2.13 Solution: (a) Using the result from Problem 2.12, the potential inside the circle can be expressed as

$$\Phi(\rho,\phi)=\frac{1}{2\pi}\int_0^{2\pi}\Phi(b,\phi')\frac{b^2-\rho^2}{b^2+\rho^2-2\rho b\cos(\phi'-\phi)}d\phi',$$

where

$$\Phi(b,\phi')=\begin{cases}V_1, & -\dfrac{\pi}{2}<\phi'<\dfrac{\pi}{2},\\ V_2, & \dfrac{\pi}{2}<\phi'<\dfrac{3\pi}{2}.\end{cases}$$

Then, the potential becomes

$$\Phi(\rho,\phi)=\frac{V_1}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{b^2-\rho^2}{b^2+\rho^2-2\rho b\cos(\phi'-\phi)}d\phi'+\frac{V_2}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{b^2-\rho^2}{b^2+\rho^2+2\rho b\cos(\phi'-\phi)}d\phi'.$$

Let $\Sigma=V_1+V_2$ and $\Delta=V_1-V_2$, the potential can be written as

$$\Phi(\rho,\phi)=\frac{\Sigma}{2}I_\Sigma+\frac{\Delta}{2}I_\Delta,$$

where

$$I_\Sigma=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{b^2-\rho^2}{b^2+\rho^2-2\rho b\cos(\phi'-\phi)}+\frac{b^2-\rho^2}{b^2+\rho^2+2\rho b\cos(\phi'-\phi)}\right)d\phi',$$

and

$$I_\Delta=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{b^2-\rho^2}{b^2+\rho^2-2\rho b\cos(\phi'-\phi)}-\frac{b^2-\rho^2}{b^2+\rho^2+2\rho b\cos(\phi'-\phi)}\right)d\phi'.$$

By a simple change of variable, the integral I_Σ can be expressed as an integral on the complex plane,

$$\begin{aligned}I_\Sigma &= \frac{1}{2\pi}\int_0^{2\pi}\frac{b^2-\rho^2}{b^2+\rho^2-2\rho b\cos\phi'}d\phi'\\ &= \frac{1}{2\pi i}\oint_{|z|=1}\frac{\rho^2-b^2}{\rho b z^2-(b^2+\rho^2)z+\rho b}dz\\ &= \frac{1}{2\pi i}\oint_{|z|=1}\left(\frac{1}{z-\rho/b}-\frac{1}{z-b/\rho}\right)dz.\end{aligned}$$

Since $\rho<b$, the integrand has a pole in the unit circle at $z=\rho/b$, while the other pole is located outside of the unit circle. Therefore, $I_\Sigma=1$, by Cauchy's theorem.

On the other hand,

$$\begin{aligned}I_\Delta &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{b^2-\rho^2}{b^2+\rho^2-2\rho b\cos(\phi'-\phi)}-\frac{b^2-\rho^2}{b^2+\rho^2+2\rho b\cos(\phi'-\phi)}\right)d\phi'\\ &= \frac{b^2-\rho^2}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{4\rho b\cos(\phi'-\phi)}{(b^2+\rho^2)^2-4\rho^2b^2\cos^2(\phi'-\phi)}d\phi'\\ &= \frac{b^2-\rho^2}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{4\rho b}{(b^2-\rho^2)^2+4\rho^2b^2\sin^2(\phi'-\phi)}d\left(\sin(\phi'-\phi)\right)\\ &= \frac{1}{2\pi}\frac{b^2-\rho^2}{\rho b}\int_{-\pi/2}^{\pi/2}\frac{d\left(\sin(\phi'-\phi)\right)}{\sin^2(\phi'-\phi)+\left(\dfrac{(b^2-\rho^2)}{2\rho b}\right)^2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\arctan\left(\frac{2\rho b}{b^2-\rho^2}\sin(\phi'-\phi)\right)\Bigg|_{-\pi/2}^{\pi/2} \\
&= \frac{2}{\pi}\arctan\left(\frac{2\rho b}{b^2-\rho^2}\cos\phi\right).
\end{aligned}
$$

Therefore,

$$
\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\arctan\left(\frac{2\rho b}{b^2-\rho^2}\cos\phi\right).
$$

(b) To determine the inner surface charge density, notice that the normal vector is pointing to the origin in the radial direction, then

$$
\begin{aligned}
\sigma &= -\varepsilon_0\left.\frac{\partial\Phi}{\partial n}\right|_{\rho=b} = \varepsilon_0\left.\frac{\partial\Phi}{\partial\rho}\right|_{\rho=b} \\
&= \varepsilon_0\frac{V_1-V_2}{\pi}\left.\frac{1}{1+\dfrac{4\rho^2b^2}{(b^2-\rho^2)^2}\cos^2\phi}\times\cos\phi\frac{\partial}{\partial\rho}\left(\frac{2\rho b}{b^2-\rho^2}\right)\right|_{\rho=b} \\
&= \varepsilon_0\frac{V_1-V_2}{\pi}\left.\frac{\cos\phi\times 2b(b^2+\rho^2)}{(b^2-\rho^2)^2+4\rho^2b^2\cos^2\phi}\right|_{\rho=b} \\
&= \frac{\varepsilon_0(V_1-V_2)}{\pi b\cos\phi}. \qquad (1)
\end{aligned}
$$