9.6 (a) In the Lorenz gauge, the scalar potential is given by $\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho(\vec{x}', t-|\vec{x}-\vec{x}'|/c)}{|\vec{x}-\vec{x}'|}$.

Since $|\vec{x}-\vec{x}'| = |\vec{x}| - \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|} = r - \vec{n}\cdot\vec{x}'$, $\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} + \frac{\vec{n}\cdot\vec{x}'}{r^2}$, we can expand the integrand as

$$\frac{\rho(\vec{x}', t-|\vec{x}-\vec{x}'|/c)}{|\vec{x}-\vec{x}'|} = \left(\rho(\vec{x}', t-r/c) + \frac{\partial\rho(\vec{x}', t-r/c)}{\partial t}\frac{\vec{n}\cdot\vec{x}'}{c}\right)\left(\frac{1}{r} + \frac{\vec{n}\cdot\vec{x}'}{r^2}\right)$$

$$= \frac{\rho_{ret}(\vec{x}')}{r} + \frac{\vec{n}\cdot(\rho_{ret}(\vec{x}')\vec{x}')}{r^2} + \frac{1}{cr}\vec{n}\cdot\frac{\partial}{\partial t}\left(\rho_{ret}(\vec{x}')\vec{x}'\right) + \cdots$$

Drop the total charge term, as it does not contribute to the radiation, then the radiating scalar potential is

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{\vec{n}}{r^2}\cdot\int d^3x' \rho_{ret}(\vec{x}')\vec{x}' + \frac{\vec{n}}{cr}\cdot\frac{\partial}{\partial t}\int d^3x' \rho_{ret}(\vec{x}')\vec{x}'\right] = \frac{1}{4\pi\epsilon_0}\left[\frac{\vec{n}\cdot\vec{p}_{ret}}{r^2} + \frac{1}{cr}\vec{n}\cdot\frac{\partial\vec{p}_{ret}}{\partial t}\right]$$

Similarly, for the vector potential, $\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}', t-|\vec{x}-\vec{x}'|/c)}{|\vec{x}-\vec{x}'|}$.

and expanding in $\vec{x}'$, we have

$$\frac{\vec{J}(\vec{x}', t-|\vec{x}-\vec{x}'|/c)}{|\vec{x}-\vec{x}'|} = \left(\vec{J}(\vec{x}', t-r/c) + \cdots\right)\left(\frac{1}{r} + \cdots\right) = \frac{\vec{J}(\vec{x}', t-r/c)}{r}$$

Then $\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi r}\int d^3x' \vec{J}(\vec{x}', t-r/c) = -\frac{\mu_0}{4\pi r}\int d^3x' \vec{x}'\left(\nabla'\cdot\vec{J}(\vec{x}', t-r/c)\right)$

$$= \frac{\mu_0}{4\pi r}\int d^3x' \vec{x}'\frac{\partial\rho(\vec{x}', t-r/c)}{\partial t} = \frac{\mu_0}{4\pi r}\frac{\partial}{\partial t}\vec{p}(t-r/c) = \frac{\mu_0}{4\pi r}\frac{\partial\vec{p}_{ret}}{\partial t}$$

(b) $\vec{B} = \nabla\times\vec{A} = \frac{\mu_0}{4\pi}\nabla\times\left(\frac{1}{r}\frac{\partial\vec{p}_{ret}}{\partial t}\right) = \frac{\mu_0}{4\pi}\left[\nabla\left(\frac{1}{r}\right)\times\frac{\partial\vec{p}_{ret}}{\partial t} + \frac{1}{r}\nabla\times\frac{\partial\vec{p}_{ret}}{\partial t}\right]$

Notice that $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{n}}{r^2}$, and

$$\nabla\times\frac{\partial\vec{p}_{ret}}{\partial t} = \nabla\times\frac{\partial\vec{p}(t-r/c)}{\partial t} = \hat{e}_i\epsilon_{ijk}\partial_j\frac{\partial p_k(t-r/c)}{\partial t} = -\epsilon_{ijk}\hat{e}_i\frac{\partial^2 p_k(t-r/c)}{\partial t^2}\frac{1}{c}\partial_j r$$

$$= -\epsilon_{ijk}\hat{e}_i\frac{x_j}{cr}\frac{\partial^2 p_k(t-r/c)}{\partial t^2} = -\frac{1}{c}\epsilon_{ijk}\hat{e}_i n_j\frac{\partial^2 p_k(t-r/c)}{\partial t^2} = -\frac{1}{c}\vec{n}\times\frac{\partial^2\vec{p}_{ret}}{\partial t^2}$$

Then, $\vec{B} = \frac{\mu_0}{4\pi}\left[-\frac{1}{r^2}\vec{n}\times\frac{\partial\vec{p}_{ret}}{\partial t} - \frac{1}{cr}\vec{n}\times\frac{\partial^2\vec{p}_{ret}}{\partial t^2}\right]$.

