Local and Robust Self testing using trapped ions

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Abstract

We provide experimental implementation of KCBS self-testing scheme. Theoretical tools are supplemented to render the results in *Physical Review Letters 122 (25), 250403* practical.

1 Introduction

1.1 Why self-testing

1.1.1 State of the art

- Bell/Device independence | Tensor structure
  - Limitation: Difficult to enforce etc
  - Curve less robust to noise
  - Math is harder
- Semi device independent
  - Easier to implement
  - EPR steering [K]; almost like blind computing
- All bipartite entangled states can be self-tested by quantum steering | EPR steering

1.1.2 Local Self testing

Motivation: Device certification.

- Computation, which is usually local, it is inconvenient also less secure to trust another party etc.

State of the art

- Vidick et al
- Kishor's et al article on self-testing. Robustness curve was given up to multiplicative constants. Not directly implementable.

1.1.3 Contextuality

Informal quick introduction

1.1.4 Contribution

Address limitations of Kishor et al's article

- theoretically obtained the robustness curve
  - Moment matrices based approach
  - First robustness curve for a non-contextuality inequality (five cycles no less!)
- experimental demonstration of local self-testing

1.1.5 Assumptions and Memory leakage :P

Self testing works under the following.

1.1.6 Circumventing the memory assumption

- Memory based
  - Quantum classical gap?
- Dimensions based measures
  - Quantum classical gap
  - log[classical dimensions]=memory; quantum dimension = quantum memory

Criticism:

BQP$\subseteq$PSPACE; that's fine we hope to find the exact curve to certify

- Poly separation or some such.

2 Background

2.1 Exclusivity graph approach to contextuality

2.2 Robust self-testing

- Definition
- Statement for KCBS [PRL]
- Robustness curve is missing! Or is it?

3 Robustness Curve

[DISCLAIMER: I am writing the following as rough notes which may ramble but should at least be consistent; in the following iteration, I hope to make improve the presentation]

What we have is some experimental data

3.1 Description

TODO:

- IID
- Restricted subspace or full subspace, Bell
- Parallel or Serial
  - When IID, parallel and serial should become identical
  - When not IID, parallel should be more general
- Clarify the issue with the sum
- NOTATIONAL
  - Difference between a "configuration" and a "realisation".

Definition 1 (Experimental Scenario)**.**

Definition 2 (Quantum Realisation)**.**

Consider the KCBS scenario, i.e. an experimental scenario which is specified by the events $e_1, e_2 \dots e_5$ whose exclusivity is given by a 5-cycle exclusivity graph. Suppose we obtain the probabilities $p_1, p_2 \dots p_5$ experimentally (and ensure that they correspond to sharp measurements). We already saw that if these values nearly saturate the quantum bound for the KCBS non-contextuality inequality, i.e. $\sum_{i=1}^5 p_i$ approaches $\sqrt{5} \approx 2.236\dots$, then we know that all quantum realisations corresponding to the experimentally obtained probabilities $\{p_i\}_{i=1}^5$ are almost equivalent to $\rho^{\text{KCBS}}, \{\Pi_i^{\text{KCBS}}\}_{i=1}^5$, up to a global isometry. To be more precise, consider any arbitrary quantum realisation given by a pure state ρ and rank one projectors $\{\Pi_i\}$ such that $\sum_{i=1}^5 p_i = 2 + \epsilon$. It was shown in [1] that then, there exists an isometry V such that

$$\left\| V \Pi_i V^\dagger - \Pi_i^{\text{KCBS}} \right\|_F \leq \mathcal{O}(\sqrt{\epsilon}) \tag{3.1}$$

for all $i \in \{1, 2 \dots 5\}$ and $\left\| \rho - \rho^{\text{KCBS}} \right\|_F \leq \mathcal{O}(\sqrt{\epsilon})$ where $\|A\|_F := \operatorname{tr}\left(\sqrt{A^\dagger A}\right)$. Despite this, without the constant hidden in Equation 3.1, one cannot obtain a robustness curve and thus one cannot apply this in practice. In this work, we remedy this problem, taking inspiration from the Bell self-testing approach. To this end, we give a slightly different statement: we show that

$$\sum_{i=1}^{5} \mathcal{F}(V \Pi_i \rho \Pi_i V^\dagger, \Pi_i^{\text{KCBS}} \rho^{\text{KCBS}} \Pi_i^{\text{KCBS}}) + \mathcal{F}(V \rho V^\dagger, \rho^{\text{KCBS}}) \geq f(\epsilon) \tag{3.2}$$

where $\mathcal{F}(A, B) := \operatorname{tr}\sqrt{\left|A^{1/2} B^{1/2}\right|}$, we only require Π_i to be projectors (such that $[\Pi_i, \Pi_j] = 0$ if (i, j) is an edge of the exclusivity graph) and allow ρ to be a mixed state. The advantage is that we are able to express $f(\epsilon)$ as a hierarchy of semi-definite programmes and compute lower bounds explicitly. While we state our result for the KCBS inequality, it readily extends to the n-cycle scenario. [TODO: if f cannot be shown to be independent of the individual p_is, then we must skip it].

