ_2 metric – therefore X is open in ρ_2.

However, we cannot formulate the condition uniformly, i.e. $(\forall\varepsilon > 0)(\exists\delta_1, \delta_2 > 0)(\forall x, y)(...)$. A counterexample is right at hand: for set $\{1/n | n \in \mathbb{N}\}$ and metric $|x - y|$ and discrete metric there cannot be such δ_1, δ_2 (the metrics are clearly equivalent, but $\forall\delta_1 > 0$ there is some point $1/n$ with non-empty δ_1 open ball, contradicting the emptiness of 1/2-ball in discrete metric).

Convergency. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in metric space. We say that x_n *converges* to x iff $(\forall\epsilon > 0)(\exists n_0)(\forall n > n_0)(\rho(x, x_n) < \epsilon)$. Notation is $x_n \to x$ or $\lim_{n\to\infty} x_n = x$, and the sequence is called *convergent*.

Convergency enables us to characterize closed sets in another way. F is closed iff $\forall$ convergent $(x_n) \subseteq F$ is $\lim x_n \in F$. To prove it, let us take a closed F, and its convergent subsequence x_n with $\lim x_n = x$. Then $\forall\varepsilon > 0$ there is x_n with $\rho(x, x_n) = \varepsilon$ which means that $\rho(x, F) = 0$ and x is in closure F. On the other hand, if $\rho(x, F) = 0$, $\forall n$ there is $x_n \in F$ with $\rho(x, x_n) < 1/n$. We can construct a subsequence of F of such x_n, and clearly $\lim x_n = x$. Therefore, $x \in F$.

Continuity. Mapping $f : (M, \rho) \to (N, \sigma)$ is called *continuos*, if $(\forall x)(\forall\varepsilon > 0)(\exists\delta > 0)(\forall y)(\rho(x, y) < \delta \Rightarrow \sigma(f(x), f(y)))$. Moreover, f is *uniformly continous* if $(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x, y)(\rho(x, y) < \delta \Rightarrow \sigma(f(x), f(y)))$, i.e. there is one uniform δ independent of x.

An example of non-uniformly continous map is x^2, one has to take δ smaller as x grows.

Now we prove some facts about continuos maps. For $f : (M, \rho) \to (N, \sigma)$ following conditions are equivalent:

- f is continous.
- Preimage of open $X \subseteq N$ is open.
- Preimage of closed $X \subseteq N$ is closed.
- For $(A \subseteq M)(f[\overline{A}] \subseteq \overline{f[A]})$.

Let X be open in N. Then, for $f(x)$ we have $B(f(x), \varepsilon > 0) \subseteq X$, and because of f beiing continuous $B(x, \delta > 0) \subseteq B^{-1}(f(x), \epsilon) \subseteq X^{-1}$, so X^{-1} is open.

If X is closed in N, then $N\backslash X$ is open and $(N\backslash X)^{-1} = M\backslash X^{-1}$ is open too, therefore X^{-1} is closed.

As $A \subseteq \overline{f[A]}^{-1}$ and preimage of closed $\overline{f[A]}^{-1}$ must be closed, we have that $\overline{A} \subseteq \overline{f[A]}^{-1}$.

Let $\varepsilon > 0$ and $A = N\backslash B(f(x), \varepsilon)$. Then $A = f[A^{-1}]$ is closed, meaning that $f[\overline{A^{-1}}] \subseteq f[A^{-1}]$ and $f[M\backslash\overline{A^{-1}}] \subseteq N\backslash A = B(f(x), \varepsilon)$. But because $\overline{A^{-1}}$ is closed, $M\backslash\overline{A^{-1}}$ is open and contains an open ball centered at x, thus completing the proof.

Morphisms. A bijective mapping f from (M, ρ) to (N, σ) is called *isometric mapping* and the spaces are *isometric*, if $\delta(x, y) = \sigma(f(x), f(y))$. If both f and f^{-1} are continuous, then f is called *homeomorphism* and the spaces are *homeomorphic*. Clearly, isometric mapping is stronger (take $\mathbb{R}$ and $(-1, 1)$, these are clearly homeomorphic by $f(x) = x/(|x| + 1)$ but cannot be isometric because distance in $\mathbb{R}$ is unbounded).

