3.7 Solution: (a) Similar to Problem 3.6, then the electrostatic potential is

$$\begin{aligned}
\Phi(\mathbf{x}) &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}-\frac{2}{|\mathbf{x}|}+\frac{1}{|\mathbf{x}+\mathbf{x}'|}\right) \\
&= \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{x_<^l}{x_>^{l+1}}\left(P_l(\cos\theta)+P_l\left(\cos(\pi-\theta)\right)\right)-\frac{q}{2\pi\varepsilon_0 x} \\
&= \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{x_<^l}{x_>^{l+1}}\left(P_l(\cos\theta)+P_l\left(-\cos\theta\right)\right)-\frac{q}{2\pi\varepsilon_0 x} \\
&= \frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{x_<^l}{x_>^{l+1}}\left(1+(-1)^l\right)P_l(\cos\theta)-\frac{q}{2\pi\varepsilon_0 x} \\
&= \frac{q}{2\pi\varepsilon_0}\sum_{j=0}^{\infty}\frac{x_<^{2j}}{x_>^{2j+1}}P_{2j}(\cos\theta)-\frac{q}{2\pi\varepsilon_0 x}.
\end{aligned}$$

In the limit of $a\to 0$ with qa^2 staying constant, only the P_2 component will remain. In this case, with $x_< = a$ and $x_> = x$, the potential becomes

$$\Phi(\mathbf{x}) = \frac{q}{2\pi\varepsilon_0}\left(\frac{1}{x}+\frac{a^2}{x^3}P_2(\cos\theta)\right)-\frac{q}{2\pi\varepsilon_0 x} = \frac{Q}{2\pi\varepsilon_0 x^3}P_2(\cos\theta).$$

(b) Due to the presence of the conducting sphere, inside it, similar to Prob. 3.6 (c), there will be extra contributions from it,

$$\Phi'(\mathbf{x}) = \frac{q}{2\pi\varepsilon_0}\sum_{j=0}^{\infty}\frac{x_<^{2j}}{x_>^{2j+1}}P_{2j}(\cos\theta)-\frac{q}{2\pi\varepsilon_0 x}+\sum_{l=0}^{\infty}A_l\frac{x^l}{b^{l+1}}P_l(\cos\theta).$$

On the sphere, $x = x_> = b$ and $x_< = a$, it can be easily verified that $A_l \equiv 0$, for odd l. For $l = 2j$ even, we will have

$$A_{2j} = -\frac{q}{2\pi\varepsilon_0}\left(\frac{a}{b}\right)^{2j}.$$

Therefore, the potential inside the sphere can be expressed as

$$\Phi'(\mathbf{x}) = \frac{q}{2\pi\varepsilon_0}\left[\left(\frac{1}{x_>}-\frac{1}{x}\right)+\sum_{j=1}^{\infty}\left(\frac{x_<^{2j}}{x_>^{2j+1}}-\frac{(ax)^{2j}}{b^{4j+1}}\right)P_{2j}(\cos\theta)\right].$$

For $x > a$, we have $x_< = a$ and $x_> = x$, the above potential becomes

$$\Phi'(\mathbf{x}) = \frac{q}{2\pi\varepsilon_0}\sum_{j=1}^{\infty}\left(\frac{a^{2j}}{x^{2j+1}}-\frac{(ax)^{2j}}{b^{4j+1}}\right)P_{2j}(\cos\theta) = \frac{q}{2\pi\varepsilon_0}\sum_{j=1}^{\infty}\frac{a^{2j}}{x^{2j+1}}\left(1-\left(\frac{x}{b}\right)^{4j+1}\right)P_{2j}(\cos\theta).$$

In the limit $a\to 0$ but $qa^2 = Q$ finite, only the $j = 1$ term will survive, and the potential becomes

$$\frac{Q}{2\pi\varepsilon_0 x^3}\left(1-\left(\frac{x}{b}\right)^5\right)P_2(\cos\theta).$$