3.5 Solution: The Green function in three-dimensional space with a grounded conducting sphere, under the Dirichlet boundary condition, is given by

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{a}{x\left|\dfrac{a^2}{x^2}\mathbf{x} - \mathbf{x}'\right|}, \tag{1}$$

where both $\mathbf{x}$ and $\mathbf{x}'$ are points inside the sphere.

In terms of spherical coordinates, the Green function, Eq. (1), can be written as

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{(x^2 + x'^2 - 2xx'\cos\gamma)^{1/2}} - \frac{1}{\left(\dfrac{x^2x'^2}{a^2} + a^2 - 2xx'\cos\gamma\right)^{1/2}},$$

where γ is the angle between $\mathbf{x}$ and $\mathbf{x}'$, and

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'),$$

with $\mathbf{x} = (x, \theta, \phi)$ and $\mathbf{x}' = (x', \theta', \phi')$ in spherical coordinates. To determine the potential inside the sphere with the potential specified on the surface, we can use Eq. (1.42) and will need the normal derivative of the Green function on the sphere. Here, since we are interested in the potential inside the sphere, the normal vector is pointing outside of the sphere, that is

$$\begin{aligned}
\left.\frac{\partial G}{\partial n'}\right|_{x'=a} &= \left.\frac{\partial G}{\partial x'}\right|_{x'=a} = \left.\left[-\frac{x' - x\cos\gamma}{(x^2 + x'^2 - 2xx'\cos\gamma)^{3/2}} + \frac{\dfrac{x^2x'}{a^2} - x\cos\gamma}{\left(\dfrac{x^2x'^2}{a^2} + a^2 - 2xx'\cos\gamma\right)^{3/2}}\right]\right|_{x'=a} \\
&= \frac{x^2 - a^2}{a\,(x^2 + x'^2 - 2xx'\cos\gamma)^{3/2}}.
\end{aligned}$$

Now, applying Eq. (1.42), the potential inside the sphere is

$$\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{1}{4\pi}\oint_{x'=a}\Phi(\mathbf{x}')\frac{\partial G}{\partial n'}da' = -\frac{1}{4\pi}\int\Phi(a, \theta', \phi')\frac{\partial G}{\partial x'}\cdot a^2 d\Omega' \\
&= \frac{a(a^2 - x^2)}{4\pi}\int\frac{V(\theta', \phi')}{(x^2 + x'^2 - 2xx'\cos\gamma)^{3/2}}d\Omega'.
\end{aligned}$$

This establishes the result in formulation (a).

To get the result in formulation (b), we can use the expansion result, Eq. (3.38),

$$\frac{1}{\mathbf{x} - \mathbf{x}'} = \sum_{l=0}^{\infty}\frac{x_<^l}{x_>^{l+1}}P_l(\cos\gamma).$$

Keep in mind that eventually we will set $x' = a$, and $x < a < a^2/x$, the Green function can be expanded as

$$G(\mathbf{x}, \mathbf{x}') = \sum_{l=0}^{\infty}\frac{x^l}{x'^{l+1}}P_l(\cos\gamma) - \frac{a}{x}\sum_{l=0}^{\infty}\frac{x'^l}{(a^2/x)^{l+1}}P_l(\cos\gamma),$$

for $\mathbf{x}'$ near the sphere. Similarly,

$$
\begin{aligned}
\left.\frac{\partial G}{\partial n'}\right|_{x'=a} &= \left.\frac{\partial G}{\partial x'}\right|_{x'=a} = \left.\left[-\sum_{l=0}^{\infty}(l+1)\frac{x^l}{x'^{l+2}}P_l(\cos\gamma) - \frac{a}{x}\sum_{l=0}^{\infty} l\frac{x'^{l-1}}{(a^2/x)^{l+1}}P_l(\cos\gamma)\right]\right|_{x'=a} \\
&= -\frac{1}{a^2}\sum_{l=0}^{\infty}(2l+1)\left(\frac{x}{a}\right)^l P_l(\cos\gamma) = -\frac{4\pi}{a^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{x}{a}\right)^l Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi),
\end{aligned}
$$

where we have used the addition theorem for spherical harmonics in the last step. Again, applying Eq. (1.42), we have

$$
\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{1}{4\pi}\oint_{x'=a}\Phi(\mathbf{x}')\frac{\partial G}{\partial n'}da' = \frac{1}{a^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int V(\theta',\phi')\left(\frac{x}{a}\right)^l Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)\cdot a^2 d\Omega' \\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\int V(\theta',\phi')Y^*_{lm}(\theta',\phi')d\Omega'\right)\left(\frac{x}{a}\right)^l Y_{lm}(\theta,\phi).
\end{aligned}
$$

By defining

$$
A_{lm} = \int V(\theta',\phi')Y^*_{lm}(\theta',\phi')d\Omega',
$$

we will obtain the result in formulation (b),

$$
\Phi(\mathbf{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}\left(\frac{x}{a}\right)^l Y_{lm}(\theta,\phi).
$$

Therefore, we have established the equivalence of the two forms of solutions for the potential inside the sphere.