Thinking Recursively

CS 106B

Programming Abstractions
Fall 2016
Stanford University
Computer Science Department

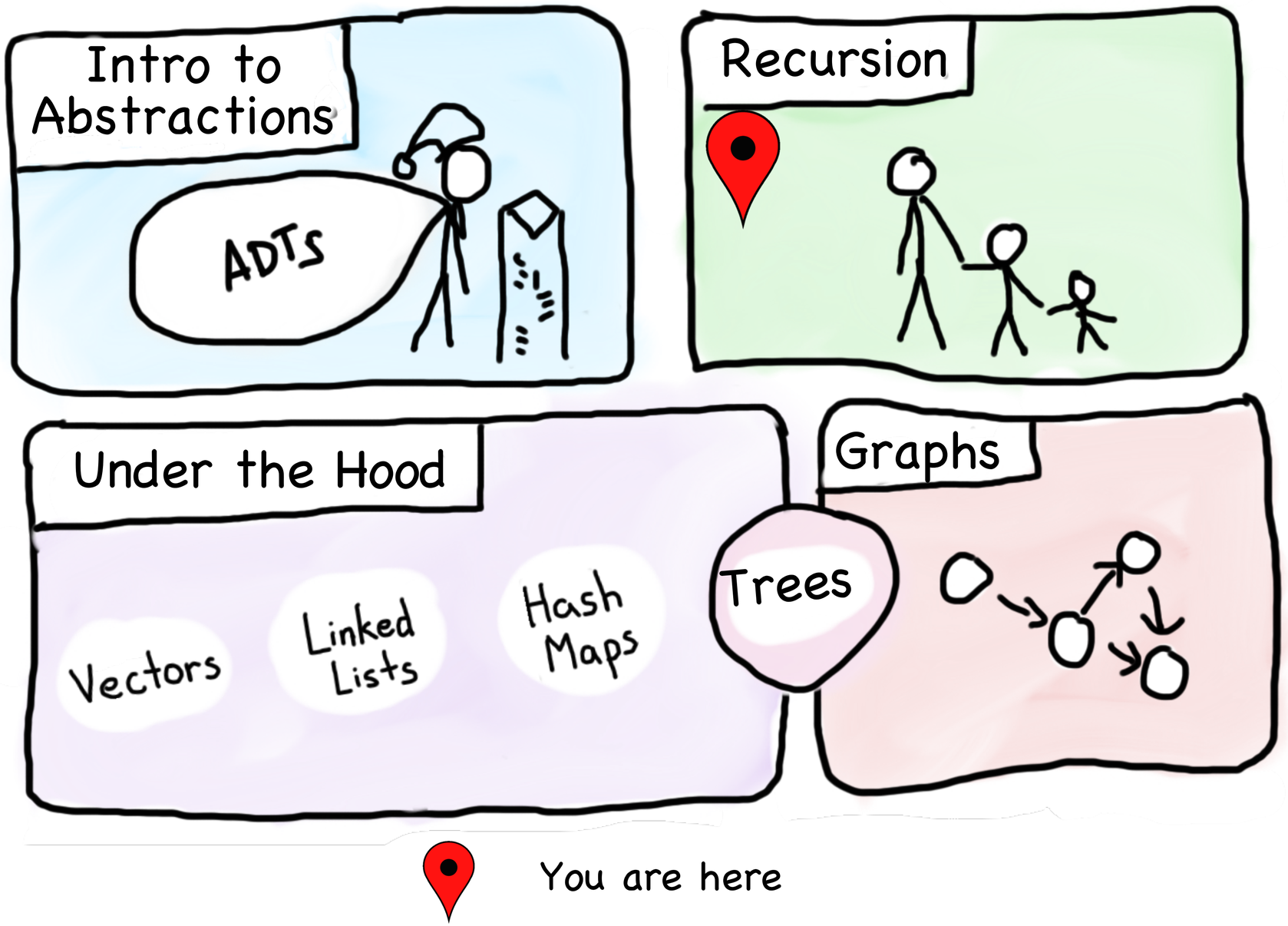
Announcements

- Assignment 2 due Saturday
- Remember no LaIR on Friday night
- Book Readings

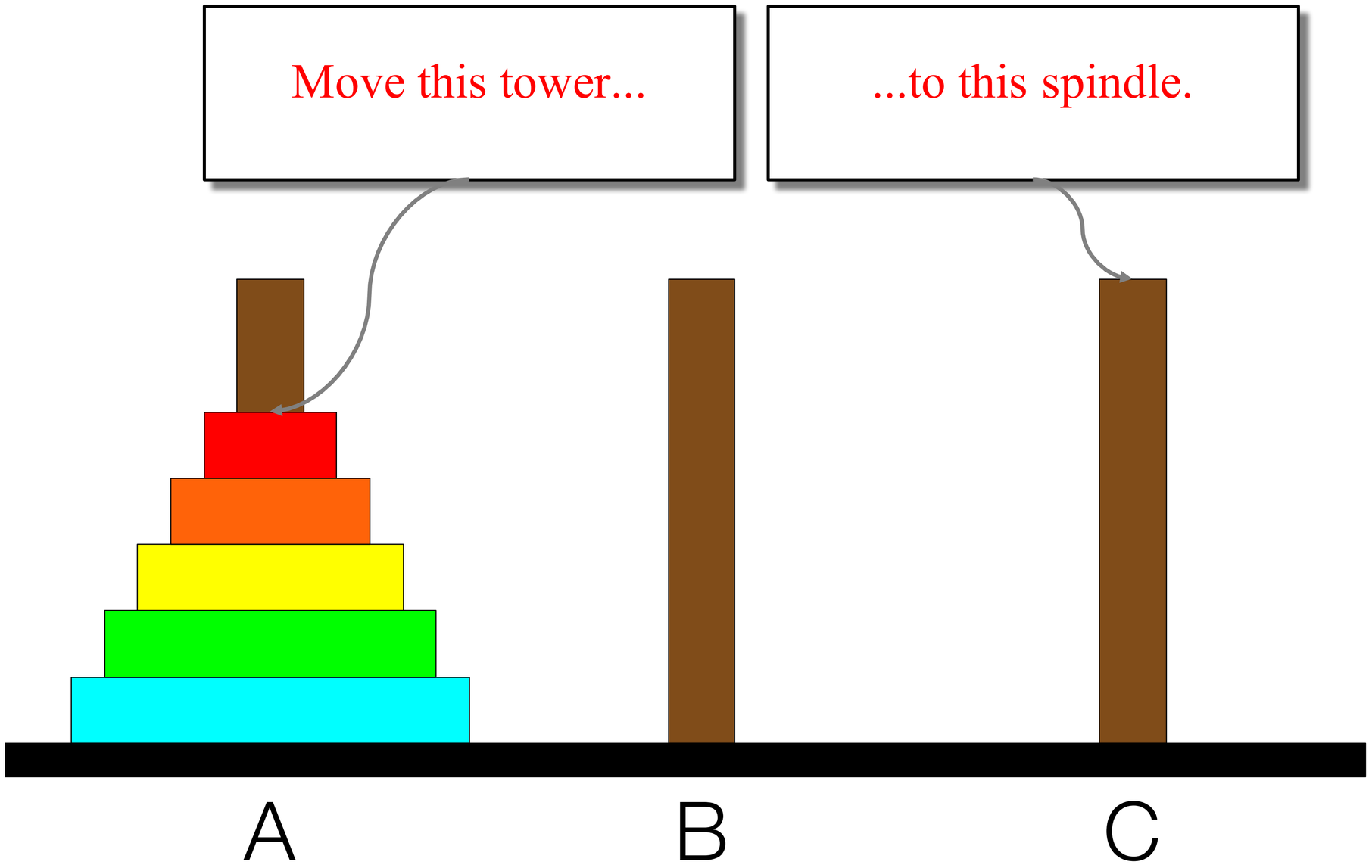
Today is an exciting day

Learn a new way to *think*

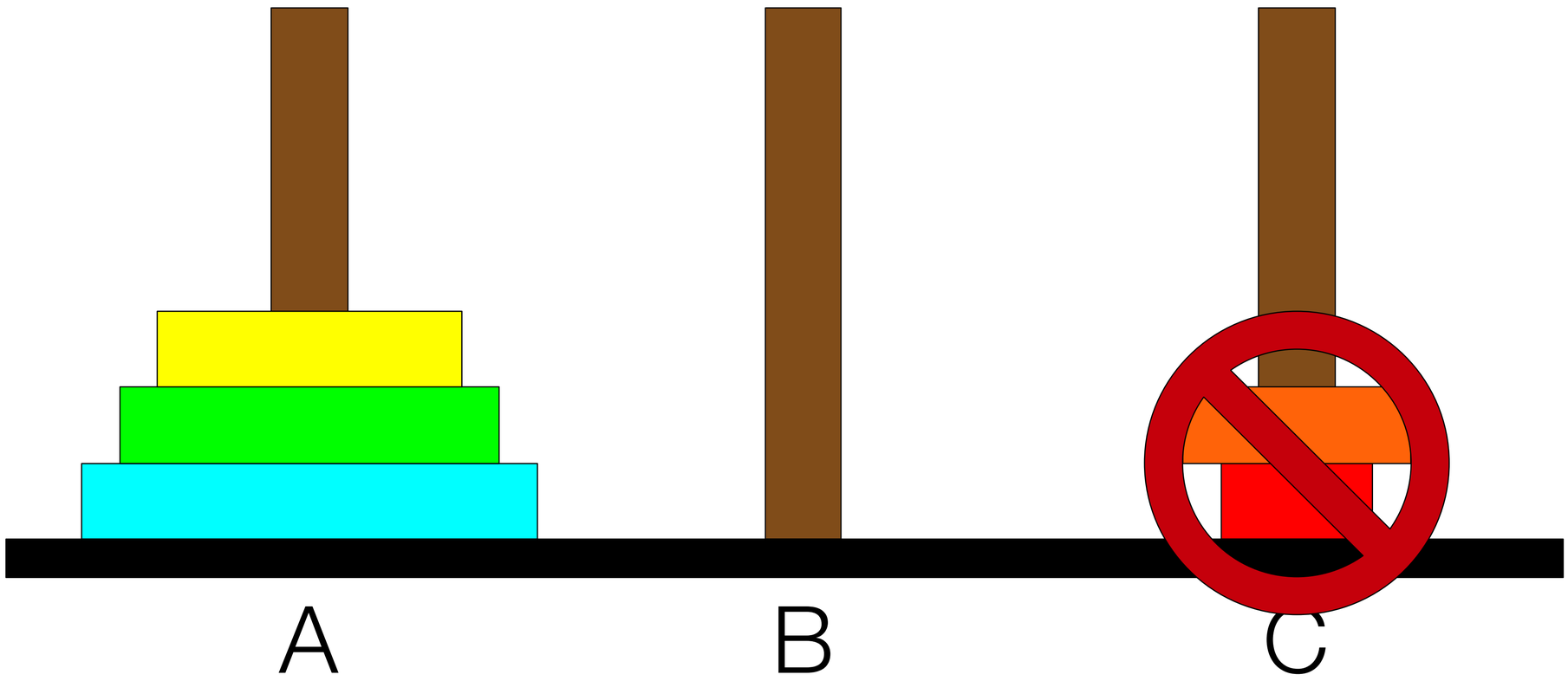
Course Syllabus



Towers of Hanoi



Towers of Hanoi



We’ve Gotten Ahead of Ourselves

Source: The Unbelievable

Start at the Beginning

Recursion Definition

Recursion:

A problem solving technique in which problems are solved by reducing them into **smaller problems** ***of the same form***.

Recursion Definition

Recursion:

A problem solving technique in which problems are solved by reducing them into **smaller problems** ***of the same form***.

???

What does that mean?

Pedagogy: Many simple examples

Operation: Understand Word

How Many Students Behind You?

How many students total are directly behind you in your "column" of the classroom?

1. You can see only the people right next to you. So you can't just look back and count.

2. But you are allowed to ask questions of the person next to you.

3. How can we solve this problem (*recursively*)

How Many Students Behind You?

```
int numStudentsBehind(Student curr) {
    if(lastInRow(curr)) {
        return 0;
    } else {
        Student personBehind = curr.getBehind();
        return numStudentsBehind(personBehind) + 1;
    }
}
```

Recursion Template

The structure of recursive functions is typically like the following:

```
recursiveFunction:
   if (test for simple case) {
      Compute the solution without recursion
   } else {
      Break the problem into subproblems of the same form
      Call recursiveFunction on each subproblem
      Reassemble the results of the subproblems
   }
}
```

Recursion Template

Every recursive algorithm involves at least 2 cases:

- **base case**: The simple case; an occurrence that can be answered directly; the case that recursive calls reduce to.

- **recursive case**: A more complex occurrence of the problem that cannot be directly answered, but can instead be described in terms of smaller occurrences of the same problem.

How Many Students Behind You?

```
int numStudentsBehind(Student curr) {
    if(lastInRow(curr)) {
        return 0;
    } else {
        Student personBehind = curr.getBehind();
        return numStudentsBehind(personBehind) + 1;
    }
}
```

Base Case

```
int numStudentsBehind(Student curr) {
    if(lastInRow(curr)) {
        return 0;
    } else {
        Student personBehind = curr.getBehind();
        return numStudentsBehind(personBehind) + 1;
    }
}
```

Recursive Case

```
int numStudentsBehind(Student curr) {
    if(lastInRow(curr)) {
        return 0;
    } else {
        Student personBehind = curr.getBehind();
        return numStudentsBehind(personBehind) + 1;
    }
}
```

Recursive Call

```
int numStudentsBehind(Student curr) {
