5.1 Solution: Let us first establish the following identity

$$\oint_{\partial S} \mathbf{A} \times d\mathbf{l} = \int_S (\nabla \cdot \mathbf{A}) d\mathbf{S} - \int_S (\nabla \mathbf{A}) \cdot d\mathbf{S}.$$

For an arbitrary constant vector $\mathbf{c}$, we have

$$\mathbf{c} \cdot \oint_{\partial S} \mathbf{A} \times d\mathbf{l} = \oint_{\partial S} (\mathbf{c} \times \mathbf{A}) \cdot d\mathbf{l}$$

Now, we can apply the Stokes theorem as

$$\oint_{\partial S} [\mathbf{c} \times \mathbf{A}] \cdot d\mathbf{l} = \int_S (\nabla \times (\mathbf{c} \times \mathbf{A})) \cdot d\mathbf{S}$$

Since

$$\nabla \times (\mathbf{c} \times \mathbf{A}) = \mathbf{c}(\nabla \cdot \mathbf{A}) - \mathbf{A}(\nabla \cdot \mathbf{c}) + (\mathbf{A} \cdot \nabla)\mathbf{c} - (\mathbf{c} \cdot \nabla)\mathbf{A},$$

and also take into account that $\mathbf{c}$ is a constant vector, we will have

$$\int_S (\nabla \times (\mathbf{c} \times \mathbf{A})) \cdot d\mathbf{S} = \int_S (\mathbf{c}(\nabla \cdot \mathbf{A}) - (\mathbf{c} \cdot \nabla)\mathbf{A}) \cdot d\mathbf{S} = \mathbf{c} \cdot \int_S ((\nabla \cdot \mathbf{A}) d\mathbf{S} - (\nabla \mathbf{A}) \cdot d\mathbf{S}),$$

which establishes the desired identity.

Apply the above identity to the expression for the magnetic induction, and notice that

$$\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} = \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right),$$

we have

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0 I}{4\pi} \oint d\mathbf{l}' \times \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \\
&= -\frac{\mu_0 I}{4\pi} \oint \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \times d\mathbf{l}' \\
&= -\frac{\mu_0 I}{4\pi} \left[\int \nabla'^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) d\mathbf{S}' - \int \nabla' \left[\nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right] \cdot d\mathbf{S} \right] \\
&= -\frac{\mu_0 I}{4\pi} \left[\int \nabla'^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) d\mathbf{S}' + \nabla \int \nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \cdot d\mathbf{S} \right],
\end{aligned}$$

where we have used the fact that $\nabla = -\nabla'$.

The first term vanishes,

$$\nabla'^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -4\pi \delta(\mathbf{x} - \mathbf{x}') = 0,$$

since $\mathbf{x} \neq \mathbf{x}'$. Also,

$$\nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \cdot d\mathbf{S}' = -d\Omega.$$

Therefore, we are left with

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \nabla \int d\Omega = \frac{\mu_0 I}{4\pi} \nabla \Omega.$$