12.2. (a) For a system with a Lagrangian $L[q(t), \dot{q}(t), t]$, the action is

$$S = \int_{t_1}^{t_2} L[q(t), \dot{q}(t), t]\, dt.$$

For another Lagrangian $L' = L + \delta L = L + \frac{d}{dt} g(q(t); t)$, the action is

$$S' = \int_{t_1}^{t_2} L'\, dt = S + [g(q(t_2); t_2) - g(q(t_1), t_1)]$$

This differs from the original action by only boundary terms, which can be viewed as constants with respect to the variation of the path. This will yield the same Euler-Lagrange equation as the original one.

(b) The Lagrangian is $\mathcal{L} = -mc^2 \sqrt{1 - \frac{u^2}{c^2}} + \frac{e}{c} \vec{u} \cdot \vec{A} - e\Phi$.

With a gauge transformation $A^\alpha \to A^\alpha + \partial^\alpha \chi$, the Lagrangian becomes

$$\mathcal{L}' = -mc^2 \sqrt{1 - \frac{u^2}{c^2}} + \frac{e}{c} \vec{u} \cdot \vec{A} - e\Phi - \frac{e}{c} \vec{u} \cdot \nabla \chi - \frac{e}{c} \frac{\partial \chi}{\partial t} = \mathcal{L} - \frac{e}{c} \frac{d\chi}{dt}.$$

i.e., the gauge transformed Lagrangian differs from the original one by a total time derivative and therefore should generate the same equations of motion. by part (a)