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Relativistic form of radiation reaction

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We present a relativistic extension of the new form which we have recently obtained for the equation of motion of a radiating electron.

We have recently presented a new approach to the problem of radiation reaction in nonrelativistic quantum electrodynamics and obtained an equation of motion which does not exhibit the problems associated with the well-known Abraham–Lorentz equation [1]. In the classical limit and in the absence of an external potential, this equation takes the form

$$M\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}=f(t)+\tau_\mathrm{e}\frac{\mathrm{d}f(t)}{\mathrm{d}t}, \tag{1}$$

where $\tau_\mathrm{e}=2e^2/3Mc^3=6\times10^{-24}$ s, M and e denote the mass and charge, respectively, and $f(t)$ is the external force. Choosing the largest value, compatible with causality, of the cut-off frequency appearing in our choice of the electron form-factor, we showed that (1) is exact.

In the relativistic case, to conform to Jackson's notation [2], we will denote the external electromagnetic field by $F^{\mu\nu}$ and the (observed) mass of the electron by m. Then our form for the relativistic generalization of (1) is

$$ma^\mu=\frac{e}{c}F^\mu{}_\kappa u^\kappa +\tau_\mathrm{e}\frac{e}{c}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}F^\mu{}_\lambda u^\lambda-\frac{1}{c^2}u^\mu u^\kappa\frac{\mathrm{d}}{\mathrm{d}\tau}F_{\kappa\lambda}u^\lambda\right), \tag{2}$$

where

$$u^\mu=\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau},\quad a^\mu=\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}, \tag{3}$$

with

$$\mathrm{d}\tau=\frac{1}{c}\sqrt{g_{\kappa\lambda}\,\mathrm{d}x^\kappa\mathrm{d}x^\lambda}=\sqrt{1-v^2/c^2}\,\mathrm{d}t =\frac{1}{\gamma}\mathrm{d}t. \tag{4}$$

Also, the field tensor has the following form when expressed in terms of the laboratory electric and magnetic fields,

$$F^{\mu\nu}=\begin{pmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & -B_z & B_y\\ E_y & B_z & 0 & -B_x\\ E_z & -B_y & B_x & 0\end{pmatrix}. \tag{5}$$

The Lorentz–Dirac (LD) form of the equation of motion is

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$$ma^\mu = \frac{e}{c} F^\mu{}_\kappa u^\kappa + m\tau_e \left(\frac{da^\mu}{d\tau} + \frac{1}{c^2} u^\mu g_{\kappa\lambda} a^\kappa a^\lambda \right)$$
$$= \frac{e}{c} F^\mu{}_\kappa u^\kappa + m\tau_e \left(\frac{da^\mu}{d\tau} - \frac{1}{c^2} u^\mu u^\kappa \frac{d}{d\tau} a_\kappa \right)$$
$$\text{(LD form)}, \tag{6}$$

where we have used the fact that $u^\mu a_\mu = 0$. It is clear that the Lorentz–Dirac equation agrees with our exact result (2) to first order in τ_e. It is also of interest to write our form of equation of motion, (2), in the three-vector form

$$m \frac{d\gamma \boldsymbol{v}}{dt} = \boldsymbol{F} + \tau_e \left[\gamma \frac{d\boldsymbol{F}}{dt} - \frac{\gamma^3}{c^2} \left(\frac{d\boldsymbol{v}}{dt} \times (\boldsymbol{v} \times \boldsymbol{F}) \right) \right], \tag{7}$$

where

$$\boldsymbol{F} = e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{8}$$

is the Lorentz force. In the same way, the Lorentz–Dirac equation can be written

$$m \frac{d\gamma \boldsymbol{v}}{dt} = \boldsymbol{F} + m\tau_e \left\{ \gamma \frac{d^2 \gamma \boldsymbol{v}}{dt^2} + \frac{\gamma^4}{c^2} \left[\left(\boldsymbol{v} \cdot \frac{d\boldsymbol{v}}{dt} \right) \frac{d\boldsymbol{v}}{dt} - \left(\frac{d\boldsymbol{v}}{dt} \right)^2 \boldsymbol{v} \right] \right\}$$
$$\text{(in LD form)}. \tag{9}$$

As an example, we consider now the case of motion in one dimension in the presence of a static electric field $\boldsymbol{E}(\boldsymbol{r}) = E(x)\hat{\boldsymbol{x}}$. With the form (7) the equations of motion become

$$dx/dt = v, \tag{10}$$

$$m \frac{d\gamma v}{dt} = eE(x) + e\tau_e \gamma v \frac{dE}{dx}. \tag{11}$$

Forming the ratio of these two equations and using the identity $v\,d\gamma v = c^2\,d\gamma$, we get the orbital equation

$$mc^2 \frac{d\gamma}{dx} = eE(x) + ec\tau_e (\gamma^2 - 1)^{1/2} \frac{dE}{dx}, \tag{12}$$

where we have used the identity $\gamma v/c = (\gamma^2 - 1)^{1/2}$.

