12.12 (a) From the identity $\frac{d}{dt}(\hat{\beta}\cdot\vec{s}) = \hat{\beta}\cdot\frac{d\vec{s}}{dt} + \frac{1}{\beta}\left[\vec{s}-\hat{\beta}(\hat{\beta}\cdot\vec{s})\right]\cdot\frac{d\vec{\beta}}{dt}$, we can calculate each term separately,

$$\hat{\beta}\cdot\frac{d\vec{s}}{dt} = \frac{e}{mc}\hat{\beta}\cdot\left\{\vec{s}\times\left[\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)\vec{B} - \left(\frac{g}{2}-1\right)\frac{\gamma}{\gamma+1}(\vec{\beta}\cdot\vec{B})\vec{\beta} - \left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)\vec{\beta}\times\vec{E}\right]\right\}$$

$$= \frac{e}{mc}\left(\frac{g}{2}-1+\frac{1}{\gamma}\right)(\hat{\beta}\times\vec{s})\cdot\vec{B} - \frac{e}{mc}\left(\frac{g}{2}-1\right)\frac{\gamma}{\gamma+1}(\vec{\beta}\cdot\vec{B})\cancelto{0}{\hat{\beta}\cdot(\vec{s}\times\vec{\beta})}$$

$$- \frac{e}{mc}\left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)(\hat{\beta}\times\vec{s})\cdot(\vec{\beta}\times\vec{E})$$

$$= \frac{e}{mc}\left(\frac{g}{2}-1\right)(\hat{\beta}\times\vec{s})\cdot\vec{B} + \frac{e}{\gamma mc}(\hat{\beta}\times\vec{s})\cdot\vec{B} - \frac{e}{mc}\left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)(\hat{\beta}\times\vec{s})\cdot(\vec{\beta}\times\vec{E})$$

and

$$\frac{1}{\beta}\left[\vec{s}-\hat{\beta}(\hat{\beta}\cdot\vec{s})\right]\cdot\frac{d\vec{\beta}}{dt} = \frac{1}{\beta}\left[\hat{\beta}\times(\vec{s}\times\hat{\beta})\right]\cdot\frac{e}{\gamma mc}\left[\vec{E}+\vec{\beta}\times\vec{B}-\vec{\beta}(\vec{\beta}\cdot\vec{E})\right]$$

$$= \frac{e}{\beta\gamma mc}\left[\hat{\beta}\times(\vec{s}\times\hat{\beta})\right]\cdot\vec{E} + \frac{e}{\beta\gamma mc}\left[\hat{\beta}\times(\vec{s}\times\hat{\beta})\right]\cdot(\vec{\beta}\times\vec{B}) - \frac{e}{\beta\gamma mc}\cancelto{0}{\left[\hat{\beta}\times(\vec{s}\times\hat{\beta})\right]\cdot\vec{\beta}(\vec{\beta}\cdot\vec{E})}$$

$$= \frac{e}{\beta\gamma mc}(\vec{s}\times\hat{\beta})\cdot(\vec{E}\times\hat{\beta}) + \frac{e}{\beta\gamma mc}\left[(\vec{s}\times\hat{\beta})\cdot\vec{B} - (\hat{\beta}\cdot\vec{B})\cancelto{0}{[(\vec{s}\times\hat{\beta})\cdot\hat{\beta}]}\right]$$

$$= \frac{e}{\beta\gamma mc}(\vec{s}\times\hat{\beta})\cdot(\vec{E}\times\hat{\beta}) + \frac{e}{\beta\gamma mc}(\vec{s}\times\hat{\beta})\cdot\vec{B}$$

Taking the sum, we now have

$$\frac{d}{dt}(\hat{\beta}\cdot\vec{s}) = \frac{e}{mc}\left(\frac{g}{2}-1\right)(\hat{\beta}\times\vec{s})\cdot\vec{B} - \frac{e}{mc}\left(\frac{g}{2}-\frac{\gamma}{\gamma+1}\right)(\hat{\beta}\times\vec{s})\cdot(\vec{\beta}\times\vec{E}) + \frac{e}{\beta\gamma mc}(\vec{s}\times\hat{\beta})\cdot(\vec{E}\times\hat{\beta})$$

$$= \frac{e}{mc}\left(\frac{g}{2}-1\right)\vec{s}\cdot(\vec{B}\times\hat{\beta}) + \frac{e}{mc}\left[\frac{1}{\beta\gamma}-\frac{g\beta}{2}+\frac{\beta\gamma}{\gamma+1}\right](\hat{\beta}\times\vec{s})\cdot(\hat{\beta}\times\vec{E})$$

$$= -\frac{e}{mc}\left(\frac{g}{2}-1\right)\vec{s}\cdot(\hat{\beta}\times\vec{B}) - \frac{e}{mc}\left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\left[(\hat{\beta}\times\vec{s})\times\hat{\beta}\right]\cdot\vec{E},$$

where we have used the fact that

$$\frac{1}{\beta\gamma}+\frac{\beta\gamma}{\gamma+1} = \frac{\gamma+1+\gamma^2\beta^2}{\beta\gamma(\gamma+1)} = \frac{\gamma+1+\frac{\beta^2}{1-\beta^2}}{\beta\gamma(\gamma+1)} = \frac{\gamma+\frac{1}{1-\beta^2}}{\beta\gamma(\gamma+1)} = \frac{\gamma+\gamma^2}{\beta\gamma(\gamma+1)} = \frac{1}{\beta}.$$

