Metric spaces

Foreword. This document contains notes from lecture by prof. P. Simon on metric spaces. All mistakes are on responsibility of authors.

Lecture 1

Metric space. Let M be a set and ρ a function $M^2 \to \mathbb{R}$. We call (M, ρ) metric space if ρ satisfies following conditions:

1) $\rho(x, y) = 0 \Leftrightarrow x = y$
2) $\rho(x, y) \geq 0$
3) $\rho(x, y) = \rho(y, x)$ *symmetry*
4) $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$ *triangle inequality*

Such a function is called *metric* (or a distance function). However, it shall be noted that conditions 1 and 4 imply the other two:

$$\rho(x, x) \leq \rho(x, z) + \rho(x, z) \Rightarrow 0 \leq 2\rho(x, z)$$

$$\rho(x, y) \leq \rho(x, x) + \rho(y, x) \Rightarrow \rho(x, y) \leq \rho(y, x);\ \rho(y, x) \leq \rho(x, y) \text{ analogically.}$$

We can make condition 4 even stronger: $\rho(x, y) \leq \max\{\rho(x, z), \rho(x, y)\}$, then ρ is called *ultrametric.*

Let us now introduce an example of a metric space. An *n-dimensional Euclidean space* is $(\mathbb{R}^n, \rho)$, where ρ is defined as $\sqrt{\sum_{i=0}^{n}(x_i - y_i)^2}$. To be sure that it is really a metric space, we should check all conditions 1-4. The only nontrivial is 4, though, so we present only its proof: for $i = 1..n$ let a_i, b_i be arbitrary real numbers. Then, for $1 \leq i < j \leq n$ we have $0 \leq (a_i b_j - a_j b_i)^2 = a_i^2 b_j^2 - 2a_i b_j a_j b_i + a_j^2 b_i^2$, therefore $2a_i b_j a_j b_i \leq a_i^2 b_j^2 + a_j^2 b_i^2$. If we sum these inequalities for all i, j; we get:

$$2 \cdot \sum_{1 \leq i < j \leq n} a_i b_j a_j b_i \leq \sum_{1 \leq i < j \leq n} a_i^2 b_j^2 + a_j^2 b_i^2$$

$$\sum_{i=1..n, j=1..n, i \neq j} a_i b_j a_j b_i \leq \sum_{i=1..n, j=1..n, i \neq j} a_i^2 b_j^2$$

$$\sum a_i b_j a_j b_i + \sum_{i=1..n} a_i^2 b_i^2 \leq \sum a_i^2 b_j^2 + \sum_{i=1..n} a_i^2 b_i^2$$

$$\left(\sum_{i=1..n} a_i b_i\right)^2 \leq \sum_{i=1..n} a_i^2 \sum_{j=1..n} b_j^2.$$

This inequality is usually called *Cauchy-Schwarz*'s. After taking the square root of the inequality and multiplying it by two, we obtain $2\sum_{i=1..n} a_i b_i \leq 2|\sum_{i=1..n} a_i b_i| \leq 2\sqrt{\sum_{i=1..n} a_i^2}\sqrt{\sum_{i=1..n} b_i^2}$. But that can be rewritten as

$$\sum_{i=1..n} (a_i + b_i)^2 \leq \left(\sqrt{\sum_{i=1..n} a_i^2} + \sqrt{\sum_{i=1..s} b_i^2}\right)^2,$$

and taking the square root again and setting $a_i = x_i - z_i,\ b_i = z_i - y_i$ gives us desired triangle inequality.

Open and closed sets. In a metric space (M, ρ) we use call $\inf\{\rho(x, y) | y \in Y \subseteq M\}$ the *distance of x to the set Y* and write it as $\rho(x, Y)$. A subset X of M is a *closed set*, if $\{x | x \in M \wedge \rho(x, X) = 0\} \subseteq X$.

A subset X of M is an *open set*, if its complement $M \setminus X$ is a closed set. A *closure* of the set X is a set $\overline{X} = \{x \in M | \rho(x, X) = 0$. Clearly $X \subseteq \overline{X}$, and $X = \overline{X}$ iff X is closed.

Closed set can be defined in an equivalent way: $\bigcap\{Y \subseteq M | Y \supseteq X \wedge Y \text{ is closed}\}$. Let us prove that this definition is equivalent to the previous one (for definition's sake, $\bigcap \emptyset = M$). First, note that $\overline{X}$ is closed: if $\rho(x, \overline{X}) = 0$, then there is $y \in \overline{X}$ so that $\rho(x, y) = 0$; and $z \in X$ so that $\rho(y, z) = 0$, triangle inequality gives us $\rho(x, X) = 0$ and by definition of closure is x in $\overline{X}$. Therefore, $\bigcap\{Y\} \subseteq \overline{X}$. On the other hand, if x is in $\overline{X}$, then by the definition of closure $\rho(x, X) = 0$ which means that $\rho(x, Y) = 0$ for all $Y \in \bigcap\{Y\}$ (because Y is superset of X, and distance can get only smaller by taking a superset). But all these Y are closed, so $x \in Y$ and $\overline{X} \subseteq \bigcap\{Y\}$.

As one can easily see, M and $\emptyset$ are both closed and open. But we can tell more about closed and open sets. If S is a family of closed sets, then $\bigcap S$ is still a closed set: let $\rho(x, \bigcap S) = 0$. But then $\rho(x, X) = 0$ for all $X \in S$ (the superset argument again), and $x \in X$ which means $x \in \bigcap S$. If R is a finite union of closed sets X_i, S is still a closed set: let $\rho(x, R) = 0$. But $\rho(x, R) = \inf\{\rho(x, X_i)\}$, and if infimum of a finite set is zero, then one of its elements must be zero, i.e. $\rho(x, X_i) = 0$ for some i. That also means that $x \in X_i$ and $x \in R$. We can not extend the union to be infinite (a counterexample is the union of all closed subintervals of $(0, 1)$ in $\mathbb{R}$). One easily gets dual characterization of open sets using DeMorgan identities (union of a family of open sets is still open, intersection of a finite set of open sets is still open).

