2.8 (a) Assume that the two lines are located at $(+c, 0)$ and $(-c, 0)$, with linear charge densities λ and $-\lambda$, respectively. Then, the potential at point (x, y) is given by

$$\Phi(x,y) = \frac{\lambda}{4\pi\varepsilon_0} \log \frac{(x+c)^2 + y^2}{(x-c)^2 + y^2}$$

Therefore, the equipotential surface can be determined by $\Phi(x,y) = V$, or

$$\frac{(x+c)^2 + y^2}{(x-c)^2 + y^2} = A, \quad \text{where } A = \exp\{4\pi\varepsilon_0 V/\lambda\}.$$

This can lead to the following results,

$$(1-A)x^2 + 2(1+A)cx + (1-A)c^2 + (1-A)y^2 = 0,$$

or $$\left[x - \frac{A+1}{A-1}c\right]^2 + y^2 = \frac{4A}{(A-1)^2}c^2.$$

Therefore, the equipotential surfaces are circular cylinders, with centers at $\left(\frac{A+1}{A-1}c, 0\right)$ and radii $R = \frac{2\sqrt{A}}{|A-1|}c$. Also, the equipotential surface reduces to the y-z plane when $A=1$, or $V=0$.

(b) For a cylinder with radius a, we can write $\frac{4A^2}{(A-1)^2} = \frac{a^2}{c^2}$, or $\left|\frac{A+1}{A-1}\right|^2 = 1 + \frac{a^2}{c^2}$,

and $\frac{A+1}{A-1} = \frac{\sqrt{c^2+a^2}}{c}$. Here, we have assumed that the center of this cylinder is on the positive x-axis. From this, we can find

$$A_1 = \frac{\sqrt{c^2+a^2}+c}{\sqrt{c^2+a^2}-c} \quad \text{and} \quad V_1 = \frac{\lambda}{4\pi\varepsilon_0}\log\left(\frac{\sqrt{c^2+a^2}+c}{\sqrt{c^2+a^2}-c}\right)$$

Similarly, for cylinder with radius b, we can assume that its center is on the negative x-axis, which leads to $-\frac{A+1}{A-1} = \frac{\sqrt{c^2+b^2}}{c}$. Then

$$A_2 = \frac{\sqrt{c^2+b^2}-c}{\sqrt{c^2+b^2}+c} \quad \text{and} \quad V_2 = \frac{\lambda}{4\pi\varepsilon_0}\log\left(\frac{\sqrt{c^2+b^2}-c}{\sqrt{c^2+b^2}+c}\right)$$

The potential difference between the two cylinders is

$$\Delta V = V_1 - V_2 = \frac{\lambda}{4\pi\varepsilon_0} \log\left(\frac{(\sqrt{c^2+b^2}+c)(\sqrt{c^2+a^2}+c)}{(\sqrt{c^2+b^2}-c)(\sqrt{c^2+a^2}-c)} \right)$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \log\left(\frac{(\sqrt{c^2+b^2}+c)(\sqrt{c^2+a^2}+c)}{ab} \right)$$

Since the two cylinders are separated d apart, we must have

$$\left(\frac{\sqrt{c^2+a^2}}{c} + \frac{\sqrt{c^2+b^2}}{c} \right) \cdot c = d, \text{ or } \sqrt{c^2+a^2} + \sqrt{c^2+b^2} = d. \qquad (1)$$

and $$(\sqrt{c^2+a^2}+c)(\sqrt{c^2+b^2}+c) = \sqrt{c^2+a^2}\sqrt{c^2+b^2} + cd + c^2.$$

Notice that $\left(\sqrt{c^2+a^2} + \sqrt{c^2+b^2}\right)^2 = a^2+b^2+2c^2+2\sqrt{c^2+a^2}\sqrt{c^2+b^2} = d^2$, then we have

$$\sqrt{c^2+a^2}\sqrt{c^2+b^2} = \frac{d^2-a^2-b^2}{2} - c^2.$$

Also, using (1), we have $c^2+a^2 = d^2+c^2+b^2-2d\sqrt{c^2+b^2}$, or $\sqrt{c^2+b^2} = \dfrac{d^2-a^2+b^2}{2d}$.

