15.6 (a) It is straightforward verification. Notice that $\vec{\beta} = \beta \vec{e}_z$, $\Delta\vec{\beta} = |\Delta\vec{\beta}| (\cos\phi \, \vec{e}_x + \sin\phi \, \vec{e}_y)$,

$\vec{n} = \sin\theta \, \vec{e}_x + \cos\theta \, \vec{e}_z$, $\vec{\epsilon}_{\parallel} = -\cos\theta \, \vec{e}_x + \sin\theta \, \vec{e}_z$, $\vec{\epsilon}_{\perp} = \vec{e}_y$. we have

$$\vec{\beta} \times \Delta\vec{\beta} = \beta |\Delta\vec{\beta}| (-\sin\phi \, \vec{e}_x + \cos\phi \, \vec{e}_y),$$

$$\vec{n} \times (\vec{\beta} \times \Delta\vec{\beta}) = \beta |\Delta\vec{\beta}| (-\cos\theta\cos\phi \, \vec{e}_x - \cos\theta\sin\phi \, \vec{e}_y + \sin\theta\cos\phi \, \vec{e}_z)$$

$$\Delta\vec{\beta} + \vec{n} \times (\vec{\beta} \times \Delta\vec{\beta}) = |\Delta\vec{\beta}| \left((1-\beta\cos\theta)\cos\phi \, \vec{e}_x + (1-\beta\cos\theta)\sin\phi \, \vec{e}_y + \beta\sin\theta\cos\phi \, \vec{e}_z \right)$$

$$\vec{\epsilon}_{\parallel} \cdot \left[\Delta\vec{\beta} + \vec{n} \times (\vec{\beta} \times \Delta\vec{\beta}) \right] = |\Delta\vec{\beta}| (\beta - \cos\theta) \cos\phi$$

$$\vec{\epsilon}_{\perp} \cdot \left[\Delta\vec{\beta} + \vec{n} \times (\vec{\beta} \times \Delta\vec{\beta}) \right] = |\Delta\vec{\beta}| (1 - \beta\cos\theta) \sin\phi .$$

(b) Since $\lim_{\omega\to 0} \frac{d^2 I}{d\omega d\Omega} = \frac{z^2 e^2}{4\pi^2 c} \left| \vec{\epsilon}^* \cdot \frac{\Delta\vec{\beta} + \vec{n} \times (\vec{\beta} \times \Delta\vec{\beta})}{(1 - \vec{n}\cdot\vec{\beta})^2} \right|^2 .$

we know
$$\lim_{\omega\to 0} \frac{d^2 I_{\parallel}}{d\omega d\Omega} = \frac{z^2 e^2}{4\pi^2 c} \frac{|\Delta\vec{\beta}|^2 (\beta - \cos\theta)^2 \cos^2\phi}{(1-\beta\cos\theta)^4},$$

and
$$\lim_{\omega\to 0} \frac{d^2 I_{\perp}}{d\omega d\Omega} = \frac{z^2 e^2}{4\pi^2 c} \frac{|\Delta\vec{\beta}|^2 \sin^2\phi}{(1-\beta\cos\theta)^2}$$

Performing the average in ϕ, $\langle \cos^2\phi \rangle = \langle \sin^2\phi \rangle = 1/2$, and in the limit $\gamma \gg 1$ and $\theta \ll 1$,

$$\beta - \cos\theta \simeq \left(1 - \frac{1}{\gamma^2}\right)^{1/2} - \left(1 - \frac{\theta^2}{2}\right) = \frac{1}{2\gamma^2} (\gamma^2\theta^2 - 1) .$$

$$1 - \beta\cos\theta \simeq 1 - \left(1 - \frac{1}{\gamma^2}\right)^{1/2} \left(1 - \frac{\theta^2}{2}\right) = \frac{1}{2\gamma^2} (1 + \gamma^2\theta^2) ,$$

then
$$\lim_{\omega\to 0} \frac{d^2 I}{d\omega d\Omega} = \lim_{\omega\to 0} \left(\frac{d^2 I_{\parallel}}{d\omega d\Omega} + \frac{d^2 I_{\perp}}{d\omega d\Omega} \right)$$

