2.14 Solution: (a) Using the result of Problem 2.12, the potential inside the cylinder has the following series form,

$$\Phi(\rho, \phi) = A_0 + \sum_{m=1}^{+\infty} \left(A_m e^{im\phi} + B_{-m} e^{-im\phi} \right) \rho^m.$$

The coefficients can be determined by the boundary condition. For the zeroth order term, it is related to the average of the potential on the side of the cylinder, and therefore is 0,

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} \Phi(b, \phi') d\phi' = 0.$$

For A_m, we have

$$\begin{aligned}
A_m &= \frac{1}{2\pi b^m} \int_0^{2\pi} \Phi(b, \phi') e^{-im\phi'} d\phi' \\
&= \frac{V}{2\pi b^m} \left(\int_0^{\pi/2} - \int_{\pi/2}^{\pi} + \int_{\pi}^{3\pi/2} - \int_{3\pi/2}^{2\pi} \right) e^{-im\phi'} d\phi' \\
&= \frac{V}{2\pi b^m} \frac{i}{m} \left(e^{-im\phi'} \Big|_0^{\pi/2} - e^{-im\phi'} \Big|_{\pi/2}^{\pi} + e^{-im\phi'} \Big|_{\pi}^{3\pi/2} - e^{-im\phi'} \Big|_{3\pi/2}^{2\pi} \right) \\
&= \frac{V}{2\pi b^m} \frac{i}{m} \left((-i)^m - 1 - (-1)^m + (-i)^m + i^m - (-1)^m - 1 + i^m \right) \\
&= \frac{V}{\pi b^m} \frac{i}{m} \left(i^m + (-i)^m - (-1)^m - 1 \right).
\end{aligned}$$

Here, $i^m + (-i)^m - (-1)^m - 1 = -4$, only when $m = 4k + 2$, for $k \geq 0$. Otherwise, it is 0. Similarly,

$$B_m = \frac{V}{\pi b^m} \frac{-i}{m} \left(i^m + (-i)^m - (-1)^m - 1 \right).$$

Put the coefficients back into the series, we will have

$$\begin{aligned}
\Phi(\rho, \phi) &= \sum_{k=0}^{+\infty} \frac{V}{\pi b^{4k+2}} \frac{i}{4k+2} \left(-4e^{i(4k+2)\phi} + 4e^{-i(4k+2)\phi} \right) \rho^{4k+2} \\
&= \frac{4V}{\pi} \sum_{k=0}^{+\infty} \left(\frac{r}{b} \right)^{4k+2} \frac{\sin\left((4k+2)\phi\right)}{2k+1}.
\end{aligned}$$

(b) Applying the Euler's identity, the potential can be written as

$$\Phi(\rho, \phi) = \frac{4V}{\pi} \mathrm{Im} \sum_{n=0}^{\infty} \frac{Z^{2n+1}}{2n+1},$$

where

$$Z = \left(\frac{\rho}{b} \right)^2 e^{2i\phi}.$$

Using the relation

$$\sum_{n=0}^{\infty} \frac{Z^{2n+1}}{2n+1} = \frac{1}{2} \log\left(\frac{1+Z}{1-Z} \right),$$

the series can be summed to

$$\Phi(\rho, \phi) = \frac{2V}{\pi} \mathrm{Im} \log\left(\frac{1+Z}{1-Z} \right) = \frac{2V}{\pi} \arctan\left(\frac{2\mathrm{Im}Z}{1 - |Z|^2} \right) = \frac{2V}{\pi} \arctan\left(\frac{2\rho^2 b^2}{b^4 - \rho^4} \sin(2\phi) \right).$$