

## Consider a spherical battery...

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## Consider a spherical battery...

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We introduce a simplified model of a "spherical battery" embedded within an infinite conducting medium, wherein the source of emf is taken to be a uniform vertical force per unit charge  $F$ , of nonelectrical nature. This model can be solved analytically for the surface charge distribution, the electric field, the electric current density, the magnetic field, the Poynting vector, and the angular momentum density. Through the net current and net power loss, a "lumped" resistance can be defined, and from the net current and the lumped resistance a lumped emf can be defined. This example explicitly illustrates that the surface charge on a battery causes it, at a distance, to behave like an electric dipole. It also provides an explicit example where the nonelectrical "field" of the battery acts only within the battery, whereas the electrostatic field of the surface charge acts both within and outside of the battery. A spherical inclusion within a material with otherwise uniform current flow is also analyzed to obtain the electric field, current density, and magnetic field. Additional geometries with current flow and surface charge, both with and without batteries, for which there exist analytic but sometimes complex solutions, are discussed. Following a discussion of voltaic cells, it is pointed out that in real voltaic cells the emf occurs at the electrode–electrolyte interface rather than throughout the volume. From that we propose a more realistic but less tractable model for a spherical battery, in which the emf acts only within two hemispherical shells.

## I. INTRODUCTION

One rarely finds, in the literature on electromagnetism, a detailed description of the electric current flow associated

with voltaic cells, commonly known as batteries.<sup>1</sup> At the simplest level, it is clear that, within a voltaic cell, the chemical emf acts as a pump for electric charge. At a somewhat more advanced level, it is understood that electric charge on the

surface of conducting wires is responsible, in part, for producing an electrostatic field that drives the current. Nevertheless, the author is not aware of any examples where one determines, analytically, both the surface charge responsible for a given current flow and the electric fields, both inside and outside the voltaic cell, produced by that surface charge. The present work considers a mathematically nontrivial model, a “spherical battery,” for which this is done. The reader should be warned, however, that the model we take, which may be thought of as employing a volume pump, is not a realistic way to model the emf of real voltaic cells, which should be thought of as employing a surface pump associated with each electrode–electrolyte interface at which the chemical reactions actually occur.<sup>2</sup>

The particular advantage of this example is that it employs the familiar spherical geometry, and requires little mathematics beyond the determination of the electric field for a cosine distribution of charge on the surface of a sphere. (This is well known to be a dipole field outside, and a uniform field inside, as discussed in Sec. II.) In addition to providing a detailed illustration of the charge distribution on the surface of a battery and the electric field it produces, this example provides a detailed illustration of how energy conservation operates, both inside and outside a model battery. It also illustrates that *any* battery has an influence on its environment, even when it is not part of a circuit, since it produces a dipole field.

A number of related situations where there are charge distributions associated with current flow have been discussed in the literature. We mention four:

(1) The charge distribution needed to cause current to move around a bend in a wire. Rosser<sup>3</sup> has provided a general estimate for the surface charge density  $\Sigma_s$  in this case. From the mere fact of current flow, throughout the wire there must be an electric field of magnitude  $E=J/\sigma$ , where  $J$  is the current density and  $\sigma$  is the electrical conductivity. At a bend in the wire, to make  $\mathbf{E}$  change direction, from dimensional considerations applied to Gauss’s law there must be a surface charge density  $\Sigma_s$  whose own electric field must be of the order of  $E$ . This is the order of the field produced by a parallel plate capacitor. Hence  $E \sim \Sigma_s/\epsilon_0$ . Thus  $\Sigma_s \sim (\epsilon_0/\sigma)J$ : to bend an appreciable current, a wire made of a good conductor requires much less charge than a geometrically identical wire made of a poor conductor. Although this is a characteristic value for  $\Sigma_s$ , note that the charge density is zero on the surface of a wire that is parallel to the field lines in the absence of the wire, as in the case of a wire that is normal to the plates of an infinite parallel plate capacitor.

(2) The charge distribution needed to cause current to flow along a long wire of radius  $a$ . Marcus<sup>4</sup> has shown that when the return wire is a perfectly conducting cylindrical shell of radius  $R$ , then  $\Sigma_s \sim (\epsilon_0/\sigma)J(z/a)\ln(R/a)$ , where  $z$  is measured along the wire. Such a linear dependence on  $z$  follows from the voltage being linear along the wire; the normal derivative, which determines both the normal component of the electric field and  $\Sigma_s$ , must also be linear in  $z$ . Although this result is characteristically true, note that the voltage on a connecting wire that is normal to the plates of an infinite parallel plate capacitor will also have a linear dependence of the potential, but *no* surface charge.

(3) The case of a conducting medium that carries a uniform current at large distances from a spherical inclusion of conductivity differing from that of the conducting medium. A well-known illustration of how current flow can be disturbed

by an obstacle, it is not usually analyzed from the viewpoint of yielding the surface charge density that causes this disturbance. (For an exception, see Zahn.<sup>5</sup>)

(4) The charge distribution in the vicinity of a battery. This has been discussed in detail by Heald<sup>6</sup> for a model in two dimensions, with a “point battery” that is part of a circular circuit. In these revealing examples (a uniform resistance and a localized resistance), the charge distribution, equipotential lines, and electric field lines are plotted.

Much of the literature on surface charge and current flow can be traced through Refs. 6–8.

