Semidefinite Programming in Quantum Theory
Lecture 5

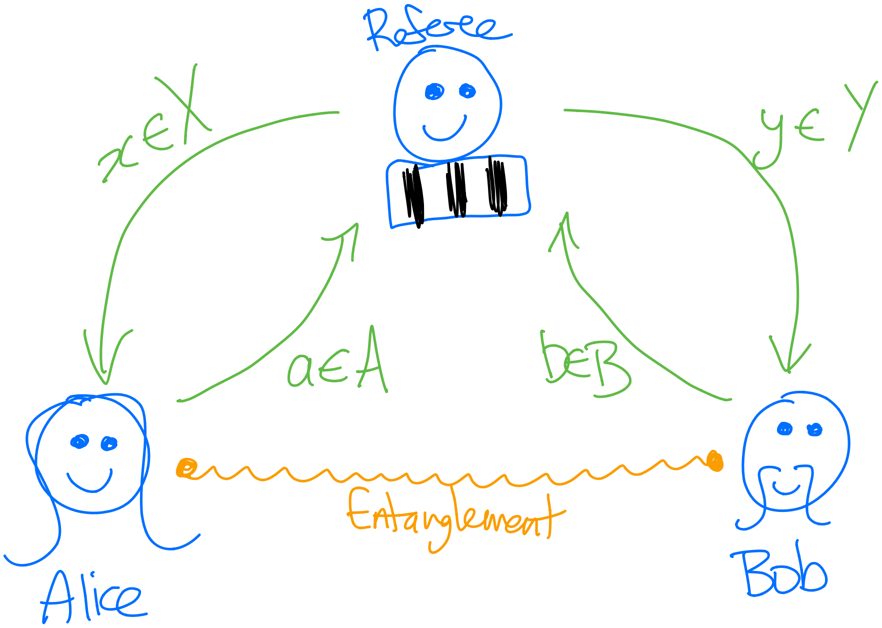
Topics covered

- Nonlocal games
- The Navascués-Pironio-Acín SDP hierarchy for quantum correlations

References

- A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations.
  Navascúes, Pironio, Acín. New J. Phys. 10, 073013 (2018)
  Arxiv: 0803.4290
- Extended nonlocal games and monogamy-of-entanglement games.
  Johnston, Mittal, Russo, Watrous. Proceedings of the Royal Society A 472(2189): 20160003, 2016.
  Arxiv: 1510.02083 (Different setting, similar proof)
- A combinatorial approach to nonlocality and contextuality.
  Acín, Fritz, Leverrier, Sainz. Comm. Math. Phys. 334(2), 533-628, 2015.
  Arxiv: 1212.4084 (Different setting, similar proof)

Nonlocal Games



- (x,y) is chosen with probability π_{xy}
- (x,y) are questions
- (a,b) are answers
- Alice & Bob win if $(abxy)$ satisfy some predicate V. I.e., $V: A \times B \times X \times Y \to \{0,1\}$.

Quantum Correlation:

- Alice has a different measurement for each $x \in X$, and each has outcomes $a \in A$. $\{P_a^x : a \in A\}$ POVM
- Bob has a different measurement for each $y \in Y$, and each has outcomes $b \in B$. $\{Q_b^y : b \in B\}$ POVM

If Alice and Bob share the quantum state $|\psi\rangle$, then

$$p(ab|xy) = \langle \psi | P_a^x \otimes Q_b^y | \psi \rangle \quad (*)$$

is the probability that Alice and Bob see the joint outcome (a,b) conditioned on being asked (x,y).

Thus, they win the game with probability

$$\sum_{x \in X} \sum_{y \in Y} \pi_{x,y} \sum_{a \in A} \sum_{b \in B} V(ab|xy)\, p(ab|xy).$$

We define the set Q where $p \in Q$ if and only if $\exists$ POVMs $\{P_a^x : a \in A\}$, $\{Q_b^y : b \in B\}$ and a quantum state $|\psi\rangle$ such that p comes from $(*)$.

<u>Definition: Entangled Value</u>: Define the vector c such that $c(ab|xy) = \pi_{xy} V(ab|xy)$. Then the entangled value of the game G is defined as

$$\omega^*(G) = \max_{p \in Q} \langle c, p \rangle.$$

In other words, Alice and Bob can win the game G with maximum probability $\omega^*(G)$ using entanglement.

