12.3 (a) From the equation of motion in the absence of a magnetic field, $\frac{d\vec{p}}{dt} = e\vec{E}$, we can express it in component form, with one parallel to the electric field and the other perpendicular to it,

i.e. $\frac{dp_\parallel}{dt} = \frac{d}{dt}\left(\frac{mu_\parallel}{\sqrt{1-\frac{u^2}{c^2}}}\right) = eE$, $\quad \frac{dp_\perp}{dt} = \frac{d}{dt}\left(\frac{mu_\perp}{\sqrt{1-\frac{u^2}{c^2}}}\right) = 0$,

where $u^2 = u_\parallel^2 + u_\perp^2$. The $p_\parallel$ component can be directly integrated, $\frac{mu_\parallel}{\sqrt{1-\frac{u^2}{c^2}}} = eEt$, as $u_\parallel(0) = 0$.

For the $p_\perp$ component, $\frac{mu_\perp}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{mv_0}{\sqrt{1-\frac{v_0^2}{c^2}}} = \gamma_0 m v_0$, where $\gamma_0 = 1/\sqrt{1-v_0^2/c^2}$. Then,

$$\frac{m^2u^2}{1-\frac{u^2}{c^2}} = (\gamma_0 m v_0)^2 + (eEt)^2, \quad \text{or} \quad \frac{u^2}{c^2} = \frac{\gamma_0^2 m^2 v_0^2 + e^2E^2t^2}{m^2c^2 + \gamma_0^2 m^2 v_0^2 + e^2E^2t^2}$$

Therefore, $u_\perp = \gamma_0 v_0 \sqrt{1-\frac{u^2}{c^2}} = \frac{\gamma_0 m v_0 c}{\sqrt{m^2c^2+\gamma_0^2m^2v_0^2+e^2E^2t^2}} = \frac{p_0c^2}{\sqrt{\mathcal{E}_0^2+(eEct)^2}}$, where $p_0 = \gamma_0 m v_0$ is the initial momentum of the particle and $\mathcal{E}_0 = \gamma_0 mc^2$ is the initial energy. Also, $u_\parallel = \frac{eEc^2t}{\sqrt{\mathcal{E}_0^2+(eEct)^2}}$.

If the particle starts from origin, then

$$x_\perp(t) = \int_0^t u_\perp(s)\,ds = \frac{p_0c}{eE}\int_0^t \frac{ds}{\sqrt{s^2+(\mathcal{E}_0/eEc)^2}} = \frac{p_0c}{eE}\operatorname{arcsinh}\left(\frac{eEct}{\mathcal{E}_0}\right).$$

and $$x_\parallel(t) = \int_0^t u_\parallel(s)\,ds = c\int_0^t \frac{s\,ds}{\sqrt{s^2+(\mathcal{E}_0/eEc)^2}} = c\left(\sqrt{t^2+\left(\frac{\mathcal{E}_0}{eEc}\right)^2} - \frac{\mathcal{E}_0}{eEc}\right) = \frac{1}{eE}\left(\sqrt{\mathcal{E}_0^2+(eEct)^2} - \mathcal{E}_0\right).$$

(b) Eliminating t from the particle trajectory, we have

$$t = \frac{\mathcal{E}_0}{eEc}\sinh\left(\frac{eE}{p_0c}x_\perp\right) = \frac{\mathcal{E}_0}{eEc}\left[\left(1+\frac{eE}{\mathcal{E}_0}x_\parallel\right)^2 - 1\right]^{1/2},$$

which leads to $\left(1+\frac{eE}{\mathcal{E}_0}x_\parallel\right)^2 = 1+\sinh^2\left(\frac{eE}{p_0c}x_\perp\right) = \cosh^2\left(\frac{eE}{p_0c}x_\perp\right)$, or $x_\parallel = \frac{\mathcal{E}_0}{eE}\left[\cosh\left(\frac{eE}{p_0c}x_\perp\right) - 1\right]$

For small time, $x_\perp \ll 1$, expand to second order, we will have $x_\parallel = \frac{eE\mathcal{E}_0}{2p_0^2c^2}x_\perp^2 = \frac{eE}{2\gamma_0 m v_0^2}x_\perp^2$, which is a parabola. For large time, $x_\perp \gg 1$, $\cosh\left(\frac{eE}{p_0c}x_\perp\right) \sim \exp\left\{\frac{eE}{p_0c}x_\perp\right\}$, and $x_\parallel = \frac{\mathcal{E}_0}{eE}\exp\left\{\frac{eE}{p_0c}x_\perp\right\}$.
The short and large time regimes can be separated by the condition that

$$\frac{eE}{p_0c}x_\perp \sim 1.$$