3.2 Overall Strategy

Definition 3 (An ideal KCBS configuration)**.** Consider a three-dimensional Hilbert space spanned by the basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Let

$$|u_l\rangle := \cos\theta\,|0\rangle + \sin\theta\sin\phi_l\,|1\rangle + \sin\theta\cos\phi_l\,|2\rangle$$

where $\phi_l := l\pi(n-1)/n$ for $1 \le l \le n$. Define

$$\begin{aligned}\left|\psi^{\mathrm{KCBS}}\right\rangle &:= |0\rangle \\ \Pi_i^{\mathrm{KCBS}} &:= |u_i\rangle\langle u_i|.\end{aligned}$$

TODO

- ~~Verify if we require that $\Pi_i\Pi_j = 0$ or that $\mathrm{tr}[\Pi_i\Pi_j\rho] = 0$. $\mathrm{tr}[\Pi_i\Pi_j\Pi_k\rho]$ (by assumption; write clearly in the previous section); [status: we handled this already]~~
- Update: say that $\rho = |\psi\rangle\langle\psi|$ without loss of generality

We may restate the aforesaid discussion more symbolically as lower bounding the value of the following objective function:

$$F := \min_{\rho,\{\Pi_i\}} \max_{V} \left[\sum_{i=1}^{5} \mathcal{F}(V\Pi_i\rho\Pi_i V^\dagger, \Pi_i^{\mathrm{KCBS}}\rho^{\mathrm{KCBS}}\Pi_i^{\mathrm{KCBS}}) + \mathcal{F}(V\rho V^\dagger, \rho^{\mathrm{KCBS}}) \right] \tag{3.3}$$

where $\rho, \{\Pi_i\}_{i=1}^5$ is a quantum realisation[1] of $\{p_i\}_{i=1}^5$, V is an isometry from $\mathcal{H}$ to $\mathcal{H}^{\mathrm{KCBS}}$, i.e. from the space on which $\rho, \{\Pi_i\}_{i=1}^5$ act/are defined to that where $\rho^{\mathrm{KCBS}}, \{\Pi_i^{\mathrm{KCBS}}\}_{i=1}^5$ act/are defined. At the broadest level, the idea is to drop the maximization over V and replace it with a particular isometry V which is expressed in terms of $\rho, \{\Pi_i\}_{i=1}^5$. Then, as we shall see, the expression for the fidelity appears as a sum of terms of the following form. Let w be a word created from the letters, $\{\mathbb{I}, \Pi_1, \Pi_2 \dots \Pi_5, \hat{P}\}$ with $\hat{P}^\dagger\hat{P} = \mathbb{I}$, $\Pi_i^2 = \Pi_i$ and $\Pi_i\Pi_j = 0$ if $(i,j) \in E(G)$, i.e. when i,j are exclusive[2]. The fidelity is a linear combination of these words, i.e. $F = \min_{\{\langle w\rangle\}} \sum_w \alpha_w \langle w\rangle$ where $\mathrm{tr}[w\rho] =: \langle w\rangle$, subject to the constraint that $\{\langle w\rangle\}_w$ corresponds to a quantum realisation. The advantage of casting the problem in this form is that one can now construct an NPA-like hierarchy. The idea is simple to state. Treat $\{\langle w\rangle\}_w$ as a vector. Denote by Q the set of all such vectors which correspond to a quantum realisation (of $\{p_i\}_{i=1}^5$). It turns out that one can impose constraints on words with k letters, for instance. Under these constraints, denote by Q_k the set that is obtained. Note that $Q_k \supseteq Q$ for it may contain vectors which don't correspond to the quantum realisation. In fact, Q_k can be characterised using semi-definite programming constraints (which in turn means they are efficiently computable). Intuitively, it is clear that $\lim_{k\to\infty} Q_k = Q$. Further, it is also clear that $F = \min_{\{\langle w\rangle\}_w \in Q} \sum_w \alpha_w \langle w\rangle \ge \min_{\{\langle w\rangle\}_w \in Q_k} \sum_w \alpha_w \langle w\rangle$ as we are minimising over a larger set on the right hand side.

3.3 Providing a lower bound on the fidelity from regularized measurement statistics

- TODO
  - ~~[done] To conclude that $\sum_{k=0}^{n-1} |k\rangle\langle k| \otimes P^{-k}$ is an isometry, we need to assume rank one projectors.~~
  - $\mathcal{G}$

[1] note that we assume Π_i are projectors as the measurements are assumed to be sharp experimentally

[2] We introduced $\hat{P}$ for completeness; its role is explained later.

For concreteness, we first consider a unitary U_{SWAP} instead of an isometry, which acts on two spaces $\mathcal{A}$ and $\mathcal{A}'$. For simplicity, suppose that the $\mathcal{A}$ register is in the state

$$\sigma \in \underbrace{\{\left|\psi^{\text{KCBS}}\right\rangle \left\langle\psi^{\text{KCBS}}\right|\} \cup \{\Pi_i^{\text{KCBS}}\}_{i=1}^5}_{:=\mathcal{S}^{\text{KCBS}}}.$$

