A GREEN'S FUNCTION SOLUTION FOR THE CASE OF LAMINAR INCOMPRESSIBLE FLOW BETWEEN NON-CONCENTRIC CIRCULAR CYLINDERS

BY

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ABSTRACT

The main result is the determination of Green's function in bi-polar coordinates for the potential equation for a non-concentric annular region. This is then used to solve Poisson's equation for the point velocity in laminar incompressible flow between non-concentric circular cylinders.

I. INTRODUCTION

An explicit formulation for the case of a concentric circular ring of Green's function for the potential equation is given by Hilbert-Courant (**1**).[2] When the bounding circles of the annular region are no longer concentric, the reflection principle used in (**1**) is no longer practical. Upon introducing bi-polar coordinates in the manner suggested in Morse-Feshbach (**2**), a fairly simple form for Green's function can be obtained. This result is then used to solve Poisson's equation for the point velocity in laminar incompressible flow between non-concentric circular cylinders. When the eccentricity of the annular region tends to zero, this velocity formulation reduces to the well-known result for a concentric annulus.

II. BI-POLAR COORDINATES

In Fig. 1, if P is distant r_1 from the fixed pole $O_1(c, 0)$ and r_2 from the pole $O_2(-c, 0)$ and the angles of the vectors O_1P and O_2P are θ_1 and θ_2, respectively, then P is defined to have the bi-polar coordinates (ξ, η) where

$$\xi = \pi - (\theta_1 - \theta_2), \quad \eta = \log(r_1/r_2). \tag{1}$$

The curves ξ = const are a family of circles passing through O_1 and O_2, the ξ-values for points on the circle segments below the x-axis being π more than the ξ-values for points on the corresponding circle segments above the x-axis.

The curves η = const are also a family of circles, having centers on the x-axis and being normal to all the ξ-circles. For $\eta < 0$, the circles lie in the right half plane and enclose O_1, while for $\eta > 0$ the circles lie in the left half plane and enclose O_2.

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

If we prefer, we may regard the xy-plane in Fig. 1 as the complex z-plane and then define a new complex variable w by $w = \xi + i\eta$. To

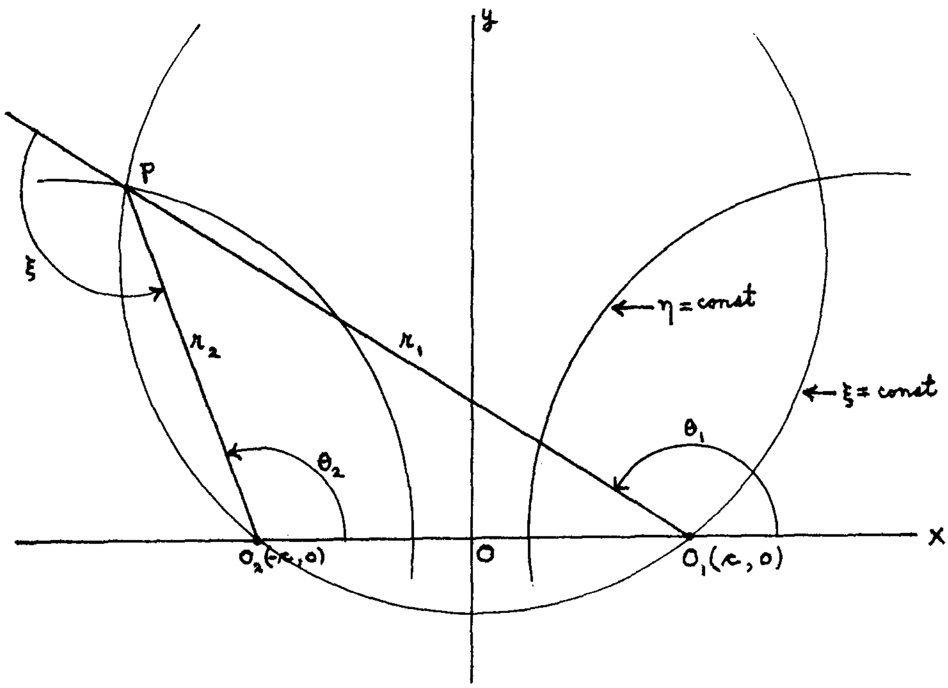

FIG. 1.

