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The momentum of light in a refracting medium

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It is shown that neither Minkowski's result, according to which the ratio of momentum to energy for a light wave in a medium of refractive index n is n/c, nor that of Abraham, who found $1/nc$, is correct. For a broad wave in a uniform medium, the correct answer is given by (2.12) with $\sigma = \frac{1}{5}$. For weak refraction it is approximately equal to the average of the Abraham and Minkowski results. Abraham's formula gives correctly the part of the momentum which resides in the electromagnetic field, but not the mechanical momentum of the medium which travels with the light pulse. Minkowski's formula gives the pseudo-momentum, a quantity of physical interest. The momentum change upon reflexion or transmission usually involves also acoustic transients, these are discussed for some simple cases.

1. Introduction

There is an extensive literature on the problem of the momentum of light in a refractive medium, starting with the work of Minkowski (1908, 1910), who found for the momentum density g of a light wave of energy density U in a medium of refractive index n

$$g = \frac{n}{c}U = |\boldsymbol{D}\times\boldsymbol{B}| = \frac{n^2}{c^2}|\boldsymbol{E}\times\boldsymbol{H}|, \tag{1.1}$$

whereas Abraham (1909, 1910) found

$$g = \frac{1}{nc}U = \frac{1}{c^2}|\boldsymbol{E}\times\boldsymbol{H}|. \tag{1.2}$$

Here c is the speed of light *in vacuo* and $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$, $\boldsymbol{H}$, are the usual Maxwell field quantities.

Of the many papers published since then, some support the Minkowski result (1.1), others the Abraham result (1.2). It will be shown in the present paper that neither of these answers can be claimed to give the total momentum density. Both relate to physically interesting quantities: the Minkowski espression (1.1) relates not to momentum, but to pseudo-momentum, in a sense to be discussed in §6, whereas Abraham's expression (1.2) gives the part of the momentum residing in the electromagnetic field, but excludes the part carried by the matter.

One of the sources of confusion is the fact that the passage of an electromagnetic wave through a material medium in general causes forces on the atoms in the medium, and that the momentum imparted to the matter does not always bear a fixed relation to the electromagnetic momentum, and is not distributed in space in the same way.

In the present paper we shall consider particularly the case of a plane wave of practically unlimited width, though of finite length. Even in this relatively simple case one has to distinguish a number of contributions to the momentum density:

(*a*) The electromagnetic momentum, given correctly by (1.2) above.

(*b*) The 'accompanying' momentum. While the wave travels in a uniform medium there is a mechanical force density proportional to the intensity gradient, so that the momentum given to the matter at the leading edge of the wave train is cancelled out at the trailing edge, so that a quantity of momentum travels with the wave.

(*c*) The 'deposited' momentum. Near an inhomogeneity in the medium, such as a reflecting surface, the force density is no longer given by a time derivative, and some momentum remains after the wave has passed. This is spread over a region on the incident side of an interface, of a thickness of the order of the length of the wave train. It will then spread with sound velocity.

(*d*) The interface impulse. If (*a*), (*b*) and (*c*) do not balance, they will leave an excess of momentum to be taken up by the interface. According to the macroscopic equations, this is given to an infinitely thin layer at the interface; in reality it is taken up by a layer of a thickness equal to a few atomic spacings. If the change in the refracting properties of the medium is gradual, there is a force proportional to the gradient of the refractive index.

It seems natural to treat (*a*) and (*b*) as constituting the total momentum of the light wave, but in any experiment involving reflecting surfaces it is essential to take account of (*c*).

The situation is more complex for a wave train of finite width, since then lateral forces arise at the sides. These affect the whole wave train if its width is less than its duration times the velocity of sound. The wave can be treated as unlimited in width, if the width is large compared to this quantity.

The general case, including that of a wave of finite width, is discussed in the recent article by Robinson (1975). For a narrow pulse the mechanical momentum density is no longer distributed in space in proportion to the electromagnetic intensity, and it is therefore impossible to define any simple quantity as the total momentum density. We conclude that the only simple case in which a definition of the total momentum is possible is that of a wide wave discussed above. One may, of course, choose to give no answer to the question of the total momentum of a light wave and to state the general form of the force density, as is done by Robinson (1975), and by Skobeltsyn (1973).

Many of the papers in the literature which attempt to find the total momentum contain errors, and this includes a paper by M. G. Burt and the present author (Burt & Peierls 1973).

I propose to take that paper, which is typical of the pro-Abraham arguments, as my starting point. The argument consisted, in essence, of three steps:

(1) An electromagnetic pulse of finite duration, in a uniform refracting medium, is a well-defined concept, as long as it can be separated from transients arising during its emission, refraction, or reflexion. Hence the relation between its momentum and its energy is well defined.

(2) The symmetry of the stress tensor requires the momentum density to equal c^{-2} times the energy flux.

(3) The energy flux must on the average be equal to the energy density times the velocity of propagation (the latter being, in the case of dispersion, the group velocity).

Abraham's result follows directly from these statements. Of these, (1) requires no comment at this stage. (2) and (3) are, separately, correct statements, but they refer to different definitions of energy transport, and it is therefore not permissible to put them together.

The point is that, during the passage of the electromagnetic wave, there are forces acting on the atoms of the medium, which may therefore acquire a non-vanishing average momentum. This was, of course, meant to be included in our discussion, which was concerned with *total* momentum. The mechanical momentum yields a momentum density ρv, where ρ is the mass density, and v the average velocity. By the symmetry of the tensor, this requires a term $\rho c^2 v$ in the energy flux. Indeed, ρc^2 is the density of rest energy, which has to be included to satisfy statement (2), which is relativistic. With this definition, (3) is no longer valid since it refers only to the additional energy carried by the light wave, and not to the rest energy of the medium, which, in any case, is not carried along with the velocity of light. The conclusions of the previous paper fail, therefore, whenever the mean velocity v of the atoms is non-zero.

In view of this finding one might now conjecture that the total momentum was given correctly by Minkowski's expression. We shall, see, however, that the usual derivation of (1.1) is also incorrect, though it yields the correct value of pseudo-momentum.

All this relates to the 'pure' wave specified in (1) above. In any experiment which attempts to measure the momentum, a change in its state will be involved, and usually this change causes momentum to be 'deposited' in the sense of (c) above, and this affects the momentum balance.

