11.12 (a) We can first expand β' as

$$\beta' = \sqrt{(\vec\beta+\delta\vec\beta_{\parallel})^2+(\delta\vec\beta_{\perp})^2} = \sqrt{\beta^2+2\beta\delta\beta_{\parallel}+\delta\beta_{\parallel}^2+\delta\beta_{\perp}^2} = \beta+\delta\beta_{\parallel}+\frac{\delta\beta^2}{2\beta}+\cdots$$

Retain only the term linear in $\delta\vec\beta$, we have

$$\frac{\tanh^{-1}\beta'}{\beta'} = \frac{\tanh^{-1}\beta}{\beta}+\left(\frac{1}{\beta(1-\beta^2)}-\frac{\tanh^{-1}\beta}{\beta^2}\right)(\beta'-\beta)+\cdots$$

$$= \frac{\tanh^{-1}\beta}{\beta}+\left(\frac{\gamma^2}{\beta}-\frac{\tanh^{-1}\beta}{\beta^2}\right)\delta\beta_{\parallel}.$$

Then,

$$\delta L = L+\delta L-L = -\left[(\vec\beta+\delta\vec\beta_{\parallel}+\delta\vec\beta_{\perp})\cdot\vec K\,\frac{\tanh^{-1}\beta'}{\beta'}-\vec\beta\cdot\vec K\,\frac{\tanh^{-1}\beta}{\beta}\right]$$

$$= -\left[\vec\beta\cdot\vec K\left(\frac{\tanh^{-1}\beta}{\beta}+\left(\frac{\gamma^2}{\beta}-\frac{\tanh^{-1}\beta}{\beta^2}\right)\delta\beta_{\parallel}\right)+\delta\vec\beta_{\parallel}\cdot\vec K\,\frac{\tanh^{-1}\beta}{\beta}\right.$$

$$\left.+\,\delta\vec\beta_{\perp}\cdot\vec K\,\frac{\tanh^{-1}\beta}{\beta}-\vec\beta\cdot\vec K\,\frac{\tanh^{-1}\beta}{\beta}\right]$$

$$= -\gamma^2\delta\vec\beta_{\parallel}\cdot\vec K-\frac{\tanh^{-1}\beta}{\beta}\delta\vec\beta_{\perp}\cdot\vec K.$$

Here, we have kept only the term linear in $\delta\vec\beta$, and also $(\vec\beta\cdot\vec K)\delta\vec\beta_{\parallel} = (\delta\vec\beta_{\parallel}\cdot\vec K)\vec\beta$, as $\delta\vec\beta_{\parallel}$ is in the same direction as $\vec\beta$

(b) It is straightforward verification.

$$C_1 = [L,\delta L] = \frac{\tanh^{-1}\beta}{\beta}\beta_i\left(\gamma^2\delta\beta_{\parallel,j}+\frac{\tanh^{-1}\beta}{\beta}\delta\beta_{\perp,j}\right)[K_i,K_j]$$

$$= -\frac{\tanh^{-1}\beta}{\beta}\varepsilon_{ijk}\beta_i\left(\gamma^2\delta\vec\beta_{\parallel}+\frac{\tanh^{-1}\beta}{\beta}\delta\vec\beta_{\perp}\right)_j S_k$$

$$= -\frac{\tanh^{-1}\beta}{\beta}\vec\beta\times\left(\gamma^2\delta\vec\beta_{\parallel}+\frac{\tanh^{-1}\beta}{\beta}\delta\vec\beta_{\perp}\right)\cdot\vec S$$

$$= -\frac{1}{\beta^2}(\tanh^{-1}\beta)^2(\vec\beta\times\delta\vec\beta_{\perp})\cdot\vec S$$

$$C_2 = [L, C_1] = \frac{\tanh^{-1}\beta}{\beta}\beta_i \cdot \frac{1}{\beta^2}(\tanh^{-1}\beta)^2 (\vec{\beta}\times\delta\vec{\beta}_\perp)_j \; [K_i, S_j]$$

$$= -\frac{1}{\beta^3}(\tanh^{-1}\beta)^3 \, \varepsilon_{ijk}\beta_i (\vec{\beta}\times\delta\vec{\beta}_\perp)_j \, K_k$$

$$= -\frac{1}{\beta^3}(\tanh^{-1}\beta)^3 \left[\vec{\beta}\times(\vec{\beta}\times\delta\vec{\beta}_\perp)\right]\cdot\vec{K}$$

$$= -\frac{1}{\beta^3}(\tanh^{-1}\beta)^3 \left[\vec{\beta}(\vec{\beta}\cdot\delta\vec{\beta}_\perp) - \delta\vec{\beta}_\perp(\vec{\beta}\cdot\vec{\beta})\right]\cdot\vec{K}, \quad (\vec{\beta}\cdot\delta\vec{\beta}_\perp = 0)$$

