2.18 Solution: (a) Using the result from Problem 2.19, the Green function inside a cylinder of radius b can be directly obtained as

$$G(\rho,\phi;\rho',\phi') = \log\left(\frac{b^2}{\rho_>^2}\right) + 2\sum_{m=1}^{\infty}\frac{\cos[m(\phi-\phi')]}{m}\rho_<^m\left(\frac{1}{\rho_>^m} - \frac{\rho_>^m}{b^{2m}}\right).$$

Using the identity

$$\log(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n},$$

we have

$$\begin{aligned}
G(\rho,\phi;\rho',\phi') &= \log\left(\frac{b^2}{\rho_>^2}\right) + 2\cdot\mathrm{Re}\left\{\sum_{m=1}^{\infty}\frac{e^{im(\phi-\phi')}}{m}\rho_<^m\left(\frac{1}{\rho_>^m} - \frac{\rho_>^m}{b^{2m}}\right)\right\} \\
&= \log\left(\frac{b^2}{\rho_>^2}\right) + 2\cdot\mathrm{Re}\left\{\sum_{m=1}^{\infty}\frac{e^{im(\phi-\phi')}}{m}\left(\left(\frac{\rho_<}{\rho_>}\right)^m - \left(\frac{\rho_<\rho_>}{b^2}\right)^m\right)\right\} \\
&= \log\left(\frac{b^2}{\rho_>^2}\right) + 2\cdot\mathrm{Re}\left\{\log\left(1-\frac{\rho_<\rho_>}{b^2}e^{i(\phi-\phi')}\right) - \log\left(1-\frac{\rho_<}{\rho_>}e^{i(\phi-\phi')}\right)\right\} \\
&= \log\left(\frac{b^2}{\rho_>^2}\right) + \log\left(1+\frac{\rho_<^2\rho_>^2}{b^4} - 2\frac{\rho_<\rho_>}{b^2}\cos(\phi-\phi')\right) \\
&\qquad - \log\left(1+\frac{\rho_<^2}{\rho_>^2} - 2\frac{\rho_<}{\rho_>}\cos(\phi-\phi')\right) \\
&= \log\left(\frac{\rho^2\rho'^2 + b^4 - 2\rho\rho' b^2\cos(\phi-\phi')}{b^2(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi'))}\right),
\end{aligned}$$

where the last equality is already symmetric in ρ and ρ'.

(b) For the potential inside the cylinder, choose the normal direction to be away from the origin, then

$$\begin{aligned}
\left.\frac{\partial G}{\partial n'}\right|_{\rho'=b} &= \left.\frac{\partial G}{\partial \rho'}\right|_{\rho'=b} \\
&= \left.\left(\frac{2\rho^2\rho' - 2\rho b^2\cos(\phi-\phi')}{\rho^2\rho'^2 + b^4 - 2\rho\rho' b^2\cos(\phi-\phi')} - \frac{2\rho' - 2\rho\cos(\phi-\phi')}{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\right)\right|_{\rho'=b} \\
&= \frac{2\rho^2 - 2\rho b\cos(\phi-\phi')}{b(\rho^2+b^2-2\rho b\cos(\phi-\phi'))} - \frac{2b-2\rho\cos(\phi-\phi')}{\rho^2+b^2-2\rho b\cos(\phi-\phi')} \\
&= \frac{2\rho^2-2b^2}{b(\rho^2+b^2-2\rho b\cos(\phi-\phi'))}.
\end{aligned}$$

Applying Eq. (1.42), the potential inside the cylinder can be expressed as

$$\begin{aligned}
\Phi(\rho,\phi) &= -\frac{1}{4\pi}\int_{\rho'=b}\Phi(\rho',\phi')\frac{\partial G}{\partial n'}da' \\
&= -\frac{1}{4\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{2\rho^2-2b^2}{b(\rho^2+b^2-2\rho b\cos(\phi-\phi'))}\cdot b\,d\phi' \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{\rho^2+b^2-2\rho b\cos(\phi-\phi')}d\phi',
\end{aligned}$$

which gives exactly the same result as Problem 2.12.

(c) Using the result from Problem 2.19, the Green function outside a cylinder of radius b can be written as

$$G(\rho,\phi;\rho',\phi') = \log\left(\frac{\rho_<^2}{b^2}\right) + 2\sum_{m=1}^{\infty}\frac{\cos[m(\phi-\phi')]}{m}\left(\rho_<^m - \frac{b^{2m}}{\rho_<^m}\right)\frac{1}{\rho_>^m}.$$

Summing the series, we will have

$$\begin{aligned}
G(\rho,\phi;\rho',\phi') &= \log\left(\frac{\rho_<^2}{b^2}\right) + 2\cdot\mathrm{Re}\left\{\sum_{m=1}^{\infty}\frac{e^{im(\phi-\phi')}}{m}\left(\left(\frac{\rho_<}{\rho_>}\right)^m - \left(\frac{b^2}{\rho_<\rho_>}\right)^m\right)\right\} \\
&= \log\left(\frac{\rho_<^2}{b^2}\right) + 2\cdot\mathrm{Re}\left\{\log\left(1-\frac{b^2}{\rho_<\rho_>}e^{i(\phi-\phi')}\right) - \log\left(1-\frac{\rho_<}{\rho_>}e^{i(\phi-\phi')}\right)\right\} \\
&= \log\left(\frac{\rho_<^2}{b^2}\right) + \log\left(1+\frac{b^4}{\rho_<^2\rho_>^2} - 2\frac{b^2}{\rho_<\rho_>}\cos(\phi-\phi')\right) \\
&\qquad - \log\left(1+\frac{\rho_<^2}{\rho_>^2} - 2\frac{\rho_<}{\rho_>}\cos(\phi-\phi')\right) \\
&= \log\left(\frac{\rho^2\rho'^2 + b^4 - 2\rho\rho' b^2\cos(\phi-\phi')}{b^2(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi'))}\right).
\end{aligned}$$

Therefore, the Green function stays the same for potential problem outside of the cylinder.