3.26 Solution: For the Green function in the form of

$$G(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty} g_l(r,r') P_l(\cos\gamma),$$

the radial part must satisfy the following differential equation,

$$\frac{1}{r'}\frac{\partial^2}{\partial r'^2}\left(r' g_l(r,r')\right) - \frac{l(l+1)}{r'^2} = -\frac{4\pi}{r'^2}\delta(r-r').$$

This equation has a general solution in the form of

$$g_l(r,r') = \frac{r_<^l}{r_>^{l+1}} + \alpha(r) r'^l + \frac{\beta(r)}{r'^{l+1}},$$

with $r_<$ ($r_>$) being the smaller (larger) of r and r'. The first term leads to the delta function, while the other terms can be used to satisfy the boundary conditions.

For Green function with Neumann boundary condition, the normal derivative of the Green function on the boundary must satisfy

$$\oint_S \frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n'} da' = -4\pi,$$

where surface integration is over the entire boundary surface, both the inner and the outer sphere in this case. For the Green function expanded with spherical harmonics, it is clear that only the $l = 0$ term can contribute to the surface integral. Then, for $l = 0$ term, we can impose the simple boundary condition as Eq. (1.45),

$$\frac{\partial g_0(r,r')}{\partial n'} = -\frac{4\pi}{S},$$

and for $l > 0$,

$$\frac{\partial g_l(r,r')}{\partial n'} = 0.$$

With these observations, we can now proceed to solve the Green function.

(a) At $r' = b$, $\partial/\partial n' = \partial/\partial r'$, and

$$\begin{aligned} 0 &= \left.\frac{\partial g_l(r,r')}{\partial n'}\right|_{r'=b} = \frac{\partial}{\partial r'}\left(\frac{r^l}{r'^{l+1}} + \alpha(r) r'^l + \frac{\beta(r)}{r'^{l+1}}\right)\Bigg|_{r'=b} \\ &= -(l+1)\frac{r^l}{b^{l+2}} + l b^{l-1}\alpha(r) - \frac{l+1}{b^{l+2}}\beta(r). \end{aligned}$$

At $r' = a$, $\partial/\partial n' = -\partial/\partial r'$, and

$$\begin{aligned} 0 &= \left.\frac{\partial g_l(r,r')}{\partial n'}\right|_{r'=a} = -\frac{\partial}{\partial r'}\left(\frac{r'^l}{r^{l+1}} + \alpha(r) r'^l + \frac{\beta(r)}{r'^{l+1}}\right)\Bigg|_{r'=a} \\ &= -\left(l\frac{a^{l-1}}{r^{l+1}} + l a^{l-1}\alpha(r) - \frac{l+1}{a^{l+2}}\beta(r)\right). \end{aligned}$$

Then, we have two equations

$$\begin{cases} l\alpha(r) - \dfrac{l+1}{b^{2l+1}}\beta(r) = (l+1)\dfrac{r^l}{b^{2l+1}}, \\ \\ l\alpha(r) - \dfrac{l+1}{a^{2l+1}}\beta(r) = -l\dfrac{1}{r^{l+1}}, \end{cases}$$

which lead to

$$\begin{cases} \alpha(r) = \dfrac{1}{b^{2l+1} - a^{2l+1}} \left(\dfrac{l+1}{l} r^l + \dfrac{a^{2l+1}}{r^{l+1}} \right), \\ \\ \beta(r) = \dfrac{1}{b^{2l+1} - a^{2l+1}} \left(a^{2l+1} r^l + \dfrac{l}{l+1} \dfrac{(ab)^{2l+1}}{r^{l+1}} \right). \end{cases}$$

For $l > 0$, we will have

$$g_l(r, r') = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{b^{2l+1} - a^{2l+1}} \left(\frac{l+1}{l} (rr')^l + \frac{l}{l+1} \frac{(ab)^{2l+1}}{(rr')^{l+1}} + a^{2l+1} \left(\frac{r'^l}{r^{l+1}} + \frac{r^l}{r'^{l+1}} \right) \right).$$

This is already symmetric in r and r'.

(b) For the inner and the outer surface, their total surface area is $S = 4\pi(a^2 + b^2)$, and the boundary condition, for $l = 0$, reads

$$\frac{\partial g_0(r, r')}{\partial n'} = -\frac{1}{a^2 + b^2}.$$

At $r' = b$, $\partial/\partial n' = \partial/\partial r'$, and

$$\left. \frac{\partial g_l(r, r')}{\partial n'} \right|_{r'=b} = \frac{\partial}{\partial r'} \left(\frac{1}{r'} + \alpha(r) + \frac{\beta(r)}{r'} \right) \Bigg|_{r'=b} = -\frac{1}{b^2} - \frac{\beta(r)}{b^2} = -\frac{1}{a^2 + b^2}.$$

At $r' = a$, $\partial/\partial n' = -\partial/\partial r'$, and

$$\left. \frac{\partial g_l(r, r')}{\partial n'} \right|_{r'=a} = -\frac{\partial}{\partial r'} \left(\frac{1}{r} + \alpha(r) + \frac{\beta(r)}{r'} \right) \Bigg|_{r'=a} = \frac{\beta(r)}{a^2} = -\frac{1}{a^2 + b^2}.$$

Both conditions lead to

$$\beta(r) = -\frac{a^2}{a^2 + b^2}.$$

Thus,

$$g_0(r, r') = \frac{1}{r_>} + \alpha(r) - \frac{a^2}{a^2 + b^2} \frac{1}{r'},$$

with an arbitrary function of $\alpha(r)$. Clearly, g_0 is not symmetric in r and r'. Following the recipe prescribed in Problem 1.14 (b), we can define a function $F(r)$ as

$$\begin{aligned} F(r) &= \frac{1}{S} \oint_S g_0(r, r') da' \\ &= \frac{1}{a^2 + b^2} \left[b^2 \left(\frac{1}{b} + \alpha(r) - \frac{a^2}{a^2 + b^2} \frac{1}{b} \right) + a^2 \left(\frac{1}{r} + \alpha(r) - \frac{a^2}{a^2 + b^2} \frac{1}{a} \right) \right] \\ &= \frac{a^2}{a^2 + b^2} \frac{1}{r} + \alpha(r) - \frac{a^3 - b^3}{(a^2 + b^2)^2}, \end{aligned}$$

and the symmetrized radial Green function for $l = 0$ is

$$g_0^S(r, r') = g_0(r, r') - F(r) = \frac{1}{r_>} - \frac{a^2}{a^2 + b^2} \left(\frac{1}{r} + \frac{1}{r'} \right) + \frac{a^3 - b^3}{(a^2 + b^2)^2}.$$