8.21 (a) For derivatives with respect to t, we have

$$\frac{\partial \psi}{\partial t} = -\frac{u(s)}{2h(s,t)^{3/2}}\frac{\partial h}{\partial t}\sin\left(\frac{\pi t}{w(s)}\right) + \frac{u(s)}{h(s,t)^{1/2}}\frac{\pi}{w(s)}\cos\left(\frac{\pi t}{w(s)}\right),$$

$$h\frac{\partial \psi}{\partial t} = -\frac{u(s)}{2h(s,t)^{1/2}}\frac{\partial h}{\partial t}\sin\left(\frac{\pi t}{w(s)}\right) + u(s)h(s,t)^{1/2}\frac{\pi}{w(s)}\cos\left(\frac{\pi t}{w(s)}\right),$$

$$\frac{\partial}{\partial t}\left(h\frac{\partial \psi}{\partial t}\right) = \frac{u(s)}{4h(s,t)^{3/2}}\left(\frac{\partial h}{\partial t}\right)^2\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u(s)}{2h(s,t)^{1/2}}\frac{\partial^2 h}{\partial t^2}\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u(s)}{2h(s,t)^{1/2}}\frac{\partial h}{\partial t}\frac{\pi}{w(s)}\cos\left(\frac{\pi t}{w(s)}\right)$$

$$+ \frac{u(s)}{2h(s,t)^{1/2}}\frac{\partial h}{\partial t}\frac{\pi}{w(s)}\cos\left(\frac{\pi t}{w(s)}\right) - u(s)h(s,t)^{1/2}\left(\frac{\pi}{w(s)}\right)^2\sin\left(\frac{\pi t}{w(s)}\right)$$

$$= \frac{u(s)}{4h(s,t)^{3/2}}\left(\frac{\partial h}{\partial t}\right)^2\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u(s)}{2h(s,t)^{1/2}}\frac{\partial^2 h}{\partial t^2}\sin\left(\frac{\pi t}{w(s)}\right) - u(s)h(s,t)^{1/2}\left(\frac{\pi}{w(s)}\right)^2\sin\left(\frac{\pi t}{w(s)}\right).$$

and finally,

$$\frac{1}{h}\frac{\partial}{\partial t}\left(h\frac{\partial \psi}{\partial t}\right) = \frac{u(s)}{4h(s,t)^{5/2}}\left(\frac{\partial h}{\partial t}\right)^2\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u(s)}{2h(s,t)^{3/2}}\frac{\partial^2 h}{\partial t^2}\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u(s)}{h(s,t)^{1/2}}\left(\frac{\pi}{w(s)}\right)^2\sin\left(\frac{\pi t}{w(s)}\right).$$

Since $h(s,t) = 1 - k(s)t$, then $\frac{\partial h}{\partial t} = -k(s)$, $\frac{\partial^2 h}{\partial t^2} = 0$. Also notice that $k(s) \simeq 0$, $w(s) \simeq a$, and their derivative can be dropped. Therefore,

$$\frac{1}{h}\frac{\partial}{\partial t}\left(h\frac{\partial \psi}{\partial t}\right) = \frac{1}{4}k(s)^2 u(s)\sin\left(\frac{\pi t}{w(s)}\right) - u(s)\left(\frac{\pi}{w(s)}\right)^2\sin\left(\frac{\pi t}{w(s)}\right).$$

Similarly, for derivation w.r.t. to s, we have

$$\frac{1}{h}\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial \psi}{\partial s}\right) = \frac{u''(s)}{h(s,t)^{5/2}}\sin\left(\frac{\pi t}{w(s)}\right) + O\left(\frac{dw}{ds}, \frac{dk}{ds}\right),$$

where we have dropped terms proportional to the s-derivative of $w(s)$ and $k(s)$. The two-dimensional wave equation now becomes

$$\frac{1}{h}\frac{\partial}{\partial t}\left(h\frac{\partial \psi}{\partial t}\right) + \frac{1}{h}\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial \psi}{\partial s}\right) + \left[k^2 + \left(\frac{\pi}{a}\right)^2\right]\psi = 0,$$

or $$u''(s)\sin\left(\frac{\pi t}{w(s)}\right) + \frac{1}{4}k(s)^2 u(s)\sin\left(\frac{\pi t}{w(s)}\right) - \left(\frac{\pi}{w(s)}\right)^2 u(s)\sin\left(\frac{\pi t}{w(s)}\right) + \left[k^2 + \left(\frac{\pi}{a}\right)^2\right]u(s)\sin\left(\frac{\pi t}{w(s)}\right) = 0,$$

which leads to $$\frac{d^2u}{ds^2} + \left[k^2 - V(s)\right]u = 0$$ with $V(s) = \pi^2\left[\frac{1}{w(s)^2} - \frac{1}{a^2}\right] - \frac{1}{4}k(s)^2$.

