2.12 Solution: The general solution to the two-dimensional potential problem can be expressed in a series form,

$$\Phi(\rho,\phi) = A_0 + B_0 \log\rho + \sum_{m\neq 0} \left(A_m \rho^m + B_m \rho^{-m}\right) e^{im\phi},$$

where the summation index m in the third term runs from $-\infty$ to $+\infty$, excluding 0. Since we are seeking regular solution inside a cylinder, coefficients for terms $\log\rho$, ρ^{-m} with $m > 0$, and ρ^m with $m < 0$ must vanish. Therefore, the series solution becomes

$$\Phi(\rho,\phi) = A_0 + \sum_{m=1}^{+\infty} \left(A_m e^{im\phi} + B_{-m} e^{-im\phi}\right) \rho^m.$$

On the surface of the cylinder, $\rho = b$, the potential is specified and must satisfy the following condition,

$$\Phi(b,\phi) = A_0 + \sum_{m=1}^{+\infty} \left(A_m e^{im\phi} + B_{-m} e^{-im\phi}\right) b^m.$$

By direct integration, the coefficients can be determined as

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi')d\phi',$$

$$A_m = \frac{1}{2\pi b^m}\int_0^{2\pi} \Phi(b,\phi')e^{-im\phi'}d\phi',$$

and

$$B_{-m} = \frac{1}{2\pi b^m}\int_0^{2\pi} \Phi(b,\phi')e^{im\phi'}d\phi'.$$

Put everything back, we have

$$\begin{aligned}
\Phi(\rho,\phi) &= \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi')\left[1 + \sum_{m=1}^{+\infty}\left(\frac{\rho}{b}\right)^m \left(e^{-im(\phi'-\phi)} + e^{im(\phi'-\phi)}\right)\right]d\phi' \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi')\left[\frac{1}{1-\rho e^{-i(\phi'-\phi)}/b} + \frac{1}{1-\rho e^{i(\phi'-\phi)}/b} - 1\right]d\phi' \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi')\left[\frac{2b^2 - 2\rho b\cos(\phi'-\phi)}{b^2+\rho^2-2\rho b\cos(\phi'-\phi)} - 1\right]d\phi' \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2\rho b\cos(\phi'-\phi)}d\phi',
\end{aligned}$$

for $\rho < b$.

For the potential in the region bounded by the cylinder and infinity, we can replace the $(\rho/b)^m$ terms with $(b/\rho)^m$ in the series to preserve the regularity of the potential at infinity. Then we can sum the series again and will obtain the potential as

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi')\frac{\rho^2-b^2}{b^2+\rho^2-2\rho b\cos(\phi'-\phi)}d\phi',$$

for $\rho > b$.