3.19 Solution: (a) Similar to Problem 1.13, we can use the Green's reciprocation theorem to find the total charge induced. Following the same argument present in the solution to Problem 1.13, we have:

(i) For the unprimed configuration, everything will be the same;

(ii) For the primed configuration, we have no charge, but at $z = z_0$, the potential is $\Phi(z_0, 0)$, and on the upper plane, $\Phi(L) = VI_{\rho\leq a}$.

Then, from the Green's reciprocation theorem

$$\int_V \rho\Phi' d^3x + \oint_S \sigma\Phi' da = \int_V \rho'\Phi d^3x + \oint_S \sigma'\Phi da,$$

the left hand side is

$$q\Phi(z_0, 0) + V \int\int_{\rho\leq a} \sigma_L(x, y)dxdy = q\Phi(z_0, 0) + VQ_L(a),$$

where σ_L and Q_L are the surface charge density and the total charge on the $z = L$ plane, and the right hand is, again, 0. Then,

$$q\Phi(z_0, 0) + VQ_L(a) = 0,$$

or,

$$Q_L(a) = -\frac{q}{V}\Phi(z_0, 0).$$

(b) For this problem, we can use the Green function from Problem 3.17 (b),

$$G(\mathbf{x}, \mathbf{x}') = 2 \sum_{m=-\infty}^{\infty} \int_0^\infty dk e^{im(\phi-\phi')} J_m(k\rho)J_m(k\rho') \frac{\sinh(kz_<)\sinh[k(L - z_>)]}{\sinh(kL)},$$

and Eq. (1.42),

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\mathbf{x}')G(\mathbf{x}, \mathbf{x}')d^3x'.$$

Here, in this problem, the charge density $\rho(\mathbf{x}')$ is a point charge located at $z' = z_0$, so the potential at $z > z_0$ can be directly written down as

$$\Phi(\mathbf{x}) = \frac{q}{2\pi\varepsilon_0} \int_0^\infty dk J_0(k\rho) \frac{\sinh(kz_0)\sinh[k(L - z)]}{\sinh(kL)},$$

which clearly vanishes at $z = L$. To determine the charge density of the upper plane, we need to evaluate the normal derivative $\partial\Phi/\partial n$, where the normal direction is in the $-\hat{z}$. Therefore,

$$\begin{aligned}
\sigma_L(\rho) &= -\varepsilon_0 \left.\frac{\partial\Phi}{\partial n}\right|_{z=L} = \varepsilon_0 \left.\frac{\partial\Phi}{\partial z}\right|_{z=L} \\
&= -\frac{q}{2\pi} \int_0^\infty dk \left.\frac{\sinh(kz_0)\cosh[k(L - z)]}{\sinh(kL)}\right|_{z=L} \cdot kJ_0(k\rho) \\
&= -\frac{q}{2\pi} \int_0^\infty dk \frac{\sinh(kz_0)}{\sinh(kL)} kJ_0(k\rho).
\end{aligned}$$

Using the identity (see, *e.g.*, Gradshteyn and Ryzhik, 7^{th} ed., p. 715, formula 6.666)

$$\int_0^\infty x^{\nu+1} \frac{\sinh(\alpha x)}{\sinh(\pi x)} J_\nu(\beta x)dx = \frac{2}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} n^{\nu+1} \sin(n\alpha)K_\nu(n\beta),$$

and making the change of integration variable, $k \to k\pi/L$, in the expression for the charge density, we have

$$\begin{aligned}
\sigma_L(\rho) &= -\frac{q\pi}{2L^2}\int_0^\infty k\frac{\sinh\left(\frac{\pi z_0}{L}k\right)}{\sinh(\pi k)}J_0\left(\frac{\pi\rho}{L}k\right)dk, \\
&= -\frac{q}{L^2}\sum_{n=1}^{\infty}(-1)^{n-1}n\sin\left(\frac{n\pi}{L}z_0\right)K_0\left(\frac{n\pi}{L}\rho\right).
\end{aligned}$$

For large argument,

$$K_\nu(x) \to \sqrt{\frac{\pi}{2x}}e^{-x}\left[1+O\left(\frac{1}{x}\right)\right],$$

we can see that the charge density falls off as $\rho^{-1/2}e^{-\pi\rho/L}$, from the first term in the series when ρ is large, while other terms falls off even more quickly, as $\rho^{-1/2}e^{-n\pi\rho/L}$, $n > 1$. The asymptotic behavior is dominated by the first term.

With the charge density, the total charge inside a circle of radius a can be directly calculated,

$$\begin{aligned}
Q_L(a) &= \int_0^{2\pi}d\phi\int_0^a d\rho\rho\sigma_L(\rho) \\
&= -q\int_0^\infty dk\frac{\sinh(kz_0)}{\sinh(kL)}k\int_0^a d\rho\rho J_0(k\rho) \\
&= -q\int_0^\infty \frac{dk}{k}\frac{\sinh(kz_0)}{\sinh(kL)}\int_0^{ka} d\lambda\lambda J_0(\lambda) \\
&= -q\int_0^\infty \frac{dk}{k}\frac{\sinh(kz_0)}{\sinh(kL)}kaJ_1(ka) \\
&= -qa\int_0^\infty dk\frac{\sinh(kz_0)}{\sinh(kL)}J_1(ka) \\
&= -q\int_0^\infty d\lambda\frac{\sinh(\lambda z_0/a)}{\sinh(\lambda L/a)}J_1(\lambda)
\end{aligned}$$

In Problem 3.18 (a), setting $\rho = 0$ and noticing $J_0(0) = 1$,

$$\Phi(z_0, 0) = V\int_0^\infty d\lambda\frac{\sinh(\lambda z_0/a)}{\sinh(\lambda L/a)}J_1(\lambda).$$

Therefore,

$$Q_L(a) = -\frac{q}{V}\Phi(z_0, 0).$$

(c) At $\rho = 0$, the charge density is

$$\begin{aligned}
\sigma(0) &= -\frac{q}{2\pi}\int_0^\infty dk\frac{\sinh(kz_0)}{\sinh(kL)}k \\
&= -\frac{q}{2\pi}\int_0^\infty dk\frac{e^{kz_0}-e^{-kz_0}}{e^{kL}-e^{-kL}}k \\
&= -\frac{q}{2\pi}\int_0^\infty dk\frac{e^{kz_0}-e^{-kz_0}}{1-e^{-2kL}}ke^{-kL}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{q}{2\pi}\sum_{n=0}^{\infty}\int_0^{\infty} dk \left[e^{kz_0} - e^{-kz_0}\right] \cdot k e^{-(2n+1)kL} \\
&= -\frac{q}{2\pi}\sum_{n=0}^{\infty}\int_0^{\infty} dk k \left[e^{-k((2n+1)L-z_0)} - e^{-k((2n+1)L+z_0)}\right] \\
&= -\frac{q}{2\pi}\sum_{n=0}^{\infty}\left[\frac{1}{((2n+1)L-z_0)^2} - \frac{1}{((2n+1)L+z_0)^2}\right] \\
&= -\frac{q}{2\pi L^2}\sum_{n>0,\ \mathrm{odd}}^{\infty}\left[\frac{1}{(n-z_0/L)^2} - \frac{1}{(n+z_0/L)^2}\right].
\end{aligned}
$$