2.3 Solution: (a) The image charges are located at $(-x_0, y_0)$, $(-x_0, -y_0)$, and $(x_0, -y_0)$ with linear charge densities $-\lambda$, λ, and $-\lambda$, respectively. Then, the potential in the first quadrant is

$$\begin{aligned}\Phi(x,y) &= \frac{\lambda}{4\pi\varepsilon_0}\left[\log\left(\frac{R^2}{(x-x_0)^2+(y-y_0)^2}\right) - \log\left(\frac{R^2}{(x+x_0)^2+(y-y_0)^2}\right)\right. \\ &\qquad \left. +\log\left(\frac{R^2}{(x+x_0)^2+(y+y_0)^2}\right) - \log\left(\frac{R^2}{(x-x_0)^2+(y+y_0)^2}\right)\right].\end{aligned}$$

On the boundary surface $x = 0$, $y \geq 0$, the potential is

$$\begin{aligned}\Phi(0,y) &= \frac{\lambda}{4\pi\varepsilon_0}\left[\log\left(\frac{R^2}{x_0^2+(y-y_0)^2}\right) - \log\left(\frac{R^2}{x_0^2+(y-y_0)^2}\right)\right. \\ &\qquad \left. +\log\left(\frac{R^2}{x_0^2+(y+y_0)^2}\right) - \log\left(\frac{R^2}{x_0^2+(y+y_0)^2}\right)\right],\end{aligned}$$

which is clearly 0. The same result can be verified for the other boundary surface, $y = 0$, $x \geq 0$.

Taking the gradient of the potential, the electric field is given by

$$\begin{aligned}\mathbf{E}(x,y) &= -\nabla\Phi(x,y) \\ &= \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{x-x_0}{(x-x_0)^2+(y-y_0)^2} - \frac{x+x_0}{(x+x_0)^2+(y-y_0)^2}\right. \\ &\qquad \left. +\frac{x+x_0}{(x+x_0)^2+(y+y_0)^2} - \frac{x-x_0}{(x-x_0)^2+(y+y_0)^2}\right]\hat{\mathbf{x}} \\ &\quad +\frac{\lambda}{2\pi\varepsilon_0}\left[\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2} - \frac{y-y_0}{(x+x_0)^2+(y-y_0)^2}\right. \\ &\qquad \left. +\frac{y+y_0}{(x+x_0)^2+(y+y_0)^2} - \frac{y+y_0}{(x-x_0)^2+(y+y_0)^2}\right]\hat{\mathbf{y}}.\end{aligned}$$

On the boundary surface $x = 0$, $y \geq 0$, the tangential electric field is the component in the y direction, or

$$\begin{aligned}\mathbf{E}_T(0,y) &= \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{y-y_0}{x_0^2+(y-y_0)^2} - \frac{y-y_0}{x_0^2+(y-y_0)^2}\right. \\ &\qquad \left. +\frac{y+y_0}{x_0^2+(y+y_0)^2} - \frac{y+y_0}{x_0^2+(y+y_0)^2}\right]\hat{\mathbf{y}},\end{aligned}$$

which is obviously 0. The same result holds for the other boundary surface.

(b) The surface charge density, per unit length in z, on the plane $y = 0$, $x > 0$ can be calculated as

$$\begin{aligned}\sigma(x,0) &= -\varepsilon_0\left.\frac{\partial\Phi}{\partial n}\right|_{y=0} = -\varepsilon_0\left.\frac{\partial\Phi}{\partial y}\right|_{y=0} \\ &= \frac{\lambda}{2\pi}\left(-\frac{y_0}{(x-x_0)^2+y_0^2} + \frac{y_0}{(x+x_0)^2+y_0^2} - \frac{y_0}{(x-x_0)^2+y_0^2} + \frac{y_0}{(x+x_0)^2+y_0^2}\right)\end{aligned}$$

$$= \quad -\frac{\lambda}{\pi}\left(\frac{y_0}{(x-x_0)^2+y_0^2}-\frac{y_0}{(x+x_0)^2+y_0^2}\right).$$

