8.4 (a) For TM wave, $E_z(\rho, \phi) = E_0 J_m\left(\frac{x_{mn}}{R}\rho\right) e^{\pm im\phi}$, where x_{mn} is the n-th solution of $J_m(x)$. The cutoff frequency is $\omega_{mn} = x_{mn}/R\sqrt{\mu\varepsilon}$.

Using Eq. (8.51), the transmitted power is

$$P = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\int_A |E_z|^2 da$$

$$= \pi E_0^2\sqrt{\frac{\varepsilon}{\mu}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\int_0^R \rho J_m\left(\frac{x_{mn}}{R}\rho\right)^2 d\rho$$

$$= \frac{\pi R^2 E_0^2}{2}\sqrt{\frac{\varepsilon}{\mu}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2} J_{m+1}(x_{mn})^2$$

From Eq. (8.59), the power loss is

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2\oint_c \frac{1}{\mu^2\omega_{mn}^2}\left|\frac{\partial E_z}{\partial \rho}\right|^2 dl$$

$$= \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 \cdot 2\pi R \cdot \frac{x_{mn}^2}{\mu^2\omega_{mn}^2 R^2} E_0^2 J_m'(x_{mn})^2$$

$$= \frac{\varepsilon}{\mu}\pi R E_0^2 \frac{1}{\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 J_m'(x_{mn})^2$$

Therefore, the attenuation constant is

$$\beta_{mn} = -\frac{1}{2P}\frac{dP}{dz} = \frac{\pi R E_0^2 \frac{\varepsilon}{\mu}\cdot\frac{1}{\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 J_m'(x_{mn})^2}{\pi R^2 E_0^2\sqrt{\frac{\varepsilon}{\mu}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2} J_{m+1}(x_{mn})^2}$$

$$= \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{R\sigma\delta}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2}\left[\frac{J_m'(x_{mn})}{J_{m+1}(x_{mn})}\right]^2 = \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{R\sigma\delta}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2}, \text{ since } J_m'(x_{mn}) = J_{m+1}(x_{mn})$$

(b) For TE wave, $H_z(\rho, \phi) = H_0 J_m\left(\frac{x_{mn}'}{R}\rho\right) e^{\pm im\phi}$, where x_{mn}' is the n-th root of $J_m'(x)$. The cutoff frequency is $\omega_{mn} = x_{mn}'/R\sqrt{\mu\varepsilon}$.

Using Eq. (8.51), the transmitted power is

$$P = \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\int_A |H_z|^2 da$$

$$= \pi H_0^2\sqrt{\frac{\mu}{\varepsilon}}\left(\frac{\omega}{\omega_{mn}}\right)^2\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\int_0^R \rho J_m\left(\frac{x_{mn}'}{R}\rho\right)^2 d\rho$$

$$= \frac{\pi R^2 H_0^2}{2} \sqrt{\frac{\mu}{\varepsilon}} \left(\frac{\omega}{\omega_{mn}}\right)^2 \left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2} \left(1-\frac{m^2}{x_{mn}'^2}\right) J_m(x_{mn}')^2$$

From Eq (8.59), the power loss is

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 \oint \left[\frac{1}{\mu\varepsilon\omega_{mn}^2}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right) |\vec{n}\times\nabla_t H_z|^2 + \frac{\omega_{mn}^2}{\omega^2}|H_z|^2 \right] dl$$

Since $\nabla_t H_z = \left(\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}\right) H_z = H_0 e^{\pm im\phi}\left[\hat{\rho}\frac{x_{mn}'}{R} J_m'\left(\frac{x_{mn}'}{R}\rho\right) \pm \hat{\phi}\frac{im}{\rho} J_m\left(\frac{x_{mn}'}{R}\rho\right)\right]$,

at the boundary $\rho = R$, the first term is simply zero, and

$$|\vec{n}\times\nabla_t H_z|^2 \Big|_{\rho=R} = \frac{m^2 H_0^2}{R^2} J_m(x_{mn}')^2 .$$

Then

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 \left[\frac{1}{\mu\varepsilon\omega_{mn}^2}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right) \cdot \frac{m^2 H_0^2}{R^2} J_m(x_{mn}')^2 + \frac{\omega_{mn}^2}{\omega^2} H_0^2 J_m(x_{mn}')^2\right] \times 2\pi R$$

$$= \pi R H_0^2 \frac{1}{\sigma\delta}\left(\frac{\omega}{\omega_{mn}}\right)^2 \left[\frac{1}{\mu\varepsilon\omega_{mn}^2}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)\frac{m^2}{R^2} + \frac{\omega_{mn}^2}{\omega^2}\right] J_m(x_{mn}')^2$$

and the attenuation constant is

$$\beta_{mn} = -\frac{1}{2P}\frac{dP}{dz} = \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{R\sigma\delta} \frac{\frac{1}{\mu\varepsilon\omega_{mn}^2}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)\frac{m^2}{R^2} + \frac{\omega_{mn}^2}{\omega^2}}{\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{1/2}\left(1-\frac{m^2}{x_{mn}'^2}\right)}$$

$$= \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{R\sigma\delta}\left(1-\frac{\omega_{mn}^2}{\omega^2}\right)^{-1/2}\left[\frac{m^2}{x_{mn}'^2 - m^2} + \frac{\omega_{mn}^2}{\omega^2}\right]$$