8.17 (a) From Eqs. (8.128) and (8.126), the transverse electric and magnetic fields are given by

$$\vec{E}_t = \frac{i}{\gamma^2}\left[k_z \nabla_t E_z - \omega\mu_0 \hat{z}\times\nabla_t H_z \right], \quad \left(\nabla_t = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} \text{ in cylindrical coordinate}\right)$$

$$(\hat{z}\times\hat{\rho} = \hat{\phi},\ \hat{z}\times\hat{\phi} = -\hat{\rho})$$

$$= \frac{i}{\gamma^2}\Big[k_z\left(\hat{\rho} A_e \gamma J_m'(\gamma\rho) + \hat{\phi}\frac{A_e}{\rho} im J_m(\gamma\rho)\right) - \omega\mu_0 \hat{z}\times\left(\hat{\rho} A_h \gamma J_m'(\gamma\rho) + \hat{\phi}\frac{A_h}{\rho} im J_m(\gamma\rho)\right)\Big] e^{im\phi}$$

$$= \frac{i}{\gamma^2}\Big[\hat{\rho}\left(k_z A_e \gamma J_m'(\gamma\rho) + \omega\mu_0 \frac{A_h}{\rho} im J_m(\gamma\rho)\right) + \hat{\phi}\left(\frac{A_e}{\rho} im J_m(\gamma\rho) - \omega\mu_0 A_h \gamma J_m'(\gamma\rho)\right)\Big] e^{im\phi}$$

and

$$\vec{H}_t = \frac{i}{\gamma^2}\left[k_z \nabla_t H_z + \omega\epsilon_0 n_1^2 \hat{z}\times\nabla_t E_z \right]$$

$$= \frac{i}{\gamma^2}\Big[\hat{\rho}\left(k_z A_h \gamma J_m'(\gamma\rho) - \omega\epsilon_0 n_1^2 \frac{A_e}{\rho} im J_m(\gamma\rho)\right) + \hat{\phi}\left(k_z \frac{A_h}{\rho} im J_m(\gamma\rho) + \omega\epsilon_0 n_1^2 A_e \gamma J_m'(\gamma\rho)\right)\Big] e^{im\phi}.$$

for $\rho < a$. The $\rho > a$ result can be obtained by the replacements $\gamma^2 \to -\beta^2$, $\gamma \to \beta$, $n_1 \to n_2$, and

$$\vec{E}_t = -\frac{i}{\beta^2}\Big[\hat{\rho}\left(k_z B_e \beta K_m'(\beta\rho) + \omega\mu_0 \frac{B_h}{\rho} im B_m(\beta\rho)\right) + \hat{\phi}\left(\frac{B_e}{\rho} im K_m(\beta\rho) - \omega\mu_0 B_h \beta K_m'(\beta\rho)\right)\Big] e^{im\phi}$$

$$\vec{H}_t = -\frac{i}{\beta^2}\Big[\hat{\rho}\left(k_z B_h \beta K_m'(\beta\rho) - \omega\epsilon_0 n_2^2 \frac{B_e}{\rho} im K_m(\beta\rho)\right) + \hat{\phi}\left(k_z \frac{B_h}{\rho} im K_m(\beta\rho) + \omega\epsilon_0 n_2^2 B_e \beta K_m'(\beta\rho)\right)\Big] e^{im\phi}.$$

Imposing the boundary condition at $\rho = a$ that the tangential components of the electric and magnetic fields should be continuous we can get the following relations,

$$E_z\big|_{\rho=a^-} = E_z\big|_{\rho=a^+} \Rightarrow A_e J_m(\gamma a) = B_e K_m(\beta a)$$

$$H_z\big|_{\rho=a^-} = H_z\big|_{\rho=a^+} \Rightarrow A_h J_m(\gamma a) = B_h K_m(\beta a)$$

$$E_\phi\Big|_{\rho=a-} = E_\phi\Big|_{\rho=a+} \Rightarrow$$

$$\frac{1}{\gamma^2}\left(k_z \frac{A_e}{a} i m J_m(\gamma a) - \omega\mu_0 A_h \gamma J_m'(\gamma a)\right) = -\frac{1}{\beta^2}\left(k_z \frac{B_e}{a} i m K_m(\beta a) - \omega\mu_0 B_h \beta K_m'(\beta a)\right)$$

$$H_\phi\Big|_{\rho=a-} = H_\phi\Big|_{\rho=a+} \Rightarrow$$

$$\frac{1}{\gamma^2}\left(k_z \frac{A_h}{a} i m J_m(\gamma a) + \omega\varepsilon_0 n_1^2 A_e \gamma J_m'(\gamma a)\right) = -\frac{1}{\beta^2}\left(k_z \frac{B_h}{a} i m K_m(\beta a) + \omega\varepsilon_0 n_2^2 B_e \beta K_m'(\beta a)\right).$$

