16.7 Since $\vec{F}_{rad} = mT\dddot{\vec{v}} = T\frac{d^2\vec{p}}{dt^2}$, the natural relativistic extension should be $F_\mu^{rad} = T\frac{d^2 p_\mu}{d\tau^2}$. Here, we have used $T = 2e^2/3mc^3$ instead of τ, in order not to confuse with the particle proper time τ. However, this extension does not satisfy the requirement $F_\mu p^\mu = 0$. Thus, we need to introduce an extra term, $F_\mu^{rad} = T\left(\frac{d^2 p_\mu}{d\tau^2} + A_\mu\right)$, to achieve the exact cancellation. The only relativistic 4-vector available is p_μ now, and A_μ must be proportional to it, i.e., $A_\mu = c p_\mu$ The constant c can be determined by the requirement, $F_\mu p^\mu = 0$, as

$$p^\mu\left(\frac{d^2 p_\mu}{d\tau^2} + c p_\mu\right) = p^\mu \frac{d^2 p_\mu}{d\tau^2} + c p^2 = 0.$$

Since $p^2 = p^\mu p_\mu = m^2c^2$, we must have $dp^2/d\tau = 2p^\mu dp_\mu/d\tau = 0$. Differentiating again, we will obtain

$$p^\mu \frac{d^2 p_\mu}{d\tau^2} + \frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau} = 0, \text{ or } p^\mu \frac{d^2 p_\mu}{d\tau^2} = -\frac{dp_\nu}{d\tau}\frac{dp^\nu}{d\tau}.$$

Then, $c = -\frac{1}{p^2} p^\mu \frac{d^2 p_\mu}{d\tau^2} = \frac{1}{m^2c^2}\frac{dp_\nu}{d\tau}\frac{dp^\nu}{d\tau}$, and

$$F_\mu^{rad} = \frac{2e^2}{3mc^3}\left[\frac{d^2 p_\mu}{d\tau^2} + \frac{p_\mu}{m^2c^2}\left(\frac{dp_\nu}{d\tau}\frac{dp^\nu}{d\tau}\right)\right]$$

16.8. (a) The equation of motion in the spatial dimension is

$$\frac{dp^1}{d\tau} = F_{ext}(\tau) + \frac{2e^2}{3mc^3}\left[\frac{d^2p^1}{d\tau^2} + \frac{p^1}{m^2c^2}\left(\frac{dp}{d\tau}\right)^2\right].$$

Here, $\left(\frac{dp}{d\tau}\right)^2 = \left(\frac{dp^0}{d\tau}\right)^2 - \left(\frac{dp^1}{d\tau}\right)^2 = \left(\frac{d}{d\tau}\sqrt{(p^1)^2+m^2c^2}\right)^2 - \left(\frac{dp^1}{d\tau}\right)^2 = \frac{(p^1)^2}{(p^1)^2+m^2c^2}\left(\frac{dp^1}{d\tau}\right)^2 - \left(\frac{dp^1}{d\tau}\right)^2$

$$= -\frac{m^2c^2}{(p^1)^2+m^2c^2}\left(\frac{dp^1}{d\tau}\right)^2$$

Let $p = p^1$, and $\dot{p} = dp^1/d\tau$, $\ddot{p} = d^2p^1/d\tau^2$, the above equation can now be written as

$$\dot{p} - \frac{2e^2}{3mc^3}\left[\ddot{p} - \frac{p\dot{p}^2}{p^2+m^2c^2}\right] = F_{ext}(\tau).$$

If we denote the ordinary Newtonian force as $f(\tau)$, and Newton's second law reads $\frac{dp}{dt} = f(\tau)$. Then $\frac{dp}{d\tau} = \frac{dp}{dt}\frac{dt}{d\tau} = \gamma\frac{dp}{dt} = \sqrt{1+\frac{p^2}{m^2c^2}}\, f(\tau)$, or $F_{ext}(\tau) = \sqrt{1+\frac{p^2}{m^2c^2}}\, f(\tau)$. The equation of motion finally becomes

$$\dot{p} - \frac{2e^2}{3mc^3}\left[\ddot{p} - \frac{p\dot{p}^2}{p^2+m^2c^2}\right] = \sqrt{1+\frac{p^2}{m^2c^2}}\, f(\tau).$$

(b) With $p = mc\sinh y$, we have $\dot{p} = \dot{y}\, mc\cosh y$, and $\ddot{p} = \ddot{y}\, mc\cosh y + \dot{y}^2\, mc\sinh y$.

The equation of motion then becomes

$$\dot{y}\, mc\cosh y - \frac{2e^2}{3mc^3}\left[\ddot{y}\, mc\cosh y + \dot{y}^2\, mc\sinh y - \frac{mc\sinh y}{m^2c^2\cosh^2 y}\cdot \dot{y}^2 m^2c^2\cosh^2 y\right] = \cosh y\, f(\tau),$$

or,
$$mc\left(\dot{y} - \frac{2e^2}{3mc^3}\ddot{y}\right) = f(\tau).$$

We can identify y as the rapidity, and cy is the velocity. Therefore, the above equation is just the Abraham-Lorentz equation. The equation of motion can be solved as in Prob 16.10, but it also requires $\dot{p}$ at $\tau=0$, so I am not going to do the solution.

