6.21 (a) From Problem 4.2, we know that the dipole's charges can be expressed formally as

$\rho(\vec{x},t) = -(\vec{p}\cdot\nabla)\delta(\vec{x}-\vec{r}_0(t))$. Using the conservation of charge, $\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$. We have

$$\frac{\partial\rho}{\partial t} = -(\vec{p}\cdot\nabla)\left[-\vec{v}\cdot\nabla\delta(\vec{x}-\vec{r}_0(t))\right] = (\vec{p}\cdot\nabla)\left[\nabla\cdot(\vec{v}\,\delta(\vec{x}-\vec{r}_0(t)))\right] = \nabla\cdot\left\{\vec{v}(\vec{p}\cdot\nabla)\delta(\vec{x}-\vec{r}_0(t))\right\}.$$

Therefore, $\vec{J}(\vec{x},t) = -\vec{v}(\vec{p}\cdot\nabla)\delta(\vec{x}-\vec{r}_0(t))$

(b) Since the dipole is moving non-relativistically, we can ignore the retardation effect and treat the potential as instantaneous. Therefore.

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|} = -\frac{1}{4\pi\epsilon_0}\int d^3x' (\vec{p}\cdot\nabla')\frac{\delta(\vec{x}'-\vec{r}_0)}{|\vec{x}-\vec{x}'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int d^3x' \delta(\vec{x}'-\vec{r}_0)\frac{\vec{p}\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot(\vec{x}-\vec{r}_0)}{|\vec{x}-\vec{r}_0|^3}$$

Now, expanding the scalar potential in $\vec{r}_0$, $|\vec{x}-\vec{r}_0|^{-3} = |\vec{x}|^{-3} + 3\vec{x}\cdot\vec{r}_0/|\vec{x}|^5$, the potential becomes

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}(\vec{p}\cdot\vec{x}-\vec{p}\cdot\vec{r}_0)\left(\frac{1}{r^3}+\frac{3\vec{x}\cdot\vec{r}_0}{r^5}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{3(\vec{p}\cdot\vec{n})(\vec{r}_0\cdot\vec{n})-\vec{p}\cdot\vec{r}_0}{r^3} - \frac{3(\vec{p}\cdot\vec{r}_0)(\vec{n}\cdot\vec{r}_0)}{r^4}\right)$$

Here, it is clear that, in addition to the dipole potential, $\frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{x}}{r^3}$. We also have an contribution from quadrupole moment,

$$\frac{1}{4\pi\epsilon_0}\frac{1}{2}\sum_{ij} Q_{ij}\frac{n_i n_j}{r^3}, \quad \text{with } Q_{ij} = 3(x_{0i}p_j + x_{0j}p_i) - 2\vec{r}_0\,\vec{p}\,\delta_{ij}.$$

Similarly, for the vector potential,

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|} = -\frac{\mu_0\vec{v}}{4\pi}\int d^3x' (\vec{p}\cdot\nabla')\frac{\delta(\vec{x}'-\vec{r}_0)}{|\vec{x}-\vec{x}'|} = \frac{\mu_0}{4\pi}\vec{v}\int d^3x' \delta(\vec{x}'-\vec{r}_0)\frac{\vec{p}\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

$$= \frac{\mu_0}{4\pi}\frac{\vec{v}[\vec{p}\cdot(\vec{x}-\vec{r}_0)]}{|\vec{x}-\vec{r}_0|^3}$$

Ignoring all the $\vec{r}_0$ terms, we have

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\frac{\vec{v}(\vec{p}\cdot\vec{x})}{r^3} = \frac{\mu_0}{4\pi}\left[\frac{1}{2}\frac{(\vec{p}\times\vec{v})\times\vec{x}}{r^3} + \frac{1}{2}\frac{\vec{p}(\vec{x}\cdot\vec{v})+\vec{v}(\vec{x}\cdot\vec{p})}{r^3}\right]$$

and therefore contains a magnetic dipole field

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{x}}{r^3}, \quad \text{with } \vec{m} = \frac{1}{2}\vec{p}\times\vec{v}.$$