We can also describe open sets in another way. We call the set $B(x, \varepsilon) = \{y \in M | \rho(x, y) < \varepsilon\}$ an open ball centered at x with radius ε or just a ball (a closed ball is the same with $\leq$). Alternatively, X is open if $(\forall x \in X)(\exists \varepsilon > 0)(B(x, \varepsilon) \subseteq X)$. Let us prove that the two definitions are equivalent. Let $\rho(x, (M \setminus X)) = 0$, then $\forall \varepsilon > 0$ is $B(x, \varepsilon) \cap (M \setminus X) \neq \emptyset$ and $x \notin X$. On the other hand, let $M \setminus X$ be closed set. Then $\forall x \in X$ is $\rho(x, M \setminus X) = \varepsilon > 0$ which means that $B(x, \varepsilon) \cap (M \setminus X) = \emptyset$ and $B(x, \varepsilon) \subseteq X$.

Topological equivalence. Two metrics ρ_1, ρ_2 on the same set are called *topologically equivalent*, if $X \subseteq M$ is open in (M, ρ_1) iff X is open in (M, ρ_2). One can also say that these metrics induce the same topology on M.

Now we present some topologically equivalent metrics on $\mathbb{R}^2$.

- $\rho_1(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
- $\rho_2(x, y) = \max\{(x_1 - y_1), (x_2 - y_2)\}$
- $\rho_3(x, y) = |x_1 - y_1| + |x_2 - y_2|$

The equivalence is immediate because open ball in any of these metrics contains (properly) open ball of the other two metrics (*image coming soon*).

Lecture 2

Topological equivalence of metrics ρ_1, ρ_2 has also an alternative form:

$$
\begin{aligned}
&(\forall x)(\forall \varepsilon > 0)(\exists \delta_1, \delta_2 > 0)(\forall y)\\
&\rho_1(x, y) < \delta_1 \Rightarrow \rho_2(x, y) < \varepsilon\\
&\rho_2(x, y) < \delta_2 \Rightarrow \rho_1(x, y) < \varepsilon,
\end{aligned}
$$

meaning that for any ball in one metric we can find a sub-ball in the other one. To prove it formally, let ρ_1, ρ_2 be topologically equivalent. Then $B_{\rho_1}(x, \varepsilon)$ is open in both metrics, so there is an ρ_2-open ball centered at x with diameter δ_2; proof for δ_1 is analogous. Now, let ρ_1, ρ_2 suffice the condition. If X is open in ρ_1, then we can find a ε-open ball centered at every $x \in X$, and due to sufficiency a δ_2 sub-ball centered at x in ρ_2 metric – therefore X is open in ρ_2.

However, we cannot formulate the condition uniformly, i.e. $(\forall\varepsilon > 0)(\exists\delta_1, \delta_2 > 0)(\forall x, y)(...)$. A counterexample is right at hand: for set $\{1/n | n \in \mathbb{N}\}$ and metric $|x - y|$ and discrete metric there cannot be such δ_1, δ_2 (the metrics are clearly equivalent, but $\forall\delta_1 > 0$ there is some point $1/n$ with non-empty δ_1 open ball, contradicting the emptiness of 1/2-ball in discrete metric).

Convergence. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in metric space. We say that x_n *converges* to x iff $(\forall\epsilon > 0)(\exists n_0)(\forall n > n_0)(\rho(x, x_n) < \epsilon)$. Notation is $x_n \to x$ or $\lim_{n\to\infty} x_n = x$, and the sequence is called *convergent.*

Convergence enables us to characterize closed sets in another way. F is closed iff $\forall$ convergent $(x_n) \subseteq F$ is $\lim x_n \in F$. To prove it, let us take a closed F, and its convergent subsequence x_n with $\lim x_n = x$. Then $\forall\varepsilon > 0$ there is x_n with $\rho(x, x_n) = \varepsilon$ which means that $\rho(x, F) = 0$ and x is in closure F. On the other hand, if $\rho(x, F) = 0$, $\forall n$ there is $x_n \in F$ with $\rho(x, x_n) < 1/n$. We can construct a subsequence of F of such x_n, and clearly $\lim x_n = x$. Therefore, $x \in F$.

Continuity. Mapping $f : (M, \rho) \to (N, \sigma)$ is called *continuous*, if $(\forall x)(\forall\varepsilon > 0)(\exists\delta > 0)(\forall y)(\rho(x, y) < \delta \Rightarrow \sigma(f(x), f(y)))$. Moreover, f is *uniformly continuous* if $(\forall\varepsilon > 0)(\exists\delta > 0)(\forall x, y)(\rho(x, y) < \delta \Rightarrow \sigma(f(x), f(y)))$, i.e. there is one uniform δ independent of x.

An example of non-uniformly continuous map is x^2, one has to take δ smaller as x grows.

Now we prove some facts about continuous maps. For $f : (M, \rho) \to (N, \sigma)$ following conditions are equivalent:

- f is continuous.
- Preimage of open $X \subseteq N$ is open.
- Preimage of closed $X \subseteq N$ is closed.
- For $(A \subseteq M)(f[\overline{A}] \subseteq \overline{f[A]})$.

Let X be open in N. Then, for $f(x)$ we have $B(f(x), \varepsilon > 0) \subseteq X$, and because of f being continuous $B(x, \delta > 0) \subseteq B^{-1}(f(x), \epsilon) \subseteq X^{-1}$, so X^{-1} is open.

If X is closed in N, then $N \setminus X$ is open and $(N \setminus X)^{-1} = M \setminus X^{-1}$ is open too, therefore X^{-1} is closed.