    if(lastInRow(curr)) {
        return 0;
    } else {
        Student personBehind = curr.getBehind();
        return numStudentsBehind(personBehind) + 1;
    }
}
```

Three Musts of Recursion

1. Your code must have a case for all valid inputs.

2. You must have a base case (makes no recursive calls).

3. When you make a recursive call it should be to a simpler instance (forward progress towards base case)

Powers

Write a recursive function that takes in a number (x) and an exponent and returns the result of x^{exp}

Powers

$$x^0 = 1$$

$$x^n = x \cdot x^{n-1}$$


Qt

Powers

Each previous call waits for the next call to finish.

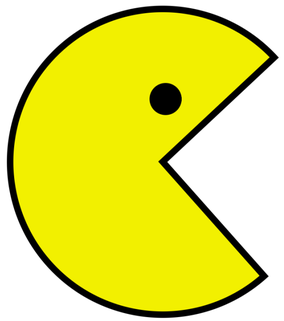
```
// first call: power(5,3)
int power(int x, int exp) {
    // second call: power(5,2)
    int power(int x, int exp) {
        // second call: power(5,1)
        int power(int x, int exp) {
            // second call: power(5,0)
            int power(int x, int exp) {
                if (exp == 0) {
                    return 1;
                } else {
                    return x * power(x, exp - 1);
                }
            }
        }
    }
}
```

Actual Powers Recursion

```
int power(int x, int exp) {
    if(exp == 0) {
        // base case
        return 1;
    } else {
        if (exp % 2 == 1) {
            // if exp is odd
            return x * power(x, exp - 1);
        } else {
            // else, if exp is even
            int y = power(x, exp / 2);
            return y * y;
        }
    }
}
```

Exponentiation by squaring

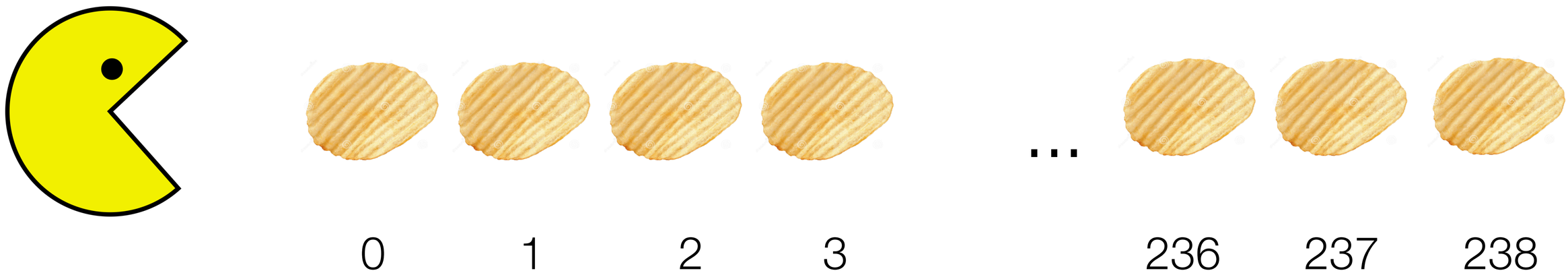
How do you eat an entire bowl of chips?



Using a For Loop

How do you eat an entire bowl of chips?

Iterative solution, for-loop:

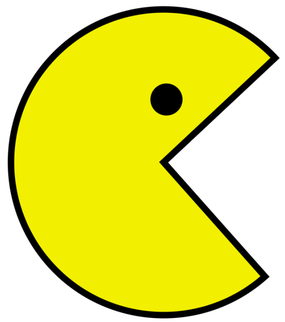


- You know exactly how many chips there are
- You start with Chip 0 and you keep eating until you eat Chip 238

Using a While Loop

How do you eat an entire bowl of chips?

Iterative solution, while-loop:

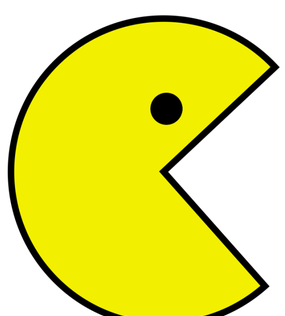


- You're not really sure how many chips there are
- But you're going to eat one chip at a time until the bowl is empty

Using Recursion

How do you eat an entire bowl of chips?

A recursive solution:



Using Recursion

How do you eat an entire bowl of chips?

A recursive solution:

- A bowl of N chips… is really: 1 chip + a bowl of (N - 1) chips
- So I'm going to eat 1 chip, then eat a bowl of (N - 1) chips

Using Recursion

How do you eat an entire bowl of chips?

A recursive solution:

- A bowl of N chips… is really: 1 chip + a bowl of (N - 1) chips
- So I'm going to eat 1 chip, then eat a bowl of (N - 1) chips
- I will keep doing this until I find that my bowl has 0 chips

Mystery Recursion

- Consider the following recursive function:

```
int mystery(int n) {
    if (n < 10) {
        return n;
    } else {
        int a = n / 10;
        int b = n % 10;
        return mystery(a + b);
    }
}
```

Q: What is the result of `mystery(648)`?

