9.8 (a) From Problem 6.10, we know the outflow of the angular momentum is

$$\frac{d\vec{L}}{dt} = -\oint \vec{n}\cdot\overleftrightarrow{M}\, da, \quad \text{where } \overleftrightarrow{M} = \overleftrightarrow{T}\times\vec{x}, \text{ where}$$

$$\overleftrightarrow{T} = \varepsilon_0\left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B})\delta_{\alpha\beta}\right]$$

Therefore, the time averaged angular momentum out flow is given by

$$\frac{d\vec{L}}{dt} = -\frac{1}{2}\mathrm{Re}\left[\int \varepsilon_0\left[(\vec{n}\cdot\vec{E})(\vec{E}^*\times\vec{n}) + c^2(\vec{n}\cdot\vec{B})(\vec{B}^*\times\vec{n}) - \frac{1}{2}(\vec{E}\cdot\vec{E}^* + c^2\vec{B}\cdot\vec{B}^*)(\vec{n}\times\vec{n})\right] r^3 d\Omega\right]$$

$$= -\frac{1}{2}\mathrm{Re}\left[\int \varepsilon_0\left[(\vec{n}\cdot\vec{E})(\vec{E}^*\times\vec{n}) + c^2(\vec{n}\cdot\vec{B})(\vec{B}^*\times\vec{n})\right] r^3 d\Omega\right],$$

where the last term is obviously zero. Due to the r^3 factor, we need to consider terms that fall as r^{-3} only. Then,

$$\frac{d\vec{L}}{dt} = -\frac{1}{2}\mathrm{Re}\left[\int\left\{\varepsilon_0\left(\frac{1}{4\pi\varepsilon_0}2(\vec{n}\cdot\vec{p})\frac{-ik}{r^2}e^{ikr}\right)\frac{k^2}{4\pi\varepsilon_0}\left[(\vec{n}\times\vec{p}^*)\times\vec{n}\right]\times\vec{n}\,\frac{e^{-ikr}}{r}\right.\right.$$

$$\left.\left. + \frac{1}{\mu_0}\left[\frac{\mu_0 c^2 k^2}{4\pi}\vec{n}\cdot(\vec{n}\times\vec{p})\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\right]\frac{\mu_0 c k^2}{4\pi}(\vec{n}\times\vec{p}^*)\times\vec{n}\,\frac{e^{-ikr}}{r}\left(1+\frac{1}{ikr}\right)\right]\right\}$$

$$\times\, r^3 d\Omega\Bigg]$$

The contribution from magnetic field is identically zero, we are left with

$$\frac{d\vec{L}}{dt} = \mathrm{Re}\left[\frac{ik^3}{16\pi^2\varepsilon_0}\int(\vec{n}\cdot\vec{p})\left[(\vec{n}\times\vec{p}^*)\times\vec{n}\right]\times\vec{n}\, d\Omega\right]$$

$$= \mathrm{Re}\left[\frac{-ik^3}{16\pi^2\varepsilon_0}\int(\vec{n}\cdot\vec{p})(\vec{n}\times\vec{p}^*)\, d\Omega\right] = \mathrm{Re}\left[\frac{-ik^3}{16\pi^2\varepsilon_0}\hat{e}_k p_i \varepsilon_{klm} p_m^* \int n_i n_l\, d\Omega\right]$$

$$= \mathrm{Re}\left[\frac{-ik^3}{16\pi^2\varepsilon_0}\hat{e}_k p_i \varepsilon_{klm} p_m^* \frac{4\pi}{3}\delta_{il}\right] = \mathrm{Re}\left[\frac{-ik^3}{12\pi\varepsilon_0}\hat{e}_k \varepsilon_{klm} p_l p_m^*\right]$$

$$= \mathrm{Re}\left[\frac{-ik^3}{12\pi\varepsilon_0}(\vec{p}\times\vec{p}^*)\right] = \frac{k^3}{12\pi\varepsilon_0}\mathrm{Im}(\vec{p}\times\vec{p}^*),$$

where we have used the relation that $\int n_i n_l\, d\Omega = \frac{4\pi}{3}\delta_{il}$. The result differs Jackson by a minus sign. We will stick with Jackson in the following.

(b) It is easy to show that $(dL/dt)/(dP/dt) = \frac{1}{\omega}$. This can be understood as the radiation is in the form of photon, whose angular momentum is related to its energy by a factor of angular frequency. In quantum mechanics, $E = \hbar\omega$, where E and $\hbar$ are the photon energy and angular momentum, respectively.

(c) The charge density can be written as $\rho(\vec{x}) = \frac{e}{r^2}\delta(r-a)\delta(\cos\theta)\delta(\phi-\omega t)$. The electric dipole can be directly calculated as

$$\vec{p}(t) = \int \vec{x}\,\rho(\vec{x})\,d^3x = \int \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix} \frac{e}{r^2}\delta(r-a)\delta(\cos\theta)\delta(\phi-\omega t)\, r^2 d\Omega = ea\begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{pmatrix}$$

$$= ea\left(\mathrm{Re}\left[e^{-i\omega t}\right],\ \mathrm{Re}\left[ie^{-i\omega t}\right],\ 0\right)$$

Therefore $\vec{p} = ea(1, i, 0)$, and $\vec{p}^* = ea(1, -i, 0)$. $\vec{p}^* \times \vec{p} = 2ie^2a^2\hat{z}$, i.e., only the z-component is non-zero, which is $dL_z/dt = e^2a^2k^3/6\pi\varepsilon_0$.

If the charge oscillates in the z-direction, then $\vec{p}$ has a fixed direction, which means that $\vec{p}^* \times \vec{p}$ is zero. In this case, there is no angular momentum outflow.

(d) For magnetic dipole radiation, only the magnetic field part will contribute. Repeat the same procedure, we can get the similar result,

$$\frac{d\vec{L}}{dt} = \frac{\mu_0 k^3}{12\pi}\,\mathrm{Im}\left(\vec{m}^* \times \vec{m}\right).$$