Section III contains an analysis of this “spherical battery” model. In addition to obtaining the surface charge density, electric field and electric current density, the magnetic field is also determined. From these, both the energy-flux (or Poynting) vector is obtained, and a detailed study of energy flux is made. The angular momentum density is also obtained. Section IV employs the results of the previous section to treat the example of Ref. 5, and extends that work by also obtaining the magnetic field for that case. Section V provides a brief discussion of voltaic cells and presents a more realistic, but less tractable, model of a spherical battery. Section VI discusses some additional geometries and provides a brief summary.

## II. A COSINE DISTRIBUTION OF CHARGE

Let us begin with a discussion of the electrical potential and electrical field produced by a surface charge density

$$\Sigma_s = C \hat{z} \cdot \hat{r} = C \cos \theta \quad (2.1)$$

placed on a sphere of radius  $a$ . This corresponds to an electric dipole moment  $\mathbf{p}$  that points along  $\hat{z}$ , with

$$\begin{aligned} \mathbf{p} &= \int dA \Sigma_s \mathbf{r} = C \int dA \hat{z} \cdot \hat{r} \mathbf{r} \\ &= C \int (a^2 \sin \theta d\theta d\phi) (\cos \theta) \hat{z} a (\cos \theta) \\ &= \frac{4\pi C a^3}{3} \hat{z}. \end{aligned} \quad (2.2)$$

It is well known that, outside a sphere, such a charge distribution produces an electrical potential that, in MKS units, is given by

$$\Phi = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} = \frac{1}{4\pi\epsilon_0} \frac{p \cos \theta}{r^2} \quad (r > a). \quad (2.3)$$

Because the tangential component of the electric field must be continuous at  $r=a$ , and for  $r < a$  must satisfy Laplace’s equation (all of the charge is on  $r=a$ ), the interior solution must have  $\cos \theta$  symmetry, and therefore must be of the form  $r \cos \theta = z$ . As a consequence,

$$\Phi = \frac{1}{4\pi\epsilon_0} \frac{p r \cos \theta}{a^3} = \frac{1}{4\pi\epsilon_0} \frac{p}{a^3} z \quad (r < a). \quad (2.4)$$

Within the sphere, the uniform electric field produced by this charge distribution is given by

$$\mathbf{E} = -\nabla \Phi = -\frac{1}{4\pi\epsilon_0} \frac{p}{a^3} \hat{z} \quad (r < a). \quad (2.5)$$

Outside the sphere, the electric field produced by this charge distribution is given by

$$\mathbf{E} = -\nabla\Phi = -\frac{1}{4\pi\epsilon_0} \frac{p}{r^3} (\hat{z} - 3 \cos \theta \hat{r}) \quad (r > a), \quad (2.6)$$

which is precisely that of a point dipole at the origin, even for observation points just outside the surface.

### III. A SPHERICAL BATTERY

Consider a spherical battery, of conductivity  $\sigma_2$ , and embedded in a material of conductivity  $\sigma_1$ . We assume that the battery produces a spatially uniform nonelectromagnetic force per unit charge of

$$\mathbf{F} = F\hat{z}, \quad (3.1)$$

so that, within the battery, the current is given by

$$\mathbf{J} = \sigma_2(\mathbf{E} + \mathbf{F}) = \sigma_2 \left( -\frac{1}{4\pi\epsilon_0} \frac{p}{a^3} + F \right) \hat{z} \quad (r < a). \quad (3.2)$$

Outside the battery, the current is given by

$$\mathbf{J} = \sigma_1 \mathbf{E} = -\sigma_1 \frac{1}{4\pi\epsilon_0} \frac{p}{a^3} (\hat{z} - 3 \cos \theta \hat{r}) \quad (r > a). \quad (3.3)$$

Note that the emf is not a uniquely defined quantity in the present case, since it is different for different locations of two leads, one on the upper and one on the lower hemisphere. For example,  $\int \mathbf{F} \cdot d\mathbf{l}$  across a vertical section gives  $Fa \cos \theta$ . In the concluding section, we discuss the issue of a more realistic (but less mathematically tractable) model for a voltaic cell.

The electric current must satisfy the boundary condition that its normal component ( $\mathbf{J} \cdot \hat{n}$ ) be continuous; this prevents charge from building up, and thus from violating the (implicit) steady-state condition. In the present case, this boundary condition is that the radial component of the current be continuous, leading to the condition that

$$\sigma_1 \left( \frac{1}{4\pi\epsilon_0} \frac{2p}{a^3} \right) \cos \theta = \sigma_2 \left( -\frac{1}{4\pi\epsilon_0} \frac{p}{a^3} + F \right) \cos \theta. \quad (3.4)$$

From this, one deduces that the strength of the induced dipole moment is

$$p = 4\pi\epsilon_0 a^3 F \frac{\sigma_2}{2\sigma_1 + \sigma_2}. \quad (3.5)$$

When the exterior is a poor conductor, one has

$$p \approx 4\pi\epsilon_0 a^3 F \quad (\sigma_1 \ll \sigma_2), \quad (3.6)$$

which implies, by Eq. (3.2), that the current is nearly zero. When the exterior is a good conductor, one has

$$p \approx \left( \frac{\sigma_2}{2\sigma_1} \right) 4\pi\epsilon_0 a^3 F \quad (\sigma_1 \gg \sigma_2), \quad (3.7)$$

and the charge on the battery is nearly completely suppressed. This corresponds to the case of a battery discharging through a good conductor. Generally, when a battery is connected to a circuit, its dipole field and surface charge can be expected to decrease (relative to the disconnected case), and the surface charge can no longer prevent current flow within the battery.