$$= \frac{z^2 e^2 |\Delta\vec{\beta}|^2}{4\pi^2 c} \left(\frac{\frac{1}{4\gamma^4} (\gamma^2\theta^2 - 1)^2 \cdot \frac{1}{2}}{\frac{1}{16\gamma^8} (1 + \gamma^2\theta^2)^4} + \frac{\frac{1}{2}}{\frac{1}{4\gamma^4} (1 + \gamma^2\theta^2)^2} \right)$$

$$= \frac{z^2 e^2 \gamma^2 |\Delta\vec{\beta}|^2}{2\pi^2 c} \left(\frac{(\gamma^2\theta^2 - 1)^2}{(1 + \gamma^2\theta^2)^4} + \frac{1}{(1 + \gamma^2\theta^2)^2} \right)$$

(c) It is easy to verify that

$$\lim_{\omega\to 0}\left(\frac{d^2 I_{\perp}}{d\omega d\Omega}+\frac{d^2 I_{\parallel}}{d\omega d\Omega}\right)=\frac{z^2e^2\gamma^4|\Delta\vec{\beta}|^2}{\pi^2 c}\frac{1+\gamma^4\theta^4}{(1+\gamma^2\theta^2)^4},\quad \lim_{\omega\to 0}\left(\frac{d^2 I_{\perp}}{d\omega d\Omega}-\frac{d^2 I_{\parallel}}{d\omega d\Omega}\right)=\frac{z^2e^2\gamma^4|\Delta\vec{\beta}|^2}{\pi^2 c}\frac{2\gamma^2\theta^2}{(1+\gamma^2\theta^2)^4},$$

and $P(\theta)=\dfrac{2\gamma^2\theta^2}{1+\gamma^4\theta^4}$. Clearly, $P(0)=0$, and the maximum is achieved at $\gamma^2\theta^2=1$, with $P_{max}=1$. This is equivalent to $\theta^2=\frac{1}{\gamma^2}$, or $1-\frac{\theta^2}{2}=1-\frac{1}{2\gamma^2}=\left(1-\frac{1}{\gamma^2}\right)^{1/2}$, which is $\cos\theta=\beta$.

(d) $$\lim_{\omega\to 0}\frac{dI}{d\omega}=\int\lim_{\omega\to 0}\frac{d^2 I}{d\omega d\Omega}\,d\Omega=2\pi\cdot\int_0^{\pi}\lim_{\omega\to 0}\frac{d^2 I}{d\omega d\Omega}\sin\theta\, d\theta$$

$$=\frac{2z^2e^2\gamma^4|\Delta\vec{\beta}|^2}{\pi c}\int_0^{+\infty}\frac{1+\gamma^4\theta^4}{(1+\gamma^2\theta^2)^4}\,\theta\, d\theta$$

$$=\frac{z^2e^2\gamma^2|\Delta\vec{\beta}|^2}{\pi c}\int_0^{+\infty}\frac{1+\gamma^4\theta^4}{(1+\gamma^2\theta^2)^4}\,d(\gamma^2\theta^2)$$

$$=\frac{z^2e^2\gamma^2|\Delta\vec{\beta}|^2}{\pi c}\int_1^{+\infty}\frac{1+(y-1)^2}{y^4}\,dy,\qquad (y=1+\gamma^2\theta^2)$$

$$=\frac{z^2e^2\gamma^2|\Delta\vec{\beta}|^2}{\pi c}\int_1^{+\infty}\frac{2-2y+y^2}{y^4}\,dy$$

$$=\frac{z^2e^2\gamma^2|\Delta\vec{\beta}|^2}{\pi c}\left(-\frac{2}{3y^3}+\frac{1}{y^2}-\frac{1}{y}\right)\Bigg|_1^{+\infty}$$

$$=\frac{2}{3}\frac{z^2e^2\gamma^2|\Delta\vec{\beta}|^2}{\pi c}$$