2.15 Solution: (a) After separation of variables, the differential equation for the potential problem in the x direction becomes

$$\frac{\partial^2 X}{\partial x^2} = -\nu^2 X.$$

Due to the Dirichlet boundary condition at $x = 0$ and $x = 1$, we must have $\nu = n\pi$, for $n > 0$, and the functions

$$\sqrt{2}\sin(n\pi x)$$

form an orthonormal set, whose completeness relation is

$$\delta(x - x') = 2\sum_{n=1}^{\infty} \sin(n\pi x)\sin(n\pi x').$$

See Problem 2.24 for more details.

Then, following the procedure described in Section 3.9, the completeness relation in the two-dimensional square region becomes

$$\delta(\mathbf{x} - \mathbf{x}') = \delta(y - y') \cdot 2\sum_{n=1}^{\infty} \sin(n\pi x)\sin(n\pi x'),$$

and the corresponding Green function must have the form of

$$G(\mathbf{x}, \mathbf{x}') = 2\sum_{n=1}^{\infty} g_n(y, y')\sin(n\pi x)\sin(n\pi x'),$$

with $g_n(y, 0) = g_n(y, 1) = 0$, the Dirichlet boundary condition dictates the vanishing of the Green function on the boundary. Applying the equation

$$\nabla^2_{\mathbf{x}'} G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}'),$$

we have

$$\sum_{n=1}^{\infty} \left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) g_n(y, y')\sin(n\pi x)\sin(n\pi x') = -4\pi\delta(y - y') \cdot \sum_{n=1}^{\infty} \sin(n\pi x)\sin(n\pi x').$$

The above equation must be valid term-wise and, therefore, we will get the equation governing the g_n terms,

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) g_n(y, y') = -4\pi\delta(y - y'), \tag{1}$$

subject to the boundary conditions $g_n(y, 0) = g_n(y, 1) = 0$.

(b) Solutions to g_n must be linear combinations of $\exp(n\pi y')$ and $\exp(-n\pi y')$. Taking into account of the boundary conditions,

$$g_n(y, y') \propto \sinh(n\pi y'),$$

for $y' < y$, and

$$g_n(y, y') \propto \sinh(n\pi(1 - y')),$$

for $y' > y$. Since the solution must satisfy the above two relations, we need to seek a solution that is symmetric in both y and y',

$$g_n(y, y') = C_n \sinh(n\pi y_<)\sinh(n\pi(1 - y_>)), \tag{2}$$

where C_n is a yet to be determined constant, and $y_<$ ($y_>$) is the smaller (larger) of y and y'.

By integrating Eq. (1) in a small interval $(y-\varepsilon, y+\varepsilon)$ near y, with $\varepsilon \to 0$,

$$\int_{y-\varepsilon}^{y+\varepsilon} \left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2 \right) g_n(y,y')dy' = -4\pi \int_{y-\varepsilon}^{y+\varepsilon} \delta(y-y')dy',$$

we arrive at the following relation connecting the derivative of g_n at y,

$$\left. \frac{\partial}{\partial y'} g_n(y,y') \right|_{y-\varepsilon}^{y+\varepsilon} = -4\pi,$$

or equivalently,

$$\left. \frac{\partial}{\partial y'} g_n(y,y') \right|_{y+\varepsilon} - \left. \frac{\partial}{\partial y'} g_n(y,y') \right|_{y-\varepsilon} = -4\pi.$$

Applying the derivative to Eq. (2), we have

$$\left. \frac{\partial}{\partial y'} g_n(y,y') \right|_{y+\varepsilon} = -n\pi C_n \sinh(n\pi y) \cosh\left[n\pi(1-y)\right],$$

and

$$\left. \frac{\partial}{\partial y'} g_n(y,y') \right|_{y-\varepsilon} = n\pi C_n \cosh\left(n\pi y\right) \sinh\left[n\pi(1-y)\right].$$

Therefore, the connection condition becomes

$$\begin{aligned} & -n\pi C_n \sinh(n\pi y) \cosh\left[n\pi(1-y)\right] - n\pi C_n \cosh\left(n\pi y\right) \sinh\left[n\pi(1-y)\right] \\ = \quad & -n\pi C_n \sinh(n\pi) = -4\pi, \end{aligned}$$

or,

$$C_n = \frac{4}{n\sinh(n\pi)}.$$

Finally, the Green function is

$$G(\mathbf{x},\mathbf{x}') = 8 \sum_{n=1}^{\infty} \frac{1}{n\sinh(n\pi)} \sin(n\pi x) \sin(n\pi x') \sinh(n\pi y_<) \sinh\left[n\pi(1-y_>)\right].$$