6.10 The torque on all the particles in a volume V is

$$\frac{d L_{mech}}{dt} = \int_V \vec{x} \times (\rho \vec{E} + \vec{J} \times \vec{B})\, d^3x$$

Following the derivation in Sect. 6.7,

$$\vec{x} \times (\rho \vec{E} + \vec{J} \times \vec{B})$$

$$= \vec{x} \times \left[\varepsilon \vec{E}(\nabla \cdot \vec{E}) - \varepsilon \vec{E} \times (\nabla \times \vec{E}) + \mu \vec{H}(\nabla \cdot \vec{H}) - \mu \vec{H} \times (\nabla \times \vec{H}) \right] - \mu \varepsilon\, \vec{x} \times \frac{\partial}{\partial t}(\vec{E} \times \vec{H})$$

Therefore, we can define the field angular momentum density as

$$\mathcal{L}_{field} = \mu \varepsilon (\vec{E} \times \vec{H}).$$

and the conservation of angular momentum becomes

$$\frac{d}{dt}\left(L_{mech} + L_{field}\right) = \frac{d}{dt}\left(\int_V (\mathcal{L}_{mech} + \mathcal{L}_{field})\, d^3x \right)$$

$$= \int_V \vec{x} \times \left[\varepsilon \vec{E}(\nabla \cdot \vec{E}) - \varepsilon \vec{E} \times (\nabla \times \vec{E}) + \mu \vec{H}(\nabla \cdot \vec{H}) - \mu \vec{H} \times (\nabla \times \vec{H}) \right] d^3x$$

$$= \int_V \varepsilon_{ijk}\, \hat{e}_i\, x_j \frac{\partial}{\partial x_l} T_{kl}\, d^3x,$$

where Einstein summation convention is assumed. Notice that

$$\varepsilon_{ijk} x_j \frac{\partial}{\partial x_l} T_{kl} = \varepsilon_{ijk}\left[\frac{\partial}{\partial x_l}(x_j T_{kl}) - T_{kl} \frac{\partial}{\partial x_l} x_j \right] = \frac{\partial}{\partial x_l}\left(\varepsilon_{ijk} x_j T_{kl} \right) - \varepsilon_{ijk} \delta_{jl} T_{kl}$$

$$= \frac{\partial}{\partial x_l}\left(\varepsilon_{ijk} x_j T_{kl} \right) - \varepsilon_{ijk} T_{kj}, \qquad (*)$$

the last term is zero, since $\overleftrightarrow{T}$ is a symmetric tensor. Therefore,

$$\frac{d}{dt} \int_V (\mathcal{L}_{mech} + \mathcal{L}_{field})\, d^3x = \int_V \nabla \cdot (\vec{x} \times \overleftrightarrow{T})\, d^3x = - \int_V \nabla \cdot \overleftrightarrow{M}\, d^3x,$$

with $\overleftrightarrow{M} = \overleftrightarrow{T} \times \vec{x}$, and the meaning for the divergence calculation and tensor indices is clear from $(*)$. Then, formally, we can have the integral relation

$$\frac{d}{dt} \int_V (\mathcal{L}_{mech} + \mathcal{L}_{field})\, d^3x + \int \vec{n} \cdot \overleftrightarrow{M}\, da = 0$$

and the corresponding differential one

$$\frac{\partial}{\partial t}(\mathcal{L}_{mech} + \mathcal{L}_{field}) + \nabla \cdot \overleftrightarrow{M} = 0$$