Explicitly, inside the battery the current is

$$\mathbf{J} = 2\sigma_2 F \left( \frac{\sigma_1}{2\sigma_1 + \sigma_2} \right) \hat{z} \quad (r < a) \quad (3.8)$$

and outside the battery the current is

$$\mathbf{J} = -\sigma_2 F \left( \frac{\sigma_1}{2\sigma_1 + \sigma_2} \right) \left( \frac{a^3}{r^3} \right) (\hat{z} - 3 \cos \theta \hat{r}) \quad (r > a). \quad (3.9)$$

From Eq. (3.8) the total current associated with the battery is

$$I = \int \mathbf{J} \cdot d\mathbf{A} = 2\pi a^2 \sigma_2 F \left( \frac{\sigma_1}{2\sigma_1 + \sigma_2} \right), \quad (3.10)$$

where the integral is most easily taken over the cross section that consists of the equator of the sphere. The same result can be obtained using Eq. (3.9) by considering cross sections consisting of the upper or lower hemisphere, or the exterior part of the equatorial plane.

It follows from Eqs. (3.2) and (3.8) that within the battery the electric field is

$$\mathbf{E} = -F \left( \frac{\sigma_2}{2\sigma_1 + \sigma_2} \right) \hat{z} \quad (r < a), \quad (3.11)$$

and from Eqs. (3.3) and (3.9) that outside the battery the electric field is

$$\mathbf{E} = -F \left( \frac{\sigma_2}{2\sigma_1 + \sigma_2} \right) \left( \frac{a^3}{r^3} \right) (\hat{z} - 3 \cos \theta \hat{r}) \quad (r > a). \quad (3.12)$$

As a check, note that  $\hat{n} \times \mathbf{E}$  is continuous at  $r = a$ .

The rate of Joule heating inside the sphere is given by

$$\mathcal{P}_{\text{in}} = \frac{1}{\sigma_2} \int J^2 dV = \frac{16\pi}{3} a^3 \frac{\sigma_1^2 \sigma_2 F^2}{(2\sigma_1 + \sigma_2)^2}. \quad (3.13)$$

The rate of Joule heating outside the sphere is given by

$$\mathcal{P}_{\text{out}} = \frac{1}{\sigma_1} \int J^2 dV = \frac{8\pi}{3} a^3 \frac{\sigma_1 \sigma_2^2 F^2}{(2\sigma_1 + \sigma_2)^2}. \quad (3.14)$$

The rate of battery discharge is

$$\mathcal{P}_{\text{bat}} = \int \mathbf{J} \cdot \mathbf{F} dV = \frac{8\pi}{3} a^3 \frac{\sigma_1 \sigma_2 F^2}{2\sigma_1 + \sigma_2}; \quad (3.15)$$

this equals the sum of the Joule heating inside [Eq. (3.13)] and outside [Eq. (3.14)] the sphere.

Using Eq. (3.10) for the current, and the sum of Eqs. (3.13) and (3.14) for the total rate of heating, one can define a total lumped resistance

$$R_{\text{tot}} = (\mathcal{P}_{\text{in}} + \mathcal{P}_{\text{out}}) / I^2 = \frac{2}{3\pi a} \frac{2\sigma_2 + \sigma_1}{\sigma_1 \sigma_2}. \quad (3.16)$$

One can also define a lumped emf  $\mathcal{E}$  by using Eqs. (3.10) and (3.16) via

$$\mathcal{E} = IR_{\text{tot}} = \frac{4}{3} a F. \quad (3.17)$$

Leads connected at arbitrary places would, typically, not yield this value for the emf. One can also think of the battery as possessing an internal resistance

$$R_{\text{int}} = \mathcal{P}_{\text{in}} / I^2 = \frac{4}{3\pi a} \frac{1}{\sigma_2}, \quad (3.18)$$

and the exterior region as possessing an external resistance

$$R_{\text{ext}} = \mathcal{P}_{\text{out}} / I^2 = \frac{2}{3\pi a} \frac{1}{\sigma_1}. \quad (3.19)$$

As expected, the sum of Eqs. (3.18) and (3.19) is Eq. (3.16).

One may also solve the differential form of Ampere's law,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad (3.20)$$

to obtain the magnetic field  $\mathbf{B}$ , under the assumption that it has only a component along  $\hat{\phi}$ , and that it is well behaved at the origin and infinity, and continuous at  $r=a$ . One finds that

$$B_{\phi} = \frac{\mu_0 \sigma_1 \sigma_2 F}{2\sigma_1 + \sigma_2} r \sin \theta \quad (r < a), \quad (3.21)$$

$$B_{\phi} = \frac{\mu_0 \sigma_1 \sigma_2 F}{2\sigma_1 + \sigma_2} \frac{a^3}{r^2} \sin \theta \quad (r > a). \quad (3.22)$$

These results may be verified by direct substitution into Eq. (3.20). They may also be obtained by use of the integral form of Ampere's law, or

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{A}, \quad (3.23)$$

where  $d\mathbf{l}$  and  $d\mathbf{A}$  are related by the right-hand rule. Use of the azimuthal symmetry of the problem applied to circles of fixed  $z$  concentric with the axis of the sphere, so  $\oint \mathbf{B} \cdot d\mathbf{l} = B 2\pi r$ , then leads to Eqs. (3.21) and (3.22), according to the placement of the circle. [For a circle within the sphere, let the surface be a disk of fixed  $z$ , and apply Eq. (3.8) to obtain  $\int \mathbf{J} \cdot d\mathbf{A}$ ; for a circle outside the sphere, let the surface be a spherical cap, and apply Eq. (3.9) to obtain  $\int \mathbf{J} \cdot d\mathbf{A}$ .]