This was pointed out in the previous paper (Burt & Peierls 1973), and we should not have been surprised, therefore, by the finding of Jones & Richards (1954) that the momentum transferred to a mirror on reflexion did not equal the difference between the momenta of the incident and the reflected wave. At the time we tried to look for corrections due to transients caused when the light entered the liquid from the air, but omitted to look for them in the reflexion process itself.

The methods used in the present paper make it possible to evaluate this momentum deposition, subject to simplifying assumptions which seem to be justified in the case of the Jones–Richards experiment, and the result is in agreement with the experimental result. The argument is only a minor extension of one given by Gordon (1973).

The plan of the present paper is as follows:

§2 discusses the forces on the atoms of the medium, and substantiates the criticism of the Abraham result outlined above. The actual value of the momentum given to the atoms depends on a coefficient, σ, which relates the field gradient at a typical atom to the gradient of the macroscopic field.

§3 determines the coefficient σ and shows it to be $\frac{1}{5}$ for a liquid or amorphous solid containing polarizable atoms, and not $\frac{1}{3}$ as is frequently assumed.

§4 discusses the customary derivation of the Minkowski result.

§5 is concerned with transients, in reflexion and transmission.

§6 discusses pseudo-momentum and a theorem by Gordon.

§7 points out a surprising conclusion in the case of a narrow light pulse in a solid medium.

2. Discussion of the momentum and of the force on the medium

On a microscopic scale, we may divide the momentum density into its parts

$$\boldsymbol{g} = \boldsymbol{g}_{\text{e.m.}} + \boldsymbol{g}_{\text{mech.}}, \tag{2.1}$$

where

$$\boldsymbol{g}_{\text{e.m.}} = \frac{1}{c^2}\boldsymbol{E}_{\text{m}} \times \boldsymbol{H}_{\text{m}}. \tag{2.2}$$

Here $\boldsymbol{E}_{\text{m}}$ and $\boldsymbol{H}_{\text{m}}$ are the microscopic fields, as distinct from the macroscopic fields $\boldsymbol{E}$ and $\boldsymbol{H}$, which occur in the macroscopic Maxwell equations. (In principle, there could be an interaction term in (2.1). However, it is easy to verify from the microscopic Maxwell equations that the sum of (2.2) and of the momentum carried by atoms is conserved, so that there is no room for an interaction term.)

For the purposes of this paper we are concerned with space and time averages. The electromagnetic momentum (2.2) is bilinear in the fields, and its average could, in principle, differ substantially from the product $\boldsymbol{E} \times \boldsymbol{B}/\mu_0 c^2$ of the macroscopic fields. However, refracting media are usually non-magnetic, i.e. their magnetic permeability differs from that of the vacuum by some 10^{-5}. Neglecting the magnetic polarization of the atoms, we may identify $\boldsymbol{H}_{\text{m}}$ with its average, $\boldsymbol{B}/\mu_0$, so that the only averaging required is that of $\boldsymbol{E}_{\text{m}}$, and this results in the macroscopic field $\boldsymbol{E}$.

We conclude that for a non-magnetic medium the electromagnetic momentum density (2.2) equals, on the average, its macroscopic form

$$\boldsymbol{g}_{\text{e.m.}} = \frac{1}{\mu_0 c^2}\boldsymbol{E} \times \boldsymbol{B} = \frac{1}{c^2}\boldsymbol{E} \times \boldsymbol{H}, \tag{2.3}$$

which equals the Abraham value (1.2). The device of avoiding complications by restricting the discussion to non-magnetic media is also used by Gordon (1973).

It remains to consider the mechanical momentum, which is determined by the force exerted by the light on the atoms of the medium. If an atom has electric dipole moment $\dot{\boldsymbol{d}}$, the force on it is

$$\boldsymbol{F} = (\boldsymbol{d}\cdot\nabla)\,\boldsymbol{E}_{\text{eff.}} + \dot{\boldsymbol{d}} \times \boldsymbol{B}_{\text{eff.}}. \tag{2.4}$$

Here the 'effective' fields $\boldsymbol{E}_{\text{eff.}}$ and $\boldsymbol{B}_{\text{eff.}}$ are the microscopic fields at the site of an atom, but excluding the field due to the atom itself. In (2.4) we have made use of the fact that the space integral of the current density due to the atom is $\dot{\boldsymbol{d}}$.

The evaluation of the second term is straightforward, since the medium is non-magnetic, and therefore $\boldsymbol{B}_{\text{eff.}}$ may be replaced by the macroscopic field, $\boldsymbol{B}$. (The correction due to the currents of the oscillating dipoles is proportional to $\dot{\boldsymbol{d}}$, and would contribute to (2.4) a term proportional to $(\dot{\boldsymbol{d}})^2$, i.e. to the square of the frequency, which goes beyond the accuracy aimed at.)

As regards the first term, one might be tempted to argue† that it should make no contribution in the case of a transverse wave, since we are interested in the momentum in the direction of propagation, while $\boldsymbol{E}$ is at right angles to it, and one might expect from symmetry $\boldsymbol{E}_{\text{eff.}}$ also to be transverse. However, while $\boldsymbol{E}_{\text{eff.}}$ is indeed transverse at the site of an atom, its gradient need not be.

Assuming for definiteness that we are interested in the force in the z direction, and that $\boldsymbol{d}$ is in the x direction, we then require $(\partial E_z/\partial x)_{\text{eff}}$. The difference between the microscopic and macroscopic fields is due to the presence of electric dipoles in the medium, and must therefore be proportional to the dipole density, or polarization, $P = (\epsilon_0 - \epsilon_0)\,E$. The most general relation consistent with this, and with symmetry, is

$$\left(\frac{\partial E_z}{\partial x}\right)_{\text{eff.}} = \frac{\partial E_z}{\partial x} + \tau\frac{\epsilon-\epsilon_0}{\epsilon_0}\frac{\partial E_z}{\partial x} + \sigma\frac{\epsilon-\epsilon_0}{\epsilon_0}\frac{\partial E_x}{\partial z}, \tag{2.5}$$

where σ and τ are numerical coefficients, and the field components on the right hand side are those of the macroscopic field. The coefficients σ and τ will be discussed in §3, and it will be shown that they are both equal to $\frac{1}{5}$.

