Concise expression of a classical radiation spectrum

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In this paper we present a concise expression of the classical electromagnetic radiation spectrum of a moving charge. It is shown to be equivalent to the often used and much more complicated form derived from the Lienard-Wiechert potentials when the observation distance R satisfies the condition $R \gg \gamma\lambda$. The expression reveals a relationship between the radiation spectrum and the motion of the radiation source. It also forms the basis of an efficient computing approach, which is of practical value in numerical calculations of the spectral output of accelerated charges. The advantages of this approach for analytical and numerical applications are discussed and the bending-magnet synchrotron radiation spectrum is calculated according to the approach.

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I. INTRODUCTION

It is well known that when a charged particle is accelerated, it radiates electromagnetic waves. One of the basic properties of interest is the radiation spectrum, defined by the energy radiated into a unit frequency interval and unit area or solid angle. The geometry of this kind of problem and the symbols used in this paper are shown in Fig. 1. Equations (1) and (2) are used to calculate this spectrum. They are well known [1] and widely used in spite of their complexity.

$$\frac{d^2I}{d\omega dA}=\frac{1}{4\pi\epsilon_0}\frac{e^2}{4\pi^2c}\left|\int_{-\infty}^{+\infty}\left\{\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2R}+\frac{(\mathbf{n}-\boldsymbol{\beta})c}{\gamma^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^2R^2}\right\}\times e^{i\omega(\tau+R(\tau)/c)}d\tau\right|^2 \tag{1}$$

and

$$\frac{d^2I}{d\omega d\Omega}=\frac{1}{4\pi\epsilon_0}\frac{e^2\omega^2}{4\pi^2c}\left|\int_{-\infty}^{+\infty}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})e^{i\omega(\tau-\mathbf{n}\cdot\mathbf{x}_e/c)}d\tau\right|^2 . \tag{2}$$

Equation (1) is exact and Eq. (2) is a simplified form valid only for the infinitely far-field region. Equation (2) is the most widely used form because of its simplicity. However, in both equations, the relationship between the charge motion and its radiation spectrum is rather obscure. In this paper we present the following expression:

$$\frac{d^2I}{d\omega dA}=\frac{\alpha\hbar\omega^4}{4\pi^2c^2}\left|\int_{-\infty}^{+\infty}\mathbf{n}(\mathrm{ret})e^{i\omega t}dt\right|^2 , \tag{3}$$

where t is the observer time and "ret" means $\mathbf{n}$ should be calculated at the retarded time τ. We will prove that this form is equivalent to the much more complicated form Eq. (1), when the observation distance $R \gg \gamma\lambda$. Moreover, the significance of this expression will be discussed in Sec. III.

The desire to use the electromagnetic radiation of moving charges has led to the construction of many synchrotron radiation light sources [2,3]. To utilize effectively this radiation one must be able to calculate its characteristics in detail [4–9], especially in synchrotron radiation research, where specially designed magnetic structures are used to control electron motion [10,2–5]. However, it turns out that the numerical spectrum calculations of nonideal trajectories are rather time consuming. As a significant result of Eq. (3), we show an alternative approach to calculate the radiation spectrum. When implemented numerically, this approach provides a very efficient computational method for spectrum calculation.

As a related exercise, we present a proof of a concise expression of the electric field of a moving charge that appeared in the Feynman lectures [11],

$$\mathbf{E}=\frac{e}{4\pi\epsilon_0}\left\{\frac{\mathbf{n}}{R^2}+\frac{R}{c}\frac{d}{dt}\frac{\mathbf{n}}{R^2}+\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}\right\}_{\mathrm{ret}} . \tag{4}$$

This form is remarkable because the third term, which describes the main radiation field, is simply a second derivative of the direction vector from the radiating charge to the observer. The formula Eq. (3), which has not been described in the literature, can be derived from

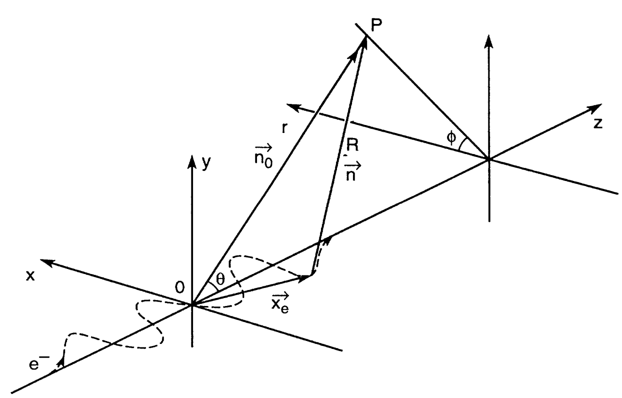

FIG. 1. Reference frame and symbols used in analysis.