These conditions can be cast into a matrix form, $\Lambda \underline{\Phi} = 0$, where

$$\Lambda = \begin{pmatrix} J_m(\gamma a) & 0 & -K_m(\beta a) & 0 \\ 0 & J_m(\gamma a) & 0 & -K_m(\beta a) \\ \frac{i m k_z}{\gamma^2 a} J_m(\gamma a) & -\frac{\omega\mu_0}{\gamma} J_m'(\gamma a) & \frac{i m k_z}{\beta^2 a} K_m(\beta a) & -\frac{\omega\mu_0}{\beta} K_m'(\beta a) \\ \frac{\omega\varepsilon_0 n_1^2}{\gamma} J_m'(\gamma a) & \frac{i m k_z}{\gamma^2 a} J_m(\gamma a) & \frac{\omega\varepsilon_0 n_2^2}{\beta} K_m'(\beta a) & \frac{i m k_z}{\beta^2 a} K_m(\beta a) \end{pmatrix}, \quad \underline{\Phi} = \begin{pmatrix} A_e \\ A_h \\ B_e \\ B_h \end{pmatrix}.$$

To have non-trivial solution for the above equation, the determinant of Λ must be 0, i.e., $\det\Lambda = 0$. After some linear algebra, we can find the eigenequation as

$$\left(\frac{n_1^2}{\gamma}\frac{J_m'(\gamma a)}{J_m(\gamma a)} + \frac{n_2^2}{\beta}\frac{K_m'(\beta a)}{K_m(\beta a)}\right)\left(\frac{1}{\gamma}\frac{J_m'(\gamma a)}{J_m(\gamma a)} + \frac{1}{\beta}\frac{K_m'(\beta a)}{K_m(\beta a)}\right) = \frac{m^2}{a^2}\left(\frac{n_1^2}{\gamma^2} + \frac{n_2^2}{\beta^2}\right)\left(\frac{1}{\gamma^2} + \frac{1}{\beta^2}\right)$$

(b) For $m=0$, the continuity of ϕ components become

$$\frac{A_h}{\gamma} J_m'(\gamma a) = -\frac{B_h}{\beta} K_m'(\beta a) \quad \text{for } E_\phi$$

and

$$\frac{n_1^2 A_e}{\gamma} J_m'(\gamma a) = -\frac{n_2^2 B_e}{\beta} K_m'(\beta a), \quad \text{for } H_\phi$$

Therefore, the field equations for electric and magnetic fields decouple. For TE mode, we have $A_h, B_h \neq 0$, and the equation becomes $\left(\frac{1}{\gamma}\frac{J_0'}{J_0} + \frac{1}{\beta}\frac{K_0'}{K_0}\right) = 0$. Similarly, for TM mode, $A_e, B_e \neq 0$, and $\left(\frac{n_1^2}{\gamma}\frac{J_0'}{J_0} + \frac{n_2^2}{\beta}\frac{K_0'}{K_0}\right) = 0$.

The cutoff frequency corresponds to the solution that is not decaying in the cladding, when $\beta^2 \to 0$.

Then, the eigen equation becomes

$$\frac{\gamma J_0(\gamma a)}{J_0'(\gamma a)} = -\beta \frac{K_0(\beta a)}{K_0'(\beta a)} = 0 \quad , \text{ for TE mode}$$

and

$$\frac{\gamma J_0(\gamma a)}{n_1^2 J_0'(\gamma a)} = -\frac{\beta K_0(\beta a)}{n_2^2 K_0'(\beta a)} = 0 \quad , \text{ for TM mode.}$$

In either case, we should have $J_0(\gamma a) = 0$, and $\gamma^2 = \frac{\omega^2}{c^2}(n_1^2 - n_2^2)$ for $\beta = 0$. Since

$$\gamma a = \frac{\omega a}{c}\sqrt{n_1^2 - n_2^2} = \frac{n_1 \omega a}{c}\sqrt{\frac{n_1^2 - n_2^2}{2n_1^2} \cdot 2} = \frac{n_1 \omega a}{c}\sqrt{2\Delta} = V,$$

we can see that the cutoff frequency will appear for the roots of $J_0(x)$, the lowest of which corresponds to $V = 2.405$.

(c)