Physics Letters B 277 (1992) 148–152
North-Holland

Electromagnetic fields of a massless particle and the eikonal ☆

R Jackiw, D. Kabat [1] and M. Ortiz [2]

Center for Theoretical Physics, Laboratory for Nuclear Science, and Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Received 10 December 1991, revised manuscript received 3 January 1992

Electromagnetic fields of a massless charged particle are described by a gauge potential that is almost everywhere a pure gauge Solution of quantum mechanical wave equations in the presence of such fields is therefore immediate and leads to a new derivation of the quantum electrodynamical eikonal approximation The electromagnetic action in the eikonal limit is localized on a contour in a two-dimensional Minkowski subspace of four-dimensional space–time The exact S-matrix of this reduced theory reproduces the eikonal approximation In this way, we apply the recent gravitational consideration of 't Hooft as well and Verlinde and Verlinde to electromagnetism

Some years ago, 't Hooft derived the eikonal approximation for the two-particle scattering amplitude arising from their gravitational interaction [1] However, he did not sum eikonal graphs (graviton exchange graphs with particle propagators approximated by neglecting virtual graviton momenta compared to particle momenta) Rather he solved quantum mechanical equations for one particle moving in the gravitational field of the other, which is taken massless, i e , external gravity is described by the Aichelburg–Sexl metric [2] Recently, this problem has been reexamined by Verlinde and Verlinde [3], and their work sparked our interest in the subject

We present here a similar approach to the electrodynamical eikonal We first determine the electromagnetic fields of a massless charged particle This is a classical physics textbook problem, because we could not locate its solution in the literature we present the formulas, which are interesting and elegant In particular, we find that the gauge potentials are pure gauges, but *different* in *different* regions of space–time The solution of a quantum mechanical wave equation is therefore obtained by exponentiating the gauge function, and the scattering amplitude is the quantum electrodynamical eikonal

To find the electromagnetic fields of a charged (e) massless point particle, we begin with those arising from a massive particle moving with constant velocity v along the $+z$-axis, and then set v to the velocity of light, which we scale to unity Our metric tensor is diag$(1, -1, -1, -1)$, and four-vectors x^μ are indexed by greek letters from the middle of the alphabet In the two-dimensional (t, z) Minkowski subspace, we index by initial greek letters $\alpha, \beta, \ldots$, also we use light-cone components $\pm \equiv (1/\sqrt{2})(0 \pm 3)$ Transverse euclidean vectors $r^i_\perp$ carry a latin index

Either by integrating the Liénard–Wiechert potentials or by Lorentz boosting the static Coulomb potential, one finds that the conserved source current

$$j^\mu_v = e n^\mu_v \delta^2(\boldsymbol{r}_\perp)\delta(z-vt), \quad n^\mu_v = (1, 0, 0, v) \tag{1}$$

gives rise to the four-vector potential

$$A^\mu_v = \frac{e}{4\pi}\frac{n^\mu_v}{R_v}, \quad R^2_v = (z-vt)^2 + (1-v^2)r^2_\perp \tag{2}$$

Since the $v=1$ limit of (2) is somewhat delicate, see below, we record first the field strengths

$$E^z_v = \frac{e(z-vt)}{4\pi}\frac{1-v^2}{R^3_v}, \quad E^i_v = \frac{er^i}{4\pi}\frac{1-v^2}{R^3_v}, \tag{3}$$

☆ This work is supported in part by funds provided by the US Department of Energy (DOE) under contract #DE-AC02-76ER03069

[1] National Science Foundation Graduate Fellow

[2] SERC fellow

$$B_v^z=0\ ,\quad B_v^i=-\frac{e\epsilon^{ij}r_\perp^j\, v}{4\pi}\frac{1-v^2}{R_v^3} \tag{3 cont'd}$$

