12.3 (a) From the equation of motion, $\frac{d\vec{p}}{dt} = e\vec{u}\cdot\vec{E}$, we have

$$\frac{dp_\parallel}{dt} = \frac{d}{dt}\left(\frac{mu_\parallel}{\sqrt{1-\frac{u^2}{c^2}}}\right) = eu_\parallel E_0, \quad \frac{dp_\perp}{dt} = \frac{d}{dt}\left(\frac{mu_\perp}{\sqrt{1-\frac{u^2}{c^2}}}\right) = 0,$$

Where $u^2 = u_\parallel^2 + u_\perp^2$. From the normal component, we know that

$$\frac{mu_\perp}{\sqrt{1-\frac{u^2}{c^2}}} = C_0 = \frac{mv_0}{\sqrt{1-v_0^2/c^2}}$$

The longitudinal component can be directly integrated,

$$\frac{mu_\parallel}{\sqrt{1-u^2/c^2}} = eE_0t. \quad \text{Then, } m^2u^2/(1-u^2/c^2) = C_0^2 + e^2E_0^2t^2, \text{ or } \frac{u^2}{c^2} = \frac{C_0^2+e^2E_0^2t^2}{m^2c^2+C_0^2+e^2E_0^2t^2}$$

Therefore,

$$u_\perp = \frac{C_0}{m}\sqrt{1-\frac{u^2}{c^2}} = \frac{C_0c}{[m^2c^2+C_0^2+e^2E_0^2t^2]^{1/2}}, \quad u_\parallel = \frac{eE_0ct}{[m^2c^2+C_0^2+e^2E_0^2t^2]^{1/2}}$$

The position of the particle, assume it starts from the origin, is

$$x_\perp(t) = \int_0^t u_\perp(s)ds = \frac{C_0c}{eE_0}\operatorname{arcsinh}\left(\frac{eE_0t}{\sqrt{m^2c^2+C_0^2}}\right), \quad x_\parallel(t) = \int_0^t u_\parallel(s)ds = \frac{c}{eE_0}\left(\sqrt{m^2c^2+C_0^2+e^2E_0^2t^2} - \sqrt{m^2c^2+C_0^2}\right).$$

Notice that $m^2c^2 + C_0^2 = m^2c^2\left(1-\frac{v_0^2}{c^2}\right)^{-1} = \gamma_0^2m^2c^2$ where $\gamma_0 = (1-v_0^2/c^2)^{-1/2}$, the position can be

Written as

$$x_\perp(t) = \frac{\gamma_0mv_0c}{eE_0}\operatorname{arcsinh}\left(\frac{eE_0}{\gamma_0mc}t\right), \quad x_\parallel(t) = \frac{\gamma_0mc^2}{eE_0}\left(\sqrt{1+\frac{e^2E_0^2}{\gamma_0^2m^2c^2}t^2} - 1\right)$$

(b) From the position of the particle, we know

$$t = \frac{\gamma_0mc}{eE_0}\sinh\left(\frac{eE_0x_\perp}{\gamma_0mv_0c}\right) = \frac{\gamma_0mc}{eE_0}\left[\left(1+\frac{eE_0x_\parallel}{\gamma_0mc^2}\right)^2 - 1\right]^{1/2},$$

Which leads to

$$\left(1+\frac{eE_0x_\parallel}{\gamma_0mc^2}\right)^2 = 1+\sinh^2\left(\frac{eE_0x_\perp}{\gamma_0mv_0c}\right) = \cosh^2\left(\frac{eE_0x_\perp}{\gamma_0mv_0c}\right), \text{ or}$$

$$x_\parallel = \frac{\gamma_0mc^2}{eE_0}\left(\cosh\left(\frac{eE_0x_\perp}{\gamma_0mv_0c}\right) - 1\right).$$

For small time, $x_\perp \ll 1$, expand cosh to second order, we have $x_\parallel = \frac{eE_0}{2\gamma_0mv_0^2}x_\perp^2$, which is a parabola.

For large time, $x_\perp \gg 1$, $\cosh\left(\frac{eE_0x_\perp}{\gamma_0mv_0c}\right) - 1 \simeq \exp\left\{\frac{eE_0x_\perp}{\gamma_0mv_0c}\right\}$, and $x_\parallel = \frac{\gamma_0mc^2}{eE_0}\exp\left\{\frac{eE_0x_\perp}{\gamma_0mv_0c}\right\}$

The short and long time regime can be identified by $\frac{eE_0x_\perp}{\gamma_0mv_0c} \sim 1$.