The Diffraction of Waves by Ribbons and by Slits

PHILIP M. MORSE AND PEARL J. RUBENSTEIN
Eastman Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts

(Received September 29, 1938)

The exact solution of the diffraction problem in elliptic cylinder coordinates is computed by the use of newly computed tables of Mathieu functions. Curves are given for the distribution-in-angle and total intensity scattered, for the diffraction by a slit or the scattering by a thin ribbon, of waves whose length is the same order of magnitude as the slit or ribbon width. The curves are for different angles of incidence, for different wave-lengths, and for two boundary conditions: zero value and zero normal gradient.

THE exact solution of the problem of the diffraction of waves by a straight edge has been obtained by Sommerfeld[1] by the use of a contour integral representation. The corresponding solution[2] for the diffraction through a slit, or by a ribbon, presents certain mathematical difficulties which can, however, be overcome by separating the wave equation in elliptic cylinder coordinates and computing the values of the resulting Mathieu function solutions. The problem has been discussed previously,[3] but few numerical solutions were computed. Since a table of the needed Mathieu functions[4] has now been completed by the authors, it is of interest to discuss the properties of these functions and the nature of the solutions of the diffraction problem.

The coordinates used are given by the equations

$$\begin{aligned} x&=(d/2)\cosh\xi\cos\varphi;\\ y&=(d/2)\sinh\xi\sin\varphi;\quad z=z. \end{aligned}\tag{1}$$

We shall consider waves whose propagation vector is in the x-y plane, so that the z coordinate may be omitted from the discussion. A possible solution of the wave equation is then $G(\xi)H(\varphi)e^{-2\pi i\nu t}$ where G and H satisfy the differential equations

$$\begin{aligned} &(d^2G/d\xi^2)+(c^2\cosh^2\xi-b)G=0;\\ &(d^2H/d\varphi^2)+(b-c^2\cos^2\varphi)H=0, \end{aligned}\tag{2}$$

in which b is the separation constant and $c=(\pi d/\lambda)$, λ being the wave-length.

The periodic solutions[5] of the equation in φ are of two types: even about $\varphi=0$, and odd about $\varphi=0$. They are possible only for certain characteristic values of b. The even functions are denoted by $Se_m(c, \cos\varphi)$ and the corresponding values of b are b_m, with the sequence in m according to increasing values of b. The corresponding odd functions and characteristic values are called $So_m(c, \cos\varphi)$ and b_m'. These functions have been chosen so that

$$\begin{aligned} &Se_m(c,1)=1;\quad [(d/d\varphi)Se_m(c,\cos\varphi)]_{\varphi=0}=0;\\ &So_m(c,1)=0;\quad [(d/d\varphi)So_m(c,\cos\varphi)]_{\varphi=0}=1. \end{aligned}\tag{3}$$

The functions are orthogonal, and their normalizing factors are given by the equations

$$\int_0^{2\pi}[Se_m]^2d\varphi=N_m;\quad \int_0^{2\pi}[So_m]^2d\varphi=N_m'.\tag{4}$$

To obtain the "radial" functions dependent on ξ, we note that the equation for ξ is the same as that for φ if we set $\varphi=i\xi$. Consequently, $Se_m(c, \cosh\xi)$ or $So_m(c, \cosh\xi)$ is a solution of Eq. (2) for the same characteristic values of b_m. It is, however, more convenient to use the

[1] Sommerfeld, Math. Ann. **47**, 317 (1896); Zeits. f. Math. u. Physik **46**, 11 (1901). Calculations have also been made using parabolic cylinder functions, see Epstein, Diss. Munich (1914), and Crudelli, Nuovo Cimento **11**, 277 (1916).

[2] Schwarzschild, Math. Ann. **55**, 177 (1902).

[3] Sieger, Ann. d. Physik **27**, 626 (1908).

