3.23 Solution: Green function, again!

(i) Let us first establish a completeness relation for Bessel function in a finite interval. For an arbitrary function satisfying certain regularity condition which we do not care here, it can be expanded with Bessel functions,

$$f(\rho) = \sum_{n=1}^{\infty} A_{mn} J_m \left(x_{mn} \frac{\rho}{a} \right),$$

for $0 < \rho < a$, and the coefficients are

$$A_{mn} = \frac{2}{a^2 J_{m+1}^2(x_{mn})} \int_0^a \rho f(\rho) J_m \left(x_{mn} \frac{\rho}{a} \right) d\rho.$$

Put this back to the original expansion,

$$f(\rho) = \int_0^a \left(\sum_{n=1}^{\infty} \frac{2}{a^2 J_{m+1}^2(x_{mn})} \rho' J_m \left(x_{mn} \frac{\rho'}{a} \right) J_m \left(x_{mn} \frac{\rho}{a} \right) \right) f(\rho') d\rho',$$

which can lead to the following completeness relation,

$$\sum_{n=1}^{\infty} \frac{2}{a^2 J_{m+1}^2(x_{mn})} J_m \left(x_{mn} \frac{\rho'}{a} \right) J_m \left(x_{mn} \frac{\rho}{a} \right) = \frac{1}{\rho} \delta(\rho - \rho').$$

Then, the three-dimensional delta function in the cylindrical coordinates becomes

$$\begin{aligned} \delta(\mathbf{x} - \mathbf{x}') &= \frac{1}{\rho} \delta(\rho - \rho') \delta(\phi - \phi') \delta(z - z') \\ &= \frac{1}{\pi} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{e^{im(\phi - \phi')}}{a^2 J_{m+1}^2(x_{mn})} J_m \left(x_{mn} \frac{\rho'}{a} \right) J_m \left(x_{mn} \frac{\rho}{a} \right) \cdot \delta(z - z'), \end{aligned}$$

and the Green function must have a similar form,

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{e^{im(\phi - \phi')}}{a^2 J_{m+1}^2(x_{mn})} J_m \left(x_{mn} \frac{\rho'}{a} \right) J_m \left(x_{mn} \frac{\rho}{a} \right) \cdot g_{mn}(z, z').$$

Applying the Laplace operator on the Green function, we have

$$\begin{aligned} \nabla_{\mathbf{x}'}^2 G(\mathbf{x}, \mathbf{x}') &= \left[\frac{1}{\rho'} \frac{\partial}{\partial \rho'} \left(\rho' \frac{\partial}{\partial \rho'} \right) + \frac{1}{\rho'^2} \frac{\partial^2}{\partial \phi'^2} + \frac{\partial^2}{\partial z'^2} \right] G(\mathbf{x}, \mathbf{x}') \\ &= \frac{1}{\pi} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{e^{im(\phi - \phi')}}{a^2 J_{m+1}^2(x_{mn})} J_m \left(x_{mn} \frac{\rho}{a} \right) J_m \left(x_{mn} \frac{\rho'}{a} \right) \\ &\qquad \times \left[\frac{\partial^2}{\partial z'^2} + \frac{1}{J_m(x_{mn}\rho'/a)} \frac{1}{\rho'} \frac{\partial}{\partial \rho'} \left(\rho' \frac{\partial}{\partial \rho'} J_m \left(x_{mn} \frac{\rho'}{a} \right) \right) - \frac{m^2}{\rho'^2} \right] g_{mn}(z, z'). \end{aligned}$$

Using the Bessel differential equation,

$$\frac{1}{\rho'} \frac{\partial}{\partial \rho'} \left(\rho' \frac{\partial}{\partial \rho'} J_m \left(x_{mn} \frac{\rho'}{a} \right) \right) + \left(\frac{x_{mn}^2}{a^2} - \frac{m^2}{\rho'^2} \right) J_m \left(x_{mn} \frac{\rho'}{a} \right) = 0,$$

we can see that

$$\nabla^2_{\mathbf{x}'} G(\mathbf{x},\mathbf{x}') = \frac{1}{\pi}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')}}{a^2 J^2_{m+1}(x_{mn})} J_m\left(x_{mn}\frac{\rho}{a}\right) J_m\left(x_{mn}\frac{\rho'}{a}\right)\left[\frac{\partial^2}{\partial z'^2} - \frac{x^2_{mn}}{a^2}\right] g_{mn}(z,z').$$