Making use of Maxwell's equation, (2.5) can be written as

$$\left(\frac{\partial E_z}{\partial x}\right)_{\text{eff.}} = \left[1 + (\sigma+\tau)\frac{\epsilon-\epsilon_0}{\epsilon_0}\right]\frac{\partial E_z}{\partial x} - \sigma\frac{\epsilon-\epsilon_0}{\epsilon_0}\frac{\partial B_y}{\partial t}, \tag{2.6}$$

and the force (2.4) becomes for a general geometry

$$\boldsymbol{F} = \left[1 + (\sigma+\tau)\frac{\epsilon-\epsilon_0}{\epsilon_0}\right](\boldsymbol{d}\cdot\nabla)\,\boldsymbol{E} - \sigma\frac{\epsilon-\epsilon_0}{\epsilon_0}\boldsymbol{d}\times\dot{\boldsymbol{B}} + \dot{\boldsymbol{d}}\times\boldsymbol{B}. \tag{2.7}$$

The force per unit volume is the sum of contribution (2.7) over all atoms in a unit volume, which again brings in the polarization of the medium

$$\frac{\mathrm{d}\boldsymbol{g}_{\text{mech.}}}{\mathrm{d}t} = \left[1 + (\sigma+\tau)\frac{\epsilon-\epsilon_0}{\epsilon_0}\right](\epsilon-\epsilon_0)\,(\boldsymbol{E}\cdot\nabla)\,\boldsymbol{E} - \sigma\frac{(\epsilon-\epsilon_0)^2}{\epsilon_0}\boldsymbol{E}\times\dot{\boldsymbol{B}} + (\epsilon-\epsilon_0)\,\dot{\boldsymbol{E}}\times\boldsymbol{B}. \tag{2.8}$$

† This view was presented by the author in a talk at the Franco-British Centenary conference in St Helier, Jersey, in April 1974. It amounts to setting $\sigma = 0$ in equations of this section.

Assume now that the electric vector is everywhere in the x direction, the magnetic vector in the y direction. Then the rate of change of the z component of mechanical momentum density (remembering that $\epsilon = n^2\epsilon_0$, and that $\epsilon_0\mu_0 = c^{-2}$,

$$\frac{\mathrm{d}g_{\mathrm{mech.},z}}{\mathrm{d}t} = \frac{n^2-1}{c^2}\dot{E}_x H_y - \sigma\frac{(n^2-1)^2}{c^2}E_x\dot{H}_y. \tag{2.9}$$

We now make the further restriction that we are dealing with a unidirectional wave, travelling, say, in the $+z$ direction. Then, from Maxwell, $H = \epsilon_0 ncE$, and

$$\dot{E}_x H_y = E_x\dot{H}_y = \frac{1}{2}\frac{d}{\mathrm{d}t}(E_x H_y). \tag{2.10}$$

In this case (2.9) takes the form

$$\frac{\mathrm{d}g_{\mathrm{mech.},z}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{n^2-1}{2c^2}[1-\sigma(n^2-1)](\boldsymbol{E}\times\boldsymbol{H})_z\right\}. \tag{2.11}$$

This result shows, first of all, that if a plane light signal is travelling into a medium at rest, the medium comes to rest again after the signal has passed. No 'wake' is left in these circumstances. There is, however, a momentum density in the medium while the light is passing. This has to be added to the electromagnetic part (2.3) to obtain the total momentum density. The quantity $g_{\mathrm{e.m.}}$, is, as we have seen, just the Abraham momentum, so that we find for the total momentum density

$$\left.\begin{aligned} g &= \left[\frac{1+n^2}{2} - \frac{\sigma}{2}(n^2-1)^2\right] g_{\mathrm{Abraham}}, \\ &= \frac{1}{2}\left[\frac{1}{n} + n - \frac{\sigma(n^2-1)^2}{n}\right] g_{\mathrm{vac}}. \end{aligned}\right\} \tag{2.12}$$

For n not too far from unity, the quadratic term is small, and, to a good approximation, the correct answer is then just the arithmetic mean of the Abraham and Minkowski results.

Figure 1 shows (2.12) as function of n for various values of σ, as well as the Abraham and Minkowski results. Of these curves, the correct one, according to §3, is the one labelled $\sigma = \frac{1}{5}$. For n exceeding about 2.8, the momentum is negative, i.e. in a direction opposite the direction of propagation.

3. The effective field

We assume in this section that the medium is a liquid or amorphous solid containing neutral polarizable atoms. The atoms are in positions with the radius vectors $\boldsymbol{r}_n$; they are assumed heavy enough for their displacement during the passage of the light signal to be negligible. We also assume, as usual, that the light wavelength is large compared to the interatomic spacings and correlation distances, and keep only the leading terms in the light wavevector.

Each atom is supposed to have electric polarizability α. The dipole moment of the nth atom is therefore

$$\boldsymbol{d}_n = \alpha \boldsymbol{E}_{\mathrm{eff.}}(\boldsymbol{r}_n). \tag{3.1}$$

We assume the exciting field to be a plane wave,

$$E_{\text{eff.}} = \boldsymbol{C}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega t} = \boldsymbol{E}+\boldsymbol{E}_{\mathrm{d}}, \tag{3.2}$$

where $\boldsymbol{E}_{\mathrm{d}}$ is the field due to the atomic dipoles. For the latter, we should, in principle, calculate a scalar and a vector potential, but it is easy to verify that the effect of the vector potential is negligible, since it depends on the current density, which is

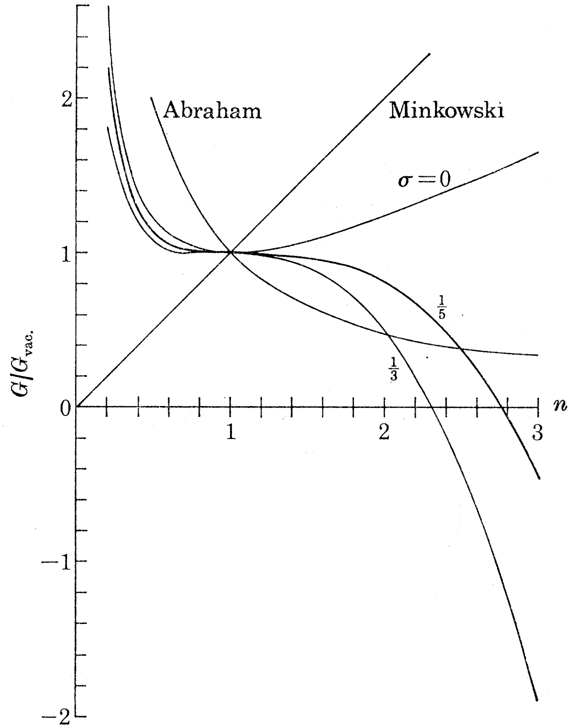

FIGURE 1. The ratio of the momentum of a light wave of given energy to the vacuum momentum for the same energy. The curves show the Minkowski and Abraham results, and our result (2.12) for various σ.

