6.18 (a) Given a point $\vec{x}' = (0, 0, z')$ on the negative z-axis, $z'<0$, $d\vec{l}' = dz'\hat{k}$. and

$\vec{x} - \vec{x}' = (x, y, z-z')$, then

$$d\vec{l}' \times (\vec{x}-\vec{x}') = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & dz' \\ x & y & z-z' \end{vmatrix} = (-y\hat{i} + x\hat{j})\, dz'.$$

The vector potential can be calculated as

$$\vec{A}(\vec{x}) = \frac{g}{4\pi}\int \frac{d\vec{l}'\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = \frac{g}{4\pi}(-y\hat{i}+x\hat{j})\int_{-\infty}^{0} \frac{dz'}{[x^2+y^2+(z-z')^2]^{3/2}}$$

$$= \frac{g}{4\pi}(-y\hat{i}+x\hat{j})\frac{1}{x^2+y^2}\left.\frac{z'-z}{(x^2+y^2+(z'-z)^2)^{1/2}}\right|_{-\infty}^{0}$$

$$= \frac{g}{4\pi}(-y\hat{i}+x\hat{j})\frac{1}{x^2+y^2}\left(1-\frac{z}{(x^2+y^2+z^2)^{1/2}}\right).$$

Using spherical coordinates, the vector potential becomes

$$\vec{A}(\vec{x}) = \frac{g}{4\pi} r\sin\theta(-\cos\phi\,\hat{i}+\sin\phi\,\hat{j})\frac{1}{r^2\sin^2\theta}\left(1-\frac{r\cos\theta}{r}\right) = \frac{g}{4\pi}\frac{-\cos\phi\,\hat{i}+\sin\phi\,\hat{j}}{r\sin\theta}(1-\cos\theta)$$

$$= \frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\hat{\phi} = \frac{g}{4\pi r}\tan\left(\frac{\theta}{2}\right)\hat{\phi}.$$

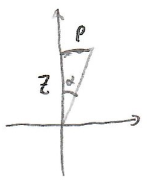
where we have used the relation $\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

(b) $$\vec{B} = \nabla\times\vec{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta} = \frac{g}{4\pi r^2}\hat{r}$$

This resembles the electric field generated by a point charge. Note that the vector potential is singular at $\theta=\pi$, or on the negative z-axis, as our integration path is along the z-axis.

Therefore, at $\theta=\pi$, the magnetic field is not well defined.

(c) For $0<\theta<\pi/2$, on the disk, for a point at $(\rho,\phi)$, the magnetic flux at this point is




$$\vec{B}\cdot\vec{n}\,da = \frac{g}{4\pi(\rho^2+z^2)}\cos\alpha\cdot\rho\,d\rho\,d\phi = \frac{gz}{4\pi(\rho^2+z^2)^{3/2}}\rho\,d\rho\,d\phi.$$ Therefore, the total magnet flux through the disk is

$$\Phi_> = \oint \vec{B}\cdot\vec{n}\,da = \int_0^{2\pi}d\phi\int_0^{R\sin\theta}d\rho\,\frac{gz\rho}{4\pi(\rho^2+z^2)^{3/2}} = -\frac{gz}{2}\left.\frac{1}{(\rho^2+z^2)^{1/2}}\right|_0^{R\sin\theta}$$

$$= \frac{g}{2}\left(1 - \frac{z}{\sqrt{z^2+R^2\sin^2\theta}}\right) = \frac{g}{2}\left(1-\frac{z}{R}\right)$$

For $z<0$, the downward flux is given by the same result, with $z$ replaced by $-z$.

Then, the upward flux is the negative of it.

$$\Phi_r = -\frac{g}{2}\left(1+\frac{z}{\sqrt{z^2+R^2\sin^2\theta}}\right) = -\frac{g}{2}\left(1+\frac{z}{R}\right)$$

(d) Since $\vec{A}$ has only $\hat{\phi}$ component, the line integral is straightforward

$$\oint \vec{A}\cdot d\vec{l} = \int_0^{2\pi} \frac{g}{4\pi R}\cdot\frac{1-\cos\theta}{\sin\theta}\cdot R\sin\theta\, d\phi$$

$$= \frac{g}{2}(1-\cos\theta) = \frac{g}{2}\left(1-\frac{z}{R}\right)$$

Comparing with the results from part (c), we can see that the magnetic flux calculated in this way agrees that with $\theta<\pi/2$, but differs a constant of $g$ for $\theta>\pi/2$.

The difference is due to the loop around the negative $z$-axis also encloses the string, which is singular in the contribution to the magnetic induction and thus the flux.