3.10 Solution: (a) Using the result from Problem 3.9, the potential inside the cylinder is

$$\begin{aligned}\Phi(\rho,\phi,z) &= \frac{V}{\pi L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{I_m(n\pi\rho/L)}{I_m(n\pi b/L)}\sin\left(\frac{n\pi}{L}z\right)e^{im\phi}\\ &\qquad\times\int_0^L dz'\sin\left(\frac{n\pi}{L}z'\right)\times\left(\int_{-\pi/2}^{\pi/2}-\int_{\pi/2}^{3\pi/2}\right)d\phi' e^{-im\phi'}.\end{aligned}$$

Direct integration shows,

$$\int_0^L dz'\sin\left(\frac{n\pi}{L}z'\right) = -\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}z'\right)\bigg|_0^{\pi} = \frac{L}{n\pi}\left(1-(-1)^n\right) = \begin{cases}\dfrac{2L}{(2k+1)\pi}, & n=2k+1,\\ 0, & n=2k,\end{cases}$$

for $k\geq 0$, and

$$\begin{aligned}&\left(\int_{-\pi/2}^{\pi/2}-\int_{\pi/2}^{3\pi/2}\right)d\phi' e^{-im\phi'}\\ =\;& \frac{i}{m}\left(e^{-im\phi'}\bigg|_{-\pi/2}^{\pi/2}-e^{-im\phi'}\bigg|_{\pi/2}^{3\pi/2}\right)\\ =\;& \frac{i}{m}\left(e^{-im\pi/2}-e^{im\pi/2}-e^{-3im\pi/2}+e^{-im\pi/2}\right)\\ =\;& \frac{2i}{m}\left((-i)^m-i^m\right)\\ =\;& \begin{cases}\dfrac{4(-1)^l}{2l+1}, & m=2l+1,\\ 0, & m=2l,\end{cases}\end{aligned}$$

for all integer l. Therefore, the potential becomes

$$\Phi(\rho,\phi,z)=\frac{8V}{\pi^2}\sum_{k=0}^{\infty}\sum_{l=-\infty}^{\infty}(-1)^l\frac{I_{2l+1}\left(\dfrac{(2k+1)\pi\rho}{L}\right)}{I_{2l+1}\left(\dfrac{(2k+1)\pi b}{L}\right)}\frac{e^{i(2l+1)\phi}}{2l+1}\frac{\sin\left(\dfrac{(2k+1)\pi z}{L}\right)}{2k+1}.$$

It is more convenient to cast the above expression into a real form. First split the sum over l,

$$\begin{aligned}\Phi(\rho,\phi,z) &= \frac{8V}{\pi^2}\sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty}(-1)^l\frac{I_{2l+1}\left(\dfrac{(2k+1)\pi\rho}{L}\right)}{I_{2l+1}\left(\dfrac{(2k+1)\pi b}{L}\right)}\frac{e^{i(2l+1)\phi}}{2l+1}+\sum_{l'=-1}^{-\infty}(-1)^{l'}\frac{I_{2l'+1}\left(\dfrac{(2k+1)\pi\rho}{L}\right)}{I_{2l'+1}\left(\dfrac{(2k+1)\pi b}{L}\right)}\frac{e^{i(2l'+1)\phi}}{2l'+1}\right)\\ &\qquad\times\frac{\sin\left(\dfrac{(2k+1)\pi z}{L}\right)}{2k+1}.\end{aligned}$$

For $l'=-l-1<0$,

$$I_{2l'+1}(x)=I_{-2l-1}(x)=I_{2l+1}(x),$$

we can rewrite the above expression as

$$\begin{aligned}
\Phi(\rho,\phi,z) &= \frac{8V}{\pi^2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{I_{2l+1}\left(\dfrac{(2k+1)\pi\rho}{L}\right)}{I_{2l+1}\left(\dfrac{(2k+1)\pi b}{L}\right)}\left((-1)^l\frac{e^{i(2l+1)\phi}}{2l+1}+(-1)^{-l-1}\frac{e^{i(-2l-1)\phi}}{-2l-1}\right)\\
&\qquad\times\frac{\sin\left(\dfrac{(2k+1)\pi z}{L}\right)}{2k+1}.\\
&= \frac{16V}{\pi^2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^l\frac{I_{2l+1}\left(\dfrac{(2k+1)\pi\rho}{L}\right)}{I_{2l+1}\left(\dfrac{(2k+1)\pi b}{L}\right)}\frac{\cos[(2l+1)\phi]}{2l+1}\frac{\sin\left(\dfrac{(2k+1)\pi z}{L}\right)}{2k+1}.
\end{aligned}$$

(b) Using the asymptotics of I_m at small value,

$$I_m(x)\sim\frac{1}{m!}\left(\frac{x}{2}\right)^m,$$

for $L \gg b$, we have

$$\frac{I_{2l+1}\left(\dfrac{(2k+1)\pi\rho}{L}\right)}{I_{2l+1}\left(\dfrac{(2k+1)\pi b}{L}\right)}\sim\left(\frac{\rho}{b}\right)^{2l+1}.$$

At $z=L/2$, we have

$$\begin{aligned}
\Phi\left(\rho,\phi,\frac{L}{2}\right) &= \frac{16V}{\pi^2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^l\left(\frac{\rho}{b}\right)^{2l+1}\frac{\cos[(2l+1)\phi]}{2l+1}\frac{\sin\left((k+\frac{1}{2})\pi\right)}{2k+1}\\
&= \frac{16V}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cdot\mathrm{Re}\left\{i\sum_{l=0}^{\infty}\frac{1}{2l+1}\left(-i\frac{\rho}{b}e^{i\phi}\right)^{2l+1}\right\}.
\end{aligned}$$

Since

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}=\frac{\pi}{4},$$

and

$$\begin{aligned}
&\mathrm{Re}\left\{i\sum_{l=0}^{\infty}\frac{1}{2l+1}\left(-i\frac{\rho}{b}e^{i\phi}\right)^{2l+1}\right\}\\
=\;& -\mathrm{Im}\left\{\sum_{l=0}^{\infty}\frac{1}{2l+1}\left(-i\frac{\rho}{b}e^{i\phi}\right)^{2l+1}\right\}\\
=\;& \mathrm{Im}\left\{\frac{1}{2}\log\left(\frac{1+i\rho e^{i\phi}/b}{1-i\rho e^{i\phi}/b}\right)\right\}\\
=\;& \frac{1}{2}\mathrm{Im}\left\{\log\left(\frac{b^2-\rho^2+2i\rho b\cos\phi}{b^2-\rho^2}\right)\right\}
\end{aligned}$$

$$= \quad \frac{1}{2}\arctan\left(\frac{2\rho b}{b^2-\rho^2}\cos\phi\right).$$

Putting things together, we have

$$\Phi\left(\rho,\phi,\frac{L}{2}\right) = \frac{2V}{\pi}\arctan\left(\frac{2\rho b}{b^2-\rho^2}\cos\phi\right),$$

which agrees with Problem 2.13. This is quite intuitive, as $L \to \infty$, at the middle point of the cylinder, the boundary effect can be ignored, and problem becomes a truly two-dimensional one.