It is also possible to obtain the rate of Joule heating from the energy flux vector (the Poynting vector)

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}. \quad (3.24)$$

With  $\mathbf{H} = \mathbf{B}/\mu_0$  in the present case, from Eqs. (3.11) and (3.21) we have, for  $r < a$ ,

$$\mathbf{S} = \sigma_1 \left( \frac{\sigma_2 F}{2\sigma_1 + \sigma_2} \right)^2 r (\hat{r} \sin^2 \theta + \hat{\theta} \sin \theta \cos \theta) \quad (r < a). \quad (3.25)$$

Explicit calculation then yields the rate of energy flow leaving unit volume for  $r < a$  to be

$$\nabla \cdot \mathbf{S} = 2\sigma_1 \left( \frac{\sigma_2 F}{2\sigma_1 + \sigma_2} \right)^2 \quad (r < a). \quad (3.26)$$

This is positive, corresponding to an outward flow of energy, since the battery is in this region. As expected from the Poynting theorem, this is the negative of the volume rate of work done by the electric field,  $-\mathbf{J} \cdot \mathbf{E}$ , which by Eq. (3.2) equals the negative of the volume rate of Joule heating plus the power density provided by the battery emf:  $-\mathbf{J} \cdot \mathbf{E} = -J^2/\sigma_2 + \mathbf{J} \cdot \mathbf{F}$ , with  $\mathbf{J}$  from Eq. (3.9).

Similarly, from Eqs. (3.12) and (3.22) we have, for  $r > a$ ,

$$\mathbf{S} = \sigma_1 \left( \frac{\sigma_2 F}{2\sigma_1 + \sigma_2} \right)^2 \frac{a^6}{r^5} (\hat{r} \sin^2 \theta - 2\hat{\theta} \sin \theta \cos \theta) \quad (r > a). \quad (3.27)$$

Explicit calculation then yields the rate of energy flow leaving unit volume to be, for  $r > a$ ,

$$\nabla \cdot \mathbf{S} = -\sigma_1 \left( \frac{\sigma_2 F}{2\sigma_1 + \sigma_2} \right)^2 \frac{a^6}{r^6} (1 + 3 \cos^2 \theta) \quad (r > a). \quad (3.28)$$

As expected from the Poynting theorem, this is the negative of the volume rate of work done by the electric field, which by Eq. (3.3) equals the negative of the volume rate of Joule heating:  $-\mathbf{J} \cdot \mathbf{E} = -J^2/\sigma_1$ , with  $\mathbf{J}$  from Eq. (3.9).

In addition, the normal (i.e., radial) component of the energy flux vector, on the surface of the sphere, takes on the value

$$\mathbf{S} \cdot \hat{r} = \sigma_1 a \left( \frac{\sigma_2 F}{2\sigma_1 + \sigma_2} \right)^2 \sin^2 \theta \quad (r = a). \quad (3.29)$$

The integral over the surface of the sphere is

$$\int \mathbf{S} \cdot d\mathbf{A} = \frac{8\pi}{3} a^3 \sigma_1 \left( \frac{\sigma_2 F}{2\sigma_1 + \sigma_2} \right)^2 \quad (r = a). \quad (3.30)$$

This is the same as the rate of Joule heating outside the sphere, given in Eq. (3.14).

One can also compute the field angular momentum density, which, for materials with  $\mathbf{B} = \mu_0 \mathbf{H}$  and  $\mathbf{D} = \epsilon_0 \mathbf{E}$  (as in the present case), is given by

$$\mathbf{L} = \frac{1}{c^2} \mathbf{r} \times \mathbf{S}, \quad (3.31)$$

where  $c$  is the speed of light. It is nonzero, but there is no net angular momentum. Since only  $B_{\phi} \neq 0$ , the only nonzero component of  $\mathbf{L}$  is given by  $L_{\phi} = -r S_{\theta}$ . Use of Eqs. (3.25) and (3.27) yields

$$\mathbf{L} = -\frac{1}{c^2} \frac{\sigma_1 \sigma_2^2 F^2}{(2\sigma_1 + \sigma_2)^2} r^2 \sin \theta \cos \theta \hat{\phi} \quad (r < a), \quad (3.32)$$

$$\mathbf{L} = \frac{2}{c^2} \frac{\sigma_1 \sigma_2^2 F^2}{(2\sigma_1 + \sigma_2)^2} \frac{a^6}{r^4} \sin \theta \cos \theta \hat{\phi} \quad (r > a). \quad (3.33)$$

There is no net angular momentum in either the upper or lower hemisphere, neither inside nor outside the sphere.

