AAPT PHYSICS EDUCATION
AMERICAN JOURNAL of PHYSICS

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Citation: Am. J. Phys. **58**, 978 (1990); doi: 10.1119/1.16260
View online: http://dx.doi.org/10.1119/1.16260
View Table of Contents: http://ajp.aapt.org/resource/1/AJPIAS/v58/i10
Published by the American Association of Physics Teachers

$$E_{\rm int} = E - \frac{1}{2}M\mathbf{V}_0^2 - \frac{1}{2}I\omega^2 = E - \frac{\mathbf{P}^2}{2M} - \frac{\mathscr{M}^2}{2I}.$$

[13]The relation (13) is a consequence of Eq. (23), because S depends only on the modulus $\mathscr{M}$ of $\mathscr{M}$. The moment of inertia, at equilibrium, is related to the entropy by

$$\frac{1}{I} = -\frac{T}{\mathscr{M}}\frac{\partial S(E,\mathscr{M},N)}{\partial \mathscr{M}}.$$

[14]One sometimes refers to "systems with negative temperature." Such "systems" only concern a particular subset of the degrees of freedom of a true system in an incomplete equilibrium state and they have no kinetic energy.

[15]This hypothesis is not essential. Due to relations (23) and (13), the argument remains valid even if the support is not rigid.

[16]This conclusion would not hold if the total momentum and angular momentum were different from zero. For instance, a nonvanishing total momentum $\mathbf{P}_{\rm tot}$ would induce a factor $\exp(-\mathbf{P}_l \cdot \mathbf{P}_{\rm tot}/kTM_{\mathscr{S}})$ in the probability p_l. But such a result is necessarily illusory: The probability of a given state of $\mathscr{S}$ cannot depend on the reference frame. The origin of this paradox is the following. If two systems exchange momentum without exchanging energy in a particular reference frame, such an exchange of momentum necessarily implies exchange of energy in another reference frame moving with respect to the first one. The question is then, in which reference frame can a so-called adiabatic separation forbid energy exchanges while allowing momentum and angular momentum exchanges? The only reasonable choice is the rest frame of the global system.

Torque and force on a magnetic dipole

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(Received 5 May 1989; accepted for publication 21 February 1990)

Recent controversies about torque and force on a magnetic dipole are discussed. Three essentially different current loop models are analyzed. Although all models yield the same expression for the torque, $\mathbf{N} = \mathbf{m} \times \mathbf{B}$, the detailed mechanisms that give rise to the torque in each case are very different. The expression for the force on a magnetic dipole is derived and analyzed for all models. The force expression is the same for all current loop models but it differs from the force on a magnetic charge dipole. The expression, obtained for the force on a current loop magnetic dipole, $\mathbf{F}_{\rm CL} = \nabla(\mathbf{m} \cdot \mathbf{B}) - (d/dt)(\mathbf{m} \times \mathbf{E}/c)$, differs from what usually appears in the educational literature. The standard "naive" calculation of the force yields the correct expression for the rate of change of the total momentum, $d\mathbf{P}/dt = \nabla(\mathbf{m} \cdot \mathbf{B})$. However, the current loop in an external electric field has an internal "hidden momentum" $\mathbf{m} \times \mathbf{E}/c$, which is not related to the motion of the center of mass of the dipole. Thus, for the force, defined as mass times acceleration, it is necessary to subtract the time derivative of this "hidden momentum."

I. INTRODUCTION

It is surprising that classical electrodynamics is still the subject of heated discussion. There are paradoxes, controversies, etc. about the torque and force acting on a magnetic dipole in an external electromagnetic field. Recently, Namias[1] presented in this Journal an explanation of the origin of the torque that is different from the one given by Bedford and Krumm.[2] Several months before, Boyer[3] discussed the force on the magnetic dipole and derived an expression that disagrees with the result obtained by Aharonov *et al.*[4,5] These are only the most recent examples of numerous controversies on this subject. The origin of this extensive polemic is the large variety of different models for a magnetic dipole that require quite different explanations for the detailed mechanisms that lead to torque and force. We shall outline the essentially different models and analyze the torque and the force expressions for all models. We shall show that Namias together with Bedford and Krumm are "both right." They gave correct explanations for *different* models of a current loop magnetic dipole. We shall argue that Boyer, following the leading textbook by Jackson,[6] missed a term in the expression for the force on a current loop magnetic dipole in a varying electromagnetic field.

