16.4 (a) Using Eq. (16.30), and also notice the expansion of the Green function for Helmholtz wave equation in spherical harmonics, Eq. (9.98),

$$\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} = 4\pi i k \sum_{l=0}^{\infty} j_l(kr_<) h_l^{(1)}(kr_>) \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi),$$

the mass $M(\omega)$ can be written as

$$M(\omega) = m_0 + \frac{2}{3c^2}\int d^3x \int d^3x' \, 4\pi i \frac{\omega}{c} \sum_{l=0}^{\infty} j_l\left(\frac{\omega a}{c}\right) h_l^{(1)}\left(\frac{\omega a}{c}\right) \sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi)\, \rho(\vec{x}')\rho(\vec{x}),$$

where $\rho(\vec{x}) = \frac{e}{4\pi a^2}\delta(r-a)$. Integration with respect to the solid angle of either $\vec{x}$ or $\vec{x}'$ will only leave the $l=0$ term in the sum, consequently, $m=0$. Therefore,

$$M(\omega) = m_0 + \frac{2e^2}{3c^2} i\frac{\omega}{c} j_0\left(\frac{\omega a}{c}\right) h_0^{(1)}\left(\frac{\omega a}{c}\right) = m_0 + \frac{2e^2}{3c^2}\cdot i\frac{\omega}{c}\frac{\sin(\omega a/c)}{\omega a/c}\frac{e^{i\omega a/c}}{i\omega a/c}$$

$$= m_0 + \frac{2e^2}{3ac^2}\frac{\sin(\omega a/c)\, e^{i\omega a/c}}{\omega a/c} = m_0 + \frac{2e^2}{3ac^2}\frac{e^{2i\omega a/c}-1}{2i\omega a/c} = m_0 + \frac{2e^2}{3ac^2}\frac{e^{i\xi}-1}{i\xi},$$

with $\xi = 2\omega a/c$. We want to express $M(\omega)$ in terms of m. From charge density expression, the form factor is

$$f(\vec{k}) = \frac{1}{e}\int \rho(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} d^3x = \frac{1}{2}\int_{-1}^{1} e^{-ika\cos\theta} d(\cos\theta) = \frac{\sin(ka)}{ka}$$

Then,
$$m = m_0 + \frac{e^2}{3\pi^2c^2}\int \frac{|f(\vec{k})|^2}{k^2} d^3k = m_0 + \frac{4e^2}{3\pi c^2}\int_0^{+\infty}\frac{\sin^2(ka)}{k^2a^2}dk = m_0 + \frac{2e^2}{3ac^2}.$$

where we have used the identity $\int_0^{+\infty}\frac{\sin^2(ka)}{k^2a^2}dk = \frac{\pi}{2a}$. Thus, $m_0 = m - \frac{2e^2}{3ac^2}$, and

$$M(\omega) = m + \frac{2e^2}{3ac^2}\left(\frac{e^{i\xi}-1}{i\xi} - 1\right) = m + \frac{2e^2}{3ac^2}\frac{e^{i\xi}-1-i\xi}{i\xi}.$$

(b) Since, $e^{i\xi} = 1 + i\xi + \frac{1}{2}(i\xi)^2 + \cdots$, to the lowest order,

$$M(\omega) = m + \frac{2e^2}{3ac^2}\cdot\frac{i\xi}{2} = m + i\omega\cdot\frac{2e^2}{3c^3} = m(1+i\omega\tau).$$

The mass $M(\omega)$ becomes 0 when $1+i\omega\tau = 0$, or $\omega\tau = i$. This is in agreement with point charge situation, where $a\to 0$. This leads to the runaway solution to the Abraham-Lorentz equation.

(c) We can equivalently consider the zeros of $i\xi M(\omega)$, which is

$$im\xi + \frac{2}{3}\frac{e^2}{ac^2}\left(e^{i\xi}-1-i\xi\right)=0, \quad \text{or} \quad i\xi + \frac{c\tau}{a}\left(e^{i\xi}-1-i\xi\right)=0$$

Let $\xi = x+iy$, the above equation can be expressed as two independent equations, from

$$ix - y + \frac{c\tau}{a}\left(e^{-y}\cos x + ie^{-y}\sin x - 1 - ix + y\right) = 0,$$

leading to

$$e^{-y}\cos x - 1 + y\left(1-a/(c\tau)\right) = 0$$

$$e^{-y}\sin x - x\left(1-a/(c\tau)\right) = 0.$$

For the second equation, we know $e^{-y} = \frac{x}{\sin x}\left(1-a/c\tau\right)$. Since $x/\sin x$ takes the minimum positive value at $x=0$ with 1, if $0<1-a/c\tau<1$ $(a>0)$, then we will have a solution with $y>0$. Therefore, we must have $1-a/c\tau<0$, or $a>c\tau$.