5.30 (a) Consider the magnetic scalar potential, Φ_m, where $\vec{H} = -\nabla\Phi_m$. By symmetry, the scalar potential must have the form,

$$\Phi_m = \sum_{m=1}^{\infty} a_m \rho^{m} \sin(m\phi), \qquad \rho < R$$

$$\Phi_m = \sum_{m=1}^{\infty} b_m \rho^{-m} \sin(m\phi), \qquad \rho > R.$$

By the continuity condition

$$\left.\frac{\partial \Phi_m}{\partial \rho}\right|_{\rho=R-} = \left.\frac{\partial \Phi_m}{\partial \rho}\right|_{\rho=R+}, \qquad \left.\frac{\partial \Phi_m}{\partial \phi}\right|_{\rho=R-} - \left.\frac{\partial \Phi_m}{\partial \phi}\right|_{\rho=R+} = K(\phi)$$

It is clear that only $m=1$ term will survive,

$$-a_1 = b_1/R^2. \qquad -a_1 + \frac{b_1}{R^2} = I/2R$$

which leads to

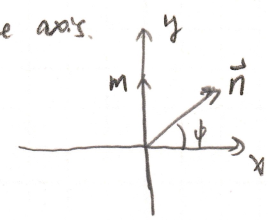
$$a_1 = -\frac{I}{4R}, \quad b_1 = \frac{IR}{4}$$

Then, inside the cylinder,

$$\vec{B} = -\mu_0 \nabla\Phi_m = \mu_0 \nabla\left(\frac{I}{4R}\rho \sin\phi\right) = \frac{\mu_0 I}{4R}\left(\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi}\right)\rho\sin\phi$$

$$= \frac{\mu_0 I}{4R}\left(\hat{\rho}\sin\phi + \hat{\phi}\cos\phi\right) = \frac{\mu_0 I}{4R}\hat{y},$$

where we have used the fact that $\hat{y} = \hat{\rho}\sin\phi + \hat{\phi}\cos\phi$. Therefore, the magnetic induction has a uniform intensity of $B_0 = \mu_0 I/4R$ and perpendicular to the axis.



Outside the cylinder.

$$\vec{B} = -\mu_0 \nabla\Phi_m = -\frac{\mu_0 IR}{4}\left(\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi}\right)\frac{\sin\phi}{\rho}$$

$$= \frac{\mu_0 IR}{4}\left(\frac{\sin\phi}{\rho^2}\hat{\rho} - \frac{\cos\phi}{\rho^2}\hat{\phi}\right) = \frac{\mu_0 IR}{4\rho^2}\left(\hat{x}\cdot 2\sin\phi\cos\phi + \hat{y}(\sin^2\phi - \cos^2\phi)\right)$$

Let $\vec{m} = \dfrac{\mu_0 IR}{4}\hat{y}$, then $\vec{B}$ can be written in the following form

$$\vec{B} = \frac{m}{\rho^2}\left[2\vec{n}(\vec{n}\cdot\vec{m}) - \vec{m}\right].$$

which has the dipole form.

(b) The magnetic energy per unit length is

$$W = \frac{1}{2\mu_0}\int |\vec{B}|^2 d^3x = \frac{1}{2\mu_0}\int_0^{2\pi} d\phi\left(\int_0^R \frac{\mu_0^2 I^2}{16R^2}\rho\, d\rho + \int_R^{+\infty}\frac{\mu_0^2 I^2 R^2}{16}\frac{\rho}{\rho^4}d\rho\right)$$

$$= \frac{\mu_0 \pi I^2}{32} + \frac{\mu_0 \pi I^2}{32} = \frac{\mu_0 \pi I^2}{16}$$

The energy inside and outside the cylinder are the same

(c) The total current in one direction is

$$I_t = \int_{-\pi/2}^{\pi/2} \frac{I \cos\phi}{2R} \cdot R\, d\phi = I$$

Then, $W = \frac{1}{2} L I_t^2$. and $L = \mu_0 \pi / 8$.