5.10 The current is given by $\vec{J}(\vec{x}) = I\,\delta(\rho-a)\,\delta(z)\,\hat{\phi} = I\,\delta(\rho-a)\,\delta(z)\,(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$.

The vector potential is given by $\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'$.

(a) Using the expansion $\frac{1}{|\vec{x}-\vec{x}'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^{+\infty} dk\; e^{im(\phi-\phi')}\cos[k(z-z')]\,I_m(k\rho_<)K_m(k\rho_>)$,

$$\vec{A}(\vec{x}) = \frac{\mu_0 I}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{+\infty} dk\int_{-\infty}^{+\infty} dz'\int_0^{+\infty}\rho'd\rho'\int_0^{2\pi} d\phi'\; e^{im(\phi-\phi')}\cos[k(z-z')]\,I_m(k\rho_<)K_m(k\rho_>)$$
$$\times\ \delta(\rho'-a)\,\delta(z')\,(-\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j})$$

$$= \frac{\mu_0 I a}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{+\infty} dk\,\cos(kz)\int_0^{2\pi} d\phi'\; e^{im(\phi-\phi')}\,I_m(k\rho_<)K_m(k\rho_>)\,(-\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j})$$

Only $m=\pm1$ terms will survive, and

$$\vec{A}(\vec{x}) = \frac{\mu_0 I a}{\pi^2}\int_0^{+\infty} dk\,\cos kz\int_0^{2\pi} d\phi'\; I_1(k\rho_<)K_1(k\rho_>)\cos(\phi-\phi')(-\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j})$$

$$= \frac{\mu_0 I a}{\pi}\int_0^{+\infty} dk\,\cos(kz)\,I_1(k\rho_<)K_1(k\rho_>)\,(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$$

which is the $\hat{\phi}$ component of the vector potential.

(b) Using the expansion $\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{m=-\infty}^{+\infty}\int_0^{+\infty} dk\; e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')\,e^{-k(z_>-z_<)}$

$$\vec{A}(\vec{x}) = \frac{\mu_0 I}{4\pi}\sum_{m=-\infty}^{+\infty}\int_0^{+\infty} dk\int_{-\infty}^{+\infty} dz'\int_0^{+\infty}\rho'd\rho'\int_0^{2\pi} d\phi'\; e^{im(\phi-\phi')}J_m(k\rho)J_m(k\rho')\,e^{-k(z_>-z_<)}$$
$$\times\ \delta(\rho'-a)\,\delta(z')\,(-\sin\phi'\,\hat{i} + \cos\phi'\,\hat{j})$$

$$= \frac{\mu_0 I a}{2\pi}\int_0^{+\infty} dk\; e^{-k|z|}\int_0^{2\pi} d\phi'\; J_1(k\rho)J_1(ka)\cos[\phi-\phi'](-\sin\phi'\,\hat{j} + \cos\phi\,\hat{j})$$

$$= \frac{\mu_0 I a}{2}\int_0^{+\infty} dk\; e^{-k|z|}J_1(k\rho)J_1(ka)\,(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$$

(c) $\vec{B} = \nabla\times\vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\phi} + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right)\hat{z}$

For (a) $B_\rho = -\frac{\partial A_\phi}{\partial z} = \frac{\mu_0 I a}{\pi}\int_0^{+\infty} k\sin(kz)\,I_1(k\rho_<)K_1(k\rho_>)\,dk$

$B_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi)$