5.9 (a) The current density in the spherical coordinates can be written as

$$\vec{J}(\vec{x}) = \hat{\phi}\,\frac{I}{r}\sin\theta\,\delta(r-d)\left[\delta(\cos\theta-\cos\theta_0)+\delta(\cos\theta+\cos\theta_0)\right],$$

which upon integration leads to the same total current as the cylindrical coordinates. Then, the internal moments are

$$m_L = -\frac{1}{L(L+1)}\int d^3x\, r^{-L-1}P_L^1(\cos\theta)\,J(r,\theta)$$

$$= -\frac{1}{L(L+1)}\int_0^{2\pi}d\phi\int_{-1}^{1}d(\cos\theta)\int_0^{+\infty}r^2dr\,\frac{1}{r^{L+1}}\frac{I}{r}\sin\theta\,\delta(r-d)\left[\delta(\cos\theta-\cos\theta_0)+\delta(\cos\theta+\cos\theta_0)\right]$$

$$\times P_L^1(\cos\theta)$$

$$= -\frac{1}{L(L+1)}\frac{I\cdot 2\pi a}{d^{L+1}}\left(P_L^1(\cos\theta_0)+P_L^1(-\cos\theta_0)\right)$$

Here, $d=\sqrt{a^2+b^2/4}$ and $a=d\sin\theta_0$. Since $P_L^m(-x)=(-1)^{L-m}P_L^m(x)$, we can see that even moments will disappear, and we are left with

$$m_1 = -\frac{I\cdot 2\pi a}{d^2}P_1^1(\cos\theta_0),\quad m_3 = -\frac{I\cdot 2\pi a}{6d^4}P_3^1(\cos\theta_0),\quad m_5 = -\frac{I\cdot 2\pi a}{15d^6}P_5^1(\cos\theta_0).$$

Using the explicit expression of the associated Legendre polynomials,

$$P_1^1(\cos\theta) = -\sin\theta,\quad P_3^1(\cos\theta) = -\frac{3}{2}\sin\theta(5\cos^2\theta-1),\quad P_5^1(\cos\theta) = -\frac{15}{8}\sin\theta(21\cos^4\theta-14\cos^2\theta-1),$$

the internal moments become

$$m_1 = -\frac{I\cdot 2\pi a}{d^2}\cdot(-\sin\theta_0) = \frac{I\cdot 2\pi a}{d^2}\cdot\frac{a}{d} = \frac{2\pi a^2 I}{d^3}$$

$$m_3 = -\frac{I\cdot 2\pi a}{6d^4}\cdot\left(-\frac{3}{2}\right)\frac{a}{d}\left(5\left(1-\frac{a^2}{d^2}\right)-1\right) = \frac{\pi a^2 I(b^2-a^2)}{2d^7},$$

$$m_5 = -\frac{I\cdot 2\pi a}{15d^6}\left(-\frac{15}{8}\right)\frac{a}{d}\left(21\left(1-\frac{a^2}{d^2}\right)^2-14\left(1-\frac{a^2}{d^2}\right)-1\right) = \frac{\pi a^2 I(b^4-6b^2a^2+2a^4)}{8d^{11}}$$

For external moments,

$$M_L = -\frac{d^L}{L(L+1)}I\cdot 2\pi a\left(P_L^1(\cos\theta_0)+P_L^1(-\cos\theta_0)\right) = d^{2L+1}m_L.$$

Therefore, $m_1 = 2\pi a^2 I$, $\quad m_3 = \frac{1}{2}\pi a^2 I (b^2 - a^2)$, $\quad m_5 = \frac{1}{8}\pi a^2 I \left(b^4 - 6b^2 a^2 + 4a^4 \right)$

(b) Using the internal multipole expansion, the magnetic induction is given by

$$\vec{B} = \nabla \times \vec{A} = \vec{e}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial r}\left(r A_\phi\right)$$

$$= \vec{e}_r \left[-\frac{\mu_0}{4\pi}\sum_L m_L r^{L-1} \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta P_L'(\cos\theta)\right) \right] - \vec{e}_\theta \left[-\frac{\mu_0}{4\pi}\sum_L m_L P_L'(\cos\theta) \frac{1}{r}\frac{\partial}{\partial r}\left(r^{L+1}\right) \right]$$

$$= \frac{\mu_0}{4\pi}\sum_L m_L r^{L-1} \left\{ \vec{e}_r L(L+1) P_L(\cos\theta) + \vec{e}_\theta (L+1) P_L'(\cos\theta) \right\} = \frac{\mu_0}{4\pi}\sum_L m_L r^{L-1} \vec{\xi}_L ,$$

where $\vec{\xi}_L = \vec{e}_r L(L+1) P_L(\cos\theta) + \vec{e}_\theta (L+1) P_L'(\cos\theta)$. For $L=1$, we know

$$\vec{\xi}_1 = 2\left(\cos\theta\, \vec{e}_r - \sin\theta\, \vec{e}_\theta\right) = 2\vec{e}_z .$$

Similarly, $\vec{\xi}_3 = \vec{e}_r \cdot 12 P_3(\cos\theta) + \vec{e}_\theta \cdot 4 P_3'(\cos\theta) = \vec{e}_r \cdot 12 \cdot \frac{1}{2}(5\cos^3\theta - 3\cos\theta) + \vec{e}_\theta \cdot 4 \cdot \frac{3}{2}\sin\theta\left(5\cos^2\theta - 1\right)$

$$\vec{\xi}_5 = \vec{e}_r \cdot 30 \cdot \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right) + \vec{e}_\theta \cdot 6 \cdot \frac{15}{8}\sin\theta\left(21x^4 - 14x^2 - 1\right).$$

Since we are interested in the magnetic induction on the z-axis, we can set $\theta = 0$ in the above formula and notice that $\vec{e}_z = \vec{e}_r$ and $z = r$, we have

$$B_z = \frac{\mu_0}{2\pi}\left(m_1 + 6 m_3 z^2 + 15 m_5 z^4\right)$$

$$= \frac{\mu_0 I a^2}{d^3}\left(1 + \frac{3(b^2 - a^2) z^2}{2d^4} + \frac{15(b^4 - 6b^2a^2 + 2a^4) z^4}{16 d^8} \right),$$

which agrees with Prob. 5.7 (b).