6.5 Solution: (a) Using the expression for the momentum of the EM field and the Maxwell equations, we have

$$\begin{aligned}
\mathbf{P}_{\text{field}} &= \frac{1}{c^2}\int (\mathbf{E}\times\mathbf{H}) d^3x &(1)\\
&= -\frac{1}{c^2}\int (\nabla\Phi\times\mathbf{H}) d^3x \\
&= -\frac{1}{c^2}\int \Big[\nabla\times(\Phi\mathbf{H}) - \Phi(\nabla\times\mathbf{H})\Big] d^3x \\
&= \frac{1}{c^2}\int \Phi\mathbf{J} d^3x - \frac{1}{c^2}\int \nabla\times(\Phi\mathbf{H}) d^3x \\
&= \frac{1}{c^2}\int \Phi\mathbf{J} d^3x - \frac{1}{c^2}\oint \mathbf{n}\times(\Phi\mathbf{H}) da, &(2)
\end{aligned}$$

where we have used $\mathbf{E} = -\nabla\Phi$, $\nabla\times\mathbf{H} = \mathbf{J}$, $\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi(\nabla\times\mathbf{a})$, and the Stokes theorem to transform the volume integral to the surface integral in the last step. The surface integral can be dropped, as long as the integrand $\Phi\mathbf{H}$ falls faster than r^{-2}, which is the case for electric and magnetic fields generated from localized charge and current, as both of them fall as r^{-2}. Therefore, we can write the EM field momentum as

$$\mathbf{P}_{\text{field}} = \frac{1}{c^2}\int \Phi\mathbf{J} d^3x. \tag{3}$$

(b) We can apply the result from part (a), and notice that near the current distribution,

$$\Phi(\mathbf{x}) = \Phi(\mathbf{0}) + \nabla\Phi(\mathbf{0})\cdot\mathbf{x} = \Phi(\mathbf{0}) - \mathbf{E}(\mathbf{0})\cdot\mathbf{x},$$

the EM field momentum becomes

$$\mathbf{P}_{\text{field}} = \frac{\Phi(\mathbf{0})}{c^2}\int \mathbf{J}(\mathbf{x}) d^3x - \frac{1}{c^2}\mathbf{E}(\mathbf{0})\cdot\int \mathbf{x}\mathbf{J}(\mathbf{x}) d^3x.$$

From section 5.6, we know that

$$\int \mathbf{J}(\mathbf{x}) d^3x = 0, \qquad \mathbf{n}\cdot\int \mathbf{x}\mathbf{J}(\mathbf{x}) d^3x = -\frac{1}{2}\mathbf{n}\times\int [\mathbf{x}\times\mathbf{J}(\mathbf{x})] d^3x = -\mathbf{n}\times\mathbf{m}.$$

Then,

$$\mathbf{P}_{\text{field}} = -\frac{1}{c^2}E(\mathbf{0})\mathbf{n}\cdot\int \mathbf{x}\mathbf{J}(\mathbf{x}) d^3x = \frac{1}{c^2}E(\mathbf{0})\mathbf{n}\times\mathbf{m} = \frac{1}{c^2}E(\mathbf{0})\times\mathbf{m}.$$

(c) For a uniform electric field, the surface integral term in Eq. (2) cannot be dropped, as $\Phi \sim r$ and $|\mathbf{H}| \sim r^-2$ by a crude estimation.

We can calculate the EM field momentum in two ways. First, using Eq. (1) and notice that the electric field is a uniform one, we can express the result as

$$\mathbf{P}_{\text{field}} = \frac{1}{c^2}\mathbf{E}_0\times\int \mathbf{H} d^3x.$$

From Section 5.6, by Eq. (5.62), we also know

$$\int \mathbf{H} d^3x = \frac{2}{3}\mathbf{m}.$$

Then,

$$\mathbf{P}_{\text{field}} = \frac{2}{3c^2}\mathbf{E}_0 \times \mathbf{m}.$$

We can also determine the EM field momentum by explicit calculation of the surface intergral. In Eq. (2), since $\Phi(\mathbf{x}) = -\mathbf{E}_0 \cdot \mathbf{x}$, the first term becomes

$$\frac{1}{c^2}\int \Phi\mathbf{J} d^3x = -\frac{1}{c^2}\mathbf{E}_0 \cdot \int \mathbf{x}\mathbf{J}(\mathbf{x}) d^3x,$$

which, from the result of part (b), is

$$\frac{1}{c^2}\int \Phi\mathbf{J} d^3x = \frac{1}{c^2}\mathbf{E}_0 \times \mathbf{m}.$$

For the second term, the surface integral, we can use the expression for the magnetic field,

$$\mathbf{H}(\mathbf{x}) = \frac{1}{4\pi}\int \mathbf{J}(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x',$$

to write it as

$$\begin{aligned}
-\frac{1}{c^2}\oint \mathbf{n} \times (\Phi\mathbf{H}) da &= -\frac{1}{4\pi c^2}\oint \mathbf{n} \times \left(\Phi(\mathbf{x}) \int \mathbf{J}(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} d^3x'\right) da \\
&= -\frac{1}{4\pi c^2}\int d^3x' \oint da \left[\mathbf{n} \times \left(\Phi(\mathbf{x})\mathbf{J}(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right)\right]. \qquad (4)
\end{aligned}$$

Since

$$\mathbf{n} \times \left(\Phi(\mathbf{x})\mathbf{J}(\mathbf{x}') \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right) = \Phi(\mathbf{x}) \left[\mathbf{J}(\mathbf{x}') \left(\mathbf{n} \cdot \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\right) - \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \left(\mathbf{n} \cdot \mathbf{J}(\mathbf{x}')\right)\right],$$

(cannot proceed further...)