3.2 Solution: (a) From symmetry and regularity considerations, the potential inside and outside of sphere can be expressed as

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta),$$

where $r_<$ and $r_>$ are the smaller and larger of r and R, respectively.

The charge density on the inner surface is

$$\sigma_<(R,\theta) = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial n}\right|_{r=R-\varepsilon} = \varepsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=R-\varepsilon} = \varepsilon_0 \sum_{l=1}^{\infty} \frac{lA_l}{R^2} P_l(\cos\theta).$$

Similarly, for the outer surface,

$$\sigma_>(R,\theta) = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial n}\right|_{r=R+\varepsilon} = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=R+\varepsilon} = \varepsilon_0 \sum_{l=0}^{\infty} \frac{(l+1)A_l}{R^2} P_l(\cos\theta).$$

Therefore, the total surface charge density is

$$\sigma(R,\theta) = \sigma_<(R,\theta) + \sigma_>(R,\theta) = \varepsilon_0 \sum_{l=0}^{\infty} \frac{(2l+1)A_l}{R^2} P_l(\cos\theta),$$

which should agree with the specified charge density. Thus, we have the relation

$$\sum_{l=0}^{\infty} (2l+1) A_l P_l(\cos\theta) = \frac{Q}{4\pi\varepsilon_0} I_{\theta>\alpha},$$

where I is the indicator function, as the charge density is vanishing near the north pole.

Integrating both sides with respect to $P_l(\cos\theta)$, we can determine the coefficients A_l as,

$$A_l = \frac{Q}{8\pi\varepsilon_0} \int_{-1}^{\cos\alpha} P_l(x)dx.$$

Since

$$P_l(x) = \frac{1}{2l+1}\left(P'_{l+1}(x) - P'_{l-1}(x)\right),$$

for $l > 0$, the integral can be exactly performed, which leads to

$$A_l = \frac{Q}{8\pi\varepsilon_0} \frac{1}{2l+1} \left(P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right),$$

where we have used the result that $P_{l+1}(-1) = P_{l-1}(-1)$. If $l = 0$, $P_0 \equiv 1$, and $A_0 = (Q/8\pi\varepsilon_0)(\cos\alpha + 1)$. Therefore, by defining $P_{-1}(\cos\alpha) \equiv -1$, we can combine these two cases and have a unified expression for the coefficients. Finally, the potential becomes

$$\Phi(r,\theta) = \frac{Q}{8\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{1}{2l+1} \left(P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right) \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta),$$

and for the potential inside the sphere,

$$\Phi(r,\theta) = \frac{Q}{8\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{1}{2l+1} \left(P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right) \frac{r^l}{R^{l+1}} P_l(\cos\theta),$$

and outside the sphere,

$$\Phi(r,\theta) = \frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left(P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right)\frac{R^l}{r^{l+1}}P_l(\cos\theta),$$

Alternatively, this problem can also be solved with direct integration. Given the charge density on the sphere, the potential inside the sphere is

$$\Phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x',$$

where

$$\rho(\mathbf{x}') = \frac{Q}{4\pi R^2}\delta(r'-R)I_{\theta\geq\alpha}.$$

Using the expansion

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi),$$

where for the potential problem inside the sphere $r_< = r$ and $r_> = R$, we have

$$\begin{aligned}\Phi(r,\theta) &= \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^{2\pi}d\phi'\int_0^{\pi}I_{\theta'\geq\alpha}\sin\theta' d\theta'\int_0^{\infty}r'^2\delta(r'-R)dr' \\ &\qquad\times\frac{1}{2l+1}\frac{r^l}{R^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \\ &= \frac{Q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^{2\pi}d\phi'\int_\alpha^{\pi}\sin\theta' d\theta'\frac{1}{4\pi}\frac{r^l}{R^{l+1}}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta')P_l^m(\cos\theta)e^{im(\phi-\phi')}.\end{aligned}$$

Only the $m=0$ term will survive the integral with respect to ϕ', and

$$\begin{aligned}\Phi(r,\theta) &= \frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}P_l(\cos\theta)\int_\alpha^{\pi}\sin\theta' P_l(\cos\theta')d\theta' \\ &= \frac{Q}{8\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{R^{l+1}}P_l(\cos\theta)\int_{-1}^{\cos\alpha}P_l(x)dx.\end{aligned}$$

This agrees with the result obtained from matching the charge density on the sphere.

(b) The electric field can be found by taking the gradient of the potential, $\mathbf{E} = -\nabla\Phi$. In spherical coordinates,

$$\begin{aligned}\nabla\Phi &= \frac{\partial\Phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta} \\ &= \sum_{l=0}^{\infty}A_l\frac{r^{l-1}}{R^{l+1}}\left(lP_l(\cos\theta)\hat{r} - \sin\theta P_l'(\cos\theta)\hat{\theta}\right).\end{aligned}$$

As $r\to 0$, only the $l=1$ term survives,

$$\mathbf{E} = -\nabla\Phi = -\frac{A_1}{R^2}\left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right).$$

Since in spherical coordinates,

$$\hat{z} = \cos\theta \hat{r} - \sin\theta \hat{\theta},$$

and

$$\frac{A_1}{R^2} = \frac{Q}{24\pi\varepsilon_0 R^2}\left[P_2(\cos\alpha) - P_0(\cos\alpha)\right] = -\frac{Q\sin^2\alpha}{16\pi\varepsilon_0 R^2},$$

therefore,

$$\mathbf{E} = \frac{Q\sin^2\alpha}{16\pi\varepsilon_0 R^2}\hat{z}.$$