6.3 (a) Let $A(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \vec{A}(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}$, we then have

$$\nabla^2 \vec{A}(\vec{x},t) = -\int \frac{d^3k}{(2\pi)^3} k^2 \vec{A}(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}$$

The original equation becomes

$$\mu\sigma \int \frac{d^3k}{(2\pi)^3} \frac{\partial \vec{A}(\vec{k},t)}{\partial t} e^{i\vec{k}\cdot\vec{x}} = -\int \frac{d^3k}{(2\pi)^3} k^2 \vec{A}(\vec{k},t) e^{i\vec{k}\cdot\vec{x}},$$

which must be valid for each $\vec{k}$, or

$$\frac{\partial \vec{A}(\vec{k},t)}{\partial t} = -\frac{k^2}{\mu\sigma} \vec{A}(\vec{k},t)$$

With initial condition $\vec{A}(\vec{k},0) = \int d^3x\, \vec{A}(\vec{x},0)\, e^{-i\vec{k}\cdot\vec{x}}$, the solution to the above first order ODE is $\quad \vec{A}(\vec{k},t) = \vec{A}(\vec{k},0) \exp\{-k^2 t/\mu\sigma\}$, with $t>0$

Then,

$$\begin{aligned}
\vec{A}(\vec{x},t) &= \int \frac{d^3k}{(2\pi)^3} \vec{A}(\vec{k},0)\, e^{-k^2t/\mu\sigma}\, e^{i\vec{k}\cdot\vec{x}} \\
&= \int \frac{d^3k}{(2\pi)^3} \int d^3x'\, \vec{A}(\vec{x}',0)\, e^{-i\vec{k}\cdot\vec{x}'}\, e^{-k^2t/\mu\sigma}\, e^{i\vec{k}\cdot\vec{x}} \\
&= \int d^3x'\, \vec{A}(\vec{x}',0) \int \frac{d^3k}{(2\pi)^3} e^{-k^2t/\mu\sigma} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \\
&= \int d^3x'\, G(\vec{x}-\vec{x}',t)\, \vec{A}(\vec{x}',0)
\end{aligned}$$

where $\quad G(\vec{x}-\vec{x}',t) = \int \frac{d^3k}{(2\pi)^3} e^{-k^2t/\mu\sigma} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$

(b) Let $G(\vec{x},t) = \int \frac{d^3k\,d\omega}{(2\pi)^4} G(\vec{k},\omega)\, e^{i\vec{k}\cdot\vec{x}-i\omega t}$,

then $\left(\frac{\partial}{\partial t} - \frac{1}{\mu\sigma}\nabla^2\right) G(\vec{x}-\vec{x}',t) = \int \frac{d^3k\,d\omega}{(2\pi)^4} \left(-i\omega + \frac{k^2}{\mu\sigma}\right) G(\vec{k},\omega)\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')-i\omega t}$

Notice that $\delta(\vec{x}-\vec{x}')\delta(t) = \int \frac{d^3k\,d\omega}{(2\pi)^4} 1\cdot e^{i\vec{k}\cdot(\vec{x}-\vec{x}')-i\omega t}$, we must have

$$\left(-i\omega + \frac{k^2}{\mu\sigma}\right) G(\vec{k},\omega) = 1,$$

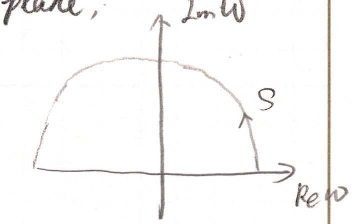
or

$$G(\vec{k},\omega) = \frac{1}{-i\omega + \frac{k^2}{\mu\sigma}},$$

to satisfy the inhomogeneous equation for the Green function. Then,

$$G(\vec{x}-\vec{x}',t) = \int \frac{d^3k\,d\omega}{(2\pi)^4} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')-i\omega t}}{-i\omega + \frac{k^2}{\mu\sigma}}$$

To perform the integral in the frequency space, we will use the contour integral method. For $t<0$, consider the contour enclosing the upper half plane, So that the integrand falls off exponentially on the semi-circle

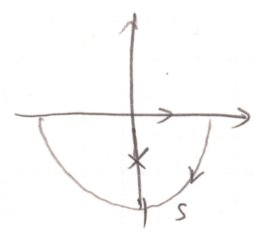


Then. $\oint_C d\omega = \int_{-\infty}^{+\infty} d\omega + \int_S d\omega$.

