5.13 The current density is given by

$$\vec{J}(r,\theta,\phi) = \sigma\omega r\sin\theta\,\delta(r-a)\,\hat{\phi} = \sigma\omega r\sin\theta\,\delta(r-a)\left(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right)$$

The vector potential can be calculated as

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x'$$

$$= \frac{\mu_0}{4\pi}\int_0^\infty r'^2dr'\int_{-1}^{1}d(\cos\theta')\int_0^{2\pi}d\phi'\;\sigma\omega r'\sin\theta'\,\delta(r'-a)\left(-\sin\phi'\,\hat{i}+\cos\phi'\,\hat{j}\right)$$

$$\times\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta')P_l^m(\cos\theta)e^{im(\phi-\phi')}\frac{r_<^l}{r_>^{l+1}}$$

Only $l=1$, $m=\pm1$ terms will contribute,

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\sigma\omega a^3\int_{-1}^{1}d(\cos\theta')\int_0^{2\pi}d\phi'\;\sin\theta'\left(-\sin\phi'\,\hat{i}+\cos\phi'\,\hat{j}\right)$$

$$\times\;\frac{1}{2}P_1^1(\cos\theta')P_1^1(\cos\theta)\times 2\cos(\phi-\phi')\times\frac{r_<}{r_>^2}$$

$$= \frac{\mu_0}{4}\sigma\omega a^3\sin\theta\left(-\sin\phi\,\hat{i}+\cos\phi\,\hat{j}\right)\int_{-1}^{1}\sin^2\theta'\,d(\cos\theta')\cdot\frac{r_<}{r_>^2}$$

$$= \frac{\mu_0\sigma\omega a^3}{3}\frac{r_<}{r_>^2}\sin\theta\,\hat{\phi}$$

For $r>a$, $\vec{A}(\vec{x}) = \dfrac{\mu_0\sigma\omega a^4}{3r^2}\sin\theta\,\hat{\phi} = \dfrac{\mu_0\sigma a^4}{3r^3}\vec{\omega}\times\vec{r}$

$r<a$, $\vec{A}(\vec{x}) = \dfrac{\mu_0\sigma\omega a}{3}r\sin\theta\,\hat{\phi} = \dfrac{\mu_0\sigma a}{3}\vec{\omega}\times\vec{r}$

$\vec{B}(\vec{x}) = \nabla\times\vec{A}(\vec{x})$

$r<a$, $\vec{B}(\vec{x}) = \dfrac{\mu_0\sigma a}{3}\nabla\times(\vec{\omega}\times\vec{r}) = \dfrac{\mu_0\sigma a}{3}\left[\vec{\omega}(\nabla\cdot\vec{r}) - (\vec{\omega}\cdot\nabla)\vec{r}\right] = \dfrac{2\mu_0\sigma a}{3}\vec{\omega}$

$r>a$, $\vec{B}(\vec{x}) = \dfrac{\mu_0\sigma a^4}{3}\nabla\times\left(\vec{\omega}\times\dfrac{\vec{r}}{r^3}\right) = \dfrac{\mu_0\sigma a^4}{3}\left[\vec{\omega}\left(\nabla\cdot\dfrac{\vec{r}}{r^3}\right) - (\vec{\omega}\cdot\nabla)\dfrac{\vec{r}}{r^3}\right]$

$$= -\frac{\mu_0\sigma a^4}{3}(\vec{\omega}\cdot\nabla)\frac{\vec{r}}{r^3} = \frac{\mu_0\sigma a^4}{3}\left[\frac{3\vec{n}(\vec{n}\cdot\vec{\omega})-\vec{\omega}}{r^3}\right]$$

$$(\vec{\omega}\cdot\nabla)\frac{\vec{r}}{r^3} = (\omega_i\partial_i)\frac{\vec{r}}{r^3} = \frac{\omega_i\delta_{ij}\vec{e}_j}{r^3} - \frac{\omega_i\vec{r}\,3r_i}{r^5} = \frac{\vec{\omega}}{r^3} - \frac{3(\vec{\omega}\cdot\vec{r})\vec{r}}{r^5}$$