11.23 (a) The energy-momentum 4-vector in the laboratory frame is $(E_{LAB} + m_2, \vec{p}_{LAB})$ and $(W, 0)$ in the center-of-mass frame. Since the norm of this 4-vector is a Lorentz invariant, we must have

$$W^2 = (E_{LAB} + m_2)^2 - |\vec{p}_{LAB}|^2 = E_{LAB}^2 - |\vec{p}_{LAB}|^2 + m_2^2 + 2m_2 E_{LAB} = m_1^2 + m_2^2 + 2m_2 E_{LAB},$$

where we have used the relation $E_{LAB}^2 - |\vec{p}_{LAB}|^2 = m_1^2$.

(b) Since the total 3-momentum in the cm frame is 0, the velocity of the cm frame is the laboratory frame must be in the total 3-momentum direction, i.e., the $\vec{p}_{LAB}$ direction. Then, using the Lorentz transformation, we have $0 = \gamma_{cm}(\vec{p}_{LAB} - \vec{\beta}_{cm}(E_{LAB} + m_2))$, or

$$\vec{\beta}_{cm} = \frac{\vec{p}_{LAB}}{m_2 + E_{LAB}}, \quad \gamma_{cm} = (1 - \beta_{cm}^2)^{-1/2} = \left(\frac{(m_2 + E_{LAB})^2 - |\vec{p}_{LAB}|^2}{(m_2 + E_{LAB})^2}\right)^{-1/2} = \frac{m_2 + E_{LAB}}{W}.$$

Then, particle 1's 3-momentum in the cm frame is

$$\vec{p}\,' = \gamma_{cm}(\vec{p}_{LAB} - \vec{\beta}_{cm} E_{LAB}) = \frac{m_2 + E_{LAB}}{W}\left(1 - \frac{E_{LAB}}{m_2 + E_{LAB}}\right)\vec{p}_{LAB} = \frac{m_2}{W}\vec{p}_{LAB}$$

(c) In the non-relativistic limit, $E_{LAB} = m_1 + \frac{p_{LAB}^2}{2m_1}$, where $p_{LAB} \ll m_1$. Then,

$$W^2 = m_1^2 + m_2^2 + 2m_2\left(m_1 + \frac{p_{LAB}^2}{2m_1}\right) = (m_1 + m_2)^2 + \frac{m_2}{m_1} p_{LAB}^2, \text{ and}$$

$$W = \left[(m_1 + m_2)^2 + \frac{m_2}{m_1} p_{LAB}^2\right]^{1/2} = (m_1 + m_2)\left[1 + \frac{m_2}{(m_1 + m_2)^2}\frac{p_{LAB}^2}{m_1}\right]^{1/2} = m_1 + m_2 + \frac{m_2}{m_1 + m_2}\frac{p_{LAB}^2}{2m_1}$$

Also,

$$\vec{\beta}_{cm} = \frac{\vec{p}_{LAB}}{m_2 + E_{LAB}} = \frac{\vec{p}_{LAB}}{m_1 + m_2}, \quad \text{and} \quad \vec{p}\,' = \frac{m_2}{W}\vec{p}_{LAB} = \frac{\vec{p}_{LAB}}{m_1 + m_2}$$