14.22 (a) I have tried to follow the procedure outlined in Prob 14.15, but the elliptic orbit makes the calculation impossible. Instead, the following comes Landau & Lifshitz, Classical Theory of Fields, section 70. We are going to use the dipole radiation results, Prob. 9.9 (a) of Jackson.

For the n-th harmonic, $\vec{p}(t) = \vec{p}_0 e^{-in\omega_0 t}$. Taking into account of $-n$ component, in Gauss unit, we have

$$P_n = 2 \cdot \frac{2}{3c^3} (n\omega_0)^4 |\vec{p}_n|^2 = \frac{4e^2}{3c^3} (n\omega_0)^4 |\vec{r}_n|^2 .$$

where $\vec{r}(t) = \vec{r}_n e^{-in\omega_0 t}$ or $\vec{r}_n = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \vec{r}(t) e^{in\omega_0 t} dt$. However, $\vec{r}_n$ is still hard to compute. Notice that $\vec{v}(t) = -in\omega_0 \vec{r}_n e^{-in\omega_0 t} = \vec{v}_n e^{-in\omega_0 t}$, or equivalently, $\vec{r}_n = \frac{i}{n\omega_0} \vec{v}_n$.

On the other hand,

$$\vec{v}_n = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} e^{in\omega_0 t} \dot{\vec{r}}(t) dt = \frac{\omega_0}{2\pi} \int_0^{2\pi} e^{in(u-\varepsilon \sin u)} d\vec{r} = \frac{\omega_0}{2\pi} \int_0^{2\pi} e^{in(u-\varepsilon\sin u)} \begin{Bmatrix} -a\sin u \\ a\sqrt{1-\varepsilon^2}\cos u \end{Bmatrix} du$$

Therefore.

$$\vec{r}_n = \frac{i}{2\pi n} \int_0^{2\pi} e^{in(u-\varepsilon\sin u)} \begin{Bmatrix} -a\sin u \\ a\sqrt{1-\varepsilon^2}\cos u \end{Bmatrix} du .$$

Using the identity $e^{ix\cos\phi} = \sum_{m=-\infty}^{+\infty} i^m J_m(x) e^{im\phi}$, we can write the exponential term in the integrand as

$$e^{in(u-\varepsilon\sin u)} = e^{inu} \exp\{-in\varepsilon\cos(u+\tfrac{\pi}{2})\} = e^{inu} \sum_{m=-\infty}^{+\infty} J_m(n\varepsilon)(-i)^m \exp\{-im(u+\pi/2)\} .$$

Then.

$$r_{nx} = -\frac{ia}{2\pi n} \int_0^{2\pi} e^{in(u-\varepsilon\sin u)} \sin u\, du = -\frac{a}{4\pi n} \int_0^{2\pi} e^{inu} \sum_{m=-\infty}^{+\infty} J_m(n\varepsilon)(-i)^m e^{-im(u+\pi/2)} (e^{iu}-e^{-iu})\, du$$

$$= -\frac{a}{2n} \left\{ J_{n+1}(n\varepsilon)(-i)^{n+1} e^{-i(n+1)\pi/2} - J_{n-1}(n\varepsilon)(-i)^{n-1} e^{-i(n-1)\pi/2} \right\}$$

$$= -\frac{a}{2n}(-1)^{n+1} \left\{ J_{n+1}(n\varepsilon) - J_{n-1}(n\varepsilon) \right\} = (-1)^n \frac{a}{n} J_n'(n\varepsilon) .$$

and

$$r_{ny} = \frac{ia\sqrt{1-\varepsilon^2}}{2n} \left\{ J_{n+1}(n\varepsilon)(-i)^{n+1} e^{-i(n+1)\pi/2} + J_{n-1}(n\varepsilon)(-i)^{n-1} e^{-i(n-1)\pi/2} \right\}$$

$$= (-1)^{n+1} \frac{ia\sqrt{1-\varepsilon^2}}{2n} \left\{ J_{n+1}(n\varepsilon) + J_{n-1}(n\varepsilon) \right\} = (-1)^{n+1} \frac{ia\sqrt{1-\varepsilon^2}}{n\varepsilon} J_n(n\varepsilon) .$$

We finally arrive at the result for the power radiated in the n-th multiple of ω_0 as

$$P_n = \frac{4e^2}{3c^3} (n\omega_0)^2 a^2 \cdot \left\{ \frac{1}{n^2} \left[(J_n'(n\varepsilon))^2 + \frac{1-\varepsilon^2}{\varepsilon^2} J_n^2(n\varepsilon) \right] \right\}$$

(b) For circular orbit, $\varepsilon \to 0$. In this limit, $J_n(n\varepsilon) \to \left(\frac{n\varepsilon}{2}\right)^n$, $J_n'(n\varepsilon) \to \frac{n}{2}\left(\frac{n\varepsilon}{2}\right)^{n-1}$,

and only $n=1$ term will contribute. Therefore,

$$P_1 = \frac{4e^2}{3c^3}\omega_0^4 a^2 \left(\frac{1}{4}+\frac{1}{4}\right) = \frac{2e^2}{3c^3}\omega_0^4 a^2 .$$

For hydrogen-like atoms, its radius for n-th eigenstate is $a = n^2 a_0/Z = \frac{n^2\hbar^2}{Zme^2}$. The fundamental frequency can be determined as

$$m\omega_0^2 a = \frac{Ze^2}{a^2}, \quad \text{or} \quad \omega_0 = \frac{Ze^2}{ma^3} = \frac{Ze^2}{m}\cdot\left(\frac{Zme^2}{n^2\hbar^2}\right)^3 = \frac{Z^4m^2e^8}{n^6\hbar^6}.$$

Thus, $\omega_0 = \frac{Z^2me^4}{n^3\hbar^3}$. The reciprocal mean life time is then given by

$$\frac{1}{\tau} = P_1/\hbar\omega_0 = \frac{2e^2}{3c^3}\frac{1}{\hbar}\omega_0^3 a^2 = \frac{2e^2}{3c^3}\frac{1}{\hbar}\left(\frac{Z^2me^4}{n^3\hbar^3}\right)^3\left(\frac{n^2\hbar^2}{Zme^2}\right)^2$$

$$= \frac{2e^2}{3c^3}\frac{1}{\hbar}\frac{Z^6m^3e^{12}}{n^9\hbar^9}\frac{n^4\hbar^4}{Z^2m^2e^4} = \frac{2}{3}\frac{Z^4me^{10}}{n^5\hbar^6c^3},$$

which agrees with Prob. 14.21 (a).