[4] To be published elsewhere. Preliminary mimeographed copies of the tables can be obtained from the Department of Physics, M. I. T. The tables were in part computed by means of the Bush Differential Analyzer; and the authors wish to thank Professor S. E. Caldwell and the staff of the Analyzer for their cooperation in this part of the task.

[5] See Stratton and Morse, Proc. Nat. Acad. **21**, 51 and 56 (1935).

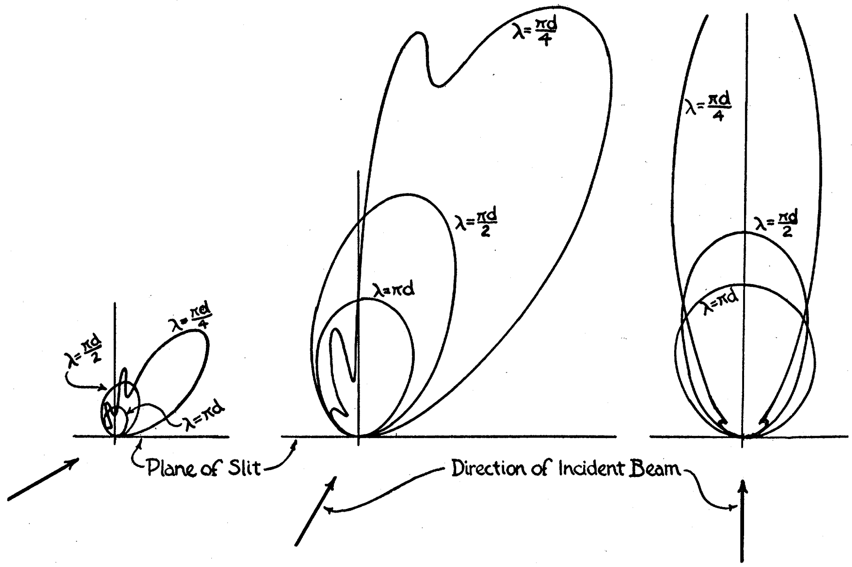

FIG. 1. Polar diagrams for distribution in angle of waves diffracted by a slit of width d (for wave function zero at boundary), or scattered by a ribbon of width d (for normal gradient zero at boundary) as function of wave-length λ and of angle of incidence of primary plane wave.

functions

$$
\begin{aligned}
Re_m{}^1(c,\rho) &= [1/(2\pi)^{\frac{1}{2}}\lambda_m]Se_m(c,\rho) \underset{\rho\to\infty}{\longrightarrow} [1/(c\rho)^{\frac{1}{2}}]\cos(c\rho-(2m+1)/4\pi),\\
Ro_m{}^1(c,\rho) &= [i/(2\pi)^{\frac{1}{2}}\lambda_m{}']So_m(c,\rho) \underset{\rho\to\infty}{\longrightarrow} [1/(c\rho)^{\frac{1}{2}}]\cos(c\rho-(2m+1)/4\pi),\\
&\qquad \rho=\cosh\xi,
\end{aligned} \tag{5}
$$

where λ_m and $\lambda_m{}'$ are constants whose values depend on the values of c. The second solutions of the equations for ξ are defined as

$$
\begin{aligned}
Re_m{}^2(c,\rho) &\underset{\rho\to\infty}{\longrightarrow} [1/(c\rho)^{\frac{1}{2}}]\sin(c\rho-(2m+1)/4),\\
Ro_m{}^2(c,\rho) &\underset{\rho\to\infty}{\longrightarrow} [1/(c\rho)^{\frac{1}{2}}]\sin(c\rho-(2m+1)/4).
\end{aligned} \tag{6}
$$

