14.13 From Eq. (14.53), $\frac{dW}{d\Omega} = \int_{-\infty}^{+\infty} |\vec{A}(t)|^2 dt$. When $\vec{A}(t)$ is periodic, it can be expanded as a Fourier series, with frequencies as the multipoles of the fundamental,

$$\vec{A}(t) = \sum_{m=-\infty}^{+\infty} \vec{A}_m e^{-im\omega_0 t}, \quad \text{with } \vec{A}_m = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \vec{A}(t) e^{im\omega_0 t} dt$$

where $T = 2\pi/\omega_0$ is the period of the motion. Then, we have

$$\frac{dW}{d\Omega} = \int_{-\infty}^{+\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \vec{A}_m^* \cdot \vec{A}_n e^{i(m-n)\omega_0 t} dt = \frac{2\pi}{\omega_0} \sum_{m=-\infty}^{\infty} |\vec{A}_m|^2$$

Therefore, the continuous frequency spectrum becomes a discrete one.

The time averaged radiation power can be expressed as $\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{T}\int_0^T \frac{dP(t)}{d\Omega} dt$, where $\frac{dP(t)}{d\Omega} = |\vec{A}(t)|^2$. Using the Fourier series representation of $\vec{A}(t)$, we have

$$\frac{1}{T}\int_0^T \frac{dP(t)}{d\Omega} dt = \frac{1}{T}\int_0^T |\vec{A}(t)|^2 dt = \frac{1}{T}\int_0^T \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \vec{A}_m^* \cdot \vec{A}_n e^{i(m-n)\omega_0 t} = \sum_{m=-\infty}^{\infty} |\vec{A}_m|^2 .$$

Since $\vec{A}(t) = \left(\frac{c}{4\pi}\right)^{1/2} [R\vec{E}]_{ret}$, following the same manipulation leading to Eq. (14.67), we have

$$\vec{A}_m = \left(\frac{e^2}{4\pi c}\right)^{1/2} \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \frac{d}{dt}\left[\frac{\vec{n}\times(\vec{n}\times\vec{\beta})}{1-\vec{\beta}\cdot\vec{n}}\right] e^{im\omega_0(t-\vec{n}\cdot\vec{x}(t)/c)} dt$$

$$= -\left(\frac{e^2}{4\pi c}\right)^{1/2} \frac{im\omega_0^2}{2\pi} \int_0^{2\pi/\omega_0} [\vec{n}\times(\vec{n}\times\vec{\beta})] e^{im\omega_0(t-\vec{n}\cdot\vec{x}(t)/c)} dt .$$

Then, $$\left\langle \frac{dP(t)}{d\Omega} \right\rangle = \sum_{m=-\infty}^{\infty} \frac{e^2 m^2 \omega_0^4}{16\pi^3 c^3} \left| \int_0^{2\pi/\omega_0} \vec{n}\times(\vec{n}\times\vec{v}) e^{im\omega_0(t-\vec{n}\cdot\vec{x}(t)/c)} dt \right|^2$$

$$= \sum_{m=1}^{\infty} \frac{e^2 m^2 \omega_0^4}{(2\pi c)^3} \left| \int_0^{2\pi/\omega_0} (\vec{n}\times\vec{v}(t)) e^{im\omega_0(t-\vec{n}\cdot\vec{x}(t)/c)} dt \right|^2 = \sum_{m=1}^{\infty} \frac{dP_m}{d\Omega},$$

with $$\frac{dP_m}{d\Omega} = \frac{e^2 m^2 \omega_0^4}{(2\pi c)^3} \left| \int_0^{2\pi/\omega_0} (\vec{n}\times\vec{v}(t)) \exp\left\{ im\omega_0 \left[t - \frac{\vec{n}\cdot\vec{x}(t)}{c} \right] \right\} \right|^2$$