Then $$c^2 = \left(\frac{d^2-a^2+b^2}{2d}\right)^2 - b^2 = \frac{d^4+a^4+b^4-2a^2d^2+2b^2d^2-2a^2b^2}{4d^2} - b^2$$

$$= \frac{d^4+a^4+b^4-2a^2d^2-2b^2d^2+2a^2b^2}{4d^2} - \frac{a^2b^2}{d^2} = \frac{(d^2-a^2-b^2)^2}{4d^2} - \frac{a^2b^2}{d^2},$$

and $$cd = \left[\frac{1}{4}(d^2-a^2-b^2)^2 - a^2b^2\right]^{1/2}.$$

Putting everything together, we will arrive at

$$(\sqrt{c^2+a^2}+c)(\sqrt{c^2+b^2}+c) = \frac{d^2-a^2-b^2}{2} + \left[\frac{1}{4}(d^2-a^2-b^2)^2 - a^2b^2\right]^{1/2},$$

and $$\Delta V = \frac{\lambda}{2\pi\varepsilon_0} \log\left(\frac{d^2-a^2-b^2}{2ab} + \sqrt{\left(\frac{d^2-a^2-b^2}{2ab}\right)^2 - 1} \right)$$

Using the function definition $\operatorname{arccosh} x = \log\left(x + \sqrt{x^2-1}\right)$, we can express the potential difference as $\Delta V = \frac{\lambda}{2\pi\varepsilon_0}\operatorname{arccosh}\left(\frac{d^2-a^2-b^2}{2ab}\right)$, and the capacitance becomes

$$C = \frac{\lambda}{\Delta V} = \frac{2\pi\varepsilon_0}{\operatorname{arccosh}\left(\frac{d^2-a^2-b^2}{2ab}\right)}$$

(c) Since $\operatorname{arccosh}\left(\frac{d^2-a^2-b^2}{2ab}\right) = \log\left(\frac{d^2-a^2-b^2}{2ab} + \sqrt{\left(\frac{d^2-a^2-b^2}{2ab}\right)^2 - 1}\right)$

$$= \log\left(\frac{d^2}{2ab}\right) + \log\left(1-\left(\frac{a}{d}\right)^2-\left(\frac{b}{d}\right)^2 + \sqrt{1+\left(\frac{a}{d}\right)^4+\left(\frac{b}{d}\right)^4-2\left(\frac{a}{d}\right)^2-2\left(\frac{b}{d}\right)^2-2\left(\frac{ab}{d^2}\right)^2}\right)$$

$$= \log\left(\frac{d^2}{2ab}\right) + \log\left(2\left(1-\left(\frac{a}{d}\right)^2-\left(\frac{b}{d}\right)^2\right)\right)$$

$$= \log\left(\frac{d^2}{ab}\right) + \log\left(1-\left(\frac{a}{d}\right)^2-\left(\frac{b}{d}\right)^2\right)$$

$$= \log\left(\frac{d^2}{ab}\right) - \left(\frac{a}{d}\right)^2-\left(\frac{b}{d}\right)^2,$$

for $\frac{a}{d}, \frac{b}{d} \ll 1$. The leading order gives $C = 2\pi\varepsilon_0 / \log(d^2/ab)$, which agrees with Prob 1.7 to the lowest order.

(d) Similar to part (b), we can determine the centers of the cylinders. However for cylinder with radius b, we need to have $\frac{A+1}{A-1} = \frac{\sqrt{c^2+b^2}}{c}$, i.e., the center now is on the positive x-axis.