II. DIFFERENT MODELS FOR MAGNETIC DIPOLE

One model, the analysis of which raises no controversy (although it is generally believed that this is *not* the correct physical model) is a magnetic dipole consisting of *magnetic charges*. The force on a magnetic dipole $\mathbf{m}$ in an electromagnetic field for the model of magnetic charges, in complete analogy with the force on an electric dipole, is given by

$$\mathbf{F}_{\rm MC} = (\mathbf{m} \cdot \nabla)\mathbf{B} - (1/c)\dot{\mathbf{m}} \times \mathbf{E}, \tag{1}$$

and the torque is given by

$$\mathbf{N} = \mathbf{m} \times \mathbf{B}. \tag{2}$$

Here and below, if not stated otherwise, the expressions are given in the rest frame of the dipole.[7]

Another possibility is that the magnetic dipole is a *current loop*. This, however, corresponds to several significantly different models. We shall specify three of them which, we believe, cover all essentially different cases. The models are

(i) A gas of charged particles constrained to move inside a *neutral* tube;

(ii) A gas of charged particles constrained to move inside a *conducting* tube;

(iii) A charged (incompressible) liquid constrained to move inside a *neutral* tube.

Since we are usually interested in an electrically neutral magnetic dipole, each model consists, in fact, of a pair of oppositely charged gases or pair of oppositely charged liquids moving in opposite directions.

Model (i) represents a general class of models in which forces constraining the motion of charged media do no work. Model (ii) represents current loops of any origin that are inside conducting shells. In this model, an external electric field does not influence the motion, nor does it influence any other property of the charged media in the current loop. In model (iii) we can see effects that arise in rotating rigid body models. The motion of the charged media inside the current loop is not influenced by an external electric field either, but, contrary to model (ii), mechanical stresses inside the loop do depend on the external electric field.

In this work we consider stationary and quasistationary situations: The electromagnetic field is varying slowly enough in space and time. Thus in model (ii) we can neglect penetration of the electric field into the conductor, and in model (iii) the liquid does not have to be *absolutely* incompressible: The velocity of sound in the liquid can be much larger than the characteristic velocities of the varying electromagnetic field without being larger than the velocity of light. Also, we assume that the size of the dipole is small enough that the variation of fields and their gradients through the volume of the dipole can be neglected.

III. THE TORQUE ON A CURRENT LOOP MAGNETIC DIPOLE

A straightforward calculation[7] of the torque in the rest frame of the magnetic dipole (we name this frame S') for any of the above models yields the same result (2) as for the magnetic charge model:

$$\mathbf{N} = \frac{1}{c}\int \mathbf{r}\times(\mathbf{J}\times\mathbf{B})\,d\tau = \mathbf{m}\times\mathbf{B}. \tag{3}$$

Namias disagrees with Bedford and Krumm about the origin of the torque on a moving magnetic dipole in an external electric field in the rest frame of the charges producing the electric field (we name this frame S). The controversy is about the following configuration (Fig. 1). A current loop is placed in the yz plane and is moving in the x direction with velocity V, with a constant external field in the z direction. In frame S there is no magnetic field and, therefore, calculations of the Lorentz force yield no torque on a current loop. However, in the rest frame of the loop (S') there is a magnetic field $\mathbf{B} = -\mathbf{V}\times\mathbf{E}/c$ which yields the torque $-(1/c)\mathbf{m}\times(\mathbf{V}\times\mathbf{E})$ acting on the loop. Indeed, the Lorentz force creates a torque by acting on charged particles that move in the two sides of the loop in opposite directions. The force on a particle with charge q and velocity $\mathbf{v}$ is

$$\mathbf{F} = -(q/c^2)\mathbf{v}\times(\mathbf{V}\times\mathbf{E}) = -(q/c^2)V\,\mathbf{v}\cdot\mathbf{E}\,\hat{\mathbf{x}}. \tag{4}$$

Finally, the particle exerts a force on the wall of the tube. Forces from all particles yield the torque $\mathbf{N} = -(1/c)\mathbf{m}\times(\mathbf{V}\times\mathbf{E})$. This is in contradiction with the "naive" analysis in the frame S that predicts zero torque. The latter analysis is inadequate, and more careful consideration involving different effects for different models has to be made.