It is straightforward to establish (for example by Fourier transforming with respect to z) that

$$\lim_{v\to 1}\frac{1-v^2}{R_v^3}=\frac{2}{r_\perp^2}\,\delta(t-z) \tag{4}$$

Thus the fields of a massless charged particle, moving in the $+z$-direction, are

$$E^z=0\ ,\quad E^i=\frac{er_\perp^i}{2\pi r_\perp^2}\,\delta(t-z)\ ,$$

$$B^z=0\ ,\quad B^i=-\frac{e\epsilon^{ij}r_\perp^j}{2\pi r_\perp^2}\,\delta(t-z)\ , \tag{5}$$

and one may verify that they satisfy the Maxwell equations with the source current

$$j^\mu=en^\mu\delta^2(\boldsymbol{r}_\perp)\delta(t-z)\ ,\quad n^\mu=(1,0,0,1) \tag{6}$$

The potentials that give rise to (5) can be chosen as

$$A_{\mathrm{I}}^0=A_{\mathrm{I}}^z=0\ ,\quad A_{\mathrm{I}}^\perp=-\frac{e}{2\pi}\,\theta(t-z)\,\nabla\ln\mu r_\perp\ , \tag{7}$$

where μ is an irrelevant parameter, setting the scale of the logarithm

For completeness, we derive the limiting forms (7) from A_v^μ in (2) First note that [#1]

$$\lim_{v\to 1}\frac{1}{R_v}=-\ln\mu^2r_\perp^2\,\delta(t-z)+\frac{1}{|t-z|}\ , \tag{8a}$$

a formula that is consistent with (4) and the relation $(\partial/\partial r_\perp^2)R_v^{-1}=-\frac{1}{2}(1-v^2)R_v^{-3}$ Here μ is a necessary regulator The validity of (8a) is seen after a Fourier transformation with respect to z, then it states

$$\lim_{v\to 1}K_0(\omega r_\perp\sqrt{1-v^2})$$

$$=-\ln\mu r_\perp+\lim_{\epsilon\to 0}\int_\epsilon^\infty\frac{\mathrm{d}z}{z}\cos z\omega \tag{8b}$$

Both the modified Bessel function K_0 and the integral give rise to singularities in the limit Upon identifying μ with $\sqrt{1-v^2}/2\epsilon$, (8a) follows We conclude therefore that

$$\lim_{v\to 1}A_v^\mu$$

$$=-\frac{en^\mu}{4\pi}\left(\ln\mu^2r_\perp^2\,\delta(t-z)-\frac{1}{|t-z|}\right) \tag{8c}$$

The last term does not contribute to the fields, it is a pure gauge, albeit its gauge function is singular at $t=z$ Thus the gauge potentials can be taken as

$$A_{\mathrm{II}}^0=A_{\mathrm{II}}^z=-\frac{e}{2\pi}\ln\mu r_\perp\,\delta(t-z)\ ,\quad A_{\mathrm{II}}^\perp=0 \tag{9}$$

They differ from (7), but of course they are related by a gauge transformation

$$A_{\mathrm{I}}^\mu=A_{\mathrm{II}}^\mu+\partial^\mu\Omega\ ,\quad \Omega=\frac{e}{2\pi}\ln\mu r_\perp\,\theta(t-z) \tag{10}$$

Both gauges will be useful in the following Expression (7) shows that A_{I}^μ vanishes at $t<z$ and is a pure gauge for $t>z$, hence it is a singular pure gauge A_{II}^μ has the advantage that the gauge potentials vanish in the transverse directions, being confined to the two-dimensional (t,z) Minkowski space

Quantum mechanical wave equations are conveniently analyzed in gauge I, eq (7) When the electromagnetic interaction of a particle with charge e' enters through the covariant derivative $\partial_\mu+ie'A_\mu$, its wave function does not interact with A_μ at early times $t<z$,

$$\psi_<=\psi_0\ , \tag{11a}$$

while at late times $t>z$,

$$\psi_>=\exp\left(-i\frac{ee'}{4\pi}\ln\mu^2r_\perp^2\right)\psi_0' \tag{11b}$$

Here ψ_0 and ψ_0' are free wave functions, related by a continuity requirement at $x^-=0$,

$$\psi_>|_{x^-=0}=\psi_<|_{x^-=0} \tag{11c}$$

Upon taking the initial wave function to be a plane wave,

$$\psi_0=\exp(-ip\ x)\ , \tag{12}$$

the derivation of the scattering amplitude follows 't Hooft to his expression with $-Gs$ replaced by $ee'/4\pi\equiv\alpha$, as was already noted by him [1] One finds for distinguishable, spinless particles