related to $\boldsymbol{d}$, and therefore contains an extra power of the frequency ω. For the scalar potential, neglecting the size of the atom for simplicity,

$$\left(\nabla^2+\frac{\omega^2}{c^2}\right)\phi(\boldsymbol{r}) = \frac{1}{\epsilon_0}\sum_n(\boldsymbol{d}_n\cdot\boldsymbol{\nabla})\,\delta(\boldsymbol{r}-\boldsymbol{r}_n) = \frac{\alpha}{\epsilon_0}\boldsymbol{C}\cdot\boldsymbol{\nabla}\sum_n\delta(\boldsymbol{r}-\boldsymbol{r}_n)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_n-\mathrm{i}\omega t}. \tag{3.3}$$

We need to find the field near a typical atom, which we shall take as the origin of coordinates. We may then replace the other atoms by their average distribution, which has density $\rho g(r)$, where ρ is the number density of atoms, and g their correlation function.

$$\left(\nabla^2+\frac{\omega^2}{c^2}\right)\phi = \frac{\alpha\rho}{\epsilon_0}(\boldsymbol{C}\cdot\boldsymbol{\nabla})\,g(r)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega t}. \tag{3.4}$$

The correlation function g equals unity at distances exceeding the atomic scale (which we assume small compared to the wavelength) and both g and $\mathrm{d}g/\mathrm{d}r$ tend to zero at $r = 0$. We may in (2.4) replace $g(r)$ by $1-f(r)$, where

$$f(r) = 1-g(r), \tag{3.5}$$

which leaves

$$\left(\nabla^2+\frac{\omega^2}{c^2}\right)\phi = -\frac{\alpha\rho}{\epsilon_0}\boldsymbol{C}\cdot\nabla f(r)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega t}+\frac{\mathrm{i}\alpha\rho}{\epsilon_0}\boldsymbol{C}\cdot\boldsymbol{k}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}-\mathrm{i}\omega t}. \tag{3.6}$$

The second term vanishes because in a transverse field $\boldsymbol{C}\cdot\boldsymbol{k}$ must vanish. (A more careful discussion would be required if there were any macroscopic charges present.)

In the first term we note that $f(r)$ is appreciable only at distances for which $\boldsymbol{k}\cdot\boldsymbol{r}$ is small, so that we may expand the exponential factor. For the same reason the second derivative on the left hand side contains a factor of the order of the inverse square of the atomic dimensions, and ω^2/c^2 is negligible in comparison with that. Writing

$$\phi = -\frac{\alpha\rho}{\epsilon_0}\mathrm{e}^{-\mathrm{i}\omega t}(\boldsymbol{C}\cdot\boldsymbol{\nabla})\,\chi(\boldsymbol{r}), \tag{3.7}$$

we have to solve the equation

$$\nabla^2\chi = f(r)\,(1+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}), \tag{3.8}$$

where terms of second and higher order in $\boldsymbol{k}\cdot\boldsymbol{r}$ have been neglected. χ should be regular at the origin and vanish at infinity. The solution is, taking $\boldsymbol{k}$ in the z direction

$$\chi = -\frac{1}{r}\int_0^r \mathrm{d}r'\,r'^2 f(r')-\int_r^\infty \mathrm{d}r' r' f(r')-\frac{\mathrm{i}kz}{3r^3}\int_0^r \mathrm{d}r' r'^4 f(r')-\frac{\mathrm{i}kz}{3}\int_1^\infty \mathrm{d}r' r' f(r'). \tag{3.9}$$

Remembering that at the origin $f = 1$, and $\mathrm{d}f/\mathrm{d}r = 0$, we find for the leading terms near the origin

$$\chi(r) \simeq \tfrac{1}{6}r^2+\tfrac{1}{10}\mathrm{i}kzr^2-\int_0^\infty (r'+\tfrac{1}{3}\mathrm{i}kr')f(r')\,\mathrm{d}r'. \tag{3.10}$$

Hence at $r = 0$,

$$\frac{\partial^2\chi}{\partial x^2} = \frac{1}{3}\frac{\partial^3\chi}{\partial x^2\,\partial z} = \tfrac{1}{5}\mathrm{i}k. \tag{3.11}$$

Hence from (2.7), taking $\boldsymbol{C}$ in the x direction

$$E_{\mathrm{d}x} = -\left(\frac{\partial\phi}{\partial x}\right)_0 = \frac{\alpha\rho}{3\epsilon_0}C_x\mathrm{e}^{-\mathrm{i}\omega t} = \frac{1}{3}\frac{\alpha\rho}{\epsilon_0}E_{\mathrm{eff.},x} = \tfrac{1}{3}\epsilon_0 P_x. \tag{3.12}$$

This is just the familiar relation leading to the Clausius–Mosotti formula.

For the gradient,

$$\left(\frac{\partial E_x}{\partial z}\right)_{\mathrm{d}} = \left(\frac{\partial E_z}{\partial x}\right)_{\mathrm{d}} = -\frac{\mathrm{i}k\alpha\rho}{5\epsilon_0}C\mathrm{e}^{-\mathrm{i}\omega t} = \frac{1}{5}\frac{1}{\epsilon_0}\frac{\partial P_x}{\partial z}, \tag{3.13}$$

which, by comparison with (2.5), shows that

$$\sigma = \tau = \tfrac{1}{5}. \tag{3.14}$$

This result seems to be in conflict with the equations given, for example, in the book by Landau & Lifshitz (1960). Their equation (56.18) can for our purposes be rewritten by assuming a non-magnetic uniform medium, and by assuming that the Clausius–Mosotti relation is valid, which requires that

$$\rho\frac{\partial\epsilon}{\partial\rho} = \epsilon-\epsilon_0+\frac{1}{3}\frac{(\epsilon-\epsilon_0)^2}{\epsilon_0}. \tag{3.15}$$

