11.11 Define $A(\lambda) = e^{\lambda(L+\delta L)} e^{-\lambda L}$, it is straightforward to show that $A(0) = I$,

$$\frac{dA(\lambda)}{d\lambda} = e^{\lambda(L+\delta L)}(L+\delta L)\, e^{-\lambda L} + e^{\lambda(L+\delta L)}(-L)\, e^{-\lambda L} = e^{\lambda(L+\delta L)}\, \delta L\, e^{-\lambda L},$$

$$\frac{d^2A(\lambda)}{d\lambda^2} = e^{\lambda(L+\delta L)}(L+\delta L)\delta L\, e^{-\lambda L} + e^{\lambda(L+\delta L)}\, \delta L(-L)\, e^{-\lambda L}$$

$$= e^{\lambda(L+\delta L)}[L, \delta L]\, e^{-\lambda L} + e^{\lambda(L+\delta L)}(\delta L)^2 e^{-\lambda L}$$

$$= e^{\lambda(L+\delta L)}[L, \delta L]\, e^{-\lambda L} + O((\delta L)^2)$$

For arbitrary n-th derivative, if we only retain the term linear in δL,

$$\frac{d^nA(\lambda)}{d\lambda^n} = e^{\lambda(L+\delta L)}\, g_n(L, \delta L)\, e^{-\lambda L},$$ then the term with one extra derivative is

$$\frac{d^{n+1}A(\lambda)}{d\lambda^{n+1}} = e^{\lambda(L+\delta L)}[L, g_n(L, \delta L)]\, e^{-\lambda L} + O(\delta L)^2),$$ and we have the recurrence relation.

$$g_1(L, \delta L) = \delta L, \quad g_2(L, \delta L) = [L, \delta L], \quad g_{n+1}(L, \delta L) = [L, g_n(L, \delta L)] = \underbrace{[L, [L, \cdots, [L, \delta L]]]}_{n}.$$

Using the Taylor series expansion, we now can write $A(\lambda)$ as

$$A(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \frac{d^nA(0)}{d\lambda^n} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} g_n(L, \delta L)$$

Setting $\lambda = 1$ will lead to

$$A = A_2 A_1^{-1} = A(1) = \sum_{n=0}^{\infty} \frac{1}{n!} g_n(L, \delta L)$$

$$= I + \delta L + \frac{1}{2!}[L, \delta L] + \frac{1}{3!}[L, [L, \delta L]] + \frac{1}{4!}[L, [L, [L, \delta L]]] + \cdots$$