3.8 Solution: (a) Comparing Eq. (3.136) from the book and the alternative form, we need to establish the following identity,

$$\log\left(\frac{2}{\sin\theta}\right) = \sum_{l=0}^{\infty} A_l P_l(\cos\theta),$$

with $A_0 = 1$, $A_{2j-1} = 0$, and

$$A_{2j} = \frac{4j+1}{2j(2j+1)},$$

for $j > 0$.

Multiplying both sides by $\sin\theta P_l(\cos\theta)$ and integrate with respect to θ between $[0, \pi]$, we have

$$A_l = \frac{2l+1}{2}\int_0^{\pi} \sin\theta \log\left(\frac{2}{\sin\theta}\right) P_l(\cos\theta) d\theta = \frac{2l+1}{2}\int_{-1}^{1} \log\left(\frac{2}{\sqrt{1-x^2}}\right) P_l(x) dx.$$

For $l = 0$, $P_0(x) \equiv 1$, the coefficient A_0 can be directly determined,

$$\begin{aligned} A_0 &= \frac{1}{2}\int_{-1}^{1} \log\left(\frac{2}{\sqrt{1-x^2}}\right) dx \\ &= \log 2 - \frac{1}{4}\Big((1+x)\log(1+x) - (1-x)\log(1-x) - 2x\Big)\Big|_{-1}^{1} \\ &= 1. \end{aligned}$$

When $l \neq 0$, using the ordinary Legendre differential equation (3.10)

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l(x)}{dx}\right] + l(l+1)P_l(x) = 0,$$

and multiplying with $\log(2/\sqrt{1-x^2})$ and integrating from -1 to 1, we have

$$\int_{-1}^{1} \log\left(\frac{2}{\sqrt{1-x^2}}\right) \frac{d}{dx}\left[(1-x^2)\frac{dP_l(x)}{dx}\right] dx + l(l+1)\int_{-1}^{1} \log\left(\frac{2}{\sqrt{1-x^2}}\right) P_l(x) dx = 0.$$

Now, the first term can be manipulated by partial integration,

$$\begin{aligned} & \int_{-1}^{1} \log\left(\frac{2}{\sqrt{1-x^2}}\right) \frac{d}{dx}\left[(1-x^2)\frac{dP_l(x)}{dx}\right] dx \\ =\ & \log\left(\frac{2}{\sqrt{1-x^2}}\right) \cdot (1-x^2)\frac{dP_l(x)}{dx}\Big|_{x=-1}^{1} - \int_{-1}^{1} \frac{d}{dx}\log\left(\frac{2}{\sqrt{1-x^2}}\right) \cdot (1-x^2)\frac{dP_l(x)}{dx} dx \\ =\ & -\int_{-1}^{1} x\frac{dP_l(x)}{dx} dx \\ =\ & -xP_l(x)|_{x=-1}^{1} + \int_{-1}^{1} P_l(x) dx \\ =\ & -P_l(1) - P_l(-1) = -1 - (-1)^l, \end{aligned}$$

which is nonzero only for even l with the value of -2. Therefore,

$$\int_{-1}^{1} \log\left(\frac{2}{\sqrt{1-x^2}}\right) P_l(x) dx = -\frac{1}{l(l+1)}\int_{-1}^{1} \log\left(\frac{2}{\sqrt{1-x^2}}\right) \frac{d}{dx}\left[(1-x^2)\frac{dP_l(x)}{dx}\right] dx$$

$$= \begin{cases} \dfrac{1}{j(2j+1)}, & l = 2j \\ 0, & l = 2j+1, \end{cases}$$

and

$$A_l = \frac{2l+1}{2}\int_{-1}^{1} \log\left(\frac{2}{\sqrt{1-x^2}}\right) P_l(x)dx = \begin{cases} \dfrac{4j+1}{2j(2j+1)}, & l = 2j \\ 0, & l = 2j+1, \end{cases}$$

for $j > 0$.

Thus, we have established the identity

$$\log\left(\frac{2}{\sin\theta}\right) = 1 + \sum_{j=1}^{\infty} \frac{4j+1}{2j(2j+1)} P_{2j}(\cos\theta),$$

which gives the desired result for the potential.

(b) From the expansion

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{\sqrt{x^2+x'^2-2xx'\cos\theta}} = \sum_{l=0}^{\infty} \frac{x_<^l}{x_>^{l+1}} P_l(\cos\theta),$$

by setting $x' = 1$ and taking the limit $x \to 1-\varepsilon$ (or, from the other side—this does not change the final result), we have

$$\frac{1}{\sqrt{2(1-\cos\theta)}} = \frac{1}{2\sin(\theta/2)} = \sum_{l=0}^{\infty} P_l(\cos\theta).$$

Similarly, perfrom the same calculation on $1/|\mathbf{x}+\mathbf{x}'|$, we have

$$\frac{1}{\sqrt{2(1+\cos\theta)}} = \frac{1}{2\cos(\theta/2)} = \sum_{l=0}^{\infty} (-1)^l P_l(\cos\theta).$$

Take the sum,

$$\frac{1}{2}\left(\frac{1}{\sin(\theta/2)} + \frac{1}{\cos(\theta/2)}\right) = \sum_{l=0}^{\infty}(1+(-1)^l)P_l(\cos\theta) = 2\sum_{j=0}^{\infty} P_{2j}(\cos\theta).$$

With the above identity, Eq. (3.137) can be written as

$$\begin{aligned}
\sigma(\theta) &= -\frac{Q}{4\pi b^2}\left[1 + \sum_{j=1}^{\infty} \frac{4j+1}{2j+1} P_{2j}(\cos\theta)\right] \\
&= -\frac{Q}{4\pi b^2}\left[1 + 2\sum_{j=1}^{\infty} P_{2j}(\cos\theta) - \sum_{j=1}^{\infty} \frac{1}{2j+1} P_{2j}(\cos\theta)\right] \\
&= -\frac{Q}{4\pi b^2}\left[2\sum_{j=0}^{\infty} P_{2j}(\cos\theta) - \sum_{j=0}^{\infty} \frac{1}{2j+1} P_{2j}(\cos\theta)\right] \\
&= -\frac{Q}{4\pi b^2}\left[\frac{1}{2}\left(\frac{1}{\sin(\theta/2)} + \frac{1}{\cos(\theta/2)}\right) - \sum_{j=0}^{\infty} \frac{1}{2j+1} P_{2j}(\cos\theta)\right].
\end{aligned}$$