[#1] Aichelburg and Sexl [2] also discuss this limit

$$f(s,t)=\frac{\Gamma(1+i\alpha)}{4\pi i\mu^2\Gamma(-i\alpha)}\left(\frac{4\mu^2}{-t}\right)^{1+i\alpha}$$
$$=\frac{\alpha}{\pi t}\frac{\Gamma(1+i\alpha)}{\Gamma(1-i\alpha)}\exp\left(i\alpha\ln\frac{4\mu^2}{-t}\right), \qquad (13)$$

where s and t are the Mandelstam variables ('t Hooft scales μ^2 to unity and apparently omits the prefactor $1/i$)

That this (and 't Hooft's gravity result) is exactly the eikonal formula, is seen be recalling the standard expression of the latter [#2]

$$f_{\text{eikonal}}(s,t)=i\int\frac{d^2b}{(2\pi)^2}\exp(i\boldsymbol{q}_\perp\cdot\boldsymbol{b})$$
$$\times\left[1-\exp\left(-iee'\int\frac{d^2k_\perp}{(2\pi)^2}\frac{\exp(i\boldsymbol{k}\cdot\boldsymbol{b})}{k_\perp^2+\mu^2}\right)\right],$$
$$t=-q_\perp^2 \qquad (14)$$

Here $1/(k_\perp^2+\mu^2)$ is the photon propagator at $k_+k_-=0$, but with an infrared regulating "mass" μ The $k_\perp$ integral leaves $(1/2\pi)K_0(\mu b)$, which for small μ is replaced by $-(1/2\pi)\ln\frac{1}{2}e^\gamma\mu b$ Absorbing $\frac{1}{2}e^\gamma$ in μ, and performing the b-integral reproduces (13)

$$f_{\text{eikonal}}(s,t)=\frac{\Gamma(1+i\alpha)}{4\pi i\mu^2\Gamma(-i\alpha)}\left(\frac{4\mu^2}{-t}\right)^{1+i\alpha} \qquad (15)$$

Next we show that high energy eikonal electrodynamics can be given an action formulation where the action is localized on a contour in the (t, z)-plane This is the analog in the present context of what has been done for Einstein gravity [3] For particles moving rapidly along the z-axis, the light cone components of the source current are functions of only one light-cone coordinate, as seen in (6)

$$j_+(x)=j_+(x^+,\boldsymbol{r}_\perp)\,,\quad j_-(x)=j_-(x^-,\boldsymbol{r}_\perp)\,, \qquad (16)$$

$$j^i(x)=0 \qquad (16\text{ cont'd})$$

For such a source current, E^z and B^z vanish We may therefore take $A^\perp=0$ and $A_\pm=\partial_\pm\Omega$, as in gauge II, eq (9) Also we work in the Landau gauge $\partial_\mu A^\mu=0$ This implies that Ω is harmonic,

$$\partial_+\partial_-\Omega=0 \qquad (17)$$

and may therefore be written as a superposition of left- and right-moving waves

$$\Omega(x)=\Omega^+(x^+,\boldsymbol{r}_\perp)+\Omega^-(x^-,\boldsymbol{r}_\perp) \qquad (18)$$

