13.5 (a) In the particle's rest frame, the screened Coulomb potential is

$$V(t',\vec{x}') = ze\frac{e^{-k_0 r'}}{r'}, \quad \text{where } r' = (x'^2+y'^2+z'^2)^{1/2}.$$ Assume that the particle is moving nonrelativistically along the x-direction with velocity v in the laboratory frame. Applying the Galileo transform, the potential in the lab frame becomes

$$V(t,\vec{x}) = ze\frac{e^{-k_0 r}}{r}, \quad \text{where } r = \left[(x-vt)^2+y^2+z^2\right]^{1/2}$$

The Fourier transform of the potential is then given by

$$\Phi(\vec{k},\omega) = \frac{1}{(2\pi)^2}\int d^3x\int dt\ V(t,\vec{x})e^{i\omega t - i\vec{k}\cdot\vec{x}},$$

$$= \frac{ze}{2\pi}\delta(\omega-\vec{k}\cdot\vec{v})\int d^3x\ \frac{e^{-i\vec{k}\cdot\vec{x}}e^{-k_0 r}}{r}$$

$$= \frac{2ze}{k_0^2+k^2}\delta(\omega-\vec{k}\cdot\vec{v})$$

where we have used the fact that $\int \frac{e^{-i\vec{k}\cdot\vec{x}}e^{-k_0 r}}{r}d^3x = \frac{4\pi}{k_0^2+k^2}$.

Using Eq. (13.26) and noticing that the particle moves nonrelativistically, $v \ll c$, the electric field is given by $\vec{E}(\vec{k},\omega) = -i\vec{k}\,\Phi(\vec{k},\omega)$, and by Eq. (13.28),

$$\vec{E}(\omega) = \frac{1}{(2\pi)^{3/2}}\int d^3k\ \vec{E}(\vec{k},\omega)e^{ik_2 b}.$$

For component of $\vec{E}$ parallel to $\vec{v}$, we have

$$E_1(\omega) = \frac{-i2ze}{(2\pi)^{3/2}}\int d^3k\ e^{ik_2 b}k_1\frac{\delta(\omega-vk_1)}{k_0^2+k^2} = \frac{-i2ze\omega}{v^2(2\pi)^{3/2}}\int_{-\infty}^{+\infty}dk_2\ e^{ik_2 b}\int_{-\infty}^{+\infty}\frac{dk_3}{k_2^2+k_3^2+\lambda^2}$$

where $\lambda^2 = k_0^2+\omega^2/v^2$. Performing the integrals in the transverse directions as in Section 13.3, we will have

$$E_1(\omega) = \frac{-2ize\omega}{v^2(2\pi)^{3/2}}\int_{-\infty}^{+\infty}\frac{\pi}{(k_2^2+\lambda^2)^{1/2}}e^{ik_2 b}dk_2 = \frac{-ize\omega}{v^2}\sqrt{\frac{2}{\pi}}K_0(\lambda b)$$

For the transverse component,

$$E_2(\omega) = \frac{-i2ze}{(2\pi)^{3/2}}\int d^3k\ e^{ik_2 b}k_2\frac{\delta(\omega-vk_1)}{k_0^2+k^2} = \frac{-i2ze}{v(2\pi)^{3/2}}\int_{-\infty}^{+\infty}\frac{\pi k_2}{(k_2^2+\lambda^2)^{1/2}}e^{ik_2 b}dk_2 = \frac{ze}{v}\sqrt{\frac{2}{\pi}}\lambda K_1(\lambda b)$$

Following the same procedure as in Prob. 13.1(b), the transverse momentum transferred to the electron is $\Delta p = \int e E_2(t)\, dt = \sqrt{2\pi}\, e\, E_2(\omega=0)$. Therefore,

$$\Delta p = \sqrt{2\pi} \cdot e \cdot \frac{ze}{v} \sqrt{\frac{2}{\pi}}\, k_0 K_1(k_0 b) = \frac{2ze^2}{v} k_0 K_1(k_0 b),$$

and the energy transfer is

$$\Delta E(b) = (\Delta p)^2/2m = \frac{2z^2e^4}{mv^2} k_0^2 K_1^2(k_0 b)$$

(b) Using equation (13.35), we have

$$\left(\frac{dE}{dx}\right)_{k_0 b<1} = 2\pi N Z \int_0^{k_0^{-1}} \Delta E(b)\, b\, db = \frac{z^2e^2}{v^2} \frac{4\pi N Z e^2}{m} \int_0^1 x K_1(x)^2 dx$$

Since $K_1(x) \sim 1/x$, for $x \to 0$, the integral diverges at the lower limit. To remedy this, we can separate the divergent part of the integral and set a cutoff. Then,

$$\int_0^1 x K_1(x)^2 dx = \int_0^1 x\left(\frac{1}{x} + K_1(x) - \frac{1}{x}\right)^2 dx = \int_0^1 \left[\frac{1}{x} + 2\left(K_1(x) - \frac{1}{x}\right) + x\left(K_1(x) - \frac{1}{x}\right)^2\right] dx$$

$$\to \int_{k_0 b_{min}}^1 \frac{dx}{x} + \int_0^1 (1 + x K_1(x))\left(K_1(x) - \frac{1}{x}\right) dx$$

$$= \log\left(\frac{1}{k_0 b_{min}}\right) - 0.5447 = \log\left(\frac{1}{1.72\, k_0 b_{min}}\right),$$

and

$$\left(\frac{dE}{dx}\right)_{k_0 b<1} = \frac{z^2e^2}{v^2} \omega_p^2 \log\left(\frac{1}{1.72\, k_0 b_{min}}\right),$$

which differs from Jackson's result by a constant.