3.25 Solution: (a) Let us focus on the charge density contribution from $\Phi^{(1)}(\rho, z)$,

$$\Phi^{(1)}(\rho, z) = \frac{(E_0 - E_1)a}{\pi}\left[\sqrt{\frac{R-\lambda}{2}} - \frac{|z|}{a}\arctan\left(\sqrt{\frac{2}{R+\lambda}}\right)\right],$$

where

$$\lambda = \frac{1}{a^2}(z^2 + \rho^2 - a^2), \quad R = \sqrt{\lambda^2 + \frac{4z^2}{a^2}}.$$

Then, for the upper surface,

$$\Delta\sigma(\rho)\bigg|_{z=0^+} = -\varepsilon_0 \frac{(E_0 - E_1)a}{\pi}\frac{\partial}{\partial z}\left[\sqrt{\frac{R-\lambda}{2}} - \frac{z}{a}\arctan\left(\sqrt{\frac{2}{R+\lambda}}\right)\right]\bigg|_{z=0^+}.$$

Notice that

$$\sqrt{\frac{2}{R+\lambda}} = \frac{a}{z}\sqrt{\frac{R-\lambda}{2}},$$

we have

$$\begin{aligned}
&\frac{\partial}{\partial z}\left[\sqrt{\frac{R-\lambda}{2}} - \frac{z}{a}\arctan\left(\sqrt{\frac{2}{R+\lambda}}\right)\right] \\
= \quad &\frac{\partial}{\partial z}\sqrt{\frac{R-\lambda}{2}} - \frac{1}{a}\arctan\left(\sqrt{\frac{2}{R+\lambda}}\right) - \frac{R+\lambda}{R+\lambda+2}\frac{\partial}{\partial z}\sqrt{\frac{R-\lambda}{2}} + \frac{1}{z}\frac{R+\lambda}{R+\lambda+2}\sqrt{\frac{R-\lambda}{2}} \\
= \quad &\frac{2}{R+\lambda+2}\frac{\partial}{\partial z}\sqrt{\frac{R-\lambda}{2}} - \frac{1}{a}\arctan\left(\sqrt{\frac{2}{R+\lambda}}\right) + \frac{1}{z}\frac{R+\lambda}{R+\lambda+2}\sqrt{\frac{R-\lambda}{2}}.
\end{aligned}$$

In the limit $z \to 0^+$, $\lambda \to \lambda_0 = (\rho^2 - a^2)/a^2$, and

$$\sqrt{\frac{R-\lambda}{2}} \to \frac{z}{\sqrt{\lambda_0}a},$$

then

$$\begin{aligned}
&\frac{\partial}{\partial z}\left[\sqrt{\frac{R-\lambda}{2}} - \frac{z}{a}\arctan\left(\sqrt{\frac{2}{R+\lambda}}\right)\right] \\
= \quad &\frac{1}{1+\lambda_0}\frac{1}{\sqrt{\lambda_0}a} - \frac{1}{a}\arctan\left(\frac{1}{\sqrt{\lambda_0}}\right) + \frac{1}{1+\lambda_0}\frac{1}{\sqrt{\lambda_0}a}, \\
= \quad &\frac{1}{\sqrt{\lambda_0}a} - \frac{1}{a}\arctan\left(\frac{1}{\sqrt{\lambda_0}}\right).
\end{aligned}$$

However,

$$\arctan\left(\frac{1}{\sqrt{\lambda_0}}\right) = \arctan\left(\frac{a}{\sqrt{\rho^2 - a^2}}\right) = \arcsin\left(\frac{a}{\rho}\right),$$

we finally have

$$\frac{\partial}{\partial z}\left[\sqrt{\frac{R-\lambda}{2}} - \frac{z}{a}\arctan\left(\sqrt{\frac{2}{R+\lambda}}\right)\right]$$

$$= \frac{1}{\sqrt{\rho^2 - a^2}} - \frac{1}{a}\arcsin\left(\frac{a}{\rho}\right).$$

Then,

$$\Delta\sigma(\rho)\bigg|_{z=0^+} = -\varepsilon_0 \frac{E_0 - E_1}{\pi}\left[\frac{a}{\sqrt{\rho^2 - a^2}} - \arcsin\left(\frac{a}{\rho}\right)\right].$$

For $z \to 0^-$, we can perform the same calculation and the result is the same.

(b) The integral can be written as

$$2\pi\varepsilon_0 \int_0^a \rho(E_0 - E_1)d\rho + 2\pi \int_a^R \rho\left[(\sigma_+ + \sigma_-) + \varepsilon_0(E_0 - E_1)\right]d\rho,$$

where the first integral is simply

$$\varepsilon_0(E_0 - E_1)\cdot \pi a^2$$

Since

$$\sigma_+ + \sigma_- = -\varepsilon_0(E_0 - E_1) - 2\varepsilon_0 \frac{E_0 - E_1}{\pi}\left[\frac{a}{\sqrt{\rho^2 - a^2}} - \arcsin\left(\frac{a}{\rho}\right)\right],$$

the second part of the integral becomes

$$-4\varepsilon_0(E_0 - E_1)\int_a^R \rho\left[\frac{a}{\sqrt{\rho^2 - a^2}} - \arcsin\left(\frac{a}{\rho}\right)\right]d\rho.$$

Integrate by parts,

$$\int_a^R \rho\arcsin\left(\frac{a}{\rho}\right)d\rho = \frac{1}{2}\rho^2\arcsin\left(\frac{a}{\rho}\right)\bigg|_{\rho=a}^R + \frac{1}{2}\int_a^R \frac{\rho a}{\sqrt{\rho^2 - a^2}}d\rho,$$

then

$$\begin{aligned}\int_a^R \rho\left[\frac{a}{\sqrt{\rho^2 - a^2}} - \arcsin\left(\frac{a}{\rho}\right)\right]d\rho &= \frac{1}{2}\int_a^R \frac{\rho a}{\sqrt{\rho^2 - a^2}} - \frac{1}{2}\rho^2\arcsin\left(\frac{a}{\rho}\right)\bigg|_{\rho=a}^R \\ &= \left(\frac{1}{2}a\sqrt{\rho^2 - a^2}d\rho - \frac{1}{2}\rho^2\arcsin\left(\frac{a}{\rho}\right)\right)\bigg|_{\rho=a}^R\end{aligned}$$

As $R \to \infty$, $\sqrt{R^2 - a^2} \to R$, and

$$\frac{1}{2}R^2\arcsin\left(\frac{a}{R}\right) \to \frac{1}{2}R^2\cdot\frac{a}{R} = \frac{1}{2}aR,$$

and, therefore,

$$-4\varepsilon_0(E_0 - E_1)\int_a^R \rho\left[\frac{a}{\sqrt{\rho^2 - a^2}} - \arcsin\left(\frac{a}{\rho}\right)\right]d\rho = -\varepsilon_0(E_0 - E_1)\cdot\pi a^2,$$

which clearly cancels the first integral and leads to a final result of 0. This is due to the fact that the total induced charge on the plate should be zero.