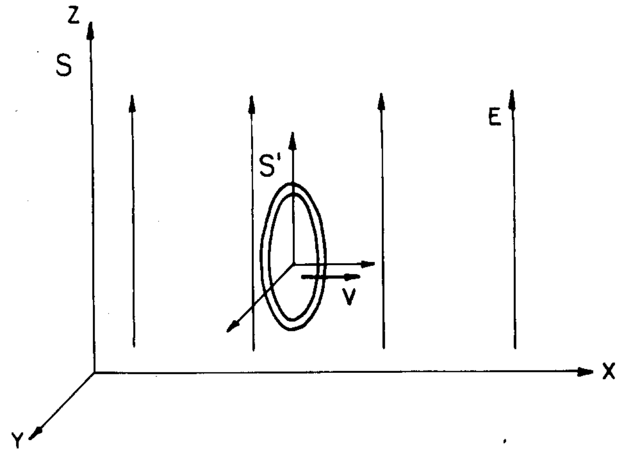

Fig. 1. A current loop moving in a constant electric field $\mathbf{E}$. In the frame S', which is the rest frame of the loop, there is a magnetic field $\mathbf{B} = -\mathbf{V}\times\mathbf{E}/c$ and, consequently, there is a torque exerted on the current loop. However, in the frame S, the rest frame of the sources of the field $\mathbf{E}$, there is no obvious reason for the torque since the magnetic field is zero, and more detailed analysis is required.

Bedford and Krumm chose model (i). In the frame S there is no magnetic field whatsoever, and therefore, there is no external force in the x direction on charged particles. Nevertheless, they found that the particles exert a force on the wall of the tube. A relativistic effect explains the issue. An external electric field accelerates the particles inside the tube changing their relativistic mass. Thus their momentum in the x direction is changing at the rate $\dot{m}V$, where $\dot{m} = (q/c^2)\mathbf{E}\cdot\mathbf{v}$ ($\mathbf{V}$ is the velocity of the loop in the frame S, and $\mathbf{v}$ is the velocity of the charged particle in the current loop in the frame S'). This change of momentum requires force exerted by the wall and, due to Newton's third law for this local (in general, nonelectromagnetic) interaction, the particles exert forces on the wall. This is exactly the force (4), and therefore, the torque in both frames S and S' is the same.

Namias chose, instead, model (ii) in which the above effect does not appear since the induced charges screen the external electric field and the charged particles do not accelerate inside the tube. He found that in this model the torque arises from the Lorentz force due to the magnetic field of the (moving) induced charges of the conducting tube. This can be seen qualitatively by the following simple rough consideration. Since the electric field inside the conductor vanishes, the potential ϕ' of the induced charges is equal to the negative of the potential of the external field $\mathbf{E}$. Therefore, the vector potential of the induced charges in the frame S, which moves with velocity $-\mathbf{V}$ relative to the rest frame of the induced charges, is $\mathbf{A}' \cong -(\mathbf{V}/c)\phi$. The magnetic field in frame S is then given by $\mathbf{B}' = \nabla\times\mathbf{A}' = (1/c)\mathbf{V}\times\nabla\phi = -(1/c)\mathbf{V}\times\mathbf{E}$. We showed that the magnetic field inside the conductor, i.e., in the location of the current, is the same in both frames S and S'. Consequently, the torque on the current loop in both frames is also the same.

Neither the magnetic field due to the induced charges nor the acceleration of the charges exists in the model (iii). Another relativistic effect is essential here. Instead of the charged particles we have a continuous charged medium;

therefore, instead of considering the force on a charged particle (4), we have to consider the x component of the force acting on a small volume $d\tau$:

$$dF_x = -(1/c^2)\rho \mathbf{v}\times(\mathbf{V}\times\mathbf{E})d\tau = -(\rho V/c^2)\mathbf{v}\cdot\mathbf{E}\,d\tau. \quad (5)$$