Feynman's expression. In Sec. II we show that Eq. (4) is identical to the more complicated, but widely used expression [1]

$$\mathbf{E}=\frac{e}{4\pi\epsilon_0}\left\{\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3Rc}+\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3R^2}\right\}_{\text{ret}} . \quad (5)$$

In Sec. III we derive Eq. (3) and show the conditions under which it is valid. Then we discuss the significance of Eq. (3) and present an efficient approach to calculate the radiation spectrum. To show how the approach works analytically, in Sec. IV, Eq. (3) is used to calculate the well-known synchrotron radiation spectrum from a bending magnet [1,12].

A brief historical review of Eq. (4) may be interesting. Though rarely used, it was first obtained by Heaviside in 1902 and rediscovered by Feynman in 1950. A proof of Eq. (4) directly from the four-vector potential of electromagnetic field was given in Ref. [13].

II. THE HEAVISIDE-FEYNMAN EXPRESSION OF ELECTRIC FIELD OF MOVING CHARGE

Here we begin with Eq. (4) from Feynman and show that it is equivalent to Eq. (5). For brevity, we drop the constant factor $e/4\pi\epsilon_0$ in this section.

The relationship between the observer's time t and the particle time τ is

$$t=\tau+R(\tau)/c . \quad (6)$$

Differentiating this equation we get

$$\frac{dt}{d\tau}=1-\mathbf{n}\cdot\boldsymbol{\beta} . \quad (7)$$

Since $1-\mathbf{n}\cdot\boldsymbol{\beta}$ is always positive, Eq. (7) guarantees there is a one-to-one mapping between t and τ. Thus we can change between them whenever necessary. From the definition of $\mathbf{n}$ and Eq. (7) we find that

$$\frac{d\mathbf{n}}{d\tau}=\frac{c}{R}\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta}) . \quad (8)$$

Using this equation, the first two terms of Eq. (4) can be changed into

$$\begin{aligned}\frac{\mathbf{n}}{R^2}+\frac{R}{c}\frac{d}{dt}\frac{\mathbf{n}}{R^2}&=\frac{\mathbf{n}}{R^2}\frac{1-\mathbf{n}\cdot\boldsymbol{\beta}}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\\&\quad+\frac{R}{c}\frac{d\tau}{dt}\left[\frac{d\mathbf{n}}{d\tau}\frac{1}{R^2}-\frac{2\mathbf{n}}{R^3}\frac{dR}{d\tau}\right]\\&=\frac{1}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})+\mathbf{n}(1+\mathbf{n}\cdot\boldsymbol{\beta})] .\end{aligned} \quad (9)$$

The third term in Eq. (4) may be expressed in particle time τ as

$$\begin{aligned}\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}&=\frac{1}{c}\frac{d\tau}{dt}\frac{d}{d\tau}\left[\frac{1}{c}\frac{d\tau}{dt}\frac{d\mathbf{n}}{d\tau}\right]\\&=\frac{1}{c}\frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})}\frac{d}{d\tau}\left[\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{(1-\mathbf{n}\cdot\boldsymbol{\beta})R}\right] .\end{aligned} \quad (10)$$

Calculating this derivative is tedious. It is accomplished by using the vector identity $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$ repeatedly, by differentiating term by term, and by collecting all β terms together. Via this process we obtain

$$\begin{aligned}\frac{d}{d\tau}\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}}&=\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}\\&\quad+\frac{1}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\left[(\dot{\mathbf{n}}\cdot\boldsymbol{\beta})\mathbf{n}+(\mathbf{n}\cdot\boldsymbol{\beta})\dot{\mathbf{n}}\right.\\&\quad\left.+\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}}(\dot{\mathbf{n}}\cdot\boldsymbol{\beta})\right] .\end{aligned} \quad (11)$$

