12.1 (a) From the Lagrangian $L = -\frac{m}{2}U_\alpha U^\alpha - \frac{q}{c}U_\alpha A^\alpha$, we have

$$\frac{\partial L}{\partial U_\alpha} = -mU^\alpha - \frac{q}{c}A^\alpha, \qquad \frac{\partial L}{\partial x_\alpha} = -\frac{q}{c}U_\beta \partial^\alpha A^\beta$$

and $$\frac{d}{d\tau}\left(\frac{\partial L}{\partial U_\alpha}\right) = -m\frac{dU^\alpha}{d\tau} - \frac{q}{c}\frac{\partial A^\alpha}{\partial x_\beta}\frac{dx_\beta}{d\tau} = -m\dot{U}^\alpha - \frac{q}{c}U_\beta \partial^\beta A^\alpha.$$

The Euler-Lagrange equation : $\frac{d}{d\tau}\left(\frac{\partial L}{\partial U_\alpha}\right) - \frac{\partial L}{\partial x_\alpha} = 0$ becomes

$$-m\frac{dU^\alpha}{d\tau} - \frac{q}{c}U_\beta\partial^\beta A^\alpha + \frac{q}{c}U_\beta\partial^\alpha A^\beta = 0,$$

or $$m\frac{dU^\alpha}{d\tau} = \frac{q}{c}F^{\alpha\beta}U_\beta, \quad \text{where } F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

(b) The conjugate 4-vector momentum is $P^\alpha = -\frac{\partial L}{\partial U_\alpha} = mU^\alpha + \frac{q}{c}A^\alpha$. and the Hamiltonian is

$$H = P^\alpha U_\alpha + L = mU_\alpha U^\alpha + \frac{q}{c}U_\alpha A^\alpha - \frac{m}{2}U_\alpha U^\alpha - \frac{q}{c}U_\alpha A^\alpha = \frac{m}{2}U_\alpha U^\alpha$$

Since by definition. $U_\alpha U^\alpha = 1$, the effective Hamiltonian is $H = \frac{m}{2}$, which is a Lorentz invariant.