3.1 Solution: The general solution to the potential problem is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta).$$

To determine the coefficients, notice the boundary conditions, we have

$$\Phi(a,\theta) = \sum_{l=0}^{\infty} \left(A_l a^l + \frac{B_l}{a^{l+1}} \right) P_l(\cos\theta) = \begin{cases} V, & 0 \le \theta \le \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} < \theta \le \pi, \end{cases}$$

and

$$\Phi(b,\theta) = \sum_{l=0}^{\infty} \left(A_l b^l + \frac{B_l}{b^{l+1}} \right) P_l(\cos\theta) = \begin{cases} 0, & 0 \le \theta \le \frac{\pi}{2}, \\ V, & \frac{\pi}{2} < \theta \le \pi. \end{cases}$$

For the inner sphere, by integrating the boundary condition with respect to the Legendre polynomial, we have

$$A_l a^l + \frac{B_l}{a^{l+1}} = \frac{2l+1}{2} V \int_0^1 P_l(x) dx.$$

Similarly, for the outer sphere,

$$A_l b^l + \frac{B_l}{b^{l+1}} = \frac{2l+1}{2} V \int_{-1}^0 P_l(x) dx = \frac{2l+1}{2} V \int_0^1 P_l(-x) dx = (-l)^l \frac{2l+1}{2} V \int_0^1 P_l(x) dx.$$

Let

$$D_l = (2l+1) \int_0^1 P_l(x) dx,$$

we can obtain the coefficients as

$$A_l = \frac{V}{2} \frac{a^{l+1} - (-1)^l b^{l+1}}{a^{2l+1} - b^{2l+1}} D_l,$$

and

$$B_l = \frac{V}{2} \frac{a^{-l} - (-1)^l b^{-l}}{a^{-(2l+1)} - b^{-(2l+1)}} D_l = -\frac{V}{2} a^{l+1} b^{l+1} \frac{b^l - (-1)^l a^l}{a^{2l+1} - b^{2l+1}} D_l,$$

which leads to the following expression for the potential,

$$\Phi(r,\theta) = \frac{V}{2} \sum_{l=0}^{\infty} \frac{\left(a^{l+1} - (-1)^l b^{l+1}\right) r^l - a^{l+1} b^{l+1} \left(b^l - (-1)^l a^l\right) r^{-(l+1)}}{a^{2l+1} - b^{2l+1}} D_l P_l(\cos\theta).$$

Now, using the identity

$$P_l(x) = \frac{1}{2l+1} \left(\frac{dP_{l+1}(x)}{dx} - \frac{dP_{l-1}(x)}{dx} \right),$$

we can show that

$$\int_0^1 P_l(x) dx = \frac{1}{2l+1} \left[P_{l+1}(x) - P_{l-1}(x) \right] \Big|_{x=0}^{1} = \frac{1}{2l+1} \left[P_{l-1}(0) - P_{l+1}(0) \right],$$

for $l > 0$. Since $P_n(0) = 0$ where n is even, only odd l terms will contribute, with

$$D_l = P_{l-1}(0) - P_{l+1}(0).$$

Meanwhile, $D_0 = 1$, $A_0 = V/2$, and $B_0 = 0$. Finally, the potential between the spheres becomes

$$\Phi(r,\theta) = \frac{V}{2} + \frac{V}{2}\sum_{l>0,\ \text{odd}} \left[P_{l-1}(0) - P_{l+1}(0)\right] \frac{\left(a^{l+1}+b^{l+1}\right) r^{l} - a^{l+1}b^{l+1}\left(a^{l}+b^{l}\right) r^{-(l+1)}}{a^{2l+1}-b^{2l+1}} P_l(\cos\theta).$$

Since

$$P_{2n}(0) = \frac{(-1)^n}{4^n}\binom{2n}{n} = \frac{(-1)^n}{4^n}\frac{(2n)!}{(n!)^2},$$

we can obtain

$$P_0(0) = 1,\ P_2(0) = -\frac{1}{2},\ P_4(0) = \frac{3}{8}.$$

So, up to $l = 4$, the potential is

$$\begin{aligned}\Phi(r,\theta) &= \frac{V}{2} \\ &+ \frac{V}{2}\cdot\frac{3}{2}\frac{\left(a^2+b^2\right) r - a^2b^2\left(a+b\right) r^{-2}}{a^3-b^3} P_1(\cos\theta) \\ &- \frac{V}{2}\cdot\frac{7}{8}\frac{\left(a^4+b^4\right) r^3 - a^4b^4\left(a^3+b^3\right) r^{-4}}{a^7-b^7} P_2(\cos\theta).\end{aligned}$$

Next, let us consider the limiting cases.

(i) For $b \to \infty$, we can rewrite the potential as

$$\Phi(r,\theta) = \frac{V}{2} + \frac{V}{2}\sum_{l>0,\ \text{odd}} \left[P_{l-1}(0) - P_{l+1}(0)\right] \frac{\left(1+(a/b)^{l+1}\right)(r/b)^{l} - \left(1+(a/b)^{l}\right)(a/r)^{-(l+1)}}{(a/b)^{2l+1}-1} P_l(\cos\theta).$$

In this limit, $a/b \to 0$ and $r/b \to 0$, we are left with

$$\Phi(r,\theta) = \frac{V}{2} + \frac{V}{2}\sum_{l>0,\ \text{odd}} \left[P_{l-1}(0) - P_{l+1}(0)\right] \left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta).$$

(ii) Similarly, for $a \to 0$, we have $a/b \to 0$ and $a/r \to 0$, and the potential is

$$\Phi(r,\theta) = \frac{V}{2} - \frac{V}{2}\sum_{l>0,\ \text{odd}} \left[P_{l-1}(0) - P_{l+1}(0)\right] \left(\frac{r}{b}\right)^{l} P_l(\cos\theta).$$