13.15 The number of transition radiation quanta emitted in the frequency interval $[\omega, \omega + d\omega)$

is given by
$$\frac{dN_\gamma}{d\nu} = \frac{1}{\hbar\omega}\frac{dI}{d\nu} = \frac{Z^2e^2}{\hbar\pi c}\frac{1}{\nu}\left[(1+2\nu^2)\log\left(1+\frac{1}{\nu^2}\right) - 2\right].$$

Then, the total number is
$$N_\gamma = \int_{1/\gamma}^{+\infty}\frac{dN_\gamma}{d\nu}d\nu = \frac{Z^2e^2}{\hbar\pi c}\int_{1/\gamma}^{+\infty}\frac{1}{\nu}\left[(1+2\nu^2)\log\left(1+\frac{1}{\nu^2}\right) - 2\right]d\nu.$$

Using the result (from Mathematica).
$$\int_a^{+\infty}\left[(1+2x^2)\log\left(1+\frac{1}{x^2}\right) - 2\right]\frac{dx}{x} = 1 - \frac{1}{2}\mathrm{Li}_2\left(-\frac{1}{a^2}\right) - (1+a^2)\log\left(1+\frac{1}{a^2}\right),$$

we know that
$$N_\gamma = \frac{Z^2e^2}{\hbar\pi c}\left[1 - \frac{1}{2}\mathrm{Li}_2(-\gamma^2) - \left(1+\frac{1}{\gamma^2}\right)\log(1+\gamma^2)\right],$$

where $\mathrm{Li}_2(x)$ is the dilogarithm function. Using its expansion at infinity,
$$\mathrm{Li}_2(x) = -\frac{1}{2}\log^2(-z) - \frac{\pi^2}{6} - \sum_{k=1}^{\infty}\frac{1}{k^2x^k},$$

we have
$$N_\gamma = \frac{Z^2e^2}{\pi\hbar c}\left[1 + \frac{1}{4}\log^2(\gamma^2) + \frac{\pi^2}{12} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2\gamma^{2k}} - \left(1+\frac{1}{\gamma^2}\right)\log(1+\gamma^2)\right]$$
$$= \frac{Z^2e^2}{\pi\hbar c}\left[1 + \log^2\gamma + \frac{\pi^2}{12} - 2\log\gamma + O\left(\frac{1}{\gamma^2}\right)\right]$$
$$= \frac{Z^2e^2}{\pi\hbar c}\left[(\log\gamma - 1)^2 + \frac{\pi^2}{12}\right].$$