9.12 The total volume of the sphere is

$$V(\beta) = \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \int_0^{R(\theta)} r^2 dr = \frac{2\pi R_0^3}{3}\int_{-1}^{1} [1+\beta P_2(\cos\theta)]^3 d(\cos\theta) = \frac{4\pi R_0^3}{3}\left(1+\frac{3\beta^2}{5}\right) = V_0\left(1+\frac{3\beta^2}{5}\right)$$

Then, the average charge density is

$$\rho(\beta) = \frac{Q}{V(\beta)} = \frac{Q}{V_0}\left(1+\frac{3\beta^2}{5}\right)^{-1} = \rho_0\left(1-\frac{3\beta^2}{5}\right) \simeq \rho_0,$$

to linear order in β, where $\rho_0 = Q/\left(\frac{4\pi R_0^3}{3}\right)$. Therefore we can calculate the multipole moments as if the charge density stays the same, while the shape of the sphere oscillates.

For the dipole moment,

$$\vec{p} = \rho_0 \int \vec{r}\, d^3x = \rho_0 \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \int_0^{R(\theta)} r^3 (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)\, dr.$$

The p_x and p_y components are clearly zero, we then only need to evaluate the z-component,

$$p_z = 2\pi\rho_0 \int_{-1}^{1} d(\cos\theta)\cdot \cos\theta\, \frac{1}{4}(1+\beta P_2(\cos\theta))^4 = 0,$$

i.e., no dipole moment to the lowest approximation.

For the quadrupole moments,

$$Q_{\alpha\beta} = \rho_0 \int (3x_\alpha x_\beta - r^2\delta_{\alpha\beta})\, d^3x.$$

It is straightforward to show that off-diagonal components are all zero. Thus, we only need to consider diagonal ones,

$$\begin{Bmatrix} Q_{xx} \\ Q_{yy} \\ Q_{zz} \end{Bmatrix} = \rho_0 \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \int_0^{R(\theta)} \begin{Bmatrix} 3\sin^2\theta\cos^2\phi - 1 \\ 3\sin^2\theta\sin^2\phi - 1 \\ 3\cos^2\theta - 1 \end{Bmatrix} r^4 dr$$

$$= 2\pi\rho_0 \int_{-1}^{1} d(\cos\theta) \int_0^{R(\theta)} \begin{Bmatrix} \frac{1}{2} - \frac{3}{2}\cos^2\theta \\ \frac{1}{2} - \frac{3}{2}\cos^2\theta \\ 3\cos^2\theta - 1 \end{Bmatrix} r^4 dr$$

$$= \frac{2\pi\rho_0}{5} R_0^5 \int_{-1}^{1} d(\cos\theta) \begin{Bmatrix} \frac{1}{2} - \frac{3}{2}\cos^2\theta \\ \frac{1}{2} - \frac{3}{2}\cos^2\theta \\ 3\cos^2\theta - 1 \end{Bmatrix} [1+\beta P_2(\cos\theta)]^5$$

$$= \frac{2\pi\rho_0 R_0^5}{5} \cdot \begin{Bmatrix} -2\beta \\ -2\beta \\ 4\beta \end{Bmatrix} = \frac{3}{5} Q R_0^2 \begin{Bmatrix} -\beta \\ -\beta \\ 2\beta \end{Bmatrix}$$

Using Eq (9.51), with $Q_0 = \frac{6QR_0^2}{5}\beta_0$, for the quadrupole radiation,

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{512\pi^2} \cdot \frac{36Q^2R_0^4}{25}\beta_0^2 \sin^2\theta\cos^2\theta = \frac{9c^2 Z_0 k^6 Q^2 R_0^4}{3200\pi^2}\beta_0^2 \sin^2\theta\cos^2\theta$$

and $$P = \frac{c^2 Z_0 k^6}{960\pi} \cdot \frac{36Q^2R_0^4}{25}\beta_0^2 = \frac{3c^2 Z_0 k^6 Q^2 R_0^4}{2000\pi}\beta_0^2$$