The term $\dot{\mathbf{n}}\cdot\boldsymbol{\beta}$ in Eq. (11) can be calculated by using Eq. (8) as

$$\begin{aligned}-\frac{R}{C}\dot{\mathbf{n}}\cdot\boldsymbol{\beta}&=[(\mathbf{n}\times\boldsymbol{\beta})\times\mathbf{n}]\cdot\boldsymbol{\beta}\\&=(\mathbf{n}\times\boldsymbol{\beta})^2\\&=\beta^2-(\mathbf{n}\cdot\boldsymbol{\beta})^2 .\end{aligned} \quad (12)$$

With these relations we can finish the differentiation in Eq. (11) and obtain

$$\begin{aligned}\frac{d}{d\tau}\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}}&=\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}\\&\quad-\frac{c}{R}\left[\frac{(\mathbf{n}\times\boldsymbol{\beta})^2}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\mathbf{n}\right.\\&\quad\left.+\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}(\beta^2-\mathbf{n}\cdot\boldsymbol{\beta})\right] .\end{aligned} \quad (13)$$

Thus the third term in Eq. (4) is

$$\begin{aligned}\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}&=\frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})c}\left[\frac{1}{R}\frac{d}{d\tau}\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}}-\frac{1}{R^2}\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\frac{dR}{d\tau}\right]\\&=\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3Rc}-\frac{1}{R^2}\left[\frac{(\mathbf{n}\times\boldsymbol{\beta})^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}\mathbf{n}+\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\left[1-\frac{1}{\gamma^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}\right]\right] .\end{aligned} \quad (14)$$

Combining Eqs. (9) and (14), and using Eq. (12), we obtain the relationship

$$\frac{\mathbf{n}}{R^2}+\frac{R}{c}\frac{d}{dt}\frac{\mathbf{n}}{R^2}+\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}=\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^3Rc}+\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3R^2}\;, \tag{15}$$

which shows that Feynman's expression, Eq. (4), is exactly equal to the generally used expression, Eq. (5). Feynman interestingly interpreted the first two terms of Eq. (4) as the static Coulomb field and the first-order correction to it. The third term gives the radiation field. Though remarkably simple, it is difficult to interpret. In the next section we will present a simple but important application of the Heaviside-Feynman expression.

III. NEW EXPRESSION OF RADIATION SPECTRUM

The primary goal of most radiation calculations is to obtain the energy spectrum. It is well known that the spectral distribution of the total energy radiated into a unit area and unit frequency range by a moving charge is given by

$$\frac{d^2I}{d\omega dA}=\frac{\epsilon_0 c}{\pi}\left|\int_{-\infty}^{+\infty}\mathbf{E}e^{i\omega t}dt\right|^2 . \tag{16}$$

The traditional way to calculate the spectral distribution is to use Eq. (5), with τ as the variable, which leads to Eq. (1), the classic expression used for radiation calculations. However, Eq. (1) is rather complicated in form. Thus the simpler form, Eq. (2), which is based on a far-field approximation, is widely used. Both Eqs. (1) and (2) are based on Eq. (5). The expression in Eq. (4) can also be used for the electric field in Eq. (16). So an alternative way to do this is to use Eq. (4) and the observer time t as the independent variable, that is,

$$\frac{d^2I}{d\omega dA}=\frac{1}{4\pi\epsilon_0}\frac{e^2c}{4\pi^2}\times\left|\int_{-\infty}^{+\infty}\left[\frac{\mathbf{n}}{R^2}+\frac{R}{c}\frac{d}{dt}\frac{\mathbf{n}}{R^2}+\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}\right]e^{i\omega t}dt\right|^2 . \tag{17}$$

To show that the first two terms in Eq. (17) are negligible, we first use Eqs. (8), (9), and the derivative theorem of the Fourier transform [14]:

$$\int_{-\infty}^{+\infty}\frac{d^m\mathbf{n}}{dt^m}e^{i\omega t}dt=(-i\omega)^m\int_{-\infty}^{+\infty}\mathbf{n}e^{i\omega t}dt \tag{18}$$

to obtain two different expressions for the integration part of Eq. (17):

$$\int_{-\infty}^{+\infty}\left[\frac{\mathbf{n}}{R^2}+\frac{R}{c}\frac{d}{dt}\frac{\mathbf{n}}{R^2}+\frac{1}{c^2}\frac{d^2\mathbf{n}}{dt^2}\right]e^{i\omega t}dt$$

$$=\int_{-\infty}^{+\infty}\left[\frac{1}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})+\mathbf{n}(1+\mathbf{n}\cdot\boldsymbol{\beta})]-\frac{i\omega}{Rc}\frac{\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\right]e^{i\omega t}dt \tag{19a}$$

$$=\int_{-\infty}^{+\infty}\left[\frac{1}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})}[\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\beta})+\mathbf{n}(1+\mathbf{n}\cdot\boldsymbol{\beta})]-\frac{\omega^2}{c^2}\mathbf{n}\right]e^{i\omega t}dt\;. \tag{19b}$$