determine the transformation between the z and w planes implied by the bi-polar relations (1), observe that

$$\frac{r_1}{r_2} = e^{\eta} = \frac{|\overrightarrow{PO} + \overrightarrow{OO_1}|}{|\overrightarrow{O_2O} + \overrightarrow{OP}|} = \frac{|c - z|}{|c + z|};$$

or,

$$e^{-\eta} = \frac{|c + z|}{|c - z|}.$$

Similarly,

$$\xi = \theta_2 - (\theta_1 - \pi) = \arg (c + z) - \arg (c - z) = \arg \left(\frac{c + z}{c - z}\right).$$

Hence, ξ and $e^{-\eta}$ are respectively the angle and the absolute value of the complex variable

$$\zeta = \frac{c + z}{c - z}. \tag{2}$$

Writing ζ in polar form

$$\zeta = \rho e^{i\xi}, \quad \rho = e^{-\eta}, \tag{3}$$

we have

$$\zeta = e^{-\eta + i\xi} = e^{i(\xi + i\eta)} = e^{iw},$$

whence

$$z = c\left(\frac{e^{iw} + 1}{e^{iw} - 1}\right). \tag{4}$$

From (2) we see that in the ζ-plane, the circle $|\zeta| = e^{-\eta}$ is the map of the circle $\left|\frac{c + z}{c - z}\right| = e^{-\eta}$, which, in the z-plane, has center at $(-c \coth \eta, 0)$ and is of radius $c \operatorname{csch} \eta$.

Applying these remarks to the non-concentric annular region of interest shown in Fig. 2, we may regard the given circles A, B, with

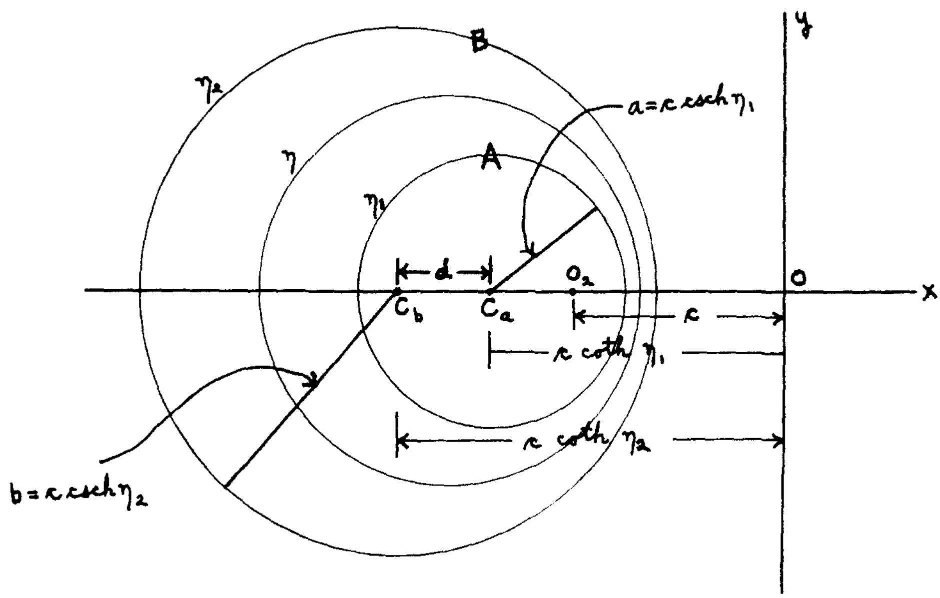

FIG. 2.