We want U_{SWAP} to map $\sigma_{\mathcal{A}} \otimes |0\rangle\langle 0|_{\mathcal{A}'}$ to $|0\rangle\langle 0|_{\mathcal{A}} \otimes \sigma_{\mathcal{A}'}$ (see Figure 3.1). Our strategy is to construct U_{SWAP} in this seemingly trivial case and then express it in terms of the state and measurement operators on $\mathcal{A}$. The rationale is that by construction, U_{SWAP} will work for the ideal case and therefore should also work for cases close to ideal. This should become clear momentarily. Note that any circuit that swaps two qutrits should let us achieve our simplified goal (because all elements of S^{KCBS} are defined on a three dimensional Hilbert space). One possible qutrit swapping unitary/circuit (a special case of the general qudit swapping unitary/circuit defined in [??], an article about self-testing Bell inequalities) may be defined as $S''_{\text{SWAP}} := TUVU$ where

$$\begin{aligned}
T &:= \mathbb{I}_{\mathcal{A}} \otimes \sum_{k=0}^{2} |-k\rangle\langle k|_{\mathcal{A}'} \\
U &:= \sum_{k=0}^{2} P_{\mathcal{A}}^{k} \otimes |k\rangle\langle k|_{\mathcal{A}'} \\
V &:= \sum_{k=0}^{2} |\bar{k}\rangle\langle\bar{k}|_{\mathcal{A}} \otimes P_{\mathcal{A}'}^{-k}
\end{aligned} \tag{3.4}$$

where $P := \sum_{i=0}^{2} \left|\overline{k+1 \mod 3}\right\rangle\left\langle\bar{k}\right|$ is a translation operator and $\{|\bar{0}\rangle, |\bar{1}\rangle, |\bar{2}\rangle\}$ is a basis for the qutrit space [TODO: fix the bar issue]. We omit the proof here (for a proof see ...). To generalise this idea and to construct an isometry, we relax the assumption that $\mathcal{A}$ is a three dimensional Hilbert space. We re-express/replace the operations in T, U, V which act on the $\mathcal{A}$ space by linear combinations of monomials in $\{\Pi_i\}_{i=1}^5$. We obtain the coefficients used in these linear combinations by assuming the space is three dimensional. The idea is simply that this map reduces to a swap operation when we re-impose the assumptions and for cases close to it, we expect it to behave appropriately. (TODO: maybe add the circuit for bell inequality to explain at some point?) We describe this procedure more precisely below.

Algorithm 4 (Constructing an isometry)**.** *Let*

- $\mathcal{A}'$ *be a three dimensional Hilbert space spanned by an orthonormal basis* $\{|0\rangle_{\mathcal{A}'}, |1\rangle_{\mathcal{A}'}, |2\rangle_{\mathcal{A}'}\}$,
- $\rho^{KCBS}, \{\Pi_i^{KCBS}\}_{i=1}^5$ *be an ideal quantum realisation (see ...) on* $\mathcal{A}'$,
- $\mathcal{A}$ *be a Hilbert space with dimension at least* 3 *containing orthonormal vectors* $\{|\bar{0}\rangle_{\mathcal{A}}, |\bar{1}\rangle_{\mathcal{A}}, |\bar{2}\rangle_{\mathcal{A}}\}$
- $\rho, \{\Pi_i\}_{i=1}^5$ *be an arbitrary quantum realisation on defined on* $\mathcal{A}$.

Define T, U, V *as in Equation 3.4 and let* $S'_{\text{SWAP}} := TUVU$ *with the following changes. Let*

$$\mathcal{W}^{KCBS} := \{\{\Pi_i^{KCBS}\}_{i=1}^5, \{\Pi_i^{KCBS}\Pi_j^{KCBS}\}_{i,j=1}^5, \dots\}$$

and

$$\mathcal{W} := \{\{\Pi_i\}_{i=1}^5, \{\Pi_i\Pi_j\}_{i,j=1}^5 \dots\}$$

be the set of "words" formed by the KCBS projectors and those of the arbitrary quantum realisation, respectively.

1. *Translation Operator:*
   (a) *Express $P_{\mathcal{A}'}$ as a linear combination of elements in $\mathcal{W}$, i.e. $P_{\mathcal{A}'} = \sum_{l^{KCBS} \in \mathcal{W}^{KCBS}} \alpha_l l^{KCBS}$.*
   (b) *Define $P_{\mathcal{A}} := \sum_{l \in \mathcal{W}} \alpha_l l$*
2. *Basis projectors: Formally replace, in V, the operators*
   (a) *$|\bar{0}\rangle \langle\bar{0}|_{\mathcal{A}}$ by Π_1,*
   (b) *$|\bar{1}\rangle \langle\bar{1}|_{\mathcal{A}}$ by Π_2 and,*
   (c) *$|\bar{2}\rangle \langle\bar{2}|_{\mathcal{A}}$ by $(\mathbb{I} - \Pi_1)(\mathbb{I} - \Pi_2)$*

   i.e. V now becomes $\Pi_1 \otimes \mathbb{I}_{\mathcal{A}'} + \Pi_2 \otimes P_{\mathcal{A}'}^{-1} + (\mathbb{I} - \Pi_1)(\mathbb{I} - \Pi_2) \otimes P_{\mathcal{A}'}^{-2}$.

