9.10 (a) From the definition of orbital magnetization, we know

$$\vec{M}(r,\theta,\varphi,t) = \frac{1}{2}\vec{r}\times\vec{J} = -i\frac{v_0 a_0}{4Z}(\vec{r}\times\hat{z})\rho(r,\theta,\varphi,t),$$

Since $\vec{r}\times\hat{r}=0$. Also,

$$\vec{r}\times\hat{z} = r\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ 0 & 0 & 1\end{vmatrix} = r(\hat{x}\sin\theta\sin\varphi - \hat{y}\sin\theta\cos\varphi) = (\hat{x}\sin\varphi - \hat{y}\cos\varphi)\cdot r\sin\theta,$$

Then, $\vec{M}(r,\theta,\varphi,t) = -i\frac{\alpha c a_0}{4}\frac{r\sin\theta}{r\cos\theta}(\hat{x}\sin\varphi-\hat{y}\cos\varphi)\rho(r,\theta,\varphi,t). = -i\frac{\alpha c a_0}{4}\tan\theta(\hat{x}\sin\varphi-\hat{y}\cos\varphi)\rho(r,\theta,\varphi,t)$

Using the charge expression, $\rho(r,\theta,\varphi,t) = \frac{2e}{\sqrt{6}a_0^4} r e^{-3r/2a_0} Y_{00} Y_{10} e^{-i\omega_0 t}$

$$= \frac{\sqrt{2}e}{4\pi a_0^4} r\cos\theta\, e^{-3r/2a_0} e^{-i\omega_0 t},$$

the magnetization becomes

$$\vec{M}(r,\theta,\varphi,t) = -i\frac{\sqrt{2}e\alpha c}{16\pi a_0^3}(y\hat{x}-x\hat{y})e^{-3r/2a_0}e^{-i\omega_0 t}$$

Direct calculation shows that

$$\nabla\cdot\vec{M} = \frac{\partial M_x}{\partial x}+\frac{\partial M_y}{\partial y}+\frac{\partial M_z}{\partial z}$$

$$= -i\frac{\sqrt{2}e\alpha c}{16\pi a_0^3}\left(-\frac{3}{2a_0}e^{-3r/2a_0}\right)\frac{1}{r}(xy-yx)e^{-i\omega_0 t}$$

$$=0.$$

Also, $\vec{r}\times\vec{M} = -i\frac{\sqrt{2}e\alpha c}{16\pi a_0^3}\left(xz\hat{x}+yz\hat{y}-(x^2+y^2)\hat{z}\right)e^{-3r/2a_0}e^{-i\omega_0 t}.$

and it is easy to show that $\nabla\cdot(\vec{r}\times\vec{M})=0$. Then, by Eqs. (9.170) and (9.172), only non-vanishing multipole moments will be

$$Q_{lm} = \int r^l Y_{lm}^* \rho\, d^3x,$$

which is non-zero only when $l=1$ and $m=0$, as $\rho(r,\theta,\varphi,t)\propto P_1(\cos\theta)$. Performing the integral,

$$Q_{10} = \int r\cdot\sqrt{\frac{3}{4\pi}}\cos\theta\cdot\frac{\sqrt{2}e}{4\pi a_0^4} r\cos\theta\, e^{-3r/2a_0} e^{-i\omega_0 t} d^3x$$

$$= \sqrt{\frac{3}{4\pi}}\frac{\sqrt{2}e}{4\pi a_0^4}e^{-i\omega_0 t}\cdot 2\pi\cdot\int_{-1}^{1}\cos^2\theta\, d(\cos\theta)\cdot\int_0^{+\infty} r^4 e^{-3r/2a_0}dr$$

$$= \sqrt{\frac{3}{4\pi}} \frac{\sqrt{2}e}{4\pi a_0^4} e^{-i\omega_0 t} \, 2\pi \cdot \frac{2}{3} \cdot \frac{24}{(3/2a_0)^5} = \frac{2^8}{3^5} \sqrt{\frac{3}{2\pi}} e a_0 e^{-i\omega_0 t}$$

Therefore $a_E(1,0) = \frac{ck^3}{3i} \sqrt{2}\, Q_{10} = -ick^3 e a_0 \frac{2^8}{3^6} \sqrt{\frac{3}{\pi}}$.

(b) Using Eq. 19.151), we have

$$\frac{dP(1,0)}{d\Omega} = \frac{Z_0}{2k^2} |a_E(1,0)|^2 |\vec{X}_{10}|^2 = \frac{Z_0}{2k^2} c^2 k^6 e^2 a_0^2 \frac{2^{16}}{3^{12}} \frac{3}{\pi} \cdot \frac{3}{8\pi} \sin^2\theta$$

$$= \frac{Z_0 c^2 k^4 e^2 a_0^2}{\pi^2} \frac{2^{12}}{3^{10}} \sin^2\theta,$$

and total power radiated is

$$P = \frac{Z_0 c^2 k^4 e^2 a_0^2}{\pi^2} \frac{2^{12}}{3^{10}} \frac{8\pi}{3} = \frac{Z_0 c^2 k^4 e^2 a_0^2}{\pi} \frac{2^{15}}{3^{11}} = \frac{2^{15}}{3^{11}} \cdot \frac{\omega_0^4 e^2 a_0^2}{\pi \varepsilon_0 c^3},$$

where we have used that $k = \omega_0/c$, and $Z_0 c = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\varepsilon_0}$. After some tedious algebra, it can be shown that $P = \left(\frac{2}{3}\right)^8 (\hbar\omega_0)\left(\frac{\alpha^4 c}{a_0}\right)$

(c) Skipped

(d) Under the assumption of circular orbit, the charge density can be expressed as

$$\rho(\vec{x}) = \frac{e}{r^2} \delta(r-2a_0)\, \delta(\cos\theta)\, \delta(\phi - \omega_0 t).$$

Then the dipole moment is

$$\vec{p} = \int \rho \vec{x}\, d^3x = \int \rho \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} d^3x = \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \int_0^{+\infty} \delta(r-2a_0)\, \delta(\cos\theta)\, \delta(\phi-\omega_0 t)\, er \begin{Bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{Bmatrix}$$

$$= 2ea_0 \begin{Bmatrix} \cos(\omega_0 t) \\ \sin(\omega_0 t) \\ 0 \end{Bmatrix} = 2ea_0 \,\mathrm{Re}\left[\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} e^{-i\omega_0 t} \right].$$

Therefore, by Eq. (9.24), the total power radiated is

$$P = \frac{Z_0 c^2 k^4}{12\pi} |\vec{p}|^2 = \frac{\omega_0^4}{12 \varepsilon_0 \pi c^3} 4e^2 a_0^2 \cdot 2 = \frac{2}{3} \frac{\omega_0^4 e^2 a_0^2}{\pi \varepsilon_0 c^3} = \frac{2}{3} \cdot \frac{3^3}{2^7} (\hbar\omega_0) \cdot \left(\frac{\alpha^4 c}{a_0}\right)$$

$$= \frac{9}{64} (\hbar\omega_0)\left(\frac{\alpha^4 c}{a_0}\right)$$

Compared with part (b), the ratio is $\frac{3^2/2^6}{2^8/3^8} = \frac{3^{10}}{2^{14}} = 3.604065$