radii a, b, respectively, as two particular members of the η-family of circles enclosing point O_2 $(-c, 0)$. If we assign A and B the values η_1 and η_2, then knowledge of a, b and d, where d is the distance between the centers C_a and C_b, is sufficient to determine η_1, η_2 and c. To see this, observe that

$$a = c \operatorname{csch} \eta_1, \quad b = c \operatorname{csch} \eta_2 \tag{5}$$

$$d = c(\coth \eta_2 - \coth \eta_1), \tag{6}$$

three equations which may be solved for the unknowns η_1, η_2, c. If we define the eccentricity ϵ and the radius ratio s by

$$\epsilon = d/(b - a), \quad s = a/b, \tag{7}$$

we find that these unknowns may be computed in the order η_2, c, η_1 from

$$\eta_2 = \cosh^{-1} \tfrac{1}{2} \left\{ \frac{1 + s}{\epsilon} + (1 - s)\epsilon \right\}, \tag{8}$$

$$c = b \sinh \eta_2 \tag{9}$$

$$\eta_1 = \sinh^{-1} (c/a). \tag{10}$$

Equation 4 yields the transformation equations

$$x = \frac{- c \sinh \eta}{\cosh \eta + \cos \xi}, \quad y = \frac{c \sin \xi}{\cosh \eta + \cos \xi}. \tag{11}$$

A useful relation between η_1 and η_2, not involving c directly, is

$$\eta_1 = \eta_2 - \log s + \log [1 - \epsilon(1 - s)e^{-\eta_2}]. \tag{12}$$

This follows directly from (5) and (6) by noting first that

$$d = b \cosh \eta_2 - a \cosh \eta_1$$

and hence,

$$\eta_1 = \cosh^{-1}\left[\frac{b}{a} \cosh \eta_2 - \frac{d}{a} \right] = \log\left[\frac{b}{a} \cosh \eta_2 - \frac{d}{a} + \sinh \eta_1 \right],$$

a form which is easily manipulated into (12).

In particular, then, we see that for small eccentricities,

$$\eta_1 - \eta_2 \doteq \log (b/a). \tag{13}$$

To effect a reduction to the concentric case, we let $d \to 0$, whence

$$\epsilon \to 0, \quad c \to \infty, \quad \eta_1 \to \infty, \quad \eta_2 \to \infty,$$

but

$$\lim (\eta_1 - \eta_2) = \log (b/a). \tag{14}$$

III. DERIVATION OF GREEN'S FUNCTION

Let (x, y) and (x_0, y_0) be distinct points of the non-concentric annular region D shown in Fig. 2. Then, it will be recalled, the Green's function in question is the function $G(x, y; x_0, y_0)$, having the following properties:

(a) $\nabla^2 G = 0$ in D for all $(x, y) \neq (x_0, y_0)$, when G is regarded as a function of (x, y).

(b) For points (x, y) on the boundaries A or B, we have $G = 0$.

(c) $G(x, y; x_0, y_0) = -\dfrac{1}{2\pi} \log \sqrt{(x-x_0)^2 + (y-y_0)^2} + G^*(x, y; x_0, y_0)$,
so that G has a logarithmic singularity at (x_0, y_0), while the function G^* is analytic in D without exception and $\nabla^2 G^* = 0$.

In terms of the complex variables $z = x + iy$, $z_0 = x_0 + iy_0$, we may write G as

$$G(z, z_0) = -\frac{1}{2\pi} \operatorname{Re} \log (z - z_0) + G^*(z, z_0), \tag{15}$$

where G^* is understood to be real-valued.

Consider now the transformation (4). If we put $u = iw = -\eta + i\xi$, then

$$z - z_0 = \frac{2c(e^{u_0} - e^u)}{(e^u - 1)(e^{u_0} - 1)}. \tag{16}$$

The circle $\eta = \eta_0$ separates the region D into the sub-domains

$$D_1\colon \eta_1 \geq \eta > \eta_0 \quad \text{and} \quad D_2\colon \eta_0 > \eta \geq \eta_2.$$

Hence in order to have convergent expansions when an infinite series representation is used for $\log (z - z_0)$, we rewrite (16) in the forms

$$\begin{aligned} z - z_0 &= 2ce^{u_0}(1 - e^{u-u_0})/(1 - e^u)(1 - e^{u_0}), \quad (\text{for } D_1) \\ &= -2ce^u[1 - e^{-(u-u_0)}]/(1 - e^u)(1 - e^{u_0}), \quad (\text{for } D_2). \end{aligned}$$

