3.24 Solution: I am not going to do the numerics, but will try to calculate the potentials as analytical as possible.

Given the Green functions, we will need the normal derivative at the upper end, with normal direction of $\hat{z}$, *i.e.*,

$$\left.\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial z'}\right|_{z'=L}.$$

(i) The Green function is given by

$$G(\mathbf{x},\mathbf{x}') = \frac{4}{a}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')}J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn}\frac{\rho'}{a}\right)}{x_{mn}J_{m+1}^2(x_{mn})\sinh\left(x_{mn}\frac{L}{a}\right)}\sinh\left(x_{mn}\frac{z_<}{a}\right)\sinh\left(x_{mn}\frac{L-z_>}{a}\right),$$

and the normal derivative is

$$\left.\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial z'}\right|_{z'=L} = -\frac{4}{a^2}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im(\phi-\phi')}J_m\left(x_{mn}\frac{\rho}{a}\right)J_m\left(x_{mn}\frac{\rho'}{a}\right)}{J_{m+1}^2(x_{mn})}\frac{\sinh\left(x_{mn}\frac{z}{a}\right)}{\sinh\left(x_{mn}\frac{L}{a}\right)}.$$

Then, the potential inside the cylinder is

$$\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{1}{4\pi}\oint\Phi(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial z'}da' \\
&= \frac{V}{\pi a^2}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{e^{im\phi}J_m\left(x_{mn}\frac{\rho}{a}\right)}{J_{m+1}^2(x_{mn})}\frac{\sinh\left(x_{mn}\frac{z}{a}\right)}{\sinh\left(x_{mn}\frac{L}{a}\right)}\int_0^{2\pi}e^{-im\phi'}d\phi'\int_0^b\rho' J_m\left(x_{mn}\frac{\rho'}{a}\right)d\rho'.
\end{aligned}$$

The integral with respect to ϕ' will leave only the $m=0$ term. The radial integral with J_0 can be exactly performed,

$$\int_0^b\rho' J_0\left(x_{0n}\frac{\rho'}{a}\right)d\rho' = \frac{a^2}{x_{0n}^2}\int_0^{x_{0n}b/a}\lambda J_0(\lambda)d\lambda = \frac{a^2}{x_{0n}^2}\cdot x_{0n}\frac{b}{a}J_1\left(x_{0n}\frac{b}{a}\right) = \frac{ab}{x_{0n}}J_1\left(x_{0n}\frac{b}{a}\right).$$

The potential now can be written as

$$\Phi(\mathbf{x}) = \frac{2b}{a}V\sum_{n=1}^{\infty}\frac{J_0\left(x_{0n}\frac{\rho}{a}\right)J_1\left(x_{0n}\frac{b}{a}\right)}{x_{0n}J_1^2(x_{0n})}\frac{\sinh\left(x_{0n}\frac{z}{a}\right)}{\sinh\left(x_{0n}\frac{L}{a}\right)}.$$

(ii) The Green function is

$$\begin{aligned}
G(\mathbf{x},\mathbf{x}') &= \frac{4}{L}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}e^{im(\phi-\phi')}\sin\left(\frac{n\pi}{L}z\right)\sin\left(\frac{n\pi}{L}z'\right)\frac{I_m\left(\frac{n\pi}{L}\rho_<\right)}{I_m\left(\frac{n\pi}{L}a\right)} \\
&\qquad\times\left[I_m\left(\frac{n\pi}{L}a\right)K_m\left(\frac{n\pi}{L}\rho_>\right)-K_m\left(\frac{n\pi}{L}a\right)I_m\left(\frac{n\pi}{L}\rho_>\right)\right],
\end{aligned}$$

and the normal derivative is

$$\left.\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial z'}\right|_{z'=L} = \frac{4\pi}{L^2}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} e^{im(\phi-\phi')}(-1)^n n\sin\left(\frac{n\pi}{L}z\right)\frac{I_m\left(\frac{n\pi}{L}\rho_<\right)}{I_m\left(\frac{n\pi}{L}a\right)} \times\left[I_m\left(\frac{n\pi}{L}a\right)K_m\left(\frac{n\pi}{L}\rho_>\right)-K_m\left(\frac{n\pi}{L}a\right)I_m\left(\frac{n\pi}{L}\rho_>\right)\right].$$

