3.27 Solution: (a) Applying Eq. (1.46), the potential between the spheres is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi}\oint_S \frac{\partial \Phi}{\partial n'} G_N(\mathbf{x}, \mathbf{x}')da'.$$

Since the normal derivative of the potential at the surface provides the electric field there, the above equation can be expressed with the electric field,

$$\Phi(\mathbf{x}) = -\frac{1}{4\pi}\oint_S E_r(b)G_N(\mathbf{x}, \mathbf{x}')da'_{out} = \frac{E_0 b^2}{4\pi}\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \;\; \cos\theta' G_N(\mathbf{x}, \mathbf{x}'),$$

where we only integrate on the outer sphere, as the electric field vanishes on the inner surface. The Green function has an expansin in spherical harmonics,

$$G(\mathbf{x}, \mathbf{x}') = \sum_{l=0}^{\infty} g_l(r, r')P_l(\cos\gamma) = G(\mathbf{x}, \mathbf{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_l(r, r')\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)P_l^m(\cos\theta')e^{im(\phi-\phi')}.$$

Integrating with respect to ϕ' leaves only the $m = 0$ term,

$$\Phi(\mathbf{x}) = \frac{E_0 b^2}{2}\sum_{l=0}^{\infty} g_l(r, b)P_l(\cos\theta)\int_0^{\pi} \sin\theta' \cos\theta' P_l(\cos\theta')d\theta',$$

which further leaves only the $l = 1$ term,

$$\Phi(\mathbf{x}) = \frac{E_0 b^2}{3} g_1(r, b)\cos\theta,$$

since

$$\int_0^{\pi} \sin\theta' \cos\theta' P_l(\cos\theta')d\theta' = \int_{-1}^{1} P_1(x)^2 dx = \frac{2}{3}.$$

Evaluating $g_1(r, r')$ at $r' = b$,

$$\begin{aligned} g_1(r, b) &= \frac{r}{b^2} + \frac{1}{b^3 - a^3}\left(2br + \frac{a^3 b}{2r^2} + a^3\left(\frac{b}{r^2} + \frac{r}{b^2}\right)\right) \\ &= \frac{3br}{b^3 - a^3}\left(1 + \frac{a^3}{2r^3}\right). \end{aligned}$$

Thus, the potential becomes

$$\Phi(\mathbf{x}) = E_0\frac{3b^3 r\cos\theta}{3(b^3 - a^3)}\left(1 + \frac{a^3}{2r^3}\right) = E_0\frac{r\cos\theta}{1 - p^3}\left(1 + \frac{a^3}{2r^3}\right),$$

with $p = a/b$.

Given the potential, the electric field can be directly calculated in the spherical coordinates,

$$E_r(r, \theta) = -\frac{\partial \Phi}{\partial r} = -E_0\frac{\cos\theta}{1 - p^3}\frac{\partial}{\partial r}\left(r + \frac{a^3}{2r^2}\right) = -E_0\frac{\cos\theta}{1 - p^3}\left(1 - \frac{a^3}{r^3}\right),$$

and

$$E_\theta(r, \theta) = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = E_0\frac{\sin\theta}{1 - p^3}\left(1 + \frac{a^3}{2r^3}\right).$$

(b) In the Cartesian coordinates, the potential is

$$\Phi(\mathbf{x}) = E_0 \frac{z}{1-p^3}\left(1 + \frac{a^3}{2(x^2+y^2+z^2)^{3/2}}\right).$$

In the cylindrical coordinates, the potential is

$$\Phi(\mathbf{x}) = E_0 \frac{z}{1-p^3}\left(1 + \frac{a^3}{2(\rho^2+z^2)^{3/2}}\right).$$

The corresponding electric field in these coordinates can be found by simple differentiation, which will not be further pursued here.