11.16 (a) In the rest frame of the medium, we have $\vec{J} = \sigma \vec{E}$. Using the field strength tensor, we can write it as $J^i = \sigma F^{i0} = \frac{\sigma}{c} F^{i\beta} U_\beta$, where $U^\beta = (c, 0)$ in the rest frame. It is natural to extend this result to the covariant form, $J^\alpha = \frac{\sigma}{c} F^{\alpha\beta} U_\beta$. However, due to the antisymmetry of $F^{\alpha\beta}$, $U_\alpha J^\alpha = \frac{\sigma}{c} F^{\alpha\beta} U_\alpha U_\beta = 0$, which is generally not true. Consider a medium with charge density at rest, where the left hand side will be $c^2 \rho$. Therefore there should be a term cancelling it. It is a clear that $\frac{1}{c^2}(U_\beta J^\beta) U^\beta$ will do. Thus, we will arrive at $J^\alpha - \frac{1}{c^2}(U_\beta J^\beta) U^\alpha = \frac{\sigma}{c} F^{\alpha\beta} U_\beta$

(b) Since $(c\rho, \vec{J})$ is a 4-vector, we know its transform property is

$$\vec{J} = \vec{J}' + \frac{\gamma-1}{\beta^2} \vec{\beta}(\vec{\beta}\cdot\vec{J}') + \gamma c\rho' \vec{\beta}, \quad c\rho = \gamma(c\rho' + \vec{\beta}\cdot\vec{J}'),$$ which is equivalent to $\gamma c\rho' = c\rho - \gamma\vec{\beta}\cdot\vec{J}'$

Therefore $$\vec{J} = \vec{J}' + \frac{\gamma-1}{\beta^2}\vec{\beta}(\vec{\beta}\cdot\vec{J}') + c\rho\vec{\beta} - \gamma\vec{\beta}(\vec{\beta}\cdot\vec{J}') = \vec{J}' + \left[\frac{\gamma-1}{\beta^2} - \gamma\right]\vec{\beta}(\vec{\beta}\cdot\vec{J}') + \rho\vec{v}$$

Now, in the rest frame of the medium, $\vec{J}' = \sigma\vec{E}'$, and $\vec{E}'$ is related to its strength in the laboratory frame as $\vec{E}' = \gamma(\vec{E} + \vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E})$, we have

$$\vec{J} = \sigma\left[\gamma(\vec{E}+\vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E}) + \left(\frac{\gamma-1}{\beta^2} - \gamma\right)\vec{\beta}\vec{\beta}\cdot\left\{\gamma(\vec{E}+\vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E})\right\}\right] + \rho\vec{v}$$

$$= \sigma\left[\gamma(\vec{E}+\vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E}) + \left(\frac{\gamma-1}{\beta^2} - \gamma\right)\vec{\beta}(\vec{\beta}\cdot\vec{E})\left(\gamma - \frac{\gamma^2\beta^2}{\gamma+1}\right)\right] + \rho\vec{v}$$

Since $\beta^2 = 1 - 1/\gamma^2$, then $\gamma - \frac{\gamma^2\beta^2}{\gamma+1} = \gamma - \frac{\gamma^2-1}{\gamma+1} = 1$, and

$$\vec{J} = \sigma\left[\gamma(\vec{E}+\vec{\beta}\times\vec{B}) + \left(\frac{\gamma-1}{\beta^2} - \frac{\gamma^2}{\gamma+1} - \gamma\right)\vec{\beta}(\vec{\beta}\cdot\vec{E})\right] + \rho\vec{v}.$$

Then, $\frac{\gamma-1}{\beta^2} = \frac{\gamma-1}{\frac{\gamma^2-1}{\gamma^2}} = \frac{\gamma^2}{\gamma+1}$. we finally arrive at

$$\vec{J} = \gamma\sigma\left[\vec{E} + \vec{\beta}\times\vec{B} - \vec{\beta}(\vec{\beta}\cdot\vec{E})\right] + \rho\vec{v}$$

(c) If the medium is initially uncharged, then $U_\alpha J^\alpha = 0$, and the equation becomes

$$J^\alpha = \frac{\sigma}{c} F^{\alpha\beta} U_\beta$$

For the spatial component, we have

$$J^i = \frac{\sigma}{c}\left(F^{i0}U^0 - F^{ij}U^j\right) = \frac{\sigma}{c}\left(E_i \gamma c - (-\varepsilon_{ijk}B_k)\gamma v_j\right) = \gamma\sigma\left(E_i + (\vec{\beta}\times\vec{B})_i\right),$$

Which leads to $\vec{J} = \gamma\sigma(\vec{E} + \vec{v}\times\vec{B})$. In the classical limit, $\gamma\to 1$, which reduces to the know result.