Then, the potential inside the cylinder can be expressed as an integral. As usual, the angular integration will leave the $m=0$ term only. Therefore,

$$\begin{aligned}\Phi(\mathbf{x}) &= -\frac{1}{4\pi}\oint\Phi(\mathbf{x}')\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial z'}da' \\ &= \frac{2\pi}{L^2}V\sum_{n=1}^{\infty}(-1)^{n-1}n\frac{\sin\left(\frac{n\pi}{L}z\right)}{I_0\left(\frac{n\pi}{L}a\right)} \\ &\quad\times\int_0^b\rho' I_0\left(\frac{n\pi}{L}\rho_<\right)\left[I_0\left(\frac{n\pi}{L}a\right)K_0\left(\frac{n\pi}{L}\rho_>\right)-K_0\left(\frac{n\pi}{L}a\right)I_0\left(\frac{n\pi}{L}\rho_>\right)\right]d\rho'.\end{aligned}$$

Similar to the previous potential problem, we can use the following identities to perfom the integral,

$$\int xI_0(x)dx = xI_1(x)+C,\qquad \int xK_0(x)dx = xK_1(x)+C.$$

We need to consider to cases.

(a) For $\rho>b$, $\rho_<=\rho'$ and $\rho_>=\rho$, and the integral becomes

$$\begin{aligned}&\int_0^b\rho' I_0\left(\frac{n\pi}{L}\rho_<\right)\left[I_0\left(\frac{n\pi}{L}a\right)K_0\left(\frac{n\pi}{L}\rho_>\right)-K_0\left(\frac{n\pi}{L}a\right)I_0\left(\frac{n\pi}{L}\rho_>\right)\right]d\rho' \\ = &\frac{Lb}{n\pi}I_1\left(\frac{n\pi}{L}b\right)\left[I_0\left(\frac{n\pi}{L}a\right)K_0\left(\frac{n\pi}{L}\rho\right)-K_0\left(\frac{n\pi}{L}a\right)I_0\left(\frac{n\pi}{L}\rho\right)\right].\end{aligned}$$

Then, the potential becomes

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{2b}{L}V\sum_{n=1}^{\infty}(-1)^{n-1}\sin\left(\frac{n\pi}{L}z\right)\frac{I_1\left(\frac{n\pi}{L}b\right)}{I_0\left(\frac{n\pi}{L}a\right)} \\ &\quad\times\left[I_0\left(\frac{n\pi}{L}a\right)K_0\left(\frac{n\pi}{L}\rho\right)-K_0\left(\frac{n\pi}{L}a\right)I_0\left(\frac{n\pi}{L}\rho\right)\right].\end{aligned}$$

(b) For $\rho<b$, we need to break the integral into two parts,

$$\begin{aligned}&\int_0^b\rho' I_0\left(\frac{n\pi}{L}\rho_<\right)\left[I_0\left(\frac{n\pi}{L}a\right)K_0\left(\frac{n\pi}{L}\rho_>\right)-K_0\left(\frac{n\pi}{L}a\right)I_0\left(\frac{n\pi}{L}\rho_>\right)\right]d\rho' \\ = &\int_0^\rho\rho' I_0\left(\frac{n\pi}{L}\rho'\right)\left[I_0\left(\frac{n\pi}{L}a\right)K_0\left(\frac{n\pi}{L}\rho\right)-K_0\left(\frac{n\pi}{L}a\right)I_0\left(\frac{n\pi}{L}\rho\right)\right]d\rho' \\ + &\int_\rho^b\rho' I_0\left(\frac{n\pi}{L}\rho\right)\left[I_0\left(\frac{n\pi}{L}a\right)K_0\left(\frac{n\pi}{L}\rho'\right)-K_0\left(\frac{n\pi}{L}a\right)I_0\left(\frac{n\pi}{L}\rho'\right)\right]d\rho' \\ = &\frac{L}{n\pi}\rho I_1\left(\frac{n\pi}{L}\rho\right)\left[I_0\left(\frac{n\pi}{L}a\right)K_0\left(\frac{n\pi}{L}\rho\right)-K_0\left(\frac{n\pi}{L}a\right)I_0\left(\frac{n\pi}{L}\rho\right)\right]\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{L}{n\pi} I_0\left(\frac{n\pi}{L}\rho\right)\left[I_0\left(\frac{n\pi}{L}a\right)\left(bK_1\left(\frac{n\pi}{L}b\right)-\rho K_1\left(\frac{n\pi}{L}\rho\right)\right)\right. \\
&\qquad\qquad \left. -K_0\left(\frac{n\pi}{L}a\right)\left(bI_1\left(\frac{n\pi}{L}b\right)-\rho I_1\left(\frac{n\pi}{L}\rho\right)\right)\right] \\
&= \quad \frac{L}{n\pi} b I_0\left(\frac{n\pi}{L}\rho\right)\left[I_0\left(\frac{n\pi}{L}a\right)K_1\left(\frac{n\pi}{L}b\right)-K_0\left(\frac{n\pi}{L}a\right)I_1\left(\frac{n\pi}{L}b\right)\right] \\
&+ \quad \frac{L}{n\pi} \rho I_0\left(\frac{n\pi}{L}a\right)\left[I_1\left(\frac{n\pi}{L}\rho\right)K_0\left(\frac{n\pi}{L}\rho\right)-I_0\left(\frac{n\pi}{L}\rho\right)K_1\left(\frac{n\pi}{L}\rho\right)\right],
\end{aligned}
$$

