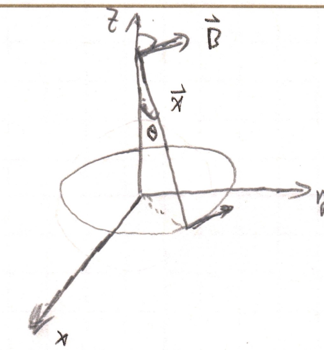
5.7 (a) For any line element on the loop, its contribution to the magnetic induction is



$$\frac{\mu_0}{4\pi}\ \frac{I\,d\vec{l}\times\vec{x}}{|\vec{x}|^3},$$

where $d\vec{l} = (-\sin\phi, \cos\phi)\, a\, d\phi$, $\vec{x} = (-a\cos\phi, -a\sin\phi, z)$.

Since $d\vec{l} \perp \vec{x}$, the magnitude of the magnetic induction is

$$\frac{\mu_0 I}{4\pi}\ \frac{a}{z^2+a^2}\, d\phi .$$

From symmetry consideration, only $\hat{z}$ component of the magnetic induction is non-zero

$$B_z = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{a}{z^2+a^2}\times\frac{a}{\sqrt{z^2+a^2}}\, d\phi = \frac{\mu_0 I a^2}{2(z^2+a^2)^{3/2}}$$

(b)
$$B_z = \frac{\mu_0 I a^2}{2}\left[\left(\left(z+\frac{b}{2}\right)^2+a^2\right)^{-3/2} + \left(\left(z-\frac{b}{2}\right)^2+a^2\right)^{-3/2}\right]$$

$$= \frac{\mu_0 I a^2}{2}\left[\left(z^2+bz+\left(a^2+\frac{b^2}{4}\right)\right)^{-3/2} + \left(z^2-bz+\left(a^2+\frac{b^2}{4}\right)\right)^{-3/2}\right]$$

$$= \frac{\mu_0 I a^2}{2d^3}\left[\left(1+\frac{bz}{d^2}+\frac{z^2}{d^2}\right)^{-3/2} + \left(1-\frac{bz}{d^2}+\frac{z^2}{d^2}\right)^{-3/2}\right], \quad \left(d^2 = a^2+b^2/4\right)$$

Using the MacLaurin expansion of

$$(1+x)^{-3/2} = 1-\frac{3}{2}x+\frac{15}{8}x^2-\frac{35x^3}{16}+\frac{315x^4}{128},$$

we have, keeping only even terms in z, as obvious from symmetry reasons,

$$\left(1+\frac{z^2\pm bz}{d^2}\right)^{-3/2} = 1-\frac{3}{2}\,\frac{z^2\pm bz}{d^2}+\frac{15}{8}\,\frac{(z^2\pm bz)^2}{d^4}-\frac{35}{16}\,\frac{(z^2\pm bz)^3}{d^6}$$

$$+\frac{315}{128}\,\frac{(z^2\pm bz)^4}{d^8}$$

$$= 1-\frac{3}{2}\,\frac{z^2}{d^2}+\frac{15}{8}\,\frac{z^4+b^2z^2}{d^4}-\frac{35}{16}\,\frac{3b^2z^4}{d^6}+\frac{315}{128}\,\frac{b^4z^4}{d^8}$$

$$= 1+\frac{z^2}{d^2}\left(\frac{15}{8}\,\frac{b^2}{d^2}-\frac{3}{2}\right)+\frac{z^4}{d^4}\left(\frac{15}{8}-\frac{105}{16}\,\frac{b^2}{d^2}+\frac{315}{128}\,\frac{b^4}{d^4}\right)$$

Since $\frac{15}{8}\frac{b^2}{d^2}-\frac{3}{2} = \frac{15b^2-12d^2}{8d^2} = \frac{15b^2-12(a^2+b^2/4)}{8d^2} = \frac{12(b^2-a^2)}{8d^2} = \frac{3(b^2-a^2)}{2d^2}$

and $\frac{15}{8}-\frac{105}{16}\frac{b^2}{d^2}+\frac{315}{128}\frac{b^4}{d^4} = \frac{240d^4-840d^2b^2+315b^4}{128d^4}$

$$= \frac{240(a^2+b^2/4)^2 - 840b^2(a^2+b^2/4)+315b^4}{128d^4}$$

$$= \frac{240a^4+120a^2b^2+15b^4-840a^2b^2-210b^4+315b^4}{128d^4}$$

$$= \frac{240a^4-720a^2b^2+120b^4}{128d^4} = \frac{15(b^4-6a^2b^2+2a^4)}{16d^4}$$

Therefore, $B_z = \frac{\mu_0 I a^2}{d^3}\left[1+\frac{3(b^2-a^2)z^2}{2d^4}+\frac{15(b^4-6a^2b^2+2a^4)z^4}{16d^8}\right]$

(c) We can write B_z as $B_z = \sigma_0+\sigma_2 z^2$, where

$$\sigma_0 = \frac{\mu_0 I a^2}{d^3}, \quad \sigma_2 = \sigma_0 \cdot \frac{3(b^2-a^2)}{2d^4}.$$

Then, $\frac{\partial B_z}{\partial z} = 2\sigma_2 z$, $\frac{\partial^2 B_z}{\partial z^2} = 2\sigma_2$. Using the results from 5.4, we have

$$B_z(\rho,z) = B_z(0,z) - \frac{\rho^2}{4}\frac{\partial^2 B_z(0,z)}{\partial z^2} = \sigma_0+\sigma_2 z^2 - \frac{\rho^2}{2}\sigma_2 = \sigma_0+\sigma_2(z^2-\rho^2/2)$$

$$B_\rho(\rho,z) = -\frac{\rho}{2}\frac{\partial B_z(0,z)}{\partial z} = -\sigma_2\rho z$$

(d) For $|z|\gg a,b$, we can expand

$$\frac{1}{|z|^3}\left(1+\frac{b}{z}+\frac{d^2}{z^2}\right)^{-3/2}, \quad \text{instead of} \quad \frac{1}{d^3}\left(1+\frac{bz}{d^2}+\frac{z^2}{d^2}\right)^{-3/2}$$

If we replace $\frac{z}{d}$ in the small z limit with $\frac{d}{|z|}$, then we will get the expansion at large $|z|$. This is equivalent to replacing d with $|z|$ in the small z expansion