9.21 From Eq. (9.137), the time-averaged angular momentum density is given by

$$\vec{m} = \frac{1}{2c^2} Re\left[\vec{r}\times(\vec{E}\times\vec{H}^*)\right],$$

whose $\hat{z}$-component is

$$m_z = \hat{z}\cdot\vec{m} = \frac{1}{2c^2} Re\left[\hat{z}\cdot\{\vec{r}\times(\vec{E}\times\vec{H}^*)\}\right] = \frac{1}{2c^2} Re\left[(\hat{z}\times\vec{r})\cdot(\vec{E}\times\vec{H}^*)\right].$$

It is straightforward to verify that $\hat{z}\times\vec{r} = r\hat{\phi}$, and

$$\vec{E}\times\vec{H}^* = \begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ E_r & E_\phi & E_z \\ H_r^* & H_\phi^* & 0 \end{vmatrix} = -\frac{k}{Z_0\beta} E_z E_\phi^* \hat{\phi} + \cdots$$

where we have omitted terms not in the $\hat{\phi}$-direction. Then,

$$m_z = \frac{1}{2c^2} Re\left[(\hat{z}\times\vec{r})\cdot(\vec{E}\times\vec{H}^*)\right] = \frac{1}{2c^2} Re\left[r\cdot\left(-\frac{k}{Z_0\beta}\right) E_z \left(-\frac{m\beta}{\gamma^2}\right)\frac{E_z^*}{r}\right] = \frac{km}{2Z_0c^2\gamma^2} J_m(\gamma r)^2.$$

The time averaged energy density is given by Eq. (9.136).

$$u = \frac{\varepsilon_0}{4}\left(\vec{E}\cdot\vec{E}^* + Z_0^2\vec{H}\cdot\vec{H}^*\right)$$

$$= \frac{\varepsilon_0}{4}\left(|E_z|^2 + \frac{m^2\beta^2}{\gamma^4}\frac{|E_z|^2}{r^2} + \frac{\beta^2}{\gamma^4}\left|\frac{\partial E_z}{\partial r}\right|^2 + \frac{k^2}{\beta^2}|E_\phi|^2 + \frac{k^2}{\beta^2}|E_r|^2\right)$$

$$= \frac{\varepsilon_0}{4}\left(J_m(\gamma r)^2 + \frac{m^2\beta^2}{\gamma^4}\frac{J_m(\gamma r)^2}{r^2} + \frac{\beta^2}{\gamma^4}\left(\frac{dJ_m(\gamma r)}{dr}\right)^2 + \frac{k^2m^2}{\gamma^4}\frac{J_m(\gamma r)^2}{r^2} + \frac{k^2}{\gamma^4}\left(\frac{dJ_m(\gamma r)}{dr}\right)^2\right)$$

The energy per unit length in the z-direction is

$$\langle u\rangle = \int u\,da$$

$$= \frac{\varepsilon_0}{4}\,2\pi\int_0^R r\left(J_m(\gamma r)^2 + \frac{m^2(k^2+\beta^2)}{\gamma^4}\frac{J_m(\gamma r)^2}{r^2} + \frac{k^2+\beta^2}{\gamma^4}\left(\frac{dJ_m(\gamma r)}{dr}\right)^2\right)dr$$

$$= \frac{\pi\varepsilon_0}{4\gamma^2}\int_0^{\gamma R} x\left(J_m(x)^2 + \frac{m^2(k^2+\beta^2)}{\gamma^2}\frac{J_m(x)^2}{x^2} + \frac{k^2+\beta^2}{\gamma^2}\left(\frac{dJ_m(x)}{dx}\right)^2\right)dx$$

This can be simplified by Bessel's equation,

$$\frac{d^2J_m(x)}{dx^2} + \frac{1}{x}\frac{dJ_m(x)}{dx} + \left(1-\frac{m^2}{x^2}\right)J_m(x) = 0, \quad \text{or} \quad \frac{d}{dx}\left[xJ_m'(x)\right] = -x\left(1-\frac{m^2}{x^2}\right)J_m(x)$$

The last term in the integral can be expressed as

$$\int_0^{\gamma R} x\left(\frac{dJ_m(x)}{dx}\right)^2 dx = \int_0^{\gamma R} x\frac{dJ_m(x)}{dx}\,d(J_m(x)) = xJ_m(x)\frac{dJ_m(x)}{dx}\Big|_0^{\gamma R} - \int_0^{\gamma R} J_m(x)\frac{d}{dx}\left(xJ_m'(x)\right)dx$$

$$= \int_0^{\gamma R} x\left(1-\frac{m^2}{x^2}\right) J_m(x)^2 dx$$

Then,
$$\langle u\rangle = \frac{\pi\varepsilon_0}{2\gamma^2}\int_0^{\gamma R} x\left(J_m(x)^2 + \frac{m^2(k^2+\beta^2)}{\gamma^2}\frac{J_m(x)^2}{x^2} + \frac{k^2+\beta^2}{\gamma^2}\left(1-\frac{m^2}{x^2}\right)J_m(x)^2\right)dx$$

$$= \frac{\pi\varepsilon_0 k^2}{\gamma^4}\int_0^{\gamma R} x J_m(x)^2 dx$$

On the other hand,
$$\langle m_z\rangle = \int m_z\, da = \frac{\pi k m}{Z_0 c^2\gamma^2}\int_0^R r J_m(\gamma r)^2 dr = \frac{\pi k m}{Z_0 c^2\gamma^4}\int_0^{\gamma R} x J_m(x)^2 dx.$$

Thus,
$$\frac{\langle m_z\rangle}{\langle u\rangle} = \frac{\pi k m/Z_0 c^2}{\pi\varepsilon_0 k^2} = \frac{m}{Z_0\varepsilon_0 c^2 k}$$

Notice that $Z_0\varepsilon_0 = \sqrt{\mu_0\varepsilon_0} = 1/c$, the final result becomes

$$\frac{\langle m_z\rangle}{\langle u\rangle} = \frac{m}{ck} = \frac{m}{\omega}$$

This is reasonable, as the EM field has an angular momentum of $m\hbar$ and energy of $\hbar\omega$.