Thus we find

$$\begin{aligned} \log (z - z_0) &= \log 2c + u_0 - \sum_{n=1}^{\infty} \frac{1}{n}[e^{n(u-u_0)} - (e^{nu} + e^{nu_0})], \quad \eta_1 \geq \eta > \eta_0 \\ &= i\pi + \log 2c + u - \sum_{n=1}^{\infty} \frac{1}{n}[e^{-n(u-u_0)} - (e^{nu} + e^{nu_0})], \quad \eta_0 > \eta \geq \eta_2. \end{aligned}$$

Therefore upon taking real parts we find

$$\begin{aligned} &-\frac{1}{2\pi} \operatorname{Re} \log (z - z_0) \\ &= \frac{1}{2\pi}\left[\eta_0 - \log 2c + \sum_{n=1}^{\infty} \frac{1}{n} \{e^{-n(\eta-\eta_0)} \cos n(\xi - \xi_0) - (e^{-n\eta} \cos n\xi + e^{-n\eta_0} \cos n\xi_0)\}\right] \end{aligned} \tag{17a}$$

when $\eta_1 \geq \eta > \eta_0$, and

$$-\frac{1}{2\pi}\operatorname{Re}\log(z - z_0)$$
$$= \frac{1}{2\pi}\left[\eta - \log 2c + \sum_{n=1}^{\infty}\frac{1}{n}\{e^{n(\eta-\eta_0)}\cos n(\xi - \xi_0) - (e^{-n\eta}\cos n\xi + e^{-n\eta_0}\cos n\xi_0)\}\right] \quad (17b)$$

when $\eta_0 > \eta \geq \eta_2$.

Returning now to Eq. 15, we denote Green's function in bi-polar coordinates by $G(\xi, \eta; \xi_0, \eta_0)$ and its non-singular component by $G^*(\xi, \eta; \xi_0, \eta_0)$. For convenience we also denote (17) by $S(\xi, \eta; \xi_0, \eta_0)$. Thus Eq. 17 becomes

$$G(\xi, \eta; \xi_0, \eta_0) = S(\xi, \eta; \xi_0, \eta_0) + G^*(\xi, \eta; \xi_0, \eta_0). \quad (18)$$

Since $\nabla^2 G^* = 0$, the most general G^* in terms of n^{th} order circular harmonics is

$$G^* = A_0\eta + B + \sum_{n=1}^{\infty}[(A_n\cos n\xi + B_n\sin n\xi)e^{-n\eta} + (C_n\cos n\xi + D_n\sin n\xi)e^{n\eta}]. \quad (19)$$

The constants in (19) can now be evaluated by noting that $G(\xi, \eta; \xi_0, \eta_0)$ must vanish when $\eta = \eta_1$ and also when $\eta = \eta_2$. We then have the conditions

$$G^*(\xi, \eta_1; \xi_0, \eta_0) = -S(\xi, \eta_1; \xi_0, \eta_0),$$
$$G^*(\xi, \eta_2; \xi_0, \eta_0) = -S(\xi, \eta_2; \xi_0, \eta_0).$$

Using these conditions, we obtain, after considerable simplification, the following form of Green's function for the domains D_1 and D_2.

For D_1: $\eta_1 \geq \eta > \eta_0 \geq \eta_2$, (20)

$$G(\xi, \eta; \xi_0, \eta_0) = \frac{1}{2\pi}\left[\frac{(\eta_1 - \eta)(\eta_0 - \eta_2)}{\eta_1 - \eta_2} + \sum_{n=1}^{\infty}\{e^{-n(\eta-\eta_0)} - H(\eta, \eta_0; \eta_1, \eta_2)\}\frac{\cos n(\xi - \xi_0)}{n}\right];$$

For D_2: $\eta_1 \geq \eta_0 > \eta \geq \eta_2$,

$$G(\xi, \eta; \xi_0, \eta_0) = \frac{1}{2\pi}\left[\frac{(\eta - \eta_2)(\eta_1 - \eta_0)}{\eta_1 - \eta_2} + \sum_{n=1}^{\infty}\{e^{n(\eta-\eta_0)} - H(\eta, \eta_0; \eta_1, \eta_2)\}\frac{\cos n(\xi - \xi_0)}{n}\right],$$

where

$$H(\eta, \eta_0; \eta_1, \eta_2) = \frac{e^{-n(\eta-\eta_2)}\sinh n(\eta_1 - \eta_0) - e^{n(\eta-\eta_1)}\sinh n(\eta_2 - \eta_0)}{\sinh n(\eta_1 - \eta_2)}. \quad (21)$$