<u>Example:</u> CHSH game has $A = B = X = Y = \{0,1\}$, $\pi_{00} = \pi_{01} = \pi_{10} = \pi_{11} = 1/4$ (each question is equally likely) and the winning predicate: $V(ab|xy) = \begin{cases} 1 & \text{if } a \oplus b = x \wedge y \\ 0 & \text{if } a \oplus b \neq x \wedge y \end{cases}$.

A very fundamental result in theoretical physics is that $\omega^*(CHSH) = \cos^2(\pi/8)$ and this value requires entanglement (it cannot be won with such large probability if Alice and Bob have shared randomness, for example).

The rest of this lecture will concentrate on characterizing the set Q as close as possible using SDPs.

NPA Hierarchy

We will slightly rewrite the definition of a quantum correlation from $p(ab|xy) = \langle\psi| P_a^x \otimes Q_b^y |\psi\rangle$ to

$$p(ab|xy) = \langle\psi| E_{xa} E_{yb} |\psi\rangle \text{ where } [E_{xa}, E_{yb}] = 0.$$

Possibly infinite dimensional — still projective measurements!

We let $\tilde{Q}$ denote these "quantum" correlations. Note: $Q \subseteq \tilde{Q}$.

We can think of $E_{xa} = P_a^x \otimes \mathbb{1}$ & $E_{yb} = \mathbb{1} \otimes Q_b^y$, but we don't have to.

Consider the matrix

$$\Gamma = \mathrm{Gram}\left(|\psi\rangle, E_{xa}|\psi\rangle, E_{yb}|\psi\rangle : \begin{matrix} x \in X, a \in A, \\ y \in Y, b \in B \end{matrix}\right) \succeq 0$$

$\mathrm{Gram}(x_1, \dots, x_n)$ is defined to be the matrix X where $X_{ij} = \langle x_i, x_j \rangle$. Note X is a Gram matrix $\iff X \in \mathrm{Pos}(\mathbb{C}^n)$.

This matrix has many interesting properties:

⓪ It has rows/columns indexed by ε, (x,a), (y,b).

empty string (for reasons that will be clear shortly!) corresponding to the "$|\psi\rangle$" row/column.

① Then $\Gamma(\varepsilon, \varepsilon) = \langle\psi|\psi\rangle = 1$.

② $\Gamma((x,a),(y,b)) = \langle\psi| E_{xa} E_{yb} |\psi\rangle = p(ab|xy)$

③ $\Gamma((x,a),(x,a)) = \langle\psi| E_{xa} E_{xa} |\psi\rangle = \langle\psi| E_{xa} |\psi\rangle = p(a|x)$

④ $\Gamma((x,a),(x,a')) = \langle\psi| \underbrace{E_{xa} E_{xa'}}_{=0} |\psi\rangle = 0$

equal — not equal

⑤ $\Gamma((x,a),\varepsilon) = \langle\psi| E_{xa} |\psi\rangle = p(a|x) = \Gamma(\varepsilon, (x,a))$.

Similarly,

(3') $\Gamma((y,b),(y,b)) = p(b|y)$

(4') $\Gamma((y,b),(y,b')) = 0$

equal   not equal

(5') $\Gamma((y,b),\varepsilon) = p(b|y) = \Gamma(\varepsilon,(y,b))$.

Thus, $p(ab|xy) \in Q \implies \exists \Gamma \succeq 0$ satisfying the above properties.

<u>However</u>, if $\Gamma \succeq 0$ satisfies the above properties for some $p(ab|xy)$ it <u>may</u> be the case that $p(ab|xy) \notin Q$.

To remedy this, we will add more constraints.

To this end, define

disjoint union

$$\Sigma_A = X \times A, \quad \Sigma_B = Y \times B, \quad \Sigma = \Sigma_A \sqcup \Sigma_B$$

Σ^k : the set of strings from alphabet Σ of length exactly k.

$\Sigma^{\leq k}$: the set of strings of length $\leq k$.

Σ^* : the set of all strings.

We have that $(x,a) \in \Sigma_a$ indexes the projection E_{xa} for Alice and (y,b) indexes E_{yb} for Bob. Since they commute, $(x,a)(y,b)$ is somehow equivalent to $(y,b)(x,a)$ (even though they are different strings).

