14.8 From Eq. (12.1), the equation of motion for a particle with mass m and charge ze in an electric field is $\frac{d}{dt}(\gamma mc\vec{\beta}) = \gamma^3 mc\vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}}) + \gamma mc\dot{\vec{\beta}} = ze\vec{E}$. Dot by $\vec{\beta}$ on both sides, we have

$$\gamma^3 mc\beta^2(\vec{\beta}\cdot\dot{\vec{\beta}}) + \gamma mc(\vec{\beta}\cdot\dot{\vec{\beta}}) = \gamma mc(\gamma^2\beta^2+1)(\vec{\beta}\cdot\dot{\vec{\beta}}) = \gamma^3 mc(\vec{\beta}\cdot\dot{\vec{\beta}}) = ze\vec{\beta}\cdot\vec{E}.$$

Therefore, $\vec{\beta}\cdot\dot{\vec{\beta}} = \frac{ze}{\gamma^3 mc}\vec{\beta}\cdot\vec{E}$, and $\dot{\vec{\beta}} = \left(ze\vec{E} - \gamma^3 mc\vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}})\right)/\gamma mc = \frac{ze}{\gamma mc}\left(\vec{E} - \vec{\beta}(\vec{\beta}\cdot\vec{E})\right)$.

Using Larmor's formula, Eq. (14.26), $P(t) = \frac{2}{3}\frac{z^2e^2}{c}\gamma^6\left[|\dot{\vec{\beta}}|^2 - |\vec{\beta}\times\dot{\vec{\beta}}|^2\right]$.

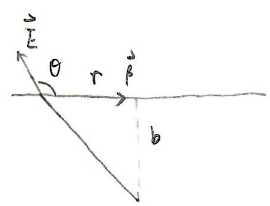
Since $|\dot{\vec{\beta}}|^2 = \frac{z^2e^2}{\gamma^2m^2c^2}\left[|\vec{E}|^2 - 2|\vec{\beta}\cdot\vec{E}|^2 + \beta^2|\vec{\beta}\cdot\vec{E}|^2\right]$, $|\vec{\beta}\times\dot{\vec{\beta}}|^2 = \frac{z^2e^2}{\gamma^2m^2c^2}|\vec{\beta}\times\vec{E}|^2 = \frac{z^2e^2}{\gamma^2m^2c^2}\left[\beta^2|\vec{E}|^2 - |\vec{\beta}\cdot\vec{E}|^2\right]$,

the Liénard result can be written as

$$P(t) = \frac{2}{3}\frac{z^4e^4}{m^2c^3}\gamma^4\left[|\vec{E}|^2 - 2|\vec{\beta}\cdot\vec{E}|^2 + \beta^2|\vec{\beta}\cdot\vec{E}|^2 - \beta^2|\vec{E}|^2 + |\vec{\beta}\cdot\vec{E}|^2\right]$$

$$= \frac{2}{3}\frac{z^4e^4}{m^2c^3}\gamma^4(1-\beta^2)\left[|\vec{E}|^2 - |\vec{\beta}\cdot\vec{E}|^2\right] = \frac{2}{3}\frac{z^4e^4}{m^2c^3}\gamma^2\left[|\vec{E}|^2 - |\vec{\beta}\cdot\vec{E}|^2\right]$$

In the configuration shown in the right, we have



$$|E| = \frac{Ze}{r^2+b^2}, \quad \vec{E}\cdot\vec{\beta} = \beta E\cos\theta = -\frac{Ze}{r^2+b^2}\frac{\beta r}{(r^2+b^2)^{1/2}}$$

then $$P(t) = \frac{2}{3}\frac{z^4Z^2e^6}{m^2c^3}\gamma^2\frac{1}{(r^2+b^2)^2}\left(1 - \frac{\beta^2r^2}{r^2+b^2}\right)$$

Integrating the power, and notice that $dt = dr/v = dr/c\beta$. we will get

$$\Delta W = \int_{-\infty}^{+\infty}P(t)\,dt = \frac{2}{3}\frac{z^4Z^2e^6}{m^2c^4\beta}\gamma^2\int_{-\infty}^{+\infty}\left[\frac{1}{(r^2+b^2)^2} - \frac{\beta^2r^2}{(r^2+b^2)^3}\right]dr$$

$$= \frac{2}{3}\frac{z^4Z^2e^6}{m^2c^4\beta}\gamma^2\left[\frac{\pi}{2b^3} - \frac{\pi\beta^2}{8b^3}\right] = \frac{2}{3}\frac{\pi z^4Z^2e^6}{m^2c^4\beta}\gamma^2\left[\frac{3}{8} + \frac{1-\beta^2}{8}\right]\frac{1}{b^3}$$

$$= \frac{2}{3}\frac{\pi z^4Z^2e^6}{m^2c^4\beta}\gamma^2\left[\frac{3}{8} + \frac{1}{8\gamma^2}\right]\frac{1}{b^3} = \frac{\pi z^4Z^2e^6}{4m^2c^4\beta}\left(\gamma^2 + \frac{1}{3}\right)\frac{1}{b^3}.$$

For non-relativistic motion, $c\beta = v_0$, $\gamma \to 1$, this reduces to the result of Prob. 14.7.