13.16 (a) The magnetic field from the neutral particle in its own rest frame is

$$\vec{B}' = \mu \frac{3\vec{n}'(\vec{n}'\cdot\hat{z}) - \hat{z}}{r'^3},$$

where $\vec{n}' = \frac{1}{r'}(\rho', 0, z')$, $r' = \sqrt{\rho'^2 + z'^2}$. Expressing the magnetic field in component form, we have

$$\vec{B}' = \frac{\mu}{r'^3}\left[\frac{3}{r'}(\rho', 0, z')\cdot\frac{z'}{r'} - (0,0,1)\right] = \frac{\mu}{r'^5}\left(3\rho' z', 0, 3z'^2 - r'^2\right).$$

or, $B'_\rho = \mu \dfrac{3\rho' z'}{[\rho'^2 + z'^2]^{5/2}}$, and $B'_z = \mu \dfrac{3z'^2 - \rho'^2}{[\rho'^2 + z'^2]^{5/2}}$

For calculations later, we only need to consider the B'_ρ component. Using the coordinates in the lab frame, $z' = \gamma(z - vt)$, the ρ-component becomes

$$B'_\rho = \mu \frac{3\rho\gamma(z - vt)}{[\rho^2 + \gamma^2(z - vt)^2]^{5/2}}.$$

and the transformed field in the lab frame is

$$B_\rho = \gamma B'_\rho = \mu \frac{3\rho\gamma^2(z - vt)}{[\rho^2 + \gamma^2(z - vt)^2]^{5/2}}$$

The electric field can be obtained by $\vec{E} = -\vec{\beta}\times\vec{B}$, and there will only be the ϕ-component.

$$E_\phi = -\beta B_\rho = -\beta\mu \frac{3\rho\gamma^2(z - vt)}{[\rho^2 + \gamma^2(z - vt)^2]^{5/2}}.$$

Now, we can perform the Fourier transform as

$$E_\phi(\vec{x}, \omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} E_\phi(\vec{x}, t)\, e^{i\omega t}\, dt = \beta\mu\left(\frac{\partial}{\partial z}\right)\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\rho\, e^{i\omega t}}{[\rho^2 + \gamma^2(z - vt)^2]^{3/2}}\, dt\right\}$$

$$= \beta\mu\left(\frac{\partial}{\partial z}\right)\left\{\frac{1}{\sqrt{2\pi}}\, e^{i\omega z/v}\int_{-\infty}^{+\infty}\frac{\rho\exp\left\{-i\frac{\omega}{\gamma v}u\right\}}{\gamma v(\rho^2 + u^2)^{3/2}}\, du\right\}$$

$$= \beta\mu\left(\frac{\partial}{\partial z}\right)\left\{\sqrt{\frac{2}{\pi}}\, e^{i\omega z/v}\int_{0}^{+\infty}\frac{\rho\cos\left(\frac{\omega u}{\gamma v}\right)}{\gamma v(\rho^2 + u^2)^{3/2}}\, du\right\}$$

$$= \frac{\beta\mu\omega}{\gamma^2 v^2}\left(\frac{\partial}{\partial z}\right)\left\{\sqrt{\frac{2}{\pi}}\, e^{i\omega z/v} K_1\left(\frac{\omega\rho}{\gamma v}\right)\right\} = \frac{\beta\mu}{\gamma Ze}\left(\frac{\partial}{\partial z}\right)\left\{\sqrt{\frac{2}{\pi}}\, e^{i\omega z/v}\frac{Ze\omega}{\gamma v^2} K_1\left(\frac{\omega\rho}{\gamma v}\right)\right\}$$

where we have used the identity $\displaystyle\int_0^{+\infty}\frac{\cos(\alpha t)}{(b^2 + t^2)^{3/2}}\, dt = \frac{\alpha}{b} K_1(\alpha b)$.

Comparing with Eqs. (13.79) and (13.80), we can see that the result for the neutral particle can be similarly obtained, with strength replacement $E_\rho \to \frac{\beta\mu}{\gamma z e}\frac{\partial}{\partial z}E_\rho$, while the electric field is in the azimuthal direction.

(b) In Fig 13.9, we have the polarization vectors as

$$\vec{\varepsilon}_a = \cos\theta\,\vec{e}_x - \sin\theta\,\vec{e}_z,\quad \vec{\varepsilon}_b = \vec{e}_y.$$

The radiation electric field has only ϕ-component, which can be expressed as

$$\vec{E}_\phi = E_\phi\hat{\phi} = -E_\phi\sin\phi\,\vec{e}_x + E_\phi\cos\phi\,\vec{e}_y = (\dots)\hat{k} - E_\phi\cos\theta\sin\phi\,\vec{\varepsilon}_a + E_\phi\cos\phi\,\vec{\varepsilon}_b,$$

where we have omitted the component in the $\hat{k}$-direction as it has no contribution to the final result. Then,

$$[\hat{k}\times\vec{E}_i]_{z=0}\times\hat{k} = \left(-E_\phi\cos\theta\sin\phi\,\vec{\varepsilon}_a + E_\phi\cos\phi\,\vec{\varepsilon}_b\right)_{z=0},$$

and $(E_\phi)_{z=0} = \frac{\beta\mu}{\gamma Ze}\frac{i\omega}{v}\sqrt{\frac{2}{\pi}}\frac{Ze\omega}{\gamma v^2}K_1\left(\frac{\omega\rho}{\gamma v}\right)$. We can evaluate $\vec{F}$ in a similar way,

$$\vec{F} = \frac{i}{\frac{\omega}{v}-k\cos\theta}\iint dx\,dy\,[\hat{k}\times\vec{E}_i]_{z=0}\times\hat{k}\,e^{-ikx\sin\theta}$$

$$= \frac{-\mu\omega/\gamma Zec}{\frac{\omega}{v}-k\cos\theta}\iint dx\,dy\,e^{-ikx\sin\theta}\left[-\cos\theta\sin\phi\,\vec{\varepsilon}_a + \cos\phi\,\vec{\varepsilon}_b\right]\sqrt{\frac{2}{\pi}}\frac{Ze\omega}{\gamma v^2}K_1\left(\frac{\omega\rho}{\gamma v}\right)$$

