2.22 Solution: (a) Similar to Eq. (2.21), the potential inside the sphere is

$$\Phi(x,\theta,\phi) = \frac{Va(a^2-x^2)}{4\pi}\int_0^{2\pi} d\phi' \int_0^1 d(\cos\theta')\left[\frac{1}{(x^2+a^2-2ax\cos\gamma)^{3/2}} - \frac{1}{(x^2+a^2+2ax\cos\gamma)^{3/2}}\right],$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$. For the potential on the positive z-axis, $\theta = 0$, and $\cos\gamma = \cos\theta'$. Then, the potential can be shown to be

$$\begin{aligned}
\Phi(z) &= \frac{Va(a^2-z^2)}{2}\int_0^1 du\left[\frac{1}{(z^2+a^2-2azu)^{3/2}} - \frac{1}{(z^2+a^2+2azu)^{3/2}}\right] \\
&= \frac{V(a^2-z^2)}{2x}\left[\frac{1}{\sqrt{z^2+a^2-2azu}} + \frac{1}{\sqrt{z^2+a^2+2azu}}\right]\Bigg|_{u=0}^{1} \\
&= \frac{V(a^2-z^2)}{2x}\left[\frac{1}{|a-z|} - \frac{1}{\sqrt{a^2+z^2}} + \frac{1}{|a+z|} - \frac{1}{\sqrt{a^2+z^2}}\right] \\
&= \frac{V(a^2-z^2)}{2x}\left[\frac{2a}{a^2-z^2} - \frac{2}{\sqrt{a^2+z^2}}\right] \\
&= V\frac{a}{z}\left[1 - \frac{a^2-z^2}{a\sqrt{a^2+z^2}}\right],
\end{aligned}$$

since $z < a$.

For potential on the negative z-axis, $\theta = 0$, and $\cos\gamma = -\cos\theta'$, the result will be the negative of that on the positive z-axis. Therefore, for $|z| < a$,

$$\Phi(z) = V\frac{a}{|z|}\left[1 - \frac{a^2-z^2}{a\sqrt{a^2+z^2}}\right].$$

(b) For points on the z-axis, the radial direction in also along the z-axis. For Eq. (2.22), the potential on the positive z-axis outside of the sphere is

$$\Phi_{out}(z) = V\left[1 - \frac{z^2-a^2}{z\sqrt{z^2+a^2}}\right].$$

Then, it is straightforward (using Mathematica) to verify that the electric field is

$$E_{r,out}(z) = -\frac{\partial\Phi_{out}(z)}{\partial z} = \frac{Va^2}{(z^2+a^2)^{3/2}}\left(3 + \frac{a^2}{z^2}\right).$$

Similarly, for electric field inside of the sphere,

$$E_{r,in}(z) = -\frac{\partial\Phi_{in}(z)}{\partial z} = -\frac{V}{a}\left[\frac{3+(a/z)^2}{(1+(z/a)^2)^{3/2}} - \frac{a^2}{z^2}\right].$$

Using the MacLaurin series

$$(1+x)^{-3/2} = 1 - \frac{3}{2}x + \frac{15}{8}x^2 + O\left(x^3\right),$$

the leading order contribution for the radial electric field inside the sphere and near the origin is

$$E_{r,in}(z) = -\frac{V}{a}\left[\left(3 - \frac{a^2}{z^2}\right)\left(1 - \frac{3}{2}\frac{z^2}{a^2} + \frac{15}{8}\frac{z^4}{a^4} + O\left(z^6\right)\right) - \frac{a^2}{z^2}\right]$$

$$
\begin{aligned}
&= -\frac{V}{a}\left[\frac{a^2}{z^2}+\frac{3}{2}-\frac{21}{8}\frac{z^2}{a^2}-\frac{a^2}{z^2}\right] \\
&= -\frac{3V}{2a}+O\left(z^2\right),
\end{aligned}
$$

and, therefore, $E_{r,in}(0) = -3V/2a$.

For electric field near the north pole, no limiting process is required. Setting $z = a$ for the expressions of the electric field inside and outside of the sphere, it is easy to show,

$$E_{r,in}(a) = -\left(\sqrt{2}-1\right)\frac{V}{a},$$

and

$$E_{r,out}(a) = \sqrt{2}\frac{V}{a}.$$