3.11 (a) Take the Bessel equation

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dJ_\nu(k\rho)}{d\rho}\right]+\left(k^2-\frac{\nu^2}{\rho^2}\right)J_\nu(k\rho)=0,$$

multiply both sides by $\rho J_\nu(k'\rho)$, and integrate from 0 to a, we will get

$$\int_0^a J_\nu(k'\rho)\frac{d}{d\rho}\left[\rho\frac{dJ_\nu(k\rho)}{d\rho}\right]d\rho+\int_0^a\left(k^2-\frac{\nu^2}{\rho^2}\right)\rho J_\nu(k'\rho)J_\nu(k\rho)\,d\rho=0.$$

Perform integration by parts on the first term, the equation becomes

$$\rho J_\nu(k'\rho)\frac{dJ_\nu(k\rho)}{d\rho}\Big|_0^a-\int_0^a\rho\frac{dJ_\nu(k'\rho)}{d\rho}\frac{dJ_\nu(k\rho)}{d\rho}d\rho+\int_0^a\left(k^2-\frac{\nu^2}{\rho^2}\right)\rho J_\nu(k'\rho)J_\nu(k\rho)\,d\rho=0.$$

Using the first boundary condition, we know $\rho J_\nu(k'\rho)\frac{dJ_\nu(k\rho)}{d\rho}\Big|_{\rho=0}=0$. The second boundary condition is $\frac{1}{J_\nu(k\rho)}\cdot\frac{dJ_\nu(k\rho)}{d\rho}\Big|_{\rho=a}=-\frac{\lambda}{a}$, or $\frac{dJ_\nu(k\rho)}{d\rho}\Big|_{\rho=a}=-\frac{\lambda}{a}J_\nu(ka)$. Then, the boundary term becomes $-\lambda J_\nu(k'a)J_\nu(ka)$. Therefore, we will get

$$-\lambda J_\nu(k'a)J_\nu(ka)-\int_0^a\rho\frac{dJ_\nu(k'\rho)}{d\rho}\frac{dJ_\nu(k\rho)}{d\rho}d\rho+\int_0^a\left(k^2-\frac{\nu^2}{\rho^2}\right)\rho J_\nu(k'\rho)J_\nu(k\rho)\,d\rho=0.$$

Exchange k and k',

$$-\lambda J_\nu(ka)J_\nu(k'a)-\int_0^a\rho\frac{dJ_\nu(k\rho)}{d\rho}\frac{dJ_\nu(k'\rho)}{d\rho}d\rho+\int_0^a\left(k'^2-\frac{\nu^2}{\rho^2}\right)\rho J_\nu(k\rho)J_\nu(k'\rho)\,d\rho=0,$$

and take the difference with the previous equation, we can see

$$(k^2-k'^2)\int_0^a\rho J_\nu(k\rho)J_\nu(k'\rho)\,d\rho=0.$$

This means, for $k\neq k'$, we must have

$$\int_0^a\rho J_\nu(k\rho)J_\nu(k'\rho)\,d\rho=0.$$

which proves the orthogonality of Bessel functions with different eigenvalues.

(b) The Bessel equation can also be written as

$$\rho\left[\rho J_\nu'(k\rho)\right]'+(k^2\rho^2-\nu^2)J_\nu(k\rho)=0,$$

where primes means derivative with respect to ρ. Multiply by $J_\nu'(k\rho)$ and integrate from 0 to a, We have

$$\int_0^a \rho J_\nu'(k\rho)\left[\rho J_\nu'(k\rho)\right]' d\rho + \int_0^a (k^2\rho^2-\nu^2) J_\nu'(k\rho) J_\nu(k\rho) d\rho = 0$$

The firs term is simply $\frac{1}{2}\left(\rho J_\nu'(k\rho)\right)^2\Big|_0^a = \frac{1}{2}a^2\left(\frac{dJ_\nu(k\rho)}{d\rho}\right)^2\Big|_{\rho=a} = \frac{1}{2}k^2a^2\left[J_\nu'(ka)\right]^2$.

