2.7 Solution: (a) By the method of image charges, the Green function can be directly written down,

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}.$$

(b) To calculate the potential in the upper half space, $z>0$, consider the normal direction of $-\hat{z}$, then

$$\left.\frac{\partial G}{\partial n'}\right|_{z'=0} = -\left.\frac{\partial G}{\partial z'}\right|_{z'=0} = -\frac{2z}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}.$$

The potential can be expressed as

$$\begin{aligned}
\Phi(x,y,z) &= -\frac{1}{4\pi}\oint \Phi(x',y',z')\frac{\partial G}{\partial n'}da' \\
&= \frac{V}{2\pi}\int_0^{2\pi} d\phi' \int_0^a \frac{z}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}\rho' d\rho' \\
&= \frac{Vz}{2\pi}\int_0^{2\pi} d\phi' \int_0^a \frac{\rho' d\rho'}{[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2]^{3/2}}.
\end{aligned}$$

(c) For $\rho=0$, *i.e.*, $x=0$ and $y=0$, the potential becomes

$$\begin{aligned}
\Phi &= Vz\int_0^a \frac{\rho' d\rho'}{[\rho'^2+z^2]^{3/2}} = \frac{Vz}{2}\int_0^a \frac{d\rho'^2}{[\rho'^2+z^2]^{3/2}} = -\left.\frac{Vz}{\sqrt{\rho'^2+z^2}}\right|_{\rho'=0}^a \\
&= V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right).
\end{aligned}$$

(d) When $\rho^2+z^2 \gg a^2$, the denominator in the potential expression can be expanded as

$$\begin{aligned}
&\left[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2\right]^{-3/2} \\
= &\quad (\rho^2+z^2)^{-3/2}\left[1+\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right]^{-3/2} \\
= &\quad (\rho^2+z^2)^{-3/2}\left[1-\frac{3}{2}\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}+\frac{15}{8}\left(\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right)^2\right].
\end{aligned}$$

Put this back into the potential and keep only those terms that are even orders of the cos function, we have

$$\begin{aligned}
\Phi &= \frac{V}{2\pi}\frac{z}{[\rho^2+z^2]^{3/2}}\int_0^{2\pi} d\phi' \int_0^a \rho' d\rho' \left[1-\frac{3}{2}\frac{\rho'^2}{\rho^2+z^2}+\frac{15}{8}\frac{\rho'^4+4\rho^2\rho'^2\cos^2(\phi-\phi')}{(\rho^2+z^2)^2}\right] \\
&= \frac{Vz}{[\rho^2+z^2]^{3/2}}\int_0^a \rho' d\rho' \left[1-\frac{3}{2}\frac{\rho'^2}{\rho^2+z^2}+\frac{15}{8}\frac{\rho'^4+2\rho^2\rho'^2}{(\rho^2+z^2)^2}\right] \\
&= \frac{Vz}{[\rho^2+z^2]^{3/2}}\left[\frac{a^2}{2}-\frac{3a^4}{8(\rho^2+z^2)}+\frac{15}{8}\left(\frac{a^6}{6(\rho^2+z^2)^2}+\frac{\rho^2a^4}{2(\rho^2+z^2)^2}\right)\right] \\
&= \frac{Va^2}{2}\frac{z}{[\rho^2+z^2]^{3/2}}\left[1-\frac{3a^2}{4(\rho^2+z^2)}+\frac{5a^2(3\rho^2+a^2)}{8(\rho^2+z^2)^2}\right].
\end{aligned}$$