We do not prove but [**CHECK] it is known that the linear combinations required in Algorithm 4 always exist, i.e. the algorithm always succeeds at constructing S'_{SWAP}. We must show that S'_{SWAP} is in fact an isometry. This is important because of the following reason. Recall that our objective was to lower bound Equation 3.3. To this end, we said we drop the maximization over all possible Vs, (for a given quantum realisation $\rho, \{\Pi_i\}_{i=1}^5$) and instead insert a specific isometry S'_{SWAP} (which is a function of $\rho, \{\Pi_i\}_{i=1}^5$). Note that this argument for lower bounding Equation 3.3 breaks if S'_{SWAP} is not an isometry. In fact, $P_{\mathcal{A}}$ as produced by the algorithm is not necessarily unitary (viz. $P_{\mathcal{A}}^\dagger P_{\mathcal{A}} = \mathbb{I}_{\mathcal{A}}$ may not hold). Following [??] we use the so-called localising matrix technique, which we discuss in some detail later. We introduce a new unitary matrix $\hat{P}_{\mathcal{A}}$ satisfying $\hat{P}P \geq 0$ where we dropped the subscript for clarity. Consider the case where P is not unitary. In that case, one can use polar decomposition to write $P = |P|U$ (not to be confused with the U above; where $|P| \geq 0$ and $U^\dagger U = \mathbb{I}$) so choosing $\hat{P} = U^\dagger$ satisfies $\hat{P}P \geq 0$. Thus for each P, the constraint can be satisfied. Consider the other case, i.e. where $P = U$ is unitary. Then,[3] $\hat{P} = U^\dagger$. Thus, in the ideal case we recover the same unitary and for the case close to ideal, we are guaranteed that there is some solution (which we expect should also work reasonably). Combining these, we can construct U_{SWAP} which is an isometry.

Lemma 5 (U_{SWAP} is indeed an Isometry)**.** *Let S'_{SWAP} be the map produced by Algorithm 4 and define S_{SWAP} to be S'_{SWAP} with $P_{\mathcal{A}}$ replaced by $\hat{P}_{\mathcal{A}}$. Then S_{SWAP} is an isometry if the following conditions hold*

$$\begin{aligned} \hat{P}_{\mathcal{A}}^\dagger \hat{P}_{\mathcal{A}} &= \mathbb{I}_{\mathcal{A}}, \\ \hat{P}_{\mathcal{A}} P_{\mathcal{A}} &\geq 0. \end{aligned}$$

Proof. It suffices to show that $\langle\psi|_{\mathcal{A}'\mathcal{A}} S_{\mathrm{SWAP}}^\dagger S_{\mathrm{SWAP}} |\psi\rangle_{\mathcal{A}'\mathcal{A}} = 1$ for all normalised $|\psi\rangle_{\mathcal{A}'\mathcal{A}}$. We express $S_{\mathrm{SWAP}} = TUVU$ where T is as in Equation 3.4, $U := \sum_{k=0}^2 \hat{P}_{\mathcal{A}}^k \otimes |\bar{k}\rangle \langle\bar{k}|_{\mathcal{A}'}$ and $V := \Pi_1 \otimes \mathbb{I}_{\mathcal{A}'} + \Pi_2 \otimes P_{\mathcal{A}'}^{-1} + (\mathbb{I} - \Pi_1)(\mathbb{I} - \Pi_2) \otimes P_{\mathcal{A}'}^{-2}$. Observe that $T^\dagger T = \mathbb{I}_{\mathcal{A}\mathcal{A}'} = U^\dagger U$ since $\hat{P}_{\mathcal{A}}^\dagger \hat{P}_{\mathcal{A}} = \mathbb{I}_{\mathcal{A}}$. Further, we have that

$$\begin{aligned} V^\dagger V &= \Pi_1 \otimes \mathbb{I}_{\mathcal{A}'} + \Pi_2 \otimes \mathbb{I}_{\mathcal{A}'} + (\mathbb{I} - \Pi_1)(\mathbb{I} - \Pi_2) \otimes \mathbb{I}_{\mathcal{A}'} \qquad \because \Pi_1\Pi_2 = 0, P_{\mathcal{A}'}^\dagger P_{\mathcal{A}'} = \mathbb{I}_{\mathcal{A}'} \\ &= \mathbb{I}_{\mathcal{A}\mathcal{A}'}. \end{aligned}$$

Hence, $S_{\mathrm{SWAP}}^\dagger S_{\mathrm{SWAP}} = \mathbb{I}_{\mathcal{A}\mathcal{A}'}$ establishing that U_{SWAP} is in fact unitary and thus also an isometry. □

We can now combine all the pieces to write the final optimisation problem we solve.

[3] [TODO: verify the reasoning once again] To see this, write the polar decomposition of $\hat{P} = \left|\hat{P}\right| E$ (where $\left|\hat{P}\right| \geq 0$ and $E^\dagger E = \mathbb{I}$). Then $\hat{P}U = \left|\hat{P}\right| EU$ which is a polar decomposition of $\hat{P}U$. The polar decomposition of a positive semi-definite matrix M always has the form $M.\mathbb{I}$. Using $M = \hat{P}U$, and identifying EU with $\mathbb{I}$, we have $E = U^\dagger$.

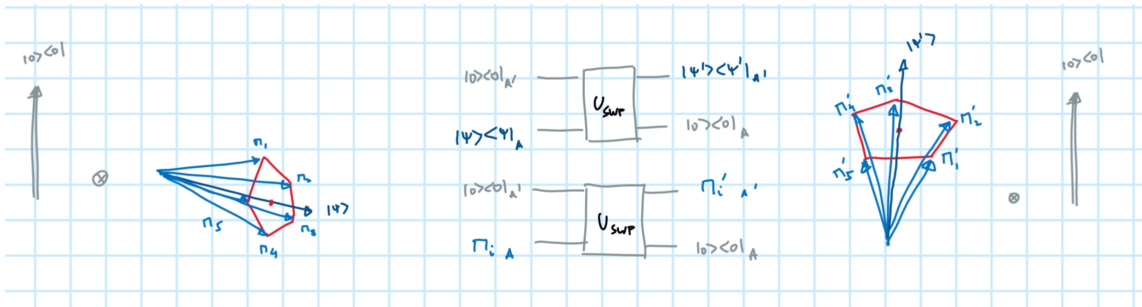

Figure 3.1: Illustration of U_{SWAP}.

