11.26 (a) Consider the elastic scattering in the cm frame. The momenta of the particles must be identical before and after the collision, i.e. $|\vec{p}| = |\vec{q}'| = \frac{m_2}{W} p_{LAB}$. Therefore.

$$E_4' = \sqrt{m_2^2 + q'^2} = m_2 \left(1 + \frac{p_{LAB}^2}{W^2}\right)^{1/2} = \frac{m_2}{W}\left(m_1^2 + m_2^2 + 2m_2 E_{LAB} + p_{LAB}^2\right)^{1/2} = \frac{m_2}{W}(m_2 + E_{LAB})$$

In the laboratory frame, the energy can be determined by Lorentz transform,

$$E_4 = \gamma_{cm}\left(E_4' + \vec{\beta}_{cm}\cdot\vec{q}'\right) = \frac{m_2 + E_{LAB}}{W}\cdot\left(\frac{m_2}{W}(m_2 + E_{LAB}) + \frac{m_2}{W}\frac{p_{LAB}^2}{m_2 + E_{LAB}}\cos\theta'\right)$$

$$= \frac{m_2}{W^2}\left((m_2 + E_{LAB})^2 + p_{LAB}^2\cos\theta'\right) = \frac{m_2}{W^2}\left(m_2^2 + 2m_2E_{LAB} + E_{LAB}^2 + p_{LAB}^2\cos\theta'\right)$$

$$= \frac{m_2}{W^2}\left(m_1^2 + m_2^2 + 2m_2E_{LAB} + p_{LAB}^2(1+\cos\theta')\right) = m_2\left(1 + \frac{p_{LAB}^2}{W^2}(1+\cos\theta')\right).$$

Thus. $\Delta E = E_4 - m_4 = E_4 - m_2 = \frac{m_2 p_{LAB}^2}{W^2}(1+\cos\theta')$

The inverse transform of the energy is given by

$$E_4' = \gamma_{cm}\left(E_4 - \vec{\beta}_{cm}\cdot\vec{p}_4\right) = \frac{m_2 + E_{LAB}}{W}\left(E_4 - \frac{p_{LAB}\cos\theta_4}{m_2 + E_{LAB}}p_4\right)$$

Using the result of E_4' in cm frame we have

$$E_4 = \sqrt{m_2^2 + p_4^2} = m_2 + \frac{p_{LAB}\cos\theta_4}{m_2 + E_{LAB}}p_4,$$

where p_4 can be solved, leading to

$$p_4 = \frac{2p_{LAB}(m_2 + E_{LAB})}{W^2 + p_{LAB}^2\sin^2\theta_4}\cos\theta_4$$

Then, $\Delta E = E_4 - m_2 = \frac{p_{LAB}\cos\theta_4}{m_2 + E_{LAB}}p_4 = \frac{2p_{LAB}^2\cos^2\theta_4}{W^2 + p_{LAB}^2\sin^2\theta_4}$

We can see that the result here is consistent with the first expression, as when $\theta_4 \to 0$, $\theta' \to 0$.

Finally. Consider the conservation of energy and momentum in the 4-vector form,

$$(E_{LAB}, \vec{p}_{LAB}) + (m_2, 0) = (E_3, \vec{p}_3) + (E_4, \vec{p}_4),$$

which is equivalent to $(E_{LAB} - E_3, \vec{p}_{LAB} - \vec{p}_3) = (E_4 - m_2, \vec{p}_4)$

Taking the norm, we have $-Q^2 = (E_4 - m_2)^2 - p_4^2 = 2m_2(m_2 - E_4)$, which leads to

$$E_4 = m_2 + \frac{Q^2}{2m_2}$$

Therefore, $\Delta E = E_4 - m_2 = \frac{Q^2}{2m_2}$

(b) From the expression $\Delta E = \frac{m_2}{W^2} p_{LAB}^2 (1+\cos\theta')$, we know that

$$\Delta E_{max} = \frac{2m_2}{W^2} p_{LAB}^2$$

Notice that $\beta_{cm} = \frac{p_{LAB}}{m_2 + E_{LAB}}$, $\gamma_{cm} = \frac{m_2 + E_{LAB}}{W}$, the maximum energy loss can be written as

$$\Delta E_{max} = 2\gamma_{cm}^2 \beta_{cm}^2 m_2$$

If $m_1 \gg m_2$, then $\beta_{cm} \simeq \frac{p_{LAB}}{E_{LAB}} = \beta$ of the incident particle. and $\gamma_{cm} \simeq \gamma$. Then,

$$\Delta E_{max} \simeq 2\beta^2\gamma^2 m_2$$

(c) Since $\beta^2 = \frac{\gamma^2 - 1}{\gamma^2}$, then $\Delta E_{max}^{(e)} = 2(\gamma_{cm}^2 - 1) m_e$.

But $E_{LAB} = \gamma m_e$, and $W^2 = 2m_e^2(1+\gamma)$, which means

$$\gamma_{cm} = \frac{m_e(1+\gamma)}{m_e\sqrt{2(1+\gamma)}} = \sqrt{\frac{1+\gamma}{2}}$$, and

$$\Delta E_{max}^{(e)} = (\gamma - 1) m_e.$$