For the electric field, $\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$. Explicitly, $\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0}{4\pi r}\frac{\partial^2 \vec{p}_{ret}}{\partial t^2}$, and

$$\nabla\Phi = \frac{1}{4\pi\varepsilon_0}\left[\nabla\left(\frac{1}{r^2}\right)\vec{n}\cdot\vec{p}_{ret} + \frac{1}{r^2}\nabla(\vec{n}\cdot\vec{p}_{ret}) + \frac{1}{c}\nabla\left(\frac{1}{r}\right)\vec{n}\cdot\frac{\partial \vec{p}_{ret}}{\partial t} + \frac{1}{cr}\nabla\left(\vec{n}\cdot\frac{\partial \vec{p}_{ret}}{\partial t}\right)\right]$$

Now, $\nabla\left(\frac{1}{r^2}\right) = -\frac{2}{r^3}\vec{n}$, $\nabla(\vec{n}\cdot\vec{p}_{ret}) = (\vec{n}\cdot\nabla)\vec{p}_{ret} + (\vec{p}_{ret}\cdot\nabla)\vec{n} + \vec{n}\times(\nabla\times\vec{p}_{ret}) + \vec{p}_{ret}\times(\nabla\times\vec{n})$

Since $(\vec{n}\cdot\nabla)\vec{p}_{ret} = n_i\partial_i\vec{p}_{ret} = -\frac{1}{c}n_i\frac{\partial \vec{p}_{ret}}{\partial t}\frac{x_i}{r} = -\frac{1}{c}\frac{\partial \vec{p}_{ret}}{\partial t}$

$(\vec{p}_{ret}\cdot\nabla)\vec{n} = \frac{1}{r}\left[\vec{p}_{ret} - \vec{n}(\vec{n}\cdot\vec{p}_{ret})\right]$, $\nabla\times\vec{n} = 0$

$\nabla\times\vec{p}_{ret} = -\frac{1}{c}\vec{n}\times\frac{\partial \vec{p}_{ret}}{\partial t}$.

we have $\nabla(\vec{n}\cdot\vec{p}_{ret}) = \frac{1}{r}\left[\vec{p}_{ret} - \vec{n}(\vec{n}\cdot\vec{p}_{ret})\right] - \frac{1}{c}\frac{\partial \vec{p}_{ret}}{\partial t} - \frac{1}{c}\vec{n}\times\left(\vec{n}\times\frac{\partial \vec{p}_{ret}}{\partial t}\right)$

Then, $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{n}}{r^2}$, and $\nabla\left(\vec{n}\cdot\frac{\partial \vec{p}_{ret}}{\partial t}\right) = \frac{1}{r}\left[\frac{\partial \vec{p}_{ret}}{\partial t} - \vec{n}\left(\vec{n}\cdot\frac{\partial \vec{p}_{ret}}{\partial t}\right)\right] - \frac{1}{c}\frac{\partial^2 \vec{p}_{ret}}{\partial t^2} - \frac{1}{c}\vec{n}\times\left(\vec{n}\times\frac{\partial^2 \vec{p}_{ret}}{\partial t^2}\right)$

Finally,
$$\nabla\Phi = \frac{1}{4\pi\varepsilon_0}\left[-\frac{2}{r^3}\vec{n}(\vec{n}\cdot\vec{p}_{ret}) + \frac{1}{r^3}\left(\vec{p}_{ret} - \vec{n}(\vec{n}\cdot\vec{p}_{ret})\right) - \frac{1}{cr^2}\frac{\partial \vec{p}_{ret}}{\partial t} - \frac{1}{cr^2}\vec{n}\times\left(\vec{n}\times\frac{\partial \vec{p}_{ret}}{\partial t}\right)\right.$$

$$\left. - \frac{1}{cr^2}\vec{n}\left(\vec{n}\cdot\frac{\partial \vec{p}_{ret}}{\partial t}\right) + \frac{1}{cr^2}\left(\frac{\partial \vec{p}_{ret}}{\partial t} - \vec{n}\left(\vec{n}\cdot\frac{\partial \vec{p}_{ret}}{\partial t}\right)\right) - \frac{1}{c^2 r}\frac{\partial^2 \vec{p}_{ret}}{\partial t^2} - \frac{1}{c^2 r}\vec{n}\times\left(\vec{n}\times\frac{\partial^2 \vec{p}_{ret}}{\partial t^2}\right)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r^3}\left(\vec{p}_{ret} - 3\vec{n}(\vec{n}\cdot\vec{p}_{ret})\right) + \frac{1}{cr^2}\left(\frac{\partial \vec{p}_{ret}}{\partial t} - 3\vec{n}\left(\vec{n}\cdot\frac{\partial \vec{p}_{ret}}{\partial t}\right)\right) - \frac{1}{c^2 r}\frac{\partial^2 \vec{p}_{ret}}{\partial t^2} - \frac{1}{c^2 r}\vec{n}\times\left(\vec{n}\times\frac{\partial^2 \vec{p}_{ret}}{\partial t^2}\right)\right]$$

Since $\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0}{4\pi r}\frac{\partial^2 \vec{p}_{ret}}{\partial t^2} = \frac{1}{4\pi\varepsilon_0}\frac{1}{c^2 r}\frac{\partial^2 \vec{p}_{ret}}{\partial t^2}$, we have

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} = \frac{1}{4\pi\varepsilon_0 r}\left[\left(1 + \frac{r}{c}\frac{\partial}{\partial t}\right)\frac{3\vec{n}(\vec{n}\cdot\vec{p}_{ret}) - \vec{p}_{ret}}{r^2} + \frac{1}{c^2}\vec{n}\times\left(\vec{n}\times\frac{\partial^2 \vec{p}_{ret}}{\partial t^2}\right)\right]$$

(c) Comparing the results from 9.5(b), it is clear and trivial to show the substitution rule.