9.9. (a) Using the result from Problem 9.6, the angular distribution of the radiation is

$$\frac{dP}{d\Omega} = r^2 \vec{n}\cdot(\vec{E}\times\vec{H}) = -\frac{1}{16\pi^2\varepsilon_0 c^3}\,\vec{n}\cdot\left\{\left[\vec{n}\times\left(\vec{n}\times\frac{\partial^2\vec{p}_{ret}}{\partial t^2}\right)\right]\times\left(\vec{n}\times\frac{\partial^2\vec{p}_{ret}}{\partial t^2}\right)\right\}$$

$$= \frac{1}{16\pi^2\varepsilon_0 c^3}\left|\vec{n}\times\left(\vec{n}\times\frac{\partial^2\vec{p}_{ret}}{\partial t^2}\right)\right|^2 = \frac{1}{16\pi^2\varepsilon_0 c^3}\left(\frac{\partial^2\vec{p}_{ret}}{\partial t^2}\right)^2\sin^2\theta.$$

Perform the angular integration,

$$P = \int\frac{dP}{d\Omega}\,d\Omega = \frac{1}{16\pi^2\varepsilon_0 c^3}\left(\frac{\partial^2\vec{p}_{ret}}{\partial t^2}\right)^2\frac{8\pi}{3} = \frac{1}{6\pi\varepsilon_0 c^3}\left(\frac{\partial^2\vec{p}_{ret}}{\partial t^2}\right)^2$$

For the angular momentum,

$$\frac{d\vec{L}}{\partial t} = -\int\varepsilon_0\left(\frac{1}{4\pi\varepsilon_0 c r^2}\,2\vec{n}\cdot\frac{\partial\vec{p}_{ret}}{\partial t}\right)\left\{\frac{1}{4\pi\varepsilon_0 c^2 r}\left[\vec{n}\times\left(\vec{n}\times\frac{\partial^2\vec{p}_{ret}}{\partial t^2}\right)\right]\times\vec{n}\right\}r^3\,d\Omega$$

$$= -\frac{1}{8\pi^2\varepsilon_0 c^3}\int\left(\vec{n}\cdot\frac{\partial\vec{p}_{ret}}{\partial t}\right)\left(\vec{n}\times\frac{\partial^2\vec{p}_{ret}}{\partial t^2}\right)d\Omega = -\frac{1}{8\pi^2\varepsilon_0 c^3}\left(\frac{\partial\vec{p}_{ret}}{\partial t}\times\frac{\partial^2\vec{p}_{ret}}{\partial t^2}\right)\frac{4\pi}{3}$$

$$= -\frac{1}{6\pi\varepsilon_0 c^3}\left(\frac{\partial\vec{p}_{ret}}{\partial t}\times\frac{\partial^2\vec{p}_{ret}}{\partial t^2}\right)$$

Again, there is a minus sign difference.

(b) For the charged particle, $\vec{p}_{ret} = e\vec{r}$. Using Newton's second law, $m\frac{d^2\vec{r}}{dt^2} = -\nabla V$. Then

$$P(t) = \frac{1}{6\pi\varepsilon_0 c^3}\left(\frac{\partial^2\vec{p}_{ret}}{\partial t^2}\right)^2 = \frac{e^2}{6\pi\varepsilon_0 m^2 c^3}\left(\frac{dV}{dr}\right)^2 = \frac{\tau}{m}\left(\frac{dV}{dr}\right)^2,$$

where $\tau = e^2/6\pi\varepsilon_0 m c^3$. Here, we have used the fact that $V(r)$ is a central field, and $(\nabla V)^2 = (dV/dr)^2$. Also, $\vec{L} = m\vec{r}\times\frac{d\vec{r}}{dt} = \frac{m}{e}\vec{r}\times\frac{d\vec{p}_{ret}}{dt}$, then $\frac{\partial\vec{p}_{ret}}{\partial t} = \frac{e}{mr^2}\vec{L}\times\vec{r}$.

and

$$\frac{\partial\vec{p}_{ret}}{\partial t}\times\frac{\partial^2\vec{p}_{ret}}{\partial t^2} = \frac{e}{mr^2}(\vec{L}\times\vec{r})\times\frac{1}{m}(-\nabla V) = \frac{e^2}{m^2r^2}\left[\vec{L}(\nabla V\cdot\vec{r}) - \vec{r}(\nabla V\cdot\vec{L})\right]$$

$$= \frac{e^2}{m^2r^2}\left(r\frac{dV}{dr}\right)\vec{L} = \frac{e^2}{m^2}\left(\frac{1}{r}\frac{dV}{dr}\right)\vec{L}.$$

Therefore

$$\frac{d\vec{L}}{dt} = \frac{e^2}{6\pi\varepsilon_0 m^2 c^3}\left(\frac{1}{r}\frac{dV}{dr}\right)\vec{L} = \frac{\tau}{m}\left(\frac{1}{r}\frac{dV}{dr}\right)\vec{L}.$$

(c) The ration is given by $\frac{\tau}{m}\left(\frac{1}{r}\frac{dv}{dr}\right) = \frac{e^2}{6\pi\varepsilon_0 m^2 c^3}\cdot\frac{e^2}{4\pi\varepsilon_0 r^3}$, for $V(r) = -e^2/4\pi\varepsilon_0 r$.

For hydrogen atom, $r \sim a_0$, which leads to

$$\frac{\tau}{m}\left(\frac{1}{r}\frac{dv}{dr}\right) \sim \frac{e^2}{6\pi\varepsilon_0 m^2 c^3}\cdot\frac{e^2}{4\pi\varepsilon_0}\cdot\frac{1}{a_0^3} = \frac{e^4}{24\pi^2\varepsilon_0^2 m^2 c^3}\cdot\frac{m^3 e^6}{64\pi^3\varepsilon_0^3\hbar^6} = \frac{m e^{10}}{1536\pi^5\varepsilon_0^5 c^3\hbar^6}$$

Since $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$, we have

$$\frac{\tau}{m}\left(\frac{1}{r}\frac{dv}{dr}\right) \sim \alpha^4 c\cdot\frac{me^2}{6\pi\varepsilon_0\hbar^2} \sim \alpha^4 c/a_0, \quad \text{where } a_0 = 4\pi\varepsilon_0\hbar^2/me^2.$$

(d) Replace the real quantity of $\vec{p}_{ret}$ with time dependence $\vec{p}e^{-i\omega t}$ and product of two real quantities with $\vec{p}$ and $\vec{p}^*$, we will get the desired result.