3.6 Solution: (a) Let $\mathbf{x}' = (0, 0, a)$ in Cartesian coordinates, then the electrostatic potential due to the two charges is

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{|\mathbf{x}+\mathbf{x}'|}\right).$$

Using the expansion

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\frac{x_<^l}{x_>^{l+1}}P_l(\cos\theta),$$

the potential can be written as

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{x_<^l}{x_>^{l+1}}\left(P_l(\cos\theta) - P_l\left(\cos(\pi-\theta)\right)\right)\\ &= \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{x_<^l}{x_>^{l+1}}\left(P_l(\cos\theta) - P_l\left(-\cos\theta\right)\right)\\ &= \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{x_<^l}{x_>^{l+1}}\left(1-(-1)^l\right)P_l(\cos\theta).\end{aligned}$$

The second term can be viewed as if we have flipped the direction of the z-axis. In the last step, we have used the relation $P_l(-x) = (-1)^l P_l(x)$. Clearly, only odd terms will contribute to the result. $x_<$ ($x_>$) is the smaller (larger) of x and a. Therefore, for $x < a$,

$$\Phi(\mathbf{x}) = \frac{q}{2\pi\varepsilon_0}\sum_{j=0}^{\infty}\frac{r^{2j+1}}{a^{2j+2}}P_{2j+1}(\cos\theta),$$

for $x > a$,

$$\Phi(\mathbf{x}) = \frac{q}{2\pi\varepsilon_0}\sum_{j=0}^{\infty}\frac{a^{2j+1}}{x^{2j+2}}P_{2j+1}(\cos\theta).$$

(b) In the limit of $a \to 0$ with qa staying constant, only the P_1 component will remain. In this case, the potential becomes

$$\Phi(\mathbf{x}) = \frac{q}{2\pi\varepsilon_0}\frac{a}{x^2}P_1(\cos\theta) = \frac{p}{4\pi\varepsilon_0}\frac{\cos\theta}{x^2}.$$

(c) In the presence of a conducting sphere, the potential will be modified with extra terms inside the sphere,

$$\Phi'(\mathbf{x}) = \frac{p}{4\pi\varepsilon_0}\frac{\cos\theta}{x^2} + \sum_{l=0}^{\infty}A_l\frac{x^l}{b^{l+1}}P_l(\cos\theta).$$

Since the sphere is grounded, the potential must vanish on the sphere, which only leaves with $l = 1$ term, *i.e.*, $A_l \equiv 0$, for $l \neq 1$. For $l = 1$,

$$A_1 = -\frac{p}{4\pi\varepsilon_0 b}.$$

Finally, the potential inside the sphere is

$$\Phi'(\mathbf{x}) = \frac{p}{4\pi\varepsilon_0}\left(\frac{1}{x^2} - \frac{x}{b^3}\right)\cos\theta = \frac{p}{4\pi\varepsilon_0 x^2}\left(1 - \left(\frac{x}{b}\right)^3\right)\cos\theta.$$