Therefore, g must satisfy the following equation,

$$\left[\frac{\partial^2}{\partial z'^2} - \frac{x^2_{mn}}{a^2}\right] g_{mn}(z,z') = -4\pi\delta(z-z'),$$

with boundary conditions $g_{mn}(z,0) = g_{mn}(z,L) = 0$. This is the same equation that we have encountered in Problem 2.15, and the solution has a symmetric form,

$$g_{mn}(z,z') = A_{mn}\sinh\left(x_{mn}\frac{z_<}{a}\right)\sinh\left(x_{mn}\frac{L-z_>}{a}\right),$$

with the unknown coefficient A that can be determined in a procedure that we have seen several times,

$$A_{mn} = \frac{4\pi a}{x_{mn}\sinh\left(x_{mn}\dfrac{L}{a}\right)}.$$

Now, the Green function becomes

$$G(\mathbf{x},\mathbf{x}') = \frac{4}{a}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')} J_m\left(x_{mn}\dfrac{\rho}{a}\right) J_m\left(x_{mn}\dfrac{\rho'}{a}\right)}{x_{mn}J^2_{m+1}(x_{mn})\sinh\left(x_{mn}\dfrac{L}{a}\right)}\sinh\left(x_{mn}\frac{z_<}{a}\right)\sinh\left(x_{mn}\frac{L-z_>}{a}\right).$$

Then, the potential for a point charge can be obtained by a direct integration with the charge density expressed in delta function,

$$\Phi(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi\varepsilon_0}\int \sigma(\mathbf{x}'')G(\mathbf{x},\mathbf{x}'')d^3x'',$$

with

$$\sigma(\mathbf{x}'') = q\delta(\mathbf{x}''-\mathbf{x}') = \frac{q}{\rho''}\delta(\rho''-\rho')\delta(\phi''-\phi')\delta(z''-z'),$$

which yields

$$\Phi(\mathbf{x},\mathbf{x}') = \frac{q}{\pi\varepsilon_0 a}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')} J_m\left(x_{mn}\dfrac{\rho}{a}\right) J_m\left(x_{mn}\dfrac{\rho'}{a}\right)}{x_{mn}J^2_{m+1}(x_{mn})\sinh\left(x_{mn}\dfrac{L}{a}\right)}\sinh\left(x_{mn}\frac{z_<}{a}\right)\sinh\left(x_{mn}\frac{L-z_>}{a}\right)$$

(ii) Let us write the delta function as

$$\begin{aligned}\delta(\mathbf{x}-\mathbf{x}') &= \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z') \\ &= \frac{1}{\pi L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi}{L}z\right)\sin\left(\frac{n\pi}{L}z'\right)\cdot\frac{1}{\rho}\delta(\rho-\rho'),\end{aligned}$$

and the corresponding Green functions is

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{\pi L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi}{L}z\right)\sin\left(\frac{n\pi}{L}z'\right)\cdot g_{mn}(\rho,\rho').$$

Applying the Laplace operator,

$$\begin{aligned}
\nabla^2_{\mathbf{x}'}G(\mathbf{x},\mathbf{x}') &= \left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right)+\frac{1}{\rho'^2}\frac{\partial^2}{\partial\phi'^2}+\frac{\partial^2}{\partial z'^2}\right]G(\mathbf{x},\mathbf{x}') \\
&= \frac{1}{\pi L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi}{L}z\right)\sin\left(\frac{n\pi}{L}z'\right) \\
&\qquad \times\left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right)-\left(\frac{n^2\pi^2}{L^2}+\frac{m^2}{\rho'^2}\right)\right]g_{mn}(\rho,\rho'),
\end{aligned}$$

which leads to the following differential equation for g,

$$\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}g_{mn}(\rho,\rho')\right)-\left(\frac{n^2\pi^2}{L^2}+\frac{m^2}{\rho'^2}\right)g_{mn}(\rho,\rho') = -\frac{4\pi}{\rho}\delta(\rho-\rho').$$