As $A \subseteq \overline{f[A]}^{-1}$ and preimage of closed $\overline{f[A]}^{-1}$ must be closed, we have that $\overline{A} \subseteq \overline{f[A]}^{-1}$.

Let $\varepsilon > 0$ and $A = N \setminus B(f(x), \varepsilon)$. Then $A = f[A^{-1}]$ is closed, meaning that $f[\overline{A^{-1}}] \subseteq f[A^{-1}]$ and $f[M \setminus \overline{A^{-1}}] \subseteq N \setminus A = B(f(x), \varepsilon)$. But because $\overline{A^{-1}}$ is closed, $M \setminus \overline{A^{-1}}$ is open and contains an open ball centered at x, thus completing the proof.

Morphisms. A bijective mapping f from (M, ρ) to (N, σ) is called *isometric mapping* and the spaces are *isometric*, if $\delta(x, y) = \sigma(f(x), f(y))$. If both f and f^{-1} are continuous, then f is called *homeomorphism* and the spaces are *homeomorphic*. Clearly, isometric mapping is stronger (take $\mathbb{R}$ and $(-1, 1)$, these are clearly homeomorphic by $f(x) = x/(|x| + 1)$ but cannot be isometric because distance in $\mathbb{R}$ is unbounded).

Let f be a homeomorphism between (M, ρ) and (N, σ), and set $\sigma_1(f(x), f(y)) = \rho(x, y)$. Then, we instantly get that σ_1 is metrics on N and f is isometric mapping between (M, ρ) and (N, σ_1). However, it also holds that σ and σ_1 are topologically equivalent – identity on N is continuous with respect to σ and σ_1, therefore open set X in one metric has open image—which is X itself—in the other.

Subspace. (N,σ) is *subspace* of (M,ρ) if $N \subseteq M$ and $\sigma = \rho \upharpoonright N^2$.

Sum. For any nonzero ordinal β and spaces $(X_\alpha, \rho_\alpha),\ \rho_\alpha(x,y) \leq 1,\ \alpha \in \beta$ we call the space $\sum(X_\alpha, \rho_\alpha) = (\langle x, \alpha\rangle, \delta)$ where $\delta(\langle x, \alpha\rangle, \langle y, \alpha) = \rho_\alpha(x,y)$ and 1 otherwise the *sum* of spaces. In the sum space, set is open iff all the corresponding α sets are open.

Product. For a nonzero ordinal $\beta \leq \omega_0$ and spaces $(X_i, \rho_i),\ \rho_i(x,y) \leq 1,\ i \in \beta$ we call the space $\prod(X_i, \rho_i) = (\langle x_i\rangle, \chi)$ a *product* space, with $\chi(\langle x_i\rangle, \langle y_i\rangle) = \sum_{i\in\beta} \rho_i(x_i, y_i)/2^i$. Convergence in the product space is equal to convergence in every i. If $\langle x_i\rangle_n \subseteq \prod X$ converges to $\langle y_i\rangle$, then $\rho_i((x_i)_n, y_i) < 2^i \chi(\langle x_i\rangle_n, y_i) < 2^i\varepsilon$ from certain n. If all $(x_i)_n$ are convergent, then we take such n_0 so that $\sum_{j=n_0} 2^{-j} < \varepsilon/2$. For all $i < n_0$, there is n_i such that $\rho_i((x_i)_{n_i}, y_i) < \varepsilon/(2n_0)$. We pick maximum of these and n_0, and we get $\chi(\langle x_i\rangle_n, \langle y_i\rangle) < n_0\varepsilon/(2n_0) + \varepsilon/2$.

For N subspace of M, $X \subseteq N$ is open in N iff $\exists$ open $Y \subseteq M$ such that $X = Y \cap N$. As X is open N, we can take a ball $B(x, \varepsilon_x)$ for every $x \in X$. Union of these balls in M is still an open set, and $(\bigcup B(x,\varepsilon_x)) \cap N = X$. On the other hand, for $x \in X$ there is $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq Y$, and $B(x,\varepsilon) \cap N \subseteq Y \cap N \subseteq X$.

Lecture 3

Totally bounded spaces. Let (M,ρ) be a metric space and $\varepsilon > 0$, then we call $N \subseteq M$ an ε-net if $\forall x, y \in M,\ x \neq y \Rightarrow \rho(x,y) \geq \varepsilon$. (M,ρ) is *totally bounded*, if every ε-net is finite. For example, $(0,1)$ is totally bounded for $\rho = |x-y|$, but $\mathbb{R}$ in such metrics is not (therefore, totally bounded space is a topological term).

If (M,ρ) is totally bounded, then any $N \subseteq M$ is totally bounded too (clearly any ε-net in N is still a net in M). Also, if $(N, \rho_N),\ N \subseteq M$ is totally bounded, then $(\overline{N}, \rho_{\overline{N}})$ is totally bounded. To see this, we take a ε-net X in $\overline{N}$. For every x in $\overline{N} \setminus N$ there is x' in N such that $\rho(x,x') < \varepsilon/3$ and call this substituted set X'. Such x' is unique for every x, since X was an ε-net. Even more, $\rho(x', y) \leq \varepsilon/3$ for any $y \in X'$ due to triangle inequality. Therefore, we can assign to any ε-net in $\overline{N}$ a ε-net in N of the same cardinality.

Let $(M_i, \rho_i)_{i\in\mathbb{N}}$ be a family of non-empty metric spaces, such that $\rho_i \leq 1$ for every i. Then $\prod_{i\in\mathbb{N}}(M_i,\rho_i) = (M,\rho)$ is totally bounded iff $\forall i \in \mathbb{N},\ (M_i,\rho_i)$ is totally bounded.

