11.28 (a) In the rest frame of the dipole, $\Phi' = \vec{p}\cdot\vec{r}'/r'^3$, and $\vec{A}' = 0$. Using the Lorentz transformation, $\vec{r}_{\parallel}' = \gamma(\vec{r}_{\parallel} - \vec{v}t)$, $\vec{r}_{\perp}' = \vec{r}_{\perp}$. We can assume K and K' have the same origin at $t = t' = 0$, then the position of the dipole in K is $\vec{x}_0(t) = \vec{v}t$. Therefore, in K, $\Phi' = \vec{p}\cdot\vec{R}/R^3$, where $\vec{R} = \vec{x} - \vec{x}_0(t)$. Since $(\Phi, \vec{A})$ is a 4-vector, we must have

$$\Phi = \gamma(\Phi' + \vec{\beta}\cdot\vec{A}') = \gamma\Phi' = \gamma\frac{\vec{p}\cdot\vec{R}}{R^3}, \quad \vec{A}_{\parallel} = \gamma(\vec{A}_{\parallel}' + \vec{\beta}\Phi') = \gamma\vec{\beta}\,\frac{\vec{p}\cdot\vec{R}}{R^3}, \quad \vec{A}_{\perp} = \vec{A}_{\perp}' = 0.$$

For small $\beta \ll 1$, $\gamma = 1 + o(\beta^2)$. Therefore, to first order in β

$$\Phi = \frac{\vec{p}\cdot\vec{R}}{R^3}, \quad \vec{A} = \vec{\beta}\,\frac{\vec{p}\cdot\vec{R}}{R^3}.$$

(b)
$$\frac{\partial\Phi}{\partial t} = \frac{\vec{p}\cdot\dot{\vec{R}}}{R^3} - \frac{3(\vec{p}\cdot\vec{R})(\dot{\vec{R}}\cdot\vec{R})}{R^5} = -\vec{v}\cdot\frac{\vec{p} - 3\vec{n}(\vec{n}\cdot\vec{p})}{R^3}, \quad \text{where } \vec{n} = \vec{R}/R.$$

$$\nabla\cdot\vec{A} = \vec{\beta}\cdot\nabla\left(\frac{\vec{p}\cdot\vec{R}}{R^3}\right) = \vec{\beta}\cdot\left[(\vec{p}\cdot\nabla)\left(\frac{\vec{R}}{R^3}\right)\right] = \vec{\beta}\cdot\left[(\vec{p}\cdot\nabla)\left(\frac{\vec{n}}{R^2}\right)\right]$$
$$= \vec{\beta}\cdot\left[\frac{1}{R^3}\left(\vec{p} - \vec{n}(\vec{n}\cdot\vec{p})\right) + \vec{n}(\vec{n}\cdot\vec{p})\frac{\partial}{\partial R}\left(\frac{1}{R^2}\right)\right] = \vec{\beta}\cdot\frac{\vec{p} - 3\vec{n}(\vec{n}\cdot\vec{p})}{R^3}$$

Therefore, $\partial_\alpha A^\alpha = \frac{1}{c}\frac{\partial\Phi}{\partial t} + \nabla\cdot\vec{A} = 0$.

(c) $\vec{E} = -\nabla\Phi = -\nabla\left(\frac{\vec{p}\cdot\vec{R}}{R^3}\right) = \frac{3\vec{n}(\vec{n}\cdot\vec{p}) - \vec{p}}{R^3}$, which can be viewed as the dipole electric field with center at $\vec{x}_0(t)$. We can also expand $\vec{n}$ and $\vec{R}$ in $\vec{x}_0(t)$, to get higher order terms, which correspond to higher order moments.

$$\vec{B} = \nabla\times\vec{A} = \nabla\left(\frac{\vec{p}\cdot\vec{R}}{R^3}\right)\times\vec{\beta} = \vec{\beta}\times\left[-\nabla\left(\frac{\vec{p}\cdot\vec{R}}{R^3}\right)\right] = \vec{\beta}\times\vec{E}.$$