2.16 Solution: Due to the Dirichlet boundary condition and also the vanishing of the potential on the boundary, the potential inside the unit square area can be expressed, by Eq. (1.42), as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \sigma(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d\mathbf{x}'.$$

From Problem 2.15, we have already obtained the series expansion of the Green function $G(\mathbf{x},\mathbf{x}')$ on the unit square area. With a uniform surface charge density of unit strength, the potentional inside the square is given by

$$\Phi(x,y) = \frac{2}{\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{\sin(n\pi x)}{n\sinh(n\pi)}\int_0^1 dx'\sin(n\pi x')\int_0^1 dy'\sinh(n\pi y_<)\sinh\left[n\pi(1-y_>)\right].$$

The integration in x' can be readily performed, yielding

$$\int_0^1 dx'\sin(n\pi x') = -\frac{1}{n\pi}\cos(n\pi x')\bigg|_0^1 = -\frac{1}{n\pi}\left((-1)^n-1\right),$$

which is only non-zero for *odd* n, with the value of $2/n\pi$. For integration in y', we need to break it into two parts,

$$\int_0^1 dy' = \int_0^y dy' + \int_y^1 dy',$$

which leads to

$$\begin{aligned}
&\int_0^1 dy'\sinh(n\pi y_<)\sinh\left[n\pi(1-y_>)\right] \\
=\;& \sinh\left[n\pi(1-y)\right]\int_0^y dy'\sinh(n\pi y') + \sinh(n\pi y)\int_y^1 dy'\sinh\left[n\pi(1-y')\right] \\
=\;& \frac{\sinh\left[n\pi(1-y)\right]}{n\pi}\cosh(n\pi y')\bigg|_0^y - \frac{\sinh(n\pi y)}{n\pi}\cosh\left[n\pi(1-y')\right]\bigg|_y^1 \\
=\;& \frac{1}{n\pi}\left(\sinh(n\pi) - \sinh\left[n\pi(1-y)\right] - \sinh(n\pi y)\right).
\end{aligned}$$

Therefore, the potential is

$$\Phi(x,y) = \frac{4}{\pi^3\varepsilon_0}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x)}{(2k+1)^3}\left[1 - \frac{\sinh\left[(2k+1)\pi(1-y)\right] + \sinh((2k+1)\pi y)}{\sinh((2k+1)\pi)}\right].$$

Using the relations

$$\sinh((2k+1)\pi) = 2\sinh((2k+1)\pi/2)\cosh((2k+1)\pi/2),$$

and

$$\sinh\left[(2k+1)\pi(1-y)\right] + \sinh((2k+1)\pi y) = 2\sinh\left(\frac{(2k+1)\pi}{2}\right)\cosh\left((2k+1)\pi\left(y-\frac{1}{2}\right)\right),$$

we will have the following result,

$$\Phi(x,y) = \frac{4}{\pi^3\varepsilon_0}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x)}{(2k+1)^3}\left[1 - \frac{\cosh\left((2k+1)\pi\left(y-1/2\right)\right)}{\cosh((2k+1)\pi/2)}\right].$$