8.13 (a) For the perturbed solution, ψ, it can be formed from linear combination of perturbed eigen mode $\psi^{(i)}$ as $\psi = \sum_{i=1}^{N} a_i \psi^{(i)}$, where the perturbed eigen mode differs from the unperturbed ones at the boundary. For the new solution, the eigen equation is $(\nabla_t^2 + \gamma^2)\psi = 0$. Apply the Green's theorem on ψ and $\psi_0^{(j)*}$, with inwardly directed norm, we have

$$\int_A \left(\psi \nabla_t^2 \psi_0^{(j)*} - \psi_0^{(j)*} \nabla_t^2 \psi \right) da = \oint_C \left(\psi_0^{(j)*} \frac{\partial \psi}{\partial n} - \psi \frac{\partial \psi_0^{(j)*}}{\partial n} \right) dl$$

The L.H.S. can be expanded as

$$\int_A \left(\psi \nabla_t^2 \psi_0^{(j)*} - \psi_0^{(j)*} \nabla_t^2 \psi \right) da = (\gamma^2 - \gamma_0^2) \int_A \psi \psi_0^{(j)*} da$$

$$= (\gamma^2 - \gamma_0^2) \sum_{i=1}^{N} a_i \int_A \psi^{(i)} \psi_0^{(j)*} da$$

Using the orthogonality of $\psi_0^{(i)}$, we have

$$\int_A \psi^{(i)} \psi_0^{(j)*} da = \int_A \psi_0^{(i)} \psi_0^{(j)*} da = \delta_{ji} \int_A |\psi_0^{(j)}|^2 da$$

The R.H.S. can be written as

$$\oint_C \left(\psi_0^{(j)*} \frac{\partial \psi}{\partial n} - \psi \frac{\partial \psi_0^{(j)*}}{\partial n} \right) dl = \sum_{i=1}^{N} a_i \oint_C \left(\psi_0^{(j)*} \frac{\partial \psi^{(i)}}{\partial n} - \psi^{(i)} \frac{\partial \psi_0^{(j)*}}{\partial n} \right) dl.$$

Define $N_j = \int_A |\psi_0^{(j)}|^2 da$, $\Delta_{ji} = \oint_C \left[\psi^{(i)} \frac{\partial \psi_0^{(j)*}}{\partial n} - \psi_0^{(j)*} \frac{\partial \psi^{(i)}}{\partial n} \right] dl$,

the the secular equation for the new solution becomes

$$\sum_{i=1}^{N} \left[(\gamma^2 - \gamma_0^2) N_j \delta_{ji} + \Delta_{ji} \right] a_i = 0$$

For large but finite conductivity, $\psi^{(i)} = f \left. \frac{\partial \psi_0^{(i)}}{\partial n} \right|_S$, and

$$\Delta_{ji} = \oint \left[f \frac{\partial \psi_0^{(i)}}{\partial n} \frac{\partial \psi_0^{(j)*}}{\partial n} - f \psi_0^{(j)*} \frac{\partial^2 \psi_0^{(i)}}{\partial n^2} \right] dl \simeq f \oint \frac{\partial \psi_0^{(i)}}{\partial n} \frac{\partial \psi_0^{(j)*}}{\partial n} dl$$

For deformation, we can follow the same argument as in 8.12 (a).

$$\Delta_{ji} = \oint_C \left[\frac{\partial \psi_0^{(i)}}{\partial n} \frac{\partial \psi_0^{(j)*}}{\partial n} - \psi_0^{(j)*} \frac{\partial^2 \psi_0^{(i)}}{\partial n^2} \right] dl$$

(b) The new surface can be parameterized as $(a\cos\phi, b\sin\phi)$. Notice that ϕ here is actually different from the circular parameterization, but the difference is quite small. Then,

$$\delta(\phi) = R - (a^2\cos^2\phi + b^2\sin^2\phi)^{1/2} = R - \sqrt{R^2 + 2R\Delta R\cos 2\phi + \Delta R^2} = -\Delta R\cos 2\phi.$$

Also, due to the boundary condition, $(\partial\psi^{(\pm)}/\partial n)_s = 0$, and

$$\frac{\partial^2\psi^{(\pm)}}{\partial n^2} = \gamma_0^2 B_0 J_1''(\gamma_0 R) e^{\pm i\phi} e^{ikz - i\omega t}.$$

Since $N_+ = \int_A |\psi^{(+)}|^2 \, da = 2\pi B_0^2 \int_0^R \rho J_1(\gamma_0\rho)^2 d\rho = \pi R^2 B_0^2 J_1(\gamma_0 R)^2 \left(1 - \frac{1}{\gamma_0^2 R^2}\right) = N$

$$N_- = N_+ = N$$

$$\Delta_{++} = -\oint_C \delta(\phi)\psi_0^{(+)*}\frac{\partial^2\psi_0^{(+)}}{\partial n^2} dl = \Delta R\, \gamma_0^2 B_0^2\, J_1(\gamma_0 R) J_1''(\gamma_0 R) \int_0^{2\pi} \cos 2\phi \, R\, d\phi = 0.$$

$$\Delta_{--} = 0$$

$$\Delta_{+-} = -\oint_C \delta(\phi)\psi_0^{(+)*}\frac{\partial^2\psi_0^{(-)}}{\partial n^2} dl = \Delta R\,\gamma_0^2 B_0^2\, J_1(\gamma_0 R) J_1''(\gamma_0 R)\int_0^{2\pi} \cos 2\phi\, e^{-2i\phi} R\, d\phi$$

$$= \pi \Delta R R \gamma_0^2 B_0^2 J_1(\gamma_0 R) J_1''(\gamma_0 R) = \Delta$$

$$\Delta_{-+} = \Delta_{+-} = \Delta$$

then the secular equation becomes

$$\begin{vmatrix} N(\gamma^2 - \gamma_0^2) & \Delta \\ \Delta & N(\gamma^2 - \gamma_0^2) \end{vmatrix} = 0, \quad \text{with solution } \gamma^2 = \gamma_0^2 \pm \frac{\Delta}{N} = \gamma_0^2\left(1 \pm \lambda\frac{\Delta R}{R}\right),$$

where $$\lambda = \frac{J_1(\gamma_0 R) J_1''(\gamma_0 R)}{J_1(\gamma_0 R)^2\left(1 - \frac{1}{\gamma_0^2 R^2}\right)} = \frac{J_1''(\gamma_0 R)}{J_1(\gamma_0 R)\left(1 - \frac{1}{\gamma_0^2 R^2}\right)}$$

From the Bessel equation, $J_1''(x) + \frac{1}{x}J_1'(x) + \left(1 - \frac{1}{x^2}\right)J_1(x) = 0$, at $x = \gamma_0 R$, $J_1'(\gamma_0 R) = 0$,

we have $J_1''(\gamma_0 R) = -\left(1 - \frac{1}{\gamma_0^2 R^2}\right)J_1(\gamma_0 R)$. Therefore, $\lambda = -1$.