The solution must be linear combinations of $I_m(n\pi\rho'/L)$ and $K_m(n\pi\rho'/L)$. For the regularity at $\rho'=0$, we cna only use $I_m(n\pi\rho'/L)$. At $\rho'=a$, for the function to vanish, the solution can be chosen to have the following form,

$$I_m\left(\frac{n\pi}{L}\rho'\right)K_m\left(\frac{n\pi}{L}a\right)-K_m\left(\frac{n\pi}{L}\rho'\right)I_m\left(\frac{n\pi}{L}a\right),$$

and the symmetric solution to the differential equation is

$$g_{mn}(\rho,\rho') = A_{mn}I_m\left(\frac{n\pi}{L}\rho_<\right)\left[I_m\left(\frac{n\pi}{L}\rho_>\right)K_m\left(\frac{n\pi}{L}a\right)-K_m\left(\frac{n\pi}{L}\rho_>\right)I_m\left(\frac{n\pi}{L}a\right)\right].$$

The coefficient A can be determined by integrating the equation near ρ,

$$\left.\frac{\partial}{\partial\rho'}g_{mn}(\rho,\rho')\right|_{\rho'=\rho+\varepsilon}-\left.\frac{\partial}{\partial\rho'}g_{mn}(\rho,\rho')\right|_{\rho'=\rho-\varepsilon} = -4\pi.$$

Since

$$\left.\frac{\partial}{\partial\rho'}g_{mn}(\rho,\rho')\right|_{\rho'=\rho+\varepsilon} = \frac{n\pi}{L}A_{mn}I_m\left(\frac{n\pi}{L}\rho\right)\left[I'_m\left(\frac{n\pi}{L}\rho\right)K_m\left(\frac{n\pi}{L}a\right)-K'_m\left(\frac{n\pi}{L}\rho\right)I_m\left(\frac{n\pi}{L}a\right)\right],$$

and

$$\left.\frac{\partial}{\partial\rho'}g_{mn}(\rho,\rho')\right|_{\rho'=\rho-\varepsilon} = \frac{n\pi}{L}A_{mn}I'_m\left(\frac{n\pi}{L}\rho\right)\left[I_m\left(\frac{n\pi}{L}\rho\right)K_m\left(\frac{n\pi}{L}a\right)-K_m\left(\frac{n\pi}{L}\rho\right)I_m\left(\frac{n\pi}{L}a\right)\right],$$

then

$$\begin{aligned}
&\left.\frac{\partial}{\partial\rho'}g_{mn}(\rho,\rho')\right|_{\rho'=\rho+\varepsilon}-\left.\frac{\partial}{\partial\rho'}g_{mn}(\rho,\rho')\right|_{\rho'=\rho-\varepsilon} \\
= \quad & \frac{n\pi}{L}A_{mn}I_m\left(\frac{n\pi}{L}a\right)\left[I'_m\left(\frac{n\pi}{L}\rho\right)K_m\left(\frac{n\pi}{L}\rho\right)-K'_m\left(\frac{n\pi}{L}\rho\right)I_m\left(\frac{n\pi}{L}\rho\right)\right]
\end{aligned}$$

$$= -\frac{n\pi}{L}A_{mn}I_m\left(\frac{n\pi}{L}a\right)W\left[I_m\left(\frac{n\pi}{L}\rho\right),K_m\left(\frac{n\pi}{L}\rho\right)\right] = \frac{A_{mn}}{\rho}I_m\left(\frac{n\pi}{L}a\right),$$

which gives

$$A_{mn} = -\frac{4\pi}{I_m\left(\frac{n\pi}{L}a\right)}.$$

Put this back into the expression for the Green function, we will have

$$\begin{aligned} G(\mathbf{x},\mathbf{x}') &= \frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}e^{im(\phi-\phi')}\sin\left(\frac{n\pi}{L}z\right)\sin\left(\frac{n\pi}{L}z'\right)\frac{I_m\left(\frac{n\pi}{L}\rho_<\right)}{I_m\left(\frac{n\pi}{L}a\right)} \\ &\quad\times\left[I_m\left(\frac{n\pi}{L}a\right)K_m\left(\frac{n\pi}{L}\rho_>\right)-K_m\left(\frac{n\pi}{L}a\right)I_m\left(\frac{n\pi}{L}\rho_>\right)\right], \end{aligned}$$

