9.17 (a) This part can be easily solved by using the results from Section 9.12, with

$$\vec{J}(\vec{x}) = \hat{r}\,\frac{I(r)}{2\pi r^2}\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right],$$

where $I(r) = I_0 \sin(2\pi r/d) = I_0 \sin(kr)$. In Eq. (9.181), the angular integration can be performed in the same way, leading

$$\int d\Omega\, Y^*_{lm}\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right] = 2\pi\,\delta_{m,0}\left[Y_{l0}(0)+Y_{l0}(\pi)\right]$$

$$= \sqrt{4\pi(2l+1)}\,,\quad \text{for } l \text{ even},\ m=0.$$

Then, $$a_E(l,0) = \frac{k}{2\pi}\left[\frac{4\pi(2l+1)}{l(l+1)}\right]^{1/2}\int_0^{d/2}\left\{-\frac{d}{dr}\left[r j_l(kr)\frac{dI}{dr}\right]\right\}dr$$

$$= \frac{kI_0}{2\pi}\left[\frac{4\pi(2l+1)}{l(l+1)}\right]^{1/2}\frac{d}{2}\,j_l\!\left(\frac{kd}{2}\right)k = \sqrt{\frac{\pi(2l+1)}{l(l+1)}}\,kI_0 j_l(\pi).$$

Since $kd/2=\pi$. The electric multipole coefficients disappear for all $m\neq 0$ and odd l.

The magnetic multipole coefficients are all zero.

We can also calculate the electric multipole moment in the long-wavelength limit. From charge conservation,

$$\rho(\vec{x}) = \frac{1}{i\omega}\frac{dI}{dr}\,\frac{\delta(\cos\theta-1)+\delta(\cos\theta+1)}{2\pi r^2}.$$

the electric multipole moment for l even is

$$Q_{l0} = \int r^l Y^*_{l0}\,\rho(\vec{x})\,d^3x$$

$$= \int_0^{2\pi}d\phi\int_{-1}^{1}d(\cos\theta)\int_0^{+\infty}r^2dr\cdot r^l\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)\,\frac{1}{i\omega}\frac{dI}{dr}\,\frac{\delta(\cos\theta-1)+\delta(\cos\theta+1)}{2\pi r^2}$$

$$= 4\pi\sqrt{\frac{2l+1}{4\pi}}\,\frac{kI_0}{i\omega}\,\frac{1}{2\pi}\int_0^{d/2} r^l\cos(kr)\,dr$$

$$= \frac{I_0}{ick^{l+1}}\sqrt{\frac{2l+1}{\pi}}\int_0^{\pi}x^l\cos x\,dx$$

and the electric multipole coefficient is

$$a_E(l,0) = \frac{ck^{l+2}}{i(2l+1)!!}\left(\frac{l+1}{l}\right)^{1/2}Q_{l0} = -\frac{kI_0}{(2l+1)!!}\sqrt{\frac{(2l+1)(l+1)}{\pi l}}\int_0^{\pi}x^l\cos x\,dx$$

(b) From the exact electric multipole coefficient, we have

$$a_E(2,0) = \sqrt{\frac{5\pi}{6}}\, k I_0 j_2(\pi) = \sqrt{\frac{5\pi}{6}}\, k I_0 \frac{3}{\pi^2} = \sqrt{\frac{15}{2\pi^3}}\, k I_0,$$

and the corresponding power radiated per solid angle is

$$\frac{dP(2,0)}{d\Omega} = \frac{Z_0}{2k^2} |a_E(2,0)|^2 |\vec{X}_{2,0}|^2 = \frac{Z_0}{2k^2} \cdot \frac{15}{2\pi^3} \cdot k^2 I_0^2 \cdot \frac{15}{8\pi} \sin^2\theta \cos^2\theta = \frac{225 Z_0 I_0^2}{32\pi^4} \sin^2\theta \cos^2\theta.$$

and the power radiated is

$$P(2,0) = \frac{Z_0}{2k^2} |a_E(2,0)|^2 = \frac{15 Z_0 I_0^2}{4\pi^3} = \frac{Z_0 I_0^2}{4\pi} \times \frac{15}{\pi^2} = \frac{Z_0 I_0^2}{4\pi} \times 1.5198178$$

For the long-wavelength approximation,

$$a_E(2,0) = \frac{k I_0}{15} \sqrt{\frac{15}{2\pi}} \cdot 2\pi = \sqrt{\frac{2\pi}{15}}\, k I_0,$$

and

$$P = \frac{Z_0}{2k^2} |a_E(2,0)|^2 = \frac{Z_0 I_0^2 \pi}{15} = \frac{Z_0 I_0^2}{4\pi} \times \frac{4\pi^2}{15} = \frac{Z_0 I_0^2}{4\pi} \times 2.6318945$$

Compare with 9.16 (b), we can see that the exact quadrupole radiation almost explains the exact total radiation, while the long-wavelength approximation is quite poor. The reason is clear, since the current system does not satisfy the condition that $kd \ll 1$, where actually $kd = 2\pi$.