In the frame S there is no electromagnetic force acting in the x direction on the charged medium, but the force that the volume $d\tau$ exerts on the wall has to be the same [Eq. (5)]. We shall show that it arises from mechanical stresses of the moving charged liquid. Mechanical stresses can be represented by the mechanical energy momentum tensor $T^{\mu\nu}$. The x component of the stress force upon the volume $d\tau$ is given by

$$dF_x = -\sum_{i=1}^{3} \partial_i T^{1i}\,d\tau. \quad (6)$$

In the rest frame of the charged moving medium the electric field yields stress in the z direction: $T^{33} = \rho E z$. This leads to $T^{30} = -(1/c)\rho z\,\mathbf{v}\cdot\mathbf{E}$ in the rest frame of the current loop S, and finally, to $T^{13} = -(V/c^2)\rho z\,\mathbf{v}\cdot\mathbf{E}$ in the frame S. (We are using the Lorentz transformation for a symmetric four-tensor omitting quadratic terms in v/c and V/c.) This yields the only contribution to the right-hand side of Eq. (6), which equals $(\rho V/c^2)\mathbf{v}\cdot\mathbf{E}\,d\tau$. The force that is finally exerted on the wall of the tube is the negative of the force acting on the volume. Since it is exactly equal to expression (5), the expressions for torque are also the same in both frames.

IV. THE FORCE ON A MAGNETIC DIPOLE AND "HIDDEN MOMENTUM"

The problem of the torque raised controversy only about the origin of the physical effect. The value of the torque can be found unambiguously in the rest frame of the magnetic dipole. Discussion of the torque in other frames has an educational character: It teaches us the importance of different subtle relativistic effects. Relativistic effects necessarily have to be taken into account when we calculate the force on a magnetic dipole. Indeed, contrary to the case of calculating the torque, there is no Lorentz frame (at least for some models of the dipole) in which the force on the dipole can be calculated as the sum of the Lorentz forces on the charged particles composing the current loop. And the controversy about the force concerns the value of the force itself rather than just the interpretation of its physical origin.

The electromagnetic (Lorentz) force acting on the current loop magnetic dipole, the force that is responsible for the change of the total mechanical momentum, is

$$\frac{d\mathbf{P}}{dt} = \frac{1}{c}\int \mathbf{J}\times\mathbf{B}\,d\tau = \nabla(\mathbf{m}\cdot\mathbf{B}). \quad (7)$$

This expression appears in textbooks as the force acting on a current loop. Recently, Boyer[3] gave a clear derivation of (7) and discussed the difference between this and Eq. (1), which yields the force on a magnetic dipole in the magnetic charge model. However, if we are interested in predictions of motion of a particle with a magnetic dipole moment, the relevant definition for force is mass times acceleration, $\mathbf{F}\equiv m\mathbf{a}$. It differs from $\mathbf{F}\equiv d\mathbf{P}/dt$ for relativistic mechanics. The origin of the differences lies in the "hidden momentum"[8-11] of a current loop. The total momentum of a magnetic dipole consists of the momentum of the center-of-mass motion (total mass m times velocity of the center of mass $\mathbf{V}_{\text{c.m.}}$) and the "hidden" momentum of the internal motion:

$$\mathbf{P} = m\mathbf{V}_{\text{c.m.}} + \mathbf{P}_{\text{hid}}. \quad (8)$$

Thus we obtain

$$\mathbf{F}\equiv m\mathbf{a} = \frac{d\mathbf{P}}{dt} - \frac{d\mathbf{P}_{\text{hid}}}{dt}. \quad (9)$$

In Sec. V we shall calculate the "hidden momentum" for different current loop models of a magnetic dipole. However, now we shall give a general argument showing the necessity of the existence of the "hidden momentum" using a simple example. We shall follow Ref. 5 starting with a lemma.[12]

Lemma: Any finite stationary distribution of matter has zero total momentum.

The total momentum is $P^i = (1/c)\int T^{i0}\,d\tau$, where $T^{\mu\nu}$ is the energy momentum tensor of the distribution that includes mechanical and electromagnetic parts. "Stationary" means that $\partial_0 T^{\mu\nu} = 0$. Due to the conservation law $\partial_\mu T^{\mu\nu} = 0$, it yields $\partial_j T^{j0} = 0$. The finiteness of the distribution yields that $T^{\mu\nu}\sim 1/r^4$ at infinity. Representation of T^{i0} as $\partial_j(x_i T^{j0}) - x_i\,\partial_j T^{j0}$ and integration by parts prove the lemma:

$$P^i = \frac{1}{c}\int T^{i0}\,d\tau = \frac{1}{c}\int [\partial_j(x_i T^{j0}) - x_i\,\partial_j T^{j0}]d\tau$$
$$= \frac{1}{c}\oint x_i T^{j0}\,dS_j = 0. \quad (10)$$

