13.10 (a) Using the expression (13.25), $\Phi(\vec{k},\omega) = \frac{2ze}{\varepsilon(\omega)} \frac{\delta(\omega - \vec{k}\cdot\vec{v})}{k^2 - \frac{\omega^2}{c^2}\varepsilon(\omega)}$, and taking the Fourier transform, we have

$$\Phi(\omega,\vec{x}) = \frac{1}{(2\pi)^{3/2}}\int d^3k\, e^{i\vec{k}\cdot\vec{x}}\,\Phi(\vec{k},\omega) = \frac{1}{(2\pi)^{3/2}}\frac{2ze}{\varepsilon(\omega)}\int dk_x\int dk_y\int dk_z\, e^{i(k_x x + k_y y + k_z z)}\frac{\delta(\omega - k_z v)}{k^2 - \frac{\omega^2}{c^2}\varepsilon(\omega)}$$

$$= \frac{ze}{v\varepsilon(\omega)}\frac{1}{\sqrt{2\pi^3}}\int dk_x\int dk_y\, e^{i\omega z/v}\frac{e^{i(k_x x + k_y y)}}{k_x^2 + k_y^2 + \frac{\omega^2}{v^2} - \frac{\omega^2}{c^2}\varepsilon(\omega)}$$

$$= \frac{ze}{v\varepsilon(\omega)}\frac{1}{\sqrt{2\pi^3}}\int_0^{2\pi} d\theta\int_0^{+\infty} k_\perp dk_\perp\,\frac{e^{ik_\perp\rho\cos\theta}}{k_\perp^2 + \frac{\omega^2}{v^2}(1-\beta^2\varepsilon(\omega))}\, e^{i\omega z/v}$$

$$= \frac{ze}{v\varepsilon(\omega)}\sqrt{\frac{2}{\pi}}\int_0^{+\infty}\frac{k_\perp J_0(k_\perp\rho)}{k_\perp^2 + \frac{\omega^2}{v^2}(1-\beta^2\varepsilon(\omega))}\,dk_\perp \cdot e^{i\omega z/v} \qquad (1)$$

$$= \frac{ze}{v\varepsilon(\omega)}\sqrt{\frac{2}{\pi}}\,K_0\!\left(\frac{|\omega|\rho}{v}\sqrt{1-\beta^2\varepsilon(\omega)}\right)e^{i\omega z/v}, \qquad (2)$$

Where we have use the series $e^{iz\cos\theta} = \sum_{n=-\infty}^{+\infty} i^n J_n(\theta)\, e^{in\theta}$ to obtain (1), and

$\int_0^{+\infty}\frac{x J_0(\alpha x)}{x^2+\beta^2}\,dx = K_0(\alpha\beta)$ to obtain (2).

(b) If $\varepsilon(\omega)$ is a constant and $1-\beta^2\varepsilon > 0$, then

$$\Phi(\vec{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega\,\Phi(\omega,\vec{x})\,e^{-i\omega t} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega\,\frac{ze}{v\varepsilon}\sqrt{\frac{2}{\pi}}\,K_0\!\left(\frac{|\omega|\rho}{v}\sqrt{1-\beta^2\varepsilon}\right)e^{i\omega(z/v - t)}$$

$$= \frac{2ze}{v\varepsilon\pi}\int_0^{+\infty} d\omega\, K_0\!\left(\frac{\omega\rho}{v}\sqrt{1-\beta^2\varepsilon}\right)\cos\!\left(\omega\left(\frac{z}{v}-t\right)\right)$$

$$= \frac{2ze}{v\varepsilon\pi}\cdot\frac{\pi}{2}\left[\frac{\rho^2}{v^2}(1-\beta^2\varepsilon) + \left(\frac{z}{v}-t\right)^2\right]^{-1/2}$$

$$= \frac{ze}{\varepsilon}\frac{1}{\left[(z-vt)^2 + \rho^2(1-\beta^2\varepsilon)\right]^{1/2}},$$

where we have used the identity $\int_0^{+\infty} dx\, K_0(\alpha x)\cos(\beta x) = \frac{\pi}{2\sqrt{\alpha^2+\beta^2}}$. Define

