9.14 (a) The current density can be written as

$$\vec{J}(\vec{x}') = \frac{I_0}{2\pi a}\delta(r'-a)\delta(\cos\theta')\cos(\omega t)\,\hat{e}_{\phi'} = \frac{I_0}{2\pi a}\delta(r'-a)\delta(\cos\theta')\cos(\omega t)\left(-\sin\phi'\hat{i}+\cos\phi'\hat{j}\right)$$

Then, $$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')\,e^{-i\vec{k}\cdot\vec{x}'}\,d^3x'$$ $\left(\vec{k} = k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),\ \vec{x}' = a(\cos\phi', \sin\phi', 0)\right)$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int_0^{2\pi}d\phi'\int_{-1}^{1}d(\cos\theta')\int_0^{+\infty}r'^2dr'\cdot\frac{I_0}{2\pi a}\delta(r'-a)\delta(\cos\theta')$$

$$\times\left(-\sin\phi'\hat{i}+\cos\phi'\hat{j}\right)\exp\{-ika\sin\theta\cos(\phi-\phi')\}$$

$$= \frac{\mu_0}{4\pi}\frac{I_0 a e^{ikr}}{r}\cdot\frac{1}{2\pi}\int_0^{2\pi}d\phi'\left(-\sin\phi'\hat{i}+\cos\phi'\hat{j}\right)\exp\{-ika\sin\theta\cos(\phi-\phi')\}$$

Using the identity for Bessell function, $e^{-iz\cos\phi} = \sum_{m=-\infty}^{+\infty}(-i)^m J_m(z)\,e^{-im\phi}$,

the integral can be evaluated as

$$\frac{1}{2\pi}\int_0^{2\pi}d\phi'\left(-\frac{e^{i\phi'}-e^{-i\phi'}}{2i}\hat{i}+\frac{e^{i\phi'}+e^{-i\phi'}}{2}\hat{j}\right)\sum_{m=-\infty}^{+\infty}(-i)^m J_m(ka\sin\theta)\,e^{-im(\phi-\phi')}$$

$$= \sum_{m=-\infty}^{+\infty}(-i)^m J_m(ka\sin\theta)\left[-\frac{1}{2i}\left(\delta_{m,-1}e^{i\phi}-\delta_{m,1}e^{-i\phi}\right)\hat{i}\right.$$

$$\left.+\frac{1}{2}\left(\delta_{m,-1}e^{i\phi}+\delta_{m,1}e^{-i\phi}\right)\hat{j}\right]$$

$$= -\frac{1}{2i}\left[(-i)^{-1}J_{-1}(ka\sin\theta)e^{i\phi}-(-i)J_1(ka\sin\theta)e^{-i\phi}\right]\hat{i}$$ $(J_{-1}(x) = -J_1(x))$

$$+\frac{1}{2}\left[(-i)^{-1}J_{-1}(ka\sin\theta)e^{i\phi}-(-i)J_1(ka\sin\theta)e^{-i\phi}\right]\hat{j}$$

$$= (-i)J_1(ka\sin\theta)\left[-\sin\phi\,\hat{i}+\cos\phi\,\hat{j}\right] = -iJ_1(ka\sin\theta)\,\hat{e}_\phi.$$

Therefore, the vector potential is given by

$$\vec{A}(\vec{x}) = -i\frac{\mu_0}{4\pi}\frac{I_0 a e^{ikr}}{r}J_1(ka\sin\theta)\,\hat{e}_\phi.$$

The magnetic field in the radiation zone is given by

$$\vec{H} = i\vec{k}\times\vec{A}/\mu_0 = -\frac{I_0}{4\pi}\frac{ka\,e^{ikr}}{r}J_1(ka\sin\theta)\,\hat{e}_\theta.$$

and the electric field is

$$\vec{E} = Z_0 \vec{H} \times \vec{n} = \frac{Z_0 I_0}{4\pi} \frac{ka e^{ikr}}{r} J_1(ka \sin\theta)\, \hat{e}_\phi$$

Then. $$\frac{dP}{d\Omega} = \frac{1}{2} \mathrm{Re}[r^2 \vec{n} \cdot \vec{E} \times \vec{H}^*] = \frac{Z_0}{2} |r\vec{H} \times \vec{n}|^2 = \frac{Z_0 I_0^2}{32\pi^2} (ka)^2 J_1(ka\sin\theta)^2.$$

(b) The radiating system does not contain any magnetization. therefore $Q'_{lm} = M'_{lm} = 0$.

Also, the current is steady, there will be no net charge, and $Q_{lm} = 0$. This leaves us with only M_{lm}. Perform integration by parts, we find

$$M_{lm} = \frac{1}{l+1} \int (\vec{r} \times \vec{J}) \cdot \nabla (r^l Y^*_{lm})\, d^3x$$

Since $\vec{r} \times \vec{J}$ is in the $\hat{z}$-direction, we must have $m=0$ for M_{lm} to be non-zero. For $l=0$, $\nabla(r^0 Y^*_{00}) = 0$, then the lowest vanishing multipole moment must start at least from $l=1$. For $l=1$,

$$\nabla(r Y^*_{10}) = \nabla\left(r \sqrt{\frac{3}{4\pi}} \frac{z}{r}\right) = \sqrt{\frac{3}{4\pi}}\, \hat{z}, \text{ Therefore.}$$

$$M_{10} = \frac{1}{2} \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \int_0^{+\infty} r^2 dr \sqrt{\frac{3}{4\pi}}\, r \frac{I_0}{2\pi a} \delta(r-a)\, \delta(\cos\theta)$$

$$= \sqrt{\frac{3}{4\pi}} \frac{I_0 a^2}{2}$$

This is consistent with the physical consideration. The loop of current clearly carries an oscillating magnetic dipole moment and the lowest multipole moment must be of that type. From M_{10}, we know

$$a_M(1,0) = \frac{ik^3}{3} \cdot \sqrt{2} \cdot \sqrt{\frac{3}{4\pi}} \frac{I_0 a^2}{2},$$

and $$\frac{dP(1,0)}{d\Omega} = \frac{Z_0}{2k^2} \cdot \frac{k^6}{9} \frac{3}{2\pi} \frac{I_0^2 a^4}{4} \cdot \frac{3}{8\pi} \sin^2\theta = \frac{Z_0 I_0^2}{128\pi^2} (ka)^4 \sin^2\theta,$$

which agrees with the exact result in the $ka \ll 1$ limit, as $J_1(x) \sim x/2$, for $x \ll 1$.