Now, consider a toroidal coil of radius R made with N current loops of area S and current I. The toroid is lying in the xy plane, and a point charge q is placed at its center (Fig. 2). Since the magnetic field is nonzero only inside the coil, the momentum of the electromagnetic field is easily estimated:

$$\mathbf{P}_{\text{em}} = \frac{1}{4\pi c}\int \mathbf{E}\times\mathbf{B}\,d\tau \cong \frac{1}{c^2}\frac{NISq}{R^2}\hat{\mathbf{z}} \cong -\frac{N}{c}\mathbf{m}\times\mathbf{E}, \quad (11)$$

where $\mathbf{m}$ is the magnetic dipole moment of each loop and $\mathbf{E}$ is an average electric field in the vicinity of the loop. At first

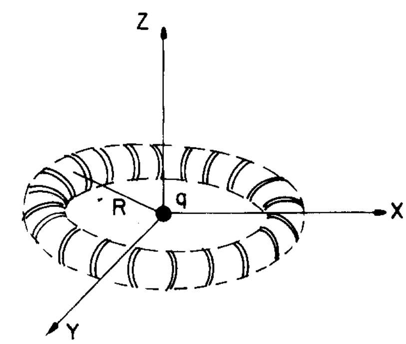

Fig. 2. A configuration of a toroidal coil with N current loops (which are not conducting wires) of area S and current I, together with a point charge q in the center. The electromagnetic field of this stationary configuration carries nonzero momentum $(1/4\pi c)\int\mathbf{E}\times\mathbf{B}\,d\tau \cong (1/c^2)(NISq/R^2)\hat{\mathbf{z}}$. Since the total momentum of stationary finite charge and current distributions has to be zero, a "hidden momentum" has to be carried by the current loops.

glance there is not any other momentum in the described system. However, it follows from the lemma that there has to be. The system is finite and stationary; therefore, there has to be an additional momentum that is equal and opposite in sign to the electromagnetic momentum (11). The static point charge does not carry momentum; therefore, each current loop has to carry a "hidden momentum":

$$\mathbf{P}_{\text{hid}} = (1/c)\mathbf{m}\times\mathbf{E}. \tag{12}$$

Another way (suggested by the referee) of seeing the necessity of "hidden momentum" without relying on the above lemma is to consider the energy flow and to use the Poynting theorem. From the conservation of energy it follows that the nonvanishing energy flow due to the electromagnetic field, whose density is given by the Poynting vector, has to be compensated by the (hidden) energy flow in the current loop. The momentum which corresponds to this hidden energy flow is the "hidden momentum." For the example of the toroidal coil with a point charge in its center, the calculations are essentially the same as above and they give once again the "hidden momentum" (12).

Let us consider a more general case:[13] A stationary current loop described by a finite current distribution $\mathbf{J}$ in an external field $\mathbf{E}$ produced by a finite static charge distribution that yields electric potential ϕ. Using a vector identity and Maxwell's equations we obtain

$$\nabla\times(\phi\mathbf{B}) = \phi\nabla\times\mathbf{B} - \mathbf{B}\times\nabla\phi = (4\pi/c)\phi\mathbf{J} + \mathbf{B}\times\mathbf{E}. \tag{13}$$

The volume integral on the left-hand side of (13) can be transformed into a surface integral and, for a finite distribution of charges and currents, it vanishes. Thus the volume integral of (13) over the whole space yields

$$\mathbf{P}_{\text{em}} = \frac{1}{4\pi c}\int \mathbf{E}\times\mathbf{B}\, d\tau = \frac{1}{c^2}\int \phi\mathbf{J}\, d\tau. \tag{14}$$

If the field $\mathbf{E}$ can be considered constant in the region of the current distribution $\mathbf{J}$, we can take only the first two terms in the Taylor decomposition of $\phi(\mathbf{r})$ around the center of the current loop. Then, using identities for a localized divergenceless current $\mathbf{J}$:[6]

$$\int \mathbf{J}(\mathbf{r})d\tau = 0, \tag{15a}$$

$$\mathbf{C}\cdot\int \mathbf{r}\, J_i\, d\tau = -\frac{1}{2}\left[\mathbf{C}\times\int \mathbf{r}\times\mathbf{J}\, d\tau\right]_i, \tag{15b}$$

we obtain, in agreement with the example of a toroidal coil, that the electromagnetic momentum of a current loop is

$$\mathbf{P}_{\text{em}} = \frac{1}{2c^2}\mathbf{E}\times\int \mathbf{r}\times\mathbf{J}\, d\tau = -\frac{1}{c}\mathbf{m}\times\mathbf{E}. \tag{16}$$

The "hidden momentum" of a loop has to be $\mathbf{P}_{\text{hid}} = -\mathbf{P}_{\text{em}}$. To convince ourselves completely of the existence of the "hidden momentum" we shall explain in Sec. V how it arises in each specific current loop model of a magnetic dipole.

