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# Hidden momentum of a relativistic fluid carrying current in an external electric field

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The parameter $\beta^2$ can be written as

$$\beta^2=2m_e(E-V_0)/\hbar^2=2m_eE'/\hbar^2\equiv\alpha'^2. \quad (16)$$

In this case the energy $E'=E-V_0>0$ must be measured from the bottom level. The resulting dispersion relation is given by

$$\cos(ak)=\cosh(\beta' b)\cos(\alpha' d) +[(\beta'^2+\alpha'^2)/2\alpha'\beta']\sinh(\beta' b)\sin(\alpha' d), \quad (17)$$

in formal agreement with Eq. (1).

In the range $E>0$, Eq. (4) will remain valid. Thus models I and II produce identical energy zone schemes for the same $b/a$ ratio, a situation which had not been encountered in the potential profile examples drawn in Fig. 1.

Contrary to the case of positive potential, for $V_0<0$ the $\delta$-potential limit describes a totally changed situation for the bound states with energies $E<0$. As can be realized, only one energy band appears, which comes from the broadening of the single eigenvalue of a $\delta$-potential well. This seems to be a further reason to avoid the Dirac comb approximation for electron states of crystals.

After this manuscript was completed, my attention was called to a recent book,[12] which contains interesting results related to those of the present paper.

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# Hidden momentum of a relativistic fluid carrying current in an external electric field


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Thirty years after the discovery of hidden momentum,[1] it still comes as a surprise to many physicists that a stationary body, i.e., a body the center of mass of which is at rest, can carry a nonvanishing mechanical linear momentum, called hidden momentum. A current-carrying nonconducting body, stationary in an external electric field $\mathbf{E}$, contains a hidden momentum $\mathbf{P}_h$, which, when the field $\mathbf{E}$ does not vary significantly over the dimensions of the body, can be expressed as[1,2]

$$\mathbf{P}_h=\frac{1}{c^2}\mathbf{m}\times\mathbf{E}, \quad (1)$$

where $\mathbf{m}$ is the magnetic dipole moment of the body. The presence of hidden momentum in a magnetic dipole must be taken into account when evaluating the force that the magnetic dipole experiences in an external electromagnetic field.[3,4] However, it can be shown easily on very general grounds that the total linear momentum of any finite stationary distribution of matter, charge and current must vanish.[5] This means that the hidden momentum $\mathbf{P}_h$ is compensated by the nonzero electromagnetic field momentum $\mathbf{P}_f$ of the static fields $\mathbf{E}$ and $\mathbf{B}$ in such a system:

$$\mathbf{P}_h=-\mathbf{P}_f=-\epsilon_0\int \mathbf{E}\times\mathbf{B}\ d^3r=-\frac{1}{c^2}\int \Phi\mathbf{J}d^3r, \quad (2)$$

where the last expression for the electromagnetic momentum $\mathbf{P}_f$ holds for the potential $\Phi$ of a localized charge distribution and a localized current density $\mathbf{J}$ such that $\mathbf{E}=-\nabla\Phi$ and $\nabla\times\mathbf{B}=\mu_0\mathbf{J}$.[6] The last expression in Eq. (2) leads easily to Eq. (1) for the hidden momentum $\mathbf{P}_h$.[7]