Algorithm 6 (The SDP for lower bounding the fidelity of a given realisation with the ideal KCBS realisation)**.** *The algorithm proceeds in two parts.*
Notation: *Let*

- $\mathcal{A}$ *represent an arbitrary Hilbert space and* $\mathcal{A}'$ *represent a three dimensional Hilbert space,*
- $\bar{\mathcal{W}}$ *be the set of "words" constructed using* $\{\Pi_i\}_{i=1}^5$, $\hat{P}$ *and* $\hat{P}^\dagger$, *i.e.* $\bar{\mathcal{W}} = \mathcal{G}(\{\Pi_i\}_{i=1}^5, \hat{P}, \hat{P}^\dagger)$, *and*
- *define* $\langle w \rangle := tr_{\mathcal{A}}(\rho w)$ *and assume* $\rho = |\psi\rangle\langle\psi|$ *so that* $\langle w \rangle = \langle\psi| w |\psi\rangle$ *for all* $w \in \mathcal{W}$.

Input:

- *(implicit)* $\Pi_i.\Pi_j = 0$ *for all* $i, j \in E(G)$ *where* G *is a five cycle graph and* E *are its edges, indexed* $[1, 2, \ldots 5]$
- *Expectation value of the KCBS operator:* $\frac{1}{2}\sum_{i,j\in E(G)} \Pr[\Pi_i = 1|\Pi_j = 0] =: c$

Evaluation | Part 1.

- *Evaluate* S_{SWAP} *as described in Lemma 5 using Algorithm 4.*
- *Define the objective function* f *as*

$$f\left(\rho, \{\Pi_i\}_{i=1}^5\right) = \sum_{i=1}^{5} \mathcal{F}\left(tr_{\mathcal{A}}\left[S_{\text{SWAP}}\left(\Pi_i \rho_{\mathcal{A}} \Pi_i \otimes |0\rangle\langle 0|_{\mathcal{A}'}\right) S_{\text{SWAP}}^\dagger\right], \Pi_i^{KCBS} \rho_{\mathcal{A}'}^{KCBS} \Pi_i^{KCBS}\right)$$
$$+ \mathcal{F}\left(tr_{\mathcal{A}}\left[S_{\text{SWAP}}\left(\rho_{\mathcal{A}} \otimes |0\rangle\langle 0|_{\mathcal{A}'}\right) S_{\text{SWAP}}^\dagger\right], \rho_{\mathcal{A}'}^{KCBS}\right)$$

  and evaluate the coefficients f_w *so that*

$$f = \sum_{w\in\mathcal{W}} f_w \langle w \rangle .$$

- *Again, from Algorithm 4, evaluate* $P_{\mathcal{A}}$ *as a linear combination of* $\langle w \rangle$.

Evaluation | Part 2. *Solve the following SDP.*

$$
\begin{aligned}
F_{noiseless} = \min_{\{\langle w\rangle\}_{w\in\mathcal{W}}} & \sum_{w\in\mathcal{W}} f_w \langle w\rangle \\
\text{s.t.} \quad & \Gamma^{(k)}(\mathbb{I}) \geq 0 && \because \textit{ all gram matrices are } \geq 0 \\
& \Gamma^{(k)}(\mathbb{I})_{v,w} = \Gamma^{(k)}(\mathbb{I})_{v',w'} && \textit{if } v^\dagger w = v'^\dagger w' \\
& \Gamma^{(k)}(\hat{P}_{\mathcal{A}} P_{\mathcal{A}}) \geq 0 && \textit{(localising matrix)} \\
& \Gamma^{(k)}(\hat{P}_{\mathcal{A}} P_{\mathcal{A}})_{v,w} = \Gamma^{(k)}(\hat{P}_{\mathcal{A}} P_{\mathcal{A}})_{v',w'} && \textit{if } v^\dagger \hat{P}_{\mathcal{A}} P_{\mathcal{A}} w = v'^\dagger \hat{P}_{\mathcal{A}} P_{\mathcal{A}} w' \\
& \sum_{i=1}^{5} \langle \Pi_i \rangle = c && \textit{(observed statistic)}
\end{aligned}
$$

where $\Gamma^{(k)}(X)$ *is a matrix which is*

- *indexed by letters* w
- *whose matrix elements are given by* $\Gamma^{(k)}(X)_{v,w} = \langle\psi| v^\dagger X w |\psi\rangle$,
- *where* k *defines the maximum number of letters that appear in the words* w *which index the matrix* $\Gamma^{(k)}$, *and,*

where in the first two equality constraints, we use the following relations

- $\Pi_i.\Pi_j = 0$ *for all* $i, j \in E(G)$ *and*
- $\hat{P}_{\mathcal{A}}^\dagger \hat{P}_{\mathcal{A}} = \hat{P}_{\mathcal{A}} \hat{P}_{\mathcal{A}}^\dagger = \mathbb{I}_{\mathcal{A}}$.

Output: $F_{noiseless}(c)$.

Remark 7. Note that $\hat{P}P \geq 0 \implies \Gamma^{(k)}(\hat{P}P) \geq 0$. This follows readily by letting $A^\dagger A = \hat{P}P$ for some A (which must exist for any positive semi-definite matrix; one can use spectral decomposition). Then $\Gamma^{(k)}(A^\dagger A)$ is a gram matrix and thus ≥ 0.