(c) To calculate the quadrupole field, use the definition,

$$\vec{E} = -\nabla\Phi = -\hat{e}_k \partial_k \left(\frac{1}{8\pi\varepsilon_0} \sum_{ij} Q_{ij} \frac{n_i n_j}{r^3} \right) = -\frac{1}{8\pi\varepsilon_0} \hat{e}_k \sum_{ij} \partial_k \left(Q_{ij} \frac{x_i x_j}{r^5} \right)$$

$$= -\frac{1}{8\pi\varepsilon_0} \hat{e}_k \sum_{ij} Q_{ij} \partial_k \left(\frac{x_i x_j}{r^5} \right).$$

Since $\partial_k \left(\frac{x_i x_j}{r^5} \right) = \frac{\delta_{ik} x_j + \delta_{jk} x_i}{r^5} - \frac{5 x_i x_j x_k}{r^7}$, we have

$$\vec{E} = -\frac{1}{8\pi\varepsilon_0} \hat{e}_k \sum_{ij} \left(\frac{\delta_{ik} x_j + \delta_{jk} x_i}{r^5} - \frac{5 x_i x_j x_k}{r^7} \right) \left[3(x_{0i} p_j + x_{0j} p_i) - 2\vec{p}\cdot\vec{r}_0 \delta_{ij} \right]$$

$$= -\frac{1}{8\pi\varepsilon_0} \hat{e}_k \left[\frac{3(x_{0k} p_j x_j + x_{0k} p_k x_i)}{r^5} - \frac{2(\vec{p}\cdot\vec{r}_0) x_k}{r^5} + \frac{3(x_{0i} p_k x_i + x_{0k} p_i x_i)}{r^5} - \frac{2(\vec{p}\cdot\vec{r}_0) x_k}{r^5} - \frac{15(x_{0i} p_j + x_{0j} p_i) x_i x_j x_k}{r^7} + \frac{10(\vec{p}\cdot\vec{r}_0) x_i x_i x_k}{r^7} \right]$$

$$= -\frac{1}{8\pi\varepsilon_0} \left(\frac{3\vec{r}_0(\vec{p}\cdot\vec{r}) + 3\vec{p}(\vec{r}\cdot\vec{r}_0)}{r^5} - \frac{2\vec{r}(\vec{p}\cdot\vec{r}_0)}{r^5} + \frac{3\vec{p}(\vec{r}\cdot\vec{r}_0) + 3\vec{r}_0(\vec{p}\cdot\vec{r})}{r^5} - \frac{2\vec{r}(\vec{p}\cdot\vec{r}_0)}{r^5} - \frac{15\vec{r}(\vec{r}\cdot\vec{r}_0)(\vec{p}\cdot\vec{r}) + 15\vec{r}(\vec{r}\cdot\vec{r}_0)(\vec{p}\cdot\vec{r})}{r^7} + \frac{10\vec{r}(\vec{p}\cdot\vec{r}_0)|\vec{r}|^2}{r^7} \right)$$

$$= \frac{1}{8\pi\varepsilon_0} \left[\frac{30\vec{n}(\vec{n}\cdot\vec{r}_0)(\vec{n}\cdot\vec{p})}{r^4} - \frac{6\vec{r}_0(\vec{n}\cdot\vec{p}) + 6\vec{p}(\vec{n}\cdot\vec{r}_0) + 6\vec{n}(\vec{p}\cdot\vec{r}_0)}{r^4} \right]$$

$$= \frac{1}{4\pi\varepsilon_0 r^4} \left[15\vec{n}(\vec{n}\cdot\vec{r}_0)(\vec{n}\cdot\vec{p}) - 3\vec{r}_0(\vec{n}\cdot\vec{p}) - 3\vec{p}(\vec{n}\cdot\vec{r}_0) - 3\vec{n}(\vec{p}\cdot\vec{r}_0) \right]$$