11.18 (a) In the rest frame of the particle, the electric and magnetic fields are given by

$$\vec{E}' = \frac{q\vec{r}'}{r'^3} = \frac{q(\vec{r}_\perp' + z'\hat{z})}{r'^3}, \quad \vec{B}' = 0$$

In the laboratory frame, $\vec{r}_\perp' = \vec{r}_\perp$, $z' = \gamma(z - vt)$, we can write the fields as

$$\vec{E}' = \frac{q(\vec{r}_\perp + \gamma(z-vt)\hat{z})}{[r_\perp^2 + \gamma^2(z-vt)^2]^{3/2}}, \quad \vec{B}' = 0$$

Applying the Lorentz transform, we know that, in the laboratory frame,

$$\vec{E} = \frac{\gamma q(\vec{r}_\perp + (z-vt)\hat{z})}{[r_\perp^2 + \gamma^2(z-vt)^2]^{3/2}}, \quad \vec{B} = \gamma\vec{\beta}\times\vec{E}' = \gamma\vec{\beta}\times\frac{q\vec{r}_\perp}{[r_\perp^2+\gamma^2(z-vt)^2]^{3/2}} = \vec{\beta}\times\vec{E}$$

The electric field can be written as

$$\vec{E} = \frac{q}{r_\perp^2}(\vec{r}_\perp + (z-vt)\hat{z})\frac{\gamma/r_\perp}{[1+(\frac{\gamma}{r_\perp})^2(z-vt)^2]^{3/2}}$$

In limit of $\beta\to 1$, $\gamma\to\infty$. In the following, we will show that $\vec{E}$ is singular with a Dirac delta function.

Consider the function $g(z) = \frac{\alpha}{2[1+\alpha^2 z^2]^{3/2}}$, whose indefinite integral is

$$\Phi(z) = \frac{1}{2}\int\frac{\alpha\, dz}{[1+\alpha^2 z^2]^{3/2}} = \frac{\alpha z}{2\sqrt{1+\alpha^2 z^2}} + C. \text{ Then}$$

$$\text{(i)} \quad \int_{-\infty}^{+\infty} g(z)dz = \lim_{z\to\infty}\left(\frac{\alpha z}{2\sqrt{1+\alpha^2 z^2}} - \frac{-\alpha z}{2\sqrt{1+\alpha^2 z^2}}\right) = 1$$

and

$$\text{(ii)} \quad \int_{\varepsilon<|z|<+\infty} g(z)dz = \frac{1}{2}\lim_{z\to\infty}\left(\frac{\alpha z}{\sqrt{1+\alpha^2 z^2}} - \frac{\alpha\varepsilon}{\sqrt{1+\alpha^2\varepsilon^2}}\right) = \frac{1}{2}\left(1 - \frac{\alpha\varepsilon}{\sqrt{1+\alpha^2\varepsilon^2}}\right)$$

As $\alpha\to\infty$,

$$\frac{\alpha\varepsilon}{\sqrt{1+\alpha^2\varepsilon^2}} = \frac{1}{\sqrt{1+\frac{1}{\alpha^2\varepsilon^2}}} = 1 - \frac{1}{2\alpha^2\varepsilon^2}.$$

Therefore

$$\lim_{\alpha\to\infty}\int_{\varepsilon<|z|<+\infty} g(z)\,dz = \frac{1}{4\alpha^2\varepsilon^2} \to 0, \text{ for any } \varepsilon>0.$$

Comparing to Theorem 2.4.1 of Fourier Analysis by Stein and Shakarchi, we know that

$$\lim_{\alpha\to\infty}\frac{\alpha z}{2[1+\alpha^2 z^2]^{3/2}} = \delta(z)$$

Thus, the electric field, in the limit $\beta\to 1$, $\gamma\to\infty$, is

$$\vec{E} = \lim_{v\to c}\frac{2q}{r_\perp^2}(\vec{r}_\perp + (z-vt)\hat{z})\frac{\gamma/r_\perp}{2[1+(\frac{\gamma}{r_\perp})^2(z-vt)^2]^{3/2}} = \frac{2q}{r_\perp^2}(\vec{r}_\perp + (z-ct)\hat{z})\,\delta(z-ct) = \frac{2q}{r_\perp^2}\vec{r}_\perp\,\delta(ct-z)$$

For the magnetic field, as $\beta \to 1$, $\vec{\beta} \to \hat{v}$, where $\hat{v}$ is in the z-direction, parallel to the velocity of the particle. Then $\vec{B} = \hat{v} \times \vec{E} = 2q \dfrac{\hat{v} \times \vec{r}_\perp}{r_\perp^2} \delta(ct-z)$.