Remark 8. TODO: explain why restricting to real $\langle\psi| w |\psi\rangle$ is enough.

We thus have an algorithm which can calculate the required lower bound on fidelity, given the observed value of the KCBS operator. In practice, however, the assumption $\Pi_i.\Pi_j = 0$ for all $i, j \in E(G)$ cannot be met. We discuss and remedy this next.

4 Connection with the experiment

KISHOR: Move these things to the introduction/preliminaries as needed; I am just writing these here to be consistent and clear.

4.1 The Ideal Case

We revisit the notion of exclusivity. Recall that we defined exclusivity at the operational level—at the level of preparations and operations (measurements are special cases with more than one outcome) which in quantum theory correspond to states and unitary evolution (or measurements). We said that two events e_1 and e_2 are exclusive if there is some measurement M for which the events correspond to different outcomes of the measurement. To a set of events $\{e_1, e_2 \dots e_5\}$ we associated what we called an exclusivity graph, G, formed by treating the events as vertices and their exclusivity relations as edges. We defined a behaviour as a map from events to probabilities such that $\Pr[e_i] + \Pr[e_j] \leq 1$ whenever $(i, j) \in E(G)$. We defined non-contextual behaviours as those which admit a non-contextual completion (details ...). We defined a quantum behaviour to be one which can be realised using a quantum state and measurements.

Definition 9 (Quantum Behaviour)**.** There exist projectors Π_i associated with each event e_i such that $\Pi_i.\Pi_j = 0$ or equivalently $\mathrm{tr}(\Pi_i.\Pi_j) = 0$ whenever $(i,j) \in E(G)$ and that $\Pr[e_i] = \mathrm{tr}[\rho\Pi_i]$ for some fixed quantum state ρ for each i.

This we already knew. Let us now restrict to our case of interest, the pentagonal exclusivity graph, i.e. the 5-cycle graph. Observe that the notion of exclusivity as defined does not exactly specify the experimental arrangement—it only assigns projectors to certain events but does not precisely specify the measurements which constitute the events (only its existence is supposed). We now describe, again at an operational level, one possible process which corresponds to the pentagonal exclusivity graph.

Definition 10 ((Operational) KCBS process)**.** Consider a preparation ρ (we use quantum symbols to be suggestive) and five measurements $\{\Pi_i^{\mathrm{exp}}\}_{i=1}^5$ with 0/1 outcomes such that $\Pi_i^{\mathrm{exp}}, \Pi_{i+1}^{\mathrm{exp}}$ are compatible for $i \in \{1, 2 \dots 5\}$; here the sum $i+1$ is modulo 5 plus 1. We define

$$e_i := (1, 0|\Pi_i^{\mathrm{exp}}, \Pi_{i+1}^{\mathrm{exp}})$$

that is the outcome is $1, 0$ when Π_i^{exp} and Π_{i+1}^{exp} are measured. We call this process the *KCBS process*. Denote by $\Pr[e_i]$ the probability assigned to the event e_i by an appropriate probabilistic model and call $(\Pr[e_i])_i$ the *behaviour* corresponding to the KCBS process.

Remark 11. Note, for instance, that e_1 and e_2 are indeed exclusive because the outcome of measuring Π_2^{exp} is 0 for e_1 and 1 for e_2.

It is not hard to see (as we shall explicitly observe) that when this KCBS process is governed by quantum theory, the resulting behaviour is just the quantum behaviour for the pentagonal exclusivity graph. We formally define quantum KCBS processes for clarity and state the aforementioned as a lemma.

Definition 12 (Quantum KCBS process)**.** A KCBS process (see Definition 10) where quantum theory governs the probabilities, $\{\Pi_i^{\mathrm{exp}}\}_i$ are projectors (not necessarily rank 1) and ρ is a density matrix both defined on an arbitrary but fixed Hilbert space.

Remark 13. We could assume, without loss of generality, that the measurements are projective (due to Naimark's theorem). However, even then, one could not deduce that the $\Pr(1, 1|\Pi_1^{\mathrm{exp}}, \Pi_1^{\mathrm{exp}}) = \mathrm{tr}(\Pi_1^{\mathrm{exp}}\Pi_1^{\mathrm{exp}}\rho\Pi_1^{\mathrm{exp}}\Pi_1^{\mathrm{exp}})$ because this would, in particular, entail that the measurement is always repeatable which it needn't be. The theorem holds, but only for one measurement; not sequential measurements as we require. Thus, the quantum KCBS process as stated, is not the most general quantum realisation of the KCBS process. In this article, we make this assumption and leave its relaxation to future work.

Lemma 14. *The set of behaviours of quantum KCBS processes equals the set of quantum behaviours for the pentagonal exclusivity graph.*

Proof. We start with showing that every behaviour of a quantum KCBS process can be cast as a quantum behaviour for the pentagonal exclusivity graph. Define

$$\Pi_i := \Pi_i^{\mathrm{exp}}(\mathbb{I} - \Pi_{i+1}^{\mathrm{exp}})\Pi_i^{\mathrm{exp}}$$

so that $(\Pi_i)_i$, ρ define a quantum behaviour such that

$$\Pr[e_i] = \mathrm{tr}[\rho\Pi_i]$$

as required. As a sanity check, note, for instance, that

$$\begin{aligned}
\Pi_1\Pi_2 &= \Pi_1^{\mathrm{exp}}(\mathbb{I} - \Pi_2^{\mathrm{exp}})\Pi_1^{\mathrm{exp}}\Pi_2^{\mathrm{exp}}(\mathbb{I} - \Pi_3^{\mathrm{exp}})\Pi_2^{\mathrm{exp}} \\
&= \Pi_1^{\mathrm{exp}}\cancelto{0}{(\mathbb{I} - \Pi_2^{\mathrm{exp}})\Pi_2^{\mathrm{exp}}}\Pi_1^{\mathrm{exp}}(\mathbb{I} - \Pi_3^{\mathrm{exp}})\Pi_2^{\mathrm{exp}} \\
&= 0
\end{aligned}$$

because $[\Pi_1^{\exp}, \Pi_2^{\exp}] = 0$ (since the measurements are compatible).

