5.35. (a) From Prob. 5.13, if the current density is in the form of $\vec{K} = \vec{j} \times \vec{r}\, \delta(r-a)$, where $\vec{j}$ points in the z-direction, then the magnetic field in the sphere is $\vec{B} = \frac{2}{3}\mu a \vec{j}$. Since the magnetic induction is a constant of B_0, we know $|\vec{j}| = \frac{3B_0}{2\mu a}$, and $\vec{K} = \frac{3B_0}{2\mu}\sin\theta\, \hat{\phi}$. The vector potential is given by

$$A_\phi = \frac{\mu|\vec{j}|a^3}{3}\frac{r_<}{r_>^3}\sin\theta = \frac{B_0 a^2}{2}\frac{r_<}{r_>^2}\sin\theta.$$

(b) With the steady current density, we can see that $A_\phi e^{i\phi}$ is proportional to $Y_{11}(\theta,\phi)$. Then, we can expand it in spherical Bessel' functions,

$$-\sqrt{\frac{3}{8\pi}}e^{i\phi}A_\phi(\vec{x}, t=0) = \int_0^{+\infty} \tilde{A}(k)\, j_1(kr)\, dk \cdot Y_{11}(\theta,\phi),$$

Where

$$\tilde{A}(k) = \frac{2k^2}{\pi}\int_0^{+\infty} \frac{B_0 a^2}{2}\frac{r_<}{r_>^2} j_1(kr)\, r^2 dr$$

$$= \frac{B_0 k^2 a^2}{\pi}\left[\frac{1}{a^2}\int_0^a r^3 j_1(kr)\,dr + a\int_a^{+\infty} j_1(kr)\,dr\right]$$

$$= \frac{B_0 k^2 a^2}{\pi}\left[\frac{1}{k^4 a^2}\int_0^{ka} x^3 j_1(x)\,dx + \frac{a}{k}\int_{ka}^{+\infty} j_1(x)\,dx\right]$$

$$= \frac{B_0 k^2 a^2}{\pi}\left[-\frac{1}{k^4 a^2}\left((k^2a^2-3)\sin(ka) + 3ka\cos ka\right) + \frac{a}{k}\frac{\sin(ka)}{ka}\right]$$

$$= \frac{3B_0 a^2}{\pi}\left(\frac{\sin(ka)}{k^2a^2} - \frac{\cos(ka)}{ka}\right) = \frac{3B_0 a^2}{\pi} j_1(ka).$$

For $t>0$, if we express the vector potential as

$$-\sqrt{\frac{3}{8\pi}}e^{i\phi}A_\phi(\vec{x},t) = \int_0^{+\infty}\tilde{A}(k,t)\, j_1(kr)\, dk \cdot Y_{11}(\theta,\phi),$$

we can arrive at $\mu\sigma\frac{\partial \tilde{A}(k,t)}{\partial t} = -k^2\tilde{A}(k,t)$, with $\tilde{A}(k,0) = \tilde{A}(k)$. Here, we have use the fact that $j_1(kr)Y_{11}(\theta,\phi)$ satisfies the Helmholtz equation,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + k^2 - \frac{l(l+1)}{r^2}\right)\psi(\vec{x}) = 0,$$

with $l=1$. It is easy to see that $\tilde{A}(k,t) = \tilde{A}(k)\exp\{-k^2t/\mu\sigma\}$. Therefore

$$A_\phi(\vec{x},t) = \frac{3B_0a^2}{\pi}\sin\theta\int_0^{+\infty} e^{-k^2t/\mu\sigma} j_1(ka)\, j_1(kr)\, dk = \frac{3B_0 a}{\pi}\sin\theta\int_0^{+\infty} e^{-k^2t/\mu\sigma a^2} j_1(k)\, j_1(kr/a)\, dk.$$

Let $\nu = 1/\mu\sigma a^2$, we finally get

$$A_\phi(\vec{x}, t) = \frac{3B_0 a}{\pi} \sin\theta \int_0^{+\infty} e^{-\nu t k^2} j_1(k) j_1(kr/a)\, dk.$$

In spherical coordinates.

$$\vec{B} = \nabla \times \vec{A} = \vec{e}_r \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial r}(r A_\phi)$$

$$= \vec{e}_r \frac{6B_0 a}{\pi r} \cos\theta \int_0^{+\infty} e^{-\nu t k^2} j_1(k) j_1(kr/a)\, dk$$

$$- \vec{e}_\theta \frac{3B_0 a}{\pi r} \sin\theta \int_0^{+\infty} e^{-\nu t k^2} j_1(k) \frac{\partial}{\partial r}\left(r j_1(kr/a)\right) dk.$$

(c) The magnetic energy is given by

$$W_m^L = \frac{1}{2}\int \vec{B}\cdot\vec{H}\,d^3x = \frac{1}{2\mu}\int (\nabla\times\vec{A})\cdot(\nabla\times\vec{A})\,d^3x$$

Notice that $\nabla\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot(\nabla\times\vec{b})$, With $\vec{a}=\vec{A}$ and $\vec{b}=\nabla\times\vec{A}$, we have

$$(\nabla\times\vec{A})\cdot(\nabla\times\vec{A}) = \nabla\cdot\left(\vec{A}\times(\nabla\times\vec{A})\right) + \vec{A}\cdot\left(\nabla\times(\nabla\times\vec{A})\right) = \nabla\cdot(\vec{A}\times(\nabla\times\vec{A})) + \vec{A}\cdot\nabla(\nabla\cdot\vec{A}) - \vec{A}\cdot\nabla^2\vec{A}$$