Making use of Maxwell's equation, their relation takes the form

$$\boldsymbol{F} = (\epsilon-\epsilon_0)(\boldsymbol{E}\cdot\nabla)\boldsymbol{E}+\frac{1}{3}\frac{(\epsilon-\epsilon_0)^2}{\epsilon_0}(\boldsymbol{E}\cdot\nabla)\boldsymbol{E}-\frac{1}{3}\frac{(\epsilon-\epsilon_0)^2}{\epsilon_0}\boldsymbol{E}\times\dot{\boldsymbol{B}}+(\epsilon-\epsilon_0)\dot{\boldsymbol{E}}\times\boldsymbol{B}. \tag{3.16}$$

Comparison with our equation (2.8) shows that Landau & Lifshitz require $\sigma = \frac{1}{3}$, $\tau = 0$. Their derivation is clearly restricted to not too high frequencies, since they neglect dispersion. One would expect, however, that their argument should apply in the optical region.

For clarification, consider the more general relation obtained by Landau & Lifshitz for isotropic solids. Substituting in their equation (56.17) from their expression (16.4) for the electrostriction terms, (34.2) for the magnetic stress, assuming $\mu = \mu_0$, no macroscopic current or charges, and a uniform medium, one obtains for the force per unit volume

$$\boldsymbol{F} = -(\tfrac{1}{2}a_1+a_2)(\boldsymbol{E}\cdot\nabla)\boldsymbol{E}+(\epsilon-\epsilon_0+a_2)\boldsymbol{E}\times\dot{\boldsymbol{B}}+(\epsilon-\epsilon_0)\dot{\boldsymbol{E}}\times\boldsymbol{B}. \tag{3.17}$$

(Note that ϵ_0 in Landau & Lifshitz, equation (16.4) is our ϵ.) Here a_1 and a_2 are coefficients determining the variation of the dielectric tensor with strain (see their (16.1)):

$$\delta\epsilon_{ik} = a_1 u_{ik}+a_2\delta_{ik}\sum_l u_{ll} \tag{3.18}$$

u_{ik} being the strain tensor.

One relation between a_1 and a_2 follows from the behaviour under a uniform compression. Then $u_{ik} = -\frac{1}{3}\delta_{ik}\delta\rho/\rho$, and therefore

$$\rho\frac{\partial\epsilon}{\partial\rho} = -(\tfrac{1}{3}a_1+a_2). \tag{3.19}$$

From Clausius–Mosotti

$$\rho\frac{\partial\epsilon}{\partial\rho} = (\epsilon-\epsilon_0)\frac{\epsilon+2\epsilon_0}{3\epsilon_0} = \epsilon-\epsilon_0+\frac{(\epsilon-\epsilon_0)^2}{3\epsilon_0}, \tag{3.20}$$

so that (3.17) now becomes

$$\boldsymbol{F} = \left[-\tfrac{1}{6}a_1+(\epsilon-\epsilon_0)+\frac{(\epsilon-\epsilon_0)^2}{3\epsilon_0}\right](\boldsymbol{E}\cdot\nabla)\boldsymbol{E} -\left[\tfrac{1}{3}a_1+\frac{(\epsilon-\epsilon_0)^2}{3\epsilon_0}\right]\boldsymbol{E}\times\dot{\boldsymbol{B}}+(\epsilon-\epsilon_0)\dot{\boldsymbol{E}}\times\boldsymbol{B}, \tag{3.21}$$

which is of the same form as our (2.8), with

$$\frac{(\epsilon-\epsilon_0)^2}{\epsilon_0}\tau = -\tfrac{1}{2}a_1\frac{(\epsilon-\epsilon_0)^2}{\epsilon_0}\sigma = \frac{(\epsilon-\epsilon_0)^2}{3\epsilon_0}+\tfrac{1}{3}a_2. \tag{3.22}$$

A further relation between a_1 and a_2 can be obtained from the remark that σ and τ must be equal. This follows from the fact that, as we have seen, the effective field near an atom is due to the dipoles on atoms at distances much less than the wavelength, and can therefore be described by a scalar potential. In that case, for example,

$$\frac{\partial E_{\mathrm{d}z}}{\partial x} \equiv \frac{\partial E_{\mathrm{d}x}}{\partial z}$$

and the equality of σ and τ follows. This determines a_1 from (3.22), and we then find that $\sigma = \tau = \frac{1}{5}$.

With this additional piece of information, the argument of Landau & Lifshitz for isotropic solids therefore leads to the answer (3.14).

For liquids, Landau & Lifshitz note that a pure shear cannot affect the state of the substance, and therefore that in (3.16) a_1 must vanish. From (3.22) this immediately leads to $\sigma = \frac{1}{3}$, $\tau = 0$. For our present purpose, when we deal with light in or near the visible region, one would not expect to find any difference between a liquid and an amorphous solid, since their configurations at any given time are similar, and the movement of the atoms during a light period is negligible. Our derivation of (3.14) was based on the assumption that the atoms can be regarded as stationary. In that case, the argument of Landau & Lifshitz, which relates the effect of the field on the stress to the effect of strain on the dielectric properties, should be applied to a strain in which the instantaneous atomic positions are scaled linearly. This makes the correlation function anisotropic, and will lead to an anisotropic dielectric tensor.

At lower frequencies, when the motion of the atoms during a cycle is no longer negligible, the results of this paper cease to apply, and for low enough frequency the Landau–Lifshitz equations for liquids will be applicable. This paper will not discuss the precise limits of the frequency range for either regime, nor the nature of the transition between them.