V. HIDDEN MOMENTUM FOR DIFFERENT CURRENT LOOP MODELS

The model (i) is a particular example of a situation in which forces that constrain the motion of charged particles inside the loop do not change the energy of particles. Therefore, the kinetic energy is determined solely by the electric potential ϕ. In relativistic mechanics the equation of conservation of energy for a particle of mass m and charge q is

$$mc^2 + q\phi = \mathscr{E}. \tag{17}$$

Let us take n as the density of particles and $\mathbf{v}$ as their velocity field. Then, taking into account (15a), we show that the mechanical "hidden" momentum of particles is, indeed, equal to minus the electromagnetic momentum (14):

$$\mathbf{P}_{\text{hid}} = \int mn\mathbf{v}\, d\tau = \frac{1}{c^2}\int (\mathscr{E} - q\phi)\frac{\mathbf{J}}{q}\, d\tau$$
$$= -\frac{1}{c^2}\int \phi\mathbf{J}\, d\tau. \tag{18}$$

In model (ii), however, since the velocity of the particles does not change along the loop, no mechanical momentum arises from the above mechanism. Also, no other source of momentum can be seen. How, then, can the total momentum vanish? The solution of this apparent inconsistency is that for this model there is no electromagnetic momentum either. There is an additional contribution to the electric field due to charges induced on the conducting tube. These induced charges ensure that the potential ϕ on the tube is constant. Then, due to (15a), the electromagnetic momentum $(1/c^2)\int\phi\mathbf{J}\, d\tau$ indeed vanishes.

In model (iii) there are no induced charges; therefore, the electromagnetic momentum (14) is not zero. However, the mechanism for hidden mechanical momentum of model (i) does not apply here. The velocity of the charged medium in the tube is not affected by the external electric field and, if the cross section of the tube does not change, the velocity is constant along the tube. Therefore, the mass density ρ_m is constant and, consequently, the "naive" mechanical momentum

$$\int \rho_m \mathbf{v}\, d\tau = \int \frac{\rho_m}{\rho}\mathbf{J}\, d\tau$$

vanishes. In model (iii) the momentum is "hidden" in a mechanical stress. There are several factors responsible for the pressure of the charged liquid. The only factor that leads to a nonzero contribution to the momentum is the effect of the electric field. In the (local) rest frame of the liquid the contribution of the electric field to the pressure can be expressed as $p = -\rho\phi$, i.e., $T^{11} = T^{22} = T^{33} = -\rho\phi$. Lorentz transformation (up to first order in v/c) of the energy momentum tensor shows that these "pressure" terms yield, in the rest frame of the tube, the density of momentum terms: $T^{i0} = -(v_i/c)\rho\phi = -(1/c)J_i\phi$. Thus we obtain also for model (iii):

$$(\mathbf{P}_{\text{hid}})_i = \frac{1}{c}\int T^{i0}\, d\tau = -\frac{1}{c^2}\left(\int \phi\mathbf{J}\, d\tau\right)_i.$$

We would like to make a remark about hidden momentum. Here, we showed that its existence is necessary for a balance of the total momentum. In stationary situations the hidden momentum is equal to minus the electromagnetic momentum. We also explained how hidden momentum arises from internal motion and stresses in the system. If the same external force arises not from an electric field but, say, from a gravitational field, the hidden momentum should be the same. Then, from the lemma of vanishing total momentum of a stationary distribution, it follows that the gravitational field has to carry the momentum that equals the negative of the hidden momentum. It may happen that calculations of hidden momentum using, for example, (18) (with gravitational potential instead of ϕ, and $\mathbf{v}\rho_m$ instead of $\mathbf{J}$) are easier than direct calculations of the momentum carried by the gravitational field.