and the potential is

$$\begin{aligned} \Phi(\mathbf{x},\mathbf{x}') &= \frac{q}{\pi\varepsilon_0 L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}e^{im(\phi-\phi')}\sin\left(\frac{n\pi}{L}z\right)\sin\left(\frac{n\pi}{L}z'\right)\frac{I_m\left(\frac{n\pi}{L}\rho_<\right)}{I_m\left(\frac{n\pi}{L}a\right)} \\ &\quad\times\left[I_m\left(\frac{n\pi}{L}a\right)K_m\left(\frac{n\pi}{L}\rho_>\right)-K_m\left(\frac{n\pi}{L}a\right)I_m\left(\frac{n\pi}{L}\rho_>\right)\right]. \end{aligned}$$

(iii) A complete orthonormal set for the geometry is

$$\psi_{mnk}(\mathbf{x}) = \frac{1}{\sqrt{2\pi}}e^{im\phi}\cdot\frac{\sqrt{2}}{aJ_{m+1}(x_{mn})}J_m\left(x_{mn}\frac{\rho}{a}\right)\cdot\sqrt{\frac{2}{L}}\sin\left(\frac{k\pi}{L}z\right).$$

Applying the Laplace operator, it can be easily shown that any function in the above set is an eigenfunction of the Laplace operator,

$$\nabla^2\psi_{mnk}(\mathbf{x}) = -\lambda_{mnk}\psi_{mnk}(\mathbf{x}),$$

with

$$\lambda_{mnk} = \left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2.$$

Then, the Green function can be constructed as

$$\begin{aligned} G(\mathbf{x},\mathbf{x}') &= 4\pi\sum_{m,n,k}\frac{\psi_{mnk}(\mathbf{x})\psi^*_{mnk}(\mathbf{x}')}{\lambda_{mnk}} \\ &= \frac{8}{La^2}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{e^{im(\phi-\phi')}\sin\left(\frac{k\pi}{L}z\right)\sin\left(\frac{k\pi}{L}z'\right)J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn}\frac{\rho'}{a}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2\right]J^2_{m+1}(x_{mn})}, \end{aligned}$$

while the potential is

$$\Phi(\mathbf{x},\mathbf{x}') = \frac{2q}{\pi\varepsilon_0 La^2}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{e^{im(\phi-\phi')}\sin\left(\frac{k\pi}{L}z\right)\sin\left(\frac{k\pi}{L}z'\right)J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn}\frac{\rho'}{a}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2\right]J^2_{m+1}(x_{mn})}.$$

(iv) Comparing the Green functions in (i) and (iii), we will immediately obtain

$$\frac{2}{La}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\pi}{L}z\right)\sin\left(\frac{k\pi}{L}z'\right)}{\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2}=\frac{\sinh\left(x_{mn}\frac{z_<}{a}\right)\sinh\left(x_{mn}\frac{L-z_>}{a}\right)}{x_{mn}\sinh\left(x_{mn}\frac{L}{a}\right)}.$$

Similarly, for (ii) and (iii),

$$\begin{aligned}&\frac{2}{a^2}\sum_{k=1}^{\infty}\frac{J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn}\frac{\rho'}{a}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{k\pi}{L}\right)^2\right]J_{m+1}^2(x_{mn})}\\ = \quad &\frac{I_m\left(\frac{n\pi}{L}\rho_<\right)}{I_m\left(\frac{n\pi}{L}a\right)}\left[I_m\left(\frac{n\pi}{L}a\right)K_m\left(\frac{n\pi}{L}\rho_>\right)-K_m\left(\frac{n\pi}{L}a\right)I_m\left(\frac{n\pi}{L}\rho_>\right)\right].\end{aligned}$$