To verify by a direct calculation that the relation $\mathbf{P}_h=-\mathbf{P}_f$ of Eq. (2) indeed holds, one has to assume a specific model for the mechanism of the current transport in the current-carrying body. Schematically, there are three different models of electric current: (i) the current constituted by a gas of charged particles circulating in a nonconducting tube, (ii) the current carried by a charged incompressible liquid circulating in a nonconducting tube, and (iii) the current in a conductor.[8] It should be possible to account for any real current in a macroscopic body by a hybrid model that would utilize, if necessary, all the above three mechanisms to the required degree.[9] In connection with hidden momentum, only models (i) and (ii) are relevant, however, as the hidden momentum, together with the electromagnetic momentum, vanishes when the current-carrying body is a conductor; this can be seen immediately from the last expression in Eq. (2), since the potential $\Phi$ is constant in a conductor and $\int \mathbf{J} d^3r = 0$ for a localized divergenceless current $\mathbf{J}$.[10] Calculations in models (i) and (ii) show that the hidden momentum (2) arises as a relativistic effect in the motion of the current carriers;[8] in model (ii), however, the hidden momentum has been obtained only to first order in $v/c$ so far.[4] It is the purpose of the present note to show that models (i) and (ii) can be subsumed in a single model of a relativistic charged ideal fluid (i.e., gas or liquid), in which the hidden momentum $\mathbf{P}_h$ is obtained exactly, and easily, and that, in the limit of negligible compressibility [i.e., in model (ii)], $\mathbf{P}_h$ can be calculated to all orders in $v/c$ using the gradient of the fluid's pressure.

The essential features of the model under consideration are as follows. A charged ideal[11] fluid flows in a stationary manner in the laboratory frame of reference, where the center of mass of the system is at rest. The fluid flow is constrained in a closed finite loop by means of a tube. The tube is electrically nonconducting, and so it does not shield off an external electric field $\mathbf{E}$, which is produced by a stationary distribution of charge localized outside the system of the fluid and tube. The field effects of the charge distribution due to the charged fluid itself are minimized by intertwining the tube with a twin nonconducting tube carrying an oppositely charged fluid flowing in the opposite direction to that of the first fluid. When the fluids are easily compressible, as in model (i) above, the external electric field will induce an overall nonzero charge density along the two tubes, as the oppositely charged fluids will move with different speeds and hence have different densities at a given point in the external field. In such a case, the electric field due to the current carriers cannot be reduced arbitrarily, unlike in model (ii) where the fluids have a negligible compressibility.[12] Nevertheless, it is assumed that such ''space-charge'' effects can be neglected so that, to a good approximation, the only electric field in the system is the external field $\mathbf{E}$. The velocity $\mathbf{v}$ of the fluid flow at any point in the laboratory frame is not assumed to be restricted to values small compared to the velocity of light, and so the model will be treated using the formalism of relativistic fluid mechanics.[13]

In the calculation that follows, it is sufficient to consider only one of the two oppositely charged fluids in the system. The components $P_i$ of the fluid momentum $\mathbf{P}$ are given in terms of the momentum density $\mathbf{g}$ as

$$P_i = \int g_i d^3r = \int \nabla \cdot (x_i \mathbf{g}) d^3r - \int x_i \nabla \cdot \mathbf{g} d^3r = -\int x_i \nabla \cdot \mathbf{g} d^3r, \tag{3}$$

where the integral over $\nabla \cdot (x_i \mathbf{g})$ is zero as it can be transformed into a surface integral of an integrand that vanishes outside the tube. In relativistic fluid mechanics, the momentum density $\mathbf{g}$ is given by the elements $T^{0i}/c$, $i=1,2,3$ of the energy-momentum four-tensor $T^{\mu\nu}$ of the fluid,[14] which satisfies the equation of motion

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = f^\mu, \tag{4}$$

where $f^\mu = (\mathbf{j}\cdot\mathbf{E}/c, \rho\mathbf{E})$ is the four-vector of the force density arising from the external electric field $\mathbf{E}$ acting on the charge density $\rho$ and current density $\mathbf{j}$ of the fluid. The time ($\mu=0$) component of Eq. (4) gives

$$\frac{\partial u}{\partial t} + \nabla \cdot (c^2 \mathbf{g}) = \mathbf{j}\cdot\mathbf{E}, \tag{5}$$

where $u = T^{00}$ is the energy density of the fluid. In a stationary flow, $\partial u/\partial t = 0$, and thus the fluid momentum (3) is calculated as

$$\mathbf{P} = -\frac{1}{c^2}\int \mathbf{r}(\mathbf{j}\cdot\mathbf{E}) d^3r = \frac{1}{c^2}\int \mathbf{r}(\mathbf{j}\cdot\nabla\Phi) d^3r = -\frac{1}{c^2}\int \Phi \mathbf{j} d^3r. \tag{6}$$