A. 8

B. 9

C. 54

D. 72

E. 648

IsPalendrome

- Write a recursive function `isPalindrome` accepts a `string` and returns `true` if it reads the same forwards as backwards.

```
isPalindrome("madam") → true
isPalindrome("racecar") → true
isPalindrome("step on no pets")  → true
isPalindrome("Java")  → false
isPalindrome("byebye")  → false
```

Three Musts of Recursion

1. Your code must have a case for all valid inputs.

2. You must have a base case (makes no recursive calls).

3. When you make a recursive call it should be to a simpler instance (forward progress towards base case)

The Recursive Insight

"The code already works!"

- **This is the most important strategy for recursion!**
- When you are writing a recursive function, ***pretend that it already works*** and use it whenever possible in the body of the function.

"Look for the subproblem of the same form"

- Before writing your recursive function, write down what it is supposed to do.
- Then see how you can express the result of the function in terms of a smaller version of the original problem.

IsPalendrome

```
// Returns true if the given string reads the same
// forwards as backwards.
// Trivially true for empty or 1-letter strings.
bool isPalindrome(const string& s) {
    if (s.length() < 2) {   // base case
        return true;
    } else {                // recursive case
        if (s[0] != s[s.length() - 1]) {
            return false;
        }
        string middle = s.substr(1, s.length() - 2);
        return isPalindrome(middle);
    }
}
```

Hailstone

Print the sequences of numbers that you take to get from N until 1, using the Hailstone (Collatz) production rules:

If n == 1, you are done.

If n is odd your next number is 3*n + 1.

If n is even your next number is n / 2.

Hailstone

```
// Couts the sequence of numbers from n to one
// produced by the Hailstone (aka Collatz)
// procedure
void hailstone(int n) {
   cout << n << endl;
   if(n == 1) {
      return;
   } else {
      if(n % 2 == 0) {
         // n is even so we repeat with n/2
         hailstone(n / 2);
      } else {
         // n is odd so we repeat with 3 * n + 1
         hailstone(3 * n + 1);
      }
   }
}
```

Are the Recursive Calls Simpler?

```
// Couts the sequence of numbers from n to one
// produced by the Hailstone (aka Collatz)
// procedure
```