Let f be a homeomorphism between (M, ρ) and (N, σ), and set $\sigma_1(f(x), f(y)) = \rho(x, y)$. Then, we instantly get that σ_1 is metrics on N and f is isometric mapping between (M, ρ) and (N, σ_1). However, it also holds that σ and σ_1 are topologically equivalent – identity on N is continuous with respect to σ and σ_1, therefore open set X in one metric has open image—which is X itself—in the other.

Subspace. (N, σ) is *subspace* of (M, ρ) if $N \subseteq M$ and $\sigma = \rho \uparrow N^2$.

Sum. For any nonzero ordinal β and spaces $(X_\alpha, \rho_\alpha),\ \rho_\alpha(x, y) \leq 1,\ \alpha \in \beta$ we call the space $\sum(X_\alpha, \rho_\alpha) = (\langle x, \alpha\rangle, \delta)$ where $\delta(\langle x, \alpha\rangle, \langle y, \alpha) = \rho_\alpha(x, y)$ and 1 otherwise the *sum* of spaces. In the sum space, set is open iff all the corresponding α sets are open.

Product. For a nonzero ordinal $\beta \leq \omega_0$ and spaces $(X_i, \rho_i),\ \rho_i(x, y) \leq 1,\ i \in \beta$ we call the space $\prod(X_i, \rho_i) = (\langle x_i\rangle, \chi)$ a *product* space, with $\chi(\langle x_i\rangle, \langle y_i\rangle) = \sum_{i\in\beta} \rho_i(x_i, y_i)/2^i$. Convergency in the product space is equal to convergency in every i. If $\langle x_i\rangle_n \subseteq \prod X$ converges to $\langle y_i\rangle$, then $\rho_i((x_i)_n, y_i) < 2^i\chi(\langle x_i\rangle_n, y_i) < 2^i\varepsilon$ from certain n. If all $(x_i)_n$ are convergent, then we take such n_0 so that $\sum_{j=n_0} 2^{-j} < \varepsilon/2$. For all $i < n_0$, there is n_i such that $\rho_i((x_i)_{n_i}, y_i) < \varepsilon/(2n_0)$. We pick maximum of these and n_0, and we get $\chi(\langle x_i\rangle_n, \langle y_i\rangle) < n_0\varepsilon/(2n_0)+\varepsilon/2$.

For N subspace of M, $X \subseteq N$ is open in N iff $\exists$ open $Y \subseteq M$ such that $X = Y \cap N$. As X is open N, we can take a ball $B(x, \varepsilon_x)$ for every $x \in X$. Union of these balls in M is still an open set, and $(\bigcup B(x, \varepsilon_x))\cap N = X$. On the other hand, for $x \in X$ there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq Y$, and $B(x, \varepsilon) \cap N \subseteq Y \cap N \subseteq X$.

Lecture 3

Totally bounded spaces. Let (M, ρ) be a metric space and $\varepsilon > 0$, then we call $N \subseteq M$ an ε-net if $\forall x, y \in M,\ x \neq y \Rightarrow \rho(x, y) \geq \varepsilon$. (M, ρ) is *totally bounded*, if every ε-net is finite. For example, $(0, 1)$ is totally bounded for $\rho = |x-y|$, but $\mathbb{R}$ in such metrics is not (therefore, totally bounded space is a topological term).

If (M, ρ) is totally bounded, then any $N \subseteq M$ is totally bounded too (clearly any ε-net in N is still a net in M). Also, if $(N, \rho_N),\ N \subseteq M$ is totally bounded, then $(\overline{N}, \rho_{\overline{N}})$ is totally bounded. To see this, we take a ε-net X in $\overline{N}$. For every x in $\overline{N}\backslash N$ there is x' in N such that $\rho(x, x') < \varepsilon/3$ and call this substitued set X'. Such x' is unique for every x, since X was an ε-net. Even more, $\rho(x', y) \leq \varepsilon/3$ for any $y \in X'$ due to triangle inequality. Therefore, we can assign to any *varepsilon*-net in $\overline{N}$ a ε-net in N of the same cardinality.