Note that  $\mathbf{L}$  is not continuous at  $r=a$  (a consequence of  $E_r$  not being continuous at  $r=a$ ). There is no contradiction with angular momentum conservation: densities of conserved quantities are *not* necessarily continuous; fluxes of conserved quantities *are*. For example, the mass density is not continuous at the interface between air and water; however, the mass flux is continuous. Similarly, Minkowski's field momentum density in a dielectric medium, proportional to  $\mathbf{D} \times \mathbf{B}$ , need not be continuous across an interface, since it is not a flux. On the other hand, the energy flux vector in a dielectric medium, proportional to  $\mathbf{E} \times \mathbf{H}$ , should be continuous; indeed, since the tangential components of  $\mathbf{E}$  and  $\mathbf{H}$  are both continuous, the normal component of  $\mathbf{E} \times \mathbf{H}$  is continuous.

#### IV. SPHERICAL INCLUSION WITHIN INFINITE CURRENT-CARRYING MEDIUM

Although this case has been considered in Ref. 5, because so many of the elements of the problem have already been developed, it is straightforward to consider it here. In addition, we calculate the magnetic field associated with the current flow, although we do not carry through the Poynting vector analysis of the previous section. Note that Ref. 5 also included the possibility of dielectric constants that differ from that of free space.

For the same geometry as above, consider that  $\mathbf{F}=0$ , but there is now a current at infinity, given by

$$\mathbf{J}_0 = J_0 \hat{z}. \quad (4.1)$$

In this case, the electric field may be thought of as being due to an applied external field

$$\mathbf{E}_0 = \frac{\mathbf{J}_0}{\sigma_1}, \quad (4.2)$$

and the electric field due to the unknown induced charge density on the surface of the sphere, obtained in the previous section. The current conservation condition then involves the electric field inside the sphere,

$$\mathbf{E} = \frac{\mathbf{J}_0}{\sigma_1} - \frac{1}{4\pi\epsilon_0} \frac{p}{a^3} \hat{z} \quad (r < a) \quad (4.3)$$

and the electric field outside the sphere,

$$\mathbf{E} = \frac{\mathbf{J}_0}{\sigma_1} - \frac{1}{4\pi\epsilon_0} \frac{p}{r^3} (\hat{z} - 3 \cos \theta \hat{r}) \quad (r > a). \quad (4.4)$$

Applying current conservation on the normal component of the currents at  $r = a$ , one obtains the condition that

$$\begin{aligned} \sigma_1 \left[ \frac{J_0}{\sigma_1} \hat{z} - \frac{1}{4\pi\epsilon_0} \frac{p}{a^3} (\hat{z} - 3 \cos \theta \hat{r}) \right] \cdot \hat{r} \\ = \sigma_2 \left( \frac{J_0}{\sigma_1} \hat{z} - \frac{1}{4\pi\epsilon_0} \frac{p}{a^3} \hat{z} \right) \cdot \hat{r}, \end{aligned} \quad (4.5)$$

which leads to the dipole moment

$$p = 4\pi\epsilon_0 \frac{J_0}{\sigma_1} a^3 \frac{\sigma_2 - \sigma_1}{2\sigma_1 + \sigma_2}. \quad (4.6)$$

Use of Eq. (4.6) in Eq. (2.1) and (2.2) leads to the surface charge density

$$\Sigma_s = 3 \frac{\epsilon_0}{\sigma_1} J_0 \frac{\sigma_2 - \sigma_1}{2\sigma_1 + \sigma_2} \cos \theta. \quad (4.7)$$

As expected, when  $\sigma_1 = \sigma_2$  there is no induced charge, and the spherical inclusion provides no disturbance. For  $\sigma_2 < \sigma_1$ , the dipole moment opposes the applied field, causing current to avoid the inclusion. (For  $\sigma_2 \ll \sigma_1$ , this corresponds to fluid flowing around an impenetrable sphere, or the magnetic field pattern around a perfect diamagnetic sphere, or the electric field pattern around a dielectric sphere of very small dielectric constant.) For  $\sigma_2 > \sigma_1$ , the dipole moment is along the applied field, causing current preferentially to pass through the inclusion. (For  $\sigma_2 \gg \sigma_1$ , this corresponds to the electric field pattern around a spherical conductor, or around a dielectric of very large dielectric constant.)

The current density is next obtained. Use of Eq. (4.6) in Eq. (4.3) yields, for  $r < a$ ,

$$\mathbf{J} = \sigma_2 \mathbf{E} = \frac{3\sigma_2 J_0}{2\sigma_1 + \sigma_2} \hat{z} \quad (r < a), \quad (4.8)$$

and use of Eq. (4.6) in Eq. (4.4) yields, for  $r > a$ ,

$$\mathbf{J} = \sigma_1 \mathbf{E} = J_0 \hat{z} - \frac{a^3}{r^3} J_0 \frac{\sigma_2 - \sigma_1}{2\sigma_2 + \sigma_1} (\hat{z} - 3 \cos \theta \hat{r}) \quad (r > a). \quad (4.9)$$

One can also determine the magnetic field. Assuming that only  $B_\phi$  is nonzero, one can obtain its value as was done in the previous section, by solving either the differential or the integral form of Ampere's law, using the current given by Eqs. (4.8) and (4.9). One then obtains, for  $r < a$ ,

$$B_\phi = \frac{3\mu_0 J_0}{2} \frac{\sigma_2}{2\sigma_1 + \sigma_2} r \sin \theta \quad (r < a) \quad (4.10)$$

and for  $r > a$ ,

$$B_\phi = \mu_0 J_0 \left( \frac{1}{2} r + \frac{a^3}{r^2} \frac{\sigma_2 - \sigma_1}{2\sigma_1 + \sigma_2} \right) \sin \theta \quad (r > a). \quad (4.11)$$

As expected, this is continuous at  $r = a$ .

