16.3. (a) The electron in an elliptic orbit has a negative energy, and the binding energy is the negative of that. Therefore, the secular change in energy is given by

$$\frac{d\mathcal{E}}{dt} = \left\langle -\frac{dE}{dt} \right\rangle = \frac{\tau}{m}\left\langle \left(\frac{dV}{dr}\right)^2 \right\rangle = \frac{\tau}{m}\cdot\frac{1}{T}\int_0^T \left(\frac{dV}{dr}\right)^2 dt,$$

where $T = \pi Z e^2 \sqrt{\frac{m}{2\mathcal{E}^3}}$ is the period of the orbit. (See. Landau & Lifshitz, Mechanics, (15.8)). The time integral over one period can be replaced by an integral in angle, $\int_0^T dt \to \int_0^{2\pi} d\theta \frac{mr^2}{L}$.

Then. $$\frac{d\mathcal{E}}{dt} = \frac{\tau}{m}\cdot\frac{1}{T}\int_0^{2\pi} \frac{mr^2}{L}\,\frac{Z^2e^4}{r^4}\,d\theta = \frac{2e^2}{3m^2c^3}\,\frac{2^{1/2}\mathcal{E}^{3/2}}{m^{1/2}}\,\frac{1}{\pi Z e^2}\,\frac{m}{L}\cdot Z^2e^4\int_0^{2\pi}\frac{d\theta}{r^2}$$

$$= \frac{2e^2}{3m^2c^3}\,\frac{2^{1/2}\mathcal{E}^{3/2}}{m^{1/2}}\,\frac{1}{\pi Z e^2}\,\frac{m}{L}\,Z^2e^4\cdot\frac{Z^2e^4m^2}{L^4}\left[2\pi + \pi\left(1-\frac{2\mathcal{E}L^2}{Z^2e^4m}\right)\right]$$

$$= \frac{2^{3/2}}{3}\,\frac{Z^3e^8m^{1/2}}{c^3}\,\frac{\mathcal{E}^{3/2}}{L^5}\left(3-\frac{2\mathcal{E}L^2}{Z^2e^4m}\right)$$

Similarly, $$\frac{dL}{dt} = -\frac{\tau}{m}\cdot\frac{1}{T}\int_0^T \frac{1}{r}\frac{dV}{dr}\,dt\cdot L$$

$$= -\frac{2e^2}{3m^2c^3}\,\frac{2^{1/2}\mathcal{E}^{3/2}}{m^{1/2}}\,\frac{1}{\pi Z e^2}\,\frac{m}{L}\cdot Ze^2\cdot L\int_0^{2\pi}\frac{d\theta}{r}$$

$$= -\frac{2e^2}{3m^2c^3}\,\frac{2^{1/2}\mathcal{E}^{3/2}}{m^{1/2}}\,\frac{1}{\pi Z e^2}\,m\cdot Ze^2\cdot\frac{Ze^2m}{L^2}\cdot 2\pi$$

$$= -\frac{2^{5/2}}{3}\,\frac{Ze^4}{m^{1/2}c^3}\,\frac{\mathcal{E}^{3/2}}{L^2}$$

(b) From part (a), we know

$$\frac{d\mathcal{E}}{dL} = -\frac{1}{2}\,\frac{Z^2e^4m}{L^3}\left(3-\frac{2\mathcal{E}L^2}{Z^2e^4m}\right) = -\frac{3}{2}\,\frac{Z^2e^4m}{L^3} + \frac{\mathcal{E}}{L}.$$

Then $$\frac{d}{dL}\left(\frac{\mathcal{E}}{L}\right) = \frac{1}{L}\frac{d\mathcal{E}}{dL} - \frac{\mathcal{E}}{L^2} = \frac{1}{L}\left(\frac{d\mathcal{E}}{dL}-\frac{\mathcal{E}}{L}\right) = -\frac{3}{2}\,\frac{Z^2e^4m}{L^4}.$$

Integrating both sides w.r.t. to L, we will arrive at

$$\frac{\mathcal{E}(L)}{L} - \frac{\mathcal{E}(L_0)}{L_0} = \frac{Z^2e^4m}{2}\left(\frac{1}{L^3}-\frac{1}{L_0^3}\right), \quad\text{or}\quad \mathcal{E}(L) = \frac{Z^2e^4m}{2L^2}\left[1-\left(\frac{L}{L_0}\right)^3\right] + \frac{\mathcal{E}_0}{L_0}L,$$

where $\mathcal{E}(L_0) = \mathcal{E}_0$.

The eccentricity of the elliptic orbit is given by, for some angular momentum,

$$e = \left(1 - \frac{2\varepsilon L^2}{Z^2 e^4 m}\right)^{1/2}.$$ Then, the change in angular momentum leads to

$$e(L) = \left(1 - \left[1 - \left(\frac{L}{L_0}\right)^3\right] - \frac{2L^2}{Z^2 e^4 m}\frac{\varepsilon_0}{L_0} L\right)^{1/2}$$

$$= \left(\left(\frac{L}{L_0}\right)^3 - \frac{2\varepsilon_0 L_0^2}{Z^2 e^4 m}\left(\frac{L}{L_0}\right)^3\right)^{1/2} = \left(\frac{L}{L_0}\right)^{3/2}\left[1 - \frac{2\varepsilon_0 L_0^2}{Z^2 e^4 m}\right]^{1/2}$$

$$= e(L_0)\left(\frac{L}{L_0}\right)^{3/2},$$

Where the eccentricity decreases as $(L/L_0)^{3/2}$. Since circular orbit has an eccentricity of 0, as time passes by, with decreasing eccentricity, the orbit will become more circular. This can also be understood as the effect of increasing binding energy. For attractive potential, the effective potential $\left(-Ze^2/r + L^2/2mr^2\right)$ has a minimum, which corresponds to a maximum binding energy and a circular orbit.