3. When you make a recursive call it should be to a simpler instance (forward progress towards base case)

Well that seems to be true…

```
}
```

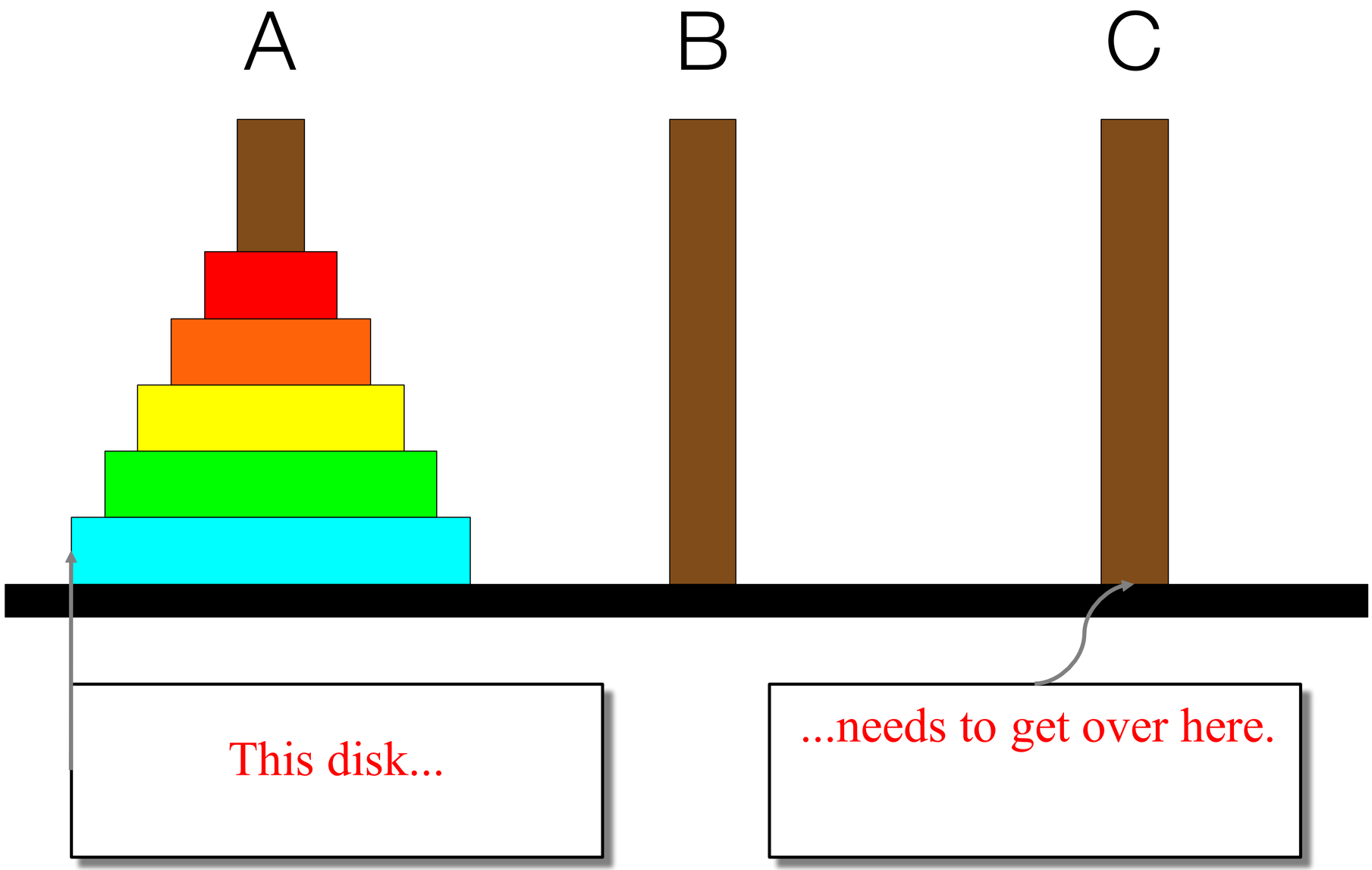
Works for numbers up to 5×10^{18}

Reward for proof

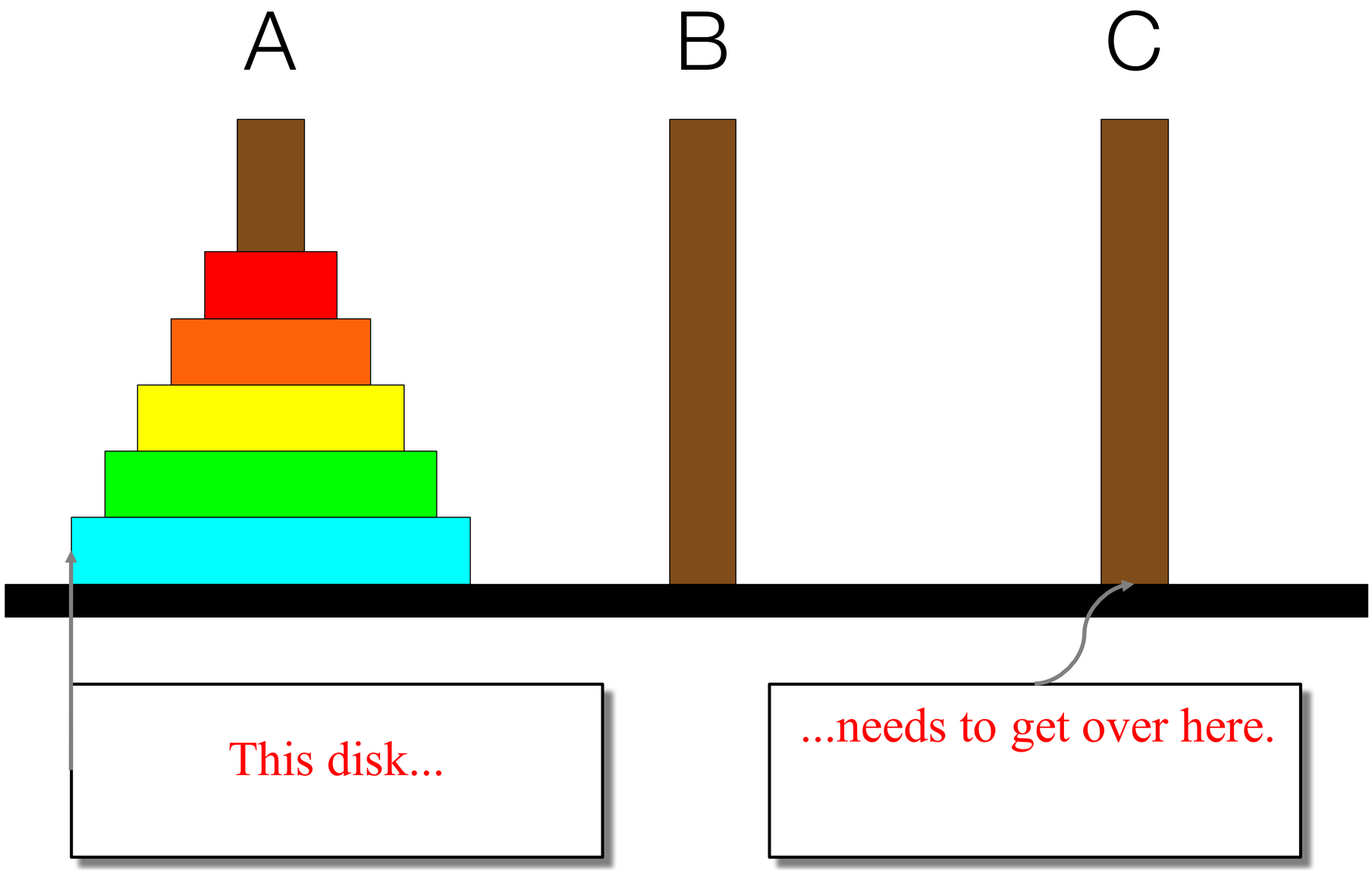
$1,400

Here we are