4. Criticism of the Minkowski result

One commonly finds in the literature (see, for example, Panofsky & Phillips 1955, and recently Ginzburg (1973) the following derivation of the Minkowski result: The Maxwell equations,

$$\left.\begin{aligned} -\operatorname{curl} \boldsymbol{E} &= \dot{\boldsymbol{B}}, \\ \operatorname{curl} \boldsymbol{H} &= \dot{\boldsymbol{D}} + \boldsymbol{j}, \end{aligned}\right\} \tag{4.1}$$

are multiplied vectorially by $\boldsymbol{D}$ and $-\boldsymbol{B}$, respectively, and the results added. This yields

$$-\boldsymbol{D} \times \operatorname{curl} \boldsymbol{E} + \operatorname{curl} \boldsymbol{H} \times \boldsymbol{B} = \boldsymbol{D} \times \dot{\boldsymbol{B}} + \dot{\boldsymbol{D}} \times \boldsymbol{B} + \boldsymbol{j} \times \boldsymbol{B}. \tag{4.2}$$

From this by elementary transformations

$$\frac{\partial}{\partial t}(\boldsymbol{D} \times \boldsymbol{B}) + \operatorname{div} \boldsymbol{\tau} + \rho \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} - \tfrac{1}{2}(\nabla \epsilon) E^2 - \tfrac{1}{2}(\nabla \mu) H^2 = 0, \tag{4.3}$$

where $\boldsymbol{\tau}$ is a tensor in three dimensions, with the components

$$\tau_{\alpha\beta} = \tfrac{1}{2}(\boldsymbol{D}\cdot\boldsymbol{E}+\boldsymbol{B}\cdot\boldsymbol{H})\,\delta_{\alpha\beta}-D_\alpha E_\beta-B_\alpha H_\beta; \quad \alpha,\beta = x,y,z. \tag{4.4}$$

In a homogeneous medium the last two terms of (4.3) are absent, and (4.3) then has the form of a conservation law. It is natural to interpret $\boldsymbol{D}\times\boldsymbol{B}$ as the momentum density and $\tau_{\alpha\beta}$ as the transport of the β component of momentum in the α direction. The equation then states that the net change in momentum equals the force on the charges and currents present. Since in the vacuum the equation reduces to the equation for momentum conservation, this interpretation is very plausible.

However, this conservation law holds only in a uniform medium, because of the terms in (4.3) containing the permeability gradients. Momentum conservation must hold even when the medium is not homogeneous, or when interfaces are present. In that case there will be forces on the medium, or on the interfaces, and in order to justify the expression for the momentum density one would have to show that the gradient terms in (4.3) give the correct force, which would have to be found by the method of §2.

There exists, however, another quantity, similar to momentum, which is conserved only in a uniform medium, namely the 'pseudo-momentum' or wave vector. This is always equal to the energy, divided by the phase velocity, and therefore equal to the Minkowski momentum in the refractive medium. The nature of this quantity will be discussed further in §6.

5. Reflexion and transmission

Consider a plane light wave in a dielectric, non-magnetic medium of refractive index n, incident normally on the plane interface with a conducting medium. We are interested in a perfect conductor, but to avoid ambiguities will assume a large but finite conductivity σ and then go to the limit.

Take the interface as $z = 0$. and the electric and magnetic fields in the x and y directions. Then the solution of Maxwell's equations is, to the leading power in the resistivity

$$\left.\begin{aligned} E_x &= I[\mathrm{e}^{\mathrm{i}k_1z-\mathrm{i}\omega t}-\mathrm{e}^{-\mathrm{i}k_1z-\mathrm{i}\omega t}],\\ B_y &= \frac{n}{c}I[\mathrm{e}^{\mathrm{i}k_1z-\mathrm{i}\omega t}+\mathrm{e}^{-\mathrm{i}k_1z-\mathrm{i}\omega t}],\end{aligned}\right\}\quad z<0 \tag{5.1}$$

$$\left.\begin{aligned} E_x &= \frac{(1-\mathrm{i})\,n\omega}{\alpha c}I\,\mathrm{e}^{(\mathrm{i}-1)\,\alpha z-\mathrm{i}\omega t},\\ B_y &= \frac{2n}{c}I\,\mathrm{e}^{(\mathrm{i}-1)\,\alpha z-\mathrm{i}\omega t},\end{aligned}\right\}\quad z>0, \tag{5.2}$$

where $\alpha^2 = \tfrac{1}{2}\mu_0\omega\sigma$, $k_1 = n\omega/c$.

The force on the conductor per unit volume is $\sigma E_x B_y$, and, remembering that we have to use in this bilinear relation the real parts of the fields in (5.2), the integral over z gives for the force per unit area

$$n^2\epsilon_0 I^2. \tag{5.3}$$

Since for the wave (5.1) the energy flux is

$$S = \tfrac{1}{2}\epsilon_0 c n^2 I^2 \tag{5.4}$$

we see that the momentum transmitted to the conductor per unit time and unit area is $2nS/c$. This is just what one would have predicted from Minkowski's result, since from (1.1) the momentum density is n^2S/c^2, and, with the speed of propagation c/n, the momentum flux would be nS/c. An equal and opposite amount is of course taken away by the reflected wave. This prediction of the Minkowski theory is in agreement with the result of Jones & Richards (1954).

However, this is greater than the sum of the actual momenta of the incident and reflected wave, since we have seen in §2 that the total momentum is less than the Minkowski value. The difference must therefore appear in the form of mechanical momentum. This can be verified by considering the forces on the atomic dipoles in the refractive medium due to the field (5.1). In order to remove ambiguity, we consider a wave train of finite duration, so that we may impose the condition that the medium was at rest before the arrival of the light signal. We therefore replace the fields (5.1) and (5.2) by a superposition of waves for a range of frequencies, taking the distribution as Gaussian, for simplicity. We may assume the spread small enough for the variation in refractive index to be negligible. Then, for $z < 0$,

$$\left.\begin{aligned} E_x &= I_0[\exp\{\tfrac{1}{2}\beta(z-c_1t)^2+\mathrm{i}k_1(z-c_1t)\}-\exp\{-\tfrac{1}{2}\beta(z+c_1t)^2-\mathrm{i}k_1(z+c_1t)\}],\\ B_y &= \frac{n}{c}I_0[\exp\{-\tfrac{1}{2}\beta(z-c_1t)^2+\mathrm{i}k_1(z-c_1t)\}+\exp\{-\tfrac{1}{2}\beta(z+c_1t)^2-\mathrm{i}k_1(z+c_1t)\}],\end{aligned}\right\} \tag{5.5}$$

where β is a measure of the spectral width of the wave train, and $c_1 = c/n$.

The energy carried by this wave train per unit area is

$$\mathscr{E} = \frac{1}{2}\sqrt{\left(\frac{\pi}{\beta}\right)}\, n^2\epsilon_0 I_0^2. \tag{5.6}$$

The force on the medium is given by (2.9). We had seen in §2 that for a unidirectional wave this reduces to (2.12), i.e. the time derivative of a quantity which vanishes before and after the passage of the wave. This still applies to the contribution, to the quadratic expression (2.9), of the terms containing only the incident, or only the reflected wave. These terms do not contribute to the momentum remaining in the medium.

