11.1 (a) First, by the homogenity of the space-time, the transformation must be linear. Otherwise, the distance measured in different frames will be different, even if any two frames are related by a simple shift. Consider two points $(0, x_1, 0, 0)$ and $(0, x_2, 0, 0)$. In another frame that is just distance d away, the coordinates are $(0, x_1', 0, 0)$ and $(0, x_2', 0, 0)$. If the transformation is $x_1' = f(x_1, y_1, z_1, t_1)$ and $x_2' = f(x_2, y_2, z_2, t_2)$ is non-linear, the distance will no longer be $|x_1 - x_2|$ any more, but rather depends on where x_1 and x_2 are located.

Now, consider two frames F and F' with F' moving with velocity v in the x-direction, relative to F, and at $t=0$, the two origines coincide. By the isotropy of space-time, the transformation in the x- and t-directions should not depend on y and z. And, by the same logic, transformation in the y- and z-directions should depend on themselves. Therefore, we should have

$$x' = f(v)(x - vt), \quad y' = \lambda(v) y, \quad z' = \lambda(v) z, \quad t' = g(v)t - h(v)x. \qquad (1)$$

Here, the form of x' is chosen so that the origin of F' is always moving with velocity v relative to F. Isotropy means if we flip the sign of x and the direction of v simultaneously, we should obtain the same transformation,

$$-x' = f(-v)(-x + vt), \quad y' = \lambda(-v) y, \quad z' = \lambda(-v) z, \quad t' = g(-v)t + h(-v)x. \qquad (2)$$

Comparing (1) and (2), we have

$$f(-v) = f(v), \quad \lambda(-v) = \lambda(v), \quad g(v) = g(-v), \quad h(v) = -h(-v).$$

Therefore, f, λ, g are even functions of v while h is odd. Then, we can write this dependence as $x' = f(v^2)(x - vt)$, $y' = \lambda(v^2) y$, $z' = \lambda(v^2) z$, $t' = g(v^2)t - v h(v^2) x$.

If we consider the inverse transform of y and z, we will get

$$y = \lambda((-v)^2) y' = \lambda(v^2) y' = (\lambda(v^2))^2 y, \quad z = (\lambda(v^2))^2 z,$$

and $\lambda(v^2) = \pm 1$. But, $\lambda(v^2) = -1$ is meaningless. Therefore, $\lambda(v^2) \equiv 1$, and the transformation now reads $x' = f(v^2)(x - vt)$, $t' = g(v^2)t - v h(v^2) t$, $y' = y$, $z' = z$. $\qquad (3)$

(b) Combine (1) and (3), we have

$$x = f(v^2)\left[x' + vt'\right] = f(v^2)\left[f(v^2)(x - vt) + v\left(g(v^2)t - vh(v^2)x\right)\right]$$

$$= f(v^2)\left[\left(f(v^2) - v^2h(v^2)\right)x - v\left(f(v^2) - g(v^2)\right)t\right],$$

Which leads to $f(v^2) \equiv g(v^2)$ and $f(v^2)^2 - v^2 f(v^2)h(v^2) = 1$

Also, $t = g(v^2)t' + vh(v^2)x' = g(v^2)\left[g(v^2)t - vh(v^2)x\right] + vh(v^2)f(v^2)(x - vt)$

$$= \left(g(v^2)^2 - v^2 f(v^2)h(v^2)\right)t - v\left(g(v^2)h(v^2) - f(v^2)h(v^2)\right)x.$$

Which leads to the same condition.

(c) Consider two successive transformations, we should have

$x_3 = f(v_2^2)(x_2 - v_2t_2)$, $x_2 = f(v_1^2)(x_1 - v_1t_1)$

$t_3 = g(v_2^2)t_2 - v_2h(v_2^2)x_2$, $t_2 = g(v_1^2)t_1 - v_1h(v_1^2)x_1$

Then, $x_3 = f(v_2^2)\left[f(v_1^2)(x_1 - v_1t_1) - v_2\left(g(v_1^2)t_1 - v_1h(v_1^2)x_1\right)\right]$

$$= f(v_2^2)\left[\left(f(v_1^2) + v_1v_2h(v_1^2)\right)x_1 - (v_1 + v_2)f(v_1^2)t_1\right].$$

$$= f(v_2^2)\left(f(v_1^2) + v_1v_2h(v_1^2)\right)\left[x_1 - \frac{v_1 + v_2}{1 + v_1v_2h(v_1^2)/f(v_1^2)}t_1\right].$$

Therefore, we must have

$$f(v_3^2) = f(v_2^2)\left(f(v_1^2) + v_1v_2h(v_1^2)\right), \quad v_3 = \frac{v_1 + v_2}{1 + v_1v_2\left(h(v_1^2)/f(v_1^2)\right)}.$$

By the universal speed limit postulate, if either v_1 or v_2 is c, v_3 should also be c,

Which means $$c = \frac{v + c}{1 + vc\left(h(v^2)/f(v^2)\right)}, \quad \text{or } h = f/c^2.$$