5.3 Solution: In cylindrical coordinates, consider a point P at $z = z_0$ on the axis. From Problem 5.1, the magnetic induction at this point from a closed loop at $z > z_0$, perpendicular to the axis and around the cross section of the solenoid, carrying a current of NI per unit length is given by

$$\mathbf{B}(z) = \frac{\mu_0 NI}{4\pi}\nabla_{z_0}\Omega(z),$$

where $\Omega(z)$ is the solid angle subtended by the closed loop at z_0. This solid angle can be exactly calculated. On the cross section, a point that is distance ρ away from the axis, has a distance of $\sqrt{\rho^2 + (z - z_0)^2}$ to the point P. Then, the solid angle element is

$$d\Omega(\rho, z) = -\nabla_z\left(\frac{1}{\sqrt{\rho^2 + (z - z_0)^2}}\right)\cdot\hat{z}da = \frac{z - z_0}{\sqrt{\rho^2 + (z - z_0)^2}}da,$$

where we have again used the result from Section 1.6. The solid angle can be found by an integration on the cross section,

$$\Omega(z) = \int_0^{2\pi} d\phi \int_0^R d\rho\ \rho d\Omega(\rho, z) = 2\pi\left(1 - \frac{z - z_0}{\sqrt{R^2 + (z - z_0)^2}}\right),$$

where R is the radius of the solenoid. Then, the magnetic induction at P can be found by simple differentiation,

$$\mathbf{B}(z) = \frac{\mu_0 NI}{4\pi}\nabla_{z_0}\Omega(z) = -\frac{\mu_0 NI}{4\pi}\nabla_z\Omega(z) = \frac{\mu_0 NI}{2}\frac{\partial}{\partial z}\left(\frac{z - z_0}{\sqrt{R^2 + (z - z_0)^2}}\right)\hat{z},$$

with only component along the axis. Now, the total magnetic induction from current loops with $z > z_0$ can be obtained by integration,

$$\begin{aligned}
\mathbf{B}_+ &= \int_{z_0}^{z_+}\mathbf{B}(z)dz = \frac{\mu_0 NI}{2}\hat{z}\int_{z_0}^{z_+}\frac{\partial}{\partial z}\left(\frac{z - z_0}{\sqrt{R^2 + (z - z_0)^2}}\right)dz \\
&= \frac{\mu_0 NI}{2}\hat{z}\frac{z - z_0}{\sqrt{R^2 + (z - z_0)^2}}\Bigg|_{z=z_0}^{z_+} \\
&= \frac{\mu_0 NI}{2}\hat{z}\frac{z_+ - z_0}{\sqrt{R^2 + (z_+ - z_0)^2}},
\end{aligned}$$

where z_+ is at the top end of the solenoid, and

$$\cos\theta_2 = \frac{z_+ - z_0}{\sqrt{R^2 + (z_+ - z_0)^2}}.$$

Therefore,

$$\mathbf{B}_+ = \hat{z}\frac{\mu_0 NI}{2}\cos\theta_2.$$

Similarly, the contribution from the part of solenoid with $z < z_0$ is

$$\mathbf{B}_- = \hat{z}\frac{\mu_0 NI}{2}\cos\theta_1.$$

The total magnetic induction is then

$$\mathbf{B} = \mathbf{B}_+ + \mathbf{B}_- = \hat{z}\frac{\mu_0 NI}{2}(\cos\theta_1 + \cos\theta_2).$$