10.2 Solution: Since the EM radiation with circular polarization but different helicity is scattered independently, the scattered EM radiation for elliptic polarization will be linear superposition of the scattered, circularly polarized radiation. In the long-wavelength limit, Eq. (10.71) becomes

$$\frac{d\sigma_{\rm sc}}{d\Omega}=\frac{2\pi}{3}\frac{k^4a^6}{1+r^2}\left|\left(\mathbf{X}_{1,1}-2i\mathbf{n}\times\mathbf{X}_{1,1}\right)+re^{i\alpha}\left(\mathbf{X}_{1,-1}+2i\mathbf{n}\times\mathbf{X}_{1,-1}\right)\right|^2.$$

By expanding the complex square, we have

$$\begin{aligned}&\left|\left(\mathbf{X}_{1,1}-2i\mathbf{n}\times\mathbf{X}_{1,1}\right)+re^{i\alpha}\left(\mathbf{X}_{1,-1}+2i\mathbf{n}\times\mathbf{X}_{1,-1}\right)\right|^2\\ =\quad&\left|\mathbf{X}_{1,1}-2i\mathbf{n}\times\mathbf{X}_{1,1}\right|^2+r^2\left|\mathbf{X}_{1,-1}+2i\mathbf{n}\times\mathbf{X}_{1,-1}\right|^2\\ &+2\mathrm{Re}\left[\left(\mathbf{X}_{1,1}-2i\mathbf{n}\times\mathbf{X}_{1,1}\right)\cdot re^{-i\alpha}\left(\mathbf{X}_{1,-1}+2i\mathbf{n}\times\mathbf{X}_{1,-1}\right)^*\right].\end{aligned}$$

Using the results between Eqs. (10.71) and (10.72), we know

$$\left|\mathbf{X}_{1,1}-2i\mathbf{n}\times\mathbf{X}_{1,1}\right|^2=\left|\mathbf{X}_{1,-1}+2i\mathbf{n}\times\mathbf{X}_{1,-1}\right|^2=\frac{15}{16\pi}(1+\cos^2\theta)-\frac{3}{2\pi}\cos\theta.$$

All we need to do now is to calculate the cross product term. To this end, we will need the explicit expression for $\mathbf{X}_{1,\pm1}$.

The defition of $\mathbf{X}_{lm}$ is

$$\mathbf{X}_{lm}=\frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm}=\frac{1}{\sqrt{l(l+1)}}\left(L_xY_{lm},L_yY_{lm},L_zY_{lm}\right)^\top.$$

Since $L_\pm=L_x\pm iL_y$, we know

$$L_x=\frac{1}{2}(L_++L_-),\quad L_x=\frac{1}{2i}(L_+-L_-).$$

Also,

$$L_\pm Y_{lm}=\sqrt{(l\mp m)(l\pm m+1)}Y_{l,m\pm1},$$

then

$$\mathbf{X}_{lm}=\frac{1}{\sqrt{l(l+1)}}\begin{pmatrix}\frac{1}{2}\left(\sqrt{(l-m)(l+m+1)}Y_{l,m+1}+\sqrt{(l+m)(l-m+1)}Y_{l,m-1}\right)\\ \frac{1}{2i}\left(\sqrt{(l-m)(l+m+1)}Y_{l,m+1}-\sqrt{(l+m)(l-m+1)}Y_{l,m-1}\right)\\ mY_{lm}\end{pmatrix}.$$

For $l=1$ and $m=1$, we have

$$\mathbf{X}_{1,1}=\left(\frac{1}{2}Y_{1,0},\frac{i}{2}Y_{1,0},\frac{1}{\sqrt{2}}Y_{1,1}\right)^\top,$$

and $l=1$ and $m=-1$,

$$\mathbf{X}_{1,-1}=\left(\frac{1}{2}Y_{1,0},-\frac{i}{2}Y_{1,0},-\frac{1}{\sqrt{2}}Y_{1,-1}\right)^\top.$$

From the identity $Y_{lm}^*=(-1)^mY_{l,-m}$, we know $Y_{1,1}^*=-Y_{l,-1}$, and $\mathbf{X}_{1,1}^*=\mathbf{X}_{1,-1}$. Here,

$$Y_{1,0}=\sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1,1}=-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}.$$

Now, we can calculate the cross product term as

$$\begin{aligned}
& \Big(\mathbf{X}_{1,1} - 2i\mathbf{n}\times\mathbf{X}_{1,1}\Big)\cdot\Big(\mathbf{X}_{1,-1} + 2i\mathbf{n}\times\mathbf{X}_{1,-1}\Big)^* \\
= \quad & \Big(\mathbf{X}_{1,1} - 2i\mathbf{n}\times\mathbf{X}_{1,1}\Big)\cdot\Big(\mathbf{X}_{1,1} - 2i\mathbf{n}\times\mathbf{X}_{1,1}\Big) \\
= \quad & \mathbf{X}_{1,1}\cdot\mathbf{X}_{1,1} - 4i\mathbf{X}_{1,1}\cdot(\mathbf{n}\times\mathbf{X}_{1,1}) - 4(\mathbf{n}\times\mathbf{X}_{1,1})\cdot(\mathbf{n}\times\mathbf{X}_{1,1}) \\
= \quad & -3\mathbf{X}_{1,1}\cdot\mathbf{X}_{1,1},
\end{aligned}$$

where we have used the fact that $\mathbf{X}_{1,1}\cdot(\mathbf{n}\times\mathbf{X}_{1,1}) = 0$ and $(\mathbf{n}\times\mathbf{X}_{1,1})\cdot(\mathbf{n}\times\mathbf{X}_{1,1}) = \mathbf{X}_{1,1}\cdot\mathbf{X}_{1,1}$. Finally,