We do, however, obtain such a contribution from the cross terms between incident and reflected wave. The contributions of the real parts of these cross terms to (2.9) amount, on the average over a cycle, to

$$\frac{n^2-1}{\mu_0c^2}\mathrm{e}^{-\beta(z^2+c_1^2t^2)}(2\beta zc_1\cos 2k_1z+2\,\omega\sin 2k_1z)[1+\sigma(n^2-1)]. \tag{5.7}$$

The momentum density given to the medium is the time integral from $-\infty$ to ∞, which is

$$\frac{n^2-1}{\mu_0c^2}\sqrt{\left(\frac{\pi}{\beta}\right)}(2\beta z\cos ak_1z+2k_1\sin 2k_1z)\,\mathrm{e}^{-\beta z^2}[1+\sigma(n^2-1)] \tag{5.8}$$

and integrating over all negative z we find the total mechanical momentum per unit area deposited in the medium

$$-\tfrac{1}{2}\epsilon_0(n^2-1)\sqrt{\left(\frac{\pi}{\beta}\right)}\,I_0^2\,\frac{n}{c}[1+\sigma(n^2-1)]. \tag{5.9}$$

By comparison with (5.6) we see that the ratio of residual momentum to incident energy is

$$-\frac{n^2-1}{n}[1+\sigma(n^2-1)]. \tag{5.10}$$

This is just the amount required to restore the momentum balance, which is the difference between the momentum transferred to the reflector, i.e. twice the Minkowski momentum of the incident wave, and the change in the total momentum of the light wave, which is twice the quantity (2.13).

This situation was in essence already discussed by Gordon (1973). For the agreement between the Minkowski result and the experiment of Jones & Richards (1954) it is essential that the (backward) momentum deposited in the medium is carried away and not imparted to the mirror.

Next consider the transmission of a wave from a medium of refractive index n_1 to one with n_2, assuming normal incidence, and choosing the axes and polarizations as in the preceding example. Then, in the first medium

$$\left.\begin{aligned} E_x &= I\exp\{-\tfrac{1}{2}\beta(z-c_1t)^2+\mathrm{i}k_1(z-c_1t)\} \\ &\qquad +R\exp\{-\tfrac{1}{2}\beta(z+c_1t)^2-\mathrm{i}k_1(z+c_1t)\}, \\ B_y &= \frac{1}{c_1}I\exp\{-\beta(z-c_1t)^2+\mathrm{i}k_1(z-c_1t)\} \\ &\qquad -\frac{1}{c_1}R\exp\{-\tfrac{1}{2}\beta(z+c_1t)^2-\mathrm{i}k_1(z+c_1t)\} \end{aligned}\right\}\quad z<0 \tag{5.11}$$

and in the second medium

$$\left.\begin{aligned} E_x &= T\exp\{-\tfrac{1}{2}\beta_2(z-c_2t)^2+\mathrm{i}k_2(z-c_2t)\}, \\ B_y &= \frac{1}{c^2}T\exp\{-\tfrac{1}{2}\beta_2(z-c_2t)^2+\mathrm{i}k_2(z-c_2t)\}, \end{aligned}\right\}\quad z>0 \tag{5.12}$$

where
$$\beta_2=\beta(n_2/n_1)^2,\quad k_1c_1=k_2c_2=\omega. \tag{5.13}$$

Continuity at $z=0$ requires, as usual,

$$R=\frac{n_1-n_2}{n_1+n_2}I,\quad T=\frac{2n}{n_1+n_2}I. \tag{5.14}$$

We may again determine from (2.9) the momentum left in the media after the passage of the light pulse. However, we can restrict ourselves to the first medium, since we saw that momentum is deposited only by virtue of the cross term between incident and reflected wave. In the second medium there is no reflected wave, and hence no cross term.

In the first medium a calculation similar to that of the preceding example gives the remaining momentum per unit volume

$$-\frac{1}{2}\sqrt{\left(\frac{\pi}{\beta}\right)}\,\epsilon_0\, I_0^2 \frac{n_1}{c}\frac{(n_1^2-1)\,(n_1-n_2)}{n_1+n_2}[1+\sigma(n_1^2-1)]\,\mathrm{e}^{-\beta z^2}\,(2\beta\cos 2k_1 z+2k_1\sin k_1 z) \quad (5.15)$$

and the integral over all negative z is

$$G_{\text{res.}} = \frac{\mathscr{E}}{c}\frac{n_1^2-1}{n_1}\frac{n_1-n_2}{n_1+n_2}[1+\sigma(n_1^2-1)], \quad (5.16)$$

where we have again used the expression (5.6) for the incident energy per unit area.

From (2.13) and (5.14) we see that the change of the momentum carried by the light wave is, per unit area,

$$G_{\text{in.}} - G_{\text{refl.}} - G_{\text{trans.}} = \frac{\mathscr{E}}{c}\frac{n_1-n_2}{n_1(n_1+n_2)}\{n_1^2-1-\sigma[n_1^4+2n_1^2n_2^2-2n_1^2-2]\}. \quad (5.17)$$

The difference between this and the momentum deposited in the first medium, (5.16),

$$\Delta G = \frac{\sigma\mathscr{E}}{c}\frac{n_1-n_2}{n_1(n_1+n_2)}(4n_1^2+1-2n_1^2n_2^2) \quad (5.18)$$

is the impulse, per unit area, given to the interface.

We see that the momentum given up by the light wave, (5.17), is transferred to the matter. Immediately after the passage of the light the quantity (5.16) is left in a layer of a thickness comparable to the length of the light pulse; the rest, (5.18), gives a force on the surface dividing the media. In the phenomenological description, this force acts on a mathematical plane, but in reality it will act on a layer a few atoms thick on either side of the interface.

Whether an experiment would respond to one or the other quantity depends of course on the nature of the measurement. The situation is particularly simple if the first medium is the vacuum. so that $n_1 = 1$. In that case (5.16) vanishes.

It is immediately evident that this must be the case, because we had seen that momentum is deposited in the medium after the passage of the light only because of the cross terms between the incident and the reflected wave. In the vacuum there are no atoms on which the wave could exert a force, and in the second medium there is no cross term. In that case of a light pulse incident from vacuum (or in practice from air) on a surface the impulsive force on the surface is directly related to the momentum change of the light.

