2.19 Solution: Similar to Problem 2.17, the Green function should be in the form of

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} g_m(\rho,\rho')e^{im(\phi-\phi')},$$

where g_m must satisfy the following differential equation,

$$\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial g_m}{\partial\rho'}\right) - \frac{m^2}{\rho'^2}g_m = -4\pi\frac{\delta(\rho-\rho')}{\rho}.$$

Due to the Dirichlet boundary condition at $\rho = b$ and $\rho = c$, g_m must have the following symmetric forms,

$$g_0 = C_0\log\left(\frac{\rho_<}{b}\right)\log\left(\frac{\rho_>}{c}\right),$$

and

$$g_m = C_m\left(\rho_<^{|m|} - \frac{b^{2|m|}}{\rho_<^{|m|}}\right)\left(\frac{1}{\rho_>^{|m|}} - \frac{\rho_>^{|m|}}{c^{2|m|}}\right),$$

for $m \neq 0$. Again, use the condition connecting the derivatives of g_m at $\rho = \rho'$, we can determine the constants as

$$C_0 = -\frac{4\pi}{\log(c/b)},$$

and

$$C_m = \frac{2\pi}{|m|}\left(1-\left(\frac{b}{c}\right)^{2|m|}\right)^{-1},$$

for $m \neq 0$. Finally, the Green function becomes

$$\begin{aligned} G(\rho,\phi;\rho',\phi') &= -2\frac{\log(\rho_</b)\log(\rho_>/c)}{\log(c/b)} + \sum_{m\neq 0}\frac{e^{im(\phi-\phi')}}{|m|\left[1-(b/c)^{2|m|}\right]}\left(\rho_<^{|m|} - \frac{b^{2|m|}}{\rho_<^{|m|}}\right)\left(\frac{1}{\rho_>^{|m|}} - \frac{\rho_>^{|m|}}{c^{2|m|}}\right) \\ &= \frac{\log(\rho_<^2/b^2)\log(c^2/\rho_>^2)}{\log(c^2/b^2)} + 2\sum_{m=1}^{\infty}\frac{\cos\left[m(\phi-\phi')\right]}{m\left[1-(b/c)^{2m}\right]}\left(\rho_<^m - \frac{b^{2m}}{\rho_<^m}\right)\left(\frac{1}{\rho_>^m} - \frac{\rho_>^m}{c^{2m}}\right). \end{aligned}$$