Then $A_3 = \frac{\sqrt{c^2+b^2}+c}{\sqrt{c^2+b^2}-c}$, and $V_3 = \frac{\lambda}{4\pi\varepsilon_0}\log\left(\frac{\sqrt{c^2+b^2}+c}{\sqrt{c^2+b^2}-c}\right)$. The potential difference is

$$\Delta V = V_1 - V_3 = \frac{\lambda}{4\pi\varepsilon_0}\log\left(\frac{(\sqrt{c^2+a^2}+c)(\sqrt{c^2+b^2}-c)}{(\sqrt{c^2+a^2}-c)(\sqrt{c^2+b^2}+c)}\right) = \frac{\lambda}{2\pi\varepsilon_0}\log\left(\frac{(\sqrt{c^2+a^2}+c)(\sqrt{c^2+b^2}-c)}{ab}\right).$$

Also, for this geometry, $\sqrt{c^2+a^2} - \sqrt{c^2+b^2} = d$, assuming $a>b$. Then

$$\left(\sqrt{c^2+a^2}+c\right)\left(\sqrt{c^2+b^2}-c\right) = \sqrt{c^2+a^2}\sqrt{c^2+b^2} - cd - c^2.$$

From $\sqrt{c^2+a^2}-\sqrt{c^2+b^2}=d$, we have

(i) $\left(\sqrt{c^2+a^2}-\sqrt{c^2+b^2}\right)^2 = 2c^2+a^2+b^2-2\sqrt{c^2+a^2}\sqrt{c^2+b^2} = d^2$.

which gives $\sqrt{c^2+a^2}\sqrt{c^2+b^2} = \dfrac{a^2+b^2-d^2}{2} + c^2$;

and (ii) $c^2+a^2 = d^2+c^2+b^2+2d\sqrt{c^2+b^2}$, or $\sqrt{c^2+b^2} = \dfrac{a^2-b^2-d^2}{2d}$

Then $c^2 = \left(\dfrac{a^2-b^2-d^2}{2d}\right)^2 - b^2 = \left(\dfrac{a^2+b^2-d^2}{2d}\right)^2 - \dfrac{a^2b^2}{d^2}$

Putting everything together, we find

$$\left(\sqrt{c^2+a^2}+c\right)\left(\sqrt{c^2+b^2}-c\right) = \frac{a^2+b^2-d^2}{2} - \sqrt{\frac{1}{4}(a^2+b^2-d^2)^2 - a^2b^2},$$

and $$\Delta V = \frac{\lambda}{2\pi\varepsilon_0}\log\left(\frac{a^2+b^2-d^2}{2ab} - \sqrt{\left(\frac{a^2+b^2-d^2}{2ab}\right)^2-1}\right)$$

Since $\Delta V<0$, and we can define capacitance as $C = \lambda/|\Delta V|$, then

$$|\Delta V| = -\frac{\lambda}{2\pi\varepsilon_0}\log\left(\frac{a^2+b^2-d^2}{2ab} - \sqrt{\left(\frac{a^2+b^2-d^2}{2ab}\right)^2-1}\right)$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\log\left(\frac{a^2+b^2-d^2}{2ab} + \sqrt{\left(\frac{a^2+b^2-d^2}{2ab}\right)^2-1}\right) = \frac{\lambda}{2\pi\varepsilon_0}\operatorname{arccosh}\left(\frac{a^2+b^2-d^2}{2ab}\right).$$

and $$C = \frac{2\pi\varepsilon_0}{\operatorname{arccosh}\left(\frac{a^2+b^2-d^2}{2ab}\right)}$$

For $d=0$, we have $\operatorname{arccosh}\left(\dfrac{a^2+b^2-d^2}{2ab}\right) = \log\left(\dfrac{a}{b}\right)$, and we will recover the well known result for concentric cylinders.

$$C = \frac{2\pi\varepsilon_0}{\log\left(\frac{a}{b}\right)}.$$