5.34 (a) From 5.10 (b), the vector potential from a loop in the x-y plane with current I is

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{2}\int_0^{+\infty} e^{-k|z|} J_1(ka) J_1(k\rho)\, dk$$

The other loop that is a distance R above the first loop has a current I in the same direction,

$$W_{12} = \int \vec{J}_1 \cdot \vec{A}_2\, d^3x = I \cdot 2\pi a \cdot \frac{\mu_0 I a}{2}\int_0^{+\infty} e^{-kR} J_1^2(ka)\, dk$$

$$= \mu_0 \pi a^2 I^2 \int_0^{+\infty} e^{-kR} J_1^2(ka)\, dk, \qquad (\vec{J}_1 = I\delta(\rho - a)\delta(z-R))$$

Then $$M_{12} = \frac{W_{12}}{I^2} = \mu_0 \pi a^2 \int_0^{+\infty} e^{-kR} J_1^2(ka)\, dk.$$

(b) Using the series expansion of $J_1(x)$, and keeping only term up to x^5,

$$J_1(x) = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384}$$

$$\int_0^{+\infty} e^{-kR}\left(\frac{ka}{2} - \frac{k^3a^3}{16} + \frac{k^5a^5}{384}\right)^2 dk = \int_0^{+\infty} e^{-kR}\left(\frac{k^2a^2}{4} - \frac{k^4a^4}{16} + \left(\frac{1}{384} + \frac{1}{256}\right)k^6a^6\right) dk$$

$$= \frac{a^2}{4}\cdot\frac{2}{R^3} - \frac{a^4}{16}\frac{24}{R^5} + \frac{5a^6}{768}\cdot\frac{720}{R^7} = \frac{1}{2a}\left(\left(\frac{a}{R}\right)^3 - 3\left(\frac{a}{R}\right)^5 + \frac{75}{8}\left(\frac{a}{R}\right)^7\right)$$

Thus. $$M_{12} = \frac{\mu_0 \pi a}{2}\left[\left(\frac{a}{R}\right)^3 - 3\left(\frac{a}{R}\right)^5 + \frac{75}{8}\left(\frac{a}{R}\right)^7\right]$$

(c) Since $\nabla_R^2 M_{12}(R) = 0$, the general solution must be

$$M_{12}(R,\theta) = \frac{\mu_0 \pi a}{2}\left[\left(\frac{a}{R}\right)^3 P_2(\cos\theta) - 3\left(\frac{a}{R}\right)^5 P_4(\cos\theta) + \frac{75}{8}\left(\frac{a}{R}\right)^7 P_6(\cos\theta)\right]$$

For the Legendre polynomial,

$$P_2(0) = -\frac{1}{2}, \quad P_4(0) = \frac{3}{8}, \quad P_6(0) = -\frac{5}{16},$$

then $$M_{12}\left(R, \frac{\pi}{2}\right) = -\frac{\mu_0 \pi a}{4}\left[\left(\frac{a}{R}\right)^3 + \frac{9}{4}\left(\frac{a}{R}\right)^5 + \frac{375}{64}\left(\frac{a}{R}\right)^7\right]$$

(d) The force is $\vec{F} = \nabla_R W_{12}$. To the leading term,

$$\vec{F} = -\frac{3\mu_0 \pi a^4}{2R^4}, \quad \text{co-axial case}$$

$$\vec{F} = \frac{3\mu_0 \pi a^4}{4R^4}. \quad \text{co-planar case}$$