8.10 (a) Using the identity $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$, we have

$$\vec{E}^*\cdot[\nabla\times(\nabla\times\vec{E})] = -\nabla\cdot[\vec{E}^*\times(\nabla\times\vec{E})] + (\nabla\times\vec{E}^*)\cdot(\nabla\times\vec{E}).$$

then the numerator becomes

$$\int_V \vec{E}^*\cdot[\nabla\times(\nabla\times\vec{E})]\,d^3x = \int_V (\nabla\times\vec{E}^*)\cdot(\nabla\times\vec{E})\,d^3x - \oint_S [\vec{E}^*\times(\nabla\times\vec{E})]\cdot\vec{n}\,da$$

Since $[\vec{E}^*\times(\nabla\times\vec{E})]\cdot\vec{n} = (\vec{n}\times\vec{E}^*)\cdot(\nabla\times\vec{E})$, where $\vec{n}\times\vec{E}^* = 0$ on the boundary, the surface integral term is identically 0. Therefore, the variational principle becomes

$$k^2 = \frac{\int_V (\nabla\times\vec{E}^*)\cdot(\nabla\times\vec{E})\,d^3x}{\int_V \vec{E}^*\cdot\vec{E}\,d^3x}$$

(b) For the TE wave, $\vec{E}_t = \dfrac{-i}{\mu\epsilon\omega^2-k^2}\,\omega\hat{z}\times\nabla_t B_z$, where

$$\nabla_t B_z = \hat{\rho}\frac{\partial}{\partial\rho}B_z + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}B_z$$

$$= B_0\sin\left(\frac{\pi z}{d}\right)\left[\hat{\rho}\,\frac{1}{R}\left(1-\frac{\rho}{R}\right)\cos\phi - \hat{\phi}\,\frac{1}{R}\left(1-\frac{\rho}{2R}\right)\sin\phi\right]$$

Then,
$$\vec{E}_t = -\frac{i\omega B_0}{(\mu\epsilon\omega^2-k^2)R}\left[\hat{\rho}\left(1-\frac{\rho}{2R}\right)\sin\phi\sin\left(\frac{\pi z}{d}\right) + \hat{\phi}\left(1-\frac{\rho}{R}\right)\cos\phi\sin\left(\frac{\pi z}{d}\right)\right]$$

where we have used $\hat{z}\times\hat{\rho} = \hat{\phi}$, and $\hat{z}\times\hat{\phi} = -\hat{\rho}$. Identifying E_0 as $-i\omega B_0/(\mu\epsilon\omega^2-k^2)R$, we can write the transverse components of $\vec{E}$ as

$$E_\rho = E_0\left(1-\frac{\rho}{2R}\right)\sin\phi\sin\left(\frac{\pi z}{d}\right), \quad E_\phi = E_0\left(1-\frac{\rho}{R}\right)\cos\phi\sin\left(\frac{\pi z}{d}\right)$$

(c) In cylindrical coordinates

$$\nabla\times\vec{E} = -\frac{\partial E_\phi}{\partial z}\hat{\rho} + \frac{\partial E_\rho}{\partial z}\hat{\phi} + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho E_\phi) - \frac{\partial}{\partial\phi}E_\rho\right)\hat{z}$$

$$= E_0\left[-\frac{\pi}{d}\left(1-\frac{\rho}{R}\right)\cos\phi\cos\left(\frac{\pi z}{d}\right)\hat{\rho} + \frac{\pi}{d}\left(1-\frac{\rho}{2R}\right)\sin\phi\cos\left(\frac{\pi z}{d}\right)\hat{\phi}\right.$$

$$\left. + \frac{1}{\rho}\left\{\left(1-\frac{2\rho}{R}\right)\cos\phi\sin\left(\frac{\pi z}{d}\right) - \left(1-\frac{\rho}{2R}\right)\cos\phi\sin\left(\frac{\pi z}{d}\right)\right\}\hat{z}\right],$$

then
$$(\nabla\times\vec{E}^*)\cdot(\nabla\times\vec{E}) = |E_0|^2\left[\frac{\pi^2}{d^2}\left(1-\frac{\rho}{R}\right)^2\cos^2\phi\cos^2\left(\frac{\pi z}{d}\right) + \frac{\pi^2}{d^2}\left(1-\frac{\rho}{2R}\right)^2\sin^2\phi\cos^2\left(\frac{\pi z}{d}\right)\right.$$

$$\left. + \frac{9}{4R^2}\cos^2\phi\sin^2\left(\frac{\pi z}{d}\right)\right]$$

Also, $\vec{E}^* \cdot \vec{E} = |E_0|^2 \left\{ \left(1-\frac{\rho}{2R}\right)^2 \sin^2\phi \sin^2\left(\frac{\pi z}{d}\right) + \left(1-\frac{\rho}{R}\right)^2 \cos^2\phi \sin^2\left(\frac{\pi z}{d}\right) \right\}$.

The integration in the numerator is

$$\frac{1}{|E_0|^2}\int_V (\nabla\times\vec{E}^*)\cdot(\nabla\times\vec{E})\, d^3x = \frac{d}{2}\cdot\pi\cdot\int_0^R \left[\frac{\pi^2}{d^2}\left(1-\frac{\rho}{R}\right)^2 + \frac{\pi^2}{d^2}\left(1-\frac{\rho}{2R}\right)^2 + \frac{9}{4R^2}\right]\rho\, d\rho$$

$$= \frac{d}{2}\cdot\pi\left[\frac{\pi^2}{d^2}\cdot\frac{R^2}{12} + \frac{\pi^2}{d^2}\cdot\frac{11}{48}R^2 + \frac{9}{8}\right] = \frac{\pi d}{2}\left[\frac{\pi^2}{d^2}\cdot\frac{5}{16}R^2 + \frac{9}{8}\right]$$

and the denominator is

$$\frac{1}{|E_0|^2}\int_V \vec{E}^*\cdot\vec{E}\, d^3x = \frac{d}{2}\cdot\pi\int_0^R\left[\left(1-\frac{\rho}{2R}\right)^2 + \left(1-\frac{\rho}{R}\right)^2\right]\rho\, d\rho = \frac{d}{2}\cdot\pi\left(\frac{1}{12}R^2 + \frac{11}{48}R^2\right) = \frac{\pi d}{2}\,\frac{5}{16}R^2$$

where we have replaced the $\sin^2(\cdots)$ and $\cos^2(\cdots)$ integration with the average. Then,

$$k^2 = \frac{\frac{\pi^2}{d^2}\cdot\frac{5}{16}R^2 + \frac{9}{32}}{\frac{5}{16}R^2} = \frac{18}{5R^2} + \frac{\pi^2}{d^2}$$

(d) The form $\vec{E}^*\cdot[\nabla\times(\nabla\times\vec{E})] = \vec{E}^*\cdot\left[\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}\right]$ is convenient in the TM case, as we only need to calculate the Laplacian of one component. When $\vec{E}$ has transverse components, $\nabla\times\vec{E}$ is more convenient.