$$\mathbf{X}_{1,1}\cdot\mathbf{X}_{1,1} = \frac{1}{2}Y_{1,1}^2 = \frac{3}{16\pi}\sin^2\theta e^{2i\phi},$$

and

$$\begin{aligned}
& 2\mathrm{Re}\left[\Big(\mathbf{X}_{1,1} - 2i\mathbf{n}\times\mathbf{X}_{1,1}\Big)\cdot re^{-i\alpha}\Big(\mathbf{X}_{1,-1} + 2i\mathbf{n}\times\mathbf{X}_{1,-1}\Big)^*\right] \\
= \quad & 2\mathrm{Re}\left[re^{-i\alpha}\cdot\frac{-9}{16\pi}\sin^2\theta e^{2i\phi}\right] \\
= \quad & -\frac{9}{8\pi}r\sin^2\theta\cos(2\phi-\alpha).
\end{aligned}$$

Putting everything together, we will have

$$\begin{aligned}
\frac{d\sigma_{\mathrm{sc}}}{d\Omega} &= \frac{2\pi}{3}\frac{k^4a^6}{1+r^2}\Bigg[\left(\frac{15}{16\pi}(1+\cos^2\theta) - \frac{3}{2\pi}\cos\theta\right) + r^2\left(\frac{15}{16\pi}(1+\cos^2\theta) - \frac{3}{2\pi}\cos\theta\right) \\
&\qquad - \frac{9}{8\pi}r\sin^2\theta\cos(2\phi-\alpha)\Bigg] \\
&= \frac{2\pi}{3}k^4a^6\left[\frac{15}{16\pi}(1+\cos^2\theta) - \frac{3}{2\pi}\cos\theta - \frac{9}{8\pi}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right] \\
&= k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right].
\end{aligned}$$

The same expression can also be obtained by applying the result from Prob. 10.1 (a). With $\mathbf{n}_0 = (0,0,1)$, $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $\epsilon_1 = (1,0,0)$, $\epsilon_2 = (0,1,0)$, $\epsilon_\pm = (\epsilon_1 \pm i\epsilon_2)/\sqrt{2} = (1,\pm i,0)/\sqrt{2}$, and

$$\epsilon = \frac{1}{\sqrt{1+r^2}}\left(\epsilon_+ + re^{i\alpha}\epsilon_-\right) = \frac{1}{\sqrt{2(1+r^2)}}(1+re^{i\alpha}, i(1-re^{i\alpha}), 0),$$

we can perform the vector products as

$$\mathbf{n}\cdot\epsilon = \frac{1}{\sqrt{2(1+r^2)}}\left((1+re^{i\alpha})\sin\theta\cos\phi + i(1-re^{i\alpha})\sin\theta\sin\phi\right),$$

$$|\mathbf{n}\cdot\epsilon|^2 = \frac{1}{2(1+r^2)}\Big((1+2r\cos\alpha+r^2)\sin^2\theta\cos^2\phi + (1-2r\cos\alpha+r^2)\sin^2\theta\sin^2\phi$$

$$
\begin{aligned}
&\qquad\qquad +4r\sin^2\theta\sin\phi\cos\phi\sin\alpha\Big) \\
&= \frac{1}{2(1+r^2)}\left((1+r^2)\sin^2\theta\cos^2\phi + 2r\sin^2\theta\cos(2\phi)\cos\alpha + 2r\sin^2\theta\sin(2\phi)\sin\alpha\right) \\
&= \frac{1}{2}\sin^2\theta + \frac{r}{1+r^2}\sin^2\theta\cos(2\phi-\alpha),
\end{aligned}
$$

and

$$
\mathbf{n}_0 \times \epsilon = \frac{1}{\sqrt{2(1+r^2)}}\left(-i(1-re^{i\alpha}), (1+re^{i\alpha}), 0\right),
$$

$$
\begin{aligned}
|\mathbf{n}\cdot(\mathbf{n_0}\times\epsilon)|^2 &= \frac{1}{2(1+r^2)}\Big((1-2r\cos\alpha+r^2)\sin^2\theta\cos^2\phi + (1+2r\cos\alpha+r^2)\sin^2\theta\sin^2\phi \\
&\qquad -4r\sin^2\theta\sin\phi\cos\phi\sin\alpha\Big) \\
&= \frac{1}{2(1+r^2)}\left((1+r^2)\sin^2\theta\cos^2\phi - 2r\sin^2\theta\cos(2\phi)\cos\alpha - 2r\sin^2\theta\sin(2\phi)\sin\alpha\right) \\
&= \frac{1}{2}\sin^2\theta - \frac{r}{1+r^2}\sin^2\theta\cos(2\phi-\alpha).
\end{aligned}
$$

Then, using Prob. 10.1 (a),

$$
\begin{aligned}
\frac{d\sigma_{\rm sc}}{d\Omega} &= k^4a^6\left[\frac{5}{4} - |\mathbf{n}\cdot\epsilon|^2 - \frac{1}{4}|\mathbf{n}\cdot(\mathbf{n_0}\times\epsilon)|^2 - \mathbf{n}\cdot\mathbf{n}_0\right] \\
&= k^4a^6\Bigg[\frac{5}{4} - \frac{1}{2}\sin^2\theta - \frac{r}{1+r^2}\sin^2\theta\cos(2\phi-\alpha) \\
&\qquad -\frac{1}{4}\left(\frac{1}{2}\sin^2\theta - \frac{r}{1+r^2}\sin^2\theta\cos(2\phi-\alpha)\right) - \cos\theta\Bigg] \\
&= k^4a^6\left[\frac{5}{4} - \frac{5}{8}\sin^2\theta - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right] \\
&= k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right].
\end{aligned}
$$