3.13 Solution: The Green function satisfying the Dirichlet boundary condition is given by Eq. (3.125),

$$G(\mathbf{x},\mathbf{x}') = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]} \left(r^l_< - \frac{a^{2l+1}}{r^{l+1}_<}\right)\left(\frac{1}{r^{l+1}_>} - \frac{r^l_>}{b^{2l+1}}\right).$$

To find the potential between the spheres, we need the normal derivative of the Green function on the surfaces of the spheres. At $r'=b$, the normal direction is pointing away from the origin, with $r_< = r$ and $r_> = r'$,

$$\begin{aligned} \left.\frac{\partial G}{\partial n'}\right|_{r'=b} &= \left.\frac{\partial G}{\partial r'}\right|_{r'=b} \\ &= 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]} \left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\frac{-(2l+1)}{b^{l+2}}. \end{aligned}$$

Similarly, at $r'=a$, the normal direction is pointing to the origin, with $r_> = r$ and $r_< = r'$,

$$\begin{aligned} \left.\frac{\partial G}{\partial n'}\right|_{r'=a} &= -\left.\frac{\partial G}{\partial r'}\right|_{r'=a} \\ &= 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]} \left[-(2l+1)a^{l-1}\right]\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right). \end{aligned}$$

The contribution to the potential from the inner sphere can now be written down as

$$\begin{aligned} \Phi_{in}(r,\theta,\phi) &= -\frac{1}{4\pi}\oint \Phi(a,\theta',\phi')\left.\frac{\partial G}{\partial n'}\right|_{r'=a} da' \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y_{lm}(\theta,\phi)}{1-\left(\frac{a}{b}\right)^{2l+1}} a^{l-1}\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\int \Phi(a,\theta',\phi')Y^*_{lm}(\theta',\phi')a^2 d\Omega' \\ &= V\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Y_{lm}(\theta,\phi)}{1-\left(\frac{a}{b}\right)^{2l+1}} a^{l+1}\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\int_0^{2\pi} d\phi'\int_0^{\pi/2} d\theta' \sin\theta' Y^*_{lm}(\theta',\phi'). \end{aligned}$$

To perform the angular integral, it is more convenient to also include $Y_{lm}(\theta,\phi)$. Then,

$$\begin{aligned} &\int_0^{2\pi} d\phi'\int_0^{\pi/2} d\theta' \sin\theta' Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \\ &= \frac{2l+1}{4\pi}\int_0^{2\pi} d\phi'\int_0^{\pi/2} d\theta' \sin\theta' \frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta')P_l^m(\cos\theta)e^{im(\phi-\phi')}. \end{aligned}$$

Only the $m=0$ term will survive the integral, and therefore,

$$\int_0^{2\pi} d\phi'\int_0^{\pi/2} d\theta' \sin\theta' Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

$$
\begin{aligned}
&= \delta_{m,0}\frac{2l+1}{2}P_l(\cos\theta)\int_0^1 dx P_l(x) \\
&= \delta_{m,0}\frac{1}{2}P_l(\cos\theta)\left[P_{l-1}(0)-P_{l+1}(0)\right],
\end{aligned}
$$

for $l > 0$. If $l = 0$, the integral is clearly $1/2$. We can define

$$
D_l = (2l+1)\int_0^1 dx P_l(x) = \begin{cases} 1, & l=0, \\ P_{l-1}(0)-P_{l+1}(0), & l>0, \end{cases}
$$

to write the angular integral as $D_l P_l(\cos\theta)/2$. Then,

$$
\Phi_{in}(r,\theta,\phi) = \frac{V}{2}\sum_{l=0}^{\infty}\frac{D_l P_l(\cos\theta)}{1-\left(\frac{a}{b}\right)^{2l+1}} a^{l+1}\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right).
$$

Similarly, for the outer sphere,

$$
\begin{aligned}
\Phi_{out}(r,\theta,\phi) &= -\frac{1}{4\pi}\oint \Phi(b,\theta',\phi')\left.\frac{\partial G}{\partial n'}\right|_{r'=b} da' \\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}(\theta,\phi)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right)\frac{1}{b^{l+2}}\int \Phi(b,\theta',\phi')Y^*_{lm}(\theta',\phi')b^2 d\Omega' \\
&= V\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}(\theta,\phi)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right)\frac{1}{b^l}\int_0^{2\pi}d\phi'\int_{\pi/2}^{\pi}d\theta'\sin\theta' Y^*_{lm}(\theta',\phi').
\end{aligned}
$$

The angular integral is, again, non-zero only for $m = 0$, and

$$
\begin{aligned}
&\int_0^{2\pi}d\phi'\int_{\pi/2}^{\pi}d\theta'\sin\theta' Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \\
&= \frac{2l+1}{4\pi}\int_0^{2\pi}d\phi'\int_{\pi/2}^{\pi}d\theta'\sin\theta'\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta')P_l^m(\cos\theta)e^{im(\phi-\phi')} \\
&= \delta_{m,0}\frac{2l+1}{2}P_l(\cos\theta)\int_{-1}^{0}dx P_l(x) \\
&= \delta_{m,0}(-1)^l\frac{2l+1}{2}P_l(\cos\theta)\int_0^1 dx P_l(x) \\
&= \delta_{m,0}(-1)^l\frac{D_l}{2}P_l(\cos\theta).
\end{aligned}
$$

The potential from the outer sphere is

$$
\Phi_{out}(r,\theta,\phi) = \frac{V}{2}\sum_{l=0}^{\infty}(-1)^l\frac{D_l P_l(\cos\theta)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right)\frac{1}{b^l}.
$$

Finally, the potetial between the spheres becomes

$$
\Phi(r,\theta,\phi) = \Phi_{in}(r,\theta,\phi)+\Phi_{out}(r,\theta,\phi)
$$

$$
\begin{aligned}
&= \frac{V}{2}\sum_{l=0}^{\infty}\frac{D_l P_l(\cos\theta)}{1-\left(\dfrac{a}{b}\right)^{2l+1}}\left[a^{l+1}\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)+\frac{(-1)^l}{b^l}\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right)\right] \\
&= \frac{V}{2}\sum_{l=0}^{\infty}\frac{D_l P_l(\cos\theta)}{1-\left(\dfrac{a}{b}\right)^{2l+1}}\left[\left(1-(-1)^l\left(\frac{a}{b}\right)^l\right)\left(\frac{a}{r}\right)^{l+1}-\left(\left(\frac{a}{b}\right)^{l+1}-(-1)^l\right)\left(\frac{r}{b}\right)^l\right],
\end{aligned}
$$

which agrees with Problem 3.1.