## V. ON VOLTAIC CELLS, AND A MORE REALISTIC MODEL

It was indicated that the present model of a voltaic cell is unrealistic, as signaled by an apparent emf that would depend on the application points of the terminals. Nevertheless, analysis of the model is revealing as an example of surface charge on a battery, of the magnetic field produced by the flowing current, and of the use of the Poynting vector to study energy flow in detail. Before discussing why the model does not apply to actual voltaic cells, we present two examples to indicate the variety of conceivable sources for  $\mathbf{F}$ .

(1) The conveyor belt in a Van de Graaf generator. This produces a "field" that is nonconservative, in the sense that its line integral over a closed path need not be zero. This is a volume effect.

(2) The field associated with the thermoelectric effect, where the chemical potential of electrons depends on temperature, thereby causing current flow until  $\mathbf{E}$  builds up enough to compensate for the effect of the gradient in the chemical potential. Because the chemical potential is charge-carrier specific—electrons, not muons—it is not a true electric field. Such a field is conservative, since it is a function of the thermodynamic state variables.<sup>9</sup> This is a volume effect.

The present model is unrealistic because, in a real voltaic cell, where chemical reactions occur, the nonelectric force per unit charge  $\mathbf{F}$ , whose integrated value is what is normally called the electromotive force (emf), is a surface effect. However, in the spherical battery model analyzed in detail in Sec. III, the emf is a volume effect. This point requires expansion. Much of what follows is relatively elementary knowledge for chemists, and may have once been the same for physicists, but discussion with students and colleagues makes it clear that no longer is the case. Examination of textbooks on electricity and magnetism, at all levels, shows that they tend to avoid detailed (or even any) discussion of voltaic cells.

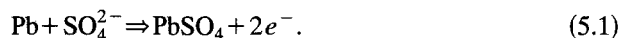
For our purposes, we will consider that there are two mutually exclusive classes of electrical conductors: those for which the conduction process is by electrons, and those for which the conduction process is by ions.<sup>10</sup> Materials in which electrons flow—ordinary conductors—are employed for the connections to a voltaic cell. Materials in which ions flow, and are the product of decomposition or recombination, are called *electrolytes*. The conductors that make contact with the electrolyte are called *electrodes*.<sup>11</sup> Usually, a voltaic cell consists of two dissimilar electrodes and an electrolyte, although sometimes there are two electrolytes and a "bridge" region separating them. When a wire is connected across the electrodes, electrons flow in the wire from the more electronegative electrode to the more electropositive one.<sup>12</sup>

That the emf is an electrode (i.e., surface) effect may be seen by the fact that voltaic cells of larger volume (e.g., a D cell, as opposed to a AA cell) do not have a larger emf; they have more "fuel," and thus run for a longer time. Only by increasing the number of electrodes (i.e., making a battery of voltaic cells), as done by Volta, with his Voltaic pile, can one obtain a larger emf from a given type of voltaic cell.

To be specific, consider a lead storage cell, familiar from the automobile battery, where the lead (Pb) electrode is nega-

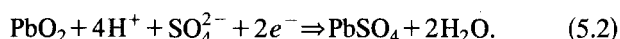
tive (this is the black terminal on the car battery) and the lead oxide ( $\text{PbO}_2$ ) electrode is positive (this is the red terminal). Our discussion is based on Ref. 13.

The net discharge process at the lead electrode is



To replace the deficit of  $\text{SO}_4^{2-}$  at this electrode, there is a tendency for an  $\text{H}_2\text{SO}_4$  to decompose near this electrode, also producing two  $\text{H}^+$  ions, which tend to diffuse away from the electrode. In addition,  $\text{SO}_4^{2-}$  and  $\text{H}_2\text{SO}_4$  from the bulk tend to diffuse toward this electrode. Thus  $\text{H}_2\text{SO}_4$  is consumed near this electrode as the battery discharges. (In charging, the same net processes occur, but in reverse. However, as charging becomes more complete, the competing process in which two  $\text{H}^+$  ions tend to combine with two electrons on this electrode, to produce  $\text{H}_2$  gas, becomes significant. It is not uncommon to find that rapidly charged batteries have  $\text{H}_2$  gas in them, produced at this electrode. This can be a safety hazard when one recharges a battery. In sealed lead-acid cells, this reaction is controlled.)

The net discharge process at the  $\text{PbO}_2$  electrode is



To replace the deficit of four  $\text{H}^+$  and one  $\text{SO}_4^{2-}$  ion, there is a tendency for  $\text{H}_2\text{SO}_4$  to decompose at this interface. In addition, the  $\text{H}_2\text{O}$  produced at this electrode tends to diffuse away, and  $\text{H}^+$ ,  $\text{SO}_4^{2-}$ , and  $\text{H}_2\text{SO}_4$  tend to diffuse toward this electrode. Thus  $\text{H}_2\text{SO}_4$  is consumed near this electrode, too, as the battery discharges. (In charging, the same net processes occur, but in reverse. However, as charging becomes more complete, the competing process in which an  $\text{H}_2\text{O}$  combines with two electrons on this electrode, to produce two  $\text{H}^+$  and half an  $\text{O}_2$ , becomes significant. In sealed lead-acid cells, this reaction is controlled.)

