3.20 Solution: (a) For this problem, we can use the Green function from Problem 3.17 (a),

$$G(\mathbf{x},\mathbf{x}') = \frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\sin\left(\frac{n\pi}{L}z\right)\sin\left(\frac{n\pi}{L}z'\right)I_m\left(\frac{n\pi}{L}\rho_<\right)K_m\left(\frac{n\pi}{L}\rho_>\right),$$

and Eq. (1.42),

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x'.$$

Here, in this problem, the charge density $\rho(\mathbf{x}')$ is a point charge located at $z' = z_0$ and $\rho' = 0$. Therefore, in the Green function, we should have $\rho_< = 0$ and $\rho_> = \rho$. Performing the integration, only $m = 0$ term will contribute. Also, notice that $I_0(0) = 1$, the potential becomes

$$\Phi(z,\rho) = \frac{q}{\pi\varepsilon_0 L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}z_0\right)\sin\left(\frac{n\pi}{L}z\right)K_0\left(\frac{n\pi}{L}\rho\right).$$

(b) For the surface charge density on the upper plate, the normal vector is in the $-\hat{z}$ direction. Therefore,

$$\begin{aligned}
\sigma_L(\rho) &= -\varepsilon_0\left.\frac{\partial\Phi}{\partial n}\right|_{z=L} = \varepsilon_0\left.\frac{\partial\Phi}{\partial z}\right|_{z=L} \\
&= \frac{q}{\pi L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}z_0\right)\cdot\frac{n\pi}{L}\cos\left.\left(\frac{n\pi}{L}z\right)\right|_{z=L}\cdot K_0\left(\frac{n\pi}{L}\rho\right) \\
&= \frac{q}{L^2}\sum_{n=1}^{\infty}(-1)^n n\sin\left(\frac{n\pi}{L}z_0\right)K_0\left(\frac{n\pi}{L}\rho\right),
\end{aligned}$$

since $\cos(n\pi) = (-1)^n$ for integer n.

Similarly, for the lower plate,

$$\begin{aligned}
\sigma_0(\rho) &= -\varepsilon_0\left.\frac{\partial\Phi}{\partial n}\right|_{z=0} = -\varepsilon_0\left.\frac{\partial\Phi}{\partial z}\right|_{z=0} \\
&= -\frac{q}{\pi L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}z_0\right)\cdot\frac{n\pi}{L}\cos\left.\left(\frac{n\pi}{L}z\right)\right|_{z=0}\cdot K_0\left(\frac{n\pi}{L}\rho\right) \\
&= -\frac{q}{L^2}\sum_{n=1}^{\infty}n\sin\left(\frac{n\pi}{L}z_0\right)K_0\left(\frac{n\pi}{L}\rho\right).
\end{aligned}$$

(c) The total charge on the upper plate can be calculated by direct integration,

$$\begin{aligned}
Q_L &= \frac{2\pi q}{L^2}\sum_{n=1}^{\infty}(-1)^n n\sin\left(\frac{n\pi}{L}z_0\right)\int_0^{\infty}\rho K_0\left(\frac{n\pi}{L}\rho\right)d\rho \\
&= \frac{2q}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi}{L}z_0\right)\int_0^{\infty}\lambda K_0(\lambda)\,d\lambda \\
&= \frac{2q}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi}{L}z_0\right),
\end{aligned}$$

since

$$\int_0^\infty \lambda K_0(\lambda) d\lambda = 1.$$

Now,

$$\begin{aligned}
Q_L &= \frac{2q}{\pi} \mathrm{Im} \left\{ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \exp\left(\frac{in\pi z_0}{L} \right) \right\} \\
&= -\frac{2q}{\pi} \mathrm{Im} \left[\log\left(1 + e^{i\pi z_0/L} \right) \right] \\
&= -\frac{2q}{\pi} \arctan\left(\frac{\sin(\pi z_0/L)}{1+\cos(\pi z_0/L)} \right) \\
&= -\frac{2q}{\pi} \arctan\left(\tan\left(\frac{\pi z_0}{2L} \right) \right) \\
&= -\frac{2q}{\pi} \cdot \frac{\pi z_0}{2L} \\
&= -\frac{z_0}{L} q.
\end{aligned}$$