5.4 (a) Let $B_\rho(\rho, z) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \alpha_n(z)$, $B_z(\rho, z) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \beta_n(z)$.

From condition $\nabla \cdot \vec{B} = 0$, we have $\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{\partial}{\partial z} B_z = 0$, or

$$\sum_{n=0}^{\infty} \frac{n+1}{n!} \rho^{n-1} \alpha_n(z) + \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \beta_n'(z) = 0$$

Rearrange terms, we will have

$$\frac{1}{\rho}\alpha_0(z) + \sum_{n=0}^{\infty} \frac{\rho^n}{n!}\left(\frac{n+2}{n+1}\alpha_{n+1}(z) + \beta_n'(z)\right) = 0$$

Since this relation is satisfied for arbitrary ρ, we must have

$$\alpha_0(z) = 0, \quad \text{and} \quad \alpha_{n+1}(z) = -\frac{n+1}{n+2}\beta_n'(z)$$

Similarly, form $\nabla \times \vec{B} = 0$, where only the $\hat{\phi}$ component is non-trivial,

$(\nabla \times \vec{B})_{\hat{\phi}} = 0$, we have $\frac{\partial}{\partial z} B_\rho - \frac{\partial}{\partial \rho} B_z = 0$, or

$$\sum_{n=0}^{\infty} \frac{\rho^n}{n!}\alpha_n'(z) - \sum_{n=1}^{\infty} \frac{\rho^{n-1}}{(n-1)!}\beta_n(z) = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{\rho^n}{n!}\left(\alpha_n'(z) - \beta_{n+1}(z)\right) = 0, \Rightarrow \beta_{n+1}(z) = \alpha_n'(z)$$

From these results, we can arrive at the following recurrence relation,

$$\beta_{n+1}(z) = \alpha_n'(z) = -\frac{n}{n+1}\beta_{n-1}''(z)$$

Also, $\beta_1(z) = \alpha_0'(z) \equiv 0$, Therefore, we should only have non-zero β's for even orders

and $\beta_{2k}(z) = (-1)^k \frac{(2k-1)!!}{(2k)!!}\beta_0^{(2k)}(z) = (-1)^k \frac{(2k-1)!!}{(2k)!!}\frac{\partial^{2k} B_z(0,z)}{\partial z^{2k}}$

Then, α has only odd terms

$$\alpha_{2k+1}(z) = -\frac{2k+1}{2k+2}\beta_{2k}'(z) = (-1)^{k+1}\frac{(2k+1)!!}{(2k+2)!!}\frac{\partial^{2k+1} B_z(0,z)}{\partial z^{2k+1}}$$

Now, $B_z(\rho, z) = \sum_{k=0}^{\infty} \frac{\rho^{2k}}{(2k)!}\beta_{2k}(z) = \sum_{k=0}^{\infty} (-1)^k \frac{\rho^{2k}}{(2k)!}\frac{(2k-1)!!}{(2k)!!}\frac{\partial^{2k} B_z(0,z)}{\partial z^{2k}}$

$$= B_z(0,z) - \frac{\rho^2}{4}\frac{\partial^2 B_z(0,z)}{\partial z^2}$$

$B_\rho(\rho, z) = \sum_{k=0}^{\infty} \frac{\rho^{2k+1}}{(2k+1)!}\alpha_{2k+1}(z) = \sum_{k=0}^{\infty} (-)^{k+1}\frac{\rho^{2k+1}}{(2k+1)!}\frac{(2k+1)!!}{(2k+2)!!}\frac{\partial^{2k+1} B_z(0,z)}{\partial z^{2k+1}}$

$$= -\frac{\rho}{2}\frac{\partial B_z(0,z)}{\partial z} + \frac{\rho^3}{16}\frac{\partial^3 B_z(0,z)}{\partial z^3}$$

(b) The terms in the series are, for B_z,

$$\gamma_{2k} = (-1)^k \frac{\rho^{2k}}{(2k)!} \frac{(2k-1)!!}{(2k+2)!!} \frac{\partial^{2k} B_z(0,z)}{\partial z^{2k}}.$$

To neglect higher order terms, we should have $\frac{\gamma_{2k+2}}{\gamma_{2k}} \ll 1$, or

$$\rho^2 \cdot \frac{\partial^{2k+2} B_z(0,z)}{\partial z^{2k+2}} \Big/ \frac{\partial^{2k} B_z(0,z)}{\partial z^{2k}} \ll 1,$$

$$\rho \ll \left[\frac{\partial^{2k+2} B_z(0,z)/\partial z^{2k+2}}{\partial^{2k} B_z(0,z)/\partial z^{2k}} \right]^{-1/2}$$