Let $(M_i, \rho_i)_{i\in\mathbb{N}}$ be a family of non-empty metric spaces, such that $\rho_i \leq 1$ for every i. Then $\prod_{i\in\mathbb{N}}(M_i, \rho_i) = (M, \rho)$ is totally bounded iff $\forall i \in \mathbb{N},\ (M_i, \rho_i)$ is totally bounded.

First, we prove the $\Leftarrow$. Let (M_i, ρ_i) totally bounded and X any ε-net in (M, ρ). We pick $i_0 \in \mathbb{N}$ such that $2^{-i_0} < \varepsilon/4$. Now, if $\overline{x} \neq \overline{y}$, there is j such that $\rho_j(x_j, y_j) \geq \varepsilon/(2i_0)$ (if not, then $\sum_0^{i_0} \rho_j(x_j, y_j)/2^j < i_0 \cdot \varepsilon/(2i_0) = \varepsilon/2$ and $\sum_{i_0+1} \rho_j(x_j, y_j)/2^j \leq \sum 1/2^j \leq \varepsilon/4$, therefore $\rho(\overline{x}, \overline{y}) < \varepsilon$). For all $i \in i_0$ we pick $\varepsilon/(8i_0)$-net X_i in M_i, that is inclusively maximall. $\forall \overline{x} \in X$ there is $f(\overline{x}) \in \prod X_i$, such that $\forall j\ \rho(x_j, f(x)_j) \leq \varepsilon/(8i_0)$. If $\overline{x} \neq \overline{y}$, then $f(\overline{x}) \neq f(\overline{y})$, because there is $j,\ \rho(x_j, y_j) \geq \varepsilon/(2i_0)$ and therefore $\rho(f(x)_j, f(y)_j) \geq \varepsilon/(4i_0)$, which means f is injective mapping into finite set, proving that X is finite.

To prove $\Rightarrow$, we just assume that there is i_0 such that (M_{i_0}, ρ_{i_0}) is not totally bounded. Then, there is ε-net in M_{i_0} that is infinite. Such net can be extended to M by picking arbitrary point in any M_j (because every M_j is nonempty), and that is an infinite $\varepsilon/2^{i_0}$-net in M.

♡

(M, ρ) is totally bounded iff $\forall\varepsilon > 0$ there is a finite cover of M with open ε-balls. If there is a cover with open balls, then any net cannot contain more than one point from any ball and cannot have greater cardinality than the cover. On the other hand, if we put an ε-ball around every point of some inclusivelly maximall ε-net, we get cover of the same cardinality.

For totally bounded (M, ρ) and $f : (M, \rho) \to (\mathbb{R}, |x - y|)$ uniformly continous, $f[M]$ is bounded subset of $\mathbb{R}$. If it were unbounded, we could assume that $\forall n \in \mathbb{N}\ \exists x_n \in X\ f(x_n) > n$. But then, $\forall n, m \in$

$\mathbb{N}$ $|f(x_n) - f(x_m)| \geq 1$. Since f is uniformly convergent, $\rho(x, y) < \delta \Rightarrow |f(x) - f(y)| < 1/2$; and for finite cover of M with $\delta/2$ balls we get that one ball cannot contain more than one x_n.

♡

Complete metric spaces. We call a sequence $\{x_n | x_n \in M, n \in \mathbb{N}\}$ a *Cauchy* sequence if $\forall \varepsilon > 0\ \exists n_0\ \forall n, m \geq n_0\ \rho(x_n, x_m) < \varepsilon$. Metric space is *complete*, if every Cauchy sequence is convergent.

In any metric space, a convergent sequence is automatically a Cauchy sequence. We can take n_0 such that $\rho(x, x_n) < \varepsilon/2$, and $\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x_m, x) < \varepsilon$.