Here, a potential $\Phi$ such that $\mathbf{E} = -\nabla\Phi$ was introduced, and the integral identity

$$\int q\mathbf{h} d^3r = -\int \mathbf{r}(\mathbf{h}\cdot\nabla q) d^3r, \tag{7}$$

which holds for any well-behaved $q$ and localized divergenceless $\mathbf{h}$,[15] used. The current density $\mathbf{j}$ of one of the two fluids contributes a half of the net current density $\mathbf{J} = 2\mathbf{j}$, and so the total mechanical momentum $\mathbf{P}_h$ of the two fluids is finally

$$\mathbf{P}_h = 2\mathbf{P} = -\frac{1}{c^2}\int \Phi \mathbf{J} d^3r. \tag{8}$$

This is the same expression for the hidden momentum $\mathbf{P}_h$ as that given in Eq. (2).

The above calculation is general and simple, as it obtains hidden momentum directly from the connection of Eq. (5) between energy flux density and momentum density demanded by special relativity, where the former quantity equals $c^2$ times the latter one, but it does not provide an insight into the physical origin of the hidden momentum of the fluid. When the fluid is a gas of charged particles, hidden momentum arises from the relativistic variation of the mass of the particles with their speed, while in a charged liquid of low compressibility, hidden momentum originates in the relativistic properties of the stress, or pressure, that is induced in the liquid by the external electric field.[4]

In the limit of negligible compressibility, the hidden momentum of a fluid flowing in a tube of constant cross section that is shaped into a rectangular loop can be calculated using its pressure $p$ as follows. The relativistic momentum density $\mathbf{g}$ of the fluid is given by[14]

$$\mathbf{g} = \frac{\gamma^2}{c^2}(u_0 + p)\mathbf{v}, \tag{9}$$

where $\mathbf{v}$ is the velocity of the fluid flow in the laboratory frame, $\gamma = (1 - |\mathbf{v}|^2/c^2)^{-1/2}$, and $u_0$ is the fluid's local rest-frame energy density. As the tube has a constant cross section, the speed $|\mathbf{v}|$ of the flow of the incompressible fluid is constant.[16] In the local rest frame, Euler's equation[17] yields for the pressure $p$, which is an invariant scalar in an ideal fluid,[18] the relation $\nabla' p = \mathbf{f}'$,[19] where $\mathbf{f}' = \rho' \mathbf{E}'$ is the external force density acting on the fluid in the rest frame (the rest-frame variables and quantities are denoted by primes). To obtain the laboratory-frame gradient of $p$ along the direction of $\mathbf{v}$, the force density's component $f'_{\parallel}$ in the direction parallel to $\mathbf{v}$ is expressed in terms of the laboratory-frame quantities, $f'_{\parallel} = \gamma[\rho E_{\parallel} - (\mathbf{j}\cdot\mathbf{E})v/c^2] = \gamma\rho(1 - v^2/c^2)E_{\parallel} = \rho E_{\parallel}/\gamma$. Projecting the equation $\nabla' p = \mathbf{f}'$ onto the direction of $\mathbf{v}$, we then get

$$\nabla'_{\parallel} p = \gamma \nabla_{\parallel} p = \frac{1}{\gamma}\rho E_{\parallel} = -\frac{1}{\gamma}\rho \nabla_{\parallel}\Phi, \tag{10}$$

where the derivative operator is Lorentz-transformed as $\nabla'_{\parallel} = \gamma[\nabla_{\parallel} + (v/c^2)\partial/\partial t]$ and the fact that $\partial p/\partial t = 0$ in the stationary flow of the fluid utilized. Multiplying Eq. (10) by $v/\gamma$, it can be written simply as

$$\mathbf{v}\cdot\nabla p = -\frac{1}{\gamma^2}\mathbf{j}\cdot\nabla\Phi, \tag{11}$$

