10.2 Solution: Since the EM radiation with circular polarization but different helicity is scattered independently, the scattered EM radiation for elliptic polarization will be linear superposition of the scattered, circularly polarized radiation. In the long-wavelength limit, Eq. (10.71) becomes

$$\frac{d\sigma_{\rm sc}}{d\Omega} = \frac{2\pi}{3}\frac{k^4a^6}{1+r^2}\left|\Big(\mathbf{X}_{1,1} - 2i\mathbf{n}\times\mathbf{X}_{1,1}\Big) + re^{i\alpha}\Big(\mathbf{X}_{1,-1} + 2i\mathbf{n}\times\mathbf{X}_{1,-1}\Big)\right|^2.$$

By expanding the complex square, we have

$$\begin{aligned}
&\left|\Big(\mathbf{X}_{1,1} - 2i\mathbf{n}\times\mathbf{X}_{1,1}\Big) + re^{i\alpha}\Big(\mathbf{X}_{1,-1} + 2i\mathbf{n}\times\mathbf{X}_{1,-1}\Big)\right|^2 \\
= \quad & \left|\mathbf{X}_{1,1} - 2i\mathbf{n}\times\mathbf{X}_{1,1}\right|^2 + r^2\left|\mathbf{X}_{1,-1} + 2i\mathbf{n}\times\mathbf{X}_{1,-1}\right|^2 \\
& +2\mathrm{Re}\left[\Big(\mathbf{X}_{1,1} - 2i\mathbf{n}\times\mathbf{X}_{1,1}\Big)\cdot re^{-i\alpha}\Big(\mathbf{X}_{1,-1} + 2i\mathbf{n}\times\mathbf{X}_{1,-1}\Big)^*\right].
\end{aligned}$$

Using the results between Eqs. (10.71) and (10.72), we know

$$\left|\mathbf{X}_{1,1} - 2i\mathbf{n}\times\mathbf{X}_{1,1}\right|^2 = \left|\mathbf{X}_{1,-1} + 2i\mathbf{n}\times\mathbf{X}_{1,-1}\right|^2 = \frac{15}{16\pi}(1+\cos^2\theta) - \frac{3}{2\pi}\cos\theta.$$

All we need to do now is to calculate the cross product term. To this end, we will need the explicit expression for $\mathbf{X}_{1,\pm1}$.

The defition of $\mathbf{X}_{lm}$ is

$$\mathbf{X}_{lm} = \frac{1}{\sqrt{l(l+1)}}\mathbf{L}Y_{lm} = \frac{1}{\sqrt{l(l+1)}}\left(L_xY_{lm}, L_yY_{lm}, L_zY_{lm}\right)^\top.$$

Since $L_\pm = L_x \pm iL_y$, we know

$$L_x = \frac{1}{2}(L_+ + L_-), \quad L_x = \frac{1}{2i}(L_+ - L_-).$$

Also,

$$L_\pm Y_{lm} = \sqrt{(l\mp m)(l\pm m+1)}Y_{l,m\pm1},$$

then

$$\mathbf{X}_{lm} = \frac{1}{\sqrt{l(l+1)}}\begin{pmatrix} \frac{1}{2}\left(\sqrt{(l-m)(l+m+1)}Y_{l,m+1} + \sqrt{(l+m)(l-m+1)}Y_{l,m-1}\right) \\ \frac{1}{2i}\left(\sqrt{(l-m)(l+m+1)}Y_{l,m+1} - \sqrt{(l+m)(l-m+1)}Y_{l,m-1}\right) \\ mY_{lm}\end{pmatrix}.$$

For $l=1$ and $m=1$, we have

$$\mathbf{X}_{1,1} = \left(\frac{1}{2}Y_{1,0}, \frac{i}{2}Y_{1,0}, \frac{1}{\sqrt{2}}Y_{1,1}\right)^\top,$$

and $l=1$ and $m=-1$,

$$\mathbf{X}_{1,-1} = \left(\frac{1}{2}Y_{1,0}, -\frac{i}{2}Y_{1,0}, -\frac{1}{\sqrt{2}}Y_{1,-1}\right)^\top.$$

