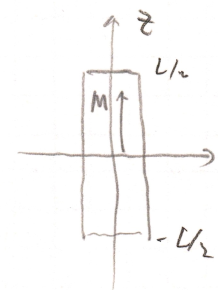
5.19 Since $\vec{M} = M\hat{z}$, which is constant inside the magnet, $\nabla\cdot\vec{M} = 0$.



The magnetic scalar potential becomes, in the cylindrical coordinates

$$\Phi_m = \frac{1}{4\pi}\oint \frac{\vec{M}\cdot\vec{n}}{|\vec{x}-\vec{x}'|}\,da'$$

$$= \frac{M}{4\pi}\int_0^{2\pi} d\phi' \int_0^a \rho' d\rho' \left[\frac{1}{[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-L/2)^2]^{1/2}} - \frac{1}{[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z+L/2)^2]^{1/2}} \right]$$

On the z-axis $\rho = 0$, the potential becomes

$$\Phi_m(z) = \frac{M}{2}\int_0^a \left[\frac{1}{[\rho'^2+(z-L/2)^2]^{1/2}} - \frac{1}{[\rho'^2+(z+L/2)^2]^{1/2}} \right] \rho' d\rho'$$

$$= \frac{M}{2}\left[\sqrt{\rho'^2+(z-L/2)^2} - \sqrt{\rho'^2+(z+L/2)^2} \right]\Bigg|_0^a$$

$$= \frac{M}{2}\left[\sqrt{a^2+(z-L/2)^2} - \sqrt{a^2+(z+L/2)^2} - \left|z-\frac{L}{2}\right| + \left|z+\frac{L}{2}\right| \right]$$

$$= \begin{cases} \frac{M}{2}\left(\sqrt{a^2+(z-L/2)^2} - \sqrt{a^2+(z+L/2)^2} + L \right), & z > L/2 \\ \frac{M}{2}\left(\sqrt{a^2+(z-L/2)^2} - \sqrt{a^2+(z+L/2)^2} - L \right), & z < -L/2 \\ \frac{M}{2}\left(\sqrt{a^2+(z-L/2)^2} - \sqrt{a^2+(z+L/2)^2} - 2z \right), & |z| < L/2 \end{cases}$$

The magnetic field then is given by

$$\vec{H}_{in} = -\nabla\Phi_m = \frac{M}{2}\left(\frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} - \frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - 2 \right)\hat{z}$$

$$\vec{H}_{out} = -\nabla\Phi_m = \frac{M}{2}\left(\frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} - \frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} \right)\hat{z}$$

The magnetic induction is

$$\vec{B}_{in} = \vec{B}_{out} = \frac{\mu_0 M}{2}\left(\frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}} - \frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} \right)\hat{z}$$

We can see that the normal component of the magnetic induction is continuous across the surface. Also, at the surface, the magnetic induction is given by

$$\frac{\mu_0 M}{2}\frac{L}{\sqrt{a^2+L^2}}$$, which will be useful later.