

1.12 Solution: In Green's theorem, Eq. (1.35), set $\phi = \Phi$ and $\psi = \Phi'$, we can obtain

$$\int_V (\Phi \nabla^2 \Phi' - \Phi' \nabla^2 \Phi) d^3x = \oint_S \left(\Phi \frac{\partial \Phi'}{\partial n} - \Phi' \frac{\partial \Phi}{\partial n} \right) da.$$

The Poisson equation states that

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}, \quad \nabla^2 \Phi' = -\frac{\rho'}{\varepsilon_0}.$$

Meanwhile,

$$\sigma = \varepsilon_0 \frac{\partial \Phi}{\partial n}, \quad \sigma' = \varepsilon_0 \frac{\partial \Phi'}{\partial n}.$$

Here, notice that the normal vector is pointing outward from the volume of interest and, therefore, the above two identities differ from usual expressions by a minus sign.

Putting things together,

$$\frac{1}{\varepsilon_0} \int_V (-\rho' \Phi + \rho \Phi') d^3x = \frac{1}{\varepsilon_0} \oint_S (\sigma' \Phi - \sigma \Phi') da.$$

Rearrange the terms, we will obtain the Green's reciprocity theorem,

$$\int_V \rho \Phi' d^3x + \oint_S \sigma \Phi' da = \int_V \rho' \Phi d^3x + \oint_S \sigma' \Phi da.$$