3.21 Solution: (a) From Problem 1.18, the capacitance of the disc can be expressed as

$$\frac{4\pi\varepsilon_0}{C} = \frac{\displaystyle\int_S da \int_S da' \sigma(\mathbf{x}) G(\mathbf{x}, \mathbf{x}') \sigma(\mathbf{x}')}{\left[\displaystyle\int_S \sigma(\mathbf{x}) da\right]^2},$$

where the surface integral is over the entire disc. Using the Green function from Problem 3.17 (b), the numerator becomes

$$\begin{aligned} & 2 \sum_{m=-\infty}^{\infty} \int_0^\infty dk \int_S da \int_S da' \ \sigma(\mathbf{x})\sigma(\mathbf{x}') e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<)\sinh[k(L-z_>)]}{\sinh(kL)} \\ = \ & 2 \sum_{m=-\infty}^{\infty} \int_0^\infty dk \int_0^{2\pi} d\phi \int_0^R d\rho \int_0^{2\pi} d\phi' \int_0^R d\rho' \ \sigma(\rho)\sigma(\rho') e^{im(\phi-\phi')} \cdot \rho J_m(k\rho) \cdot \rho' J_m(k\rho') \\ & \times \frac{\sinh(kd)\sinh[k(L-d)]}{\sinh(kL)}. \end{aligned}$$

Integral over ϕ or ϕ' will leave only the $m = 0$ term. Also, for $d \ll L$,

$$\frac{\sinh(kd)\sinh[k(L-d)]}{\sinh(kL)} = \frac{1}{2} \frac{(e^{kd} - e^{-kd})(e^{k(L-d)} - e^{-k(L-d)})}{e^{kL} - e^{-kL}} = \frac{1}{2}\left(1 - e^{-2kd}\right).$$

Therefore, the numerator can be written as

$$4\pi^2 \cdot \int_0^\infty dk \ \left(1 - e^{-2kd}\right) \left[\int_0^R \rho\sigma(\rho) J_0(k\rho) d\rho\right]^2.$$

In the meantime, the denominator is

$$4\pi^2 \cdot \left[\int_0^R \rho\sigma(\rho) d\rho\right]^2$$

Thus, the capacitance is

$$\frac{4\pi\varepsilon_0}{C} = \int_0^\infty dk \ \left(1 - e^{-2kd}\right) \frac{\left[\displaystyle\int_0^R \rho\sigma(\rho) J_0(k\rho) d\rho\right]^2}{\left[\displaystyle\int_0^R \rho\sigma(\rho) d\rho\right]^2}.$$

(b) For constant charge density, $\sigma(\rho) \equiv \sigma$, the integral over ρ in the numerator is

$$\int_0^R \rho\sigma(\rho) J_0(k\rho) d\rho = \sigma \int_0^R \rho J_0(k\rho) d\rho = \frac{\sigma}{k^2} \int_0^{kR} \lambda J_0(\lambda) d\lambda = \frac{\sigma}{k^2} \cdot kR J_1(kR) = \frac{\sigma R}{k} J_1(kR),$$

while the integral in the denominator is simply

$$\int_0^R \rho\sigma(\rho) d\rho = \sigma \int_0^R \rho d\rho = \frac{\sigma R^2}{2}.$$

In this case, the capacitance is

$$\frac{4\pi\varepsilon_0}{C} = 4\int_0^\infty \left(1 - e^{-2kd}\right) \frac{J_1(kR)^2}{(kR)^2} dk = \frac{4}{R}\int_0^\infty \left(1 - e^{-2\lambda d/R}\right) \frac{J_1(\lambda)^2}{\lambda^2} d\lambda.$$

For $d \ll R$, expand the integral to the first order in d, we have

$$\frac{4\pi\varepsilon_0}{C} = \frac{8d}{R^2}\int_0^\infty \frac{J_1(\lambda)^2}{\lambda} d\lambda = \frac{4d}{R^2},$$

where we have used the result

$$\int_0^\infty \frac{J_1(\lambda)^2}{\lambda} d\lambda = \frac{1}{2},$$

and the capacitance is

$$C = \varepsilon_0 \frac{\pi R^2}{d}.$$

This is just the capacitance for the capacitor with infinite parallel plates, as when $d \ll R$, the disc can be viewed as infinitely large.

