2.17 Solution: (a) The Green function in three-dimensional space is

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|},$$

or equivalently, in Cartesian coordinates,

$$G(x,y,z;x',y',z') = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}.$$

Using the identity

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x+\sqrt{x^2+a^2}\right)+C,$$

integrating the three-dimensional Green function in $z'-z$ from $-Z$ to Z, we have

$$\begin{aligned}
&\int_{-Z}^{Z} \frac{d(z'-z)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\
= \quad & \log\left(\frac{\sqrt{(x-x')^2+(y-y')^2+Z^2}+Z}{\sqrt{(x-x')^2+(y-y')^2+Z^2}-Z}\right) \\
= \quad & 2\log\left(\sqrt{(x-x')^2+(y-y')^2+Z^2}+Z\right) - \log\left((x-x')^2+(y-y')^2\right). \qquad (1)
\end{aligned}$$

When Z is large, the leading order correction is proportional to

$$\frac{(x-x')^2+(y-y')^2}{Z^2}$$

and can be ignored compared to $\log\left((x-x')^2+(y-y')^2\right)$. Therefore, the two-dimensional free-space Green function becomes

$$\begin{aligned}
G(x,y;x',y') &= -\log\left((x-x')^2+(y-y')^2\right) \\
&= -\log\left(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')\right).
\end{aligned}$$

(b) For two-dimensional problem, the completeness relation in the angular direction is

$$\delta(\phi-\phi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')},$$

and the two-dimensional Dirac delta function can therefore be written as

$$\delta(\boldsymbol{\rho}-\boldsymbol{\rho}') = \frac{1}{\rho}\delta(\rho-\rho')\cdot\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}.$$

Thus, the Green function must have the following form,

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} g_m(\rho,\rho')e^{im(\phi-\phi')}.$$

Applying the Laplacian operator on the Green function leads to the Dirac delta function,

$$\nabla^2_{\boldsymbol{\rho}'}G(\boldsymbol{\rho},\boldsymbol{\rho}') = -4\pi\delta(\boldsymbol{\rho}-\boldsymbol{\rho}'),$$

which is equivalent to

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\left[\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right)-\frac{m^2}{\rho'^2}\right]g_n(\rho,\rho')e^{im(\phi-\phi')}=-4\pi\frac{\delta(\rho-\rho')}{\rho}\cdot\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}.$$

The above equation must be valid term-wise, which gives the following differential equation for g_m,

$$\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right)-\frac{m^2}{\rho'^2}g_m=-4\pi\frac{\delta(\rho-\rho')}{\rho}. \tag{2}$$

(c) Solution for Eq. (2) must be linear combinations of ρ'^m and ρ'^{-m}, for $m\neq 0$. For $\rho'<\rho$, the regularity of the solution at $\rho'=0$ dictates that

$$g_m\propto\begin{cases}\rho'^m, & m>0,\\ \rho'^{-m}, & m<0.\end{cases}$$

Similarly, for $\rho'>\rho$, the regularity at $\rho'\to\infty$ requires that

$$g_m\propto\begin{cases}\rho'^{-m}, & m>0,\\ \rho'^{m}, & m<0.\end{cases}$$

Since the solution must be symmetric in ρ and ρ', the solution we are seeking should be in the form of

$$g_m(\rho,\rho')=C_m\left(\frac{\rho_<}{\rho_>}\right)^{|m|}, \tag{3}$$

where C_m is a yet to be determined constant, and $\rho_<$ ($\rho_>$) is the smaller (larger) of ρ and ρ'.

With Eq. (3), we can now multiply both sides of Eq. (2) with ρ' and integrate on a small interval around ρ. This leads to an equation that connects the derivatives of g_m on both sides of ρ,

$$\rho'\frac{\partial}{\partial\rho'}g_m(\rho,\rho')\bigg|_{\rho+\varepsilon}-\rho'\frac{\partial}{\partial\rho'}g_m(\rho,\rho')\bigg|_{\rho-\varepsilon}=-4\pi,$$

which gives

$$C_m=\frac{2\pi}{|m|}.$$

For $m=0$, there is some ambiguity and arbitrariness in pikcing a form for g_0, with $\log\rho_<$, $\log\rho_>$, and $\log(\rho_</\rho_>)$ are all valid choices. To recover the form given in the book, let us choose

$$g_0=C_0\log\rho_>.$$

By the connecting relation of the derivatives, it can be easily shown that

$$C_0=-4\pi.$$

Putting all the coefficients together, the Green function can be written as

$$G(\rho,\phi;\rho',\phi') \;=\; -2\log\rho_>+\sum_{m\neq 0}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^{|m|}e^{im(\phi-\phi')}$$

$$= \quad -\log\rho_>^2 + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos\left[m(\phi-\phi')\right].$$

The second term can be summed. Using the identity

$$\log(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n},$$

the second term can be expressed as

$$\begin{aligned}
& 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos\left[m(\phi-\phi')\right] \\
= \quad & -2\cdot\mathrm{Re}\left\{\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m e^{im(\phi-\phi')}\right\} \\
= \quad & -2\cdot\mathrm{Re}\left\{\log\left(1-\frac{\rho_<}{\rho_>}e^{im(\phi-\phi')}\right)\right\} \\
= \quad & -\log\left(1+\frac{\rho_<^2}{\rho_>^2}-2\frac{\rho_<}{\rho_>}\cos(\phi-\phi')\right).
\end{aligned}$$

Combine with the first term, we can recover the Green function of part (a).