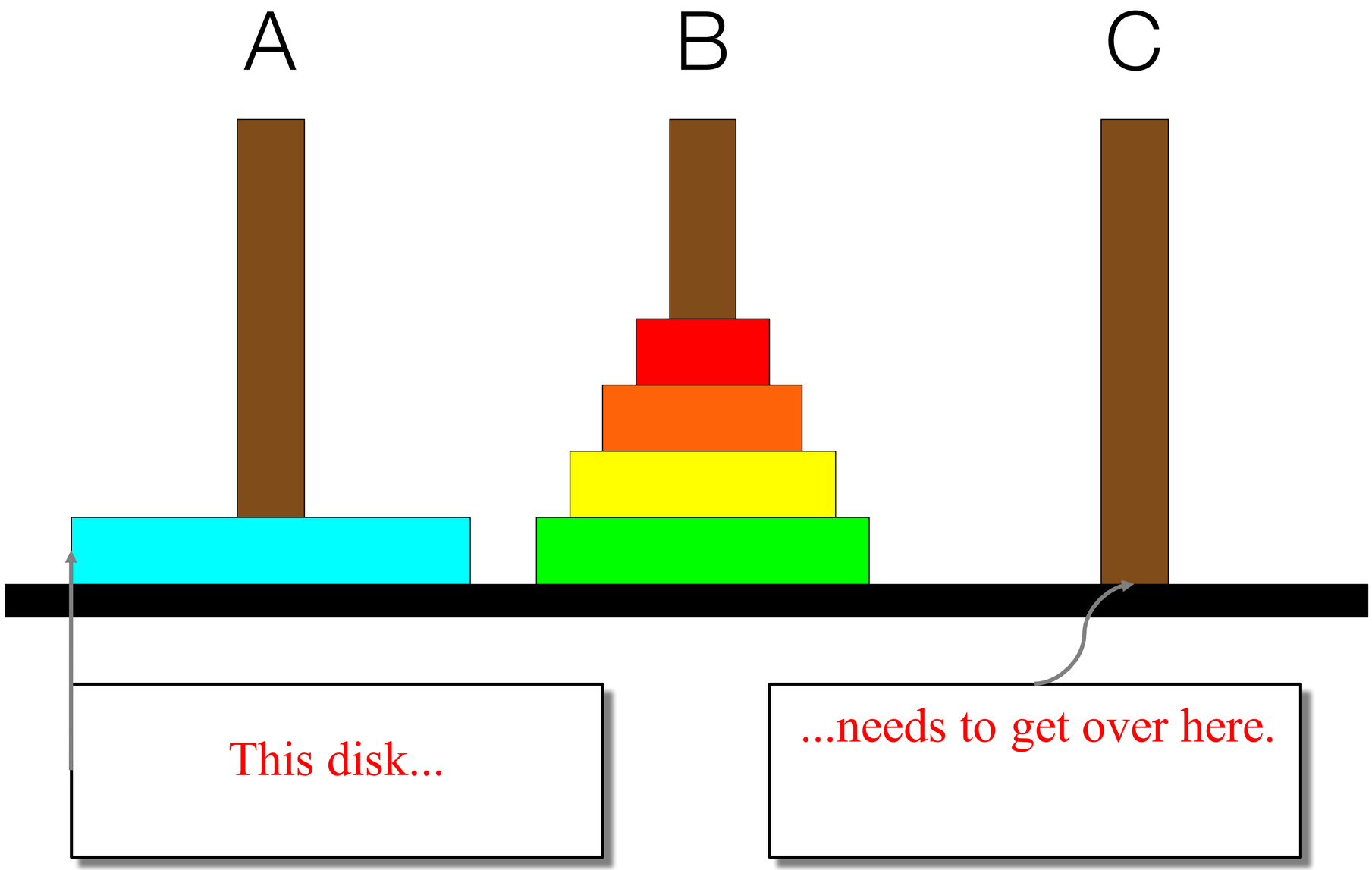
Towers of Hanoi



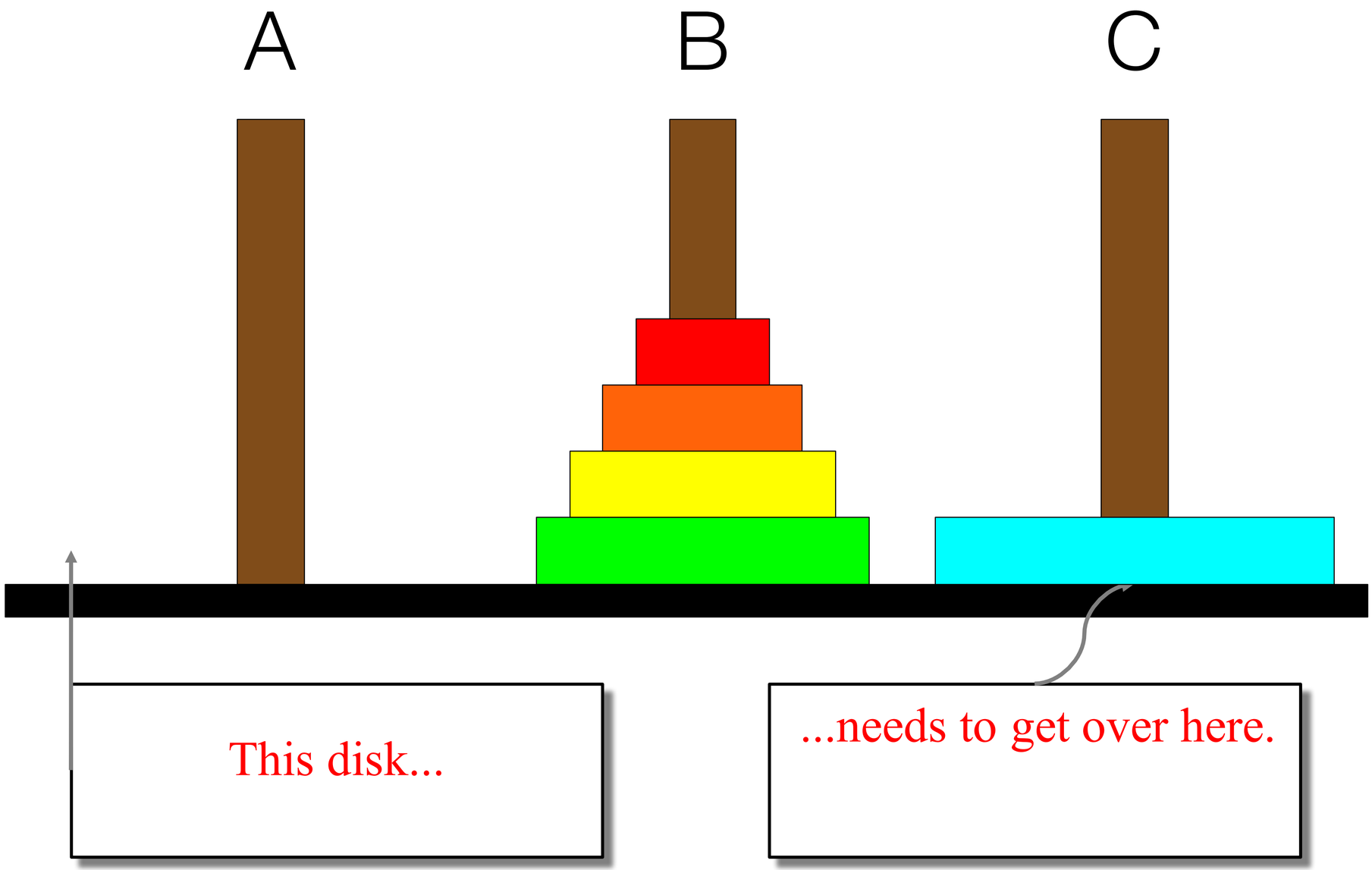
Towers of Hanoi Insight



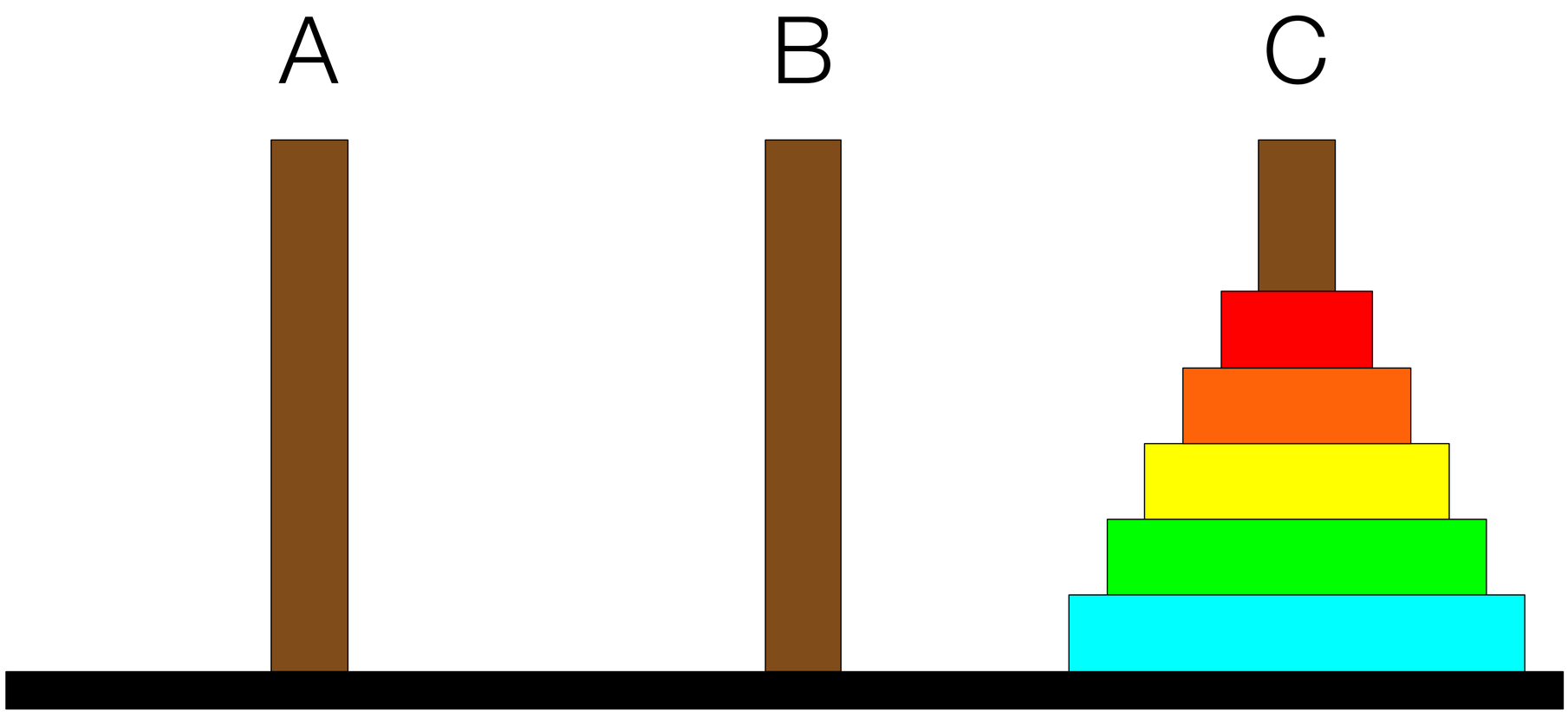
Towers of Hanoi Insight



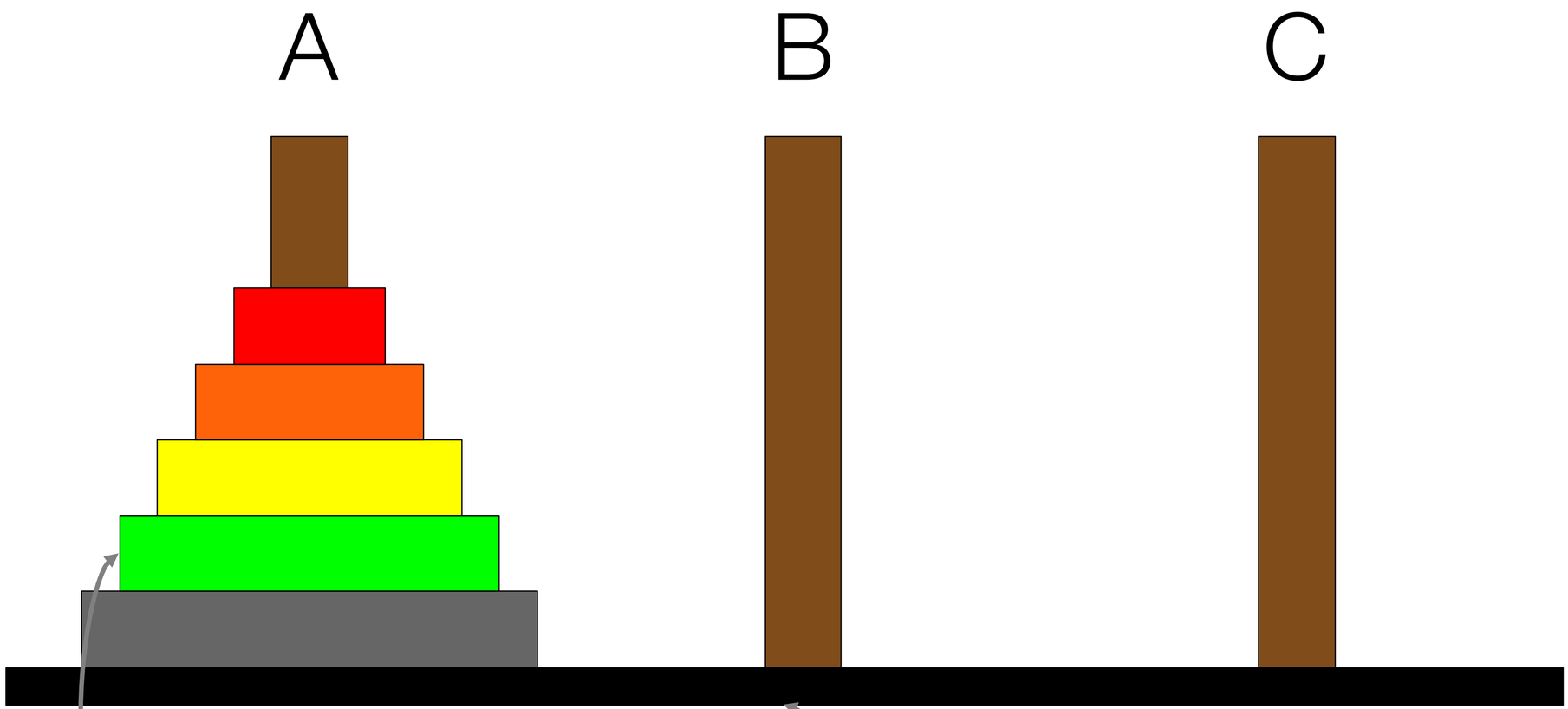
Towers of Hanoi Insight



Towers of Hanoi Insight