$\Gamma = (1-\beta^2\varepsilon)^{-1/2}$, we can write the potential as

$$\Phi(\vec{x},t) = \frac{ze\Gamma}{\varepsilon\left[\rho^2 + \Gamma^2(z-vt)^2\right]^{1/2}}.$$

This result is almost identical to (11.152), except the inclusion of the dielectric constant, which redefines γ. The result also reduces to (11.152) as $\varepsilon \to 1$

(c) When $\beta^2\varepsilon > 1$, we have

$$\Phi(\vec{x},\omega) = \frac{ze}{v\varepsilon(\omega)}\sqrt{\frac{2}{\pi}}\, e^{i\omega z/v} K_0\left(\pm \frac{i|\omega|\rho}{v}\sqrt{\beta^2\varepsilon - 1}\right)$$

for $\omega \gtrless 0$. The choice of the phase factor is to ensure that $\Phi(\vec{x},\omega)^* = \Phi(\vec{x},-\omega)$. Notice that $K_0(\pm i x) = \frac{\pi}{2}\left[-N_0(x) \mp i J_0(x)\right]$, we have

$$\Phi(\vec{x},\omega) = \frac{ze}{v\varepsilon(\omega)}\sqrt{\frac{\pi}{2}}\, e^{i\omega z/v}\left[-N_0\left(\frac{|\omega|\rho}{v}\sqrt{\beta^2\varepsilon-1}\right) \mp i J_0\left(\frac{|\omega|\rho}{v}\sqrt{\beta^2\varepsilon-1}\right)\right] \quad \omega \gtrless 0$$

Applying the Fourier transform, the scaler potential can be obtained as

$$\Phi(\vec{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \Phi(\vec{x},\omega) e^{-i\omega t}\, d\omega$$

$$= \frac{ze}{2v\varepsilon}\left\{\int_0^{+\infty}\left[-N_0\left(\frac{\omega\rho}{v}\sqrt{\beta^2\varepsilon-1}\right) - iJ_0\left(\frac{\omega\rho}{v}\sqrt{\beta^2\varepsilon-1}\right)\right] e^{i\omega(z/v - t)}\, d\omega \right.$$

$$\left. + \int_{-\infty}^{0}\left[-N_0\left(-\frac{\omega\rho}{v}\sqrt{\beta^2\varepsilon-1}\right) + iJ_0\left(-\frac{\omega\rho}{v}\sqrt{\beta^2\varepsilon-1}\right)\right] e^{i\omega(z/v - t)}\, d\omega \right\}$$

$$= \frac{ze}{v\varepsilon}\int_0^{+\infty}\left[-N_0\left(\frac{\omega\rho}{v}\sqrt{\beta^2\varepsilon-1}\right)\cos\left(\omega\left(\frac{z}{v}-t\right)\right) + J_0\left(\frac{\omega\rho}{v}\sqrt{\beta^2\varepsilon-1}\right)\sin\left(\omega\left(\frac{z}{v}-t\right)\right)\right] d\omega$$

Using the identities

$$\int_0^{+\infty} J_0(ax)\sin(bx)\,dx = \begin{cases} 0, & 0<b<a \\ \frac{1}{\sqrt{b^2-a^2}}, & 0<a<b \end{cases} \qquad \int_0^{+\infty} N_0(ax)\cos(bx)\,dx = \begin{cases} 0, & 0<b<a \\ -\frac{1}{\sqrt{b^2-a^2}}, & 0<a<b, \end{cases}$$

we have
$$\Phi(\vec{x},t) = \frac{2ze}{v\varepsilon}\left[\left(\frac{z}{v}-t\right)^2 - \frac{\rho^2}{v^2}(\beta^2\varepsilon-1)\right]^{-1/2} = \frac{2ze/\varepsilon}{\sqrt{(z-vt)^2 - (\beta^2\varepsilon-1)\rho^2}} \qquad (*)$$

for $z - vt > \sqrt{\beta^2\varepsilon-1}\,\rho$, and 0 otherwise. This is consistent with the result for vector potential in section 13.4, where the potential is zero outside the Cherenkov cone and non-zero inside. Also, from Eq. (13.25), if the $\varepsilon(\omega)$ is a constant, the scalar and vector potentials are related by $\vec{A}(\vec{x},t) = \varepsilon\vec{\beta}\,\Phi(\vec{x},t)$, which is also satisfied, comparing $(*)$ and Eq. (13.51).