(b) Write the equation in the form of

$$-\frac{d^2 u}{ds^2} + V(s)u = k^2 u,$$

and notice that

$$V(s) = \begin{cases} 0, & |s| > R\theta/2, \\ -\frac{1}{4R^2}, & |s| \le R\theta/2, \end{cases}$$

then the problem reduces to the finite depth potential well problem in quantum mechanics.

For the bound state, outside of the potential, the solution decays exponentially, as $k^2 < 0$. Let $k^2 = -q_0^2$, $q_0 > 0$, and for $s > 0$, we have $u(s) \propto e^{-q_0 s}$. Inside the potential, there is always an even-parity solution, $u(s) \propto \cos(q_1 s)$, where $q_1 = \left(\frac{1}{4R^2} + k^2\right)^{1/2}$ $= \left(\frac{1}{4R^2} - q_0^2\right)^{1/2}$. Using the continuity condition at $s = R\theta/2$, we have

$$\left.\frac{u'}{u}\right|_{s+} = \left.\frac{u'}{u}\right|_{s-}, \quad \text{or} \quad q_0 = q_1 \tan(q_1 R\theta/2).$$

Since $\tan x \approx x$ for $x \ll 1$, we can write the eigenequation as

$$q_1^2 = \frac{2q_0}{R\theta}, \quad \text{or} \quad \frac{1}{4R^2} - q_0^2 = \frac{2q_0}{R\theta}.$$

which gives $q_0 = \frac{1}{2}\left[\left(\frac{4}{R^2\theta^2} + \frac{1}{R^2}\right)^{1/2} - \frac{2}{R\theta}\right] = \frac{1}{R\theta}\left[\left(1 + \frac{\theta^2}{4}\right)^{1/2} - 1\right] = \frac{\theta}{8R}$

Therefore, the bound state energy is

$$W_0^2 = \left(\frac{\pi c}{a}\right)^2 + k^2 c^2 = \left(\frac{\pi c}{a}\right)^2 - q_0^2 c^2 = \left(\frac{\pi c}{a}\right)^2 - \left(\frac{\theta}{8R}\right)^2 c^2$$

$$= \left(\frac{\pi c}{a}\right)^2\left[1 - \left(\frac{\theta a}{8\pi R}\right)^2\right]$$

8.21. (a) From the ansatz for the solution, we have

$$\frac{\partial\psi}{\partial t} = -\frac{u(s)}{2h(s,t)^{3/2}}\frac{\partial h}{\partial t}\sin\left(\frac{\pi t}{w(s)}\right) + \frac{u(s)}{h(s,t)^{1/2}}\frac{\pi}{w(s)}\cos\left(\frac{\pi t}{w(s)}\right),$$

$$h\frac{\partial\psi}{\partial t} = -\frac{u(s)}{2h(s,t)^{1/2}}\frac{\partial h}{\partial t}\sin\left(\frac{\pi t}{w(s)}\right) + u(s)\,h(s,t)^{1/2}\frac{\pi}{w(s)}\cos\left(\frac{\pi t}{w(s)}\right).$$

$$\frac{\partial}{\partial t}\left(h\frac{\partial\psi}{\partial t}\right) = \frac{u(s)}{4h(s,t)^{3/2}}\left(\frac{\partial h}{\partial t}\right)^2\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u(s)}{2h(s,t)^{1/2}}\frac{\partial^2 h}{\partial t^2}\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u(s)}{2h(s,t)^{1/2}}\frac{\partial h}{\partial t}\frac{\pi}{w(s)}\cos\left(\frac{\pi t}{w(s)}\right)$$

$$+\frac{u(s)}{2h(s,t)^{1/2}}\frac{\partial h}{\partial t}\frac{\pi}{w(s)}\cos\left(\frac{\pi t}{w(s)}\right) - u(s)\,h(s,t)^{1/2}\left(\frac{\pi}{w(s)}\right)^2\sin\left(\frac{\pi t}{w(s)}\right).$$