Upon integration, the first term becomes a surface integral and is zero. Also, $\nabla\cdot\vec{A}=0$, as $\vec{A}$ has only a ϕ-component which does not depend on ϕ. Finally, $\nabla^2\vec{A} = \mu\sigma\frac{\partial\vec{A}}{\partial t}$. The magnetic energy now becomes

$$W_m = -\frac{1}{2\mu}\int \vec{A}\cdot\mu\sigma\frac{\partial\vec{A}}{\partial t}\,d^3x.$$

Using result of part (b), we can write the magnetic energy as

$$W_m = -\frac{1}{2\mu}\int d^3x\cdot\frac{3B_0a}{\pi}\sin\theta\int_0^{+\infty} e^{-\nu t k_1^2} j_1(k_1)\,j_1(k_1r/a)\,dk_1$$

$$\mu\sigma\frac{\partial}{\partial t}\left(\frac{3B_0a}{\pi}\sin\theta\int_0^{+\infty} e^{-\nu t k_2^2} j_1(k_2)\,j_1(k_2r/a)\,dk_2\right)$$

$$= \frac{1}{2\mu}\int dk_1\int dk_2\,\frac{9B_0^2a^2}{\pi^2}\,\mu\sigma\nu k_2^2\,j_1(k_1)j_1(k_2)\,e^{-\nu t(k_1^2+k_2^2)}\int d^3x\,\sin^2\theta\cdot j_1(k_1r/a)\,j_1(k_2r/a) \quad (I)$$

The last integral gives

$$\int d^3x\,\sin^2\theta\,j_1(k_1r/a)\,j_1(k_2r/a) = 2\pi\int_{-1}^{1}\sin^2\theta\,d(\cos\theta)\int_0^{+\infty} r^2 j_1(k_1r/a)\,j_1(k_2r/a)\,dr$$

$$= 2\pi\cdot\frac{4}{3}\cdot\frac{\pi}{2(k_1/a)^2}\delta\left(\frac{k_1}{a}-\frac{k_2}{a}\right) = \frac{4\pi^2}{3}\frac{a^3}{k_1^2}\delta(k_1-k_2)$$

Then, the integral (I) becomes, noticing that $\nu = 1/\mu\sigma a^2$,

$$(I) = \frac{1}{2\mu}\cdot\frac{9B_0^2a^2}{\pi^2}\cdot\frac{4\pi^2}{3}\int dk_1\int dk_2\cdot\frac{k_2^2}{a^2}\cdot\frac{a^3}{k_1^2}\delta(k_1-k_2)\;j_1(k_1)j_1(k_2)\,e^{-\nu t(k_1^2+k_2^2)}$$

$$= \frac{6B_0^2a^3}{\mu}\int_0^{+\infty} e^{-2\nu t k^2}\left[j_1(k)\right]^2dk$$

By a variable change, $k = u/\sqrt{2\nu t}$, the integral can be expressed as

$$W_m = \frac{6B_0^2a^3}{\mu}\cdot\frac{1}{\sqrt{2\nu t}}\int_0^{+\infty} e^{-u^2}\left[j_1\left(\frac{u}{\sqrt{2\nu t}}\right)\right]^2du.$$

For $x\ll1$; $j_1(x)\sim x/3$. Then, in the limit $\nu t\gg1$, the integral asymptotically becomes

$$W_m \to \frac{6B_0^2 a^3}{\mu}\frac{1}{(2\nu t)^{3/2}}\int_0^{+\infty} e^{-u^2}\frac{u^2}{9}\,du = \frac{6B_0^2 a^3}{\mu}\frac{1}{2\sqrt{2}(\nu t)^{3/2}}\frac{\sqrt{\pi}}{36} = \frac{\sqrt{2\pi}\,B_0^2 a^3}{24\mu(\nu t)^{3/2}}.$$

We could also cut off the integral at $\sim 1/\sqrt{\nu t}$ and treat $\exp\{-\nu t k^2\}$ as 1 in this integration region. We will get the same asymptotic behavior, but different prefactor

(d) Applying the same procedure as in part (c) to the vector potential, we will have

$$A_\phi = \frac{3B_0 a}{\pi}\sin\theta\frac{1}{\sqrt{\nu t}}\int_0^{+\infty} e^{-u^2} j_1\left(\frac{u}{\sqrt{\nu t}}\right) j_1\left(\frac{u}{\sqrt{\nu t}}\frac{r}{a}\right)du$$

$$= \frac{3B_0}{\pi}\sin\theta\frac{r}{(\nu t)^{3/2}}\int_0^{+\infty} e^{-u^2}\frac{u^2}{9}\,du = \frac{B_0}{12\sqrt{\pi}(\nu t)^{3/2}}\,r\sin\theta,$$

Which has the same asymptotic behavior as the magnetic energy. The magnetic field is

$$\vec{B} = \nabla\times\vec{A} = \vec{e}_r\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)$$

$$= \frac{B_0}{6\sqrt{\pi}(\nu t)^{3/2}}\left(\vec{e}_r\cos\theta - \vec{e}_\theta\sin\theta\right) = \frac{B_0}{6\sqrt{\pi}(\nu t)^{3/2}}\vec{e}_z,$$

Which is constant. This is valid as long as $\nu t \gg 1$, therefore in the sphere with radius $R = a\sqrt{\nu t} \gg a$.

For distance larger than R, we cannot apply the approximation here. and the magnetic field is still the same. This can be understood as when we turn off the current, the decay of the magnetic field needs time to propagate.