First, we prove the $\Leftarrow$. Let (M_i,ρ_i) totally bounded and X any ε-net in (M,ρ). We pick $i_0 \in \mathbb{N}$ such that $2^{-i_0} < \varepsilon/4$. Now, if $\overline{x} \neq \overline{y}$, there is j such that $\rho_j(x_j, y_j) \geq \varepsilon/(2i_0)$ (if not, then $\sum_0^{i_0} \rho_j(x_j,y_j)/2^j < i_0 \cdot \varepsilon/(2i_0) = \varepsilon/2$ and $\sum_{i_0+1} \rho_j(x_j,y_j)/2^j \leq \sum 1/2^j \leq \varepsilon/4$, therefore $\rho(\overline{x},\overline{y}) < \varepsilon$). For all $i \in i_0$ we pick $\varepsilon/(8i_0)$-net X_i in M_i, that is inclusively maximal. $\forall \overline{x} \in X$ there is $f(\overline{x}) \in \prod X_i$, such that $\forall j\ \rho(x_j, f(x)_j) \leq \varepsilon/(8i_0)$. If $\overline{x} \neq \overline{y}$, then $f(\overline{x}) \neq f(\overline{y})$, because there is $j,\ \rho(x_j,y_j) \geq \varepsilon/(2i_0)$ and therefore $\rho(f(x)_j, f(y)_j) \geq \varepsilon/(4i_0)$, which means f is injective mapping into finite set, proving that X is finite.

To prove $\Rightarrow$, we just assume that there is i_0 such that (M_{i_0}, ρ_{i_0}) is not totally bounded. Then, there is ε-net in M_{i_0} that is infinite. Such net can be extended to M by picking arbitrary point in any M_j (because every M_j is nonempty), and that is an infinite $\varepsilon/2^{i_0}$-net in M.

♡

(M,ρ) is totally bounded iff $\forall\varepsilon > 0$ there is a finite cover of M with open ε-balls. If there is a cover with open balls, then any net cannot contain more than one point from any ball and cannot have greater cardinality than the cover. On the other hand, if we put an ε-ball around every point of some inclusively maximal ε-net, we get cover of the same cardinality.

For totally bounded (M,ρ) and $f:\ (M,\rho) \to (\mathbb{R}, |x-y|)$ uniformly continuous, $f[M]$ is bounded subset of $\mathbb{R}$. If it were unbounded, we could assume that $\forall n \in \mathbb{N}\ \exists x_n \in X\ f(x_n) > n$. But then, $\forall n, m \in$

$\mathbb{N}$ $|f(x_n) - f(x_m)| \geq 1$. Since f is uniformly convergent, $\rho(x, y) < \delta \Rightarrow |f(x) - f(y)| < 1/2$; and for finite cover of M with $\delta/2$ balls we get that one ball cannot contain more than one x_n.

♡

Complete metric spaces. We call a sequence $\{x_n | x_n \in M, n \in \mathbb{N}\}$ a *Cauchy* sequence if $\forall \varepsilon > 0\ \exists n_0\ \forall n, m \geq n_0\ \rho(x_n, x_m) < \varepsilon$. Metric space is *complete*, if every Cauchy sequence is convergent.

In any metric space, a convergent sequence is automatically a Cauchy sequence. We can take n_0 such that $\rho(x, x_n) < \varepsilon/2$, and $\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x_m, x) < \varepsilon$.

Lecture 4

`coming soon?`

Lecture 5

`... damn cryptic ...`

Dense sets. A set $X \subseteq M$ is *dense* in (M, ρ) if $\overline{X} = M$. Set X is of type G_δ if $X = \bigcap_{i \in \mathbb{N}} Y_i$ where Y_i are open sets. Set X is of type F_σ if $X = \bigcup_{i \in \mathbb{N}} Y_i$ where Y_i are closed sets. Clearly, X is of type G_δ iff $\overline{X}$ is of type F_σ.

Strange corollary (?) : every metric space can be isometrically embedded onto dense subspace of complete metric space (?).

Baire theorem. Let (M, ρ) be complete metric space, $\{U_n; n \in \mathbb{N}\}$ family of open dense subsets of M. Then $\bigcap U_n$ is dense in M.

Proof: we just need to show that $\bigcap U_n \cap G \neq \emptyset$ for any non-empty open G. First, $G \cap U_1$ is non-empty and open. Let $x_1 \in G \cap U_1$ and $\varepsilon < 1$, so that $B_\rho(x_1, \varepsilon) \subseteq G \cap U_1$. By induction, we have constructed $x_n,\ \varepsilon < 1/n,\ B_\rho(x_n, \varepsilon_n) \subseteq G \cap \bigcap_n U_i$. As $B_\rho(x_n, \varepsilon_n)$ is open, we can pick $x_{n+1} \in B_\rho(x_n, \varepsilon_n/2) \cap U_{n+1}$ and put $\varepsilon_{n+1} = \min\{1/(n+1), (\varepsilon_n - \rho(x_n, x_{n+1}))/2\}$. Then we have $B_\rho(x_n, \varepsilon_n) \subseteq \overline{B_\rho(x_n, \varepsilon_n)} \subseteq B_\rho(x_{n-1}, \varepsilon_{n-1})$, therefore $\bigcap B_\rho(x_n, \varepsilon_n) = \bigcap \overline{B_\rho(x_n, \varepsilon_n)}$ and by Cantor's theorem from previous lecture there is a point in $\bigcap B_\rho(x_n, \varepsilon_n) \subseteq G \cap \bigcap U_n$.

♡

As a strange corollary (?) : there is a continuous function that has no derivate.

Let Y be a G_δ type subset in (M, ρ). Then, there is a metric σ so that:

1) (Y, σ) is complete.
2) (Y, σ) is topologically equivalent to $(Y, \rho \upharpoonright Y^2)$

(this is supposed to hold in the other direction too?)

