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The nonvanishing components of the Ricci tensor R^{μ}_{ν} are found to be

$$R^0_0 = -\exp(-\lambda)(\nu''/2+\nu'^2/4-\lambda'\nu'/4+\omega'\nu'/2),$$

$$R^1_1 = -\exp(-\lambda)(\nu''/2+\nu'^2/4-\lambda'\nu'/4+\omega'' +\omega'^2/2-\omega'\lambda'/2), \qquad \text{(A3)}$$

$$R^2_2 = R^3_3 = -\exp(-\lambda)(\omega''/2+\omega'^2/2-\omega'\lambda'/4+\omega'\nu'/4) +\exp(-\omega).$$

The curvature scalar is

$$R_c = -\exp(-\lambda)(\nu''+\nu'^2/2-\lambda'\nu'/2+\omega'\nu'-\omega'\lambda' +2\omega''+3\omega^2/2)+2\exp(-\omega). \qquad \text{(A4)}$$

The nonvanishing components of the Einstein tensor are

$$G^0_0 = -\exp(-\omega)+\exp(-\lambda)(\omega''+3\omega'^2/4-\omega'\lambda'/2),$$

$$G^1_1 = -\exp(-\omega)+\exp(-\lambda)(\omega'\nu'/2+\omega'^2/4), \qquad \text{(A5)}$$

$$G^2_2 = G^3_3 = \exp(-\lambda)(\nu''/2+\nu'^2/4-\lambda'\nu'/4 +\omega'\nu'/4-\omega'\lambda'/4+\omega''/2+\omega'^2/4).$$

The calculation of the scalar curvature of the two-dimensional surface that is given by Eq. (4) proceeds as follows. We write the two-dimensional metric as $ds^2 = g_{ik}dx^i dx^k$ with $x^1=R$, $x^2=\phi$, $g_{11}=\exp\lambda(R)=(R^2+b^2)/(R^2+a^2)$, $g_{22}=\exp\omega(R)=R^2+a^2$, and $g_{12}=g_{21}=0$. The nonvanishing independent connection coefficients are found to be Γ^1_{11}, Γ^1_{22}, and Γ^2_{12}; they are the same as the ones given in Eq. (A1). There is only one independent component of the curvature tensor. It is R^1_{212}, and is the same as that given in Eq. (A2). Contracting the curvature tensor twice to obtain the scalar curvature gives Eq. (4).

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Proof that the Neumann Green's function in electrostatics can be symmetrized

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We prove by construction that the Green's function satisfying the Neumann boundary conditions in electrostatic problems can be symmetrized. An illustrative example is given.

I. INTRODUCTION

It is well known that the Green's function satisfying the Dirichlet boundary conditions in electrostatic problems is symmetric in its arguments. The symmetry property is often very useful in constructing an explicit representation of the Green's function. It is stated in Jackson[1] that, for a Green's function satisfying the Neumann boundary conditions, the symmetry is not automatic but can be imposed as a separate requirement. However, an explicit proof that Neumann Green's function can indeed be symmetrized does not appear to be readily available in published references. Here, we offer such a proof, and present an illustrative example.

II. PROOF

Following the discussion in Jackson,[1] we consider the electrostatic boundary value problem in a volume V bounded by a surface S. The Green's function satisfies the following equation for $\mathbf{x}$ and $\mathbf{x}'$ in V:

$$\nabla'^2 G_{D,N}(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}'). \qquad (1)$$

We distinguish two different Green's functions G_D and G_N. For a Dirichlet problem,

$$G_D(\mathbf{x},\mathbf{x}')=0 \quad \text{for } \mathbf{x}' \text{ on } S. \qquad (2)$$

For a Neumann problem, we must satisfy the Gauss theorem constraint,

$$\oint_S \frac{\partial G}{\partial n'} da' = -4\pi.$$

The simplest way to satisfy the requirement is to impose

$$\frac{\partial G_N}{\partial n'}(\mathbf{x},\mathbf{x}') = -\frac{4\pi}{S} \quad \text{for } \mathbf{x}' \text{ on } S \text{ and } \mathbf{x} \text{ within } V. \tag{3}$$

Here $\partial/\partial n'$ is the normal derivative at the surface S directed outwards from inside the volume V, and S in Eq. (3) is also the total area of the boundary surface. The solution to the Neumann boundary value problem is then

$$\Phi(\mathbf{x}) = \langle\Phi\rangle_S + \int_V \rho(\mathbf{x}') G_N(\mathbf{x},\mathbf{x}') d^3\mathbf{x}' + \frac{1}{4\pi}\int_S \frac{\partial\Phi}{\partial n'} G_N. \tag{4}$$

The symmetry property of the Dirichlet Green's function G_D can be proved by means of the second Green's identity

$$\int_V [\varphi(\mathbf{y})\nabla^2\psi(\mathbf{y}) - \psi(\mathbf{y})\nabla^2\varphi(\mathbf{y})] d^3\mathbf{y} = \oint \left(\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right) da. \tag{5}$$

Thus by setting $\varphi = G_D(\mathbf{x},\mathbf{y})$ and $\psi = G_D(\mathbf{x}',\mathbf{y})$ in the above, one obtains readily $G_D(\mathbf{x},\mathbf{x}') = G_D(\mathbf{x}',\mathbf{x})$.

