3.17 Solution: (a) The Dirac delta function in the cylindrical coordinates can be expressed as

$$\delta(\mathbf{x}-\mathbf{x}') = \frac{1}{\pi L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi}{L}z\right)\sin\left(\frac{n\pi}{L}z'\right)\cdot\frac{1}{\rho}\delta(\rho-\rho').$$

Therefore, the Green function must have a similar form,

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{\pi L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi}{L}z\right)\sin\left(\frac{n\pi}{L}z'\right) g_{mn}(\rho,\rho').$$

Applying the Lapacian on the Green function, we will have

$$\begin{aligned}
\nabla^2_{\mathbf{x}'}G(\mathbf{x},\mathbf{x}') &= \frac{1}{\rho'}\frac{d}{d\rho'}\left[\rho'\frac{dG}{d\rho'}\right]+\frac{1}{\rho'^2}\frac{\partial^2 G}{\partial\phi'^2}+\frac{\partial^2 G}{\partial z'^2}\\
&= \frac{1}{\pi L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi}{L}z\right)\sin\left(\frac{n\pi}{L}z'\right)\\
&\qquad\cdot\left[\frac{1}{\rho'}\frac{d}{d\rho'}\left(\rho'\frac{dg_{mn}}{d\rho'}\right)-\left(\left(\frac{n\pi}{L}\right)^2+\frac{m^2}{\rho'^2}\right)g_{mn}\right]\\
&= -4\pi\cdot\frac{1}{\pi L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi}{L}z\right)\sin\left(\frac{n\pi}{L}z'\right)\cdot\frac{1}{\rho}\delta(\rho-\rho').
\end{aligned}$$

The above identity must be valid term-wise, which means

$$\frac{1}{\rho'}\frac{d}{d\rho'}\left(\rho'\frac{dg_{mn}}{d\rho'}\right)-\left(\left(\frac{n\pi}{L}\right)^2+\frac{m^2}{\rho'^2}\right)g_{mn} = -\frac{4\pi}{\rho}\delta(\rho-\rho').$$

Now, we can apply the usual procedure to determine the unknown function g, which must be a linear combination of $I_m(n\pi\rho'/L)$ and $K_m(n\pi\rho'/L)$. For $\rho'<\rho$, in order to have a finite solution as $\rho'\to 0$, the solution must be proportional to $I_m(n\pi\rho'/L)$. Similarly, for $\rho'>\rho$, the solution must be proportional to $K_m(n\pi\rho'/L)$. The solution is also symmetric in ρ and ρ'. Therefore,

$$g_{mn}(\rho,\rho') = A_{mn}I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right).$$

To determine A_{mn}, multiply both sides of the differential equation governing g by ρ' and integrate, we have

$$\left.\frac{\partial g_{mn}(\rho,\rho')}{\partial\rho'}\right|_{\rho'=\rho+\varepsilon}-\left.\frac{\partial g_{mn}(\rho,\rho')}{\partial\rho'}\right|_{\rho'=\rho-\varepsilon} = -\frac{4\pi}{\rho}.$$

Since

$$\left.\frac{\partial g_{mn}(\rho,\rho')}{\partial\rho'}\right|_{\rho'=\rho+\varepsilon} = A_{mn}\frac{\partial}{\partial\rho'}\left(I_m\left(\frac{n\pi}{L}\rho\right)K_m\left(\frac{n\pi}{L}\rho'\right)\right)\Bigg|_{\rho'=\rho+\varepsilon} = A_{mn}\frac{n\pi}{L}I_m\left(\frac{n\pi}{L}\rho\right)K_m'\left(\frac{n\pi}{L}\rho\right),$$

and

$$\left.\frac{\partial g_{mn}(\rho,\rho')}{\partial\rho'}\right|_{\rho'=\rho-\varepsilon} = A_{mn}\frac{\partial}{\partial\rho'}\left(I_m\left(\frac{n\pi}{L}\rho'\right)K_m\left(\frac{n\pi}{L}\rho\right)\right)\Bigg|_{\rho'=\rho-\varepsilon} = A_{mn}\frac{n\pi}{L}I_m'\left(\frac{n\pi}{L}\rho\right)K_m\left(\frac{n\pi}{L}\rho\right),$$

