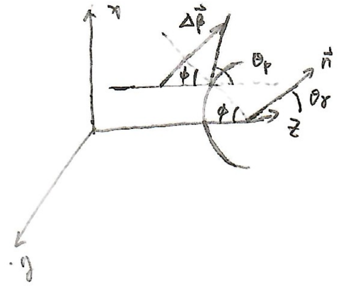
15.2 Assume that the particle is moving in the z-direction and the center of the hard sphere lies on the z-axis. The elastic differential cross section from the hard sphere is $d\sigma_p = \frac{1}{4} R^2 d\Omega_p$. For the particle moving in the xz plane, its velocity change after the collision with the hard sphere is



$$\Delta\vec{\beta} = 2\beta \sin\phi \left(\sin\phi\, \vec{e}_z + \cos\phi\, \vec{e}_x \right) = 2\beta \cos\left(\frac{\theta_p}{2}\right) \left(\cos\left(\frac{\theta_p}{2}\right) \vec{e}_z + \sin\left(\frac{\theta_p}{2}\right) \vec{e}_x \right)$$

$$= \beta\left((\cos^2\theta_p + 1)\, \vec{e}_z + \sin(\theta_p)\, \vec{e}_x \right),$$

since $\phi = (\pi - \theta_p)/2$. However, if the particle path is rotated from the xz-plane by an angle of ϕ_p, the velocity change will gain non-zero y-component, which becomes

$$\Delta\vec{\beta} = \beta\left((\cos\theta_p + 1)\, \vec{e}_z + \sin\theta_p \cos\phi_p\, \vec{e}_x + \sin\theta_p \sin\phi_p\, \vec{e}_y \right).$$

Then, the photon cross section can be calculated with Eq. (15.7).

$$\frac{d^3\sigma}{d\Omega_p\, d(\hbar\omega)\, d\Omega_\gamma} = \sum_i \frac{e^2}{4\pi^2 c} \frac{1}{\hbar^2\omega} \left| \vec{\varepsilon}_i^{\,*} \cdot \Delta\vec{\beta} \right|^2 \frac{d\sigma_p}{d\Omega_p}$$

where the two polarizations for a direction that is θ measured from z-axis are

$$\vec{\varepsilon}_1 = -\sin\theta\, \vec{e}_z + \cos\theta\, \vec{e}_x \qquad \vec{\varepsilon}_2 = \vec{e}_y .$$

The photon scattering cross section is

$$\frac{d^3\sigma}{d\Omega_p\, d(\hbar\omega)\, d\Omega_\gamma} = \frac{e^2}{16\pi^2 c} \frac{\beta^2}{\hbar^2\omega} R^2 \left[\left(-\sin\theta (\cos\theta_p + 1) + \cos\theta \sin\theta_p \cos\phi_p \right)^2 + \sin^2\theta_p \sin^2\phi_p \right]$$

Upon integrating the solid angle for the elastic scattering cross section, only terms that are squares of the trignometric functions of θ_p and ϕ_p will contribute, i.e.

$$\int \cos^2\theta_p\, d\Omega_p = \frac{4\pi}{3} \qquad \int d\Omega_p = 4\pi, \quad \int \sin^2\theta_p \cos^2\phi_p\, d\Omega_p = \int \sin^2\theta_p \sin^2\phi_p\, d\Omega_p = \frac{4\pi}{3}.$$

Therefore, we are left with

$$\frac{d^2\sigma}{d(\hbar\omega)\, d\Omega_\gamma} = \int \frac{d^3\sigma}{d\Omega_p\, d(\hbar\omega)\, d\Omega_\gamma}\, d\Omega_p = \frac{R^2}{16\pi^2} \frac{e^2}{\hbar c} \frac{\beta^2}{\hbar\omega} \left[\sin^2\theta \left(\frac{4\pi}{3} + 4\pi \right) + \cos^2\theta \cdot \frac{4\pi}{3} + \frac{4\pi}{3} \right]$$

$$= \frac{R^2}{16\pi^2} \frac{e^2}{\hbar c} \frac{\beta^2}{\hbar\omega} \left[\frac{8\pi}{3} + \frac{12\pi}{3} \sin^2\theta \right] = \frac{R^2}{12\pi} \frac{e^2}{\hbar c} \frac{\beta^2}{\hbar\omega} \left(2 + 3\cos^2\theta \right).$$

The total bremsstrahlung cross section is given by

$$\frac{d\sigma}{d(\hbar\omega)} = \int \frac{d^2\sigma}{d(\hbar\omega)d\Omega_\gamma} d\Omega_\gamma = \frac{R^2}{12\pi} \frac{e^2}{\hbar c} \frac{\beta^2}{\hbar\omega} \left(8\pi + 3 \cdot \frac{8\pi}{3} \right) = \frac{4R^2}{3} \frac{e^2}{\hbar c} \frac{\beta^2}{\hbar\omega} .$$

In classical mechanics, the elastic collision happens instantaneously, therefore $\omega\tau \to 0$ always holds.

To ignore the phase factor contribution, we therefore need to have $\omega R/c \ll 1$.