3.16 Solution: (a) Take the Bessel equation

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dJ_\nu(k\rho)}{d\rho}\right]+\left(k^2-\frac{\nu^2}{\rho^2}\right)J_\nu(k\rho)=0,$$

multiply by $\rho J_\nu(k'\rho)$, and integrate with respect to ρ from 0 to ∞, we will have

$$\int_0^\infty J_\nu(k'\rho)\frac{d}{d\rho}\left[\rho\frac{dJ_\nu(k\rho)}{d\rho}\right]d\rho+\int_0^\infty\left(k^2-\frac{\nu^2}{\rho^2}\right)\rho J_\nu(k'\rho)J_\nu(k\rho)d\rho=0.$$

The first term can be manipulated with partial integration, which leads to

$$-\int_0^\infty \rho\frac{dJ_\nu(k'\rho)}{d\rho}\frac{dJ_\nu(k\rho)}{d\rho}d\rho+\int_0^\infty\left(k^2-\frac{\nu^2}{\rho^2}\right)\rho J_\nu(k'\rho)J_\nu(k\rho)d\rho=0.$$

Exchange k and k',

$$-\int_0^\infty \rho\frac{dJ_\nu(k'\rho)}{d\rho}\frac{dJ_\nu(k\rho)}{d\rho}d\rho+\int_0^\infty\left(k'^2-\frac{\nu^2}{\rho^2}\right)\rho J_\nu(k'\rho)J_\nu(k\rho)d\rho=0,$$

and take the difference between the two identities, we are left with

$$(k^2-k'^2)\int_0^\infty \rho J_\nu(k'\rho)J_\nu(k\rho)d\rho=0.$$

Clearly, the integral

$$\int_0^\infty \rho J_\nu(k'\rho)J_\nu(k\rho)d\rho=0,$$

if $k\neq k'$, while for $k=k'$, the integral diverges. Therefore, the integral must be proportional to the Dirac delta function, *i.e.*,

$$\int_0^\infty \rho J_\nu(k'\rho)J_\nu(k\rho)d\rho=C\delta(k-k'),$$

where C is a constant that needs to be determined.

Integrate both sides with respect to k', we will have

$$\begin{aligned}
C &= \int_0^\infty d\rho\rho J_\nu(k\rho)\cdot\int_0^\infty dk' J_\nu(k'\rho)\\
&= \int_0^\infty d\rho J_\nu(k\rho)\cdot\int_0^\infty dk' J_\nu(k')\\
&= \int_0^\infty d\rho J_\nu(k\rho)\\
&= \frac{1}{k}\int_0^\infty d\rho J_\nu(\rho)\\
&= \frac{1}{k},
\end{aligned}$$

where we have used the following result twice,

$$\int_0^\infty d\rho J_\nu(\rho)=1,$$

for $\mathrm{Re}(\nu) \geq -1$. Thus, we have established the completeness relation, or the Hankel transform,

$$\int_0^\infty \rho J_\nu(k'\rho) J_\nu(k\rho) d\rho = \frac{1}{k}\delta(k - k').$$

Make the substitution, $\rho \to k$, $k \to \rho$, $k' \to \rho'$, we will get the equivalent relation,

$$\int_0^\infty k J_\nu(k\rho') J_\nu(k\rho) dk = \frac{1}{\rho}\delta(\rho - \rho').$$

The proof presented here is quite hand-waving, but should suffice from a physicist's perspective.

(b) With the help of the Hankel transform, the Dirac delta function in the cylindrical coordinates can be expressed as

$$\delta(\mathbf{x} - \mathbf{x}') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dk e^{im(\phi-\phi')} \cdot k J_m(k\rho') J_m(k\rho) \cdot \delta(z - z').$$

Therefore, the Green function must have a similar form,

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dk e^{im(\phi-\phi')} \cdot k J_m(k\rho') J_m(k\rho) \cdot g_m(k; z, z').$$

Applying the Lapacian on the Green function, we will have

$$\begin{aligned}
\nabla^2_{\mathbf{x}'} G(\mathbf{x}, \mathbf{x}') &= \frac{1}{\rho'}\frac{d}{d\rho'}\left[\rho'\frac{dG}{d\rho'}\right] + \frac{1}{\rho'^2}\frac{\partial^2 G}{\partial \phi'^2} + \frac{\partial^2 G}{\partial z'^2} \\
&= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dk\ e^{im(\phi-\phi')} \cdot k J_m(k\rho') J_m(k\rho) \left(\frac{\partial^2 g_m(k; z, z')}{\partial z'^2} - k'^2 g_m(k; z, z')\right) \\
&= -4\pi \cdot \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dk\ e^{im(\phi-\phi')} \cdot k J_m(k\rho') J_m(k\rho) \cdot \delta(z - z').
\end{aligned}$$