For $n_1 = 1$ and not too large n_2, (5.18) is negative, i.e. there is a loss of momentum in the direction of the incident wave, and the force on the surface is inward. For n_2 greater than about 2.6, the expression changes sign, and there should be an outward force on the surface. However, such large indices of refraction can be found only when there is strong dispersion, and in that case one should examine the arguments carefully in case there are important corrections due to the dispersion.

It should be stressed, however, that our result assumes a wave of infinite width.

It should be valid provided the width of the pulse is large compared to its length, times the ratio of sound to light velocity. Our result is therefore, not in contradiction with the experiment of Ashkin & Dziedzic (1973), since they were using a narrow light pulse. A discussion of the effect of lateral forces was given by Gordon (1973), who found that the experiment agreed with the theory of the lateral forces.

6. Some remarks about pseudo-momentum

Several authors, including Gordon (1973) and Blount (unpublished memorandum) have pointed out that the Minkowski quantity (1.1) gives the density of pseudo-momentum. Since the concept of pseudo-momentum is much less familiar than actual momentum, a few comments may be helpful.

Momentum conservation is associated with the invariance of the laws of physics under a displacement of the origin of the coordinates. The operator for total momentum, P_x, therefore has in quantum theory the property that, for any operator, A,

$$[P_x, A] = -\mathrm{i}\hbar \frac{\partial A}{\partial x}, \tag{6.1}$$

where the derivative on the right hand side implies an infinitesimal change of all x coordinates. If the Hamiltonian H, is independent of the origin of the coordinates,

$$[P_x, H] = 0, \tag{6.2}$$

which, as usual, implies the constancy in time of P_x. In classical theory the same argument applies, with the commutators replaced by Poisson brackets.

The conservation of pseudo-momentum, or wave vector, is associated with the invariance of the laws against a displacement of all physical parameters from one point of the medium to another. Thus, under the pseudo-momentum operation Ka, where K is the vector of pseudo-momentum, and a a small distance, we do not move the atoms by an amount a, but only replace the displacement, or the dielectric moment, or any other relevant quantity of an atom, by that of an atom a distance a away.

For this transformation to leave the physical laws unchanged, all the properties of the medium must be uniform. Because of the atomic structure of the medium, a cannot be made smaller than the lattice spacing, and this is related to the possibility of Umklapp processes in crystals, which, however, are of no importance for problems which, like the present, involve only long waves.

In liquids, or amorphous solids, a has to be kept rather larger than the atomic spacing, so that it is sufficient to consider averages of the electric and mechanical parameters over regions containing many atoms. If the light wavelength is too short to justify this, the fluctuations in the medium will give rise to dissipative effects.

The classical example is that of a lattice wave (phonon) in an harmonic crystal, which has pseudo-momentum $K = \hbar k$ per phonon, but real momentum zero. The momentum of the crystal is carried by the degree of freedom with $k = 0$, corresponding to a uniform displacement of all atoms, for which there is no restoring force in a free crystal, and the phonon concept fails. This statement is sometimes regarded with surprise in view of the fact that phonons often behave as if they had momentum $\hbar k$. Consider, for example, the generation of a phonon by the scattering of a neutron, a common enough experiment today. In this process momentum is conserved, and the crystal must acquire (or if it is supported, transmit to its support) a momentum equal to that lost by the neutron. In addition, pseudo-momentum is also conserved, because the process may be thought of as taking place far enough from any boundary or other inhomogeneity. Therefore, the pseudo-momentum lost by the neutron equals that acquired by the phonon, i.e. $\hbar k$. The neutron propagates through the solid practically as a free particle, and its momentum and pseudo-momentum are equal. Putting both conservation laws together, we have therefore shown that the crystal receives a momentum equal to the pseudo-momentum of the phonon which is produced. This is carried by the centre-of-mass degree of freedom of the crystal, or, if the neutron was originally localized, in a long-wave acoustic disturbance.

Returning to the problem of light in a refractive medium, the result that the momentum given to a mirror in reflexion equals the change of pseudo-momentum of the light, with the balance made up by an acoustic disturbance, makes one look for a reason why in this case also pseudo-momentum should be conserved.

Gordon (1973) proves a theorem that, in certain circumstances, the right answer can be obtained by assuming conservation of pseudo-momentum. The requirement is that there exists a surface surrounding the object on which the force is to be found, such that outside of that surface one is dealing with a homogeneous medium, and on the surface the force density vanishes.

This, however, would seem to be just the condition for K to satisfy a conservation law of the type (6.2). The operation Ka can be thought of in two steps: first, all excitations (atomic motions, polarization, etc.) are displaced in the space outside the excluded volume. Secondly, the bounding surface is displaced by a. The first step causes no change in the Hamiltonian, because the medium is assumed uniform, and the effect of the second is proportional to the appropriate component of the force density, integrated over the surface.

The argument sketched in the last few paragraphs is, of course, just a reformulation of Gordon's theorem, and would not have been obvious without his paper. It is given here because it fits in well with the presentation of this paper.

7. The case of a solid

In this paper the light wave has been assumed unlimited laterally. Gordon (1973) has discussed the case of a light beam of finite width in a gas or liquid. What happens if a pulse of finite width and finite duration passes through a large refracting solid? In the first place, each atom receives a small forward velocity as the wave front passes, and an equal and opposite acceleration in the tail. In a gas, the net result would be a small displacement of all the atoms over which the pulse has passed.

In a solid, this clearly is not a stationary situation, because shear forces between the displaced and the other atoms will be involved. In order to solve this problem we have to consider the equations of elasticity, with the rapid displacement of the atoms as a source term.

But the structure of these equations is precisely that of the equations for the Cherenkov effect, with the light pulse playing the part of the charged particle, and sound waves appearing in place of Cherenkov electromagnetic radiation. Since the mechanical pulse caused by the light travels with a velocity vastly greater than that of sound, the Cherenkov angle is practically $\frac{1}{2}\pi$.

No quantitative estimate of the intensity of this 'Cherenkov sound' is yet available, and it can be expected to be very small. Nevertheless, the conclusion that a light pulse of finite dimensions in a large uniform solid must continuously emit sound waves, seems unexpected.

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