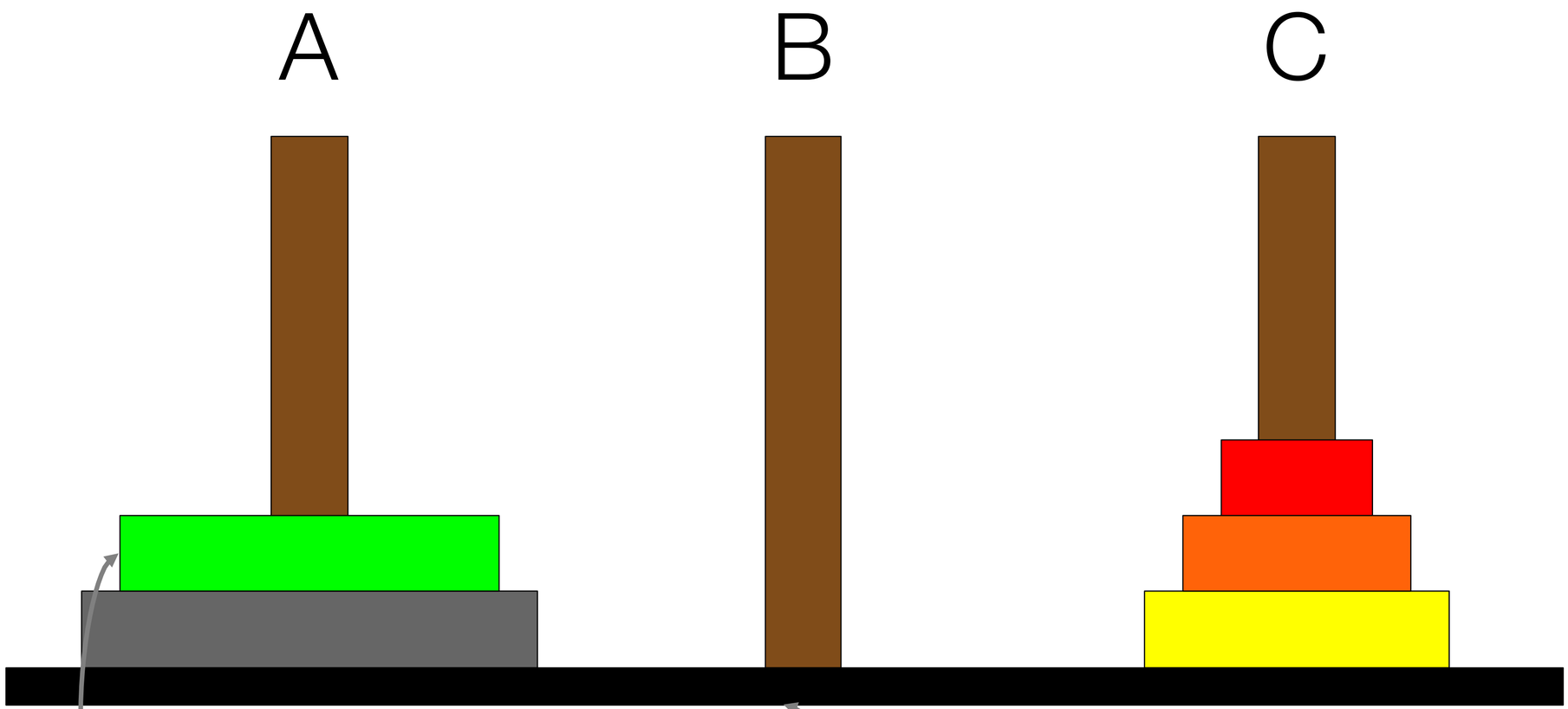
Towers of Hanoi Insight



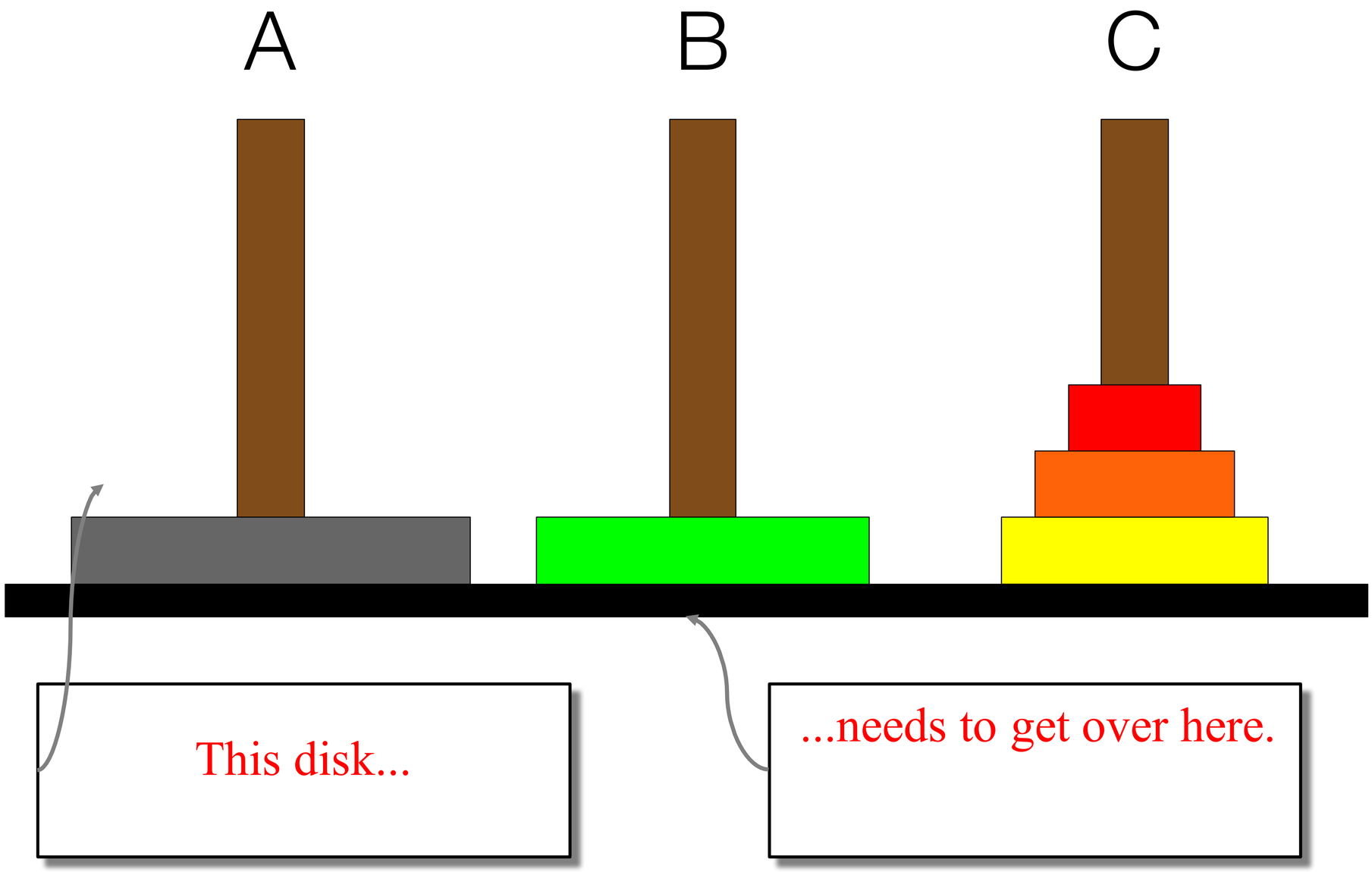
This disk...

...needs to get over here.

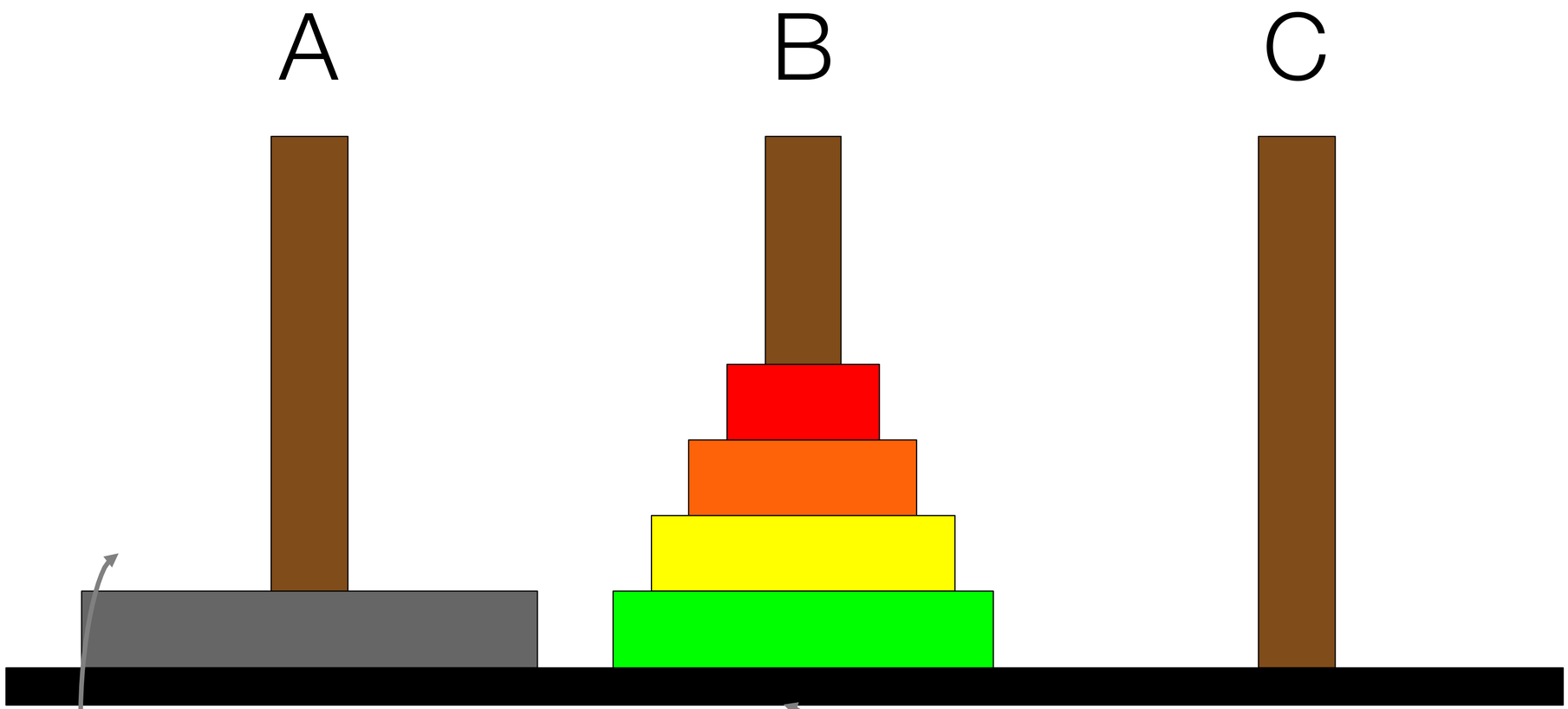
Towers of Hanoi Insight



Towers of Hanoi Insight



Towers of Hanoi Insight



This disk...

...needs to get over here.