which leads to

$$A_{mn}\frac{n\pi}{L}W\left[I_m\left(\frac{n\pi}{L}\rho\right),K_m\left(\frac{n\pi}{L}\rho\right)\right] = -\frac{4\pi}{\rho},$$

where W is the Wronskian. For modified Bessel functions,

$$W\left[I_m(x), K_m(x)\right] = -\frac{1}{x},$$

we can find

$$A_{mn}\frac{n\pi}{L}\cdot\left(-\frac{1}{n\pi\rho/L}\right) = -\frac{4\pi}{\rho},$$

or

$$A_{mn} = 4\pi.$$

Now, the Green function reads

$$G(\mathbf{x},\mathbf{x}') = \frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi}{L}z\right)\sin\left(\frac{n\pi}{L}z'\right)I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right).$$

(b) With the help of the Hankel transform, the Dirac delta function in the cylindrical coordinates can be expressed as

$$\delta(\mathbf{x}-\mathbf{x}') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk e^{im(\phi-\phi')}\cdot kJ_m(k\rho')J_m(k\rho)\cdot\delta(z-z').$$

Therefore, the Green function must have a similar form,

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk e^{im(\phi-\phi')}\cdot kJ_m(k\rho')J_m(k\rho)\cdot g_m(k;z,z').$$

Applying the Lapacian on the Green function, we will have

$$\begin{aligned}
\nabla^2_{\mathbf{x}'}G(\mathbf{x},\mathbf{x}') &= \frac{1}{\rho'}\frac{d}{d\rho'}\left[\rho'\frac{dG}{d\rho'}\right] + \frac{1}{\rho'^2}\frac{\partial^2 G}{\partial\phi'^2} + \frac{\partial^2 G}{\partial z'^2} \\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\ e^{im(\phi-\phi')}\cdot kJ_m(k\rho')J_m(k\rho)\left(\frac{\partial^2 g_m(k;z,z')}{\partial z'^2} - k'^2 g_m(k;z,z')\right) \\
&= -4\pi\cdot\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\ e^{im(\phi-\phi')}\cdot kJ_m(k\rho')J_m(k\rho)\cdot\delta(z-z').
\end{aligned}$$

The above identity must be valid term-wise, which means

$$\frac{\partial^2 g_m(k;z,z')}{\partial z'^2} - k^2 g_m(k;z,z') = -4\pi\delta(z-z').$$

Now, we can apply the usual procedure to determine the unknown function g, which must be a linear combination of $e^{kz'}$ and $e^{-kz'}$. For $z' < z$, the Green function vanishes as $z' \to 0$, the solution must be proportional to $\sinh(kz')$. Similarly, for $z' > z$, the solution must be proportional to $\sinh[k(L-z')]$. The solution is also symmetric in z and z'. Therefore,

$$g_m(k;z,z') = A_m(k)\sinh(kz_<)\sinh[k(L-z_>)].$$

To determine $A_m(k)$, integrate both sides of the differential equation governing g, we have

$$\left.\frac{\partial g_m(k;z,z')}{\partial z'}\right|_{z'=z+\varepsilon} - \left.\frac{\partial g_m(k;z,z')}{\partial z'}\right|_{z'=z-\varepsilon} = -4\pi.$$

Since

$$\left.\frac{\partial g_m(k;z,z')}{\partial z'}\right|_{z'=z+\varepsilon} = A_m(k)\ \frac{\partial}{\partial z'}\left(\sinh(kz)\sinh[k(L-z')]\right)\Big|_{z'=z+\varepsilon} = -kA_m(k)\sinh(kz)\cosh[k(L-z)],$$

and

$$\left.\frac{\partial g_m(k;z,z')}{\partial z'}\right|_{z'=z-\varepsilon} = A_m(k)\ \frac{\partial}{\partial z'}\left(\sinh(kz')\sinh[k(L-z)]\right)\Big|_{z'=z+\varepsilon} = kA_m(k)\cosh(kz)\sinh[k(L-z)],$$

which leads to

$$-kA_m(k)\left(\sinh(kz)\cosh[k(L-z)] + \cosh(kz)\sinh[k(L-z)]\right) = -kA_m(k)\sinh(kL) = -4\pi,$$

or

$$A_m(k) = \frac{4\pi}{k\sinh(kL)}.$$

With the knowledge of the coefficient, the Green function becomes

$$G(\mathbf{x},\mathbf{x}') = 2\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\ e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')\frac{\sinh(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)}.$$