From the identity $Y^*_{lm} = (-1)^mY_{l,-m}$, we know $Y^*_{1,1} = -Y_{l,-1}$, and $\mathbf{X}^*_{1,1} = \mathbf{X}_{1,-1}$. Here,

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1,1} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}.$$

Now, we can calculate the cross product term as

$$\begin{aligned}
& \Big(\mathbf{X}_{1,1} - 2i\mathbf{n}\times\mathbf{X}_{1,1}\Big)\cdot\Big(\mathbf{X}_{1,-1} + 2i\mathbf{n}\times\mathbf{X}_{1,-1}\Big)^* \\
= \quad & \Big(\mathbf{X}_{1,1} - 2i\mathbf{n}\times\mathbf{X}_{1,1}\Big)\cdot\Big(\mathbf{X}_{1,1} - 2i\mathbf{n}\times\mathbf{X}_{1,1}\Big) \\
= \quad & \mathbf{X}_{1,1}\cdot\mathbf{X}_{1,1} - 4i\mathbf{X}_{1,1}\cdot(\mathbf{n}\times\mathbf{X}_{1,1}) - 4(\mathbf{n}\times\mathbf{X}_{1,1})\cdot(\mathbf{n}\times\mathbf{X}_{1,1}) \\
= \quad & -3\mathbf{X}_{1,1}\cdot\mathbf{X}_{1,1},
\end{aligned}$$

where we have used the fact that $\mathbf{X}_{1,1}\cdot(\mathbf{n}\times\mathbf{X}_{1,1}) = 0$ and $(\mathbf{n}\times\mathbf{X}_{1,1})\cdot(\mathbf{n}\times\mathbf{X}_{1,1}) = \mathbf{X}_{1,1}\cdot\mathbf{X}_{1,1}$. Finally,

$$\mathbf{X}_{1,1}\cdot\mathbf{X}_{1,1} = \frac{1}{2}Y_{1,1}^2 = \frac{3}{16\pi}\sin^2\theta e^{2i\phi},$$

and

$$\begin{aligned}
& 2\mathrm{Re}\left[\Big(\mathbf{X}_{1,1} - 2i\mathbf{n}\times\mathbf{X}_{1,1}\Big)\cdot re^{-i\alpha}\Big(\mathbf{X}_{1,-1} + 2i\mathbf{n}\times\mathbf{X}_{1,-1}\Big)^*\right] \\
= \quad & 2\mathrm{Re}\left[re^{-i\alpha}\cdot\frac{-9}{16\pi}\sin^2\theta e^{2i\phi}\right] \\
= \quad & -\frac{9}{8\pi}r\sin^2\theta\cos(2\phi-\alpha).
\end{aligned}$$

Putting everything together, we will have

$$\begin{aligned}
\frac{d\sigma_{\rm sc}}{d\Omega} \quad = \quad & \frac{2\pi}{3}\frac{k^4a^6}{1+r^2}\Bigg[\left(\frac{15}{16\pi}(1+\cos^2\theta) - \frac{3}{2\pi}\cos\theta\right) + r^2\left(\frac{15}{16\pi}(1+\cos^2\theta) - \frac{3}{2\pi}\cos\theta\right) \\
& \qquad -\frac{9}{8\pi}r\sin^2\theta\cos(2\phi-\alpha)\Bigg] \\
= \quad & \frac{2\pi}{3}k^4a^6\left[\frac{15}{16\pi}(1+\cos^2\theta) - \frac{3}{2\pi}\cos\theta - \frac{9}{8\pi}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right] \\
= \quad & k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right].
\end{aligned}$$

