7.26. A strange convention is chosen in performing the Fourier transform, but the final result after integration in momentum space does not depend on the convention chosen.

(a) The charge density in the real space is $\rho(\vec{x},t) = Ze\,\delta(\vec{x}-\vec{v}t)$, assuming the charged particle starts at $\vec{x}=0$. Then,

$$\rho(\vec{q},\omega) = \int \frac{d^3x\,dt}{(2\pi)^4}\, \rho(\vec{x},t)\, e^{-i(\vec{q}\cdot\vec{x}-\omega t)} = Ze \int \frac{dt}{(2\pi)^4}\, e^{-i(\vec{q}\cdot\vec{v}-\omega)t}$$

$$= \frac{Ze}{(2\pi)^3}\,\delta(\omega-\vec{q}\cdot\vec{v})$$

(b) The Coulomb's law states that $\nabla\cdot\vec{D}(\vec{x},t) = \rho(\vec{x},t)$. Also,

$$\vec{D}(\vec{x},t) = \int \frac{d^3x'dt'}{(2\pi)^4}\, \varepsilon(\vec{x}-\vec{x}', t-t')\, \vec{E}(\vec{x}',t') = -\int \frac{d^3x'dt'}{(2\pi)^4}\, \varepsilon(\vec{x}-\vec{x}',t-t')\, \nabla'\phi(\vec{x}',t')$$

Introduce the Fourier transform,

$$\vec{D}(\vec{x},t) = -\int \frac{d^3x'dt'}{(2\pi)^4} \int d^3q_1 d\omega_1\, \varepsilon(\vec{q}_1,\omega_1)\, e^{i\vec{q}_1\cdot(\vec{x}-\vec{x}')-i\omega_1(t-t')}$$
$$\times\left(\nabla' \int d^3q_2 d\omega_2\, \phi(\vec{q}_2,\omega_2)\, e^{i\vec{q}_2\cdot\vec{x}'-i\omega_2 t'}\right)$$

$$= -\int d^3q_1 d\omega_1 \int d^3q_2 d\omega_2\, (i\vec{q}_2)\, \varepsilon(\vec{q}_1,\omega_1)\, \phi(\vec{q}_2,\omega_2)\, e^{i\vec{q}_1\cdot\vec{x}-i\omega_1 t}$$
$$\times \int \frac{d^3x'dt'}{(2\pi)^4}\, e^{i(\vec{q}_2-\vec{q}_1)\cdot\vec{x}'-i(\omega_2-\omega_1)t'}$$

$$= -\int d^3q_1 d\omega_1 \int d^3q_2 d\omega_2\, (i\vec{q}_2)\, \varepsilon(\vec{q}_1,\omega_1)\, \phi(\vec{q}_2,\omega_2)\, \delta(\vec{q}_1-\vec{q}_2)\, \delta(\omega_1-\omega_2)\, e^{i\vec{q}_1\cdot\vec{x}-i\omega_1 t}$$

$$= -\int d^3q\, d\omega\, e^{i\vec{q}\cdot\vec{x}-i\omega t}\, i\vec{q}\, \varepsilon(\vec{q},\omega)\, \phi(\vec{q},\omega)$$

then, $\nabla\cdot\vec{D}(\vec{x},t) = \int d^3q\, d\omega\, e^{i\vec{q}\cdot\vec{x}-i\omega t}\, q^2\varepsilon(\vec{q},\omega)\,\phi(\vec{q},\omega)$.

Also, $\rho(\vec{x},t) = \int d^3q\, d\omega\, e^{i\vec{q}\cdot\vec{x}-i\omega t}\, \rho(\vec{q},\omega)$. Therefore, for each component,

$$q^2\varepsilon(\vec{q},\omega)\phi(\vec{q},\omega) = \rho(\vec{q},\omega), \quad \text{or} \quad \phi(\vec{q},\omega) = \frac{\rho(\vec{q},\omega)}{q^2\varepsilon(\vec{q},\omega)}$$

(e) $-\frac{dW}{dt} = -\int \vec{J}\cdot\vec{E}\, d^3x = \int \vec{J}\cdot\nabla\Phi\, d^3x = \int [\nabla(\Phi\vec{J}) - \Phi(\nabla\cdot\vec{J})]\, d^3x$

$$= \int \Phi \frac{\partial\rho}{\partial t} d^3x = \int d^3x \int d^3q_1 d\omega_1\, \phi(\vec{q}_1,\omega_1)\, e^{i(\vec{q}_1\cdot\vec{x} - i\omega_1 t)}$$

$$\times \frac{\partial}{\partial t}\int d^3q_2 d\omega_2\, \rho(\vec{q}_2,\omega_2)\, e^{i(\vec{q}_2\cdot\vec{x} - \omega_2 t)}$$

$$= \int d^3q_1 d\omega_1 \int d^3q_2 d\omega_2 (-i\omega_2)\phi(\vec{q}_1,\omega_1)\rho(\vec{q}_2,\omega_2)\int d^3x\, e^{i(\vec{q}_1+\vec{q}_2)\cdot\vec{x} - i(\omega_1+\omega_2)t}$$

$$= \int d^3q\, d\omega_1 d\omega_2 (-i\omega_2)\phi(\vec{q},\omega_1)\rho(-\vec{q},\omega_2)\cdot(2\pi)^3 e^{-i(\omega_1+\omega_2)t}$$

$$= (2\pi)^3 \int d^3q\, d\omega_1 d\omega_2 (-i\omega_2)\frac{\rho(\vec{q},\omega_1)}{q^2\epsilon(\vec{q},\omega_1)}\rho(-\vec{q},\omega_2)\, e^{-i(\omega_1+\omega_2)t}$$

$$= \frac{Z^2e^2}{(2\pi)^3}\int d^3q\, d\omega_1 d\omega_2 (-i\omega_2)\frac{1}{q^2\epsilon(\vec{q},\omega_1)}\delta(\vec{q}\cdot\vec{v} - \omega_1)\,\delta(-\vec{q}\cdot\vec{v} - \omega_2)\, e^{-i(\omega_1+\omega_2)t}$$

$$= \frac{Z^2e^2}{(2\pi)^3}\int d^3q\, d\omega\, (i\omega)\frac{1}{q^2\epsilon(\vec{q},\omega)}\delta(\omega - \vec{q}\cdot\vec{v})$$

Only the real part will contribute to the energy loss. Therefore,

$$-\frac{dW}{dt} = \frac{Z^2e^2}{(2\pi)^3}\int d^3q\, d\omega\, \frac{\omega}{q^2} \mathrm{Im}\left[\frac{1}{\epsilon(\vec{q},\omega)}\right]\delta(\omega - \vec{q}\cdot\vec{v})$$

Since $\mathrm{Im}[\epsilon(q,\omega)]$ is odd in ω, we can write the final result as

$$-\frac{dW}{dt} = \frac{Z^2e^2}{4\pi^3}\int \frac{d^3q}{q^2}\int_0^{+\infty} d\omega\, \omega\, \mathrm{Im}\left[\frac{1}{\epsilon(\vec{q},\omega)}\right]\delta(\omega - \vec{q}\cdot\vec{v})$$