5.31 (a) Similar to the previous problem, the magnetic scalar potential is given by

$$\Phi_m = a_1 \rho \sin\phi, \qquad \rho < R$$

$$\Phi_m = (b_1 \rho + c_1/\rho)\sin\phi, \qquad R < \rho < R'$$

Here, we have only retained the non-zero term. The continuity condition leads to

$$\begin{cases} a_1 = b_1 - c_1/R^2 \\ -a_1 + b_1 + c_1/R^2 = \dfrac{NI}{2R} \\ b_1 + c_1/R'^2 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = -\dfrac{NIR}{4}\left(\dfrac{1}{R^2} + \dfrac{1}{R'^2}\right) \\ b_1 = -\dfrac{NIR}{4R'^2} \\ c_1 = \dfrac{NIR}{4} \end{cases}$$

For $\rho < R$,

$$\vec{B} = -\mu_0 \nabla \Phi_m = \frac{\mu_0 NIR}{4}\left(\frac{1}{R^2} + \frac{1}{R'^2}\right)\left(\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi}\right)\rho \sin\phi$$

$$= \frac{\mu_0 NI}{4R}\left(1 + \frac{R^2}{R'^2}\right)(\hat{\rho}\sin\phi + \hat{\phi}\cos\phi) = \frac{\mu_0 NI}{4R}\left(1 + \frac{R^2}{R'^2}\right)\hat{y}$$

(b) For $\rho < R$,

$$W_{in} = \frac{1}{2\mu_0}\int |\vec{B}|^2 d^2x = \frac{1}{2\mu_0}\frac{\mu_0^2 N^2 I^2}{16R^2}\left(1 + \frac{R^2}{R'^2}\right)^2 \cdot \pi R^2 = \frac{\mu_0 \pi N^2 I^2}{32}\left(1 + \frac{R^2}{R'^2}\right)^2$$

Which enhances the energy inside, compared to the case without the iron.

For $\rho > R$,

$$\vec{B} = -\mu_0 \nabla \Phi_m = \frac{\mu_0 NIR}{4}\left(\hat{\rho}\frac{\partial}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi}\right)\left(\frac{\rho}{R'^2} - \frac{1}{\rho}\right)\sin\phi$$

$$= \frac{\mu_0 NIR}{4}\left[\hat{\rho}\left(\frac{1}{R'^2} + \frac{1}{\rho^2}\right)\sin\phi + \hat{\phi}\left(\frac{1}{R'^2} - \frac{1}{\rho^2}\right)\cos\phi\right]$$

and

$$W_{out} = \frac{1}{2\mu_0}\int |\vec{B}|^2 d^2x = \frac{\mu_0 \pi N^2 I^2 R^2}{16}\int_R^{R'}\left(\frac{1}{\rho^4} + \frac{1}{R'^4}\right)\rho\, d\rho$$

$$= \frac{\mu_0 \pi N^2 I^2 R^2}{32}\left(\frac{1}{R'^2} - \frac{R^2}{R'^4} - \frac{1}{R'^2} + \frac{1}{R^2}\right) = \frac{\mu_0 \pi N^2 I^2}{32}\left(1 - \left(\frac{R}{R'}\right)^4\right)$$

(c) $$W = W_{in} + W_{out} = \frac{\mu_0 \pi N^2 I^2}{16}\left(1 + \left(\frac{R}{R'}\right)^2\right) = \frac{1}{2}LI^2$$

Thus, $$L = \frac{\mu_0 \pi N^2}{8}\left(1 + \left(\frac{R}{R'}\right)^2\right)$$