Proof: let us take open sets $U_n \subseteq X$, $\bigcap U_n = Y$, and set $F_n = X \setminus U_n$. Then F_n is closed, $F_n \cap Y = \emptyset$, $X \setminus Y = \bigcup F_n$. Without loss of generality $F_n \neq \emptyset$, $\forall x, y\ \rho(x, y) \leq 1$ (this can be assumed, because for complete metric ρ on X we can take $\rho'(x, y) = \max\{1, \rho(x, y)\}$ which is topologically equivalent to ρ and has same Cauchy sequences).

Lecture 6

`the Lavrentieff's theorem is kind of incomplete...`

Theorem. Let (M,ρ), (N,σ) be metric spaces, N is complete, $A \subseteq M$ be dense in M. If $f : A \to N$ is continuous, then there is a G_δ set $B \supseteq A$ and $F : B \to N$, continuous function $F \supseteq f$.

Proof: set $O_n = \bigcup\{G,\ G \subseteq M \text{ is open}, \forall x,y \in G \cap A\ \sigma(f(x),f(y) \le 1/n\}$. We see that every O_n is open (as a union of open sets G), but it also holds that $A \subseteq O_n$ – we pick $x \in A$, then $\exists\delta > 0\ \rho(x,y) < \delta \Rightarrow \sigma(f(x),f(y) \le 1/(2n)$ (since f is continuous), therefore for any $y,y' \in B_\delta(x)$ is $\sigma(f(y),f(y')) \le \sigma(f(y),f(x)) + \sigma(f(y'),f(x)) \le 1/n$. We set $B = \bigcap O_n$, then B is G_δ set containing A. Now, we proceed with construction of F:

- for $z \in A$ set $F(z) = f(z)$,
- for $z \in B \setminus A$ pick a sequence $(x_n^z,\ n \in \mathbb{N},\ x_n^z \in A)$ such that this sequence converges to z (we can do this since A is dense), and set $F(z) = \lim_{n\to\infty} f(x_n^z)$. To be sure that $F(z)$ is well-defined, we need to check that $f(x_n^z)$ is Cauchy: as f is continous, there is δ such that $\rho(x,y) \le \delta \Rightarrow \sigma(f(x),f(y))$. x_n^z itself is convergent and therefore Cauchy, so we can pick n_0 such that $\forall m,n > n_0\ \rho(x_m^z,x_n^z) < \delta$ and $f(x_m^z,x_n^z) < \varepsilon$.

Clearly is $F \supseteq f$, so we just have to check that F is continuous. We have an ε and seek the right δ. Let us pick arbitrary $z = (x_n^z)$, $y = (x_m^y)$ (if either y or z are already in A, we set the sequence identical), and bound distance of their images: $\sigma(f(z),f(y)) \le \sigma(f(z),f(x_n^z)) + \sigma(f(x_n^z),f(x_m^y)) + \sigma(f(x_m^y),f(y))$. As $f(x_n^z)$ converges to $f(z)$, we can pick n_0 such that $\sigma(f(z),f(x_{n_0..}^z))$ is $< \varepsilon/3$, same for $\sigma(f(x_m^y),f(y))$. If $\rho(z,y)$ is lesser than some δ, then we can pick n_1 and m_1 such that $\rho(x_{n_1..}^z,x_{m_1..}^y) < 3\delta$. Since f is continuous, we can pick δ_1 such that $\rho(a,b) < \delta_1 \Rightarrow \sigma(f(a),f(b)) < \varepsilon/3$, and then we set $\delta = \delta_1/3$; and our bounding is done.

♡

Theorem (Lavrentieff). (?) Let M,N be complete metric spaces, $A \subseteq M$ and $B \subseteq N$ dense sets, $f : A \to B$ homeomorphism, then there are G_δ sets $C \supseteq A$ and $D \supseteq B$ and homeomorphism $F : C \to D$, $f \subseteq F$.

Proof: Let g be the inverse of f, from previous theorem we know that there are extensions $F_0 : C_0 \to Y$ and $G_0 : D_0 \to X$ with $C_0 \supseteq A$ and $D_0 \supseteq B$ G_δ-sets. Set $C = C_0 \cap G_0^{-1}[D_0]$ and $D = D_0 \cap F_0^{-1}[C_0]$, both are still G_δ sets; clearly with $A \subseteq C$ and $B \subseteq D$. Now, we show that $G_0(F_0(x)) = x$ for every $x \in C$ (which implies $F_0[C] \subseteq D$, since $F_0[C] \subseteq G_0^{-1}[C] \subseteq G_0^{-1}[C_0] = D$). `missing shit`. Symmetrically, we can assume that $F_0(G_0(x)) = x\ \forall x \in D$ and $G_0[D] \subseteq C$; therefore $G_0[D] = C$ and $F_0[C] = D$. If we set $F = F_0 \upharpoonright C$ and $G = G_0 \upharpoonright D$, both remain continuous and are mutually inverse, therefore F is homeomorphism.

