16.11 (a) From our experience in Prob. 16.10, we know that we need to consider the preacceleration effect of the electric force. Using the integro-differential equation,

$$m\dot{v}(t) = \int_0^{+\infty} F(t+\tau s)\, e^{-s}\, ds,$$

and also the condition that the force $F(x)$ is only non-zero when $0<x<T$, the equation becomes

$$\dot{v}(t) = \alpha \int_{-t/\tau}^{(T-t)/\tau} e^{-s} ds = \alpha e^{t/\tau}\left(1-e^{-T/\tau}\right).$$

Here we have used the fact that the radiation reaction effect is small and T' is not very different from T. Then, the velocity of the particle at $t=0$ is given by

$$v(0) = v_0 + \int_{-\infty}^{0} \dot{v}(u)\, du = v_0 + \alpha\tau\left(1-e^{-T/\tau}\right).$$

For $t>0$, it is straightforward to show that

$$\dot{v}(t) = \alpha\int_0^{(T-t)/\tau} e^{-s} ds = \alpha\left(1-e^{-(T-t)/\tau}\right),$$

$$v(t) = v(0) + \int_0^t \dot{v}(u)\,du = v_0 + \alpha\tau\left(1-e^{-T/\tau}\right) + \alpha t + \alpha\tau\left(e^{-T/\tau} - e^{-(T-t)/\tau}\right)$$

$$x(t) = \int_0^t v(u)\,du = v_0 t + \frac{\alpha}{2}t^2 + \alpha\tau t + \alpha\tau^2\left(e^{-T/\tau} - e^{-(T-t)/\tau}\right)$$

(b) The particle exits the gap at $t=T'$, which can be determined by

$$d = v_0 T' + \frac{\alpha}{2}T'^2 + \alpha\tau T' + \alpha\tau^2\left(e^{-T/\tau} - e^{-(T-T')/\tau}\right).$$

Keep only term up to first order in τ, we have

$$\frac{\alpha}{2}T'^2 + (v_0+\alpha\tau)T' - d = 0,$$

which has a solution as

$$T' = \sqrt{\left(\frac{v_0}{\alpha}+\tau\right)^2 + \frac{2d}{\alpha}} - \left(\frac{v_0}{\alpha}+\tau\right) = T - \frac{\tau T}{\left[\left(\frac{v_0}{\alpha}\right)^2 + 2\left(\frac{d}{\alpha}\right)\right]^{1/2}},$$

where the last equality is obtained from Taylor expansion. Since

$$v_1 = \sqrt{v_0^2 + 2\alpha d}, \qquad v_1 - v_0 = \alpha T.$$

we finally have $\quad T' = T - \dfrac{\alpha\tau T}{v_1} = T - \tau\left(1-\dfrac{v_0}{v_1}\right).$

Then. $v_1' = v(T') = v_0 + \alpha T' + \alpha\tau\left(1 - e^{-(T-T')/\tau}\right)$

$$\simeq v_0 + \alpha T' = v_0 + \alpha T - \alpha\tau\left(1 - \frac{v_0}{v_1}\right) = v_1 - \frac{\alpha^2\tau}{v_1}T.$$

(c) I will only check for the 1st order approximation. The kinetic energy change is

$$\Delta T = \frac{1}{2}m(v_1'^2 - v_0^2) = \frac{1}{2}m(v_1^2 - v_0^2 - 2\alpha^2\tau T) = m\alpha d - m\alpha^2\tau T.$$

and the radiated energy is

$$\Delta E = \tau m \int_0^T \dot{v}(t)^2\,dt = \tau m\alpha^2 T.$$

The work done by the electric field is $\Delta W = m\alpha d$. Therefore, to first order,

$$\Delta W = \Delta T + \Delta E.$$