It should be noted that the series

$$\sum_{n=1}^{\infty}\frac{1}{n}e^{-n(\eta-\eta_0)}\cos n(\xi - \xi_0), \quad \sum_{n=1}^{\infty}\frac{1}{n}e^{n(\eta-\eta_0)}\cos n(\xi - \xi_0)$$

diverge when $(\xi_0, \eta_0) = (\xi, \eta)$, as they should, since $G(\xi, \eta; \xi_0, \eta_0)$, considered as a function of (ξ, η), has a logarithmic-type singularity at (ξ_0, η_0). It is also readily verified that $G(\xi, \eta_i; \xi_0, \eta_0) \equiv 0, \quad i = 1, 2.$

In examining Eqs. 20 one notes that the symmetry property,

$$G(\xi, \eta; \xi_0, \eta_0) \equiv G(\xi_0, \eta_0; \xi, \eta),$$

which is characteristic of Green's functions, is not readily apparent. This is easily remedied by noting that

$$e^{-n(\eta-\eta_0)} - H \equiv \frac{2 \sinh n(\eta_1 - \eta) \sinh n(\eta_0 - \eta_2)}{\sinh n(\eta_1 - \eta_2)},$$

$$e^{n(\eta-\eta_0)} - H \equiv \frac{2 \sinh n(\eta - \eta_2) \sinh n(\eta_1 - \eta_0)}{\sinh n(\eta_1 - \eta_2)}.$$

Thus, the Green's function takes on the compact, symmetrical and rather elegant form

For D_1: $\eta_1 \geq \eta > \eta_0 \geq \eta_2, \quad G \equiv G_1, \quad$ where (22)

$$G_1(\xi, \eta; \xi_0, \eta_0) = \frac{1}{2\pi}\left[\frac{(\eta_1 - \eta)(\eta_0 - \eta_2)}{\eta_1 - \eta_2} + 2\sum_{n=1}^{\infty} \frac{\sinh n(\eta_1 - \eta) \sinh n(\eta_0 - \eta_2)}{n \sinh n(\eta_1 - \eta_2)} \cos n(\xi - \xi_0)\right];$$

For D_2: $\eta_1 \geq \eta_0 > \eta \geq \eta_2, \quad G \equiv G_2, \quad$ where

$$G_2(\xi, \eta; \xi_0, \eta_0) = \frac{1}{2\pi}\left[\frac{(\eta - \eta_2)(\eta_1 - \eta_0)}{\eta_1 - \eta_2} + 2\sum_{n=1}^{\infty} \frac{\sinh n(\eta_1 - \eta_0) \sinh n(\eta - \eta_2)}{n \sinh n(\eta_1 - \eta_2)} \cos n(\xi - \xi_0)\right].$$

IV. POINT VELOCITY IN LAMINAR FLOW

The laminar flow of an incompressible fluid of viscosity μ through the region bounded by non-concentric circular cylinders is described by Poisson's equation

$$\nabla^2 v = -\frac{1}{\mu}\frac{dp}{dL}, \tag{23}$$

where v is the point velocity at (ξ, η) in a normal cross section D and dp/dL is the pressure gradient at D in the direction of the flow.

Let $\frac{1}{\mu} dp/dL$ be denoted by the constant K. Then stated as a boundary value problem, we wish to solve

$$\nabla^2 v = -K, \tag{24}$$

subject to the conditions $v(\xi, \eta_1) = 0$, $v(\xi, \eta_2) = 0$, where η_1, η_2 characterize the boundary circles A, B of the region D.

