4.1 Solution: (a) The charge density, in spherical coordinates, is

$$\rho(\mathbf{x}) = \frac{q}{r^2}\delta(r-a)\delta(\cos\theta)\left[\delta(\phi) + \delta\left(\phi - \frac{\pi}{2}\right) - \delta(\phi - \pi) - \delta\left(\phi - \frac{3\pi}{2}\right)\right].$$

By the definition of the multiple moments,

$$\begin{aligned} q_{lm} &= \int Y^*_{lm}(\theta,\phi) r^l \rho(\mathbf{x}) d^3x \\ &= qa^l\left[Y^*_{lm}\left(\frac{\pi}{2},0\right) + Y^*_{lm}\left(\frac{\pi}{2},\frac{\pi}{2}\right) - Y^*_{lm}\left(\frac{\pi}{2},\pi\right) - Y^*_{lm}\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\right] \\ &= qa^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)\left[1+(-i)^m - (-1)^m - i^m\right]. \end{aligned}$$

The sum involving powers of m is only non-zero for $m = 4k \pm 1$, where k is an integer, *i.e.*,

$$1 + (-i)^m - (-1)^m - i^m = \begin{cases} 2(1-i), & m = 4k+1, \\ 2(1+i), & m = 4k-1, \\ 0, & \text{otherwise.} \end{cases}$$

Now, let us check the multipole moments for different l.

(i) $l = 0$. In this case, m can only be 0. Therefore, $q_{00} = 0$.

(ii) $l = 1$. We need to consider $m = \pm 1$. For $m = 1$, we have

$$q_{11} = qa\sqrt{\frac{3}{4\pi}}P_1^1(0)\cdot 2(1-i) = -qa(1-i)\sqrt{\frac{3}{2\pi}},$$

and for $m = -1$,

$$q_{1,-1} = -q^*_{11} = qa(1+i)\sqrt{\frac{3}{2\pi}}.$$

(iii) $l = 2$. For $m = \pm 1$, $P_2^{\pm 1}(0) = 0$, so $q_{2,m} = 0$ for all m for -2 to 2.

(iv) $l = 3$. We need consider $m = \pm 1$ and $m = \pm 3$. For $m = 3$,

$$q_{33} = qa^3\sqrt{\frac{7}{2880\pi}}P_3^3(0)\cdot 2(1+i) = -qa^3(1+i)\sqrt{\frac{35}{16\pi}},$$

and

$$q_{3,-3} = -q^*_{33} = qa^3(1-i)\sqrt{\frac{35}{16\pi}}.$$

For $m = 1$,

$$q_{31} = qa^3\sqrt{\frac{7}{48\pi}}P_3^1(0)\cdot 2(1-i) = qa^3(1-i)\sqrt{\frac{21}{16\pi}},$$

and

$$q_{3,-1} = -q^*_{31} = -qa^3(1+i)\sqrt{\frac{21}{16\pi}}$$

(b) The charge density, in spherical coordinates, is

$$\rho(\mathbf{x}) = \frac{q}{2\pi r^2}\left[\delta(r-a)\delta(\cos\theta - 1) + \delta(r-a)\delta(\cos\theta+1)\right] - \frac{q}{2\pi r^2}\delta(r).$$

By the definition of the multiple moments,

$$\begin{aligned} q_{lm} &= \int Y_{lm}^*(\theta,\phi) r^l \rho(\mathbf{x}) d^3x \\ &= \delta_{m,0} q a^l \sqrt{\frac{2l+1}{4\pi}} \left[P_l(1) - 2\delta_{l,0} + P_l(-1) \right], \end{aligned}$$

i.e., the multiple moments vanish for all $m = 0$ component, as the charge configuration here is axial symmetric. We can see that the non-vanishing moments are

$$q_{2l,0} = q a^{2l} \sqrt{\frac{4l+1}{\pi}},$$

for $l > 0$, and the first two non-vanishing moments are

$$q_{20} = q a^2 \sqrt{\frac{5}{\pi}}, \quad q_{40} = q a^4 \sqrt{\frac{9}{\pi}}.$$

(c) The multipole expansion is

$$\Phi(\mathbf{x}) = \frac{1}{\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}},$$

for the charge configuration in (b), the multipole expansion is

$$\begin{aligned} \Phi(\mathbf{x}) &= \frac{1}{\varepsilon_0} \sum_{l=1}^{\infty} \frac{1}{4l+1} q a^{2l} \sqrt{\frac{4l+1}{\pi}} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}} = \frac{1}{\varepsilon_0} \sum_{l=1}^{\infty} \frac{q a^{2l}}{\sqrt{(4l+1)\pi}} \frac{Y_{2l,0}(\theta,\phi)}{r^{2l+1}} \\ &= \frac{q}{2\pi\varepsilon_0} \sum_{l=1}^{\infty} \frac{a^{2l}}{r^{2l+1}} P_{2l}(\cos\theta), \end{aligned}$$

since

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta).$$

(d) Directly calculating from Coulomb's law, similar to Problem 3.7, the potential is

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0} \left(\frac{1}{|\mathbf{x} - a\hat{z}|} - \frac{1}{|\mathbf{x}|} + \frac{1}{|\mathbf{x} + a\hat{z}|} \right).$$

For $|\mathbf{x}| = r > a$, using Legendre polynomial, we have

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{a^l}{r^{l+1}} \left(P_l(\cos\theta) + P_l(-\cos\theta) \right) - \frac{q}{2\pi\varepsilon_0 r}.$$

The odd term drops out and also the $l = 0$ term will cancel the last one from the charge at origin. Therefore, we will arrive at the same expression as (c).