The form of the source current (16) allows introducing functions $k^-(x^-,\boldsymbol{r}_\perp)$ and $k^+(x^+,\boldsymbol{r}_\perp)$ defined by $j_-(x^-,\boldsymbol{r}_\perp)=\partial_-k^-(x^-,\boldsymbol{r}_\perp)$ and $j_+(x^+,\boldsymbol{r}_\perp)=\partial_+k^+(x^+,\boldsymbol{r}_\perp)$ That is

$$j^\alpha=\epsilon^{\alpha\beta}\,\partial_\beta k\,, \qquad (19)$$

where

$$k(x)=k^+(x^+,\boldsymbol{r}_\perp)-k^-(x^-,\boldsymbol{r}_\perp) \qquad (20)$$

Note that writing j in the form (19) insures current conservation, $\partial_\alpha j^\alpha=0$

The electromagnetic Lagrange density in the presence of an external current is given in the eikonal limit by

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-j^\mu A_\mu$$
$$=\tfrac{1}{2}\partial_i\partial_\alpha\Omega\,\partial_i\partial^\alpha\Omega-j^\alpha\,\partial_\alpha\Omega \qquad (21)$$

As a consequence of Landau gauge (17) and current conservation, this is a total x^α-derivative Introducing the decomposition into left- and right-moving waves, $\mathscr{L}$ may be written as

$$\mathscr{L}=-\tfrac{1}{2}\partial_-\Omega^-\;\nabla^2\partial_+\Omega^+-\tfrac{1}{2}\partial_+\Omega^+\;\nabla^2\partial_-\Omega^-$$
$$-\partial_+k^+\;\partial_-\Omega^--\partial_-k^-\;\partial_+\Omega^+$$
$$=-\partial_-(\tfrac{1}{2}\Omega^-\;\nabla^2\partial_+\Omega^++\partial_+k^+\Omega_-)$$
$$-\partial_+(\tfrac{1}{2}\Omega^+\;\nabla^2\partial_-\Omega^-+\partial_-k^-\Omega^+) \qquad (22)$$

Upon integrating over the entire transverse $\boldsymbol{r}_\perp$-space and a closed surface in the two-dimensional (t, z) Minkowski space, the action is given by the boundary contribution

[#2] See e g Fried [4] In transcribing Fried's equation (9 13) into our (14), the following changes must be made He deals with fermions whose electromagnetic vertex $\bar{u}(p_i)\gamma^\mu u(p_f)$ is approximated by p^μ/m, $p_i^\mu\approx p_f^\mu\sim p^\mu$, we consider bosons with vertex $p_i^\mu+p_f^\mu\sim 2p^\mu$ Consequently, his amplitude carries a factor $p\,p'/m^2\sim s/2m^2$, while the corresponding factor in ours should be $4p\,p'\sim 2s$, thus one must multiply (9 13) by $4m^2$ Also he omits the standard kinematical factor, which for bosons is $(2\pi)^{-2}(2E_iE_i'2E_f2E_f')^{-1/2}\sim 1/8\pi^2 s$ Therefore a factor of $m^2/2\pi^2 s$ converts Fried's amplitude (9 13) to ours and also his $\gamma(s)$ is set to 1, for large s

$$I(\Omega, k)=\int \mathrm{d}^4x\, \mathscr{L}=\oint \mathrm{d}\tau \int \mathrm{d}^2r_\perp \times (\tfrac{1}{2}\Omega^- \nabla^2\Omega^+ - \tfrac{1}{2}\Omega^+\nabla^2\Omega^- + k^+\Omega^- - k^-\Omega^+) \quad (23)$$

Here all quantities are evaluated along a closed contour $x^\alpha(\tau)$ bounding the surface in the (t, z)-plane An overdot denotes τ-differentiation The equations of motion,

$$\nabla^2\Omega^+ = -k^+, \quad \nabla^2\Omega^- = -k^-, \quad (24)$$

may be immediately integrated

$$\Omega^+(x^+, \boldsymbol{r}_\perp) = -\frac{1}{\nabla^2} k^+(x^+, \boldsymbol{r}_\perp),$$

$$\Omega^-(x^-, \boldsymbol{r}_\perp) = -\frac{1}{\nabla^2} k^-(x^-, \boldsymbol{r}_\perp) \quad (25)$$