Clearly, a proper description of what happens within the bulk of a battery must include ionic and neutral molecular densities and currents. The equation of motion for each density must involve: (1) the divergence of the associated current (the ionic currents are driven by the electrical force and by concentration gradients, whereas the neutral molecular currents are driven only by concentration gradients); and (2) source terms due to chemical reactions. Solution of the equations of motion leads to an ionic screening region, of a few atomic dimensions, near each electrode. The net effect is that there is no emf, except at the electrode–electrolyte interface, where the full electrochemical action must be considered to occur. One should not think in terms of a field  $\mathbf{F}$ , but rather in terms of two emfs, one associated with each electrode. Indeed, electrochemical discussions of voltaic cells consider them to consist of two half-cells.<sup>14,15</sup>

Thus, a more realistic model of a spherical battery would consist of two hemispherical shells, the upper one providing an emf  $\mathcal{E}_1$  driving current radially outward, and the lower one providing an emf  $\mathcal{E}_2$  driving current radially outward. The net emf would be  $\mathcal{E}_1 - \mathcal{E}_2$  and, being a surface effect, it would be independent of the placement of the terminals on the upper and lower hemispheres.

## VI. DISCUSSION AND CONCLUSIONS

The sphere-related examples given here have the advantage that they build on the well-known problem of a conducting sphere in a uniform applied electric field, which sets up a

$\cos \theta$  distribution of charge on its surface. Hence that result can be immediately applied to the present examples.

Analogous cylindrical geometries can also be employed, building on the problem of a cylindrical conductor in a uniform applied electric field normal to its axis. Thus one can also study current flow in the presence of a cylindrical battery with uniform electrochemical field normal to its axis, or a cylindrical inclusion normal to the current flow.

Because, by Sec. V the emf is a surface effect, the two-dimensional “point” battery on a circle, discussed in Ref. 6, is not as idealized as it may have seemed at first sight. It can be converted to a more realistic model by considering two such point batteries of emfs  $\mathcal{E}_1$  and  $\mathcal{E}_2$ , separated by a small annular interior region representing the electrolyte, and connected on the exterior by a larger annular region representing the external circuit. It is likely that the resulting solution is solvable, but in the form of an infinite series, as found in Ref. 6 for the point battery with a lumped resistor.

We have found two other geometries that permit closed-form solutions. An infinite sequence of current-carrying stacked cylinders of different conductivities (such as segments of copper and aluminum wire) is solvable. In addition, a modification of the two-dimensional geometry of Ref. 6, in which one gives the battery a finite angular range, is solvable. However, both of these geometries involve infinite series, and therefore are less desirable than the simple spherical and simple cylindrical geometries. Nevertheless, they may have a place in the graduate curriculum.

Finally, as discussed in Sec. V, our model of a voltaic cell is a drastic oversimplification. A proper treatment of the chemistry is decidedly nontrivial, and requires sophisticated thermodynamic analysis.<sup>14,15</sup> Discussion of such phenomena is unusual in the physics literature, but a treatment of current flow through a concentration cell (in which metal is dissolved from one plate and deposited on the other when a voltage is applied) may be found in Landau and Lifshitz.<sup>16</sup>

## ACKNOWLEDGMENTS

I would like to acknowledge valuable suggestions from Chia-Ren Hu, David Jackson, Sanjeev Mukerjee, and Robert Webb.

<sup>1</sup>Historical usage changes (sometimes surprisingly rapidly, as in the short-lived 1980s usage by children of *transformer* for a toy that would “transform” from an automobile to a warrior), and history is often lost. Therefore, it may be worth remarking on the history of the word *battery*, which seems to have been introduced to electrical terminology by Benjamin Franklin (cc. 1748), who placed Leyden jars both in series and in parallel; in Franklin’s time, as today, a row of guns on a warship was referred to as a battery. (Note that, slightly before Franklin, Daniel Gralath employed Leyden jars, but only in parallel.) Yet another example of change in usage relates to the Leyden jar. Used for storing (or “condensing”) electricity, the Leyden jar inspired the word *condenser*; in modern terminology, we would now call the Leyden jar the first two-plate *capacitor*. Until the recent development of electronic ignitions for automobiles, which replaced ignition systems that utilized the discharge of “condensers,” use of the word *condenser* as a synonym for *capacitor* would elicit knowing responses from the automobile literate.

