**2.25** Solution: (a) The Dirac delta function on the two-dimensional surface with restricted angular range $[0, \beta]$ can be written as

$$\delta(\boldsymbol{\rho}-\boldsymbol{\rho}') = \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi') = \frac{1}{\rho}\delta(\rho-\rho')\cdot\frac{2}{\beta}\sum_{m=1}^{\infty}\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta),$$

and the corresponding Green function must have a similar form,

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{2}{\beta}\sum_{m=1}^{\infty} g_m(\rho,\rho')\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta). \tag{1}$$

Applying the Laplace operator, we have

$$\nabla^2_{\boldsymbol{\rho}'}G(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial G}{\partial\rho'}\right) + \frac{1}{\rho'^2}\frac{\partial^2 G}{\partial\phi'^2} = -4\pi\delta(\boldsymbol{\rho}-\boldsymbol{\rho}').$$

With Eq. (1), we have

$$\begin{aligned}\sum_{m=1}^{\infty}&\left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right)-\frac{m^2\pi^2}{\beta^2}\right]g_m(\rho,\rho')\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta)\\ &= -4\pi\cdot\frac{1}{\rho}\delta(\rho-\rho')\cdot\sum_{m=1}^{\infty}\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta).\end{aligned}$$

The above equation must be valid term-wise, which leads to a differential equation for $g_m$,

$$\left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right)-\frac{m^2\pi^2}{\beta^2}\right]g_m(\rho,\rho') = -4\pi\cdot\frac{1}{\rho}\delta(\rho-\rho').$$

Now, we can apply the familiar procedure to determine $g_m$. First, solutions to $g_m$ must be linear combinations of $\rho'^{m\pi/\beta}$ and $\rho'^{-m\pi/\beta}$. For $\rho' < \rho$, regularity at $\rho' = 0$ indicates that we can only choose the $\rho'^{m\pi/\beta}$ solution, or

$$g_m(\rho,\rho') \propto \rho'^{m\pi/\beta}.$$

Similarly, for $\rho' > \rho$,

$$g_m(\rho,\rho') \propto \rho'^{-m\pi/\beta}.$$

The final solution must be symmetric in $\rho$ and $\rho'$, which means

$$g_m(\rho,\rho') = C_m\left(\frac{\rho_<}{\rho_>}\right)^{m\pi/\beta}.$$

where $\rho_<$ ($\rho_>$) is the smaller (larger) of $\rho$ and $\rho'$. Integrate the differential equation governing $g_m$ in a small interval around $\rho$, we can obtain a relation that connects the derivative of $g_m$ on both side of $\rho$,

$$\rho'\frac{\partial}{\partial\rho'}g_m(\rho,\rho')\Big|_{\rho-\varepsilon}^{\rho+\varepsilon} = -4\pi.$$

For $\rho' = \rho+\varepsilon$,

$$\rho'\frac{\partial}{\partial\rho'}g_m(\rho,\rho')\Big|_{\rho'=\rho+\varepsilon} = C_m\,\rho'\frac{\partial}{\partial\rho'}\left(\frac{\rho}{\rho'}\right)^{m\pi/\beta}\Bigg|_{\rho'=\rho+\varepsilon} = -C_m\frac{m\pi}{\beta},$$

and for $\rho' = \rho - \varepsilon$,

$$\rho' \frac{\partial}{\partial \rho'} g_m(\rho, \rho')\bigg|_{\rho'=\rho-\varepsilon} = C_m\, \rho' \frac{\partial}{\partial \rho'} \left(\frac{\rho'}{\rho}\right)^{m\pi/\beta}\Bigg|_{\rho'=\rho-\varepsilon} = C_m \frac{m\pi}{\beta},$$

which leads to

$$\rho' \frac{\partial}{\partial \rho'} g_m(\rho, \rho')\bigg|_{\rho-\varepsilon}^{\rho+\varepsilon} = -2C_m \frac{m\pi}{\beta} = -4\pi,$$

or,

$$C_m = \frac{2\beta}{m}.$$

Put the solution of $g_m$ back into Eq. (1), the Green function becomes

$$G(\rho, \phi; \rho', \phi') = 4 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_<}{\rho_>}\right)^{m\pi/\beta} \sin(m\pi\phi/\beta) \sin(m\pi\phi'/\beta).$$

(b) Using the identity

$$\sin(m\pi\phi/\beta) \sin(m\pi\phi'/\beta) = \frac{1}{2} \left(\cos\left[m(\phi - \phi')/\beta\right] - \cos\left[m(\phi + \phi')/\beta\right]\right),$$

the Green function can be expressed as

$$\begin{aligned}
G(\rho, \phi; \rho', \phi') &= 2 \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_<}{\rho_>}\right)^{m\pi/\beta} \left(\cos\left[m(\phi - \phi')/\beta\right] - \cos\left[m(\phi + \phi')/\beta\right]\right) \\
&= 2 \cdot \mathrm{Re}\left\{\sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho_<}{\rho_>}\right)^{m\pi/\beta} \left(e^{im(\phi-\phi')/\beta} - e^{im(\phi+\phi')/\beta}\right)\right\} \\
&= -2 \cdot \mathrm{Re}\left\{\log\left(1 - \left(\frac{\rho_<}{\rho_>}\right)^{\pi/\beta} e^{i(\phi-\phi')/\beta}\right) - \log\left(1 - \left(\frac{\rho_<}{\rho_>}\right)^{\pi/\beta} e^{i(\phi+\phi')/\beta}\right)\right\} \\
&= \log\left(1 + \left(\frac{\rho_<}{\rho_>}\right)^{2\pi/\beta} - 2\left(\frac{\rho_<}{\rho_>}\right)^{\pi/\beta} \cos(\phi + \phi')\right) \\
&\qquad - \log\left(1 + \left(\frac{\rho_<}{\rho_>}\right)^{2\pi/\beta} - 2\left(\frac{\rho_<}{\rho_>}\right)^{\pi/\beta} \cos(\phi - \phi')\right) \\
&= \log\left(\frac{(\rho)^{2\pi/\beta} + (\rho')^{2\pi/\beta} - 2(\rho\rho')^{\pi/\beta} \cos(\phi + \phi')}{(\rho)^{2\pi/\beta} + (\rho')^{2\pi/\beta} - 2(\rho\rho')^{\pi/\beta} \cos(\phi - \phi')}\right),
\end{aligned}$$

where we have used the MacLaurin series for $\log(1 - x)$,

$$\log(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$