We now look at quantities like:

$$\langle \psi | E_{x_1a_1} E_{y_1b_1} E_{x_2a_2} E_{x_3a_3} \cdots E_{y_nb_n} | \psi \rangle.$$

Does this help?

Define the relation:

- $s\sigma t \sim s\sigma\sigma t \quad \forall s,t \in \Sigma^*, \sigma \in \Sigma$ (corresponds to $E_{xa}E_{xa} = E_{xa}$, $E_{yb}E_{yb} = E_{yb}$)
- $s\sigma\tau t \sim s\tau\sigma t \quad \forall s,t \in \Sigma^*, \sigma \in \Sigma_A, \tau \in \Sigma_B$ (corresponds to $E_{xa}E_{yb} = E_{yb}E_{xa}$)

We use this to define an equivalence relation on Σ^*.

Suppose $p(ab|xy) \in Q$. Define the function

$$\phi : \Sigma^* \to \mathbb{C}$$

where, for $s = s_1 \cdots s_k \in \Sigma^k$, we define

$$\phi(s) = \langle \psi | E_{s_1} \cdots E_{s_k} | \psi \rangle.$$

Notice that it satisfies:

① $\forall s,t \in \Sigma^*$, we have

$$\sum_{a \in A} \phi(s\,(x,a)\,t) = \phi(st) \quad \& \quad \sum_{b \in B} \phi(s(y,b)t) = \phi(st)$$

since $\sum_a E_{xa} = \mathbb{1}$ & $\sum_b E_{yb} = \mathbb{1}$.

② $\forall s,t \in \Sigma^*$, we have

$$\phi(s\,(x,a)(x,a')\,t) = 0 \quad \& \quad \phi(s(y,b)(y,b')t) = 0$$

for every $x \in X$, $a \neq a' \in A$, $y \in Y$, $b \neq b' \in B$.

③ $\forall s,t \in \Sigma^*$ satisfying $s \sim t$, we have

$$\phi(s) = \phi(t).$$

We will say any function that satisfies ①–③ is <u>admissable.</u>

<u>Def'n:</u> We say that $\Gamma_k \in \mathrm{Pos}(\mathbb{C}^{\Sigma^{\leq k}})$ is k-th order admissible if there exists $\phi : \Sigma^{\leq 2k} \to \mathbb{C}$ such that

$$\Gamma(s,t) = \phi(s^R t)$$

reverse of the string. $\text{Jamie}^R = \text{eimaJ}$

<u>Def'n:</u> Q_k is the set of $p(ab|xy)$ such that $\exists \Gamma_k$ k-th order admissible and $\Gamma_k((x,a),(y,b)) = p(ab|xy)$.

$$\alpha_k = \sup t$$
Γ_k satisfies ①, ②, ③
$\Gamma_k((x,a),(y,b)) = p(ab|xy)$
$\Gamma_k \geq t\mathbb{I}$

$p(ab|xy) \in Q_k$
$\iff \alpha_k \geq 0.$

Fun fact: $\tilde{Q} \subseteq \bigcap_{k=1}^{\infty} Q_k$. But, do we have equality?

Convergence

Let $p(ab|xy) \in \bigcap_{k=1}^{\infty} Q_k$. Then $\exists\, \Gamma_k \geq 0$ k-th order admissible such that $\Gamma_k((x,a),(y,b)) = p(ab|xy)$, $\forall k \in \mathbb{N}$.

Define $\tilde{\Gamma}_k \in \mathrm{Pos}(\Sigma^*)$ by padding Γ_k with 0's.

$$\text{E.g. } \tilde{\Gamma}_k = \left[\begin{array}{c|c} \Gamma_k & 0 \\ \hline 0 & 0 \end{array}\right] \geq 0.$$

Tychonoff's Theorem: Any collection of compact topological spaces is compact with respect to the product topology.

