16.6 (a) $f(\vec{k}) = \frac{1}{e}\int \rho(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} d^3x = \frac{1}{4\pi a^2}\int \frac{e^{-r/a}}{r} e^{-i\vec{k}\cdot\vec{x}} d^3x.$

Using the well-known result $\int \frac{e^{-\alpha r}}{r} e^{-i\vec{k}\cdot\vec{x}} d^3x = \frac{4\pi}{k^2+\alpha^2}$, we then have

$$f(\vec{k}) = \frac{1}{a^2}\cdot\frac{1}{\frac{1}{a^2}+k^2} = \frac{1}{1+k^2a^2}.$$

(b) $$m = m_0 + \frac{e^2}{3\pi^2c^2}\int \frac{|f(\vec{k})|^2}{k^2} d^3k = m_0 + \frac{4e^2}{3\pi c^2}\int_0^{+\infty} \frac{dk}{[1+k^2a^2]^2}$$

Since $$\int_0^{+\infty}\frac{dk}{[1+k^2a^2]^2} \overset{k=\tan\theta/a}{=\!=\!=} \int_0^{\pi/2}\frac{1}{[1+\tan^2\theta]^2}\frac{d\theta}{a\cos^2\theta} = \int_0^{\pi/2}\frac{\cos^2\theta}{a}d\theta = \frac{\pi}{4a},$$

then $$m = m_0 + \frac{e^2}{3ac^2} = m_0 + \frac{mc\tau}{2a}$$

(c) $$M(\omega) = m + \frac{e^2\omega^2}{3\pi^2c^4}\int d^3k \frac{|f(\vec{k})|^2}{k^2[k^2-(\omega/c)^2]} = m + \frac{4e^2\omega^2}{3\pi c^4}\int_0^{+\infty}\frac{dk}{[1+k^2a^2]^2[k^2-(\omega/c)^2]}$$

The integral can be performed in Mathematica, $\int_0^{+\infty}\frac{dk}{(1+k^2a^2)^2[k^2-(\omega/c)^2]} = -\frac{\pi(a\omega/c+2i)}{4\omega/c\,[a\omega/c+i]^2}$

and $M(\omega) = m - \frac{e^2}{3c^3}\frac{\omega(a\omega/c+2i)}{(a\omega/c+i)^2} = m\left(1-\frac{\tau c}{2a}\frac{(a\omega/c)(a\omega/c+2i)}{(a\omega/c+i)^2}\right)$. The zero can be

determined by $\left(\frac{a\omega}{c}+i\right)^2 - \frac{\tau c}{2a}\frac{a\omega}{c}\left(\frac{a\omega}{c}+2i\right) = \left(1-\frac{\tau c}{2a}\right)\left(\frac{a\omega}{c}\right)^2 + 2i\left(1-\frac{\tau c}{2a}\right)\left(\frac{a\omega}{c}\right) - 1 = 0.$

Which leads to $\frac{a\omega}{c} = -i\left[1\pm\sqrt{1-(1-c\tau/2a)^{-1}}\right]$, or $\omega\tau = -i(c\tau/a)\left[1\pm\sqrt{1-(1-c\tau/2a)^{-1}}\right]$

(d) In terms of m_0, $M(\omega) = m_0 + m\frac{\tau c}{2a}\left(1-\frac{(a\omega/c)(a\omega/c+2i)}{(a\omega/c+i)^2}\right) = m_0 - \frac{m}{(a\omega/c+i)^2}$

For $m_0>0$, the zeros are located at $\omega\tau = \frac{c\tau}{a}\left[-i\pm\left(\frac{m}{m_0}\right)^{1/2}\right]$

For $m_0<0$, the zeros are at $\omega\tau = -i\frac{c\tau}{a}\left[1\pm\left(-\frac{m}{m_0}\right)^{1/2}\right]$