Finally, notice that $(\hat{\beta}\times\vec{s})\times\hat{\beta} = \vec{s}-\hat{\beta}(\hat{\beta}\cdot\vec{s}) = \vec{s}-\vec{s}_{\parallel} = \vec{s}_{\perp}$, where $\vec{s}_{\parallel}$ and $\vec{s}_{\perp}$ are the components of $\vec{s}$ parallel and perpendicular to $\hat{\beta}$, and also that only $\vec{s}_{\perp}$ contribute to $\vec{s}\cdot(\hat{\beta}\times\vec{B})$, we can write the result as

$$\frac{d}{dt}(\hat{\beta}\cdot\vec{s}) = -\frac{e}{mc}\vec{s}_{\perp}\cdot\left[\left(\frac{g}{2}-1\right)\hat{\beta}\times\vec{B} + \left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\vec{E}\right].$$

(b) In the configuration, $\hat{\beta}\cdot\vec{s} = s\cos\theta$ and $\vec{s}_\perp = s\sin\theta\,\hat{n}$. Then.

$$\frac{d}{dt}(\hat{\beta}\cdot\vec{s}) = -s\sin\theta\frac{d\theta}{dt} = -\frac{e}{mc}s\sin\theta\,\hat{n}\cdot\left[\left(\frac{g}{2}-1\right)\hat{\beta}\times\vec{B} + \left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\vec{E}\right].$$

or
$$\frac{d\theta}{dt} = \frac{e}{mc}\left[\left(\frac{g}{2}-1\right)\hat{n}\cdot(\hat{\beta}\times\vec{B}) + \left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\hat{n}\cdot\vec{E}\right]$$

(c) For this configuration, $\vec{E} = -\beta B\hat{n}$, which can be easily verified. Therefore.

$$\frac{d\theta}{dt} = \frac{e}{mc}\left[\left(\frac{g}{2}-1\right)(\hat{n}\times\hat{\beta})\cdot\vec{B} - \left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\beta B\right] = \frac{eB}{mc}\left[\frac{g}{2}-1-\frac{g\beta^2}{2}+1\right] = \frac{geB}{2mc}(1-\beta^2)$$

$$= \frac{geB}{2\gamma^2 mc} = \frac{g}{2\gamma}\omega_B.$$

(d) The equation can be verified by matrix multiplication. Notice that

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -\vec{E} \\ \vec{E} & -\varepsilon_{ijk}B_k \end{pmatrix},$$ we can write $L_\alpha F^{\alpha\beta}N_\beta$ as

$$L_\alpha F^{\alpha\beta}N_\beta = (\gamma\beta, -\gamma\hat{\beta})\begin{pmatrix} 0 & -\vec{E} \\ \vec{E} & -\varepsilon_{ijk}B_k \end{pmatrix}\begin{pmatrix} 0 \\ -\hat{n} \end{pmatrix} = \gamma\beta\,\hat{n}\cdot\vec{E} - \gamma\hat{\beta}\cdot(\hat{n}\times\vec{B}) = \gamma\beta\hat{n}\cdot\vec{E} + \gamma\hat{n}\cdot(\hat{\beta}\times\vec{B}).$$

Similarly. $$U_\alpha F^{\alpha\beta}N_\beta = (\gamma c, -\gamma c\vec{\beta})\begin{pmatrix} 0 & -\vec{E} \\ \vec{E} & -\varepsilon_{ijk}B_k \end{pmatrix}\begin{pmatrix} 0 \\ -\hat{n} \end{pmatrix} = \gamma c(\hat{n}\cdot\vec{E}) + \gamma c\,\hat{n}\cdot(\vec{\beta}\times\vec{B}).$$

Then. $$\left(\frac{g}{2}L_\alpha - \frac{1}{v}U_\alpha\right)F^{\alpha\beta}N_\beta = \frac{g}{2}\left(\gamma\beta\hat{n}\cdot\vec{E} + \gamma\hat{n}\cdot(\hat{\beta}\times\vec{B})\right) - \frac{\gamma c}{v}\left(\hat{n}\cdot\vec{E} + \hat{n}\cdot(\vec{\beta}\times\vec{B})\right)$$

$$= \gamma\left[\left(\frac{g}{2}-\frac{c\beta}{v}\right)\hat{n}\cdot(\hat{\beta}\times\vec{B}) + \left(\frac{g\beta}{2}-\frac{c}{v}\right)\hat{n}\cdot\vec{E}\right]$$

$$= \gamma\left[\left(\frac{g}{2}-1\right)\hat{n}\cdot(\hat{\beta}\times\vec{B}) + \left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\hat{n}\cdot\vec{E}\right]$$

Notice that $dt/d\tau = \gamma$, we have

$$\frac{d\theta}{d\tau} = \frac{d\theta}{dt}\frac{dt}{d\tau} = \gamma\cdot\frac{e}{mc}\left[\left(\frac{g}{2}-1\right)\hat{n}\cdot(\hat{\beta}\times\vec{B}) + \left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\hat{n}\cdot\vec{E}\right]$$

$$= \frac{e}{mc}\left[\frac{g}{2}L_\alpha - \frac{1}{v}U_\alpha\right]F^{\alpha\beta}N_\beta.$$