At $\rho=1$ or $\xi=0$ (that is, along a ribbon of width d in the x-z plane, between the lines $y=0$, $x=\pm(d/2)$), the values and slopes of the functions are

$$
\begin{aligned}
&Re_m{}^1(c,1)=1/(2\pi)^{\frac{1}{2}}\lambda_m; \quad Re_m{}^2(c,1)=-(2\pi)^{\frac{1}{2}}\mu_m;\\
&Ro_m{}^1(c,1)=0; \quad Ro_m{}^2(c,1)=-(2\pi)^{\frac{1}{2}}\lambda_m{}';\\
&[(d/d\xi)Re_m{}^1(c,\cosh\xi)]_{\xi=0}=0;\\
&[(d/d\xi)Re_m{}^2(c,\cosh\xi)]_{\xi=0}=(2\pi)^{\frac{1}{2}}\lambda_m;\\
&[(d/d\xi)Ro_m{}^1(c,\cosh\xi)]_{\xi=0}=1/(2\pi)^{\frac{1}{2}}\lambda_m{}';\\
&[(d/d\xi)Ro_m{}^2(c,\cosh\xi)]_{\xi=0}=(2\pi)^{\frac{1}{2}}\mu_m{}'.
\end{aligned} \tag{7}
$$

The constants λ and μ are given in the tables.[4]

It can be shown that a plane wave with propagation vector $k=(2\pi/\lambda)$ in the x-y plane and at an angle u with the x axis is given by

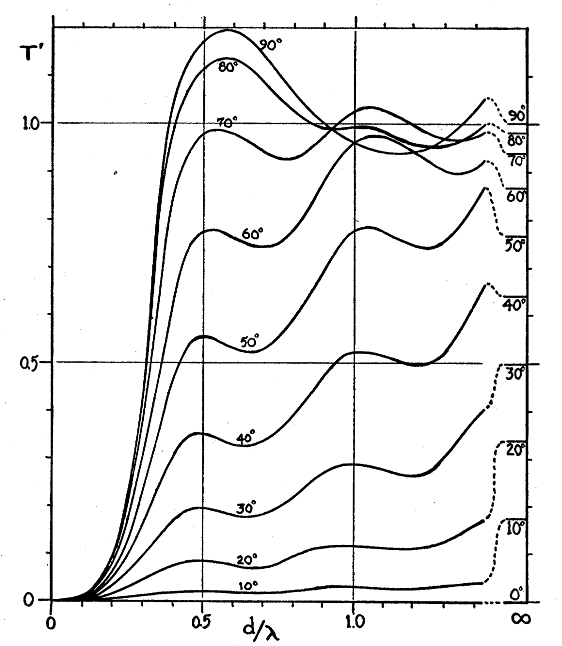

FIG. 2. Total transmission factor corresponding to the case of Fig. 1. Factor T' is the ratio between the total intensity diffracted or scattered and that required by geometrical optics for normal incidence.