From Eq. (19a), if $R \gg \lambda\!\!\!^{-}=c/\omega$, we can omit the first term because it has the same direction as the third term and has a negligible magnitude. Similarly, from Eq. (19b), we see that the second term will be negligible also if the ratio of the second term to the third term,

$$\frac{1+\mathbf{n}\cdot\boldsymbol{\beta}}{1-\mathbf{n}\cdot\boldsymbol{\beta}}\left[\frac{c}{R\omega}\right]^2\le\frac{2}{1-\beta}\left[\frac{\lambda\!\!\!^{-}}{R}\right]^2\le\left[\frac{2\gamma\lambda\!\!\!^{-}}{R}\right]^2, \tag{20}$$

is small, which is true when $R\gg 2\gamma\lambda\!\!\!^{-}$. From these arguments we conclude that if the observation distance satisfies the condition

$$R\gg\gamma\lambda\!\!\!^{-}\;, \tag{21}$$

then the first two terms in Eq. (17) are negligible. The third term is a complete second derivative of the vector $\mathbf{n}$; according to Eq. (18), we obtain the rather simple form [15], Eq. (3).

As a preliminary check of Eq. (3), we integrate it over frequency and get

$$\begin{aligned}\frac{dI}{dA}&=\int_0^{\infty}\frac{d^2I}{dAd\omega}d\omega\\&=\frac{\alpha\hbar}{4\pi^2c^2}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}dt'\mathbf{n}(t)\cdot\mathbf{n}(t')\tfrac{1}{2}\int_{-\infty}^{+\infty}d\omega\,\omega^4e^{i\omega(t-t')}\\&=\frac{\alpha\hbar}{8\pi^2c^2}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}d\tau\,\mathbf{n}(t)\cdot\mathbf{n}(t+\tau)\int_{-\infty}^{+\infty}\omega^4e^{-i\omega\tau}d\omega\\&=\frac{\alpha\hbar}{8\pi^2c^2}\int_{-\infty}^{+\infty}dt\,\mathbf{n}(t)\cdot\left[2\pi\frac{d^4}{d\tau^4}\mathbf{n}(t+\tau)\right]_{\tau=0}=\frac{\alpha\hbar}{4\pi c^2}\int_{-\infty}^{+\infty}dt\,\mathbf{n}\cdot\frac{d^4\mathbf{n}}{dt^4}=\frac{\alpha\hbar}{4\pi c^2}\int_{-\infty}^{+\infty}dt\,\ddot{\mathbf{n}}\cdot\ddot{\mathbf{n}}=\int_{-\infty}^{+\infty}(c\epsilon_0\mathbf{E}^2)dt,\end{aligned} \tag{22}$$

which is the expected result, the electromagnetic flux density integrated over time. Reference [9] gives numerical verification of Eq. (3) and the related integration boundary problem concerned in numerical applications.

We have proved that the concise expression, Eq. (3), is equivalent to the much more complicated standard expression Eq. (1) if Eq. (21) is satisfied, which is true in most practical applications. Both expressions represent the spectrum of the radiation field generated by a moving charge. Equation (1) works with the retarded time τ. Though complicated in form, it is often convenient for analytical calculations due to the difficulty explained in Sec. IV. Equation (3) works with the observer time t. It is very concise and of significance to both theoretical representation and practical computation of the radiation spectrum.

In addition to its conciseness, Eq. (3) reveals an important relationship between the trajectory of a charge and its radiation spectrum. The physical meaning of the direction vector $\mathbf{n}$ is quite clear and, according to Eq. (3), the radiation spectrum is just the Fourier transform of $\mathbf{n}$. This understanding allows one to obtain a great deal of insight into the properties of radiation from knowledge of the Fourier transform and the particle trajectory. For example, in an undulator, electrons undulate periodically along a straight orbit [5]. According to Eq. (3), it is evident that the radiation will be linearly polarized if the electron moves in a plane and elliptically polarized if the electron moves in a spiral. Moreover, from the properties of the Fourier transform we know that the spectrum will consist of peaks having the same width, which is inversely proportional to the number of periods of the device.