The second term can be manipulated by integration by parts,

$$\int_0^a (k^2\rho^2-\nu^2) J_\nu'(k\rho) J_\nu(k\rho) d\rho = \frac{1}{2}\int_0^a (k^2\rho^2-\nu^2) d\left(J_\nu(k\rho)^2\right)$$

$$= \frac{1}{2}(k^2a^2-\nu^2) J_\nu(ka)^2 - k^2\int_0^a \rho J_\nu(k\rho)^2 d\rho$$

Therefore, $\int_0^a \rho J_\nu(k\rho)^2 d\rho = \frac{a^2}{2}\left(J_\nu'(ka)\right)^2 + \frac{a^2}{2}\left(1-\frac{\nu^2}{k^2a^2}\right) J_\nu(ka)^2$

$$= \frac{a^2}{2}\left[\left(1-\frac{\nu^2}{y_{\nu n}^2}\right) J_\nu(y_{\nu n})^2 + \left(\frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}}\right)^2\right]$$

Then, for the Bessel-Fourier series,

$$f(\rho) = \sum_{n=1}^{\infty} A_n J_\nu\left(\frac{y_{\nu n}}{a}\rho\right),$$

we can determine A_n by multiplying both sides by $\rho J_\nu\left(\frac{y_{\nu n}\rho}{a}\right)$ and integrating from 0 to a.

By the second boundary condition, we can relate the derivate of the Bessel function to the function.

Specifically, we know $\frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}} = -\frac{\lambda}{y_{\nu n}} J_\nu(y_{\nu n})$ and $J_\nu(y_{\nu n}) = -\frac{y_{\nu n}}{\lambda}\frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}}$. Then,

$$\left(1-\frac{\nu^2}{y_{\nu n}^2}\right) J_\nu(y_{\nu n})^2 + \left(\frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}}\right)^2 = \left(1-\frac{\nu^2}{y_{\nu n}^2}\right) J_\nu(y_{\nu n})^2 + \frac{\lambda^2}{y_{\nu n}^2} J_\nu(y_{\nu n})^2 = \left(1+\frac{\lambda^2-\nu^2}{y_{\nu n}^2}\right) J_\nu(y_{\nu n})^2$$

$$= \left(1-\frac{\nu^2}{y_{\nu n}^2}\right)\frac{y_{\nu n}^2}{\lambda^2}\left(\frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}}\right)^2 + \left(\frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}}\right)^2$$

$$= \left(1+\frac{y_{\nu n}^2-\nu^2}{\lambda^2}\right)\left(\frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}}\right)^2$$

From the identity $\frac{dJ_\nu(x)}{dx} = \frac{1}{2}\left(J_{\nu+1}(x) - J_{\nu-1}(x)\right)$, we have

$$\left(\frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}}\right)^2 = \frac{1}{4}\left(J_{\nu+1}(y_{\nu n}) - J_{\nu-1}(y_{\nu n})\right)^2 = \frac{1}{4}\left(J_{\nu+1}(y_{\nu n}) + J_{\nu-1}(y_{\nu n})\right)^2 - J_{\nu+1}(y_{\nu n}) J_{\nu-1}(y_{\nu n})$$

$$= \frac{1}{4} \cdot \frac{4\nu^2}{y_{\nu n}^2} J_\nu(y_{\nu n})^2 - J_{\nu+1}(y_{\nu n}) J_{\nu-1}(y_{\nu n}).$$

$$= \frac{\nu^2}{y_{\nu n}^2} J_\nu(y_{\nu n})^2 - J_{\nu+1}(y_{\nu n}) J_{\nu-1}(y_{\nu n}).$$

Where we have use $J_{\nu+1}(x) + J_{\nu-1}(x) = \frac{2\nu}{x} J_\nu(x)$. Then.

$$\left(1 - \frac{\nu^2}{y_{\nu n}^2}\right) J_\nu(y_{\nu n})^2 + \left(\frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}}\right)^2 = J_\nu(y_{\nu n})^2 - J_{\nu+1}(y_{\nu n}) J_{\nu-1}(y_{\nu n}).$$

For $\lambda \to \infty$, the second boundary condition becomes $J_\nu(x) = 0$. Its normalization becomes

$$\left(\frac{dJ_\nu(y_{\nu n})}{dy_{\nu n}}\right)^2 = \left(J_{\nu+1}(y_{\nu n})\right)^2 = \left(J_{\nu-1}(y_{\nu n})\right)^2,$$ as discussed in the text.

In the above proof, we have assumed $\rho J_\nu'(k\rho)\Big|_{\rho=0} = 0$, and $J_\nu(k\rho)\Big|_{\rho=a} = 0$. This can be seen from the first boundary condition. Since the Bessel function is regular at $\rho = 0$, we must have $J_\nu(k\rho) = O(\rho^\alpha)$, with $\alpha > 0$, and $\frac{dJ_\nu(k\rho)}{d\rho} = O(\rho^{\alpha-1})$. Then. first boundary condition requires that $\alpha > 0$. From this asymptotic behavior. we will obtain the necessary vanishing lower boundary values.