It is readily verified that this solution also follows from the full Maxwell equations $\partial_\mu F^{\mu\nu} = j^\nu$ with the eikonal form (16) for the source current

Because the action (23) is first order in τ, its symplectic structure leads to the quantum commutator [5]

$$[\Omega^+(x^+(\tau), \boldsymbol{r}_\perp), \Omega^-(x^-(\tau), \boldsymbol{r}'_\perp)] = \frac{1}{2\pi} \ln|\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp| \quad (26)$$

To rederive the scattering amplitude (13) in this formalism, one must calculate the S-matrix for the scattering process by computing the expectation value of the operator

$$V = \exp\left(\mathrm{i} \oint \mathrm{d}\tau \int \mathrm{d}^2r_\perp\, (k^+\Omega^- - k^-\Omega^+)\right)$$

The relevant functional integral, corresponding to an integration over the unobserved electromagnetic degrees of freedom, is given by

$$\langle V \rangle = \frac{\int \mathscr{D}\Omega \exp[\mathrm{i}I(\Omega, k)]}{\int \mathscr{D}\Omega \exp[\mathrm{i}I(\Omega, 0)]}, \quad (27)$$

where $I(\Omega, 0)$ is the kinetic part of the action (23) Since the latter is quadratic, the functional integral evaluates to its classical saddle-point value Substituting (25) in (23) leaves

$$\langle V \rangle = \exp\Bigg(\frac{\mathrm{i}}{4\pi} \oint \mathrm{d}\tau \int \mathrm{d}^2r_\perp \times \int \mathrm{d}^2r'_\perp [k^+(\tau, \boldsymbol{r}_\perp) \ln|\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp| k^-(\tau, \boldsymbol{r}'_\perp) - k^-(\tau, \boldsymbol{r}_\perp) \ln|\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp| \dot{k}^+(\tau, \boldsymbol{r}'_\perp)]\Bigg) \quad (28)$$

For the two-particle scattering problem, we take

$$k = e\theta(x^+ - x^{(1)+})\delta(\boldsymbol{r}_\perp - \boldsymbol{r}_\perp^{(1)}) + e'\theta(x^- - x^{(2)-})\delta(\boldsymbol{r}_\perp - \boldsymbol{r}_\perp^{(2)}), \quad (29)$$

which corresponds to the current for a right- and left-moving charged particle, with an impact parameter $\boldsymbol{r}_\perp^{(1)} - \boldsymbol{r}_\perp^{(2)}$ Inserting the above expression for k into (28) and performing the three integrations yields the scattering matrix

$$S_{12} = \exp(\mathrm{i}\,\alpha \ln|\boldsymbol{r}_\perp^{(1)} - \boldsymbol{r}_\perp^{(2)}|^2) \quad (30)$$

To see explicitly that this scattering matrix leads to the conditions (11), it is sufficient to rewrite S_{12} in terms of the total momentum and angular momentum of the system The former is, of course,

$$P^\mu \equiv p_1^\mu + p_2^\mu \quad (31)$$

An expression for the latter is constructed from the four-vector [3]

$$J_\mu \equiv -\frac{1}{\sqrt{P^2}} \epsilon_{\mu\alpha\beta\gamma} p_1^\alpha p_2^\beta (x_1 - x_2)^\gamma, \quad (32a)$$

which in the center-of-mass frame ($\boldsymbol{p}_1 = -\boldsymbol{p}_2$) has no time component, and its space component is the total angular momentum

$$\boldsymbol{J} = \boldsymbol{L} \equiv \boldsymbol{r}_1 \times \boldsymbol{p}_1 + \boldsymbol{r}_2 \times \boldsymbol{p}_2 \quad (32b)$$

Hence, $\boldsymbol{L}^2 = -J^2$, and for $p_1^2 = p_2^2 = 0$, S_{12} may be written as

$$S_{12} = \left(-\frac{J^2}{\frac{1}{4}P^2}\right)^{\mathrm{i}\alpha}, \quad (33)$$

which is the gravitational result [3], with $-Gs$ replaced by α We see that for free partial waves the action of the scattering matrix is equivalent to (11), and so conclude that S_{12} reproduces the scattering amplitude (13)