VI. THE FORCE ON A CURRENT LOOP MAGNETIC DIPOLE

We showed that in current loop models of a magnetic dipole without conductors there is a "hidden momentum" that is not related to the motion of the center of mass of the dipole, and therefore the force ($\mathbf{F}\equiv m\mathbf{a}$) on such a dipole [see (9), (7), and (12)] is

$$\mathbf{F}_{\rm CL} = \nabla(\mathbf{m}\cdot\mathbf{B}) - \frac{d}{dt}\frac{\mathbf{m}\times\mathbf{E}}{c}. \tag{19}$$

In model (ii) there is no "hidden momentum," but we believe that the force on the current loop in this model is also, essentially, represented by Eq. (19). One obvious difference is the existence of the purely electric force of the electric field acting on the induced electric charges. We should not consider this as a force on a magnetic dipole since it does not depend on the current in the loop. We believe that the relevant, current-dependent part of the force is modified relative to the textbook expression $\nabla(\mathbf{m}\cdot\mathbf{B})$ by the same term $-(d/dt)(\mathbf{m}\times\mathbf{E}/c)$. The expression $\nabla(\mathbf{m}\cdot\mathbf{B})$ was obtained without taking into account the current and changing electric field of induced charges which appear in a changing external electric field. Therefore, the correction can be found as a total contribution of the current-dependent forces due to the induced charges. The magnetic force on the current of the induced charges due to magnetic field of the current loop also has to be taken into account. Newton's third law says that the total contribution of current–current forces vanishes, so that the correction to the expression $\nabla(\mathbf{m}\cdot\mathbf{B})$ arises only from the force on the current of magnetic dipole due to the magnetic field whose source is the changing electric field of the induced charges. In general, it is not easy to calculate. We have calculated it for a simplified version of model (ii) (which we discuss in the next paragraph), and obtained the same correction $-(d/dt)(\mathbf{m}\times\mathbf{E}/c)$. This led us to believe that all current loop models of a magnetic dipole yield the same expression (19) for the force in an electromagnetic field. We challenge the reader to provide a proof for an arbitrary shape of the conductor in model (ii); this will be a valuable addition to this work.

The simplified version of model (ii) is a current loop inside a conducting sphere. The electric field due to the induced charges inside the sphere is just the negative of the external electric field. Assuming that the external field is constant in space in the region of the sphere and taking into account the symmetry of the problem we calculated the magnetic field whose source is the changing electric field of the induced charges: $\mathbf{B} = (1/2c)\mathbf{r}\times(\partial\mathbf{E}/\partial t)$. Therefore, the correction to force expression (7) is, indeed, the expected one:

$$\frac{1}{c}\int \mathbf{J}\times\mathbf{B}\,d\tau = \frac{1}{2c^2}\int \mathbf{J}\times\left(\mathbf{r}\times\frac{\partial\mathbf{E}}{\partial t}\right)d\tau$$
$$= \frac{1}{2c^2}\frac{\partial\mathbf{E}}{\partial t}\times\int \mathbf{r}\times\mathbf{J}\,d\tau = -\frac{d}{dt}\frac{\mathbf{m}\times\mathbf{E}}{c}. \tag{20}$$

In order to get the second equality in (20) we had to perform some algebraic manipulations based on (15a), and the last equality we got by neglecting the time derivative of the magnetic moment $\dot{\mathbf{m}}$. However, we can see how our result may be correct even without the assumption $\dot{\mathbf{m}} = 0$. Indeed, changing the current in the loop creates an electric field that acts on the induced charges. This force also has to be, and has not yet been, taken into account.

One more argument for the existence of the second term in Eq. (19) is the fact that without it we run into a paradox of the type discussed by Shockley and James.[8] Consider a superconducting toroidal coil with a given current and a point charge in its center (Fig. 2) which are placed in a free space at rest. Now, let us assume that the coil ceases to be superconducting and its current is gradually changed to zero. Making an approximation of a thin coil it is easy to calculate the impulse delivered to the point charge due to the induced electric field. However, if the field on the current loop is given by $\nabla(\mathbf{m}\cdot\mathbf{B})$, then there is no impulse delivered to the coil in this process. Since the radiation in this situation may be neglected, we get a contradiction with the law of conservation of momentum of a closed system. We leave it as an exercise for the reader to verify that including the second term of Eq. (19) resolves the paradox.

Our conclusion is that the force on the magnetic dipole in an external electromagnetic field is the same for all current loop models. The force is given by (19),

$$\mathbf{F}_{\rm CL} = \nabla(\mathbf{m}\cdot\mathbf{B}) - \frac{d}{dt}\frac{\mathbf{m}\times\mathbf{E}}{c}.$$

The second term, $-(d/dt)(\mathbf{m}\times\mathbf{E}/c)$, is missing in all (as far as I know) textbooks on electromagnetic theory. For most of the textbooks, however, this is not a mistake since they discuss the force on current loop in the chapter on magnetostatistics, and in the static situation the "missing" term vanishes.

