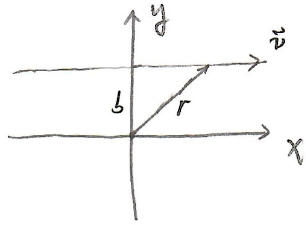
15.9 (a) For a screened Coulomb potential, $V(r) = Zze^2 e^{-\alpha r}/r$, the force on the particle is given by



$$|\nabla V| = Zze^2\left(\alpha + \frac{1}{r}\right)\frac{e^{-\alpha r}}{r}.$$

In its componentwise form, the Newton's second law is

$$m\ddot{x}(t) = Zze^2\left(\alpha + \frac{1}{r}\right)\frac{e^{-\alpha r}}{r}\frac{x}{r},$$

$$m\ddot{y}(t) = Zze^2\left(\alpha + \frac{1}{r}\right)\frac{e^{-\alpha r}}{r}\frac{b}{r},$$

where $r = \sqrt{x^2+b^2}$, assuming straight path of the particle. To apply the dipole approximation, we need the Fourier transform of the accelerations,

$$\ddot{x}(\omega) = \int_{-\infty}^{+\infty}\ddot{x}(t)e^{i\omega t}dt = \frac{Zze^2}{m}\int_{-\infty}^{+\infty}\left(\alpha + \frac{1}{r}\right)\frac{e^{-\alpha r}}{r}\frac{x}{r}e^{i\omega t}dt$$

$$= \frac{\alpha Zze^2}{m}\int_{-\infty}^{+\infty}\left(\frac{1}{x^2+b^2} + \frac{1}{\alpha(x^2+b^2)^{3/2}}\right)\exp\left\{-\alpha\sqrt{x^2+b^2}\right\}x\,e^{i\omega t}dt$$

Since $x = vt$ for a straight path, and notice the parity of the integrand, we have

$$\ddot{x}(\omega) = \frac{i\alpha Zze^2}{mv}\int_{-\infty}^{+\infty}\left(\frac{1}{x^2+b^2} + \frac{1}{\alpha(x^2+b^2)^{3/2}}\right)\exp\left\{-\alpha\sqrt{x^2+b^2}\right\}x\sin\left(\frac{\omega}{v}x\right)dx$$

Similarly,

$$\ddot{y}(\omega) = \frac{\alpha b Zze^2}{mv}\int_{-\infty}^{+\infty}\left(\frac{1}{x^2+b^2} + \frac{1}{\alpha(x^2+b^2)^{3/2}}\right)\exp\left\{-\alpha\sqrt{x^2+b^2}\right\}\cos\left(\frac{\omega}{v}x\right)dx.$$

Using the integration identities (see Gradshteyn and Ryzhik, 8^{th} ed., 3.914.5 and 3.914.10, pg. 522),

$$\int_0^{+\infty}\left(\frac{1}{\alpha(x^2+b^2)^{3/2}} + \frac{1}{x^2+b^2}\right)e^{-\alpha\sqrt{x^2+b^2}}\cos(\gamma x)\,dx = \frac{\sqrt{\alpha^2+\gamma^2}}{\alpha b}K_1\left(b\sqrt{\alpha^2+\gamma^2}\right),$$

$$\int_0^{+\infty}\left(\frac{1}{\alpha(x^2+b^2)^{3/2}} + \frac{1}{x^2+b^2}\right)e^{-\alpha\sqrt{x^2+b^2}}x\sin(\gamma x)\,dx = \frac{\gamma}{\alpha}K_0\left(b\sqrt{\alpha^2+\gamma^2}\right),$$

we will get

$$\ddot{x}(\omega) = \frac{2i\omega Zze^2}{mv^2}K_0\left(b\sqrt{\alpha^2+\frac{\omega^2}{v^2}}\right),\quad \ddot{y}(\omega) = \frac{2\sqrt{\alpha^2+\omega^2/v^2}\,Zze^2}{mv}K_1\left(b\sqrt{\alpha^2+\omega^2/v^2}\right)$$

From the dipole approximation, $\frac{dI}{d\omega}(\omega,b) = \frac{2}{3\pi c^3}\left|\ddot{\vec{d}}(\omega)\right|^2$, where $\ddot{\vec{d}}(\omega) = ze\left(\ddot{x}(\omega), \ddot{y}(\omega)\right)$

Then,
$$\frac{dI}{d\omega}(\omega,b) = \frac{8}{3\pi c^3}\left(\frac{Zz^2e^3}{mv}\right)^2\left[\frac{\omega^2}{v^2}K_0^2\left(b\sqrt{\alpha^2+\frac{\omega^2}{v^2}}\right) + \left(\alpha^2+\frac{\omega^2}{v^2}\right)K_1^2\left(b\sqrt{\alpha^2+\frac{\omega^2}{v^2}}\right)\right]$$

For $\omega \gg v/b$, $b\sqrt{\alpha^2+\omega^2/v^2} \to \omega b/v \gg 1$ From the asymptotic behavior of the modified Bessel function, $K_\nu(x) \sim e^{-x}$, $x \to \infty$, we can see that the radiated energy is almost negligible. On the other hand, for $\omega \ll v/b$, we can safely drop terms with ω^2/v^2, and will obtain

$$\frac{dI}{d\omega}(\omega,b) = \frac{8}{3\pi c^3}\left(\frac{Zz^2e^3}{mv}\right)^2\alpha^2K_1(\alpha b) = \frac{8}{3\pi}\frac{Z^2e^2}{c}\left(\frac{z^2e^2}{mc^2}\right)^2\left(\frac{c}{v}\right)^2\alpha^2K_1^2(\alpha b).$$

(b) The radiation cross section is

$$\frac{d\chi}{d\omega} = \int_{b_{min}}^{b_{max}}\frac{dI}{d\omega}(\omega,b)\,2\pi b\,db = \frac{16}{3}\frac{Z^2e^2}{c}\left(\frac{z^2e^2}{mc^2}\right)^2\left(\frac{c}{v}\right)^2\int_{b_{min}}^{b_{max}}\alpha^2K_1^2(\alpha b)\,b\,db$$

$$= \frac{16}{3}\frac{Z^2e^2}{c}\left(\frac{z^2e^2}{mc^2}\right)^2\left(\frac{c}{v}\right)^2\int_{\alpha b_{min}}^{\alpha b_{max}}xK_1^2(x)\,dx.$$

Since
$$\int xK_1^2(x)\,dx = \frac{1}{2}x^2\left(K_1(x)^2 - K_0(x)K_2(x)\right) + \text{const.}$$
$$= \frac{1}{2}x^2\left[K_1(x)^2 - K_0(x)\left(K_0(x) + \frac{2}{x}K_1(x)\right)\right]$$
$$= -\frac{1}{2}x^2\left[K_0(x)^2 - K_1(x)^2 + \frac{2}{x}K_0(x)K_1(x)\right],$$

then
$$\frac{d\chi}{d\omega} = -\frac{16}{3}\frac{Z^2e^2}{c}\left(\frac{z^2e^2}{mc^2}\right)^2\left(\frac{c}{v}\right)^2\left\{\frac{x^2}{2}\left[K_0(x)^2 - K_1(x)^2 + \frac{2}{x}K_0(x)K_1(x)\right]\right\}\Bigg|_{\alpha b_{min}}^{\alpha b_{max}}$$