For a Neumann problem the Green's function $G_N(\mathbf{x},\mathbf{x})$ is, in general, not symmetric in $\mathbf{x}$ and $\mathbf{x}'$. However, we can show that a symmetrized Green's function G_N^S can always be constructed. For this purpose, set $\varphi = G_N(\mathbf{x},\mathbf{y})$ and $\psi = G_N(\mathbf{x}',\mathbf{y})$ in Eq. (5) to obtain

$$G_N(\mathbf{x},\mathbf{x}') - G_N(\mathbf{x}',\mathbf{x}) = F(\mathbf{x}) - F(\mathbf{x}'), \tag{6}$$

where

$$F(\mathbf{x}) = \frac{1}{S}\oint_S G_N(\mathbf{x},\mathbf{y}) da_y \tag{7}$$

apart from the possibility of an added constant. Equation (6) implies that the combination

$$G_N^S(\mathbf{x},\mathbf{x}') = G_N(\mathbf{x},\mathbf{x}') - F(\mathbf{x}) \tag{8}$$

is symmetric in $\mathbf{x}$ and $\mathbf{x}'$. Further, since $G_N^S(\mathbf{x},\mathbf{x}')$ differs from $G_N(\mathbf{x},\mathbf{x}')$ by a function $F(\mathbf{x})$ that depends only on $\mathbf{x}$, it satisfies both Eqs. (1) and (3). Therefore $G_N^S(\mathbf{x},\mathbf{x}')$ is a Neumann Green's function which is symmetric in $\mathbf{x}$ and $\mathbf{x}'$. One might be concerned that the additional $F(\mathbf{x})$ changes the solution, Eq. (4). However, Gauss' law saves the day because the added contribution to the potential from $F(\mathbf{x})$ is

$$\Delta\Phi(\mathbf{x}) = F(\mathbf{x})\left(\int_V \rho(\mathbf{x}') d^3\mathbf{x}' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'} da'\right).$$

The first integral is the total charge within V, while the second is the negative of the total electric flux leaving V (divided by 4π); the sum vanishes. We note that the function $F(\mathbf{x})$ defined by Eq. (7) is what is needed to make the Green's function symmetric, but that any function of $\mathbf{x}$ can be added to $G_N(\mathbf{x},\mathbf{x}')$ without affecting the result for the potential.

III. AN EXAMPLE

As an example, we consider the Neumann Green's function for the volume V between two concentric spheres of radii a and b ($a<b$). We have the following expansion in spherical harmonics:

$$G_N(\mathbf{x},\mathbf{x}') = \sum_{l=0}^{\infty} g_l(r,r') P_l(\cos\gamma), \tag{9}$$

where r and r' are the radial components of $\mathbf{x}$ and $\mathbf{x}'$, respectively, and γ is the angle between $\mathbf{x}$ and $\mathbf{x}'$. The function g_l can be written in the following form:

$$g_l(r,r') = \frac{r_<^l}{r_>^{l+1}} + \alpha_l(r) r'^l + \beta_l(r)\frac{1}{r'^{l+1}}. \tag{10}$$

Here $r_<$ ($r_>$) is the smaller (greater) of r and r'. The first term in the right-hand side of Eq. (10) gives rise to the delta when inserted to Eq. (1). The unknown functions $\alpha_l(r)$ and $\beta_l(r)$ are to be determined from the boundary condition. The requirement Eq. (3) involves only $g_0(r,r')$, the spherically symmetric term. The boundary conditions at $r'=b$ and $r'=a$ are, respectively,

$$\left.\frac{\partial g_0}{\partial n'}\right|_{r'=b} = \frac{\partial}{\partial r'}\left(\frac{1+\beta_0(r)}{r'}\right)\Bigg|_{r'=b} = -\frac{1}{a^2+b^2}, \tag{11}$$

$$\left.\frac{\partial g_0}{\partial n'}\right|_{r'=a} = -\frac{\partial}{\partial r'}\left(\frac{1}{r} + \frac{\beta_0(r)}{r'}\right)\Bigg|_{r'=a} = -\frac{1}{a^2+b^2}. \tag{12}$$

It is easy to see that both Eqs. (11) and (12) lead to

$$\beta_0(r) = -\frac{a^2}{a^2+b^2}. \tag{13}$$

Thus we have

$$g_0(r,r') = \frac{1}{r_>} + \alpha_0(r) - \frac{a^2}{a^2+b^2}\frac{1}{r'}. \tag{14}$$

The function $\alpha_0(r)$ is left undetermined, and the function g_0 is in general not symmetric in r and r'. As noted above, the form of $a_0(r)$ is of no consequence for the solution of the potential problem.

