8.6 (a) For TM mode, $E_z(\rho,\phi) = E_0 J_m(x_{mn}\rho/R)e^{\pm im\phi}$, and the resonant frequency is

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\varepsilon}}\left(\frac{x_{mn}^2}{R^2}+\frac{p^2\pi^2}{L^2}\right)^{1/2},$$ with $p \geq 0$ and x_{mn} being the n-th root of $J_m(x)$.

Similarly, for TE mode, $H_z(\rho,\phi) = H_0 J_m(x'_{mn}\rho/R)e^{\pm im\phi}$, and

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\varepsilon}}\left(\frac{x'^2_{mn}}{R^2}+\frac{p^2\pi^2}{L^2}\right)^{1/2},$$ with $p > 0$ and x'_{mn} being the n-th root of $J'_m(x)$.

(b) The lowest resonant frequency is achieved with TM_{010} mode. For this mode, the stored energy is

$$U = \frac{L}{2}\varepsilon\int_A |E_z|^2 da = \pi\varepsilon L E_0^2\int_0^R \rho J_0\left(x_{01}\frac{\rho}{R}\right)^2 d\rho = \frac{\varepsilon L E_0^2}{2}\pi R^2 J_1(x_{01})^2.$$

and the power loss is

$$P_{loss} = \frac{1}{2\sigma\delta}\left[\oint_c dl\int_0^d dz\,|\vec{n}\times\vec{H}|^2_{sides} + 2\int_A da\,|\vec{n}\times\vec{H}|^2_{ends}\right]$$

Since $$\nabla_t E_z = \left(\hat{\rho}\frac{\partial}{\partial\rho}+\hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}\right)E_z = E_0 e^{\pm im\phi}\left[\hat{\rho}\frac{x_{mn}}{R}J'_m\left(x_{mn}\frac{\rho}{R}\right)+\hat{\phi}\frac{im}{\rho}J_m\left(x_{mn}\frac{\rho}{R}\right)\right]$$

$$= E_0\frac{x_{01}}{R}J'_0\left(x_{01}\frac{\rho}{R}\right)\hat{\rho}.\quad \text{for } m=0,\ n=1,$$

the magnetic field is

$$\vec{H}_t = \frac{i}{\mu\omega}E_0\frac{x_{01}}{R}J'_0\left(x_{01}\frac{\rho}{R}\right)\hat{\phi},\quad \text{for } p=0.$$

The power loss now becomes

$$P_{loss} = \frac{1}{2\sigma\delta}\frac{E_0^2}{\mu^2\omega_{010}^2}\frac{x_{01}^2}{R^2}\left[2\pi R\cdot L\cdot J'_0(x_{01})^2 + 4\pi\int_0^R \rho J'_0\left(x_{01}\frac{\rho}{R}\right)^2 d\rho\right]$$

$$= \frac{1}{2\sigma\delta}\frac{E_0^2}{\mu^2\omega_{010}^2}\frac{x_{01}^2}{R^2}\left[2\pi R\cdot L\,J_1(x_{01})^2 + 4\pi R^2\int_0^1 tJ_1(x_{01}t)^2 dt\right]$$

$$= \frac{1}{2\sigma\delta}\frac{E_0^2}{\mu^2\omega_{010}^2}\frac{x_{01}^2}{R^2}2\pi R\,J_1(x_{01})^2(R+L). = \frac{1}{2\sigma\delta}\frac{\varepsilon}{\mu}E_0^2\cdot 2\pi R\,J_1(x_{01})^2(R+L)$$

Where we have used the identity

$$\int_0^1 x J_1(ax)^2 dx = \frac{1}{2}\left(J_0(a)^2 - \frac{2J_1(a)J_0(a)}{a} + J_1(a)^2\right)$$

and $J_0(a) = 0$ for $a = x_{01}$. Then,

$$Q = \omega_{010}\frac{U}{P_{loss}} = \omega_{010}\cdot\mu\sigma\delta\frac{L}{2\left(1+\frac{L}{R}\right)} = \mu\sqrt{2\sigma\omega_{010}/\mu_c}\frac{L}{2\left(1+\frac{L}{R}\right)} = \frac{\mu}{\mu_c}\frac{L}{\sqrt{\frac{2}{\mu_c\sigma\omega_{010}}}}\frac{1}{1+\frac{L}{R}} = \frac{\mu}{\mu_c}\frac{L}{\delta}\frac{1}{1+\frac{L}{R}}$$