Since we now have the Green's function available, the solution can be written down directly. Thus,

$$v(\xi,\eta) = K\int\int_D G(\xi,\eta;\xi_0,\eta_0)dA \tag{25}$$

where the element of area $dA = dxdy = J(x, y/\xi_0, \eta_0)d\xi_0 d\eta_0$, the Jacobian being computed from the transformation Eqs. 11. We find

$$dA = \frac{c^2 d\xi_0 d\eta_0}{(\cosh\eta_0 + \cos\xi_0)^2}, \tag{26}$$

so that the solution (25), expressed as an iterated integral, becomes

$$v(\xi,\eta) = Kc^2\int_{\eta_1}^{\eta_2}\int_0^{2\pi}\frac{G(\xi,\eta;\xi_0,\eta_0)d\xi_0 d\eta_0}{(\cosh\eta_0+\cos\xi_0)^2}. \tag{27}$$

In terms of G_1 and G_2 as defined in (22), this may in turn be written

$$v(\xi,\eta) = Kc^2\int_{\eta_1}^{\eta}\int_0^{2\pi}\frac{G_2(\xi,\eta;\xi_0,\eta_0)d\xi_0 d\eta_0}{(\cosh\eta_0+\cos\xi_0)^2} + Kc^2\int_{\eta}^{\eta_2}\int_0^{2\pi}\frac{G_1(\xi,\eta;\xi_0,\eta_0)d\xi_0 d\eta_0}{(\cosh\eta_0+\cos\xi_0)^2}. \tag{28}$$

We write (28) as

$$v(\xi,\eta) = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \frac{Kc^2(\eta-\eta_2)}{2\pi(\eta_1-\eta_2)}\int_{\eta_1}^{\eta}\int_0^{2\pi}\frac{\eta_1-\eta_0}{(\cosh\eta_0+\cos\xi_0)^2}d\xi_0 d\eta_0, \tag{30}$$

$$I_2 = \frac{Kc^2(\eta_1-\eta)}{2\pi(\eta_1-\eta_2)}\int_{\eta}^{\eta_2}\int_0^{2\pi}\frac{\eta_0-\eta_2}{(\cosh\eta_0+\cos\xi_0)^2}d\xi_0 d\eta_0, \tag{31}$$

$$I_3 = \frac{Kc^2}{\pi}\sum_{n=1}^{\infty}\int_{\eta_1}^{\eta}\int_0^{2\pi}\frac{\sinh n(\eta_1-\eta_0)\sinh n(\eta-\eta_2)\cos n(\xi-\xi_0)d\xi_0 d\eta_0}{n\sinh n(\eta_1-\eta_2)(\cosh\eta_0+\cos\xi_0)^2}, \tag{32}$$

$$I_4 = \frac{Kc^2}{\pi}\sum_{n=1}^{\infty}\int_{\eta}^{\eta_2}\int_0^{2\pi}\frac{\sinh n(\eta_1-\eta)\sinh n(\eta_0-\eta_2)\cos n(\xi-\xi_0)d\xi_0 d\eta_0}{n\sinh n(\eta_1-\eta_2)(\cosh\eta_0+\cos\xi_0)^2}. \tag{33}$$

The integrals I_1 and I_2 are readily evaluated and their sum is

$$I_1 + I_2 = -\frac{Kcd}{2}\left(\frac{\eta_1-\eta}{\eta_1-\eta_2}\right) + \frac{Kc^2}{2}(\coth\eta - \coth\eta_1). \tag{34}$$

The integration with respect to ξ_0 in (32) and (33) is a matter of evaluating the integral

$$I^* = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos n(\xi - \xi_0)}{(\cosh \eta_0 + \cos \xi_0)^2}\, d\xi_0. \tag{35}$$

Employing symmetry considerations and putting $z = e^{i\xi_0}$, we may write (35) as a contour integral,

$$I^* = \frac{2\cos n\xi}{\pi i}\oint \frac{z^{n+1}dz}{(z + e^{\eta_0})^2(z + e^{-\eta_0})^2}, \tag{36}$$

the integration being taken around the unit circle $|z| = 1$. Since $\eta_0 > 0$ the integrand in (36) has a pole of order 2 at $-e^{-\eta_0}$. Thus we may write

$$I^* = 4\cos n\xi \left\{ \frac{d}{dz}\left[\frac{z^{n+1}}{(z + e^{\eta_0})^2}\right]\right\}_{z = -e^{-\eta_0}},$$

where

$$I^* = (-1)^n e^{-n\eta_0}\left[\frac{\cosh \eta_0 + n \sinh \eta_0}{\sinh^3 \eta_0}\right]\cos n\xi. \tag{37}$$

