8.12 (a) Here, we choose the normal direction to be inwardly pointing. Given the unperturbed solution, the perturbed mode is given by $\psi = \psi_0 + \frac{\partial \psi_0}{\partial n}\delta(x,y)$, where $\delta(x,y)$ is positive if the deformation is away from the surface. For the new mode, the eigenequation is $(\nabla_t^2 + \gamma^2)\psi = 0$.

Applying the Green's theorem on ψ and ψ_0, we have

$$\int_{S_0} (\psi \nabla_t^2 \psi_0^* - \psi_0^* \nabla_t^2 \psi)\, da = \oint_{C_0} \left(\psi_0^* \frac{\partial \psi}{\partial n} - \psi \frac{\partial \psi_0^*}{\partial n}\right) dl.$$

with inward normal direction. The L.H.S. of the equation is

$$(\gamma^2 - \gamma_0^2)\int_{S_0} \psi\psi_0^*\, da = (\gamma^2 - \gamma_0^2)\int_{S_0}\left(\psi_0 + \frac{\partial \psi_0}{\partial n}\delta(x,y)\right)\psi_0^*\, da \simeq (\gamma^2-\gamma_0^2)\int_{S_0}|\psi_0|^2 da,$$

The R.H.S. becomes

$$\oint_{C_0}\left[\psi_0^*\frac{\partial \psi_0}{\partial n} + \psi_0^* \frac{\partial^2 \psi_0}{\partial n^2}\delta(x,y) - \psi_0 \frac{\partial \psi_0^*}{\partial n} - \left|\frac{\partial \psi_0}{\partial n}\right|^2 \delta(x,y)\right] dl$$

$$= -\oint_{C_0} \delta(x,y)\left[\left|\frac{\partial \psi_0}{\partial n}\right|^2 - \psi_0^* \frac{\partial^2 \psi_0}{\partial n^2}\right] dl,$$

where terms in the form of $\psi_0^* \frac{\partial \psi_0}{\partial n}$ and $\psi_0 \frac{\partial \psi_0^*}{\partial n}$ vanish due to the boundary condition.

Then,

$$\gamma^2 - \gamma_0^2 = -\frac{\oint_{C_0}\delta(x,y)\left[\left|\frac{\partial \psi_0}{\partial n}\right|^2 - \psi_0^*\frac{\partial^2\psi_0}{\partial n^2}\right] dl}{\int_{S_0}|\psi_0|^2 da}$$

(b) It is straightforward to show that, for TM_{mn} and TE_{mn} modes,

$$\int_{S_0}|\psi_0|^2 da = \begin{cases} ab/4, & m\neq 0,\ n\neq 0 \\ ab/2, & m=0 \text{ or } n=0 \end{cases}$$

Now, for TM_{mn} mode, $m\neq 0$, $n \neq 0$, and $\psi_0(x,y) = \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$ Only the first term in the numerator will contribute. For this configuration,

$$\delta(x,y) = \frac{y}{b}\delta$$

and

$$\frac{\partial \psi_0}{\partial n} = \begin{cases} \left.\frac{\partial \psi_0}{\partial x}\right|_{x=0} = \frac{m\pi}{a}\sin\left(\frac{n\pi y}{b}\right). \\ -\left.\frac{\partial \psi_0}{\partial x}\right|_{x=a} = -\frac{m\pi}{a}\sin\left(\frac{n\pi y}{b}\right) \end{cases}$$

The line integral is given by

$$\oint_{C_0} \delta(x,y) \left|\frac{\partial \psi_0}{\partial n}\right|^2 dl = 2\int_0^b \frac{y}{b}\delta \cdot \frac{m^2\pi^2}{a^2} \cdot \sin^2\left(\frac{n\pi y}{b}\right) dy = \frac{2\delta}{b} \cdot \frac{m^2\pi^2}{a^2} \cdot \frac{b^2}{4} = \frac{b\pi^2\delta}{2a^2} m^2.$$

The correction is

$$\gamma^2 - \gamma_0^2 = -\frac{b\pi^2\delta}{2a^2} m^2 \Big/ \frac{ab}{4} = -\frac{2\pi^2\delta}{a^3} m^2,$$

and $\gamma^2 = \gamma_0^2 - \frac{2\pi^2\delta}{a^3} = \left(\frac{1}{a^2} + \frac{1}{b^2}\right)\pi^2 - \frac{2\delta}{a^3}\pi^2$, for $TM_{1,1}$ mode.

For $TE_{1,0}$ mode, $\psi_0(x,y) = \cos\left(\frac{\pi x}{a}\right)$, and

$$\frac{\partial^2 \psi}{\partial n^2} = \frac{\partial^2 \psi}{\partial x^2} = -\frac{\pi^2}{a^2}\cos\left(\frac{\pi x}{a}\right) = \begin{cases} -\pi^2/a^2, & x = 0 \\ \pi^2/a^2, & x = a \end{cases}$$

Also, $\psi_0\big|_{x=0} = 1$, $\psi_0\big|_{x=a} = -1$, the line integral is

$$\oint_{C_0} \delta(x,y)\, \psi_0^* \frac{\partial^2 \psi_0}{\partial n^2}\, dl = -2\int_0^b \frac{y}{b}\delta \cdot \frac{\pi^2}{a^2}\, dy = -\frac{b\pi^2\delta}{a^2}.$$

And the correction is

$$\gamma^2 - \gamma_0^2 = -\frac{b\pi^2\delta}{a^2} \Big/ \frac{ab}{2} = -\frac{2\pi^2\delta}{a^3}.$$

and $\gamma^2 = \gamma_0^2 - \frac{2\pi^2\delta}{a^3} = \left(\frac{1}{a^2} - \frac{2\delta}{a^3}\right)\pi^2$