5.11 The loop with normal having spherical angles θ_0 and ϕ_0 can be viewed as rotating a loop lying in the x-y plane by first rotating θ_0 around y-axis, and ϕ_0 around z-axis. Then, any point or direction originally in the x-y plane can be determined by multiplying the vector with the rotation matrix, which is given by

$$T = \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 & 0 \\ \sin\phi_0 & \cos\phi_0 & 0 \\ & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_0 & 0 & \sin\theta_0 \\ 0 & 1 & 0 \\ -\sin\theta_0 & 0 & \cos\theta_0 \end{pmatrix}.$$

Point $a(\cos\phi, \sin\phi, 0)$ becomes

$$u(\phi) = a\begin{pmatrix} \cos\phi\cos\theta_0\cos\phi_0 - \sin\phi\sin\phi_0 \\ \cos\phi\cos\theta_0\sin\phi_0 + \sin\phi\cos\phi_0 \\ -\cos\phi\,\sin\theta_0 \end{pmatrix} = a\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Direction $(-\sin\phi, \cos\phi, 0)$ becomes

$$v(\phi) = \begin{pmatrix} -\sin\phi\cos\theta_0\cos\phi_0 - \cos\phi\sin\phi_0 \\ -\sin\phi\cos\theta_0\sin\phi_0 + \cos\phi\cos\phi_0 \\ \sin\phi\sin\theta_0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\vec{F} = \int \vec{J}(\vec{x})\times\vec{B}(\vec{x})\,d^3x = I\int d\vec{l}\times\vec{B}(\vec{x})d^3x = Ia\int_0^{2\pi}\vec{v}\times\vec{B}(\vec{u}(\phi))\,d\phi$$

$$= IaB_0\int_0^{2\pi}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ 1+\beta a u_2 & 1+\beta a u_1 & 0 \end{vmatrix} d\phi$$

$$= IaB_0\int_0^{2\pi}\Big[-(1+\beta a u_1)v_3\hat{i} + (1+\beta a u_2)v_3\hat{j} + \big[(1+\beta a u_1)v_1 - (1+\beta a u_2)v_2\big]\hat{k}\Big]d\phi$$

First order term in $\cos\phi$ and $\sin\phi$ will drop out

$$\vec{F} = \beta I a^2 B_0\int_0^{2\pi}\left(-u_1v_3\hat{i} + u_2v_3\hat{j} + (u_1v_1 - u_2v_2)\hat{k}\right)d\phi$$

Further, $\sin\phi\cos\phi$ terms will also drop out

$$\vec{F} = \beta I a^2 B_0\int_0^{2\pi}\Big(\sin^2\phi\sin\theta_0\sin\phi_0\hat{i} + \sin^2\phi\sin\theta_0\cos\phi_0\hat{j}$$
$$(\cos 2\phi\cos\theta_0\sin\phi_0\cos\phi_0 + \cos 2\phi\cos\theta_0\sin\phi_0\cos\phi_0)\hat{k}\,d\phi$$

$$= \beta I\pi a^2 B_0\left(\sin\theta_0\cos\phi_0\,\hat{i} + \sin\theta_0\cos\phi_0\,\hat{j}\right)$$

After rotation, $\hat{z}$ direction becomes $\hat{n} = (\sin\theta_0 \cos\phi_0, \sin\theta_0 \sin\phi_0, \cos\theta_0)$.

Then $\vec{F} = \nabla(\vec{m}\cdot\vec{B}) = I\pi a^2 \nabla(\hat{n}\cdot\vec{B})$

$$= I\pi a^2 B_0 \nabla\Big((1+\beta y)\sin\theta_0\cos\phi_0 + (1+\beta x)\sin\theta_0\sin\phi_0 \Big)$$

$$= \beta I\pi a^2 B\Big(\sin\theta_0 \sin\phi_0 \hat{i} + \sin\theta_0\cos\phi_0 \hat{j}\Big)$$

Which agrees with exact calculation

(b) $\vec{N} = \vec{m}\times\vec{B}(0) = I\pi a^2 \hat{n}\times\vec{B}(0)$

$$= I\pi a^2 B \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin\theta_0\cos\phi_0 & \sin\theta_0\sin\phi_0 & \cos\theta_0 \\ 1 & 1 & \end{vmatrix} = I\pi a^2 B\Big(-\cos\theta_0\hat{i} + \cos\theta_0\hat{j} + \sin\theta_0(\cos\phi_0 - \sin\phi_0)\hat{k}\Big)$$