♡

Compactness. We say that metric space (M,ρ) is *compact*, if for each cover $\mathcal{C}$ of M by open balls there is a finite sub-cover $\mathcal{C}' \subseteq \mathcal{C}$ of M. Following statements are equivalent:

a) M is compact.

b) Every family $\mathcal{F}$ of closed sets, such that any finite subset of $\mathcal{F}$ has nonempty intersection, has nonempy intersection.

c) Every infinite subset X of M has a limit point (a point x such that any ball around x contains $y \in X,\ y \ne x$).

d) Every sequence has a convergent subsequence.

e) Every continuous function $f : M \to \mathbb{R}$ is bounded.

f) Every continuous function $f : M \to \mathbb{R}$ attains its supremum and infimum.

g) For every countable cover of M there is a finite subcover.

e $\Rightarrow$ f: let f be continuous not attaining its sup – $t = \sup f(x) \Rightarrow \neg\exists x_0,\ f(x_0) = t$. Define $g : (-\infty, t) \to \mathbb{R},\ g(y) = 1/(t-y)$, g is continous, $g \circ f$ is continous but unbounded.

f $\Rightarrow$ e: let f be continuous attaining its supremum and infimum. But that means that $f[M] \subseteq [f(\inf), f(\sup)]$ and f is bounded.

a $\Rightarrow$ b: let $\bigcap \mathcal{F} = \emptyset$, then $\{M \setminus F,\ F \in \mathcal{F}\}$ is open cover of M. This implies there is a finite open sub-cover – $M = M \setminus \bigcap F_{1..n}$, meaning that $\bigcap F_{1..n} = \emptyset$.

b $\Rightarrow$ a: let $\mathcal{U}$ be a set of open balls without any finite sub-cover of M. Then, $\mathcal{F} = \{M \setminus U,\ U \in \mathcal{U}\}$ has the finite-intersection property and therefore $\bigcap \mathcal{F} \neq \emptyset$, meaning that $\mathcal{U}$ did not cover U.

a $\Rightarrow$ c: let X be a subset of M without limit point. Then for all $x \in M$ there is an open set $G_x \ni x$ with finite intersection with M. But such G_x form cover of M, therefore there is a finite sub-cover H_x, so that $\bigcup H_x = M \supseteq X$ and each $X \cap H_x$ is finite, therefore X is finite.

Lecture 7

`couple of theorems are missing...`

c $\Rightarrow$ d: for a sequence x_n either there is such $r \in M$ so that $x_n = r$ for infinitely many n's or every value is present only finitely-times. In the first case, we can simply pick all x_n that equal r to get convergent subsequence; in the second case, we construct set X as a range of x_n. X is therefore infinite and has a limit point x, i.e. for every m we select x_m so that $\rho(x_m, x) < 1/m$ and get converget subsequence.

d $\Rightarrow$ e: let f be unbounded, i.e. $\forall n\ \exists x_n\ |f(x_n)| > n$, and such a sequence x_n cannot have convergent subsequence, since every subsequence is unbounded.

a $\Rightarrow$ g: trivial, since g is special case of a.

e $\Rightarrow$ c: let X be a set without limit point, i.e. $\forall x\ \exists\varepsilon\ |B(x,\varepsilon) \cap X| < \infty$. Specially, we can say that $X = \{x_i\}$, $\forall i\ \exists\varepsilon_i\ B(x_i, \varepsilon_i) \cap X = \{x_i\}$. If we set $\delta_i = \min\{\varepsilon/3, 1/n\}$, then $B(x_i, \delta_i)$ are pairwise disjoint and $\forall x \in M\ \exists\varepsilon(x)\ B(x, \varepsilon)$ intersects only finitely many $B(x_i, \delta_i)$. Now, we define $f_n(x) = \rho(x, X \backslash B(x_n, \delta_n))$ – this f is continous, as it is defined from metrics; for $x \notin X$ equals 0 and $f(x_i) \geq \delta_n$. Then, $F(x) = \sum n \cdot 1/\delta_n \cdot f_n(x)$ is continous as a sum of continous functions (note that $\forall x\ F(x)$ is determined only by finitely many $f_n(x)$ since $f_n(x)$ is non-zero only for $x \in B(x_i, \delta_i)$); and therefore has to be bounded. But as $F(x_n) \geq n$, we get that X was finite.

c $\Rightarrow$ a: let $\mathcal{C}$ is open cover of M. First, we prove an auxilliary lemma:

$$(\exists\varepsilon > 0)(\forall x \in M)(\exists C \in \mathcal{C})(B(x,\varepsilon) \subseteq C) \qquad (*)$$

Proof: by contradiction, for $\varepsilon_i = 1/i$ we get points x_i such that $B(x_i, \varepsilon_i)$ is not subset of any set in cover. But then $X = \{x_i\}$ is an infinite set without limit point – since limit point l would have to be in some C, we could pick ε so that $B(l, \varepsilon) \subseteq C$, $B(l, \varepsilon/2) \cap X$ is still infinite, and for all x_i in such intersections is $B(x_i, \varepsilon/2) \subseteq C$, contradicting that $B(x_i, 1/i)$ is not subset of any cover for $1/i < \varepsilon/2$.

Now, we pick ε from $(*)$ and arbitrary x_1, for which there is $C_1 \in \mathcal{C}$ so that $B(x_1, \varepsilon) \subseteq C_1$. Now, having constructed $C_{1..i-1}$ we proceed constructing C_i – if $M \neq \bigcup C_{1..i-1}$, we pick $x_{i+1} \in M \setminus \bigcup C_{1..i-1}$, for which there is $C_i \in \mathcal{C}$ so that $B(x_i, \varepsilon) \subseteq C_i$, this C_i is clearly $\neq C_{1..i-1}$. However, if such sequence of x_i is infite, we have an infinite ε-net (since every x_i is not in $B(x_{1..i-1}, \varepsilon)$), which contradicts something (`what?`). Therefore, sequence is finite, and $C_{1..i}$ form a finite subcover of M.

g $\Rightarrow$ `to be finished`

♡

Theorem. Metric space is compact iff it is complete and totally bounded.

Proof: Any compact space has to be totally bounded, since we can pick cover $\{B(x,\varepsilon); x \in M\}$ which has a finite subcover S. A ε-net cannot have more than one point in each of balls in S, therefore has to be finite. Any compact space has to be complete, since a non-complete space `something with Cantor theorem`.

Now for the other implication, `to be finished`.

