3.14 Solution: (a) The linear charge density can be determined by the condition

$$\lambda \int_{-d}^{d} (d^2 - r^2) dr = \frac{4}{3}\lambda d^3 = Q,$$

or

$$\lambda = \frac{3Q}{4d^3}.$$

In the spherical coordinates, the density can be expressed as

$$\sigma(\mathbf{x}) = \frac{3Q}{4d^3}\frac{1}{2\pi r^2}\left[\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\right].$$

Then, the potential inside the sphere can be obtained from Eq. (1.46),

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \sigma(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d^3x' = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \int_0^d r'^2 dr' \; \sigma(\mathbf{x}')G(\mathbf{x},\mathbf{x}'),$$

with the Green function given by Eq. (3.125), where $a = 0$,

$$\begin{aligned} G(\mathbf{x},\mathbf{x}') &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) r^l_< \left(\frac{1}{r^{l+1}_>} - \frac{r^l_>}{b^{2l+1}}\right) \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{(l-m)!}{(l+m)!} P^m_l(\theta,\phi)P^m_l(\theta',\phi')e^{im(\phi-\phi')} r^l_< \left(\frac{1}{r^{l+1}_>} - \frac{r^l_>}{b^{2l+1}}\right). \end{aligned}$$

Integration with respect to ϕ' will leave $m = 0$ term only with a result of 2π. Integration with respect to θ', due to the Dirac delta functions, will leave a sum, $P_l(1) + P_l(-1)$. Therefore,

$$\begin{aligned} \Phi(\mathbf{x}) &= \frac{1}{4\pi\varepsilon_0}\frac{3Q}{4d^3}\sum_{l=0}^{\infty}\left[P_l(1) + P_l(-1)\right] P_l(\cos\theta) \int_0^d (d^2 - r'^2) r^l_< \left(\frac{1}{r^{l+1}_>} - \frac{r^l_>}{b^{2l+1}}\right) dr' \\ &= \frac{3Q}{8\pi\varepsilon_0 d^3}\sum_{l=0,\text{ even}}^{\infty} P_l(\cos\theta)\int_0^d (d^2 - r'^2) r^l_< \left(\frac{1}{r^{l+1}_>} - \frac{r^l_>}{b^{2l+1}}\right) dr'. \end{aligned}$$

Now, we need to consider two cases for the location of the point that we are interested in.
(i) $r > d$. In this case, $r_< = r'$ and $r_> = r$, and the radial integral is simply

$$\begin{aligned} I &= \int_0^d (d^2 - r'^2) r^l_< \left(\frac{1}{r^{l+1}_>} - \frac{r^l_>}{b^{2l+1}}\right) dr' \\ &= \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\int_0^d (d^2 - r'^2) r'^l dr' \\ &= \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\left(\frac{d^2 r'^{l+1}}{l+1} - \frac{r'^{l+3}}{l+3}\right)\Bigg|_{r'=0}^{d} \\ &= \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\frac{2d^{l+3}}{(l+1)(l+3)}, \end{aligned}$$

and the potential is

$$\Phi(\mathbf{x}) = \frac{3Q}{4\pi\varepsilon_0}\sum_{l=0,\text{ even}}^{\infty}\frac{d^l}{(l+1)(l+3)}\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right) P_l(\cos\theta).$$

(ii) $r < d$. The radial integral has to be broken into two parts, as has been done in Section 3.10. The calculation is tedious, cumbersome, and not very informative. I will just omit it here. Just need to be careful that, for $l = 0$ and $l = 2$, there are log terms due to the integration of $1/r'$.

(b) For the surface charge density on the inner surface of the shell, the normal direction is $-\hat{r}$ and also $r = b > d$. Then,

$$
\begin{aligned}
\sigma &= -\varepsilon_0 \frac{\partial \Phi}{\partial n}\bigg|_{r=b} = \varepsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=b} \\
&= -\frac{3Q}{4\pi} \sum_{l=0,\ \text{even}}^{\infty} \frac{d^l}{(l+1)(l+3)} \left(\frac{l+1}{r^{l+2}} + \frac{l r^{l-1}}{b^{2l+1}} \right) P_l(\cos\theta) \bigg|_{r=b} \\
&= -\frac{3Q}{4\pi b^2} \sum_{l=0,\ \text{even}}^{\infty} \frac{2l+1}{(l+1)(l+3)} \left(\frac{d}{b} \right)^l P_l(\cos\theta).
\end{aligned}
$$

(c) For $d \ll b$, we can rewrite the potential inside the sphere as

$$
\Phi(\mathbf{x}) = \frac{3Q}{4\pi\varepsilon_0 b} \sum_{l=0,\ \text{even}}^{\infty} \frac{1}{(l+1)(l+3)} \left(\frac{d}{b} \right)^l \left(\left(\frac{b}{r} \right)^{l+1} - \left(\frac{r}{b} \right)^l \right) P_l(\cos\theta).
$$

Only the $l = 0$ term will survive, which leads to

$$
\Phi(\mathbf{x}) = \frac{Q}{4\pi\varepsilon_0} \left(\frac{1}{r} - \frac{1}{b} \right).
$$

This is the potential for a point charge inside a conducting sphere.

The charge density can be determined in a similar way,

$$
\sigma = -\frac{Q}{4\pi b^2}.
$$

The charge in the sphere is completely shielded due to the equal but opposite amount of charge induced on the sphere, and the induced sphere is distributed evenly at the surface.