8.9 (a) Consider the variation of k^2 with respect to E,

$$\delta(k^2) = \frac{\int_V [\delta\vec{E}^* \cdot (\nabla\times(\nabla\times\vec{E})) + \vec{E}^* \cdot (\nabla\times(\nabla\times\delta\vec{E}))]\, d^3x}{\int_V \vec{E}^*\cdot\vec{E}\, d^3x} - \frac{\int_V [\delta\vec{E}^*\cdot\vec{E} + \vec{E}^*\cdot\delta\vec{E}]\, d^3x}{\left(\int_V \vec{E}^*\cdot\vec{E}\, d^3x\right)^2} \times (\text{numerator})$$

Since $\vec{E}^*\cdot(\nabla\times(\nabla\times\delta\vec{E})) = (\nabla\times\vec{E}^*)\cdot(\nabla\times\delta\vec{E}) - \nabla\cdot(\vec{E}^*\times(\nabla\times\delta\vec{E}))$

$$= \nabla\cdot(\delta\vec{E}\times(\nabla\times\vec{E}^*)) + \delta\vec{E}\cdot(\nabla\times(\nabla\times\vec{E}^*)) - \nabla\cdot(\vec{E}^*\times(\nabla\times\delta\vec{E})),$$

the intergal in the first numerator becomes

$$\int_V \vec{E}^*\cdot(\nabla\times(\nabla\times\delta\vec{E}))\, d^3x = \oint_S [\delta\vec{E}\times(\nabla\times\vec{E}^*) - \vec{E}^*\times(\nabla\times\delta\vec{E})]\cdot\vec{n}\, da + \int_V \delta\vec{E}\cdot(\nabla\times(\nabla\times\vec{E}^*))\, d^3x$$

Notice that $(\delta\vec{E}\times(\nabla\times\vec{E}^*))\cdot\vec{n} = (\vec{n}\times\delta\vec{E})\cdot(\nabla\times\vec{E}^*)$ and similarly for the second surface integral, the two surface integrals are simply zero, due to the boundary conditions $\vec{n}\times\vec{E} = 0$ and $\vec{n}\times\delta\vec{E} = 0$. Then,

$$\delta k^2 = \frac{\int_V [\delta\vec{E}^*\cdot(\nabla\times(\nabla\times\vec{E})) + \delta\vec{E}\cdot(\nabla\times(\nabla\times\vec{E}^*))]\, d^3x}{\int_V \vec{E}^*\cdot\vec{E}\, d^3x}$$

$$- \frac{\int_V [\delta\vec{E}^*\cdot\vec{E} + \vec{E}^*\cdot\delta\vec{E}]\, d^3x \cdot \int_V [\vec{E}^*\cdot(\nabla\times(\nabla\times\vec{E}))]\, d^3x}{\left(\int_V \vec{E}^*\cdot\vec{E}\, d^3x\right)^2}$$

$$= \frac{k^2\int_V (\delta\vec{E}^*\cdot\vec{E} + \delta\vec{E}\cdot\vec{E}^*)\, d^3x}{\int_V \vec{E}^*\cdot\vec{E}\, d^3x} - \frac{k^2\int_V (\delta\vec{E}^*\cdot\vec{E} + \vec{E}^*\cdot\delta\vec{E})\, d^3x \cdot \int_V \vec{E}^*\cdot\vec{E}\, d^3x}{\left(\int_V \vec{E}^*\cdot\vec{E}\, d^3x\right)^2}$$

$$= 0,$$

where we have used the ~~relation~~ $\nabla\times(\nabla\times\vec{E}) = k^2\vec{E}$. Therefore, k^2 is staionary with respect to the variation of $\vec{E}$.