For $l>0$, one obtains by analogous calculation ($\partial g_l/\partial r' = 0$ at $r'=a,b$),

$$\alpha_l(r) = \frac{1}{b^{2l+1} - a^{2l+1}}\left[\left(\frac{l+1}{l}\right) r^l + \frac{a^{2l+1}}{r^{l+1}}\right], \tag{15}$$

$$\beta_l(r) = \frac{1}{b^{2l+1} - a^{2l+1}}\left(a^{2l+1} r^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{r^{l+1}}\right), \tag{16}$$

$$g_l(r,r') = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{b^{2l+1} - a^{2l+1}}\left[\frac{l+1}{l}(rr')^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{(rr')^{l+1}} + a^{2l+1}\left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}}\right)\right]. \tag{17}$$

Note that $g_l(r,r')$ for $l>0$ is symmetric in r and r'.

The integral in Eq. (7) defining $F(\mathbf{x})$ receives contributions only from the term $l=0$. We find

$$F(\mathbf{x})=\frac{1}{a^2+b^2}\left[a^2\left(\frac{1}{r}+\alpha_0(r)-\frac{a}{a^2+b^2}\right)\right.$$
$$\left.+b^2\left(\frac{1}{b}+\alpha_0(r)-\frac{a^2}{a^2+b^2}\frac{1}{b}\right)\right]. \tag{18}$$

The symmetrized Green function Eq. (7) becomes in this case

$$G_N^S(\mathbf{x},\mathbf{x}')=\sum_{l=0}^{\infty} g_l^S(r,r')P_l(\cos\gamma), \tag{19}$$

where

$$g_0^S(r,r')=\frac{1}{r_>}-\frac{a^2}{a^2+b^2}\left(\frac{1}{r}+\frac{1}{r'}\right)+\frac{a^3-b^3}{(a^2+b^2)^2}, \tag{20}$$

$$g_l^S(r,r')=g_l(r,r') \quad \text{for } l>0. \tag{21}$$

Note that the left-hand side of Eq. (20) is explicitly symmetric. The last term $(a^3-b^3)/(a^2+b^2)^2$ is a constant and thus can be omitted.

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[1] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), Sec. 1.10, pp. 43–45.

WHY DOES THE ELECTRON CARE ABOUT OUR MATHEMATICS?

[My experience with the Millikan oil drop experiment] brought home to me as nothing else could the truth of Einstein's remark, "One may say the eternal mystery of the world is its comprehensibility." Here was I, sitting at my desk for weeks on end, doing the most elaborate and sophisticated calculations to figure out how an electron should behave. And here was the electron on my little oil drop, knowing quite well how to behave without waiting for the result of my calculation. How could one seriously believe that the electron really cared about my calculation, one way or the other? And yet the experiments at Columbia showed that it did care. Somehow or other, all this complicated mathematics that I was scribbling established rules that the electron on the oil drop was bound to follow. We know that this is so. Why it is so, why the electron pays attention to our mathematics, is a mystery that even Einstein could not fathom.

Freeman Dyson, *Disturbing the Universe* (Harper, New York, 1979), pp. 49–50.

KEPLER'S FIRST LAW—A *SINGLE* CART-FUL OF DUNG

> "The conclusion is quite simply that the planet's path is not a circle—it curves inward on both sides and outward again at opposite ends. Such a curve is called an oval. The orbit is not a circle, but an oval figure."—Kepler

Why indeed an oval? There is something in the perfect symmetry of spheres and circles which has a deep, reassuring appeal to the unconscious—otherwise it could not have survived two millennia. The oval lacks all such archetypal appeal. It has an arbitrary form. It distorts that eternal dream of the harmony of the spheres, which lay at the origin of the whole quest. Who art thou, Johann Kepler, to destroy divine symmetry? All he has to say in his own defence is, that having cleared the stable of astronomy of cycles and spirals, he left behind him "only a single cart-ful of dung:" his oval.

Arthur Koestler, *The Sleepwalkers* (1959). (Reprinted by Grosset & Dunlap, New York, 1963), p. 329.