♡

Corollary. *Closed subset of compact space is compact; product of compact spaces is compact.*

Theorem (Heine–Borel–Lebesgue). *Closed bounded subset of* $\mathbb{R}^n$ *is compact.*

Proof: For arbitrary K closed and bounded, there are a, b such that $a < b$ and $K \subseteq [a,b]^n$. But $[a,b]^n$ is complete and totally bounded, therefore it is compact, therefore its closed subset is compact.

♡

Theorem. *For metric space* (N,σ), *compact metric space* (M,ρ) *and* $f : M \to N$ *which is continuous and onto is* (N,σ) *compact.*

Proof: Let G_i be open cover of N, because of continuity of f we have that $H = \{f^{-1}[G_i]\}$ is a set of open sets, and since f is onto is H an open cover of M. Therefore we have H' a finite subcover, and its image is a finite subset of G and still an open cover of N.

♡

Theorem. *Let* (M,ρ) *be a compact space,* f *continous; then* f *is uniformly continous.*

Proof: $(\forall\varepsilon > 0)(\forall x)(\exists\delta_x)$ such that $f[B(x,\delta_x)]$ is open. Clearly, $B(x,\delta_x/2)$ form an open cover, therefore there is its finite subcover S. Take $\delta = \min\{\delta_x/2; B(x,\delta_x/2) \in S\}$, and watch: $\rho(x,y) < \delta \Rightarrow x, y \in B(z,\delta_z),\ B(z,\delta_z/2) \in S \Rightarrow \sigma(f(x),f(y)) \le \sigma(f(x),f(z)) + \sigma(f(y),f(z)) \le 2\varepsilon$.

♡

Lecture 8

Theorem. *For* (M,ρ) *compact metric space,* (N,σ) *metric space and* $f : M \to N$ *continuous is* $f[M]$ *closed subset of* N.

Proof: We already know that continuous image of compact space is a compact space, and also that compact space is totally bounded and complete. Therefore if there was a point in $x \in \overline{f[X]} \setminus f[X]$, we could construct a Cauchy sequence $(x_n | x_n \in f[M], \sigma(x_n,x) < 1/n)$ with limit x, which wouldn't have a limit in $f[X]$ – a contradiction.

♡

Corollary. *For compact metric space* (M,ρ), *metric space* (N,σ) *and* f *continuous bijection, we have that* f *is homeomorphism.*

Proof: If $X \subseteq N$ is closed, then $(f^{-1})^{-1}[X] = f[X]$ is closed due to previous theorem, therefore f^{-1} is continuous.

♡

Cantor Set. In $\mathbb{R}$, we say that for

$$C_0 = [0,1], C_i = C_{i-1} \setminus \bigcup_{j=0}^{3^{i-1}} (j \cdot 3^{-i+1} + 3^{-i}, j \cdot 3^{-i+1} + 2 \cdot 3^{-i})$$

is $\mathcal{C} = \bigcup C_i$ the *Cantor set.* This set has many funny properties, for example it consist only of isolated points and therefore has no interior (no open ball is subset of it) and is closed (and compact, as we are in $\mathbb{R}$).

Alternatively, we can say that $\mathcal{C}$ consists of points in $[0,1]$ that have no 1 in their ternary expansion. This argument implies that cardinality of $\mathcal{C}$ is continuum. This ternary expansion gives us more: if we consider the space $(\{0,1\}^{\omega_0}, \iota)$ – where $\iota((x_n),(y_n)) = \sum_i [x_i \neq y_i] \cdot 2^{-i}$, and $f : \mathcal{C} \to \{0,1\}^{\omega_0}$, $f(x) = x/2$; we get that $\mathcal{C}$ is bijection and continuous – because

We can construct a similiar set, by removing subintervals at $(1/5, 2/5)$ and $(3/5, 4/5)$ instead of $(1/3, 2/3)$. Later, we shall see that this set is homeomorphic to $\mathcal{C}$. With more dimensions there is more fun. If we take unit square and remove the middle subrectangle (height 1, width 1/3) in a simillar manner; we get a 2-dimensional $\mathcal{C}$. Such a set has still big cardinality and no interior, and has area 0 (total sum of area of removed squares is 1, the area of unit square). On the other hand, if the removed subrectangle has width $0 < \alpha < 1/3$, the area is nonzero and resulting set still doesn't have any interior.

Theorem. *Every compact space is continous image of* $\mathcal{C}$.

Remark: One shall note that every compact space has cardinality $\leq \mathcal{C}$, since `to be finished.`

Proof: `any volunteers?`

♡

`And here again, something is missing.`

Lecture 9

`Well, is incomplete all over.`
`I'm not even embarrassed for another missing lecture note.`

Lecture 10

Definition. $K_x \subseteq M$ is a *quasicomponent* of (M, δ) in x, if $K_x = \bigcap\{Z \subseteq M; x \in Z$ and Z is both closed and open set $\}$.

Theorem. Every point of M is contained in exactly one quasicomponent.

Proof: Let $x \in K_x$, $y \in K_y$. If $K_x \neq K_y$, we can wlog assume that $t \in K_x \setminus K_y$. That means that there is Z' *cao* such that $y \in Z'$ and $t \notin Z'$, and $\forall Z$ *cao* $x \in Z \Rightarrow t \in Z$. However, $M \setminus Z'$ is also *cao*, and because $t \in M \setminus Z'$, x is also in $M \setminus Z'$ (otherwise t would be in a set and in its complement). Therefore, $K_x \subseteq M \setminus Z'$ and $K_y \subseteq Z'$, implying that $K_x \cap K_y$ and any two different quasicomponents have empty intersection.

