12.15 (a) In Lorenz gauge, the Maxwell equation is $\Box A^\alpha + \mu^2 A^\alpha = \frac{4\pi}{c} J^\alpha$, which in the static limit reduces to $\nabla^2 A^\alpha - \mu^2 A^\alpha = -\frac{4\pi}{c} J^\alpha$. For the vector potential, we have $\vec{J} = c\nabla\times\vec{m} = -c\,\vec{m}\times\nabla f$, the equation becomes $\nabla^2\vec{A} - \mu^2\vec{A} = 4\pi\,\vec{m}\times\nabla f$. It is straightforward to show that the solution to the Helmholtz equation $\nabla^2\Phi - \mu^2\Phi = 4\pi g$ is $\Phi(\vec{x}) = -\int g(\vec{x}')\,\frac{e^{-\mu|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,d^3x'$. Then, the solution for the vector potential is $\vec{A} = -\vec{m}\times\int \nabla' f(\vec{x}')\,\frac{e^{-\mu|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,d^3x'$. Integrate by parts, and notice that $\nabla = -\nabla'$, we have

$$\vec{A}(\vec{x}) = -\vec{m}\times\nabla\int f(\vec{x}')\,\frac{e^{-\mu|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,d^3x'.$$

(b) For $f(\vec{x}) = \delta(\vec{x})$, we have $\vec{A}(\vec{x}) = -\vec{m}\times\nabla\left(\frac{e^{-\mu r}}{r}\right)$, and the magnetic field is

$$\vec{B}(\vec{x}) = \nabla\times\vec{A}(\vec{x}) = -\nabla\times\left[\vec{m}\times\nabla\left(\frac{e^{-\mu r}}{r}\right)\right] = -\vec{m}\,\nabla^2\left(\frac{e^{-\mu r}}{r}\right) + (\vec{m}\cdot\nabla)\nabla\left(\frac{e^{-\mu r}}{r}\right).$$

Since $\nabla\left(\frac{e^{-\mu r}}{r}\right) = -\frac{\mu e^{-\mu r}}{r}\frac{\vec{r}}{r} - \frac{e^{-\mu r}}{r^2}\frac{\vec{r}}{r} = -\vec{n}\left[\left(\frac{\mu}{r}+\frac{1}{r^2}\right)e^{-\mu r}\right]$,

using the relation $(\vec{a}\cdot\nabla)\vec{n}f(r) = \frac{f(r)}{r}\left[\vec{a} - \vec{n}(\vec{n}\cdot\vec{a})\right] + \vec{n}(\vec{n}\cdot\vec{a})\frac{\partial f}{\partial r}$, we have

$$(\vec{m}\cdot\nabla)\nabla\left(\frac{e^{-\mu r}}{r}\right) = -(\vec{m}\cdot\nabla)\vec{n}\left[\left(\frac{\mu}{r}+\frac{1}{r^2}\right)e^{-\mu r}\right] = -\left[\left(\frac{\mu}{r^2}+\frac{1}{r^3}\right)e^{-\mu r}\left(\vec{m}-\vec{n}(\vec{n}\cdot\vec{m})\right) - \vec{n}(\vec{n}\cdot\vec{m})\left(\frac{\mu^2}{r}+\frac{2\mu}{r^2}+\frac{2}{r^3}\right)e^{-\mu r}\right]$$

$$= -e^{-\mu r}\left[\left(\frac{\mu}{r^2}+\frac{1}{r^3}\right)\vec{m} - \left(\frac{\mu^2}{r}+\frac{3\mu}{r^2}+\frac{3}{r^3}\right)\vec{n}(\vec{n}\cdot\vec{m})\right],$$

with $f(r) = \left(\frac{\mu}{r}+\frac{1}{r^2}\right)e^{-\mu r}$. Also, $\nabla^2\left(\frac{e^{-\mu r}}{r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(e^{-\mu r}\right) = \frac{\mu^2}{r}e^{-\mu r}$, then

$$\vec{B}(\vec{x}) = -\vec{m}\,\frac{\mu^2}{r}e^{-\mu r} - e^{-\mu r}\left(\frac{\mu}{r^2}+\frac{1}{r^3}\right)\vec{m} + e^{-\mu r}\left(\frac{\mu^2}{r}+\frac{3\mu}{r^2}+\frac{3}{r^3}\right)\vec{n}(\vec{n}\cdot\vec{m})$$

$$= e^{-\mu r}\left(\frac{\mu^2}{r}+\frac{3\mu}{r^2}+\frac{3}{r^3}\right)\vec{n}(\vec{n}\cdot\vec{m}) - e^{-\mu r}\left(\frac{\mu^2}{3r}+\frac{\mu}{r^2}+\frac{1}{r^3}\right)\vec{m} - \frac{2}{3}\mu^2\frac{e^{-\mu r}}{r}\vec{m}$$

$$= \left[3\vec{n}(\vec{n}\cdot\vec{m}) - \vec{m}\right]\left(1+\mu r+\frac{\mu^2 r^2}{3}\right)\frac{e^{-\mu r}}{r^3} - \frac{2}{3}\mu^2\frac{e^{-\mu r}}{r}\vec{m}.$$