1.14 Solution: (a) In Green's theorem, Eq. (1.35), set $\phi = G_D(\mathbf{x}, \mathbf{y})$ and $\psi = G_D(\mathbf{x}', \mathbf{y})$, we can obtain

$$\begin{aligned}
&\int_V \left(G_D(\mathbf{x}, \mathbf{y})\nabla^2_{\mathbf{y}} G_D(\mathbf{x}', \mathbf{y}) - G_D(\mathbf{x}', \mathbf{y})\nabla^2_{\mathbf{y}} G_D(\mathbf{x}, \mathbf{y})\right) d^3y \\
= \quad &\oint_S \left(G_D(\mathbf{x}, \mathbf{y})\frac{\partial G_D(\mathbf{x}', \mathbf{y})}{\partial n} - G_D(\mathbf{x}', \mathbf{y})\frac{\partial G_D(\mathbf{x}, \mathbf{y})}{\partial n}\right) da_y.
\end{aligned}$$

The right hand side is identically zero, since the Green function with Dirichlet boundary conditions will vanish on the boundary. On the left hand side, using the Poisson equation,

$$\nabla^2_{\mathbf{y}} G_D(\mathbf{x}, \mathbf{y}) = -4\pi\delta(\mathbf{x} - \mathbf{y}), \quad \nabla^2_{\mathbf{y}} G_D(\mathbf{x}', \mathbf{y}) = -4\pi\delta(\mathbf{x}' - \mathbf{y}),$$

we have

$$-4\pi\left(G_D(\mathbf{x}, \mathbf{x}') - G_D(\mathbf{x}', \mathbf{x})\right) = 0, \tag{1}$$

or

$$G_D(\mathbf{x}, \mathbf{x}') = G_D(\mathbf{x}', \mathbf{x}),$$

which means that the Green function satisfying the Dirichlet boundary conditions must be symmetric in its arguments.

(b) For Green functions with Neumann boundary conditions, the normal derivative on the boundary is specified,

$$\left.\frac{\partial G_N(\mathbf{x}, \mathbf{y})}{\partial n}\right|_{\mathbf{y}\in a} = -\frac{4\pi}{S}.$$

Therefore, following the same argument leading to Eq. (1),

$$-4\pi\left(G_N(\mathbf{x}, \mathbf{x}') - G_N(\mathbf{x}', \mathbf{x})\right) = -\frac{4\pi}{S}\oint_S \left(G_N(\mathbf{x}, \mathbf{y}) - G_N(\mathbf{x}', \mathbf{y})\right) da_y.$$

Moving this around, we can see that

$$G_N(\mathbf{x}, \mathbf{x}') - \frac{1}{S}\oint_S G_N(\mathbf{x}, \mathbf{y}) da_y = G_N(\mathbf{x}', \mathbf{x}) - \frac{1}{S}\oint_S G_N(\mathbf{x}', \mathbf{y}) da_y.$$

So, $G_N(\mathbf{x}, \mathbf{x}')$ is not symmetric in its arguments, but $G_N(\mathbf{x}, \mathbf{x}') - F(\mathbf{x})$ with

$$F(\mathbf{x}) = \frac{1}{S}\oint_S G_N(\mathbf{x}, \mathbf{y}) da_y$$

is.

(c) With the addition of $F(\mathbf{x})$ to the Green function, Eq. (1.46) gets additional contributions from the volume and the surface integrations,

$$\Delta\Phi = -\frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')F(\mathbf{x})d^3x' - \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'}F(\mathbf{x})da' = -F(\mathbf{x})\left[\frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'}da'\right],$$

since $F(\mathbf{x})$ does not depend on $\mathbf{x}'$. The second term can be rewritten as

$$\frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'}da' = \frac{1}{4\pi}\oint_S \nabla\Phi\cdot\mathbf{n}'da'.$$

Using Gauss theorem, it becomes

$$\frac{1}{4\pi}\oint_S \nabla\Phi\cdot\mathbf{n}'da' = \frac{1}{4\pi}\int_V \nabla^2\Phi d^3x' = -\frac{1}{4\pi\varepsilon_0}\int_V \rho(x')d^3x',$$

which cancels the first term, rendering

$$\Delta\Phi = 0.$$

This is means that the inclusion of $F(\mathbf{x})$ has no effect on the potential.