Consider the application to a field localized in space with the particle incident from the left. Then, if the constant τ_e were zero, we could integrate to get

$$mc^2 \gamma(x) = mc^2 \gamma(-\infty) + e \int_{-\infty}^{x} dx' E(x'), \tag{13}$$

which is just the statement of energy conservation; the term on the left-hand-side is the energy at position x which is equal to the energy at the beginning ($x = -\infty$) plus the energy gained from the field. To get a corresponding result for nonzero τ_e we must integrate the differential equation (12). To illustrate how this is accomplished we consider the example

$$E(x) = \tfrac{1}{2} E_0 \{\tanh(x/a) + \tanh[(L-x)/a]\}, \tag{14}$$

where $a \ll L$. Thus, from $x = \infty$ to 0, the field slowly builds up from a zero value. It reaches its maximum value of E in the region of $x = a$ and remains constant at this value until the region of $x = L - a$ is reached, at which point the field falls off smoothly to zero. Then, if we put as a new dependent variable

$$Y(x) = \gamma(x) - \gamma(-\infty) - \frac{e}{mc^2} \int_{-\infty}^{x} dx' E(x')$$
$$= \gamma(x) - \gamma(-\infty)$$
$$- \frac{eE_0 L}{2mc^2} \left[\frac{a}{L} \log\left(\frac{\cosh(x/a)}{\cosh[(L-x)/a]} \right) + 1 \right], \tag{15}$$

and introduce the dimensionless dependent variable

$$X = x/L, \tag{16}$$

then (12) takes the form

$$\frac{dY}{dX} = \frac{c\tau_e}{a} \{[Y + \gamma(-\infty) + \Gamma(X)]^2 - 1\}^{1/2} D(X), \tag{17}$$

where

$$\Gamma(X) = \frac{1}{2} \frac{eE_0 L}{mc^2} \left[\frac{a}{L} \log\left(\frac{\cosh[(L/a)X]}{\cosh[(L/a)(X-1)]} \right) + 1 \right] \tag{18}$$

and

$$D(X) = \frac{eE_0 L}{mc^2} \left(\frac{1}{1 + \cosh[2(L/a)X]} - \frac{1}{1 + \cosh[2(L/a)(X-1)]} \right). \tag{19}$$

It is clear from (13) and (15) that $Y(x)$ is equal to the negative of the energy radiated. For a selection of parameters, we integrate (17) numerically and plot $Y(x)$ as a function of $X = x/L$.

It is clear from fig. 1 that, for most of its journey through the field, the electron is actually gaining en-

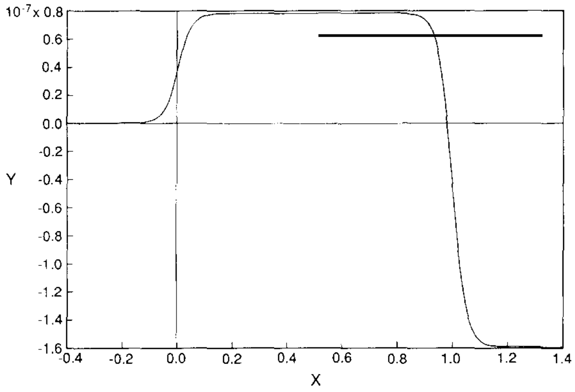

Fig. 1. The negative of the energy radiated, $Y(x)$, versus $X=x/L$ for an electron in a spatially-varying electric field $E(x)$ given by eq. (14), for a selection of parameters given by $\gamma(-\infty)=1.01$, $eE_0L/mc^2=0.1$ and $a/L=0.05$ and $c\tau_e/a=0.001$.

ergy from the field. However, in the region $x \geqslant L$, where the field is rapidly charging, the energy loss due to radiation is greater than the energy gained from the field.

For one-dimensional motion with a sharp discontinuity of the field, we find the conditions across the jump by rewriting (12) in the form

$$mc^2 \frac{\mathrm{d}}{\mathrm{d}x} \log(\gamma^2-1)^{1/2}$$

$$= e(\gamma^2-1)^{-1/2}E(x) + ec\tau_e \frac{\mathrm{d}E}{\mathrm{d}x} . \tag{20}$$

Integrating over an infinitesimal interval across the jump, we find

$$[mc^2 \log(\gamma^2-1)^{1/2} - ec\tau_e E]_{\text{jump}} = 0 , \tag{21}$$

where the notation indicates that the quantity within the square brackets is constant across the jump. We can write this condition in an interesting way if we express it as a condition on the velocity across the jump. If v_0 is the velocity just before the jump and v_1 is the velocity just after the jump, then (21) can be written

$$v_1 = \frac{v_0+u}{1+v_0u/c^2} , \tag{22}$$

where

$$u = \mathrm{ctanh}(e\Delta E\tau_e/mc) , \tag{23}$$

with ΔE the jump in the field. We recognize relation (22) as the relativistic law for addition of velocities.

With this result we can give an exact solution of the orbital equation for example (14) in the limit $a \to 0$, that is for

$$\begin{aligned} E(x) &= 0, \quad -\infty < x < 0 , \\ &= E_0, \quad 0 < x < L , \\ &= 0, \quad L < x < \infty . \end{aligned} \tag{24}$$

Then the solution of (12) can be written

$$\begin{aligned} \gamma(x) &= \gamma_0, && -\infty < x < 0 , \\ &= \gamma_1 + \frac{eE_0L}{mc^2} X, && 0 < x < L , \\ &= \gamma_2, && L < x < \infty , \end{aligned} \tag{25}$$

where

$$\begin{aligned} \gamma_1 &= \cosh[\cosh^{-1}(\gamma_0) + \tau_e eE_0/mc] , \\ \gamma_2 &= \cosh[\cosh^{-1}(\gamma_1 + eE_0L/mc^2) - \tau_e eE_0/mc] . \end{aligned} \tag{26}$$

The corresponding solution of (17) then becomes

$$\begin{aligned} Y(X) &= 0, && -\infty < X < 0 , \\ &= \gamma_1 - \gamma_0, && 0 < X < 1 , \\ &= \gamma_2 - \gamma_0 - \frac{eE_0L}{mc^2} , && 1 < X < \infty . \end{aligned} \tag{27}$$

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