Another important result of Eq. (3) is an alternative way to calculate the radiation spectrum. Instead of working with the complicated Eq. (1), one can calculate the direction vector $\mathbf{n}(t)$ in the observer time frame and then do a Fourier transformation. As an illustration of this approach we will obtain the bending-magnet synchrotron radiation spectrum in the next section using Eq. (3). However, the most important advantage of this approach is in numerical calculation of the radiation spectrum. Because $\mathbf{n}(t)$ can be easily calculated numerically and the Fourier transformation can be done very efficiently with the well-known fast-Fourier-transform (FFT) method [14], such an approach provides the most efficient computational method to calculate the radiation spectrum of an arbitrarily moving charge. It is of practical value in synchrotron radiation research, where numerical methods have to be used to compute the radiation spectrum because of the nonideal trajectory resulting from the magnetic field errors. The computational method presented here could be several orders faster than a straightforward integration of Eq. (1). Moreover, Eq. (3) is applicable to the near-field case [6,7], where numerical methods are usually necessary because of the mathematical complexity. In addition, only the trajectory $\mathbf{n}$ is needed in Eq. (3) instead of $\mathbf{n},\boldsymbol{\beta},\dot{\boldsymbol{\beta}}$ in Eq. (1), so the requirement for storage of velocity and acceleration is removed when using Eq. (3) to calculate spectrum. Also one does not need to calculate the derivative of velocity, which is an error-sensitive process. In conclusion, the present computational approach has fundamental importance for efficient radiation spectrum calculation and is of practical value in synchrotron radiation research. A detailed discussion of the numerical aspects of the present expression and its application in insertion device synchrotron radiation calculations appears in Ref. [9].

IV. BENDING-MAGNET SYNCHROTRON RADIATION SPECTRUM

The synchrotron radiation from a bending magnet is produced by a charged particle moving along a circular trajectory under the influence of a uniform magnetic field. We use the Cartesian coordinate system shown in Fig. 2 and assume that the charge moves in the x-z plane with a trajectory radius R_0 and a circular frequency ω_0. So we can write the trajectory as

$$\begin{aligned} x &= R_0(\cos\omega_0\tau - 1)\ , \\ y &= 0\ , \\ z &= R_0\sin\omega_0\tau\ , \\ \beta &= \omega_0 R_0/c \simeq 1\ . \end{aligned} \tag{23}$$

If the observer is located at $(0,r\theta,r)$, we can write the direction vector as

$$\mathbf{n} = \frac{1}{R(\tau)}(R_0(1-\cos\omega_0\tau), r\theta, r - R_0\sin\omega_0\tau)\ , \tag{24}$$

where the distance between the observer and charge is

$$\begin{aligned} R(\tau) &= (1+\theta^2)^{1/2}\left\{\left[r - \frac{R_0\sin\omega_0\tau}{1+\theta^2}\right]^2 + \left[\frac{2R_0\sin(\omega_0\tau/2)}{(1+\theta^2)^{1/2}}\right]^2 - \left[\frac{R_0\sin\omega_0\tau}{1+\theta^2}\right]^2\right\} \\ &= (1+\theta^2)^{1/2}\left[r - \frac{R_0\sin\omega_0\tau}{1+\theta^2}\right] + o\left[\frac{R_0\omega_0\tau}{r}\right]\ . \end{aligned} \tag{25}$$

Because $\omega_0\tau \ll 1$ and $R_0 \le r$, the higher-order terms can be omitted. Therefore, to the lowest order of $\omega_0\tau$, Eq. (24) can be written as

$$\mathbf{n} = \left[\frac{R_0}{r}\frac{1}{2}(\omega_0\tau)^2, \theta + \frac{R_0}{r}\theta\omega_0\tau, 1\right]\ . \tag{26}$$

To get the function $\tau(t)$ used in Eq. (3), we have to solve Eq. (6). Usually it is difficult to derive an analytical function for $\tau(t)$. This is the main factor limiting the use of Eq. (3) in analytical calculations. However, in the present case we are able to get a sufficiently accurate solution. Using the above expression for $R(\tau)$ and dropping the

