3.22 Solution: This problem can be solved by repeating the procedure described in the solution of Problem 2.25, with a slight modification for $\rho = a$ boundary.

The Dirac delta function on the two-dimensional surface with restricted angular range $[0, \beta]$ can be written as

$$\delta(\boldsymbol{\rho} - \boldsymbol{\rho}') = \frac{1}{\rho}\delta(\rho - \rho')\delta(\phi - \phi') = \frac{1}{\rho}\delta(\rho - \rho') \cdot \frac{2}{\beta}\sum_{m=1}^{\infty} \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right),$$

and the corresponding Green function must have a similar form,

$$G(\boldsymbol{\rho}, \boldsymbol{\rho}') = \frac{2}{\beta}\sum_{m=1}^{\infty} g_m(\rho, \rho') \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right). \tag{1}$$

Applying the Laplace operator, we have

$$\nabla^2_{\boldsymbol{\rho}'} G(\boldsymbol{\rho}, \boldsymbol{\rho}') = \frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial G}{\partial\rho'}\right) + \frac{1}{\rho'^2}\frac{\partial^2 G}{\partial\phi'^2} = -4\pi\delta(\boldsymbol{\rho} - \boldsymbol{\rho}').$$

With Eq. (1), we have

$$\begin{aligned}\sum_{m=1}^{\infty} &\left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right) - \frac{m^2\pi^2}{\beta^2}\right] g_m(\rho, \rho') \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right) \\ &= -4\pi \cdot \frac{1}{\rho}\delta(\rho - \rho') \cdot \sum_{m=1}^{\infty} \sin\left(\frac{m\pi\phi}{\beta}\right) \sin\left(\frac{m\pi\phi'}{\beta}\right).\end{aligned}$$

The above equation must be valid term-wise, which leads to a differential equation for g_m,

$$\left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right) - \frac{m^2\pi^2}{\beta^2}\right] g_m(\rho, \rho') = -4\pi \cdot \frac{1}{\rho}\delta(\rho - \rho').$$

Now, we can apply the familiar procedure to determine g_m. First, solutions to g_m must be linear combinations of $\rho'^{m\pi/\beta}$ and $\rho'^{-m\pi/\beta}$. For $\rho' < \rho$, regularity at $\rho' = 0$ indicates that we can only choose the $\rho'^{m\pi/\beta}$ solution, or

$$g_m(\rho, \rho') \propto \rho'^{m\pi/\beta}.$$

Similarly, for $\rho' > \rho$, taking into account of the fact that $g_m(\rho, a) = 0$,

$$g_m(\rho, \rho') \propto \rho'^{m\pi/\beta} - \frac{a^{2m\pi/\beta}}{\rho'^{m\pi/\beta}}.$$

The final solution must be symmetric in ρ and ρ', which means

$$g_m(\rho, \rho') = A_m \rho_<^{m\pi/\beta}\left(\rho_>^{m\pi/\beta} - \frac{a^{2m\pi/\beta}}{\rho_>^{m\pi/\beta}}\right).$$

where $\rho_<$ ($\rho_>$) is the smaller (larger) of ρ and ρ'. Integrate the differential equation governing g_m in a small interval around ρ, we can obtain a relation that connects the derivative of g_m on both side of ρ,

$$\rho'\frac{\partial}{\partial\rho'} g_m(\rho, \rho')\bigg|_{\rho-\varepsilon}^{\rho+\varepsilon} = -4\pi.$$

For $\rho' = \rho + \varepsilon$,

$$\rho' \frac{\partial}{\partial \rho'} g_m(\rho, \rho') \bigg|_{\rho'=\rho+\varepsilon} = A_m \, \rho' \frac{\partial}{\partial \rho'} \left[\rho^{m\pi/\beta} \left(\rho'^{m\pi/\beta} - \frac{a^{2m\pi/\beta}}{\rho'^{m\pi/\beta}} \right) \right] \bigg|_{\rho'=\rho+\varepsilon} = A_m \frac{m\pi}{\beta} \left(\rho^{2m\pi/\beta} + a^{2m\pi/\beta} \right),$$

and for $\rho' = \rho - \varepsilon$,

$$\rho' \frac{\partial}{\partial \rho'} g_m(\rho, \rho') \bigg|_{\rho'=\rho-\varepsilon} = A_m \, \rho' \frac{\partial}{\partial \rho'} \left[\rho'^{m\pi/\beta} \left(\rho^{m\pi/\beta} - \frac{a^{2m\pi/\beta}}{\rho^{m\pi/\beta}} \right) \right] \bigg|_{\rho'=\rho-\varepsilon} = A_m \frac{m\pi}{\beta} \left(\rho^{2m\pi/\beta} - a^{2m\pi/\beta} \right),$$

which leads to

$$\rho' \frac{\partial}{\partial \rho'} g_m(\rho, \rho') \bigg|_{\rho-\varepsilon}^{\rho+\varepsilon} = 2A_m \frac{m\pi}{\beta} a^{2m\pi/\beta} = -4\pi,$$

or,

$$A_m = -\frac{2\beta}{m} a^{-2m\pi/\beta}.$$

Put the solution of g_m back into Eq. (1), the Green function becomes

$$\begin{aligned} G(\rho, \phi; \rho', \phi') &= -\sum_{m=1}^{\infty} \frac{4}{m} a^{-2m\pi/\beta} \rho_<^{m\pi/\beta} \left(\rho_>^{m\pi/\beta} - \frac{a^{2m\pi/\beta}}{\rho_>^{m\pi/\beta}} \right) \sin\left(\frac{m\pi\phi}{\beta} \right) \sin\left(\frac{m\pi\phi'}{\beta} \right) \\ &= \sum_{m=1}^{\infty} \frac{4}{m} \rho_<^{m\pi/\beta} \left(\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{a^{2m\pi/\beta}} \right) \sin\left(\frac{m\pi\phi}{\beta} \right) \sin\left(\frac{m\pi\phi'}{\beta} \right). \end{aligned}$$