The above identity must be valid term-wise, which means

$$\frac{\partial^2 g_m(k; z, z')}{\partial z'^2} - k^2 g_m(k; z, z') = -4\pi\delta(z - z').$$

Now, we can apply the usual procedure to determine the unknown function g, which must be a linear combination of $e^{kz'}$ and $e^{-kz'}$. For $z' < z$, in order to have a finite solution as $z' \to -\infty$, the solution must be proportional to $e^{kz'}$. Similarly, for $z' > z$, the solution must be proportional to $e^{-kz'}$. The solution is also symmetric in z and z'. Therefore,

$$g_m(k; z, z') = A_m(k) e^{-k(z_> - z_<)}.$$

To determine $A_m(k)$, integrate both sides of the differential equation governing g, we have

$$\left.\frac{\partial g_m(k; z, z')}{\partial z'}\right|_{z'=z+\varepsilon} - \left.\frac{\partial g_m(k; z, z')}{\partial z'}\right|_{z'=z-\varepsilon} = -4\pi.$$

Since

$$\left.\frac{\partial g_m(k; z, z')}{\partial z'}\right|_{z'=z+\varepsilon} = A_m(k)\ \frac{\partial}{\partial z'} e^{-k(z'-z)}\Bigg|_{z'=z+\varepsilon} = -k A_m(k),$$

and

$$\left.\frac{\partial g_m(k;z,z')}{\partial z'}\right|_{z'=z-\varepsilon} = A_m(k)\ \frac{\partial}{\partial z'}e^{-k(z-z')}\Big|_{z'=z+\varepsilon} = kA_m(k),$$

which leads to $-2kA_m(k) = -4\pi$, or

$$A_m(k) = \frac{2\pi}{k}.$$

With the knowledge of the coefficient, the Green function becomes

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\ e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')e^{-k(z_>-z_<)}, \tag{1}$$

which is the expansion we are seeking.

(c) In the expansion from part (b), set $\rho' = 0$ and $z' = 0$, we have

$$\frac{1}{\sqrt{\rho^2+z^2}} = \sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\ e^{im(\phi-\phi')}J_m(0)J_m(k\rho)e^{-k(z_>-z_<)}.$$

Since $J_m(0) = 1$ for $m = 0$, and 0 otherwise, only the $m = 0$ term survives. Also, if $z > z' = 0$, $e^{-k(z_>-z_<)} = e^{-kz}$, and if $z < z' = 0$, $e^{-k(z_>-z_<)} = e^{kz}$, which means

$$e^{-k(z_>-z_<)} = e^{-k|z|}.$$

Combine all these observations, we have the following expansion,

$$\frac{1}{\sqrt{\rho^2+z^2}} = \int_0^{\infty} J_0(k\rho)e^{-k|z|}dk. \tag{2}$$

Instead, if we set $z' = 0$ and $\phi' = 0$ while keep ρ' non-zero, the Green function expansion (1) becomes

$$\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi+z^2}} = \sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\ e^{im\phi}J_m(k\rho)J_m(k\rho')e^{-k|z|}.$$

Meanwhile, the left hand side of the above expansion can also be expanded with (2),

$$\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi+z^2}} = \int_0^{\infty} J_0(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi})e^{-k|z|}dk.$$

Comparing the two equivalent expansions, we can establish the following identity,

$$J_0(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi}) = \sum_{m=-\infty}^{\infty} e^{im\phi}J_m(k\rho)J_m(k\rho').$$

In the generating function of the Bessel function,

$$\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right) = \sum_{n=-\infty}^{\infty} J_n(x)t^n,$$

set $t = ie^{i\phi}$, it can be shown

$$\exp\left(\frac{x}{2}\left(ie^{i\phi}+ie^{-i\phi}\right)\right) = \exp\left(ix\cos\phi\right) = \sum_{n=-\infty}^{\infty} J_n(x)i^n e^{in\phi}.$$

Let $x = k\rho$, we will obtain the desired identity,

$$\exp(ik\rho\cos\phi) = \sum_{n=-\infty}^{\infty} J_n(k\rho) i^n e^{in\phi}.$$

(d) Using the last expression from part (c), it is straightforward to show, by inverse Fourier transform, that

$$J_m(x) = \frac{1}{2\pi i^m} \int_0^{2\pi} e^{ix\cos\phi - im\phi} d\phi.$$

This brings back some fond memories about my grad school days. One of my Ph.D. projects was to study the oscillations observed in electric resistance in some materials. Key to the explanation of the phenomenon is that the solution of the Boltzmann equation contains cosine terms of other cosine terms, which can be expanded with the identities here to obtain the oscillating Bessel functions. So much for the rambling.