10.1 (a) The crucial identity for this problem is that, for a unit vector $\vec{n}$ and the two polarization vectors associated with it, $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$, we have

$$\sum_{\lambda=1,2} \epsilon_{\lambda,i}^* \epsilon_{\lambda,j} = \delta_{ij} - n_i n_j .$$

With this knowledge, and also using Einstein summation convention, the differential scattering cross section before averaging over outgoing polarization, is

$$\frac{d\sigma}{d\Omega}(\vec{n}_0, \vec{\epsilon}_0; \vec{n}, \vec{\epsilon}) = k^4 a^6 \left| \vec{\epsilon}^* \cdot \vec{\epsilon}_0 - \frac{1}{2}(\vec{n}\times\vec{\epsilon}^*)\cdot(\vec{n}_0 \times \vec{\epsilon}_0) \right|^2$$

$$= k^4 a^6 \left| \vec{\epsilon}^* \cdot \vec{\epsilon}_0 - \frac{1}{2}(\vec{n}\cdot\vec{n}_0)(\vec{\epsilon}^*\cdot\vec{\epsilon}_0) + \frac{1}{2}(\vec{n}\cdot\vec{\epsilon}_0)(\vec{\epsilon}^*\cdot\vec{n}_0) \right|^2$$

$$= k^4 a^6 \left| \left[1 - \frac{1}{2}(\vec{n}\cdot\vec{n}_0)\right](\vec{\epsilon}^*\cdot\vec{\epsilon}_0) + \frac{1}{2}(\vec{n}\cdot\vec{\epsilon}_0)(\vec{\epsilon}^*\cdot\vec{n}_0) \right|^2$$

$$= k^4 a^6 \left\{ \left[1 - \frac{1}{2}(\vec{n}\cdot\vec{n}_0)\right]^2 |\vec{\epsilon}^*\cdot\vec{\epsilon}_0|^2 + \left[1 - \frac{1}{2}(\vec{n}\cdot\vec{n}_0)\right](\vec{n}\cdot\vec{\epsilon}_0)(\vec{\epsilon}^*\cdot\vec{\epsilon}_0)(\vec{\epsilon}\cdot\vec{n}_0) \right.$$

$$\left. + \frac{1}{4}|\vec{n}\cdot\vec{\epsilon}_0|^2 |\vec{\epsilon}^*\cdot\vec{n}_0|^2 \right\}.$$

Using the above identity, we can find that

$$\sum_\lambda (\vec{\epsilon}_\lambda \cdot \vec{a})(\vec{\epsilon}_\lambda^* \cdot \vec{b}) = \sum_\lambda \epsilon_{\lambda i} \epsilon_{\lambda j}^* a_i b_j = (\delta_{ij} - n_i n_j) a_i b_j = \vec{a}\cdot\vec{b} - (\vec{n}\cdot\vec{a})(\vec{n}\cdot\vec{b}).$$

Applying to the differential scattering cross section, we can get

$$\frac{d\sigma}{d\Omega}(\vec{\epsilon}_0, \vec{n}_0, \vec{n}) = \sum_\lambda \frac{d\sigma}{d\Omega}(\vec{n}_0, \vec{\epsilon}_0; \vec{n}, \vec{\epsilon}_\lambda)$$

$$= k^4 a^6 \left\{ \left[1 - \frac{1}{2}(\vec{n}\cdot\vec{n}_0)\right]^2 \left(1 - |\vec{n}\cdot\vec{\epsilon}_0|^2\right) + \left[1 - \frac{1}{2}(\vec{n}\cdot\vec{n}_0)\right](\vec{n}\cdot\vec{\epsilon}_0)\left[(\vec{n}_0\cdot\vec{\epsilon}_0) - (\vec{n}\cdot\vec{n}_0)(\vec{n}\cdot\vec{\epsilon}_0)\right] \right.$$

$$\left. + \frac{1}{4}|\vec{n}\cdot\vec{\epsilon}_0|^2 \left(1 - |\vec{n}\cdot\vec{n}_0|^2\right) \right\}$$

Notice that $\vec{n}_0 \cdot \vec{\epsilon}_0 = 0$, the above equation can be slightly simplified as

$$\frac{d\sigma}{d\Omega}(\vec{\epsilon}_0, \vec{n}_0, \vec{n}) = k^4 a^6 \left\{ \left(1 - (\vec{n}\cdot\vec{n}_0) + \frac{1}{4}|\vec{n}\cdot\vec{n}_0|^2\right)\left(1 - |\vec{n}\cdot\vec{\epsilon}_0|^2\right) - \left(1 - \frac{1}{2}(\vec{n}\cdot\vec{n}_0)\right)(\vec{n}\cdot\vec{n}_0)|\vec{n}\cdot\vec{\epsilon}_0|^2 \right.$$

$$\left. + \frac{1}{4}|\vec{n}\cdot\vec{\epsilon}_0|^2\left(1 - |\vec{n}\cdot\vec{n}_0|^2\right) \right\}$$