♡

Example. As previous theorem remarked some resemblance between components and quasicomponents, now we present that these two terms indeed differ. Imagine $M \subseteq \mathbb{R}^2$, $M = \{[0,0],[0,1]\} \cup \{1/n\} \times [0,1]$. If $C \subseteq M$ is a connected set containing $[0,0]$, then there can't be any other point in C since $[0,0]$ itself is a *cao* subset of C, therefore C is a component. On the other hand, let $Z \subseteq M$ be *cao* set containing $[0,1]$. Z is open and therefore $\exists \varepsilon > 0$ $B([0,0],\varepsilon) \cap M \subseteq Z$ and $\exists n_0$ $\forall n > n_0$ $[1/n, 0] \in Z$. But $\{1/n\} \times [0,1]$ is connected *cao* set intersecting Z thus $\{1/n\} \times [0,1] \subseteq Z$ (otherwise the intersection would form a proper *cao* subset of $\{1/n\} \times [0,1]$). Specially, $[1/n, 1] \in Z$, and because Z is closed, we have $[0,1] \in Z$. A quasicomponent of $[0,0]$ therefore contains $[0,1]$ also, unlike the component.

Theorem. If K_x is open quasicomponent, then K_x is component.

Proof: As K_x is intersection of closed sets, it is indeed closed, and due to assumption *cao*. If K_x is not connected, then there are nonempty disjoint open sets U, V such that $K_x = U \cup V$. Wlog $x \in U$, but U is open in M too and $M \setminus V$ is closed in M, implying that $U = K_x \setminus V = (M \setminus V) \cap K_x$ is closed and therefore *cao*. But x can't be in proper *cao* subset of K_x.

♡

Corollary. For M with finitely many quasicomponents is every quasicomponent a component too.

Theorem. If K_x is quasicomponent of compact space, then K_x is component.

Proof: Let K_x be diconnected, $K_x = F \cup H$, F and H are disjoint nonempty closed sets. Set $U = \{y \in M; \delta(y,H) > \delta(y,F)\}$ and $V = \{y \in M; \delta(y,F) > \delta(y,F)\}$, these are clearly disjoint, open, and supersets of F and H respectively.

> **Lemma.** $\exists Z \subset M$, $x \in Z$ *cao*, $Z \subseteq U \cap V$. If there were no such Z, then the set $S = \bigcap\{P \setminus (U \cup V)$, P is *cao* containing $x\}$ is an intersection of nonempty closed sets where every two sets have nonempty intersection (`why?`), due to compactness is S nonempty and there is $y \in S$ that should be in K_x too, but it is not even in $U \cup V \supset K_x$.

Wlog assume that $x \in Z \cap U$; also $Z \cap U = Z \setminus V$ is closed (since substraction of open set from closed set yields closed set) and open (since both Z and U are open) containing x, but $K_x \not\supseteq U \cap Z$ since $K_x \cap V \neq \emptyset$; a contradiction.

♡

Definition. *Continuum* is a connected compact metric space. If a continuum consists of a single point, we call it degenerated, otherwise we do not. Unless explicitly stated, all present continuums are not degenerated.

Theorem. Let $A_0 \supseteq A_1 \supseteq \ldots$ be a sequence of continuums, then $\bigcap A_i$ is a continuum itself.

Proof: Due to compactness of A_0, we have that $M = \bigcap A_i$ is an intersection of nonempty closed sets, where $\bigcap_j A_i = A_j \neq \emptyset$, and therefore nonempty. As intersection of closed sets it is itself closed and therefore compact. Assume for contradiction that $M = F \cup H$ where F and H are nonempty disjoint closed subsets of M. As in previous proof, we construct sets $U = \{x \in A_0; \delta(x,F) < \delta(x,H)\}$ and $V = \{x \in A_0; \delta(x,H) < \delta(x,F)\}$.

> **Lemma.** $\exists n$ such that $A_n \subseteq U \cup V$. If not, $\{A_i \setminus (U \cup V)\}$ is decreasing sequence of nonempty closed (since substraction of open set from closed yields closed set) sets in compact A_0 therefore its intersection contains a y, but $U \cup V$ was supposed to be superset of $\bigcap A_i$.

But existence of such A_n as claimed in lemma leads to contradiction, since A_n is not connected (both $A_n \setminus U$ and $A_n \setminus V$ are nonempty and closed, and form a disjoint partition of A_n).

♡

Example. $A_i = ([0,1] \times [0,1/n]) \setminus (0,1) \times \{0\}$. All A_i are connected but their intersection is not (because A_i are not compact).

Theorem. $M \subseteq \mathbb{R}$ is continuum, iff $M = [a,b]$.

Proof: Let M be continuum, then it is compact and identical map to R attains its max b and min a, therefore $M \subseteq [a, b]$. But any $a < c < b$ has to be in M since otherwise M would not be connected. On the other hand, any $[a, b]$ is clearly continuum.

♡

Definition. Border of $X \subseteq M$ is defined as $\mathrm{bd}(X) = \{x \in M; \delta(x, X) = \delta(x, M \setminus X) = 0\}$. Equivalently, $x \in \mathrm{bd}(X) \equiv \forall \varepsilon > 0\ B(x, \varepsilon) \cap X \neq \emptyset \wedge B(x, \varepsilon) \cap X \setminus M \neq \emptyset$; or as $\mathrm{bd}(X) = \overline{X} \cap \overline{M \setminus X}$.

Theorem. Let M be continuum, F closed nonempty proper subset of X, and K a component of F. Then $K \cap \mathrm{bd}(F) \neq \emptyset$.

Proof: Assume for contradiction that $\exists K,\ K \cap \mathrm{bd}(F) = \emptyset$. Then $U = F \setminus \mathrm{bd}(F) = F \setminus \overline{(M \setminus F)}$ is open set (`why?`) and $K \subseteq U$. As F is compact, K quasicomponent and $\exists Z \subseteq F$ *cao*, $K \subseteq Z$, $Z \cap \mathrm{bd}\, F = \emptyset$ (`really?`). As Z is open in F, it is also open in U which is open in M and therefore Z is open in M. On the other hand, Z is closed in F and in M also, contradicting conectedness of M.

♡