3.18 Solution: (a) Similar to Problem 3.12, the general form of the solution that vanishes on the $z = 0$ plane is

$$\Phi(\rho, \phi, z) = \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk A_m(k) J_m(k\rho) \sinh(kz) e^{im\phi}.$$

On the $z = L$ plane, the potential is specified,

$$V I_{\rho \leq a} = \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk A_m(k) J_m(k\rho) \sinh(kL) e^{im\phi}.$$

Clearly, only the $m = 0$ term will contribute and the coefficient $A_0(k)$ can be determined as

$$\begin{aligned}
V \int_0^a \rho J_0(k\rho) d\rho &= \int_0^{\infty} dk' A_0(k') \sinh(k'L) \cdot \left(\int_0^{\infty} \rho J_0(k\rho) J_0(k'\rho) d\rho \right) \\
&= \int_0^{\infty} dk' A_0(k') \sinh(k'L) \cdot \frac{1}{k} \delta(k - k') \\
&= \frac{A_0(k)}{k} \sinh(kL),
\end{aligned}$$

where we have used the Hankel transform

$$\frac{1}{k} \delta(k - k') = \int_0^{\infty} \rho J_\nu(k\rho) J_\nu(k'\rho) d\rho.$$

Using the fact that

$$\frac{d}{dx} \left(x J_1(x) \right) = x J_0(x),$$

the above integral can be exactly performed,

$$A_0(k) = \frac{V}{k \sinh(kL)} x J_1(x) \Big|_{x=0}^{ka} = \frac{Va}{\sinh(kL)} J_1(ka).$$

Therefore, the potential above the plane is

$$\Phi(\rho, \phi, z) = Va \int_0^{\infty} J_1(ka) J_0(k\rho) \frac{\sinh(kz)}{\sinh(kL)} dk.$$

With a change of integration variable $\lambda = ka$, we will obtain the desired form

$$\Phi(\rho, \phi, z) = V \int_0^{\infty} J_1(\lambda) J_0(\lambda\rho/a) \frac{\sinh(\lambda z/a)}{\sinh(\lambda L/a)} dk.$$

(b) With $a \to \infty$, we can expand J_0 and sinh functions with MacLaurin series,

$$J_0\left(\frac{\lambda\rho}{a}\right) = 1 - \frac{\lambda^2 \rho^2}{4a^2} + \cdots,$$

$$\sinh\left(\frac{\lambda z}{a}\right) = \frac{\lambda z}{a} + \frac{1}{6} \left(\frac{\lambda z}{a}\right)^3 + \cdots,$$

and

$$\sinh\left(\frac{\lambda z}{L}\right) = \frac{\lambda z}{L} + \frac{1}{6} \left(\frac{\lambda z}{L}\right)^3 + \cdots.$$

Then,

$$
\begin{aligned}
\Phi(\rho, z) &= V \int_0^\infty d\lambda J_1(\lambda) \left(1 - \frac{\lambda^2 \rho^2}{4a^2} + \cdots\right) \frac{\dfrac{\lambda z}{a} + \dfrac{1}{6}\left(\dfrac{\lambda z}{a}\right)^3 + \cdots}{\dfrac{\lambda z}{L} + \dfrac{1}{6}\left(\dfrac{\lambda z}{L}\right)^3 + \cdots} \\
&= V \frac{z}{L} \int_0^\infty d\lambda J_1(\lambda) \left[1 + \frac{\lambda^2}{a^2}\left(\frac{z^2 - L^2}{6} - \frac{\rho^2}{4}\right) + \cdots\right],
\end{aligned}
$$

where the higher order terms are proportional to the even powers of λ. The first term integrates to 1, since

$$\int_0^\infty J_\nu(\lambda) d\lambda = 1,$$

for $\mathrm{Re}(\nu) \geq -1$. So, the leading order contribution to the potential is

$$\Phi^{(0)}(\rho, z) = V \frac{z}{L}.$$

This is expected, as $a \to \infty$, the whole system approaches an infinite capacitor with parallel plates, where the potential between the plates drop linearly from one to the other.

For higher order terms, we need to perform the integration

$$\int_0^\infty \lambda^{2n} J_1(\lambda) d\lambda,$$

with $n > 0$. This integral diverges, which means that the problem cannot be solved perturbatively.