<sup>2</sup>Electrochemistry employs a more exacting terminology than the present paper. Consider an ordinary flashlight battery, to which the present paper employs the terminology *voltaic cell*. In electrochemical terminology, when discharging, the battery would be called a *voltaic cell*, but the same battery, when charging, would be called an *electrolytic cell*. In general, if a cell is run in such a way that the current flow is not in the direction of spontaneous current flow, the terminology *electrolytic cell* is employed. Thus when an external emf drives a normally inert cell (i.e., one with two identical copper electrodes in a uniform  $\text{CuSO}_4$  electrolyte solution, so one



- electrode is plated and the other is depleted), the cell is called *electrolytic*.
- <sup>3</sup>W. G. V. Rosser, "Magnitudes of surface charge distributions associated with electric current flow," *Am. J. Phys.* **38**, 265–266 (1970).
- <sup>4</sup>A. Marcus, "The electric field associated with a steady current in a long cylindrical conductor," *Am. J. Phys.* **9**, 225–226 (1941).
- <sup>5</sup>M. Zahn, *Electromagnetic Field Theory: A Problem Solving Approach* (Wiley, New York, 1979), see pp. 288–290.
- <sup>6</sup>Mark A. Heald, "Electric fields and charges in elementary circuits," *Am. J. Phys.* **52**, 522–526 (1984).
- <sup>7</sup>J. M. Aguirregabiria, A. Hernández, and M. Rivas, "An example of surface charge distribution on conductors carrying steady currents," *Am. J. Phys.* **60**, 138–141 (1992).
- <sup>8</sup>O. D. Jefimenko, *Electricity and Magnetism* (Appleton-Century-Crofts, New York, 1966); a second edition was published by Electret Scientific Co., Star City, West Virginia.
- <sup>9</sup>Thus in general, a nonelectric field can be either conservative or nonconservative. (When there is only one type of charge carrier, as is often the case, the conservative part of the nonelectric field is often treated as being part of the electrostatic field.) Similarly, as is well-known, true electric fields  $\mathbf{E}$  (in the sense that the force on any particle with charge  $q$ , due to  $\mathbf{E}$ , is given by  $q\mathbf{E}$ ) can be either conservative or nonconservative. As is well known, the electric field produced by static electric charge is conservative, whereas the electric field induced by a time-varying magnetic field is nonconservative.
- <sup>10</sup>In certain salts of the alkali metals, at low temperatures the lattice is essentially rigid, and conduction only takes place via thermally excited electrons. At higher temperatures, however, the small alkali atoms can "melt," breaking free of the lattice and engaging in ionic conduction within the structure of the ions that remain in the lattice. Such materials are called superionic conductors.
- <sup>11</sup>An ordinary conductor is always part of the electrode, even when the electrode is  $\text{H}_2$  gas bubbling over a Pt electrode. In this case the  $\text{H}_2$  gas provides a source of ions, and the Pt provides a source of electrons.
- <sup>12</sup>When an electron reaches one electrode and attaches to an ion, it participates in a *reduction* reaction; when an electron leaves an ion and attaches to an electrode, it participates in an *oxidation* reaction. Adding an electron hardly sounds like a lessening, as would be suggested by the term reduction. However, when an oxidized material (such as rust,  $\text{Fe}_2\text{O}_3$ ) is heated (sometimes in the presence of a reducing agent, such as  $\text{CO}$ ), it releases oxygen, and becomes Fe, with a *reduced* weight. It is in this sense that *reduction* was first defined, before the discovery of the electron and the theory of chemical bonding.
- <sup>13</sup>*Handbook of Batteries and Fuel Cells*, edited by David Linden (McGraw-Hill, New York, 1984), see Chaps. 14 and 15.
- <sup>14</sup>P. W. Atkins, *Physical Chemistry* (Freeman, San Francisco, 1982).
- <sup>15</sup>V. S. Bagotsky, *Fundamentals of Electrochemistry* (Plenum, New York, 1993).
- <sup>16</sup>L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, 2nd ed. (Pergamon, Oxford, 1984), see pp. 103–104.

## Graphical representations of Fraunhofer interference and diffraction

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Traditional textbook presentations of Fraunhofer interference and diffraction are usually limited to a few discrete examples of one, two, and sometimes more sources, in which the analysis yields graphs of intensity versus distance  $y$  on a screen that is at a large distance  $D$  in the  $x$  direction from the sources. This paper presents two separate sets of graphical representations that can assist students and teachers in improving their understanding of the Fraunhofer phenomena. The polar intensity plots stress the angular nature of the interference phenomena, and stress the fact that the phenomena need not be limited to a narrow angular range around the forward direction. The three-dimensional plots illustrate the unity of diffraction and interference by examining the patterns when one makes a continuous variation of the geometrical parameters that characterize the sources.

### I. INTRODUCTION

To see diffraction one must have waves passing through a slit and then traveling on to a screen on which observations are made. The width of the slit can be a few wavelengths. In order to see Fraunhofer diffraction, the distance from the slit to the observing screen must be much larger than either the wavelength  $\lambda$  of the waves and the width of the slit. For small distances to the screen one observes Fresnel diffraction. Slater and Frank give a series of diagrams showing the intensity pattern on a screen very close to the slit, followed by a series of patterns as the distance from the slit to the screen is increased until the distance is sufficiently large so

that Fraunhofer diffraction (FD) is observed.<sup>1</sup> The series of diagrams illustrate the transition from Fresnel diffraction to FD.

In introductory physics texts, the discussions of FD usually appear in the chapters on physical optics.<sup>2–5</sup> More complete discussions and examples of the FD phenomena are covered in intermediate optics texts.<sup>6–9</sup>

In the introductory texts it is common to start with a derivation of the interference pattern when one has coherent radiation of wavelength  $\lambda$  from two point sources that are separated by a distance  $d$  which is a few wavelengths. As shown in Fig. 1, points on a distant screen are described by different values of the angle  $\theta$ . The paths from the two sources to any point on the distant screen have a path differ-