13.12 (a) From Eq. (13.78), the contribution to the radiation field from the medium in the $z>0$ region is given by $\vec{E}_{rad,z>0} = \frac{e^{ikr}}{r}\left(\frac{-\omega_2^2}{4\pi c^2}\right)\vec{F}_{z>0}$, with, Eq. (13.83)

$$\vec{F}_{z>0} = \vec{\varepsilon}_a 4\sqrt{2\pi}\,\frac{ze}{c}\left(\frac{c}{\omega_-}\right)^2 \gamma\sqrt{\eta}\left(\frac{1}{1+\eta} - \frac{1}{1+\frac{1}{\nu^2}+\eta}\right).$$ Therefore.

$$\vec{E}_{rad,z>0} = \vec{\varepsilon}_a\left(-\sqrt{\frac{2}{\pi}}\right)\frac{ze\theta}{c}\left(\frac{1}{\frac{1}{\gamma^2}+\theta^2} - \frac{1}{\frac{1}{\gamma^2}+\frac{\omega_2^2}{\omega^2}+\theta^2}\right)\frac{e^{ikr}}{r}$$

Similarly, in the $z<0$ region.

$$\vec{E}_{rad,z<0} = \vec{\varepsilon}_a\sqrt{\frac{2}{\pi}}\frac{ze\theta}{c}\left(\frac{1}{\frac{1}{\gamma^2}+\theta^2} - \frac{1}{\frac{1}{\gamma^2}+\frac{\omega_1^2}{\omega^2}+\theta^2}\right)\frac{e^{ikr}}{r}$$

Which can be obtained by a change of variable $\vec{x}\to-\vec{x}$ in the calculation for $\vec{F}$.

then, the total radiation field becomes

$$\vec{E}_{rad} = \vec{E}_{rad,z>0} + \vec{E}_{rad,z<0} = \vec{\varepsilon}_a\sqrt{\frac{2}{\pi}}\frac{ze\theta}{c}\left(\frac{1}{\frac{1}{\gamma^2}+\frac{\omega_2^2}{\omega^2}+\theta^2} - \frac{1}{\frac{1}{\gamma^2}+\frac{\omega_1^2}{\omega^2}+\theta^2}\right),$$

and $$\frac{d^2I}{d\omega\,d\Omega} = \frac{z^2e^2\theta^2}{\pi^2 c}\left|\frac{1}{\frac{1}{\gamma^2}+\frac{\omega_1^2}{\omega^2}+\theta^2} - \frac{1}{\frac{1}{\gamma^2}+\frac{\omega_2^2}{\omega^2}+\theta^2}\right|^2.$$

(b) The total energy radiated can be directly integrated

$$I = \int_0^{+\infty}d\omega\int d\Omega\,\frac{d^2I}{d\omega\,d\Omega} = \frac{2z^2e^2}{\pi c}\int_0^{+\infty}d\omega\int_0^{+\infty}\theta^3 d\theta\left(\frac{1}{\frac{1}{\gamma^2}+\frac{\omega_1^2}{\omega^2}+\theta^2} - \frac{1}{\frac{1}{\gamma^2}+\frac{\omega_2^2}{\omega^2}+\theta^2}\right)^2$$

$$= \frac{z^2e^2}{\pi c}\int_0^{+\infty}d\omega\int_0^{+\infty}\theta^2\left(\frac{1}{\frac{1}{\gamma^2}+\frac{\omega_1^2}{\omega^2}+\theta^2} - \frac{1}{\frac{1}{\gamma^2}+\frac{\omega_2^2}{\omega^2}+\theta^2}\right)^2 d(\theta^2)$$

Integration with respect to θ is

$$\int_0^{+\infty}\theta^2\left(\frac{1}{\frac{1}{\gamma^2}+\frac{\omega_1^2}{\omega^2}+\theta^2} - \frac{1}{\frac{1}{\gamma^2}+\frac{\omega_2^2}{\omega^2}+\theta^2}\right)^2 d(\theta^2) = \frac{\omega_1^2+\omega_2^2+2\omega^2/\gamma^2}{\omega_1^2-\omega_2^2}\log\left(\frac{\omega_1^2+\omega^2/\gamma^2}{\omega_2^2+\omega^2/\gamma^2}\right) - 2$$

Next, perform the integration w.r.t. ω, we will have

$$\int_0^{+\infty}\left[\frac{\omega_1^2+\omega_2^2+2\omega^2/\gamma^2}{\omega_1^2-\omega_2^2}\log\left(\frac{\omega_1^2+\omega^2/\gamma^2}{\omega_2^2+\omega^2/\gamma^2}\right)-2\right]d\omega = \gamma\int_0^{+\infty}\left[\frac{\omega_1^2+\omega_2^2+2y^2}{\omega_1^2-\omega_2^2}\log\left(\frac{\omega_1^2+y^2}{\omega_2^2+y^2}\right)-2\right]dy$$

$$= \frac{\gamma}{3(\omega_1^2-\omega_2^2)}\Bigg[2\left(\omega_1^3+3\omega_1\omega_2^2\right)\arctan\left(\frac{y}{\omega_1}\right) - 2\left(3\omega_1^2\omega_2+\omega_2^3\right)\arctan\left(\frac{y}{\omega_2}\right)$$
$$+\, y\left((3\omega_1^2+3\omega_2^2+2y^2)\log\left(\frac{\omega_1^2+y^2}{\omega_2^2+y^2}\right) - 2(\omega_1^2-\omega_2^2)\right)\Bigg]\Bigg|_0^{+\infty}$$

$$= \frac{\gamma}{3(\omega_1^2-\omega_2^2)}\Big[\pi\left(\omega_1^3-3\omega_1^2\omega_2+3\omega_1\omega_2^2-\omega_2^3\right)$$

$$+\lim_{y\to\infty} y\left((3\omega_1^2+3\omega_2^2+2y^2)\log\left(\frac{\omega_1^2+y^2}{\omega_2^2+y^2}\right)-2(\omega_1^2-\omega_2^2)\right)\Big]$$

The last term can be expand at $y\to\infty$ as

$$[3(\omega_1^2+\omega_2^2)+2y^2]\log\left(\frac{\omega_1^2+y^2}{\omega_2^2+y^2}\right)-2(\omega_1^2-\omega_2^2)$$

$$=[3(\omega_1^2+\omega_2^2)+2y^2]\left(\frac{\omega_1^2-\omega_2^2}{y^2}-\frac{\omega_1^4-\omega_2^4}{2y^4}\right)-2(\omega_1^2-\omega_2^2)$$

$$=2(\omega_1^2-\omega_2^2)+\frac{2(\omega_1^4-\omega_2^4)}{y^2}+O\left(\frac{1}{y^4}\right)-2(\omega_1^2-\omega_2^2)$$

$$=\frac{2(\omega_1^4-\omega_2^4)}{y^2}+O\left(\frac{1}{y^4}\right),$$

and the last term will vanish. We are then left with

$$I=\frac{z^2e^2}{\pi c}\cdot\frac{\gamma\pi(\omega_1-\omega_2)^3}{3(\omega_1^2-\omega_2^2)}=\frac{z^2e^2}{3c}\cdot\frac{(\omega_1-\omega_2)^2}{\omega_1+\omega_2}\cdot\gamma$$