9.2 The charge density can be expressed in the spherical coordinates as

$$\rho(\vec{x}) = \frac{q}{r^2}\delta(r-R)F(\cos\theta)\left[\delta(\phi-\omega t)-\delta(\phi-\tfrac{\pi}{2}-\omega t)+\delta(\phi-\pi-\omega t)-\delta(\phi-\tfrac{3\pi}{2}-\omega t)\right]$$

where $R=a/\sqrt{2}$. It can be easily shown that there's no electric dipole and magnetic dipole for this charge distribution. Therefore, we need to consider electric quadrupole to the lowest order contribution to the radiation. The quadrupole is

$$Q_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2)\rho(\vec{x})\,d^3x.$$

If either α or β is the z component, the quadrupole moment will vanish, due to the $F(\cos\theta)$ term.

Consider the diagonal moments.

$$Q_{xx} = \int (3x^2-r^2)\rho(\vec{x})\,d^3x$$

$$= \int_0^{+\infty} r^2 dr \int_{-1}^{1} d(\cos\theta)\int_0^{2\pi} d\phi\,(3\sin^2\theta\cos^2\phi-1)\,q\,\delta(r-R)\delta(\cos\theta)$$

$$\times\left[\delta(\phi-\omega t)-\delta(\phi-\tfrac{\pi}{2}-\omega t)+\delta(\phi-\pi-\omega t)-\delta(\phi-\tfrac{3\pi}{2}-\omega t)\right]$$

$$= qR^2\cdot 3\left(\cos^2(\omega t)-\cos^2(\tfrac{\pi}{2}+\omega t)+\cos^2(\pi+\omega t)-\cos^2(\tfrac{3\pi}{2}+\omega t)\right)$$

$$= 6qR^2\cos(2\omega t) = \mathrm{Re}\left[6qR^2e^{-i2\omega t}\right]$$

Similarly, $Q_{yy} = -Q_{xx} = -\mathrm{Re}\left[6qR^2e^{-i2\omega t}\right]$.

For the off-diagonal component.

$$Q_{xy} = \int 3xy\,\rho(\vec{x})\,d^3x$$

$$= qR^2\cdot 3\left(\cos(\omega t)\sin(\omega t)-\cos(\tfrac{\pi}{2}+\omega t)\sin(\tfrac{\pi}{2}+\omega t)+\cos(\pi+\omega t)\sin(\pi+\omega t)-\cos(\tfrac{3\pi}{2}+\omega t)\sin(\tfrac{3\pi}{2}+\omega t)\right)$$

$$= 6qR^2\sin(2\omega t) = \mathrm{Re}\left[-6iqR^2e^{-i2\omega t}\right]$$

and $Q_{yx}=Q_{xy}$. From the above result, we can see that the radiation frequency is 2ω.

Drop the time dependence, we have $\overleftrightarrow{Q} = 6qR^2\begin{pmatrix}1 & -i & 0\\ -i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$. and

$$\vec{Q}(\vec{n}) = \overleftrightarrow{Q}\cdot\vec{n} = 6qR^2\begin{pmatrix}1 & -i & 0\\ -i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix} = 6qR^2\begin{pmatrix}\sin\theta\, e^{-i\phi}\\ -i\sin\theta\, e^{i\phi}\\ 0\end{pmatrix}$$

The total power radiated is then

$$P = \frac{c^2 Z_0 k^6}{1440\pi} \sum_{\alpha\beta} |Q_{\alpha\beta}|^2 = \frac{c^2 Z_0 k^6}{1440\pi} 144 q^2 R^4 = \frac{c^2 Z_0}{10\pi} \frac{64\omega^6}{c^4} \cdot \frac{q^2 a^4}{4}$$

$$= \frac{8 Z_0 q_e^2 a^4 \omega^6}{5\pi c^4}$$