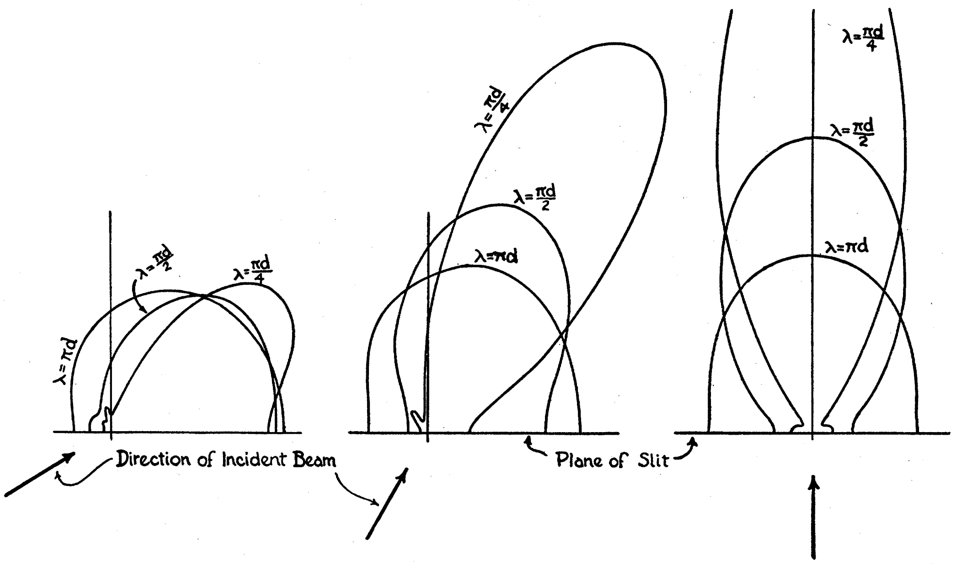

FIG. 3. Polar diagrams for waves scattered by a ribbon (for wave function zero at ribbon) or diffracted by a slit (for normal gradient zero at boundary).

the following series

$$
\begin{aligned}
e^{ik(x\cos u+y\sin u)}&=e^{ic(\cosh\xi\cos\varphi\cos u+\sinh\xi\sin\varphi\sin u)}\\
&=(8\pi)^{\frac{1}{2}}\sum_m i^m\big[(1/N_m)Se_m(c,\cos u)\\
&\qquad\times Se_m(c,\cos\varphi)Re_m{}^1(c,\cosh\xi)\\
&\qquad+(1/N_m')So_m(c,\cos u)\\
&\qquad\qquad\times So_m(c,\cos\varphi)Ro_m{}^1(c,\cosh\xi)\big],
\end{aligned}\tag{8}
$$

where $c=(kd/2)=(\pi d/\lambda)$.

By the use of the properties given in Eqs. (3)–(8), it is possible to compute the scattering from an elliptic cylinder whose axes correspond to the lines $y=0$, $x=\pm(d/2)$, and in particular from the ribbon corresponding to the limiting case of a cylinder of zero thickness, $\xi=0$. If the boundary condition is that the normal gradient of the wave function ψ is zero at the ribbon, the correct solution for a plane wave incident at an angle u to the ribbon, plus the necessary scattered wave, is

$$
\begin{aligned}
\psi=e^{ik(x\cos u+y\sin u)}&+(8\pi)^{\frac{1}{2}}\sum_m(i^m/N_m')\sin\gamma_m' e^{i\gamma_m'}\\
&\times So_m(c,\cos u)So_m(c,\cos\varphi)\\
&\times[Ro_m{}^1(c,\cosh\xi)+iRo_m{}^2(c,\cosh\xi)],
\end{aligned}\tag{9}
$$

where $\cot\gamma_m'=2\pi\lambda_m'\mu_m'$. The angles γ are given in the tables.[4] The intensity scattered at an angle φ to the plane of the ribbon per unit incident intensity therefore becomes $(d/r)I'$ at large distances from the ribbon, where

$$
\begin{aligned}
I'=(4\pi/c)\sum_{m,\,n}&(1/N_m'N_n')\sin\gamma_m'\sin\gamma_n'\\
&\times\cos(\gamma_n'-\gamma_m')So_m(c,\cos u)So_n(c,\cos u)\\
&\times So_m(c,\cos\varphi)So_n(c,\cos\varphi)
\end{aligned}\tag{10}
$$

and r is the distance from the center line of the ribbon. This angular distribution is the correct solution of the problem of the scattering of sound waves from a solid ribbon of width d, and also of that of the scattering of electromagnetic waves polarized so that the magnetic vector is parallel to the ribbon axis. Further investigation shows that this distribution is also correct for the diffraction of waves through a slit of width d. The boundary surface in this case is given by $\varphi=0$ and $\varphi=\pi$; and the boundary condition requiring the result (10) is that the wave function go to zero at the surface, corresponding to electromagnetic waves with their electric vector parallel to the slit axis. Eq. (10) then gives the intensity of the diffracted wave resulting when a plane wave of unit intensity is incident on the other side of the slit, directed at an angle u to the plane of the slit.