We now show that every quantum behaviour can be realised as a quantum KCBS process. Suppose the quantum behaviour is defined using $(\Pi_i)_i, \rho$ and let $\Pi_i^{\exp} := \Pi_i$. It follows that $(\Pi_i^{\exp})_i, \rho$ define a quantum KCBS process because $[\Pi_i^{\exp}, \Pi_j^{\exp}] = 0$ for all $(i,j) \in E(G)$ (since $\Pi_i.\Pi_j = 0$) where G is the pentagonal graph and $\mathrm{tr}(\rho\Pi_i^{\exp}(\mathbb{I} - \Pi_i^{\exp})\Pi_i^{\exp}) = \mathrm{tr}(\rho\Pi_i^{\exp}) = \Pr[e_i]$. □

The analogous statement for the classical case also holds.

Lemma 15. *The set of behaviours of classical KCBS processes equals the set of non-contextual behaviours for the pentagonal exclusivity graph.*

Proof. Idea: TODO: Kishor has to complete this; I hope it is trivial

- $[\Pi_i^{\exp}, \Pi_j^{\exp}] = 0$ for all $i, j \in V(G)$; $|\psi\rangle$ is arbitrary.
- $|\psi\rangle$ s.t. $\Pi_i |\psi\rangle = |\psi\rangle$.

□

To summarise, we defined the KCBS process to bridge the gap between the exclusivity graph approach and the experiment being performed in the lab. Note that both are, initially, defined at the operational level. We saw that they give rise to the same set of classical and quantum behaviours. The advantage of defining the KCBS process is two-fold. (1) It moves us closer to a device independent description and, as we shall see, (2) it facilitates the handling of experimental imperfections insofar as the enforcement of the underlying assumptions is concerned. [TODO: swap 1 and 2] It also clarifies that the fidelity we calculate is for the abstract projectors $(\Pi_i)_i$ which are related to, but not exactly equal to, the projectors $(\Pi_i^{\exp})_i$ which model the quantum KCBS process, i.e. the experiment (under the sharpness assumption; see Remark 13).

4.2 Experimental Imperfections

Two issues can plague an experimental realisation of a quantum KCBS process (see Definition 12). First, as already alluded to in Remark 13, the measurements may not be exactly repeatable which means that one cannot assume them to be projectors. In this work, we do not address this problem. Second, the measurements may not be exactly compatible, which may manifest as $p(a, b|x, y)$ being only approximately equal to $p(b, a|y, x)$. In the following, we discuss one way of handling this imperfection.

4.3 SDP with non-orthogonal projectors

Suppose we are given a quantum KCBS process (see Definition 12) and to it we associate a quantum behaviour (see Definition 9) as in the proof of Lemma 14, i.e. we identify $\Pi_i = \Pi_i^{\exp}(\mathbb{I} - \Pi_j^{\exp})\Pi_i^{\exp}$. We saw (again in the proof) that $[\Pi_i^{\exp}, \Pi_j^{\exp}] = 0 \implies \Pi_i.\Pi_j = 0$. It is easy to see that $\|\Pi_i.\Pi_j\| \leq \left\|[\Pi_i^{\exp}, \Pi_j^{\exp}\right\|$ by using the triangle inequality and noting that $\|\Pi_i^{\exp}\| \leq 1$ (and the norm we use is the one induced by the vector norm). We now lower bound the result of the SDP in Algorithm 6 when the requirement $\Pi_i.\Pi_j = 0$ is replaced with $\|\Pi_i.\Pi_j\| \leq \epsilon$.

Definition 16. Let $F_{\mathrm{nonO}}(c)$ be the output of Algorithm 6 where "Evaluation, part 2" (the SDP part) is changed as follows (by non-O we mean non-orthogonal).

- The first equality constraint, $\Gamma^{(k)}(\mathbb{I})_{v,w} = \Gamma^{(k)}(\mathbb{I})_{v',w'}$ is imposed when $v^\dagger w = v'^\dagger w'$ as stated except that we do not assume $\Pi_i.\Pi_j = 0$.
- Similarly, the second constraint, $\Gamma^{(k)}(\hat{P}_\mathcal{A} P_\mathcal{A})_{v,w} = \Gamma^{(k)}(\hat{P}_\mathcal{A} P_\mathcal{A})_{v',w'}$ is also imposed when $v^\dagger \hat{P}_\mathcal{A} P_\mathcal{A} w = v'^\dagger \hat{P}_\mathcal{A} P_\mathcal{A} w'$ as stated except that we do not assume $\Pi_i.\Pi_j = 0$.

