3.9 Solution: We want to find the potential inside the cylinder. In order that Φ be single valued and vanish on both end surfaces, we must have

$$Q(\phi) = e^{im\phi},$$

and

$$Z(z) = \sin\left(\frac{n\pi}{L}z\right),$$

with m being the integers and n positive integers. The radial factor is

$$R(\rho) = C_I I_m\left(\frac{n\pi}{L}\rho\right) + C_K K_m\left(\frac{n\pi}{L}\rho\right),$$

where I and K are modified Bessel functions of the first and the second kind, respectively. If the potential is finite at $\rho = 0$, $C_K = 0$. Therefore, the general form of the solution is

$$\Phi(\rho, \phi, z) = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} A_{nm} \sin\left(\frac{n\pi}{L}z\right) I_m\left(\frac{n\pi}{L}\rho\right) e^{im\phi}.$$

On the surface of the cylinder, we are given the potential $V(\phi, z)$,

$$V(\phi, z) = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} A_{nm} \sin\left(\frac{n\pi}{L}z\right) I_m\left(\frac{n\pi}{L}b\right) e^{im\phi}.$$

The coefficients can be determined as

$$A_{nm} = \frac{1}{\pi L I_m(n\pi b/L)} \int_0^{2\pi} d\phi \int_0^L dz V(\phi, z) \sin\left(\frac{n\pi}{L}z\right) e^{-im\phi},$$

and the potential inside the cylinder becomes

$$\Phi(\rho, \phi, z) = \frac{1}{\pi L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{I_m(n\pi\rho/L)}{I_m(n\pi b/L)} \int_0^{2\pi} d\phi' \int_0^L dz' V(\phi', z') \sin\left(\frac{n\pi}{L}z\right) \sin\left(\frac{n\pi}{L}z'\right) e^{im(\phi-\phi')}.$$

The potential is real, since $I_{-m}(x) \equiv I_m(x)$ for integer m. We will keep the current form of the solution for the ease of the calculation in next problem.