Using (37) the remaining integrations in (32), (33) are easily made. Combining the results for I_3, I_4 so obtained with (34), we obtain the result

$$\begin{aligned} v(\xi, \eta) = &-\frac{Kcd}{2}\left(\frac{\eta_1 - \eta}{\eta_1 - \eta_2}\right) + \frac{Kc^2}{2}(\coth \eta - \coth \eta_1) \\ &+ Kcd \sum_{n=1}^{\infty} (-1)^n e^{-n\eta_1} \frac{\sinh n(\eta - \eta_2)}{\sinh n(\eta_1 - \eta_2)} \cos n\xi \\ &- Kc^2\left[\sum_{n=1}^{\infty} (-1)^n e^{-n\eta}\cos n\xi\right](\coth \eta_2 - \coth \eta). \end{aligned} \tag{38}$$

It is easily verified that $v(\xi, \eta_1) \equiv v(\xi, \eta_2) \equiv 0$ if one uses relation (6).

The form (38) may be simplified somewhat by using the following relation, derived in (**2**, p. 1215).

$$1 + 2\sum_{n=1}^{\infty} (-1)^n e^{-n\eta}\cos n\xi = \frac{\sinh \eta}{\cosh \eta + \cos \xi}.$$

With additional simplifications we can then show that

$$\begin{aligned} v(\xi, \eta) = &-\frac{Kcd}{2}\left(\frac{\eta_1 - \eta}{\eta_1 - \eta_2}\right) + \frac{Kc^2}{2}\frac{\sinh \eta(\coth \eta - \coth \eta_1)}{\cosh \eta + \cos \xi} \\ &- Kcd \sum_{n=1}^{\infty} (-1)^n e^{-n\eta_2} \frac{\sinh n(\eta_1 - \eta)}{\sinh n(\eta_1 - \eta_2)} \cos n\xi. \end{aligned} \tag{39}$$

If we let R be the radius of the circle with parameter η, then using (14) we see that as the annular eccentricity $\epsilon \to 0$,

$$\frac{\eta_1 - \eta}{\eta_1 - \eta_2} \to \frac{\log (R/a)}{\log (b/a)}.$$

We note also that, although $c \to \infty$ and $d \to 0$ the product

$$cd \to \tfrac{1}{2}(b^2 - a^2).$$

This follows since

$$\begin{aligned} cd &= c^2[\coth \eta_2 - \coth \eta_1] \\ &= c^2\left[\sqrt{1 + \frac{b^2}{c^2}} - \sqrt{1 + \frac{a^2}{c^2}}\right] = \frac{b^2 - a^2}{\sqrt{1 + \frac{b^2}{c^2}} + \sqrt{1 + \frac{a^2}{c^2}}}. \end{aligned}$$

Thus

$$-\frac{Kcd}{2}\left(\frac{\eta_1 - \eta}{\eta_1 - \eta_2}\right) \to -\frac{K}{4}(b^2 - a^2)\frac{\log (R/a)}{\log (b/a)}. \tag{40}$$

Similarly we can show that as $\epsilon \to 0$

$$\frac{Kc^2}{2}\frac{\sinh \eta(\coth \eta - \coth \eta_1)}{\cosh \eta + \cos \xi} \to \frac{K}{4}(R^2 - a^2).$$

Finally, we note that

$$\left[\sum_{n=1}^{\infty} (-1)^n e^{-n\eta_2} \cos n\xi \frac{\sinh n(\eta_1 - \eta)}{\sinh n(\eta_1 - \eta_2)}\right] < \frac{e^{-\eta_2}}{1 - e^{-\eta_2}} \to 0.$$

Thus, in the limit as $\epsilon \to 0$, the form (39) tends towards the well-known solution for the concentric annulus, namely,

$$v(x, y) = -\frac{K}{4}\left[R^2 - a^2 - (b^2 - a^2)\frac{\log (R/a)}{\log (b/a)}\right].$$

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