5.8 $\vec{J}(\vec{x}) = \hat{\phi} J(r,\theta)$.

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x$$

$$= \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int d^3x' \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi) J(r',\theta')(-\sin\phi'\hat{i}+\cos\phi'\hat{j})$$

$$= \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int d^3x' \frac{r_<^l}{r_>^{l+1}}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta')P_l^m(\cos\theta)e^{im(\phi-\phi')}$$

$$\times J(r',\theta')(-\sin\phi'\hat{i}+\cos\phi'\hat{j})$$

Only $m=\pm1$ terms will contribute. Also.

$$P_l^{-1}(x) = -\frac{(l-1)!}{(l+1)!}P_l^1(x) = -\frac{1}{l(l+1)}P_l^1(x)$$

then $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}\int d^3x' \frac{r_<^l}{r_>^{l+1}} J(r',\theta')(-\sin\phi'\hat{i}+\cos\phi'\hat{j})$

$$\times\left(\frac{1}{l(l+1)}P_l^1(\cos\theta')P_l^1(\cos\theta)e^{i(\phi-\phi')} + l(l+1)\frac{1}{l(l+1)}P_l^1(\cos\theta')\frac{1}{l(l+1)}P_l^1(\cos\theta)e^{-i(\phi-\phi')}\right)$$

$$= \frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}\int d^3x' \frac{r_<^l}{r_>^{l+1}} J(r',\theta')(-\sin\phi'\hat{i}+\cos\phi'\hat{j})$$

$$\times\frac{1}{l(l+1)}P_l^1(\cos\theta')P_l^1(\cos\theta)\, 2\cos(\phi-\phi')$$

$$= \frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}\int_0^{+\infty} r'^2dr'\int_{-1}^{1}d(\cos\theta')\int_0^{2\pi}d\phi' \frac{r_<^l}{r_>^{l+1}}J(r',\theta')(-\sin\phi'\hat{i}+\cos\phi'\hat{j})$$

$$\times\frac{1}{l(l+1)}P_l^1(\cos\theta')P_l^1(\cos\theta)\, 2\cos(\phi-\phi')$$

Since $\int_0^{2\pi}d\phi'\cos(\phi-\phi')\sin\phi' = \pi\sin\phi$, $\int_0^{2\pi}d\phi'\cos(\phi-\phi')\cos\phi' = \pi\cos\phi$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}P_l^1(\cos\theta)\int_0^{+\infty}r'^2dr'\int_{-1}^{+1}d(\cos\theta')\frac{r_<^l}{r_>^{l+1}}J(r',\theta')$$

$$\times\frac{1}{l(l+1)}P_l^1(\cos\theta')\,2\pi(-\sin\phi\hat{i}+\cos\phi\hat{j})$$

Or. $A_\phi = \frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}\frac{1}{l(l+1)}P_l^1(\cos\theta)\int_0^{+\infty}r'^2dr'\int_{-1}^{1}d(\cos\theta')\int_0^{2\pi}d\phi'\frac{r_<^l}{r_>^{l+1}}J(r',\theta')P_l^1(\cos\theta')$

Interior: $A_\phi(r,\theta) = -\frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}m_l r^l P_l^1(\cos\theta)$, $m_l = -\frac{1}{l(l+1)}\int d^3x\, r^{-l-1}P_l^1(\cos\theta)J(r,\theta)$

Exterior: $A_\phi(r,\theta) = -\frac{\mu_0}{4\pi}\sum_{l=1}^{\infty}\mu_l r^{-l-1}P_l^1(\cos\theta)$, $\mu_l = -\frac{1}{l(l+1)}\int d^3x\, r^l P_l^1(\cos\theta)J(r,\theta)$