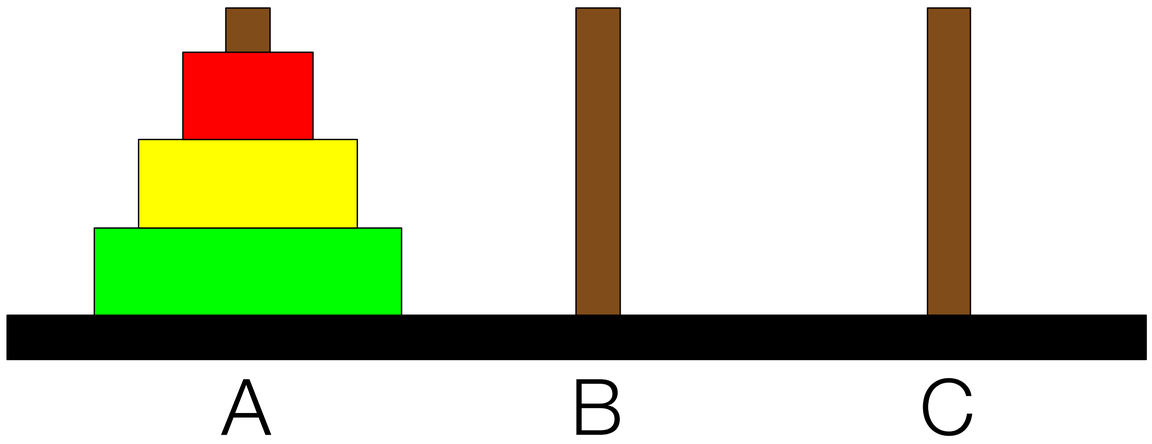
Base Case

- We need to find a very simple case that we can solve directly in order for the recursion to work.

- If the tower has size one, we can just move that single disk from the source to the destination.


Qt

```
int main() {
    moveTower(3, 'a', 'c', 'b');
}
```



```
int main() {
    r
}
```

```
void moveTower(int n, char from, char to, char temp) {
    i
}
```

```
void moveTower(int n, char from, char to, char temp) {
    if (n == 1) {
        moveSingleDisk(from, to);
    } else {
        moveTower(n – 1, from, temp, to);
        moveSingleDisk(from, to);
        moveTower(n – 1, temp, to, from);
    }
}
```

n	2	from	b	to	c	temp	a

The Solution

```
void moveTower(int n, char from, char to, char temp) {
    if (n == 1) {
        moveSingleDisk(from, to);
    } else {
        moveTower(n – 1, from, temp, to);
        moveSingleDisk(from, to);
        moveTower(n – 1, temp, to, from);
    }
}
```

This problem is very hard to solve “iteratively”

Welcome to the world of recursion

Your Brain is Recursive

There is a pathway in your brain for imagination

Step 1: Imagine your life at the end of CS106B

Step 1: Imagine your life at the end of CS106B

Step 2: Imagine one thing you could do next

Step 2: Imagine one thing you could do next

Step 3: Imagine your life after that choice

Executes the same “function” with different inputs

Step 3: Imagine your life after that choice

Executes the same “function” with different inputs

Corballis, M.C. The Recursive Mind. Princeton University Press, 2011.

The End

Why Recursion?

1. Great style
2. Some things are naturally recursive
3. Master of control flow

More Practice

```
int mystery(int n) {
    if (n < 10) {
        return (10 * n) + n;
    } else {
        int a = mystery(n / 10);
        int b = mystery(n % 10);
        return (100 * a) + b;
    }
}
```

Q: What is the result of `mystery(348)` ?

A. 3828

B. 348348

C. 334488

D. 3408

E. none of the above

```
// call 1:    348
int mystery(int n) {
  if (n < 10) {
    return (10 * n) + n;
  } else {
    int a = mystery(n /
10);
    int b = mystery(n %
10);
    return (100 * a) + b;
  }
}
```

```
// call 2a:    34
int mystery(int n) {
  if (n < 10) {
    return (10 * n) + n;
  } else {
```

```
// call 2b:     8
int mystery(int n) {
  if (n < 10) {
    return (10 * n) + n;
  } else {
mystery(n /

mystery(n %

100 * a) + b;
```

```
// call 3a:     3
int mystery(int n) {
  if (n < 10) {
    return (10 * n) + n;
  } else {
    int a = mystery(n /
10);
    int b = mystery(n %
10);
    return (100 * a) + b;
  }
}
```

```
// call 3b:     4
int mystery(int n) {
  if (n < 10) {
    return (10 * n) + n;
  } else {
    int a = mystery(n /
10);
    int b = mystery(n %
10);
    return (100 * a) + b;
  }
}
```

- Write a recursive function `evenDigits` that accepts an integer and returns a new number containing only the even digits, in the same order. If there are no even digits, return 0.
  - Example: `evenDigits(8342116)` returns `8426`
  - Example: `evenDigits(40109)` returns `400`
  - Example: `evenDigits(8)` returns `8`
  - Example: `evenDigits(-163505)` returns `-60`
  - Example: `evenDigits(35179)` returns `0`

```
// Returns a new integer containing only the even-valued
// digits from the given integer, in the same order.
// Returns 0 if there are no even digits.
int evenDigits(int n) {
    if (n < 0) {
        return -evenDigits(-n);
    } else if (n == 0) {
        return 0;
    } else if (n % 2 == 0) {
        return 10 * evenDigits(n / 10) + n % 10;
    } else {
        return evenDigits(n / 10);
    }
}
```

– What is a very easy power to compute without a loop?
– How is the task of computing exponents *self-similar* ?

```
int power(int base, int exp) {
    if (???) {
        // base case; no recursive calls needed
        ...
    } else {
        // recursive case
        ...
    }
}
```

- Each previous call waits for the next call to finish.
  - `cout << power(5, 3) << endl;`

```
// first call:   5         3
int power(int base, int exp) {
    if
    }

    }
}
```

```
// second call:  5         2
int power(int base, int exp) {
    if
        } (
    }
}
```

```
// third call:   5         1
int power(int base, int exp) {
    if (exp == 1) {
        return base;   // 5
    } else {
        return base * power(base, exp - 1);
    }
}
```

- Recursion is about solving a small piece of a large problem.
  - What is 69743 in binary?
    - Do we know *anything* about its representation in binary?
  - Case analysis:
    - What is/are easy numbers to print in binary?
    - Can we express a larger number in terms of a smaller number(s)?

- Suppose we are examining some arbitrary integer N.
  - if N's binary representation is **10010101011**
  - `(N / 2)`'s binary representation is **1001010101**
  - `(N % 2)`'s binary representation is **1**

  - What can we infer from this relationship?

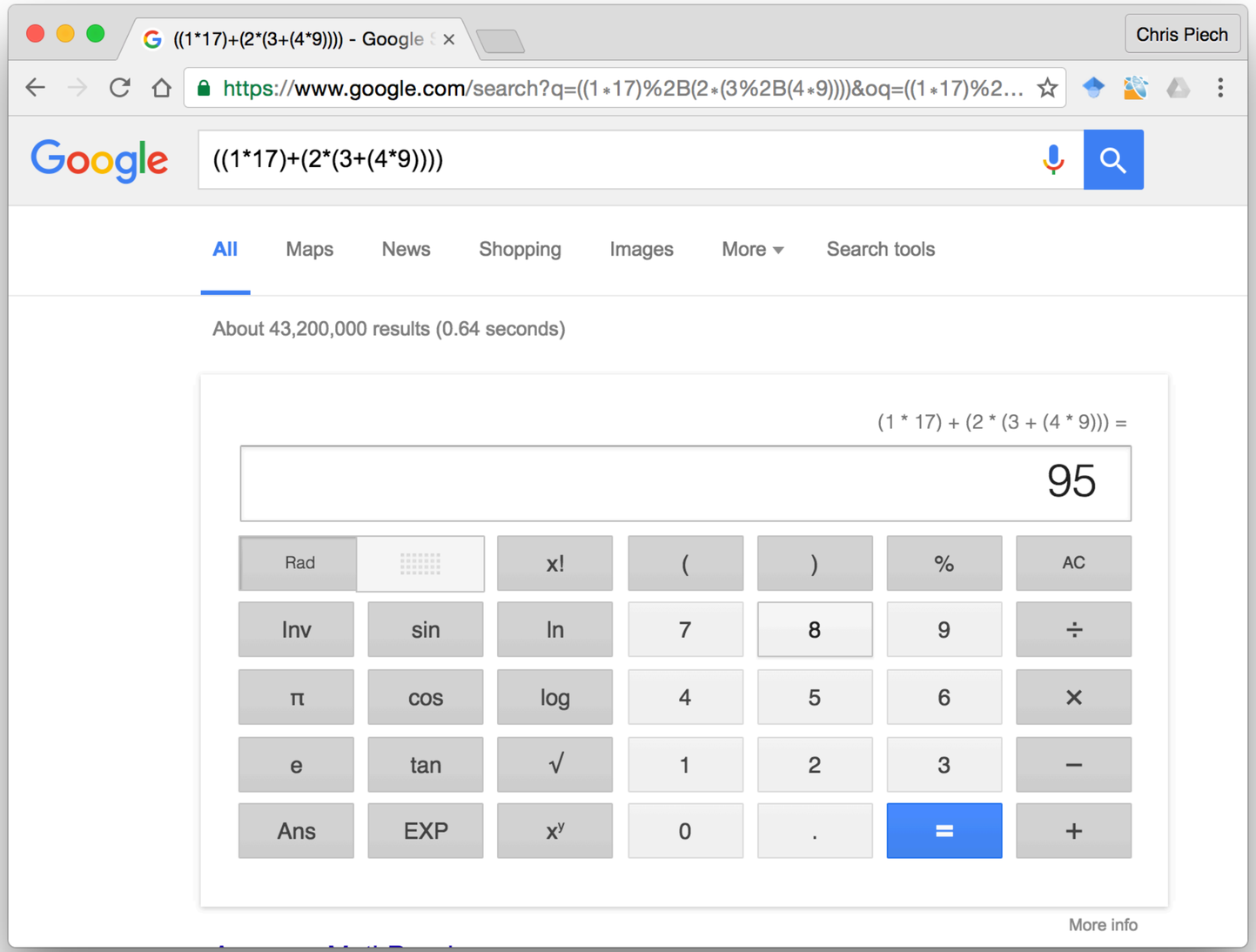
```
// Prints the given integer's binary representation.
// Precondition: n >= 0
void printBinary(int n) {
    if (n < 2) {
        // base case; same as base 10
        cout << n;
    } else {
        // recursive case; break number apart
        printBinary(n / 2);
        printBinary(n % 2);
    }
}
```

