7.22) From Eq. (7.120), we have

$$\mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{2}{\pi} P\int_0^{+\infty} \frac{\omega' \,\mathrm{Im}[\varepsilon(\omega')/\varepsilon_0]}{\omega'^2 - \omega^2}\, d\omega'$$

(a) For $\mathrm{Im}(\varepsilon(\omega)/\varepsilon_0) = \lambda[\theta(\omega-\omega_1) - \theta(\omega-\omega_2)]$, $\omega_2 > \omega_1 > 0$.

$$\mathrm{Re}(\varepsilon(\omega)/\varepsilon_0) = 1 + \frac{2}{\pi} P\int_{\omega_1}^{\omega_2} \frac{\lambda\omega'}{\omega'^2 - \omega^2}\, d\omega'$$

If $\omega^2 < \omega_1^2$ or $\omega^2 > \omega_2^2$, it is straightforward to show that

$$\mathrm{Re}(\varepsilon(\omega)/\varepsilon_0) = 1 + \frac{\lambda}{\pi}\log\left(\frac{\omega_2^2 - \omega^2}{\omega_1^2 - \omega^2}\right).$$

For the case of $\omega_1^2 < \omega^2 < \omega_2^2$, we have to take the principal value of the integral.

$$\mathrm{Re}(\varepsilon(\omega)/\varepsilon_0) = 1 + \frac{2\lambda}{\pi}\left(\int_{\omega_1}^{\omega-\varepsilon} + \int_{\omega+\varepsilon}^{\omega_2}\right) \frac{\omega'}{\omega'^2 - \omega^2}\, d\omega'$$

$$= 1 + \frac{\lambda}{\pi}\left(\log\left(\frac{\omega^2 - (\omega-\varepsilon)^2}{\omega^2 - \omega_1^2}\right) + \log\left(\frac{\omega_2^2 - \omega^2}{(\omega+\varepsilon)^2 - \omega^2}\right)\right)$$

$$= 1 + \frac{\lambda}{\pi}\log\left(\frac{\omega_2^2 - \omega^2}{\omega^2 - \omega_1^2}\right)$$

Therefore,

$$\mathrm{Re}(\varepsilon(\omega)/\varepsilon_0) = 1 + \frac{\lambda}{\pi}\log\left|\frac{\omega_2^2 - \omega^2}{\omega_1^2 - \omega^2}\right|.$$

(b) For $\mathrm{Im}(\varepsilon(\omega)/\varepsilon_0) = \dfrac{\lambda\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$,

$$\mathrm{Re}(\varepsilon(\omega)/\varepsilon_0) = 1 + \frac{1}{\pi} P\int_{-\infty}^{+\infty} \frac{\mathrm{Im}(\varepsilon(\omega')/\varepsilon_0)}{\omega' - \omega}\, d\omega' = 1 + \frac{\lambda}{\pi} P\int_{-\infty}^{+\infty} \frac{1}{\omega'-\omega}\, \frac{\gamma\omega'}{(\omega_0^2 - \omega'^2)^2 + \gamma^2\omega'^2}\, d\omega'$$

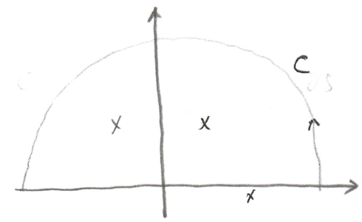
$$= 1 + \frac{\lambda}{\pi} P\int_{-\infty}^{+\infty} \frac{1}{2i}\,\frac{1}{\omega'-\omega}\left(\frac{1}{\omega'^2 - i\gamma\omega' - \omega_0^2} - \frac{1}{\omega'^2 + i\gamma\omega' - \omega_0^2}\right) d\omega'$$

Assume $\gamma > 0$, we will perform the integration with residues.

Consider the function $f(z) = \frac{1}{2i}\,\frac{1}{z-\omega+i0^+}\,\frac{1}{z^2-i\gamma z-\omega_0^2} = \frac{1}{2i}\,\frac{1}{z-\omega+i0^+}\,\frac{1}{(z-z_+)(z-z_-)}$

Where $z_\pm = \frac{i\gamma}{2} \pm \nu_0$, $\nu_0^2 = \omega_0^2 - \gamma^2/4$. Choose the integration contour as in the figure, then

$\oint f(z)\,dz = \int_{-\infty}^{+\infty} f(z)\,dz + \int_C f(z)\,dz$. The last term for the integration along the semi-circle with $R\to\infty$ will drop out. The contour integral can be evaluated with Cauchy's ~~theorem~~.



$$\oint f(z)\,dz = 2\pi i\cdot\left(\mathrm{Res}\, f(z)\Big|_{z=z_+} + \mathrm{Res}\, f(z)\Big|_{z=z_-}\right)$$

$$= \pi\left(\frac{1}{z_+-\omega}\,\frac{1}{z_+-z_-} + \frac{1}{z_--\omega}\,\frac{1}{z_--z_+}\right) = \frac{-\pi}{(z_+-\omega)(z_--\omega)} = \frac{\pi}{\omega_0^2+i\gamma\omega-\omega^2}$$

On the other side,

$$\int_{-\infty}^{+\infty} f(z)\,dz = P\int_{-\infty}^{+\infty} f(z)\,dz + \frac{1}{2i}\int_{-\infty}^{+\infty}(-i\pi)\,\delta(z-\omega)\,\frac{1}{z^2-i\gamma z-\omega_0^2}\,dz$$

$$= P\int_{-\infty}^{+\infty} f(z)\,dz - \frac{\pi}{2}\,\frac{1}{\omega^2-i\gamma\omega-\omega_0^2}$$

Therefor, $P\int_{-\infty}^{+\infty} f(z)\,dz = \frac{\pi}{2}\,\frac{1}{\omega_0^2+i\gamma\omega-\omega^2}$

Similarly, let $g(z) = -\frac{1}{2i}\,\frac{1}{z-\omega+i0^+}\,\frac{1}{z^2+i\gamma z-\omega_0^2}$, and use ~~the~~ same contour as $f(z)$. Clearly, the contour integral $\oint g(z)\,dz$ is 0, as all the poles are in the lower half plane. Then,

$$\int_{-\infty}^{+\infty} g(z)\,dz = P\int_{-\infty}^{+\infty} g(z)\,dz - \frac{1}{2i}\int_{-\infty}^{+\infty}(-i\pi)\,\delta(z-\omega)\,\frac{1}{z^2+i\gamma z-\omega_0^2}\,dz$$

$$= P\int_{-\infty}^{+\infty} g(z)\,dz + \frac{\pi}{2}\,\frac{1}{\omega^2+i\gamma\omega-\omega_0^2} = 0$$

or $P\int_{-\infty}^{+\infty} g(z)\,dz = \frac{\pi}{2}\,\frac{1}{\omega_0^2-i\gamma\omega-\omega^2}$

Finally, $\mathrm{Re}(\epsilon(\omega)/\epsilon_0) = 1 + \frac{\lambda}{\pi}\,P\int_{-\infty}^{+\infty}(f(z)+g(z))\,dz = 1 + \frac{\lambda}{\pi}\cdot\frac{\pi}{2}\left(\frac{1}{\omega_0^2+i\gamma\omega-\omega^2} + \frac{1}{\omega_0^2-i\gamma\omega-\omega^2}\right)$

$$= 1 + \frac{\lambda(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$