(b) Using the Maxwell equation, we can show that

$$\rho = \frac{1}{4\pi} \nabla \cdot \vec{E} = \frac{q}{2\pi} \nabla_\perp \cdot \left(\frac{\vec{r}_\perp}{r_\perp^2} \right) \delta(ct-z).$$

It's straightforward to verify that, for $r_\perp \neq 0$, $\nabla_\perp \cdot \left(\frac{\vec{r}_\perp}{r_\perp^2} \right) = 0$. On the other hand, for any circle enclosing $\vec{r}_\perp = 0$, we have

$$\int \nabla_\perp \cdot \left(\frac{\vec{r}_\perp}{r_\perp^2} \right) dS = \oint \frac{\vec{r}_\perp}{r_\perp^2} \cdot \vec{n}\, dl = \int_0^{2\pi} \frac{\vec{r}_\perp}{r_\perp^2} \cdot \vec{r}_\perp d\theta = 2\pi$$

Therefore, symbolically, $\nabla \cdot \left(\frac{\vec{r}_\perp}{r_\perp^2} \right) = 2\pi \delta^{(2)}(\vec{r}_\perp)$, and $\rho = q \delta^{(2)}(\vec{r}_\perp) \delta(ct-z)$

For the 3-current, $\frac{4\pi}{c} \vec{J} = \nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$. Since

$$\nabla \times \vec{B} = \nabla \times \left(2q \frac{\hat{v} \times \vec{r}_\perp}{r_\perp^2} \delta(ct-z) \right) = 2q \left(\nabla \delta(ct-z) \times \frac{\hat{v} \times \vec{r}_\perp}{r_\perp^2} + \delta(ct-z) \nabla \times \frac{\hat{v} \times \vec{r}_\perp}{r_\perp^2} \right)$$

$$= 2q \left(-\delta'(ct-z) \hat{v} \times \frac{\hat{v} \times \vec{r}_\perp}{r_\perp^2} + \delta(ct-z) \hat{v} \left(\nabla \cdot \frac{\vec{r}_\perp}{r_\perp^2} \right) \right)$$

$$= 2q \left(\frac{\vec{r}_\perp}{r_\perp^2} \delta'(ct-z) + 2\pi \hat{v} \delta^{(2)}(\vec{r}_\perp) \delta(ct-z) \right),$$

and $\dfrac{\partial \vec{E}}{\partial t} = 2qc \dfrac{\vec{r}_\perp}{r_\perp^2} \delta'(ct-z)$, then

$$\frac{4\pi}{c} \vec{J} = \nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = 4\pi q \hat{v} \delta^{(2)}(\vec{r}_\perp) \delta(ct-z), \Rightarrow \vec{J} = qc\hat{v} \delta^{(2)}(\vec{r}_\perp) \delta(ct-z).$$

Finally, $J^\alpha = (c\rho, \vec{J}) = (1, \hat{v}) qc \delta^{(2)}(\vec{r}_\perp) \delta(ct-z) = qcv^\alpha \delta^{(2)}(\vec{r}_\perp) \delta(ct-z)$,

where $v^\alpha = (1, \hat{v})$

(c) From the first gauge, we have

$$\vec{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \vec{A}_1}{\partial t} = -\left(\hat{z} \frac{\partial}{\partial z} + \nabla_\perp \right) A_1^0 - \frac{1}{c} \hat{z} \frac{\partial A_1^z}{\partial t}$$

$$= -2q \hat{z} \delta'(ct-z) + 2q \frac{\vec{r}_\perp}{r_\perp^2} \delta(ct-z) + 2q \hat{z} \delta'(ct-z) = 2q \frac{\vec{r}_\perp}{r_\perp^2} \delta(ct-z)$$

$$\vec{B} = \nabla \times \vec{A}_1 = \nabla \times (\hat{z} A_1^z) = \nabla A_1^z \times \hat{z} = \nabla_\perp A_1^z \times \hat{z} = -2q \frac{\vec{r}_\perp}{r_\perp^2} \delta(ct-z) \times \hat{z} = 2q \frac{\hat{v} \times \vec{r}_\perp}{r_\perp^2} \delta(ct-z).$$

For the second gauge,

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}_2}{\partial t} = -\frac{1}{c} \frac{\partial \vec{A}_{2,\perp}}{\partial t} = 2q \delta(ct-z) \nabla_\perp \log(\lambda r_\perp) = 2q \frac{\vec{r}_\perp}{r_\perp^2} \delta(ct-z)$$

$$\vec{B} = \nabla \times \vec{A}_2 = \nabla \times \vec{A}_{2,\perp} = \left(\hat{z} \frac{\partial}{\partial z} + \nabla_\perp \right) \times \left(-2q\, \Theta(ct-z) \nabla_\perp \log(\lambda r_\perp) \right) = -2q \frac{\partial}{\partial z} \Theta(ct-z)\, \hat{z} \times \nabla_\perp \log(\lambda r_\perp)$$

$$= 2q\,\delta(ct-z)\,\hat{z}\times\nabla_\perp \log(\lambda r_\perp) = 2q\,\frac{\hat{v}\times\vec{r}_\perp}{r_\perp^2}\,\delta(ct-z)$$

The difference between the two gauges is

$$\Delta A^\alpha = A_1^\alpha - A_2^\alpha = \left(-2q\,\delta(ct-z)\log(\lambda r_\perp),\ 2q\,\Theta(ct-z)\nabla_\perp \log(\lambda r_\perp),\ -2q\,\delta(ct-z)\log(\lambda r_\perp)\right)$$

It can be easily verified that $\Delta A^\alpha = \partial^\mu \chi$, where

$$\chi = -2q\,\Theta(ct-z)\log(\lambda r_\perp)$$