3.12 Solution: (a) The general form of the solution is

$$\Phi(\rho,\phi,z)=\sum_{m=-\infty}^{\infty}\int_0^\infty dk A_m(k)J_m(k\rho)e^{-kz}e^{im\phi}.$$

On the plane, the potential is specified,

$$VI_{\rho\le a}=\sum_{m=-\infty}^{\infty}\int_0^\infty dk A_m(k)J_m(k\rho)e^{im\phi}.$$

The coefficients $A_m(k)$ can be determined by first multiplying both sides with $e^{-im\phi}$ and integrating with respect to ϕ. Then, only the $m=0$ term will survive,

$$VI_{\rho\le a}=\int_0^\infty dk A_0(k)J_0(k\rho).$$

Next, using the Hankel transform,

$$\frac{1}{k}\delta(k-k')=\int_0^\infty \rho J_\nu(k\rho)J_\nu(k'\rho)d\rho,$$

we have

$$V\int_0^a \rho J_0(k\rho)d\rho=\int_0^\infty dk' A_0(k')\cdot\left(\int_0^\infty \rho J_0(k\rho)J_0(k'\rho)d\rho\right)=\int_0^\infty dk' A_0(k')\cdot\frac{1}{k}\delta(k-k')=\frac{A_0(k)}{k},$$

and

$$A_0(k)=Vk\int_0^a \rho J_0(k\rho)d\rho=\frac{V}{k}\int_0^{ka} xJ_0(x)dx.$$

Using the fact that

$$\frac{d}{dx}\left(xJ_1(x)\right)=xJ_0(x),$$

the above integral can be exactly performed,

$$A_0(k)=\frac{V}{k}xJ_1(x)\Big|_{x=0}^{ka}=VaJ_1(ka).$$

Therefore, the potential above the plane is

$$\Phi(\rho,\phi,z)=Va\int_0^\infty J_1(ka)J_0(k\rho)e^{-kz}dk.$$

It is not very obvious at this moment how the solution will reduce to the specified potential on the $z=0$ plane, where

$$\Phi(\rho,\phi,0)=Va\int_0^\infty J_1(ka)J_0(k\rho)dk.$$

Using the identity (see, *e.g.*, Gradshteyn and Ryzhik, 7^{th} ed., p. 660, formula 6.512.3),

$$\int_0^\infty J_\mu(at)J_{\mu-1}(bt)dt=\begin{cases} b^{\mu-1}/a^\mu, & b<a,\\ 1/2b, & b=a,\\ 0, & b>a,\end{cases}$$

it can be shown that, with $\mu = 1$,

$$\int_0^\infty J_1(ka)J_0(k\rho)dk = \begin{cases} 1/a, & 0 < \rho < a, \\ 1/2a, & \rho = a, \\ 0, & \rho > a, \end{cases}$$

which gives the specified potential, except for the point on the edge of the disk. This is a common phenomenon of generalized Fourier transform, where the Bessel-Fourier series can only reproduce the mid-value at a discountinuous point.

(b) For the potential at $\rho = 0$, we have

$$\Phi_0(z) = Va\int_0^\infty J_1(ka)e^{-kz}dk,$$

since $J_0(0) = 1$. Then,

$$\begin{aligned}
\Phi_0(z) &= V\int_0^\infty J_1(\lambda)e^{-\lambda z/a}d\lambda \\
&= -V\int_0^\infty J_0'(\lambda)e^{-\lambda z/a}d\lambda \qquad (\text{since } J_1(\lambda) = -J_0'(\lambda)) \\
&= -V\left[e^{-\lambda z/a}J_0(\lambda)\Big|_{\lambda=0}^\infty + \frac{z}{a}\int_0^\infty J_0(\lambda)e^{-\lambda z/a}d\lambda\right] \\
&= V\left[1 - \frac{z}{a}\int_0^\infty J_0(\lambda)e^{-\lambda z/a}d\lambda\right] \\
&= V\left(1 - \frac{z}{a}\frac{1}{\sqrt{1+z^2/a^2}}\right) \\
&= V\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right),
\end{aligned}$$

where we have use the identity (see, *e.g.*, Gradshteyn and Ryzhik, 7^{th} ed., p. 695, formula 6.611.1)

$$\int_0^\infty e^{-\alpha x}J_\nu(bx)dx = \frac{b^{-\nu}\left(\sqrt{\alpha^2+b^2}-\alpha\right)^\nu}{\sqrt{\alpha^2+b^2}}.$$

(c) For the potential at $\rho = a$, we have

$$\begin{aligned}
\Phi_a(z) &= Va\int_0^\infty J_1(ka)J_0(ka)e^{-kz}dk \\
&= V\int_0^\infty J_1(\lambda)J_0(\lambda)e^{-\lambda z/a}d\lambda \\
&= -V\int_0^\infty J_0'(\lambda)J_0(\lambda)e^{-\lambda z/a}d\lambda \qquad (\text{since } J_1(\lambda) = -J_0'(\lambda)) \\
&= -\frac{V}{2}\int_0^\infty e^{-\lambda z/a}d\left(J_0(\lambda)^2\right) \\
&= -\frac{V}{2}\left[e^{-\lambda z/a}J_0(\lambda)^2\Big|_{\lambda=0}^\infty + \frac{z}{a}\int_0^\infty J_0(\lambda)^2e^{-\lambda z/a}d\lambda\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{V}{2}\left[1-\frac{z}{a}\cdot\frac{2}{\pi\sqrt{4+z^2/a^2}}K\left(\frac{2}{\sqrt{4+z^2/a^2}}\right)\right] \\
&= \frac{V}{2}\left[1-\frac{kz}{\pi a}K\left(k\right)\right],
\end{aligned}
$$

with $k = 2a/\sqrt{z^2+4a^2}$. Here, we have used the identity (see, *e.g.*, Gradshteyn and Ryzhik, 7^{th} ed., p. 696, formula 6.612.4),

$$
\int_0^\infty e^{-\alpha x} J_0(bx)^2 dx = \frac{2}{\pi\sqrt{\alpha^2+4b^2}}K\left(\frac{2b}{\pi\sqrt{\alpha^2+4b^2}}\right),
$$

where $K(x)$ is the complete elliptic integral of the first kind.