$$= \frac{u(s)}{4h(s,t)^{3/2}}\left(\frac{\partial h}{\partial t}\right)^2\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u(s)}{2h(s,t)^{1/2}}\frac{\partial^2 h}{\partial t^2}\sin\left(\frac{\pi t}{w(s)}\right) - u(s)\,h(s,t)^{1/2}\left(\frac{\pi}{w(s)}\right)^2\sin\left(\frac{\pi t}{w(s)}\right)$$

$$\frac{1}{h}\frac{\partial}{\partial t}\left(h\frac{\partial\psi}{\partial t}\right) = \frac{u(s)}{4h(s,t)^{5/2}}\left(\frac{\partial h}{\partial t}\right)^2\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u(s)}{2h(s,t)^{3/2}}\frac{\partial^2 h}{\partial t^2}\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u(s)}{h(s,t)^{1/2}}\left(\frac{\pi}{w(s)}\right)^2\sin\left(\frac{\pi t}{w(s)}\right),$$

and
$$\frac{\partial\psi}{\partial s} = \frac{u'(s)}{h(s,t)^{1/2}}\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u(s)}{2h(s,t)^{3/2}}\frac{\partial h}{\partial s}\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u(s)}{h(s,t)^{1/2}}\frac{\pi t}{w(s)^2}\frac{dw}{ds}\cos\left(\frac{\pi t}{w(s)}\right)$$

$$\frac{1}{h}\frac{\partial\psi}{\partial s} = \frac{u'(s)}{h(s,t)^{3/2}}\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u(s)}{2h(s,t)^{5/2}}\frac{\partial h}{\partial s}\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u(s)}{h(s,t)^{3/2}}\frac{\pi t}{w(s)^2}\frac{dw}{ds}\cos\left(\frac{\pi t}{w(s)}\right)$$

$$\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial\psi}{\partial s}\right) = \frac{u''(s)}{h(s,t)^{3/2}}\sin\left(\frac{\pi t}{w(s)}\right) - \frac{3u'(s)}{2h(s,t)^{5/2}}\frac{\partial h}{\partial s}\sin\left(\frac{\pi t}{w(s)}\right) - \frac{u'(s)}{h(s,t)^{3/2}}\frac{\pi t}{w(s)^2}\frac{dw}{ds}\cos\left(\frac{\pi t}{w(s)}\right)$$

$$-\frac{u'(s)}{2h(s,t)^{5/2}}\frac{\partial h}{\partial s}\sin\left(\frac{\pi t}{w(s)}\right) + \frac{5u(s)}{4h(s,t)^{7/2}}\left(\frac{\partial h}{\partial s}\right)^2\sin\left(\frac{\pi t}{w(s)}\right)$$

$$-\frac{u(s)}{2h(s,t)^{5/2}}\frac{\partial^2 h}{\partial s^2}\sin\left(\frac{\pi t}{w(s)}\right) + \frac{u(s)}{2h(s,t)^{5/2}}\frac{\partial h}{\partial s}\frac{\pi t}{w(s)^2}\frac{dw}{ds}\cos\left(\frac{\pi t}{w(s)}\right)$$

$$-\frac{u'(s)}{h(s,t)^{3/2}}\frac{\pi t}{w(s)^2}\frac{dw}{ds}\cos\left(\frac{\pi t}{w(s)}\right) + \frac{3u(s)}{2h(s,t)^{5/2}}\frac{\partial h}{\partial s}\frac{\pi t}{w(s)^2}\frac{dw}{ds}\cos\left(\frac{\pi t}{w(s)}\right) + \frac{u'(s)}{h(s,t)^{3/2}}\frac{2\pi t}{w(s)^3}\left(\frac{dw}{ds}\right)^2\cos\left(\frac{\pi t}{w(s)}\right)$$

$$-\frac{u'(s)}{h(s,t)^{3/2}}\frac{\pi t}{w(s)^2}\frac{d^2w}{ds^2}\cos\left(\frac{\pi t}{w(s)}\right) - \frac{u'(s)}{h(s,t)^{3/2}}\left(\frac{\pi t}{w(s)^2}\right)^2\left(\frac{dw}{ds}\right)^2\sin\left(\frac{\pi t}{w(s)}\right)$$