7.16 (a) From $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ and $\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$ where no current is present, we have

$$\nabla\times(\nabla\times\vec{E}) = -\nabla\times\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla\times\vec{B}) = -\mu_0\frac{\partial}{\partial t}(\nabla\times\vec{H}) = -\mu_0\frac{\partial^2\vec{D}}{\partial t}$$

Assuming $\vec{E}\propto e^{i\vec{k}\cdot\vec{x}-i\omega t}$ and consequently, $\vec{D}\propto e^{i\vec{k}\cdot\vec{x}-i\omega t}$, the above equation becomes

$$i\vec{k}\times(i\vec{k}\times\vec{E}) = \mu_0\omega^2\vec{D} \quad , \text{ or } \quad \vec{k}\times(\vec{k}\times\vec{E}) + \mu_0\omega^2\vec{D} = 0.$$

(b) Using the identity $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$, the above equation can be transformed

to $\vec{k}(\vec{k}\cdot\vec{E}) - k^2\vec{E} + \mu_0\omega^2\vec{D} = 0$. Let $\vec{k} = k\vec{n}$, the equation can be rewritten

as $\vec{n}(\vec{n}\cdot\vec{E}) - \vec{E} + \mu_0 v^2\vec{D} = 0$, where $v = \omega/k$.

Since $D_i = \epsilon_i E_i$, $i = 1,2,3$, the above equation has the following component-wise from

$$n_i(\vec{n}\cdot\vec{E}) - E_i + \frac{v^2}{v_i^2}E_i = 0,$$

or

$$E_i = \frac{n_i(\vec{n}\cdot\vec{E})}{1 - v^2/v_i^2}$$

Multiply both sides by n_i and take the sum, we are left with

$$\vec{n}\cdot\vec{E} = \sum_{i=1}^{3}\frac{n_i^2}{1-v^2/v_i^2}(\vec{n}\cdot\vec{E}), \quad \text{or} \quad \sum_{i=1}^{3}\frac{n_i^2}{1-v^2/v_i^2} = 1 = \sum_{i=1}^{3} n_i^2$$

Therefore, $\sum_{i=1}^{3} n_i^2\left(\frac{1}{1-v^2/v_i^2} - 1\right) = 0, \Rightarrow \sum_{i=1}^{3} n_i^2\frac{v^2}{v_i^2-v^2} = 0$

which leads to $$\sum_{i=1}^{3}\frac{n_i^2}{v^2 - v_i^2} = 0$$

The Fresnel condition is equivalent to a quadratic equation in v^2,

$$v^4 - \left(n_1^2(v_2^2+v_3^2) + n_2^2(v_3^2+v_1^2) + n_3^2(v_1^2+v_2^2)\right)v^2 + n_1^2v_2^2v_3^2 + n_2^2v_3^2v_1^2 + n_3^2v_1^2v_2^2 = 0$$

This has two distinct solutions, as is clear from the coefficients.

(c) The component wise equation can also be written as

$$n_i \sum_j n_j \frac{D_j}{\varepsilon_j} - \frac{D_i}{\varepsilon_i} + \mu_0 v^2 D_i = 0 \quad \Rightarrow \quad n_i \sum_j n_j v_j^2 D_j - v_i^2 D_i + v^2 D_i = 0$$

Multiply by n_i and sum, noticing $\sum n_i^2 = 1$, we have $\vec{n} \cdot \vec{D} = 0$, which means that the two distinct modes in the displacement are transverse. For the two modes $\vec{D}_a$ and $\vec{D}_b$, the component wise equations are

$$n_i \sum n_j v_j^2 D_{aj} - v_i^2 D_{ai} + v_a^2 D_{ai} = 0 \qquad (1)$$

$$n_i \sum n_j v_j^2 D_{bj} - v_i^2 D_{bi} + v_b^2 D_{bi} = 0 \qquad (2)$$

Multiply (1) by D_{bi} and sum, and (2) by D_{ai} and sum, we have

$$\left(\sum_i n_i D_{bi}\right) \cdot \left(\sum_j n_j v_j^2 D_{aj}\right) - \sum_i v_i^2 D_{ai} D_{bi} + v_a^2 D_{ai} D_{bi} = 0 \qquad (3)$$

$$\left(\sum_i n_i D_{ai}\right) \cdot \left(\sum_j n_j v_j^2 D_{bj}\right) - \sum_i v_i^2 D_{ai} D_{bi} + v_b^2 D_{ai} D_{bi} = 0 \qquad (4)$$

Take the difference between (3) and (4) and use the transversality of $\vec{D}$, we are left with $(v_a^2 - v_b^2)\, \vec{D}_a \cdot \vec{D}_b = 0$

Since v_a^2 and v_b^2 are distinct, we must have $\vec{D}_a \cdot \vec{D}_b = 0$