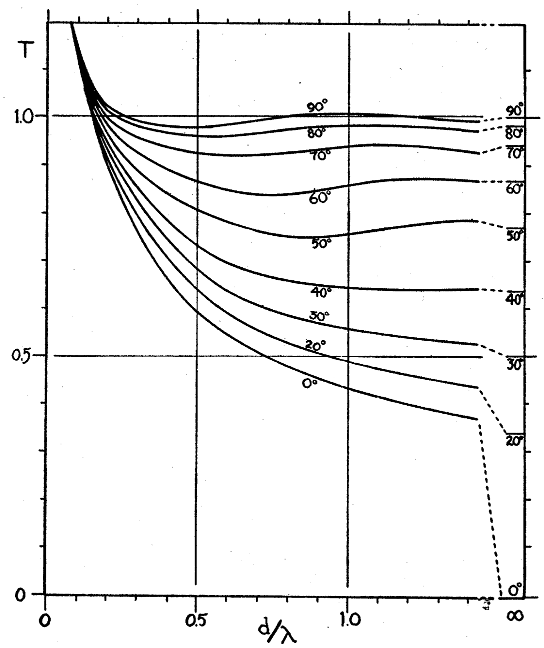

FIG. 4. Corresponding curves for total transmission factor T for the boundary conditions similar to Fig. 3, for different wave-lengths λ and different angles of incidence u.

The total energy scattered by a ribbon in all directions per unit length of ribbon, per unit primary intensity, is obtained by integrating $I'd$ over φ from 0 to 2π, the result being $2dT'$, where

$$T'=(2\pi/c)\sum_m(1/N_m')\sin^2\gamma_m'[So_m(c,\cos u)]^2. \quad (11)$$

The quantity $2T'$ is thus the ratio of the amount of energy actually scattered by the ribbon to the amount of energy which geometrical optics predicts would be reflected by the ribbon at normal incidence. Similarly, for the opposite polarization, T' is the ratio of the energy actually transmitted through the slit to the energy which geometrical optics predicts would be transmitted at normal incidence. The difference of the factor of 2 results from the fact that the wave from the slit covers only the back half of the plane, whereas the wave scattered from the ribbon is on both sides, and at short wave-lengths must be responsible for the reflected wave *and also* for interference with the plane wave behind the ribbon to cause the shadow. This reciprocity between slit and ribbon is an aspect of Babinet's theorem.

The series for I' and T' converge very rapidly for small values of $c=(\pi d/\lambda)$; only the first three or four terms are needed as long as c is less than 4. This range of c covers, however, most of the range not covered by the usual approximate diffraction theory. This range is definitely useful for acoustics and may find some use in the case of short radio waves, although it is not particularly interesting for optics. Curves of I' and T' are given in Figs. 1 and 2. The effects of resonance, for $d=(n\lambda/2)$, are quite noticeable in Fig. 2.

For the complementary boundary conditions of zero value at the surface of a ribbon or zero gradient at the boundary of a slit, the corresponding series for I and T are

$$I=(4\pi/c)\sum_{m,n}(1/N_mN_n)\sin\gamma_m\sin\gamma_n$$
$$\times\cos(\gamma_n-\gamma_m)Se_m(c,\cos u)Se_n(c,\cos u)$$
$$\times Se_m(c,\cos\varphi)Se_n(c,\cos\varphi), \quad (12)$$
$$T=(2\pi/c)\sum_m(1/N_m)\sin^2\gamma_m[Se_m(c,\cos u)]^2.$$

These results are useful for calculating the diffraction through a slit of sound waves, or of electromagnetic waves with the magnetic vector parallel to the slit axis; or for calculating the scattering from a ribbon of electromagnetic waves with the electric vector parallel to the ribbon axis. Curves of I and T are given in Figs. 3 and 4. The difference is marked between these curves and those for the other boundary condition; very little resonance effect is noticeable in Fig. 4.

Several other applications of these functions can be made in the field of acoustics. Some of the properties of the ribbon microphone and the ribbon loudspeaker can be studied; and also the absorption of sound by a strip of absorbing material placed in a reflecting wall can be treated. Reports on these applications will be published elsewhere.