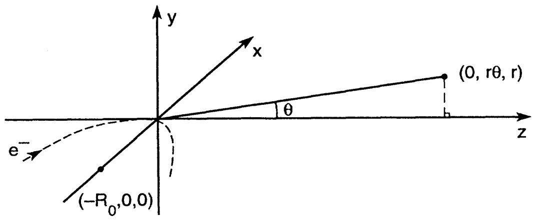

FIG. 2. Coordinate system used in Sec. IV.

constant term we get

$$t=\tau-\frac{R_0}{c}(1+\theta^2)^{-1/2}\sin(\omega_0\tau)$$
$$=\frac{1}{2\gamma^2}[1+(\gamma\theta)^2]\tau+\frac{1}{6\omega_0}(\omega_0\tau)^3+o((\omega_0\tau)^3)\ , \qquad (27)$$

where the last identity of Eq. (23) is used. As in previous derivations [1], we just keep up to the third-order term of $\omega_0\tau$,

$$(\omega_0\tau)^3+3\gamma^{-2}[1+(\gamma\theta)^2]\omega_0\tau-6\omega_0 t=0\ , \qquad (28)$$

yielding

$$\omega_0\tau=\gamma^{-1}[1+(\gamma\theta)^2]^{1/2}\{[(\eta^2+1)^{1/2}+\eta]^{1/3}-[(\eta^2+1)^{1/2}-\eta]^{1/3}\}\ ,$$
$$\eta=3\omega_0\gamma^3[1+(\gamma\theta)^2]^{-3/2}t\ . \qquad (29)$$

Defining

$$\omega_c=\tfrac{3}{2}\gamma^3\omega_0\ ,$$
$$\xi=\frac{\omega}{2\omega_c}[1+(\gamma\theta)^2]^{3/2}\ , \qquad (30)$$

we have $\omega t=\xi\eta$. Using Eqs. (30), (23), (26), and (29), and dropping constant terms, Eq. (3) becomes

$$\frac{d^2I}{d\omega d\Omega}=\frac{\alpha\hbar}{4\pi^2}\left|\frac{\omega}{\omega_0}\xi\int_{-\infty}^{+\infty}(\tfrac{1}{2}(\omega_0\tau)^2,\theta\omega_0\tau)e^{i\xi\eta}d\eta\right|^2\ . \qquad (31)$$

The x component in the absolute square is

$$\frac{\omega}{\omega_0}\xi\tfrac{1}{2}\int_{-\infty}^{+\infty}(\omega_0\tau)^2e^{i\xi\eta}d\eta=\frac{\omega}{\omega_0}\xi\gamma^{-2}[1+(\gamma\theta)^2]\int_0^{\infty}\{[(\eta^2+1)^{1/2}+\eta]^{1/3}-[(\eta^2+1)^{1/2}-\eta]^{1/3}\}^2\cos\xi\eta\,d\eta$$
$$=\frac{\omega}{\omega_c}\gamma[1+(\gamma\theta)^2]\tfrac{3}{2}\xi\frac{2}{\sqrt{3}}\frac{1}{\xi}K_{2/3}(\xi)\ . \qquad (32)$$

So,

$$\frac{d^2I_x}{d\omega d\Omega}=\frac{3\alpha\hbar}{4\pi^2}\gamma^2\left[\frac{\omega}{\omega_c}\right]^2[1+(\gamma\theta)^2]^2K_{2/3}^2(\xi)\ . \qquad (33)$$

Similarly, the y component is

$$\frac{\omega}{\omega_0}\xi\theta\int_{-\infty}^{+\infty}\omega_0\tau e^{i\xi\eta}d\eta=\frac{\omega}{\omega_0}\theta\gamma^{-1}[1+(\gamma\theta)^2]^{1/2}\xi 2i\int_0^{\infty}\{[(\eta^2+1)^{1/2}+\eta]^{1/3}-[(\eta^2+1)^{1/2}-\eta]^{1/3}\}\sin\xi\eta\,d\eta$$
$$=-i\sqrt{3}\frac{\omega}{\omega_c}\gamma\gamma\theta[1+(\gamma\theta)^2]^{1/2}K_{1/3}(\xi) \qquad (34)$$

and

$$\frac{d^2I_y}{d\omega d\Omega}=\frac{3\alpha\hbar}{4\pi^2}\gamma^2\left[\frac{\omega}{\omega_c}\right]^2(\gamma\theta)^2[1+(\gamma\theta)^2]K_{1/3}^2(\xi)\ . \qquad (35)$$

We see that the y component of the electric field is retarded in phase by $\pi/2$ relative to the x component when $\theta>0$, i.e., above the orbit plane. Equations (33) and (35) are the same as the standard results. Detailed calculations of the Fourier transform in Eqs. (32) and (34) are shown in the Appendix.

