3.3 Solution: (c) Let us first establish the relation between the total charge and the (constant) potential on the disc. From the charge density

$$\sigma(\rho) = \frac{\lambda}{\sqrt{R^2 - \rho^2}},$$

the total charge on the disc is

$$Q = \int_S \sigma(\mathbf{x}) da = \lambda \int_0^{2\pi} d\phi \int_0^R d\rho \; \frac{\rho}{\sqrt{R^2 - \rho^2}} = 2\pi R\lambda,$$

which yields

$$\lambda = \frac{Q}{2\pi R}.$$

Now, consider the potential on the z-axis from the charged disc, which can be directly calculated from the denfinition,

$$\Phi(z) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\sigma(\mathbf{x})}{\sqrt{\rho^2 + z^2}} da = \frac{Q}{8\pi^2\varepsilon_0 R} \int_0^{2\pi} d\phi \int_0^R d\rho \; \frac{\rho}{\sqrt{(R^2 - \rho^2)(\rho^2 + z^2)}}.$$

This integral can be easily done, by first making a change of integration variable, $\rho = R\sin\theta$, which leads to

$$\Phi(z) = \frac{Q}{4\pi\varepsilon_0} \int_0^{\pi/2} \frac{\sin\theta}{\sqrt{R^2 \sin^2\theta + z^2}} d\theta.$$

The last integral is elementary, and we will finally get

$$\Phi(z) = \frac{Q}{4\pi\varepsilon_0 R} \arctan\left(\frac{R}{|z|}\right).$$

Set $z = 0$, we can get the potential at the center of the disc as

$$\Phi(\mathbf{0}) = \frac{Q}{4\pi\varepsilon_0 R} \cdot \frac{\pi}{2} = \frac{Q}{8\varepsilon_0 R}.$$

Therefore, the capacitance is

$$C = \frac{Q}{V} = \frac{Q}{\Phi(\mathbf{0})} = 8\varepsilon_0 R,$$

with $Q = 8\varepsilon_0 RV$. Technically, we are cheating here, as we have not established the potential over the entire disc, and rather are relying on the fact that the disc is a conductor and thus should have the same potential everywhere, be it the center or the edge. We will demonstrate *a posteriori* that the potential is indeed uniform on the disc.

(a) Now we have the potential on the z-axis, we can apply the trick as described in Section 3.3. For the potential on the z-axis, we have

$$\Phi(z) = \frac{Q}{4\pi\varepsilon_0 R} \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \left(\frac{R}{|z|}\right)^{2j+1} = \frac{2V}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \left(\frac{R}{|z|}\right)^{2j+1}.$$

Then, for points above the disc, *i.e.*, $z > 0$, with $r > R$, the potential can be obtained as

$$\Phi(\mathbf{x}) = \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{R}{r}\right)^{2l+1} P_{2l}(\cos\theta).$$

Similarly, for points below the disc,

$$\Phi(\mathbf{x}) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l+1}P_{2l}(-\cos\theta) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l+1}P_{2l}(\cos\theta),$$

from the parity of the Legendre polynomial.

Now, let us check the potential on the edge of the disc by setting $r = R$ and $\cos\theta = 0$. Then, the potential is

$$\Phi_e = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}P_{2l}(0).$$

Using the generating function of Legendre polynomials,

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty}P_l(x)t^l,$$

we have

$$\frac{1}{\sqrt{1-t^2}} = \sum_{l=0}^{\infty}P_l(0)(it)^l = \sum_{l=0}^{\infty}P_{2l}(0)(it)^{2l} = \sum_{l=0}^{\infty}(-1)^l P_{2l}(0)t^{2l}.$$

Performing a simple integration on both sides,

$$\frac{\pi}{2} = \int_0^1\frac{dt}{\sqrt{1-t^2}} = \sum_{l=0}^{\infty}(-1)^l P_{2l}(0)\int_0^1 t^{2l}dt = \sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}P_{2l}(0),$$

which can show that the potential on the edge is indeed equal to V.

(b) For $r < R$, we cannot directly use the Taylor expansion of the arctan function, as r will appear in the denominator, rather than the numerator, in contrast to the expected result. However, notice that

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2},$$

we can express the potential on the z-axis alternatively as

$$\Phi(z) = \frac{2V}{\pi}\left(\frac{\pi}{2} - \arctan\left(\frac{|z|}{R}\right)\right) = V - \frac{2V}{\pi}\arctan\left(\frac{|z|}{R}\right).$$

Now, we can use the Taylor expansion of the arctan function and will obtain

$$\Phi(z) = V - \frac{2V}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\left(\frac{|z|}{R}\right)^{2j+1}.$$

For points above the disc with $r < R$, we have

$$\Phi(\mathbf{x}) = V - \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{r}{R}\right)^{2l+1}P_{2l+1}(\cos\theta),$$

and points below the disc,

$$\Phi(\mathbf{x}) = V - \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{r}{R}\right)^{2l+1}P_{2l+1}(-\cos\theta) = V + \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{r}{R}\right)^{2l+1}P_{2l+1}(\cos\theta),$$

again by the parity of the Legendre polynomial. It is clear that, on the disc, $\cos\theta = 0$, and $P_{2l+1}(0) = 0$. Therefore, the potential on the disc is also V.