Fun fact: $\tilde{\Gamma}_k \in \mathbb{B}^{\mathbb{N} \times \mathbb{N}}$ (each entry has modulus bounded by 1)
(This isn't too hard to prove) $\quad (z \in \mathbb{B} \iff |z| \leq 1)$

Tychonoff's Theorem implies $\{\tilde{\Gamma}_k : k \in \mathbb{N}\}$ has a convergent subsequence.
Let Γ_∞ be the limit point of this subsequence (this converges point-wise).
Thus $\Gamma_\infty((x,a),(y,b)) = p(ab|xy)$. It can also be shown that $\exists$ admissible function ϕ such that $\Gamma_\infty(s,t) = \phi(s^R t)$.

How do we use Γ_∞ to define a commuting operator strategy?

First, we need a Hilbert space! We start with the vector space $V = \mathbb{C}^{\Sigma^*}$ and define the inner product on basis states

$$\langle e_s, e_t \rangle = \Gamma_\infty[s,t]$$

and extend linearly. Now, this inner product may not be positive definite (i.e., there might be $v \neq 0$ such that $\langle v, v \rangle = 0$).

Define

$$K = \{v : \langle v, v \rangle = 0\}$$ It's a subspace!

Define $V' = \underbrace{V \bmod K}_{\text{Quotient Vector Space}}$ $\quad v_1 \sim v_2$ if $v_1 - v_2 \in K$.

To make V' "complete", i.e., a Hilbert Space, we take it's Cauchy Completion.

(Basically, we just "force" V' to be complete.)

Define $\mathcal{H}$ to be the Cauchy Completion of V', and this is the Hilbert space we will use.

Define the operators: E_{xa} (for $(x,a) \in \Sigma_A$) and E_{yb} (for $(y,b) \in \Sigma_B$) as:

$E_{xa} e_s = e_{(x,a)s}$ $\forall s \in \Sigma^*$ and

$E_{yb} e_s = e_{(y,b)s}$ $\forall s \in \Sigma^*$ (and extend linearly).

Lemma: E_{xa} & E_{yb} are self-adjoint.

Proof: $\langle e_s, E_{xa} e_t \rangle = \langle e_s, e_{(x,a)t} \rangle = \Gamma(s, (x,a)t) = \phi(s^R (x,a) t)$

$\langle E_{xa} e_s, e_t \rangle = \langle e_{(x,a)s}, e_t \rangle = \Gamma((x,a)s, t) = \phi(s^R (x,a) t)$. $\square$

Lemma: E_{xa} & E_{yb} are idempotent.

Proof: We want that $E_{xa}e_s - E_{xa}E_{xa}e_s \in O_{\mathcal{H}}$.

$\langle E_{xa}e_s - E_{xa}E_{xa}e_s, E_{xa}e_s - E_{xa}E_{xa}e_s \rangle$

$= \langle e_{(x,a)s} - e_{(x,a)(x,a)s}, e_{(x,a)s} - e_{(x,a)(x,a)s} \rangle$

$= \phi(s^R(x,a)(x,a)s) - \phi(s^R(x,a)(x,a)(x,a)s) - \phi(s^R(x,a)(x,a)(x,a)s)$
$\quad + \phi(s^R(x,a)(x,a)(x,a)(x,a)s)$

$= 0.$

Thus, $(E_{xa} - E_{xa}E_{xa})e_s = 0 \quad \forall s \in \Sigma^* \Rightarrow E_{xa} - E_{xa}E_{xa} = 0$ (operator)
$\Rightarrow E_{xa} = E_{xa}E_{xa}.$ $\square$

Lemma: $E_{xa}E_{x,a'} = 0$ for $a \neq a'$.

Proof: $\langle E_{xa}E_{xa'}e_s, E_{xa}E_{xa'}e_s \rangle = \phi(s^R(x,a')(x,a)(x,a)(x,a')s) = 0.$

Thus, $E_{xa}E_{xa'}e_s = 0 \quad \forall s \in \Sigma^* \Rightarrow E_{xa}E_{xa'} = 0.$ $\square$

Lemma: $\sum_a E_{xa} = \mathbb{1}_{\mathcal{H}}$ and $\sum_b E_{yb} = \mathbb{1}_{\mathcal{H}}$.

Proof: $\langle \sum_a E_{xa}e_s - e_s, \sum_a E_{xa}e_s - e_s \rangle = \langle \sum_a e_{(x,a)s} - e_s, \sum_a e_{(x,a)s} - e_s \rangle$

$= \sum_{a,a'} \phi(s^R(x,a)(x,a')s) - \sum_a \phi(s^R(x,a)s)$
$- \sum_{a'} \phi(s^R(x,a')s) + \phi(s^R s)$
$= 0.$ $\square$

Thus, E_{xa} (and E_{yb}) are projective measurements on $\mathcal{H}$.

