1.5 Solution: Since the potential is spherically symmetric, applying the Laplace operator, we have

$$\begin{aligned}
\nabla^2\Phi &= \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\frac{\partial^2}{\partial r^2}\left[e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)\right] \\
&= \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\frac{\partial}{\partial r}\left[-\frac{\alpha}{2}e^{-\alpha r}(1+\alpha r)\right] \\
&= \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\frac{\alpha^3}{2}re^{-\alpha r} \\
&= \frac{q\alpha^3}{8\pi\varepsilon_0}e^{-\alpha r},
\end{aligned}$$

for $r \neq 0$. As $r \to 0$, the potential becomes singular and behaves as

$$\Phi \to \frac{q}{4\pi\varepsilon_0}\frac{1}{r}.$$

Therefore,

$$\nabla^2\Phi \to -\frac{q}{\varepsilon_0}\delta(r), \quad r \to 0.$$

Combine these results, the charge distribution is then given by

$$\rho = -\varepsilon_0\nabla^2\Phi = q\delta(r) - \frac{q\alpha^3}{8\pi}e^{-\alpha r}.$$

The charge distribution consists of two parts, a singular one and a regular one. The singular part represents the positive charge of the hydrogen nucleus, located at the origin. The regular part comes from the electron surrounding the nucleus, which screens the positive charge and leads to a potential that drops exponentially in distance. It can be easily shown, that the total charge from the regular distribution is $-q$ from the electron, as

$$-\frac{q\alpha^3}{8\pi}\int e^{-\alpha r}dV = -\frac{q\alpha^3}{2}\int_0^{+\infty} r^2 e^{-\alpha r}dr = -q,$$

where we have used the fact that

$$\int_0^{+\infty} x^{\beta-1}e^{-\alpha x}dx = \frac{\Gamma(\beta)}{\alpha^\beta}.$$