2.23 Solution: (a) We can consider the potential from the two planes with constant potentials separately, and later obtain the full result by superposition.

For the plane at $z = a$, the potential is given by Eq. (2.57),

$$\Phi_a(\mathbf{x}) = \sum_{m,n=1}^{\infty} A_{mn} \sin(\alpha_m x) \sin(\alpha_n y) \sinh(\beta_{mn} z),$$

where

$$\alpha_m = \frac{m\pi}{a}, \quad \beta_{mn} = \frac{\pi}{a}\sqrt{m^2+n^2}.$$

The coefficients A_{mn} can be determined by the boundary condition and is given by Eq. (2.58),

$$A_{mn} = \frac{4V}{a^2 \sinh(\beta_{mn} a)} \int_0^a \sin(\alpha_m x) dx \int_0^a \sin(\alpha_n y) dy.$$

The integral can be exactly evaluated,

$$\int_0^a \sin(\alpha_m x) dx = \frac{a}{m\pi}\left(1 - (-1)^m\right),$$

which is only non-zero for odd m. Therefore, the potentail becomes

$$\Phi_a(\mathbf{x}) = \frac{16V}{\pi^2} \sum_{m,n \text{ odd}}^{\infty} \frac{1}{mn \sinh(\beta_{mn} a)} \sin(\alpha_m x) \sin(\alpha_n y) \sinh(\beta_{mn} z).$$

Similarly, the contribution from the plane at $z = 0$ is

$$\Phi_a(\mathbf{x}) = \frac{16V}{\pi^2} \sum_{m,n \text{ odd}}^{\infty} \frac{1}{mn \sinh(\beta_{mn} a)} \sin(\alpha_m x) \sin(\alpha_n y) \sinh[\beta_{mn}(a-z)],$$

and the total potential inside the cube is

$$\Phi(\mathbf{x}) = \frac{16V}{\pi^2} \sum_{m,n \text{ odd}}^{\infty} \frac{1}{mn \sinh(\beta_{mn} a)} \sin(\alpha_m x) \sin(\alpha_n y) \left(\sinh(\beta_{mn} z) + \sinh[\beta_{mn}(a-z)]\right).$$

(b) I am not going to do the numerics, but from the result of Problem 2.28, the potential at the center of the cube should be the average of the potentials on the sides of the cube, which is $V/3$.

(c) For the charge density calculation, the normal directions is $-\hat{z}$, and

$$\begin{aligned}
\sigma_a(x,y) &= -\varepsilon_0 \left.\frac{\partial \Phi}{\partial n}\right|_{z=a} = \varepsilon_0 \left.\frac{\partial \Phi}{\partial z}\right|_{z=a} \\
&= \varepsilon_0 \frac{16V}{\pi^2} \sum_{m,n \text{ odd}}^{\infty} \frac{\beta_{mn}}{mn \sinh(\beta_{mn} a)} \sin(\alpha_m x) \sin(\alpha_n y) \left(\cosh(\beta_{mn} z) - \cosh[\beta_{mn}(a-z)]\right)\Bigg|_{z=a} \\
&= \varepsilon_0 \frac{16V}{\pi^2} \sum_{m,n \text{ odd}}^{\infty} \frac{\beta_{mn}}{mn} \sin(\alpha_m x) \sin(\alpha_n y) \frac{\cosh(\beta_{mn} a) - 1}{\sinh(\beta_{mn} a)} \\
&= \varepsilon_0 \frac{16V}{\pi^2} \sum_{m,n \text{ odd}}^{\infty} \frac{\beta_{mn}}{mn} \sin(\alpha_m x) \sin(\alpha_n y) \tanh\left(\frac{\beta_{mn} a}{2}\right).
\end{aligned}$$