Define $|\psi\rangle = e_\varepsilon$ (the vector $|0\rangle$, basically). Then we have

$$\langle \psi | E_{xa}E_{yb} | \psi \rangle = \langle e_\varepsilon, E_{xa}E_{yb}e_\varepsilon \rangle$$
$$= \langle E_{xa}e_\varepsilon, E_{yb}e_\varepsilon \rangle$$
$$= \Gamma_\infty((x,a),(y,b))$$
$$= p(ab|xy)$$

Thus, $p(ab|xy) \in \tilde{Q}$. $\square$

Example: The CHSH correlation (for $A=B=X=Y=\{0,1\}$) is defined as

$$p(ab|xy)=\begin{cases}\frac{\cos^2(\pi/8)}{2} & \text{if } a\oplus b=xy\\ \frac{\sin^2(\pi/8)}{2} & \text{if } a\oplus b\neq xy.\end{cases}$$

To see if $p\in Q_1$, we can solve the following SDP:

$$\alpha_1=\sup\ t$$

$p(a|x)=1/2$ $\quad\cos^2(\pi/8)$ $\quad\sin^2(\pi/8)$

$$\begin{array}{c c}
 & \begin{array}{ccccc} \quad & x{=}0 & x{=}1 & y{=}0 & y{=}1 \end{array}\\
\begin{array}{c} \varepsilon \\ x{=}0 \\ \\ x{=}1 \\ \\ y{=}0 \\ \\ y{=}1 \\ \\ \end{array} &
\frac{1}{2}\left[\begin{array}{c|cc|cc|cc|cc}
2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ \hline
1 & 1 & 0 & * & * & c^2 & s^2 & c^2 & s^2\\
1 & 0 & 1 & * & * & s^2 & c^2 & s^2 & c^2\\ \hline
1 & * & * & 1 & 0 & c^2 & s^2 & s^2 & c^2\\
1 & * & * & 0 & 1 & s^2 & c^2 & c^2 & s^2\\ \hline
1 & c^2 & s^2 & c^2 & s^2 & 1 & 0 & * & *\\
1 & s^2 & c^2 & s^2 & c^2 & 0 & 1 & * & *\\ \hline
1 & c^2 & s^2 & s^2 & c^2 & * & * & 1 & 0\\
1 & s^2 & c^2 & c^2 & s^2 & * & * & 0 & 1
\end{array}\right]\succeq tI
\end{array}$$

We have $\alpha_1\geq 0$ (a solution can be found numerically, or just use the strategy we all know and love). Thus, $p\in Q_1$, as expected.

Example: The PR-BOX correlation (for $A=B=X=Y=\{0,1\}$) is defined as:

$$p(ab|xy)=\begin{cases}1/2 & \text{if } a\oplus b=xy\\ 0 & \text{if } a\oplus b\neq xy\end{cases}$$

$$\alpha_1=\sup\ t$$

$$X=\frac{1}{2}\left[\begin{array}{c|cc|cc|cc|cc}
2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ \hline
1 & 1 & 0 & \multicolumn{2}{c|}{*} & 1 & 0 & 1 & 0\\
1 & 0 & 1 & & & 0 & 1 & 0 & 1\\ \hline
1 & \multicolumn{2}{c|}{*} & 1 & 0 & 1 & 0 & 0 & 1\\
1 & & & 0 & 1 & 0 & 1 & 1 & 0\\ \hline
1 & 1 & 0 & 1 & 0 & 1 & 0 & \multicolumn{2}{c}{*}\\
1 & 0 & 1 & 0 & 1 & 0 & 1 & & \\ \hline
1 & 1 & 0 & 0 & 1 & \multicolumn{2}{c|}{*} & 1 & 0\\
1 & 0 & 1 & 1 & 0 & & & 0 & 1
\end{array}\right]\succeq tI$$

Solving this numerically (which doesn't constitute a proper proof!) yields $\alpha_1\approx -1.48$.

Thus, $p(ab|xy)\notin Q_1$

$\Rightarrow p(ab|xy)\notin Q.$