The integration for polarization $\vec{\varepsilon}_a$ will vanish, as the integrand is odd in y. Then,

$$\vec{F} = \vec{\varepsilon}_b\,\frac{-\mu\omega/\gamma Zec}{\frac{\omega}{v}-k\cos\theta}\iint dx\,dy\,e^{-ikx\sin\theta}\frac{x}{\sqrt{x^2+y^2}}\sqrt{\frac{2}{\pi}}\frac{Ze\omega}{\gamma v^2}K_1\left(\frac{\omega\rho}{\gamma v}\right)$$

$$= \vec{\varepsilon}_b\left(-\frac{\mu\omega}{\gamma Zec}\right)\frac{2\sqrt{2\pi}\,Ze\sin\theta\cdot k}{v\left(\frac{\omega}{v}-k\cos\theta\right)\left(\frac{\omega^2}{\gamma^2v^2}+k^2\sin^2\theta\right)}.$$

where we have followed the same procedure leading to Eq. (13.82). In the same limit for Eq. (13.83), $k \sim k\cos\theta \cdot \frac{\omega^2}{v^2\gamma^2}$. Therefore, $\vec{F}$ is different from Eq. (13.83) in magnitude by a factor of $\left(-\frac{\mu\omega}{\gamma Zec}\right)$, and the intensity distributions in angle and frequency are given by Eqs. (13.84) and (13.85), multiplied by $(\mu\omega/\gamma Zec)^2$.

(c) For electron, $Z=1$. Using the result from part (b), we have

$$\frac{dI_\mu(\nu)}{dI_e(\nu)} = \left(\frac{\mu\omega}{e\gamma c}\right)^2 = \left(\frac{\mu}{\mu_B}\right)^2 \nu^2 \left(\frac{\mu_B\omega_p}{ec}\right)^2.$$

Since $\mu_B = e\hbar/2m_e c$, we have $\frac{\mu_B}{ec^2} = \frac{\hbar}{2m_e c^2}$. Also, the Bohr radius is $a_0 = \frac{\hbar}{m_e c\alpha}$.

then $m_e c = \frac{\hbar}{a_0\alpha}$, and $\frac{\mu_B}{ec} = \frac{\hbar}{2\hbar c/a_0\alpha}$. The fine structure constant is given by

$\alpha = \frac{e^2}{\hbar c}$, we can finally have

$$\frac{\mu_B}{ec} = \hbar \cdot \frac{\alpha^2}{2} \cdot \frac{1}{e^2/a_0} = \frac{\alpha^2}{2}\frac{\hbar}{\hbar\omega_0}.$$

and

$$\frac{dI_\mu(\nu)}{dI_e(\nu)} = \frac{\alpha^4}{4}\left(\frac{\mu}{\mu_B}\right)^2\left(\frac{\hbar\omega_p}{\hbar\omega_0}\right)^2 \nu^2$$

(d) From Eq. (13.84), for the magnetic moment,

$$\frac{dI_\mu}{d\nu} = \frac{\mu^2\omega^2}{\gamma^2c^2}\frac{\gamma\omega_p}{\pi c}\left[(1+2\nu^2)\log\left(1+\frac{1}{\nu^2}\right)-2\right] = \frac{\mu^2\gamma\omega_p^3}{\pi c^3}\nu^2\left[(1+2\nu^2)\log\left(1+\frac{1}{\nu^2}\right)-2\right].$$

and

$$I_\mu = \int_0^{\nu_{max}} \frac{dI_\mu}{d\nu}\,d\nu$$

$$= \frac{1}{15}\left(2\arctan(\nu_{max}) + (6\nu_{max}^2+5)\nu_{max}^3\log\left(1+\frac{1}{\nu_{max}^2}\right) - 6\nu_{max}^3 - 2\nu_{max}\right)\cdot\frac{\mu^2\gamma\omega_p^3}{\pi c^3}$$

For electron, $I_e = \int_0^{\nu_{max}} \frac{dI_e}{d\nu}\,d\nu$

$$= \frac{1}{3}\left(2\arctan(\nu_{max}) + (2\nu_{max}^2+3)\nu_{max}\log\left(1+\frac{1}{\nu_{max}^2}\right) - 2\nu_{max}\right)\cdot\frac{e^2\gamma\omega_p}{\pi c}$$

Therefore

$$\frac{I_\mu}{I_e} = \frac{1}{5}\left(\frac{\mu\omega_p}{ec}\right)^2 G(\nu_{max}) = \frac{1}{5}\left(\frac{\mu}{\mu_B}\right)^2\left(\frac{\mu_B\omega_p}{ec}\right)^2 G(\nu_{max}) = \frac{\alpha^4}{20}\left(\frac{\mu}{\mu_B}\right)^2\left(\frac{\hbar\omega_p}{\hbar\omega_0}\right)^2 G(\nu_{max}).$$

where

$$G(x) = \frac{2\arctan(x) + (6x^2+5)x^3\log(1+1/x^2) - 6x^3 - 2x}{2\arctan(x) + (2x^2+3)x\log(1+1/x^2) - 2x}.$$

We have used Mathematica to perform the integral, and part (c) to express the result with dimensionless constants.