The same expression can also be obtained by applying the result from Prob. 10.1 (a). With $\mathbf{n}_0 = (0,0,1)$, $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $\boldsymbol{\epsilon}_1 = (1,0,0)$, $\boldsymbol{\epsilon}_2 = (0,1,0)$, $\boldsymbol{\epsilon}_\pm = (\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2)/\sqrt{2} = (1,\pm i,0)/\sqrt{2}$, and

$$\boldsymbol{\epsilon} = \frac{1}{\sqrt{1+r^2}}\left(\boldsymbol{\epsilon}_+ + re^{i\alpha}\boldsymbol{\epsilon}_-\right) = \frac{1}{\sqrt{2(1+r^2)}}(1+re^{i\alpha}, i(1-re^{i\alpha}), 0),$$

we can perform the vector products as

$$\mathbf{n}\cdot\boldsymbol{\epsilon} = \frac{1}{\sqrt{2(1+r^2)}}\left((1+re^{i\alpha})\sin\theta\cos\phi + i(1-re^{i\alpha})\sin\theta\sin\phi\right),$$

$$|\mathbf{n}\cdot\boldsymbol{\epsilon}|^2 \quad = \quad \frac{1}{2(1+r^2)}\Big((1+2r\cos\alpha+r^2)\sin^2\theta\cos^2\phi + (1-2r\cos\alpha+r^2)\sin^2\theta\sin^2\phi$$

$$+4r\sin^2\theta\sin\phi\cos\phi\sin\alpha\Big)$$

$$\begin{aligned}
&= \frac{1}{2(1+r^2)}\left((1+r^2)\sin^2\theta\cos^2\phi + 2r\sin^2\theta\cos(2\phi)\cos\alpha + 2r\sin^2\theta\sin(2\phi)\sin\alpha\right) \\
&= \frac{1}{2}\sin^2\theta + \frac{r}{1+r^2}\sin^2\theta\cos(2\phi-\alpha),
\end{aligned}$$

and

$$\mathbf{n}_0\times\boldsymbol{\epsilon} = \frac{1}{\sqrt{2(1+r^2)}}\left(-i(1-re^{i\alpha}), (1+re^{i\alpha}), 0\right),$$

$$\begin{aligned}
|\mathbf{n}\cdot(\mathbf{n_0}\times\boldsymbol{\epsilon})|^2 &= \frac{1}{2(1+r^2)}\Big((1-2r\cos\alpha+r^2)\sin^2\theta\cos^2\phi + (1+2r\cos\alpha+r^2)\sin^2\theta\sin^2\phi \\
&\qquad -4r\sin^2\theta\sin\phi\cos\phi\sin\alpha\Big) \\
&= \frac{1}{2(1+r^2)}\left((1+r^2)\sin^2\theta\cos^2\phi - 2r\sin^2\theta\cos(2\phi)\cos\alpha - 2r\sin^2\theta\sin(2\phi)\sin\alpha\right) \\
&= \frac{1}{2}\sin^2\theta - \frac{r}{1+r^2}\sin^2\theta\cos(2\phi-\alpha).
\end{aligned}$$

Then, using Prob. 10.1 (a),

$$\begin{aligned}
\frac{d\sigma_{\text{sc}}}{d\Omega} &= k^4a^6\left[\frac{5}{4} - |\mathbf{n}\cdot\boldsymbol{\epsilon}|^2 - \frac{1}{4}|\mathbf{n}\cdot(\mathbf{n_0}\times\boldsymbol{\epsilon})|^2 - \mathbf{n}\cdot\mathbf{n}_0\right] \\
&= k^4a^6\Bigg[\frac{5}{4} - \frac{1}{2}\sin^2\theta - \frac{r}{1+r^2}\sin^2\theta\cos(2\phi-\alpha) \\
&\qquad -\frac{1}{4}\left(\frac{1}{2}\sin^2\theta - \frac{r}{1+r^2}\sin^2\theta\cos(2\phi-\alpha)\right) - \cos\theta\Bigg] \\
&= k^4a^6\left[\frac{5}{4} - \frac{5}{8}\sin^2\theta - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right] \\
&= k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right].
\end{aligned}$$