It is known that the eikonal approximation gives the correct high-energy ($s \to \infty$, $t/s \to 0$) behavior of perturbative graphs when vector mesons are exchanged, but it fails with scalar meson exchange because there the eikonal contributions do not dominate non-eikonal effects [6] With vector exchange, one-meson emission vertices carry an additional factor of $\sqrt{s}$ and this serves to enhance the eikonal contribution over the non-eikonal Thus one may expect that also with graviton exchange, where the single graviton emission vertex is enhanced by s, the eikonal contributions should be dominant, in particular, the (non-renormalizable) infinities are formally subdominant at large s While these remarks support the reliability of 't Hooft's formula, an explicit check would be welcome [7] Finally, we mention that the approximations in the eikonal approach to quantum electrodynamics are closely related to those used in deriving the low-energy behavior of photons This is similarly true for gravitons, exact photon low-energy theorems may be extended to the gravitational case [8], where they enjoy a universal validity, similar to 't Hooft's scattering amplitude

The eikonal approximation is also valid for non-abelian vector meson exchange, where it produces essentially abelian results [9] In the context of the present investigation, this is seen by taking the non-abelian single-particle current to be

$$\mathcal{J}^{\mu}(x)=Q(t)j^{\mu}(x)\,, \tag{34}$$

where $\mathcal{J}^{\mu}$ and Q are in the Lie algebra, j^{μ} also depends on the particle path $x^{\mu}(t)$ and is conserved, as in (1) and (6) In order that $\mathcal{J}^{\mu}$ be covariantly conserved, Q must satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(t)+[x^{\mu}(t)A_{\mu}(x(t)),Q(t)]=0 \tag{35}$$

When the eikonal *ansatz* is made

$$E^{x}=0,\quad B^{z}=0,\quad A_{\pm}=g^{-1}\,\partial_{\pm}g,\quad A^{\perp}=0 \tag{36}$$

one find that

$$Q(t)=g^{-1}Q_0 g\,, \tag{37}$$

where g is a group element and Q_0 is constant It is then easily seen that extending the previous eikonal argument to the non-abelian case amounts to an embedding of the abelian results in the Q_0-"direction" It remains an open question whether an essentially non-abelian eikonal approximation can be formulated

Note added

We have been informed by P Aichelburg, whom we thank, that the field strengths (5) have been previously given by refs [10,11]

Professor T D Lee encouraged us to develop these ideas and we thank him

References

[1] G 't Hooft, Phys Lett B 198 (1987) 61,
see also D Amati, N Ciafaloni and G Veneziano, Phys Lett B 197 (1987) 81
[2] P Aichelburg and R Sexl, Gen Rev Grav 2 (1971) 303,
T Dray and G 't Hooft, Nucl Phys B 253 (1985) 173
[3] H Verlinde and E Verlinde, Princeton preprint PUPT-1279 (September 1991)
[4] H Fried, Functional methods and models in quantum field theory (MIT Press, Cambridge, MA, 1972)
[5] L Faddeev and R Jackiw, Phys Rev Lett 60 (1988) 1692
[6] G Tiktopoulos and S Trieman, Phys Rev D 3 (1971) 1037,
E Eichten and R Jackiw, Phys Rev D 4 (1971) 439
[7] D Kabat and M Ortiz, in preparation,
see also D Amati, N Ciafaloni and G Veneziano, Nucl Phys B 347 (1990) 550
[8] D Gross and R Jackiw, Phys Rev 166 (1968) 1287,
R Jackiw and L Soloviev, Phys Rev 173 (1968) 1485
[9] J Cornwall and G Tiktopoulos, Phys Rev D 15 (1977) 2937
[10] I Robinson and K Rózga, J Math Phys 25 (1984) 499
[11] P Aichelburg and F Embacher, in Gravitation and geometry, eds W Rindler and A Trautman (Bibliopolis, Naples, 1987)