(b) For $E_z = E_0 \cos(\pi\rho/2R)$, using the identity $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$, and also noticing $\nabla\cdot\vec{E}=0$, we have

$$\nabla\times(\nabla\times\vec{E}) = -\nabla^2\vec{E} = -(\nabla^2 E_z)\hat{z} = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right)\hat{z} = \frac{\pi}{2R\rho}E_0\sin\left(\frac{\pi\rho}{2R}\right) + \left(\frac{\pi}{2R}\right)^2 E_0\cos\left(\frac{\pi\rho}{2R}\right)$$

Then, $\int_V \vec{E}^*\cdot[\nabla\times(\nabla\times\vec{E})]\,d^3x = d\cdot 2\pi E_0^2\int_0^R\left[\frac{\pi}{2R}\sin\left(\frac{\pi\rho}{2R}\right)\cos\left(\frac{\pi\rho}{2R}\right) + \left(\frac{\pi}{2R}\right)^2\rho\cos^2\left(\frac{\pi\rho}{2R}\right)\right]d\rho$

Also, $\int_V \vec{E}^*\cdot\vec{E}\,d^3x = d\cdot 2\pi E_0^2\int_0^R\rho\cos^2\left(\frac{\pi\rho}{2R}\right)d\rho$, we have

$$k^2 = \int_0^R\left[\frac{\pi}{4R}\sin\left(\frac{\pi\rho}{R}\right) + \left(\frac{\pi}{2R}\right)^2\rho\cos^2\left(\frac{\pi\rho}{2R}\right)\right]d\rho \Big/ \int_0^R\rho\cos^2\left(\frac{\pi\rho}{2R}\right)d\rho$$

Since $\int_0^R\frac{\pi}{4R}\sin\left(\frac{\pi\rho}{R}\right)d\rho = \frac{1}{2}$, $\int_0^R\left(\frac{\pi}{2R}\right)^2\rho\cos^2\left(\frac{\pi\rho}{2R}\right)d\rho = \frac{\pi^2}{16}-\frac{1}{4}$,

we have $$k^2 = \left(\frac{\pi}{2R}\right)^2\frac{\frac{\pi^2}{16}+\frac{1}{4}}{\frac{\pi^2}{16}-\frac{1}{4}} = \left(\frac{\pi}{2R}\right)^2\frac{\pi^2+4}{\pi^2-4} \Rightarrow kR = \frac{\pi}{2}\sqrt{\frac{\pi^2+4}{\pi^2-4}} = 2.4146,$$

which is larger than the first root of $J_0(x)$, $x_{01} = 2.4048$.

(c) For the trial function,

$$\nabla\times(\nabla\times\vec{E}) = -\frac{4\alpha}{R^2} + \frac{16(1+\alpha)}{R^2}\left(\frac{\rho}{R}\right)^2.$$

Then $$\int_V \vec{E}^*\cdot(\nabla\times(\nabla\times\vec{E}))\,d^3x = d\cdot 2\pi\int_0^R\rho\left[1+\alpha\left(\frac{\rho}{R}\right)^2-(1+\alpha)\left(\frac{\rho}{R}\right)^4\right]\left[-\frac{4\alpha}{R^2}+\frac{16(1+\alpha)}{R^2}\left(\frac{\rho}{R}\right)^2\right]d\rho$$

$$= 2\pi d\cdot\frac{1}{3}(\alpha^2+4\alpha+6)$$

and $$\int_V \vec{E}^*\cdot\vec{E}\,d^3x = 2\pi d\int_0^R\rho\left[1+\alpha\left(\frac{\rho}{R}\right)^2-(1+\alpha)\left(\frac{\rho}{R}\right)^4\right]^2d\rho = 2\pi d\cdot\frac{R^2}{60}(\alpha^2+7\alpha+16),$$

which gives

$$k^2R^2 = \frac{20(\alpha^2+4\alpha+6)}{\alpha^2+7\alpha+16}.$$

The minimum value can be found at $\alpha = \frac{1}{3}(\sqrt{34}-10)$, while the corresponding minimum is $k^2R^2 = \frac{8}{3}(8-\sqrt{34})$, or $kR = \left[\frac{8}{3}(8-\sqrt{36})\right]^{1/2} = 2.6050$.

Compared to part (b), the value is lower and closer to the true value