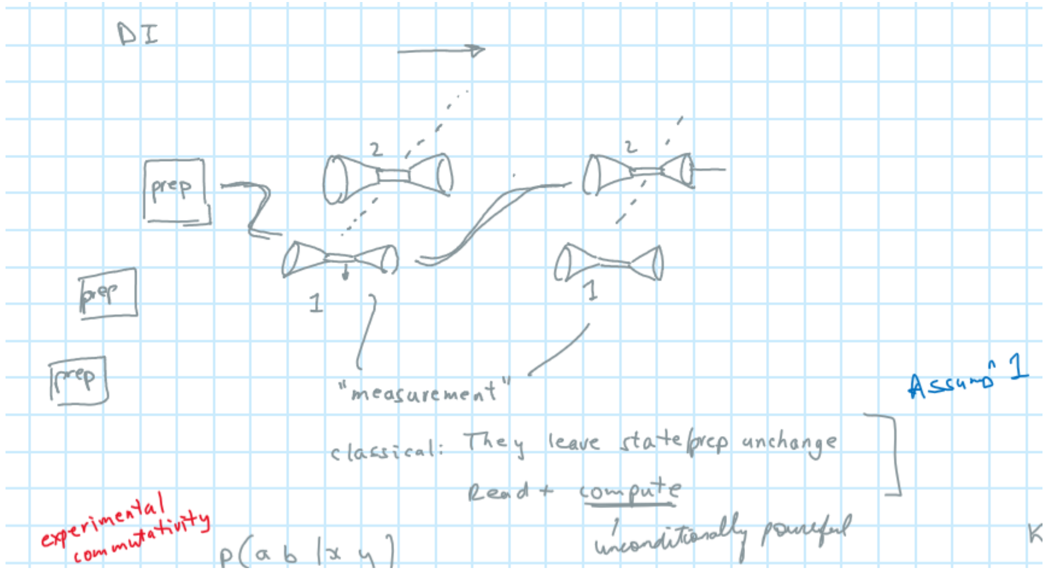

Figure 4.1: A KCBS Device

Lemma 17. *Let $F_{noiseless}(c)$ be the output of Algorithm 6 on the input c and let $F_{nonO}(c)$ be as described in Definition 16. Suppose $\|\Pi_i.\Pi_j\| \le \epsilon$.*

We now obtain an expression for ... [** I am here **]

4.4 Towards a device independent perspective

We attempt a rephrasing of the KCBS process in a device independent language.

Definition 18 (A KCBS Device)**.** An (ϵ, δ)–KCBS Device has three parts.

(i) Input module (this is supposed to correspond to the preparation ρ).

(ii) Reader modules labelled X1, X2 . . . X5, Y1, Y2, . . . Y5 which when applied on an input module, output 0/1 (these correspond to the measurements $\{\Pi_i\}_{i=1}^5$).

The device takes as input an ordered pair $(x, y) \in \{1, 2, \dots 5\} \times \{1, 2 \dots 5\}$ and returns two ordered bits (a, b) generated by the following procedure.

1. The input module read using the reader module labelled Xx and its output is returned as a.

2. The input module is subsequently read using the reader module labelled Yy and its output is returned as b.

Denote by $p(a, b|x, y)$ the probability (over identical copies of the device) of obtaining the output (a, b) given (x, y) as the input. The device satisfies, what we call the (ϵ, δ) condition, i.e.

$$|p(a, b|x, y) - p(b, a|y, x)| \le \epsilon \quad \forall\, a, b \in \{0, 1\} \quad \forall\, x, y \in \{1, 2 \dots 5\}$$
$$|p(a, a|x, x) - p(a|x)| \le \delta \quad \forall\, a \in \{0, 1\} \quad \forall x \in \{1, 2 \dots 5\}$$

for all possible input modules.

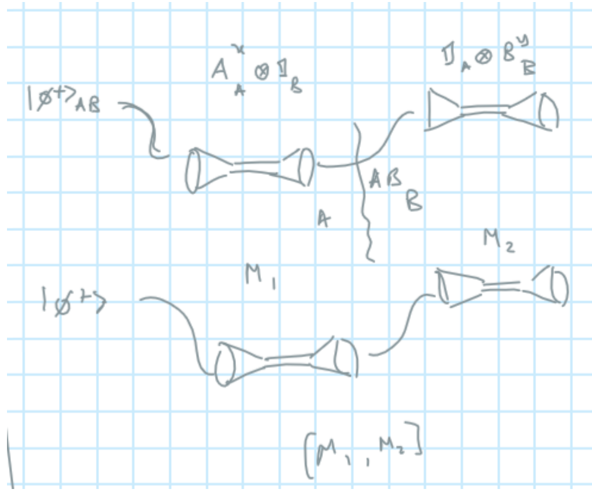

Figure 4.2: Bell as a special case; allows for direct enforcement of the assumptions

A few remarks are in order. This definition is not very useful from a device independent point of view as it is infeasible to imagine that all possible input modules are tested. Further, we require a device independent condition which, in quantum theory, should reduce to the requirement that $(\Pi_i^{\text{exp}})_i$ in fact behave like measurements and in the classical case, should reduce to the requirement that $(\Pi_i^{\text{exp}})_i$ do not influence the state $\rho = |\psi\rangle\langle\psi|$ being measured (we can then take convex combinations of these states).

5 Experimental results

The experimental implementation corresponding to the work in ref [1] was carried on.

6 Discussion

7 Conclusions

References

[1] Kishor Bharti, Maharshi Ray, Antonios Varvitsiotis, Naqueeb Ahmad Warsi, AdÃ¡n Cabello, and Leong-Chuan Kwek. Robust self-testing of quantum systems via noncontextuality inequalities.