(c) The total charge can be found by a direct integration,

$$\begin{aligned}
Q_x &= \int_0^\infty \sigma(x,0)dx \\
&= -\frac{\lambda}{\pi}\int_0^\infty \left(\frac{y_0}{(x-x_0)^2+y_0^2}-\frac{y_0}{(x+x_0)^2+y_0^2}\right)dx \\
&= -\frac{\lambda}{\pi}\left(\arctan\left(\frac{x-x_0}{y_0}\right)-\arctan\left(\frac{x+x_0}{y_0}\right)\right)\Bigg|_{x=0}^{\infty} \\
&= -\frac{\lambda}{\pi}\left(\frac{\pi}{2}-\arctan\left(-\frac{x_0}{y_0}\right)-\frac{\pi}{2}+\arctan\left(\frac{x_0}{y_0}\right)\right) \\
&= -\frac{2\lambda}{\pi}\arctan\left(\frac{x_0}{y_0}\right).
\end{aligned}$$

(d) Expand the logarithm in the Green function for $\rho \gg \rho_0$,

$$\begin{aligned}
& \log\left((x-x_0)^2+(y-y_0)^2\right) \\
= \quad & \log\left(\rho^2+\rho_0^2-2(xx_0+yy_0)\right) \\
= \quad & \log\rho^2+\log\left(1+\frac{\rho_0^2}{\rho^2}-\frac{2(xx_0+yy_0)}{\rho^2}\right) \\
= \quad & \log\rho^2+\frac{\rho_0^2}{\rho^2}-\frac{2(xx_0+yy_0)}{\rho^2}-\frac{1}{2}\left(\frac{\rho_0^2}{\rho^2}-\frac{2(xx_0+yy_0)}{\rho^2}\right)^2 \\
= \quad & \log\rho^2+\frac{\rho_0^2}{\rho^2}-\frac{2(xx_0+yy_0)}{\rho^2}-\frac{\rho_0^4-4\rho_0^2(xx_0+yy_0)+4(x^2x_0^2+y^2y_0^2+2(xx_0)(yy_0))}{2\rho^4},
\end{aligned}$$

$$\begin{aligned}
& \log\left((x+x_0)^2+(y-y_0)^2\right) \\
= \quad & \log\rho^2+\frac{\rho_0^2}{\rho^2}+\frac{2(xx_0-yy_0)}{\rho^2}-\frac{\rho_0^4+4\rho_0^2(xx_0-yy_0)+4(x^2x_0^2+y^2y_0^2-2(xx_0)(yy_0))}{2\rho^4},
\end{aligned}$$

$$\begin{aligned}
& \log\left((x-x_0)^2+(y+y_0)^2\right) \\
= \quad & \log\rho^2+\frac{\rho_0^2}{\rho^2}-\frac{2(xx_0-yy_0)}{\rho^2}-\frac{\rho_0^4-4\rho_0^2(xx_0-yy_0)+4(x^2x_0^2+y^2y_0^2-2(xx_0)(yy_0))}{2\rho^4},
\end{aligned}$$

$$\begin{aligned}
& \log\left((x+x_0)^2+(y+y_0)^2\right) \\
= \quad & \log\rho^2+\frac{\rho_0^2}{\rho^2}+\frac{2(xx_0+yy_0)}{\rho^2}-\frac{\rho_0^4+4\rho_0^2(xx_0+yy_0)+4(x^2x_0^2+y^2y_0^2+2(xx_0)(yy_0))}{2\rho^4},
\end{aligned}$$

In the fourth order terms, ignore those explicitly involving ρ_0, it can be shown by direct summation, that

$$\Phi \to \frac{4\lambda}{\pi\varepsilon_0}\frac{(x_0y_0)(xy)}{\rho^4}.$$