and the integral is

$$
\begin{aligned}
\Phi(\mathbf{x}) &= \frac{2}{L} V \sum_{n=1}^{\infty} (-1)^{n-1} \sin\left(\frac{n\pi}{L}z\right) \\
&\quad \times \left[b \frac{I_0\left(\frac{n\pi}{L}b\right)}{I_0\left(\frac{n\pi}{L}a\right)} \left[I_0\left(\frac{n\pi}{L}a\right)K_1\left(\frac{n\pi}{L}b\right)-K_0\left(\frac{n\pi}{L}a\right)I_1\left(\frac{n\pi}{L}b\right)\right]\right. \\
&\qquad \left. +\rho\left[I_1\left(\frac{n\pi}{L}\rho\right)K_0\left(\frac{n\pi}{L}\rho\right)-I_0\left(\frac{n\pi}{L}\rho\right)K_1\left(\frac{n\pi}{L}\rho\right)\right]\right].
\end{aligned}
$$

(iii) The Green function is

$$
G(\mathbf{x},\mathbf{x}') = \frac{8}{La^2} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{e^{im(\phi-\phi')} \sin\left(\frac{k\pi}{L}z\right) \sin\left(\frac{k\pi}{L}z'\right) J_m\left(x_{mn}\frac{\rho}{a}\right) J_m\left(x_{mn}\frac{\rho'}{a}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2 + \left(\frac{k\pi}{L}\right)^2\right] J_{m+1}^2(x_{mn})},
$$

and the normal derivative is

$$
\left.\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial z'}\right|_{z'=L} = \frac{8\pi}{L^2a^2} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (-1)^k k \frac{e^{im(\phi-\phi')} \sin\left(\frac{k\pi}{L}z\right) J_m\left(x_{mn}\frac{\rho}{a}\right) J_m\left(x_{mn}\frac{\rho'}{a}\right)}{\left[\left(\frac{x_{mn}}{a}\right)^2 + \left(\frac{k\pi}{L}\right)^2\right] J_{m+1}^2(x_{mn})}
$$

Doing the integral, keep only the $m = 0$ term,

$$
\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{1}{4\pi} \oint \Phi(\mathbf{x}') \frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial z'} da' \\
&= \frac{4\pi}{L^2a^2} V \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{k-1} k \frac{\sin\left(\frac{k\pi}{L}z\right) J_0\left(x_{0n}\frac{\rho}{a}\right)}{\left[\left(\frac{x_{0n}}{a}\right)^2 + \left(\frac{k\pi}{L}\right)^2\right] J_1^2(x_{0n})} \int_0^b \rho' J_0\left(x_{0n}\frac{\rho'}{a}\right) d\rho' \\
&= \frac{4b\pi}{L^2a} V \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{k-1} k \sin\left(\frac{k\pi}{L}z\right) \frac{J_0\left(x_{0n}\frac{\rho}{a}\right) J_1\left(x_{0n}\frac{b}{a}\right)}{\left[\left(\frac{x_{0n}}{a}\right)^2 + \left(\frac{k\pi}{L}\right)^2\right] x_{0n} J_1^2(x_{0n})}.
\end{aligned}
$$