11.25 (a) Using the fact that the norm of energy-momentum 4-vector is a Lorentz invariant, we have

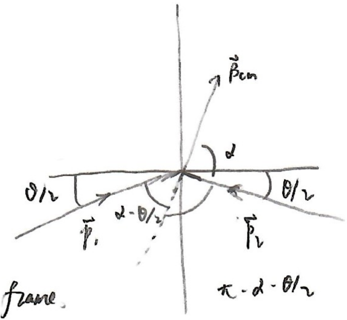
$$W^2 = (\vec{E}_1+\vec{E}_2)^2 - |\vec{p}_1+\vec{p}_2|^2 = m_1^2+m_2^2+2E_1E_2-2\vec{p}_1\cdot\vec{p}_2.$$

For relativistic particles, $E=\sqrt{p^2+m^2} = p+\frac{m^2}{2p}$. Also, the angle between $\vec{p}_1$ and $\vec{p}_2$ is $\pi-\theta$. Therefore,

$$W^2 = m_1^2+m_2^2+2\left(p_1+\frac{m_1^2}{2p_1}\right)\left(p_2+\frac{m_2^2}{2p_2}\right)+2p_1p_2\cos\theta$$

$$= m_1^2+m_2^2+\frac{m_1^2}{p_1}+\frac{m_2^2}{p_2}+2p_1p_2(1+\cos\theta)$$

$$= 4p_1p_2\cos^2\left(\frac{\theta}{2}\right)+(p_1+p_2)\left(\frac{m_1^2}{p_1}+\frac{m_2^2}{p_2}\right).$$




(b) The total 3-momentum in the CM frame is 0. Then, due to the Lorentz transform in the transverse direction, we know that the transverse momentum should be 0 in the laboratory frame.

or. $p_1\sin(\alpha-\frac{\theta}{2}) = p_2\sin(\pi-\alpha-\theta/2)$.

Which is equivalent to $p_1\sin\alpha\cos\frac{\theta}{2} - p_1\cos\alpha\sin\frac{\theta}{2} = p_2\sin\alpha\cos\frac{\theta}{2}+p_2\cos\alpha\sin\frac{\theta}{2}$.

$$\Rightarrow (p_1-p_2)\sin\alpha\cos\frac{\theta}{2} = (p_1+p_2)\cos\alpha\sin\frac{\theta}{2} \quad\Rightarrow\quad \tan\alpha = \frac{p_1+p_2}{p_1-p_2}\tan\frac{\theta}{2}$$

In the longitudinal direction,

$$0=\gamma_{cm}\left(p_1\cos(\alpha-\frac{\theta}{2})+p_2\cos(\pi-\alpha-\frac{\theta}{2})-\beta_{cm}(E_1+E_2)\right)$$

$$\Rightarrow \beta_{cm} = \frac{1}{E_1+E_2}\left(p_1\cos\left(\alpha-\frac{\theta}{2}\right)-p_2\cos\left(\alpha+\frac{\theta}{2}\right)\right) = \frac{1}{E_1+E_2}\left((p_1-p_2)\cos\alpha\cos\frac{\theta}{2}+(p_1+p_2)\sin\alpha\sin\frac{\theta}{2}\right)$$

Using earlier result, $(p_1-p_2)\cos\frac{\theta}{2} = (p_1+p_2)\sin\frac{\theta}{2}\frac{\cos\alpha}{\sin\alpha}$. Then

$$\beta_{cm} = \frac{1}{E_1+E_2}(p_1+p_2)\sin\frac{\theta}{2}\left(\frac{\cos\alpha}{\sin\alpha}+\sin\alpha\right) = \frac{(p_1+p_2)\sin\frac{\theta}{2}}{(E_1+E_2)\sin\alpha}$$

(c) For the configuration, $\theta\to 0$, and $\alpha\to\frac{p_1+p_2}{p_1-p_2}\frac{\theta}{2}$. Then,

$$\beta_{cm}\to\frac{p_1+p_2}{E_1+E_2}\cdot\frac{1}{\frac{p_1+p_2}{p_1-p_2}} = \frac{p_1-p_2}{E_1+E_2}.$$

But. $p_1=p_{LAB}$, $p_2=0$, $E_1=E_{LAB}$, $E_2=m_2$, we have $\beta_{cm}=\frac{p_{LAB}}{E_{LAB}+m_2}$. Also, $\alpha\to 0$, $\beta_{cm}$ is in the same direction as $\vec{p}_1$. Thus, $\vec{\beta}_{cm} = \frac{\vec{p}_{LAB}}{E_{LAB}+m_2}$