1.10 Solution: From Eq. (1.36), without loss of generality, set $\mathbf{x} = 0$,

$$\Phi(\mathbf{0}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\mathbf{x}')}{R} d^3x' + \frac{1}{4\pi} \oint_S \left[\frac{1}{R} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left(\frac{1}{R} \right) \right] da', \tag{1}$$

with $R = |\mathbf{x}'| = x'$. On the right hand side of Eq. (1), the first term vanishes, as there is no charge in the space of interest, *i.e.*, $\rho(\mathbf{x}') \equiv 0$. Then,

$$\Phi(\mathbf{0}) = \frac{1}{4\pi} \oint_S \left[\frac{1}{R} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left(\frac{1}{R} \right) \right] da'. \tag{2}$$

To evaluation Eq. (2), consider a sphere enclosing the origin with radius r. The first term can be written as

$$\frac{1}{4\pi} \oint_S \frac{1}{R} \frac{\partial \Phi}{\partial n'} da' = \frac{1}{4\pi r} \oint_S \nabla \Phi \cdot \mathbf{n}' da' = \frac{1}{4\pi r} \int_V \nabla^2 \Phi d^3x' = -\frac{1}{4\pi\varepsilon_0 r} \int_V \rho(\mathbf{x}') d^3x' = 0.$$

Here, we have used the fact that $R = r$ on the surface of the sphere. Also, the surface integral evaluates to 0 by the application of divergence theorem and noticing that there is no charge inside the sphere.

For the second term of Eq. (2), the normal vector points away from the origin, and on the surface of the sphere,

$$\frac{\partial}{\partial n'} \left(\frac{1}{R} \right) = \frac{\partial}{\partial r} \left(\frac{1}{r} \right) = -\frac{1}{r^2}.$$

The surface integral can be directly evaluated,

$$-\frac{1}{4\pi} \oint_S \Phi \frac{\partial}{\partial n'} \left(\frac{1}{R} \right) da' = \frac{1}{4\pi} \oint_S \frac{\Phi}{r^2} da' = \frac{1}{4\pi} \int \frac{\Phi}{r^2} r^2 d\Omega' = \frac{1}{4\pi} \int \Phi d\Omega',$$

which is the average of the potential on the surface of the enclosing sphere. Therefore, we have established the mean value theorem.