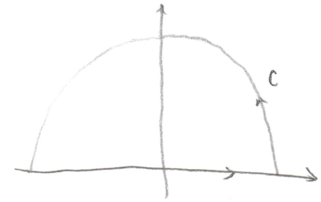
7.23 Consider the function $\varepsilon(\omega) - i\sigma/\omega$, which is analytic in the upper half plane.

Define $f(z) = \frac{1}{z-\omega+i0^+}\left(\frac{\varepsilon(z)}{\varepsilon_0} - 1 - \frac{i\sigma}{\varepsilon_0 z}\right)$ perform the integral along the contour shown in the figure. Clearly, $\oint_c f(z)\,dz = 0$, as no poles exist in the upper half plane, and the integral on the large semi-circle at infinity is also 0. Then,



$$\oint_c f(z)\,dz = 0 = \int_{-\infty}^{+\infty} f(z)\,dz = P\int_{-\infty}^{+\infty} \frac{1}{z-\omega}\left(\frac{\varepsilon(z)}{\varepsilon_0} - 1 - \frac{i\sigma}{\varepsilon_0 z}\right)dz$$

$$- i\pi \int_{-\infty}^{+\infty} \delta(z-\omega)\left(\frac{\varepsilon(z)}{\varepsilon_0} - 1 - \frac{i\sigma}{\varepsilon_0 z}\right)dz$$

$$= P\int_{-\infty}^{+\infty} \frac{1}{z-\omega}\left(\frac{\varepsilon(z)}{\varepsilon_0} - 1 - \frac{i\sigma}{\varepsilon_0 z}\right)dz - i\pi\left(\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 - \frac{i\sigma}{\varepsilon_0\omega}\right)$$

$$= P\int_{-\infty}^{+\infty} \frac{1}{z-\omega}\left[\mathrm{Re}\left(\frac{\varepsilon(z)}{\varepsilon_0}\right) - 1 + i\left(\mathrm{Im}\left(\frac{\varepsilon(z)}{\varepsilon_0}\right) - \frac{\sigma}{\varepsilon_0 z}\right)\right]dz$$

$$+ \pi\left[\mathrm{Im}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right) - \frac{\sigma}{\varepsilon_0\omega} - i\left(\mathrm{Re}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right) - 1\right)\right]$$

Therefore, $\mathrm{Re}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right) - 1 = \frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{1}{z-\omega}\left[\mathrm{Im}\left(\frac{\varepsilon(z)}{\varepsilon_0}\right) - \frac{\sigma}{\varepsilon_0 z}\right]dz$

and $\mathrm{Im}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right) - \frac{\sigma}{\varepsilon_0\omega} = -\frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{1}{z-\omega}\left[\mathrm{Re}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right) - 1\right]dz.$

Notice that $P\int_{-\infty}^{+\infty} \frac{1}{z-\omega}\frac{1}{z}\,dz = 0$, the real part of the KK relation becomes

$\mathrm{Re}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right) - 1 = \frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{\mathrm{Im}(\varepsilon(z)/\varepsilon_0)}{z-\omega}\,dz$. Using the oddness of the imaginary part of $\varepsilon(\omega)$, we can see that this identical to the original KK relation. The imaginary part is given by

$$\mathrm{Im}\left(\frac{\varepsilon(\omega)}{\varepsilon_0}\right) = \frac{\sigma}{\varepsilon_0\omega} - \frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{\mathrm{Re}[\varepsilon(\omega)/\varepsilon_0 - 1]}{z-\omega}\,dz$$

or $$\mathrm{Im}\,\varepsilon(\omega) = \frac{\sigma}{\omega} - \frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{\mathrm{Re}[\varepsilon(\omega) - \varepsilon_0]}{z-\omega}\,dz.$$