$$= k^4 a^6 \Big\{ 1 - |\vec{n}\cdot\vec{\varepsilon}_0|^2 - |\vec{n}\cdot\vec{n}_0| + (\vec{n}\cdot\vec{n}_0)(\vec{n}\cdot\vec{\varepsilon}_0)^2 + \frac{1}{4}|\vec{n}\cdot\vec{n}_0|^2 - \frac{1}{4}|\vec{n}\cdot\vec{n}_0|^2|\vec{n}\cdot\vec{\varepsilon}_0|^2$$

$$- |\vec{n}\cdot\vec{n}_0||\vec{n}\cdot\vec{\varepsilon}_0|^2 + \frac{1}{2}|\vec{n}\cdot\vec{n}_0|^2|\vec{n}\cdot\vec{\varepsilon}_0|^2$$

$$+ \frac{1}{4}|\vec{n}\cdot\vec{\varepsilon}_0|^2 - \frac{1}{4}|\vec{n}\cdot\vec{\varepsilon}_0|^2|\vec{n}\cdot\vec{n}_0|^2 \Big\}$$

$$= k^4 a^6 \Big\{ 1 - \vec{n}\cdot\vec{n}_0 - \frac{3}{4}|\vec{n}\cdot\vec{\varepsilon}_0|^2 + \frac{1}{4}|\vec{n}\cdot\vec{n}_0|^2 \Big\}$$

This can be further manipulated to the desired form by noting that

$$|\vec{n}\cdot\vec{\varepsilon}_0|^2 + |\vec{n}\cdot\vec{n}_0|^2 + |\vec{n}\cdot(\vec{n}_0\times\vec{\varepsilon}_0)|^2 = 1,$$

as $\vec{n}_0, \vec{\varepsilon}_0$, and $\vec{n}_0\times\vec{\varepsilon}_0$ form a complete set of base. Then.

$$\frac{d\sigma}{d\Omega}(\vec{\varepsilon}_0, \vec{n}_0, \vec{n}) = k^4 a^6 \Big\{ 1 - \vec{n}\cdot\vec{n}_0 - |\vec{n}\cdot\vec{\varepsilon}_0|^2 + \frac{1}{4}\left(|\vec{n}\cdot\vec{\varepsilon}_0|^2 + |\vec{n}\cdot\vec{n}_0|^2\right) \Big\}$$

$$= k^4 a^6 \Big\{ 1 - \vec{n}\cdot\vec{n}_0 - |\vec{n}\cdot\vec{\varepsilon}_0|^2 + \frac{1}{4}\left(1 - |\vec{n}\cdot(\vec{n}_0\times\vec{\varepsilon}_0)|^2\right) \Big\}$$

$$= k^4 a^6 \left[\frac{5}{4} - |\vec{\varepsilon}_0\cdot\vec{n}|^2 - \frac{1}{4}|\vec{n}\cdot(\vec{n}_0\times\vec{\varepsilon}_0)|^2 - \vec{n}\cdot\vec{n}_0 \right]$$

(b) For linearly polarized incident radiation, we can choose $\vec{n}_0 = (0,0,1), \vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

and $\vec{\varepsilon}_0 = (1,0,0)$. Then

$$\vec{n}\cdot\vec{n}_0 = \cos\theta, \quad \vec{n}\cdot\vec{\varepsilon}_0 = \sin\theta\cos\phi \quad \vec{n}\cdot(\vec{n}_0\times\vec{\varepsilon}_0) = \sin\theta\sin\phi$$

and $$\frac{d\sigma}{d\Omega}(\vec{\varepsilon}_0, \vec{n}_0, \vec{n}) = k^4 a^6 \Big\{ \frac{5}{4} - \sin^2\theta\cos^2\phi - \frac{1}{4}\sin^2\theta\sin^2\phi - \cos\theta \Big\}$$

$$= k^4 a^6 \Big\{ \frac{5}{4} - \frac{1}{2}\sin^2\theta(\cos 2\phi + 1) - \frac{1}{8}\sin^2\theta(1 - \cos 2\phi) - \cos\theta \Big\}$$

$$= k^4 a^6 \Big\{ \frac{5}{4} - \frac{5}{8}\sin^2\theta - \frac{3}{8}\sin^2\theta\cos 2\phi - \cos\theta \Big\}$$

$$= k^4 a^6 \Big\{ \frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos 2\phi \Big\}.$$

(c) From part (b), we have

$$\frac{d\sigma(\theta=\pi/2,\phi=0)/d\Omega}{d\sigma(\theta=\pi/2,\phi=\pi/2)/d\Omega} = \frac{\frac{5}{8}-\frac{3}{8}}{\frac{5}{8}+\frac{3}{8}} = \frac{1}{4}.$$

For $\theta=\pi/2$, $\phi=0$, the incident electric field will induce electric current in the polarization direction. However, electric dipole does not radiate in its dipole moment direction. Therefore, the radiation in the polarization direction is only due to the magnetic dipole. On the other hand, magnetic dipole does not contribute perpendicular to this polarization. For $\theta=\pi/2$, $\phi=\pi/2$, we are only getting electric dipole radiation. From Eq. (10.16), we can see that magnetic dipole radiation strength is only 1/4 of that of the electric dipole.