For $d \gg R$, we only need to retain the constant term,

$$\frac{4\pi\varepsilon_0}{C} = \frac{4}{R}\int_0^\infty \frac{J_1(\lambda)^2}{\lambda^2} d\lambda = \frac{16}{3\pi R},$$

where we have used the result

$$\int_0^\infty \frac{J_1(\lambda)^2}{\lambda^2} d\lambda = \frac{4}{3\pi}.$$

The exact capacitance, as shown in Problem 3.3 (c), is $4\pi\varepsilon_0/C_0 = \pi/2R$, and comparing our result here, we have

$$\frac{C}{C_0} = \frac{3\pi^2}{32} = 0.9252.$$

Clearly, the charge density assumption is not exact, which gives a lower capacitance, as discussed in Problem 1.18 (b).

(c) Consider the following trial density,

$$\sigma(\rho) = \sigma_0 + \frac{\sigma_1}{\sqrt{R^2 - \rho^2}}.$$

For the denominator, the integration is easy,

$$\int_0^R \rho\sigma(\rho) d\rho = \int_0^R \left(\sigma_0\rho + \frac{\sigma_1\rho}{\sqrt{R^2 - \rho^2}}\right) d\rho = \frac{\sigma_0 R^2}{2} + \sigma_1 R,$$

while for the numerator, the constant terms gives, as shown in the last part,

$$\frac{\sigma_0 R}{k} J_1(kR).$$

For the exact density part,

$$\sigma_1 \int_0^R \frac{\rho}{\sqrt{R^2 - \rho^2}} J_0(k\rho) d\rho$$

$$
\begin{aligned}
&= \sigma_1 R \int_0^1 \frac{x}{\sqrt{1-x^2}} J_0(kRx)dx \\
&= \sigma_1 R \cdot \frac{\sin(kR)}{kR} \\
&= \sigma_1 \frac{\sin(kR)}{k},
\end{aligned}
$$

where we have used (see, *e.g.*, Gradshteyn and Ryzhik, 7^{th} ed., p. 675, formula 6.554.2)

$$\int_0^1 \frac{xJ_0(xy)}{\sqrt{1-x^2}}dx = \frac{\sin y}{y}.$$

Therefore,

$$\frac{4\pi\varepsilon_0}{C} = \int_0^\infty \left(1-e^{-2kd}\right) \left(\frac{\sigma_0 \frac{J_1(kR)}{k} + \sigma_1 \frac{\sin(kR)}{kR}}{\frac{\sigma_0}{2} R + \sigma_1} \right)^2 dk.$$

Calculation after this point can get quite cumbersome, as we need to evaluate the integrals in order to perform the variation to determine the coefficients of σ_0 and σ_1 to maximize the capacitance. Rather, with the prior knowledge of the form of the exact density, I will just simply calculate the capacitance with it. To this end, we can set $\sigma_0 = 0$ in the above expression,

$$\frac{4\pi\varepsilon_0}{C} = \int_0^\infty \left(1-e^{-2kd}\right) \left(\frac{\sin(kR)}{kR}\right)^2 dk = \frac{1}{R}\int_0^\infty \left(1-e^{-2\lambda d/R}\right) \left(\frac{\sin\lambda}{\lambda}\right)^2 d\lambda.$$

In the limit of $d \gg R$, keep only the constant term, and used the well-known identity

$$\int_0^\infty \left(\frac{\sin\lambda}{\lambda}\right)^2 d\lambda = \frac{\pi}{2},$$

we will have

$$\frac{4\pi\varepsilon_0}{C} = \frac{\pi}{2R},$$

which is the result from Problem 3.3 (c).