$$= -\frac{1}{\beta}(\tanh^{-1}\beta)^3 \, \delta\vec{\beta}_\perp\cdot\vec{K}$$

$$= (\tanh^{-1}\beta)^2 \delta L_\perp, \qquad \left(\delta L_\perp = -\frac{\tanh^{-1}\beta}{\beta}\delta\vec{\beta}_\perp\cdot\vec{K}\right).$$

$$C_3 = [L, C_2] = \frac{\tanh^{-1}\beta}{\beta}\beta_i \cdot \frac{(\tanh^{-1}\beta)^3}{\beta}\delta\beta_{\perp,j} \; [K_i, K_j]$$

$$= -\frac{(\tanh^{-1}\beta)^4}{\beta^2}\varepsilon_{ijk}\beta_i\delta\beta_{\perp,j} S_k = -\frac{(\tanh^{-1}\beta)^4}{\beta^2}(\vec{\beta}\times\delta\vec{\beta}_\perp)\cdot\vec{S}$$

$$= (\tanh^{-1}\beta)^2 C_1,$$

$$C_4 = [L, C_3] = (\tanh^{-1}\beta)^2 [L, C_1] = (\tanh^{-1}\beta)^4 \delta L_\perp.$$

(c) $A_T = A_2 A_1^{-1} = \exp\{L+\delta L\}\exp\{-L\}$

$$= I + \delta L + \frac{1}{2!}[L,\delta L] + \frac{1}{3!}[L,[L,\delta L]] + \frac{1}{4!}[L,[L,[L,\delta L]]] + \cdots$$

$$= I + \delta L + \frac{1}{2!}C_1 + \frac{1}{3!}C_2 + \frac{1}{4!}C_3 + \frac{1}{5!}C_4$$

$$= I + \delta L_\parallel + \delta L_\perp + \frac{(\tanh^{-1}\beta)^2}{2!}B + \frac{1}{3!}(\tanh^{-1}\beta)^2\delta L_\perp + \frac{(\tanh^{-1}\beta)^4}{4}B + \frac{1}{5!}(\tanh^{-1}\beta)^4\delta L_\perp + \cdots$$

$$= I - B + \delta L_\parallel + B\sum_{n=0}^{\infty}\frac{1}{(2n)!}(\tanh^{-1}\beta)^{2n} + D\cdot\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(\tanh^{-1}\beta)^{2n+1}$$

$$= I + \delta L_\parallel + B\left(\cosh(\tanh^{-1}\beta) - 1\right) + D\cdot\sinh(\tanh^{-1}\beta).$$

Here, $\delta L_\parallel = \delta L - \delta L_\perp = -\gamma^2\delta\vec{\beta}_\parallel\cdot\vec{K}$, and $C_1 = (\tanh^{-1}\beta)^2 B$, with

$B = -\frac{1}{\beta^2}(\vec{\beta}\times\delta\vec{\beta}_\perp)\cdot\vec{S}$. Also $\delta L_\perp = \tanh^{-1}\beta\cdot D$, with $D = -\frac{1}{\beta}\delta\vec{\beta}_\perp\cdot\vec{K}$

Finally, notice that $\cosh(\tanh^{-1}\beta) = \dfrac{1}{\sqrt{1-\tanh^2(\tanh^{-1}\beta)}} = \dfrac{1}{\sqrt{1-\beta^2}} = \gamma$, and

$\sinh(\tanh^{-1}\beta) = \tanh(\tanh^{-1}\beta)\cdot\cosh(\tanh^{-1}\beta) = \gamma\beta$, we have

$$A_T = I + \delta L_{\parallel} + B(\gamma - 1) + \gamma\beta D$$

$$= I - (\gamma^2 \delta\vec{\beta}_{\parallel}\cdot\vec{k} + \gamma\delta\vec{\beta}_{\perp}\cdot\vec{k}) - \frac{\gamma-1}{\beta^2}(\vec{\beta}\times\delta\vec{\beta}_{\perp})\cdot\vec{s}$$

Since $\beta^2 = 1 - \frac{1}{\gamma^2}$, and $\dfrac{\gamma-1}{\beta^2} = \dfrac{\gamma-1}{\frac{\gamma^2-1}{\gamma^2}} = \dfrac{\gamma^2}{\gamma+1}$, and we have

$$A_T = I - (\gamma^2\delta\vec{\beta}_{\parallel}\cdot\vec{k} + \gamma\delta\vec{\beta}_{\perp}\cdot\vec{k}) - \frac{\gamma^2}{\gamma+1}(\vec{\beta}\times\delta\vec{\beta}_{\perp})\cdot\vec{s}.$$