12.6 Let us first consider the case with parallel electric and magnetic fields configuration, then try to reduce the general problem to this special case.

(b) The component wise equation of motion is, assuming the fields are in the z-direction,

$$\frac{dP_x}{dt} = \frac{e}{c} v_y B, \quad \frac{dP_y}{dt} = -\frac{e}{c} v_x B, \quad \frac{dP_z}{dt} = eE$$

From Prob. 12.3. we know that the trajectory in the z-direction is $z(t) = \frac{\mathcal{E}_0}{eE}\left(\sqrt{1+\left(\frac{eEct}{\mathcal{E}_0}\right)^2} - 1\right)$.

Where $\mathcal{E}_0 = \gamma_0 mc^2 = mc^2/\sqrt{1-v_0^2/c^2}$ is the initial energy and v_0 is the initial velocity perpendicular to the z-direction. Also, $\gamma = \frac{1}{mc^2}\sqrt{\mathcal{E}_0^2 + (eEct)^2} = \frac{\gamma_0}{\mathcal{E}_0}\sqrt{\mathcal{E}_0^2 + (eEct)^2}$. From the x- and y-components, we have

$\frac{d}{dt}(P_x + iP_y) = -i\frac{eB}{\gamma mc}(P_x + iP_y)$. As $P_\perp = \sqrt{P_x^2 + P_y^2}$ is a constant, we can write

$P_x + iP_y = P_\perp e^{-i\phi}$, then the xy-components becomes $\frac{d\phi}{dt} = \frac{eBc}{\sqrt{\mathcal{E}_0^2 + (eEct)^2}}$.

This can be directly integrated, leading to $\phi = \frac{B}{E}\operatorname{arcsinh}\left(\frac{eEct}{\mathcal{E}_0}\right)$ or $ct = \frac{\mathcal{E}_0}{eE}\sinh\left(\frac{E}{B}\phi\right)$

Using this parametrization, we can see, $z = \frac{\mathcal{E}_0}{eE}\left(\cosh\left(\frac{E}{B}\phi\right) - 1\right)$.

Also, $P_x = P_\perp\cos\phi$, $P_y = -P_\perp\sin\phi$. Then

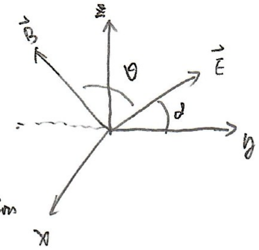
$x = \int_0^t v_x ds = \int_0^t \frac{P_x}{\gamma m} ds = \frac{P_\perp}{m}\int_0^\phi \frac{\cos\psi}{\gamma}\frac{ds}{d\psi}d\psi = \frac{P_\perp c}{eB}\sin\phi$, and similarly

$y = \frac{P_\perp c}{eB}\cos\phi$. Here, we have chosen the reference frame so that the initial position of the particle has been absorbed into the integration constants. Since $\mathcal{E}_0^2 = P_\perp^2 c^2 + m^2c^4$, and drop the constant in z, we have

$$z = \frac{mc^2}{eE}\sqrt{1+(P_\perp/mc)^2}\cosh\left(\frac{E}{B}\phi\right), \quad ct = \frac{mc^2}{eE}\sqrt{1+(P_\perp/mc)^2}\sinh\left(\frac{E}{B}\phi\right).$$

Define $A = P_\perp/mc$, $R = \frac{mc^2}{eB}$, $\rho = E/B$, then we can write the solution as

$$x = AR\sin\phi, \quad y = AR\cos\phi, \quad z = \frac{R}{\rho}\sqrt{1+A^2}\cosh(\rho\phi), \quad ct = \frac{R}{\rho}\sqrt{1+A^2}\sinh(\rho\phi).$$



(a) Suppose in the lab frame, the electric and magnetic fields lie in the yz-plane, making an angle of θ. For $\theta = \pi/2$, we can use the result of Prob. 12.5. Therefore, we will only consider $\theta \neq \pi/2$. Consider a reference frame moving in the x-direction with velocity $v = \beta c$. In this frame, we know

$$E_y' = \gamma(E_y - \beta B_z) = \gamma(E\cos\alpha - \beta B\sin(\alpha+\theta)). \quad B_y' = \gamma(B_y + \beta E_z) = \gamma(B\cos(\alpha+\theta) + \beta E\sin\alpha)$$

If we can make these y-components vanish, we will have the field configuration as in part (b). Therefore, by solving two equations, $E\cos\alpha - \beta B\sin(\alpha+\theta) = 0$ and $B\cos(\alpha+\theta) + \beta E\sin\alpha = 0$, we can determine the angle α and velocity β, to make the electric and magnetic fields parallel.

However, in part (b), we have assumed that the z-component of the particle velocity is 0. If in the frame where $\vec{E} \parallel \vec{B}$, this component is not zero, we can move to another reference frame that moves with velocity $v' = \beta' c$ in the z-direction. By the Lorentz transformation, $\vec{E} \parallel \vec{B}$ in this frame. Thus, we could make two successive transformation to reduce the problem to that in part (b). Solution to the original problem can be obtained by inverse transformation.