8.7 (a) Let $u_l(r) = r f_l(r)$, then Eq. (8.103) becomes

$$\frac{d^2 f_l(r)}{dr^2} + \frac{2}{r}\frac{df_l(r)}{dr} + \left[\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2}\right] f_l(r) = 0.$$

The solutions of $f_l(r)$ are linear combinations of spherical Bessel functions.

$$f_l(r) = A_l j_l(kr) + B_l n_l(kr),$$

with $k = \omega/c$. Then, $u_l(r) = r\left(A_l j_l(kr) + B_l n_l(kr)\right)$. The boundary condition of

$\frac{du_l(r)}{dr} = 0$ at $r = a$ and $r = b$ leads to

$$A_l\left(j_l(ka) + ka\, j_l'(ka)\right) + B_l\left(n_l(ka) + ka\, n_l'(ka)\right) = 0$$

$$A_l\left(j_l(kb) + kb\, j_l'(kb)\right) + B_l\left(n_l(kb) + kb\, n_l'(kb)\right) = 0$$

For these equations to have non-trivial solutions, we must have

$$\begin{vmatrix} j_l(ka) + ka\, j_l'(ka) & n_l(ka) + kb\, n_l'(ka) \\ j_l(kb) + bb\, j_l'(kb) & n_l(kb) + kb\, n_l'(kb) \end{vmatrix} = 0,$$

or equivalently,

$$\frac{j_l(ka) + ka\, j_l'(ka)}{n_l(ka) + ka\, n_l'(ka)} = \frac{j_l(kb) + kb\, j_l'(kb)}{n_l(kb) + kb\, n_l'(kb)}$$

(b) For convenience, the transendental equation can also be writtens as

$$\frac{\left.\frac{d}{dx}\left(x j_l(x)\right)\right|_{x=ka}}{\left.\frac{d}{dx}\left(x n_l(x)\right)\right|_{x=ka}} = \frac{\left.\frac{d}{dx}\left(x j_l(x)\right)\right|_{x=kb}}{\left.\frac{d}{dx}\left(x n_l(x)\right)\right|_{x=kb}}$$

Using the identity

$$\frac{d}{dx}\left(x z_l(x)\right) = x z_{l-1}(x) - l z_l(x),$$

for both j_l and n_l, and for $l = 1$, we have

$$\frac{d}{dx}\left(x j_1(x)\right) = x j_0(x) - j_1(x) = \sin x - \frac{\sin x}{x^2} + \frac{\cos x}{x}$$

$$\frac{d}{dx}\left(x n_1(x)\right) = x n_0(x) - n_1(x) = -\cos x + \frac{\cos x}{x^2} + \frac{\sin x}{x}$$

The transcendental equation now becomes

$$\frac{(ka)^2 \sin(ka) + ka\cos(ka) - \sin(ka)}{(ka)^2\cos(ka) - ka\sin(ka) - \cos(ka)} = \frac{(kb)^2\sin(kb) + kb\cos(kb) - \sin(kb)}{(kb)^2\cos(kb) - kb\sin(kb) - \cos(kb)}$$

or $\left((ka)^2-1\right)\left((kb)^2-1\right)\sin(ka)\cos(kb)+ka\left((kb)^2-1\right)\cos(ka)\cos(kb)$

$$-kb\left((ka)^2-1\right)\sin(ka)\sin(kb)-ka\cdot kb\cos(ka)\sin(kb)$$

$$=\left((ka)^2-1\right)\left((kb)^2-1\right)\cos(ka)\sin(kb)-ka\left((kb)^2-1\right)\sin(ka)\sin(kb)$$

$$+kb\left((ka)^2-1\right)\cos(ka)\cos(kb)-ka\cdot kb\sin(ka)\cos(kb)$$

Rearranging terms, we have

$$\left((ka)^2-1\right)\left((kb)^2-1\right)\sin\left(k(a-b)\right)+ka\cdot kb\cdot\sin\left(k(a-b)\right)$$

$$=-ka\left((kb)^2-1\right)\cos\left(k(a-b)\right)+kb\left((ka)^2-1\right)\cos\left(k(a-b)\right).$$

Let $b=a+h$, we have

$$\tan(kh)=\frac{ka\left((kb)^2-1\right)-kb\left((ka)^2-1\right)}{\left((ka)^2-1\right)\left((kb)^2-1\right)+ka\cdot kb}=\frac{ka\cdot kb\cdot(kb-ka)+(kb-ka)}{ka\cdot kb+\left((ka)^2-1\right)\left((kb)^2-1\right)}$$

$$=kh\cdot\frac{ka\cdot kb+1}{ka\cdot kb+\left((ka)^2-1\right)\left((kb)^2-1\right)}=kh\cdot\frac{k^2+\frac{1}{ab}}{k^2+ab\left(k^2-\frac{1}{a^2}\right)\left(k^2-\frac{1}{b^2}\right)}$$

Finally, $$\frac{\tan(kh)}{kh}=\frac{k^2+\frac{1}{ab}}{k^2+ab\left(k^2-\frac{1}{a^2}\right)\left(k^2-\frac{1}{b^2}\right)}$$

(c) For $h\ll a$, $kh\ll 1$. We can approximate the L.H.S. of the equation above as 1, and the resulting equation is

$$k^2+\frac{1}{ab}=k^2+ab\left(k^2-\frac{1}{a^2}\right)\left(k^2-\frac{1}{b^2}\right)\Rightarrow k^4-k^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=0$$

The non-trivial solution leads to

$$k=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{1/2}\Rightarrow \omega_1\simeq\sqrt{2}\,\frac{c}{a}.$$

which is $\sqrt{l(l+1)}\,c/a$ for $l=1$. Since $b=a+h$, and $b^{-2}=a^{-2}-2ha^{-3}$, to the next order,

$$k=\left(\frac{1}{a^2}+\frac{1}{a^2}-\frac{2h}{a^3}\right)^{1/2}=\frac{\sqrt{2}}{a}\left(1-\frac{h}{a}\right)^{1/2}\simeq\frac{\sqrt{2}}{a}\left(1-\frac{h}{2a}\right)$$