16.9. (a) From Prob. 16.7, we know $\frac{dp_\nu}{d\tau}\frac{dp^\nu}{d\tau} = -p_\nu \frac{d^2p^\nu}{d\tau^2}$. Then,

$$F_\mu^{rad} = \frac{2e^2}{3mc^3}\left[\frac{d^2p_\mu}{d\tau} - \frac{p_\mu p_\nu}{m^2c^2}\frac{d^2p^\nu}{d\tau^2}\right] = \frac{2e^2}{3mc^3}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2c^2}\right)\frac{d^2p^\nu}{d\tau^2}.$$

(b) From the condition $F\cdot p = 0$, we know

$$F^0\cdot \gamma mc - \vec{F}\cdot\vec{p} = 0, \quad \text{or} \quad F^0 = \vec{F}\cdot\vec{\beta}.$$

Using the relativistic generalization of the radiation force,

$$F_\mu^{rad} = \tau\left[\frac{d^2p_\mu}{d\tau^2} + \frac{p_\mu}{m^2c^2}\frac{dp_\nu}{d\tau}\frac{dp^\nu}{d\tau}\right].$$

and replace $dp_\mu/d\tau$ with F_μ^{ext}, we will have

$$F_\mu^{rad} = \tau\left(\frac{dF_\mu^{ext}}{d\tau} + \frac{p_\mu}{m^2c^2}F_\nu^{ext}\frac{dp^\nu}{d\tau}\right)$$

For the spatial part, $\vec{F}^{rad} = \tau\left(\frac{d\vec{F}^{ext}}{d\tau} + \frac{\vec{p}}{m^2c^2}F_\nu^{ext}\frac{dp^\nu}{d\tau}\right)$.

Since $\vec{F}^{ext} = \gamma\vec{F}$, we have

$$\frac{d\vec{F}^{ext}}{d\tau} = \gamma\frac{d}{dt}(\gamma\vec{F}) = \gamma^2\frac{d\vec{F}}{dt} + \gamma\vec{F}\frac{d\gamma}{dt} = \gamma^2\frac{d\vec{F}}{dt} + \gamma^4\vec{F}(\vec{\beta}\cdot\dot{\vec{\beta}}), \quad \text{where } \dot{\vec{\beta}} = \frac{1}{c}\frac{d\vec{v}}{dt}.$$

From Eq. (14.12), we also know

$$\frac{d(\gamma m v^\mu)}{d\tau} = m\left(c\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}}),\ c\gamma^2\dot{\vec{\beta}} + c\gamma^4\vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}})\right),$$

then

$$F_\nu^{ext}\frac{dp^\nu}{d\tau} = m\left(F^{0,ext}\dot{p}^0 - \vec{F}^{ext}\cdot\dot{\vec{p}}\right)$$

$$= m\left(F^{0,ext}\cdot c\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}}) - c\gamma^2\vec{F}^{ext}\cdot\dot{\vec{\beta}} - c\gamma^4(\vec{F}^{ext}\cdot\vec{\beta})(\vec{\beta}\cdot\dot{\vec{\beta}})\right)$$

$$= m\left((\vec{F}^{ext}\cdot\vec{\beta})\cdot c\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}}) - c\gamma^2\vec{F}^{ext}\cdot\dot{\vec{\beta}} - c\gamma^4(\vec{F}^{ext}\cdot\vec{\beta})(\vec{\beta}\cdot\dot{\vec{\beta}})\right)$$

$$= -mc\gamma^2\vec{F}^{ext}\cdot\dot{\vec{\beta}} = -mc\gamma^3\vec{F}\cdot\dot{\vec{\beta}}$$

Putting everything together, the radiation force is

$$\vec{F}^{rad} = \tau\left[\gamma^2\frac{d\vec{F}}{dt} + \gamma^4\vec{F}(\vec{\beta}\cdot\dot{\vec{\beta}}) - \frac{\gamma m\vec{v}}{m^2c^2}mc\gamma^3\vec{F}\cdot\dot{\vec{\beta}}\right]$$

$$= \tau\left[\gamma^2\frac{d\vec{F}}{dt} + \frac{\gamma^4}{c^2}\vec{F}\left(\vec{v}\cdot\frac{d\vec{v}}{dt}\right) - \frac{\gamma^4}{c^2}\vec{v}\left(\vec{F}\cdot\frac{d\vec{v}}{dt}\right)\right]$$

$$= \tau\left[\gamma^2\frac{d\vec{F}}{dt} - \frac{\gamma^4}{c^2}\frac{d\vec{v}}{dt}\times(\vec{v}\times\vec{F})\right]$$

Then,

$$\frac{d\vec{p}}{d\tau} = \gamma\frac{d\vec{p}}{dt} = \vec{F}^{ext} + \vec{F}^{rad} = \gamma\vec{F} + \tau\left[\gamma^2\frac{d\vec{F}}{dt} - \frac{\gamma^4}{c^2}\frac{d\vec{v}}{dt}\times(\vec{v}\times\vec{F})\right],$$

and finally.

$$\frac{d\vec{p}}{dt} = \vec{F} + \tau\left[\gamma\frac{d\vec{F}}{dt} - \frac{\gamma^3}{c^2}\frac{d\vec{v}}{dt}\times(\vec{v}\times\vec{F})\right].$$