8.14 (a) From the invariant $\bar{n} = n(x_{max}) = n(0)\,\mathrm{sech}(\alpha x_{max}) = n(0)\cos\theta(0)$, we have $\mathrm{sech}(\alpha x_{max}) = \cos\theta(0)$. Then, the trajectory $z(x)$ is given by

$$z(x) = \bar{n}\int_0^x \frac{du}{\sqrt{n(u)^2 - \bar{n}^2}} = \cos\theta(0)\int_0^x \frac{du}{\sqrt{\mathrm{sech}^2(\alpha u) - \mathrm{sech}^2(\alpha x_{max})}}$$

$$= \cos\theta(0)\int_0^x \frac{\cosh(\alpha x_{max})\cosh(\alpha u)}{\sqrt{\cosh^2(\alpha x_{max}) - \cosh^2(\alpha u)}}\,du$$

$$= \int_0^x \frac{\cosh(\alpha u)}{\sqrt{\cosh^2(\alpha x_{max}) - 1 - \sinh^2(\alpha u)}}\,du = \frac{1}{\alpha}\int_0^x \frac{d(\sinh(\alpha u))}{\sqrt{\sinh^2(\alpha x_{max}) - \sinh^2(\alpha u)}}$$

$$= \frac{1}{\alpha}\arcsin\left(\frac{\sinh(\alpha x)}{\sinh(\alpha x_{max})}\right),$$

or equivalently, $\sinh(\alpha x) = \sinh(\alpha x_{max})\sin(\alpha z)$,

$$\Rightarrow \quad \alpha x = \mathrm{arcsinh}\left[\sinh(\alpha x_{max})\sin(\alpha z)\right].$$

(b) In part (a), set $x = x_{max}$,

$$z = \bar{n}\int_0^{x_{max}} \frac{dx}{\sqrt{n^2(x) - \bar{n}^2}} = \frac{\pi}{2\alpha},$$

which is independent of $\bar{n}$.

(c) $$L_{opt} = \int_0^{x_{max}} n(x)\,ds = \int_0^{x_{max}} \frac{n^2(x)}{\sqrt{n^2(x) - \bar{n}^2}}\,dx$$

$$= n(0)\int_0^{x_{max}} \frac{\mathrm{sech}^2(\alpha x)}{\left[\mathrm{sech}^2(\alpha x) - \mathrm{sech}^2(\alpha x_{max})\right]^{1/2}}\,dx$$

$$= n(0)\cosh(\alpha x_{max})\int_0^{x_{max}} \frac{\mathrm{sech}(\alpha x)\,dx}{\left[\cosh^2(\alpha x_{max}) - \cosh^2(\alpha x)\right]^{1/2}}$$

$$= \frac{n(0)\cosh(\alpha x_{max})}{\alpha}\int_0^{x_{max}} \frac{d(\sinh(\alpha x))}{\cosh^2(\alpha x)\sqrt{\sinh^2(\alpha x_{max}) - \sinh^2(\alpha x)}}$$

Let $a = \sinh(\alpha x_{max})$, and $\sinh(\alpha x) = a\sin\theta$, the integral becomes

$$\int_0^{x_{max}} \frac{d(\sinh(\alpha x))}{(1+\sinh^2(\alpha x))\sqrt{\sinh^2(\alpha x_{max}) - \sinh^2(\alpha x)}}$$

$$= \int_0^{\pi/2} \frac{a\cos\theta \, d\theta}{(1+a^2\sin^2\theta)\cdot a\cos\theta} = \int_0^{\pi/2} \frac{d\theta}{1+a^2\sin^2\theta}$$

$$= \frac{\pi}{2\sqrt{1+a^2}} = \frac{\pi}{2\cosh(\alpha x_{max})}$$

Then,

$$L_{opt} = \frac{n(0)\cosh(\alpha x_{max})}{\alpha} \cdot \frac{\pi}{2\cosh(\alpha x_{max})} = n(0)\frac{\pi}{2\alpha} = n(0) Z.$$