Since the integrand has no pole in the upper half plane. $\oint_C d\omega = 0$.

Meanwhile, $\int_S d\omega = 0$, as $e^{-i\omega t}$ falls off exponentially. then we have the integral along the frequency variable is 0. Therefore, the Green's function is zero for $t<0$

For $t>0$.



$$\oint_c \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{-i\omega+\frac{k^2}{\mu\sigma}} = (-2\pi i)\times i\times e^{-k^2t/\mu\sigma}\cdot\frac{1}{2\pi}$$

$$= e^{-k^2t/\mu\sigma}$$

Again, $\int_S d\omega = 0$, we have

$$\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{-i\omega+k^2/\mu\sigma} = e^{-k^2t/\mu\sigma}$$

Then. $G(\vec{x}-\vec{x}',t) = \int \frac{d^3k\,d\omega}{(2\pi)^4}\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')-i\omega t}}{-i\omega+k^2/\mu\sigma} = \int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')-k^2t/\mu\sigma}$, for $t>0$.

(c)

$$= \frac{1}{4\pi^2}\int_0^{+\infty}k^2dk\int_{-1}^{1}d(\cos\theta)\,e^{ik|\vec{x}-\vec{x}'|\cos\theta-k^2t/\mu\sigma}$$

$$= \frac{1}{4\pi^2}\int_0^{+\infty}k\,dk\,\frac{e^{ik|\vec{x}-\vec{x}'|}-e^{-ik|\vec{x}-\vec{x}'|}}{i|\vec{x}-\vec{x}'|}e^{-k^2t/\mu\sigma}$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\frac{e^{ik|\vec{x}-\vec{x}'|-k^2t/\mu\sigma}}{i|\vec{x}-\vec{x}'|}k\,dk$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\frac{k}{i|\vec{x}-\vec{x}'|}\exp\left\{-\frac{t}{\mu\sigma}\left(k-\frac{i\mu\sigma}{2t}|\vec{x}-\vec{x}'|\right)^2-\frac{\mu\sigma}{4t}|\vec{x}-\vec{x}'|^2\right\}dk$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\frac{1}{i|\vec{x}-\vec{x}'|}\left(k-\frac{i\mu\sigma}{2t}|\vec{x}-\vec{x}'|+\frac{i\mu\sigma}{2t}|\vec{x}-\vec{x}'|\right)$$
$$\times\exp\left\{-\frac{t}{\mu\sigma}\left(k-\frac{i\mu\sigma}{2t}|\vec{x}-\vec{x}'|\right)^2-\frac{\mu\sigma}{4t}|\vec{x}-\vec{x}'|^2\right\}dk$$

$$= \frac{1}{4\pi^2} \exp\left\{-\frac{\mu\sigma}{4t}|\vec{x}-\vec{x}'|^2\right\} \int_{-\infty}^{+\infty} \left(\frac{k}{i|\vec{x}-\vec{x}'|} + \frac{\mu\sigma}{2t} \right) e^{-k^2 t/\mu\sigma}\, dk$$

$$= \frac{1}{4\pi^2} \exp\left\{-\frac{\mu\sigma}{4t}|\vec{x}-\vec{x}'|^2\right\} \frac{\mu\sigma}{2t} \sqrt{\frac{\mu\sigma\pi}{t}}$$

$$= \frac{1}{8\pi^{3/2}} \left(\frac{\mu\sigma}{t}\right)^{3/2} \exp\left\{-\frac{\mu\sigma}{4t}|\vec{x}-\vec{x}'|^2\right\} = \left(\frac{\mu\sigma}{4\pi t}\right)^{3/2} \exp\left\{-\frac{\mu\sigma}{4t}|\vec{x}-\vec{x}'|^2\right\}$$

The Green function is only non-zero for $t>0$, Finally.

$$G(\vec{x}-\vec{x}';t) = \Theta(t) \left(\frac{\mu\sigma}{4\pi t}\right)^{3/2} \exp\left\{-\frac{\mu\sigma}{4t}|\vec{x}-\vec{x}'|^2\right\}$$