6.23 (a) From the gauge transformation,

$$\vec{\Pi}_e' = \vec{\Pi}_e + \mu_0 \nabla\times\vec{G} - \nabla g, \quad \vec{\Pi}_m' = \vec{\Pi}_m - \mu\frac{\partial\vec{G}}{\partial t},$$

We can see that

$$\nabla^2\vec{\Pi}_e' = \nabla^2\vec{\Pi}_e + \mu_0\nabla\times(\nabla^2\vec{G}) - \nabla(\nabla^2 g), \quad \frac{\partial^2\vec{\Pi}_e'}{\partial t^2} = \frac{\partial^2\vec{\Pi}_e}{\partial t^2} + \mu_0\nabla\times\left(\frac{\partial^2\vec{G}}{\partial t^2}\right) - \nabla\left(\frac{\partial^2 g}{\partial t^2}\right),$$

$$\nabla^2\vec{\Pi}_m' = \nabla^2\vec{\Pi}_m - \mu\frac{\partial}{\partial t}(\nabla^2\vec{G}), \quad \frac{\partial^2\vec{\Pi}_m'}{\partial t^2} = \frac{\partial^2\vec{\Pi}_m}{\partial t^2} - \mu\frac{\partial}{\partial t}\left(\frac{\partial^2\vec{G}}{\partial t^2}\right).$$

Therefore,

$$\nabla^2\vec{\Pi}_e = \nabla^2\vec{\Pi}_e' - \mu_0\nabla\times(\nabla^2\vec{G}) + \nabla(\nabla^2 g), \quad \frac{\partial^2\vec{\Pi}_e}{\partial t^2} = \frac{\partial^2\vec{\Pi}_e'}{\partial t^2} - \mu_0\nabla\times\left(\frac{\partial^2\vec{G}}{\partial t^2}\right) + \nabla\left(\frac{\partial^2 g}{\partial t^2}\right)$$

$$\nabla^2\vec{\Pi}_m = \nabla^2\vec{\Pi}_m' + \mu\frac{\partial}{\partial t}(\nabla^2\vec{G}), \quad \frac{\partial^2\vec{\Pi}_m}{\partial t^2} = \frac{\partial^2\vec{\Pi}_m'}{\partial t^2} + \mu\frac{\partial}{\partial t}\left(\frac{\partial^2\vec{G}}{\partial t^2}\right)$$

Then, Eq. (6.168) becomes

$$\mu\epsilon\frac{\partial^2\vec{\Pi}_e}{\partial t^2} - \nabla^2\vec{\Pi}_e = \mu\epsilon\frac{\partial^2\vec{\Pi}_e'}{\partial t^2} - \nabla^2\vec{\Pi}_e' - \mu_0\nabla\times\left(\mu\epsilon\frac{\partial^2\vec{G}}{\partial t^2}\right) + \mu_0\nabla\times(\nabla^2\vec{G}) + \nabla\left(\mu\epsilon\frac{\partial^2 g}{\partial t^2}\right) - \nabla(\nabla^2 g)$$

$$= \vec{P}_{ext} - \frac{\mu_0}{\mu}\nabla\times\vec{V}$$

$$\mu\epsilon\frac{\partial^2\vec{\Pi}_m}{\partial t^2} - \nabla^2\vec{\Pi}_m = \mu\epsilon\frac{\partial^2\vec{\Pi}_m'}{\partial t^2} - \nabla^2\vec{\Pi}_m' + \mu\frac{\partial}{\partial t}\left(\mu\epsilon\frac{\partial^2\vec{G}}{\partial t^2}\right) - \mu\frac{\partial}{\partial t}(\nabla^2\vec{G}) = \vec{M}_{ext} + \frac{\partial\vec{V}}{\partial t} + \nabla\left(\frac{\partial\xi}{\partial t}\right).$$

If the gauge functions satisfy the conditions

$$\left(\mu\epsilon\frac{\partial^2}{\partial t^2} - \nabla^2\right)\begin{Bmatrix}\vec{G}\\ g\end{Bmatrix} = \begin{Bmatrix}\frac{1}{\mu}(\vec{V}+\nabla\xi)\\ 0\end{Bmatrix}$$

We can see that $\vec{\Pi}_e'$ and $\vec{\Pi}_m'$ satisfy Eq. (6.168) without $\vec{V}$ and ξ.

(b) Since $\vec{A} = \mu\frac{\partial\vec{\Pi}_e}{\partial t} + \mu_0\nabla\times\vec{\Pi}_m$, we have

$$\vec{A}' = \mu\frac{\partial\vec{\Pi}_e'}{\partial t} + \mu_0\nabla\times\vec{\Pi}_m' = \mu\left(\frac{\partial\vec{\Pi}_e}{\partial t} + \mu_0\nabla\times\left(\frac{\partial\vec{G}}{\partial t}\right) - \nabla\left(\frac{\partial g}{\partial t}\right)\right) + \mu_0\nabla\times\left(\vec{\Pi}_m - \mu\frac{\partial\vec{G}}{\partial t}\right)$$

$$= \mu\frac{\partial\vec{\Pi}_e}{\partial t} + \mu_0\nabla\times\vec{\Pi}_m - \mu\nabla\left(\frac{\partial g}{\partial t}\right) = \vec{A} - \nabla\Lambda,$$

where $\Lambda = \mu\frac{\partial g}{\partial t}$. We can do the same calculation for Φ. Therefore, the gauge transformation on the Hertz vectors is equivalent to gauge transformations on $\vec{A}$ and Φ.