For magnetic charge model the force is given by (1), $\mathbf{F}_{\rm MC} = (\mathbf{m}\cdot\nabla)\mathbf{B} - (1/c)\dot{\mathbf{m}}\times\mathbf{E}$. Using vector identity $\nabla(\mathbf{m}\cdot\mathbf{B}) = (\mathbf{m}\cdot\nabla)\mathbf{B} + \mathbf{m}\times(\nabla\times\mathbf{B})$ we can see the difference between the force expressions in current loop and magnetic charge models:

$$\mathbf{F}_{\rm CL} = \mathbf{F}_{\rm MC} + (4\pi/c)\mathbf{m}\times\mathbf{J}. \tag{21}$$

If the region is free of current, then the force expressions are the same.

VII. THE NATURE OF THE NEUTRON MAGNETIC MOMENT

The calculations of the force on a neutral particle with magnetic moment based on Dirac's equation yield, in classical limit, expression (19); see, for example, recent work by Anandan.[14] This supports experimental results on scattering of neutrons from ferromagnetic materials[15] which resolved the controversy about the nature of the magnetic dipole moment of the neutron in favor of current loop models versus the magnetic charge model. The experiment, however, does not resolve the conflict between this result and a claim by Boyer that the force on a neutron is given neither by (1) nor by (19), but by the expression $\nabla(\mathbf{m}\cdot\mathbf{B})$. The latter, as well as (19) includes, relative to (1), the term $(4\pi/c)\,\mathbf{m}\times\mathbf{J}$ which is responsible for differences in scattering of differently polarized neutrons. An experiment, in which the acceleration of a neutron is measured in a region with nonzero displacement current (such as the field between the plates of a capacitor that is being charged), should resolve the controversy. According to Boyer, the term $(1/c)\mathbf{m}\times d\,\mathbf{E}/dt$ does, and we claim, does not, appear in the force expression for a neutron [in addition to Eq. (1)].

The force on a neutron plays an important role in understanding the Aharonov–Casher effect, which is a shift of an interference pattern for neutrons passing around a line of charge. Aharonov *et al.*[4,5] claim that neutrons with magnetic moment parallel to the line of charge do not experience a force (do not accelerate). And the effect, which was recently observed,[16] is a nonlocal topological phenomenon dual to the Aharonov–Bohm effect. Boyer,[17] however, claims that the neutron does experience a force and, according to his calculations, the shift predicted by Aharonov and Casher is due to a classical lag arising from the force on neutrons. In the Aharonov–Casher configuration the neutron moves in a region free of current; and the magnetic field (in the rest frame of the neutron) causes no torque since it is parallel to the neutron's magnetic moment. Thus the force expression is $\mathbf{F} = (\mathbf{m}\cdot\nabla)\mathbf{B}$, and, due to the translational symmetry along the line of charge, it yields zero force on neutrons.

ACKNOWLEDGMENTS

I am most grateful to Y. Aharonov and P. Pearle for introducing me to the problem, and I wish to thank them, J. Anandan, and J. Knight for helpful discussions. I deeply appreciate the useful correspondence with M. G. Calkin. I am also grateful to the referees whose criticisms led to a considerable improvement.

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The exactly solvable Thomas–Fermi atom in two dimensions

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(Received 18 October 1989; accepted for publication 14 February 1990)

The mathematical model of the two-dimensional Thomas–Fermi atom is solved analytically. The electrons interact with each other and with the nucleus through logarithmic potentials that have at least one arbitrary length parameter that sets the zero of energy. The most stable configuration of the system is shown to be a positive ion.

Atoms are simply described in the well-known Thomas–Fermi model[1,2] in terms of a screening function that obeys a universal nonlinear equation. This nonlinear equation has different boundary conditions for atoms and positive ions, and is easily solved numerically.[3,4] It is amusing and instructive that in two dimensions, where the Poisson equation is obeyed by a logarithmic potential, the Thomas–Fermi model is exactly solvable for atoms as well as "ions." It is the purpose of this note to describe this mathematical model that is simple enough to be given as a graduate exercise in a quantum mechanics course.

Consider the Hamiltonian with an infinitely heavy point