V. CONCLUSIONS

The classical radiation spectrum of moving charges can be expressed in the form

$$\frac{d^2I}{d\omega dA}=\frac{\alpha\hbar\omega^4}{4\pi^2c^2}\left|\int_{-\infty}^{+\infty}\mathbf{n}(\mathrm{ret})e^{i\omega t}dt\right|^2\ ,$$

provided that the observation distance $R\gg\gamma\lambda$. It is a clearer and more concise relationship between radiation properties and the trajectory of the radiating charge. Therefore one can get more intuitive understanding of the radiation properties from knowledge of the particle trajectory and the Fourier transform. We also developed an alternative approach to calculate the classical radiation of moving charges. This approach may not make analytical calculations easier because of the difficulty of getting an analytical function $\tau(t)$. But, it does simplify numerical calculations significantly when the FFT is applicable. The approach is a very efficient computational method to calculate radiation spectra, and of practical value in synchrotron radiation research.

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APPENDIX

To calculate the Fourier transforms in Eqs. (32) and (34), we introduce two symbols:

$$\eta_\pm=(\eta^2+1)^{1/2}\pm\eta \tag{A1}$$

and notice that $\eta_+\eta_-=1$ and $\eta_++\eta_-=2(\eta^2+1)^{1/2}$. The Fourier transform in Eq. (32) is

$$\begin{aligned}\int_0^\infty (\eta_+^{1/3}-\eta_-^{1/3})^2\cos\xi\eta\, d\eta &= \int_0^\infty (\eta_+^{2/3}+\eta_-^{2/3}-2)\cos\xi\eta d\eta \\ &= \int_0^\infty \frac{(\eta_+^{2/3}+\eta_-^{2/3})(\eta_++\eta_-)}{2(\eta^2+1)^{1/2}}\cos\xi\eta\, d\eta-2\pi\delta(\xi) \\ &= \frac{1}{2}\int_0^\infty \left[\frac{\eta_+^{5/3}+\eta_-^{5/3}}{(\eta^2+1)^{1/2}}+\frac{\eta_+^{1/3}+\eta_-^{1/3}}{(\eta^2+1)^{1/2}}\right]\cos\xi\eta\, d\eta-2\pi\delta(\xi) \\ &= \cos\frac{\pi}{6}[K_{2/3-1}(\xi)-K_{2/3+1}(\xi)]-2\pi\delta(\xi) \\ &= \frac{2}{\sqrt{3}}\frac{1}{\xi}K_{2/3}(\xi)-2\pi\delta(\xi)\ ,\end{aligned} \tag{A2}$$

where the identities [16,17]

$$\int_0^\infty \frac{[(x^2+\beta^2)^{1/2}+x]^\nu+[(x^2+\beta^2)^{1/2}-x]^\nu}{(x^2+\beta^2)^{1/2}}\cos\alpha x\, dx=2\beta^\nu\cos\frac{\nu\pi}{2}K_\nu(\alpha\beta)\ , \tag{A3}$$

$$K_{\nu-1}(\xi)-K_{\nu+1}(\xi)=\frac{2\nu}{\xi}K_\nu(\xi)\ , \tag{A4}$$

$$K_{-\nu}(\xi)=K_\nu(\xi) \tag{A5}$$

are used. Similarly, the Fourier transform in Eq. (34) is found with the identity [16]

$$\int_0^\infty \frac{[(x^2+\beta^2)^{1/2}+x]^\nu-[(x^2+\beta^2)^{1/2}-x]^\nu}{(x^2+\beta^2)^{1/2}}\sin\alpha x\, dx=2\beta^\nu\sin\frac{\nu\pi}{2}K_\nu(\alpha\beta)\ . \tag{A6}$$

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[15] It is interesting to notice that **n** can be viewed as the momentum direction of the photons received by the observer while $\hbar\omega/c$ is the magnitude of the momentum. So we can express Eq. (3) as $d^2I/d\omega dA=(\alpha/\hbar)|F[\dot{\mathbf{p}}]|^2$, where $\dot{\mathbf{p}}$ is the observed rate of change of the photon momentum, and **F**[ ] represents the Fourier transform.
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