6.19 (a) Space inversion can be achieved in spherical coordinates by setting $\theta \to \pi - \theta$ and $\phi \to -\phi$. Then, the only non-zero component of the vector potential becomes

$$\vec{A} = \frac{g}{4\pi r}\frac{1-\cos\theta}{\sin\theta}\hat{\phi} \to \frac{g}{4\pi r}\frac{1-\cos(\pi-\theta)}{\sin(\pi-\theta)}(-\hat{\phi}) = -\frac{g}{4\pi r}\frac{1+\cos\theta}{\sin\theta}\hat{\phi}.$$

By taking the curl of the vector potential, it can be shown that it leads to the same magnetic field as in the previous problem, except that now the vector potential has a singularity at $\theta = 0$.

(b) $\delta\vec{A} = \vec{A}' - \vec{A} = -\frac{g}{2\pi r}\frac{1}{\sin\theta}\hat{\phi}$.

Using the gradient formula in spherical coordinates, $\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}$,

we can see that by setting $\chi = -\frac{g}{2\pi}\phi$, we can write $\delta\vec{A} = \nabla\chi$.

(c) Consider the two definitions of the vector potential. The first one,

$\vec{A}(\vec{x}) = \frac{g}{4\pi}\int_L \frac{d\vec{l}' \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}$, where L is from $-\infty$ to 0 along the negative z-axis.

The other one, $\vec{A}'(\vec{x}) = \frac{g}{4\pi}\int_{L'} \frac{d\vec{l}' \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}$, where L' is from ∞ to 0 along the positive z-axis. The difference,

$$\delta\vec{A} = \vec{A} - \vec{A}' = \frac{g}{4\pi}\int_{-\infty}^{+\infty} \frac{d\vec{l}' \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} = \frac{g}{2\pi}\frac{-y\hat{i} + x\hat{j}}{x^2 + y^2} = \frac{1}{r\sin\theta}\frac{g}{2\pi}\hat{\phi} = \nabla\chi,$$

where $\chi = \frac{g}{2\pi}\phi$. Therefore, the gauge function comes from the choice of integration contour.