since $v\nabla_{\parallel} = \mathbf{v}\cdot\nabla$ and $\rho\mathbf{v} = \mathbf{j}$. As the charge density $\rho$, rest-frame energy density $u_0$ and Lorentz factor $\gamma$ are all constant in the incompressible fluid flowing with a constant speed, $\nabla\cdot(\gamma^2 u_0 \mathbf{v}) = (\gamma^2 u_0/\rho)\nabla\cdot\mathbf{j} = 0$ and so only the second term in the momentum density (9) can contribute to the fluid momentum. Using Eq. (11) and the integral identity (7), the fluid momentum $\mathbf{P}$ is then obtained, on integrating the momentum density (9), as

$$\mathbf{P} = \frac{\gamma^2}{c^2}\int p\mathbf{v}d^3r = -\frac{\gamma^2}{c^2}\int \mathbf{r}(\mathbf{v}\cdot\nabla p)d^3r$$

$$= \frac{1}{c^2}\int \mathbf{r}(\mathbf{j}\cdot\nabla\Phi)d^3r = -\frac{1}{c^2}\int \Phi\mathbf{j}d^3r, \tag{12}$$

which agrees fully with the last expression of Eq. (6).


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[6]See, for example, M. G. Calkin, Ref. 5.

[7]See, for example, L. Vaidman, Ref. 4.

[8]Aharanov, Pearle and Vaidman, Ref. 5; L. Vaidman, Ref. 4.

[9]This includes the current realized by two counter-rotating oppositely charged solid disks of Shockley and James (Ref. 1), as the essential features of this mechanism can be covered in model (ii) by two counter-circulating oppositely charged liquids.

[10]As a result of the net force between the current induced in the conducting body by an external electric field varying slowly with time and the current that is responsible for the body's dipole moment, the net force on the magnetic dipole in a conducting body is the same as if the body contained the hidden momentum of Eq. (1), see V. Hnizdo, "Comment on 'Torque and force on a magnetic dipole,' " Am. J. Phys. **60**, 279–280 (1992).

[11]An ideal fluid has a negligible thermal conductivity and viscosity; it also does not support any shear stresses.

[12]The author is indebted to E. Comay of Tel Aviv University for bringing this point to his attention.

[13]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1987), 2nd ed., Secs. 133 and 134.

[14]L. D. Landau and E. M. Lifshitz, Ref. 13, Sec. 133.

[15]This can be proved separately for each component, using Eq. (3) with $g_i = qh_i$ and the fact that $\nabla\cdot(q\mathbf{h}) = \mathbf{h}\cdot\nabla q$ for a divergenceless $\mathbf{h}$.

[16]Except in the short bends of the rectangularly shaped loop; such regions of nonuniform speed can be made negligibly small compared to the straight sections of the tube.

[17]L. D. Landau and E. M. Lifshitz, Ref. 13, Sec. 2.

[18]C. Møller, *The Theory of Relativity* (Clarendon, Oxford, 1972), 2nd ed., pp. 191–192.

[19]Assuming not only that $\mathbf{v}' = 0$ but also that $\partial\mathbf{v}'/\partial t' = 0$, which holds in the local rest frame except at the bends of the tube.


## VACATIONS ARE FOR RESEARCH

I am now engaged during the college vacation in repeating some of my older experiments and in preparing for the press the results I have obtained on the subject of induction during the four or five years past. My college duties are such that I can do nothing in the way of investigation during the term and at the end of the vacation I often find my self in the midst of an interesting course of experiments which I am obliged to put aside and before I can return to them my mind has become occupied with other objects. In this way I have gone on accumulating almost a volume of new results which will require much time and labour to prepare for publication. Indeed I find so much pleasure in the prosecution of these researches that the publication of them in comparison becomes a task. I have become careless of reputation and have suffered a number of my results to be rediscovered abroad merely from the reluctance I feel to the trouble of preparing them for the press.

Joseph Henry, letter to Benjamin Peirce (1845), in *The Papers of Joseph Henry*, edited by Marc Rothenberg (Smithsonian Institution Press, Washington, 1992), Vol. 6, pp. 358–359.