– Can we eliminate the precondition and deal with negatives?

```
// Prints the given integer's binary representation.
void printBinary(int n) {
    if (n < 0) {
        // recursive case for negative numbers
        cout << "-";
        printBinary(-n);
    } else if (n < 2) {
        // base case; same as base 10
        cout << n << endl;
    } else {
        // recursive case; break number apart
        printBinary(n / 2);
        printBinary(n % 2);
    }
```

((1+3)*(2*(4+1)))

Google Search I'm Feeling Lucky


((1*17)+(2*(3+(4*9)))) - Google
Chris Piech
https://www.google.com/search?q=((1*17)%2B(2*(3%2B(4*9))))&oq=((1*17)%2...
Google
((1*17)+(2*(3+(4*9))))
All
Maps
News
Shopping
Images
More
Search tools
About 43,200,000 results (0.64 seconds)
(1 * 17) + (2 * (3 + (4 * 9))) =
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x^y
0
.
=
+
More info

“((1+3)*(2*(4+1)))”

95


LELAND STANFORD JUNIOR UNIVERSITY
DIE LUFT DER FREIHEIT WEHT
1891

Challenge

Implement a function which evaluates an expression string:

“((1+3)*(2*(4+1)))”

or

“(7+6)”

or

“(((4*(1+2))+6)*7)”

Only * or +

Fully parenthesized

Motivating Problem

Implement a function which evaluates an expression:

```
((1+3)*(2*(4+1)))
```

or

```
(7+6)
```

or

```
(((4*(1+2))+6)*7)
```

Only * or +

Fully parenthesized

Task

Implement **`int evaluate(string & expression)`**:

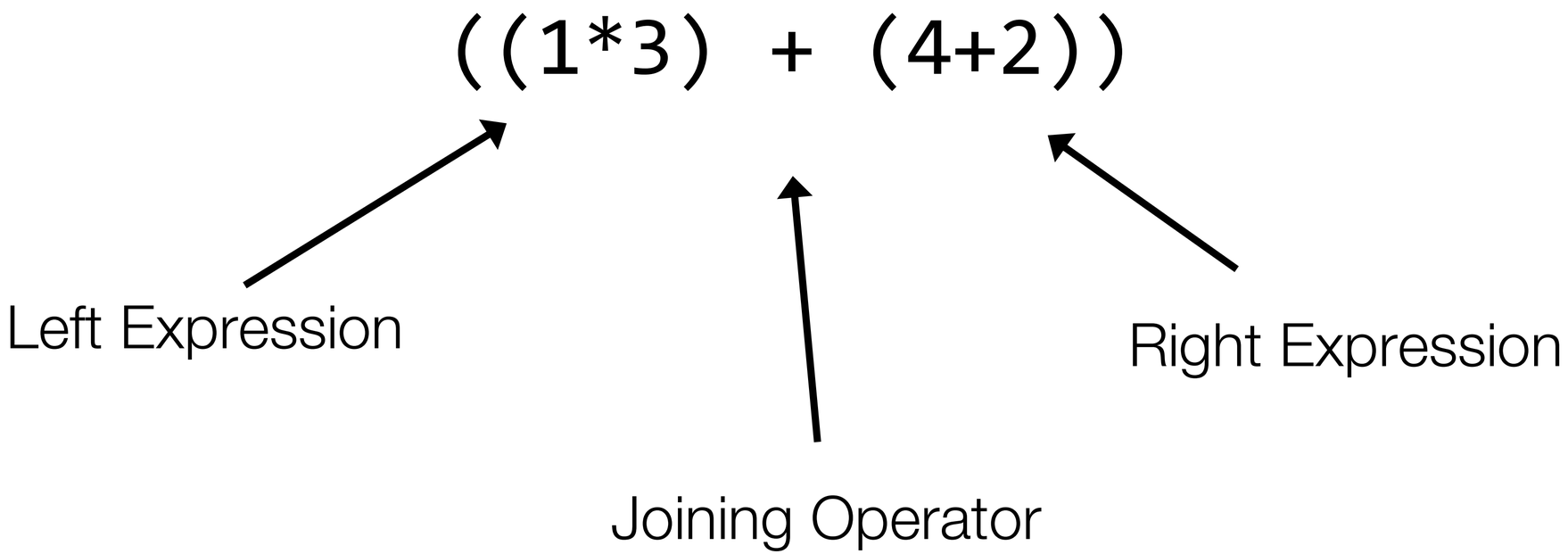
- Takes a parameter of format `"((1+3)*(2*(4+1)))"`
- **Fully parenthesized, no whitespace**
- Only multiplication and addition
- Returns the integer result of the expression, e.g. `40`

Assume you have a helper function:

- `indexOfOperator(string& exp);`

  which returns the index of the next operator to be evaluated

Anatomy of an Expression



It is Recursive

```
((1*3) + (4+2))
```

The big instance of this problem is:

```
((1*3) + (4+2))
```

The smaller instances are:

`(1*3)` and `(4+2)`

What's the algorithm for solving expressions?

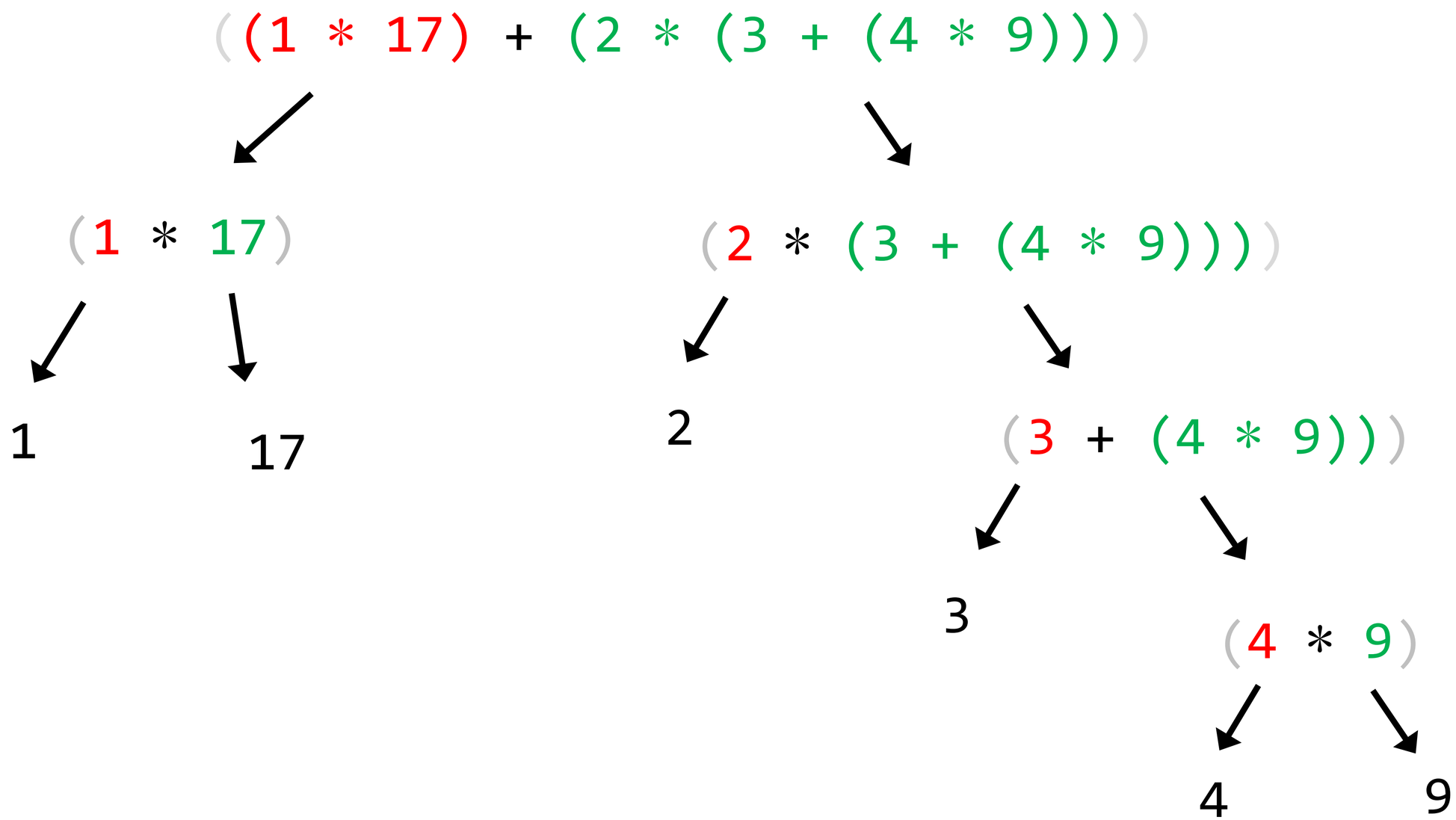
It is Recursive

```
((1*3) + (4+2))
```

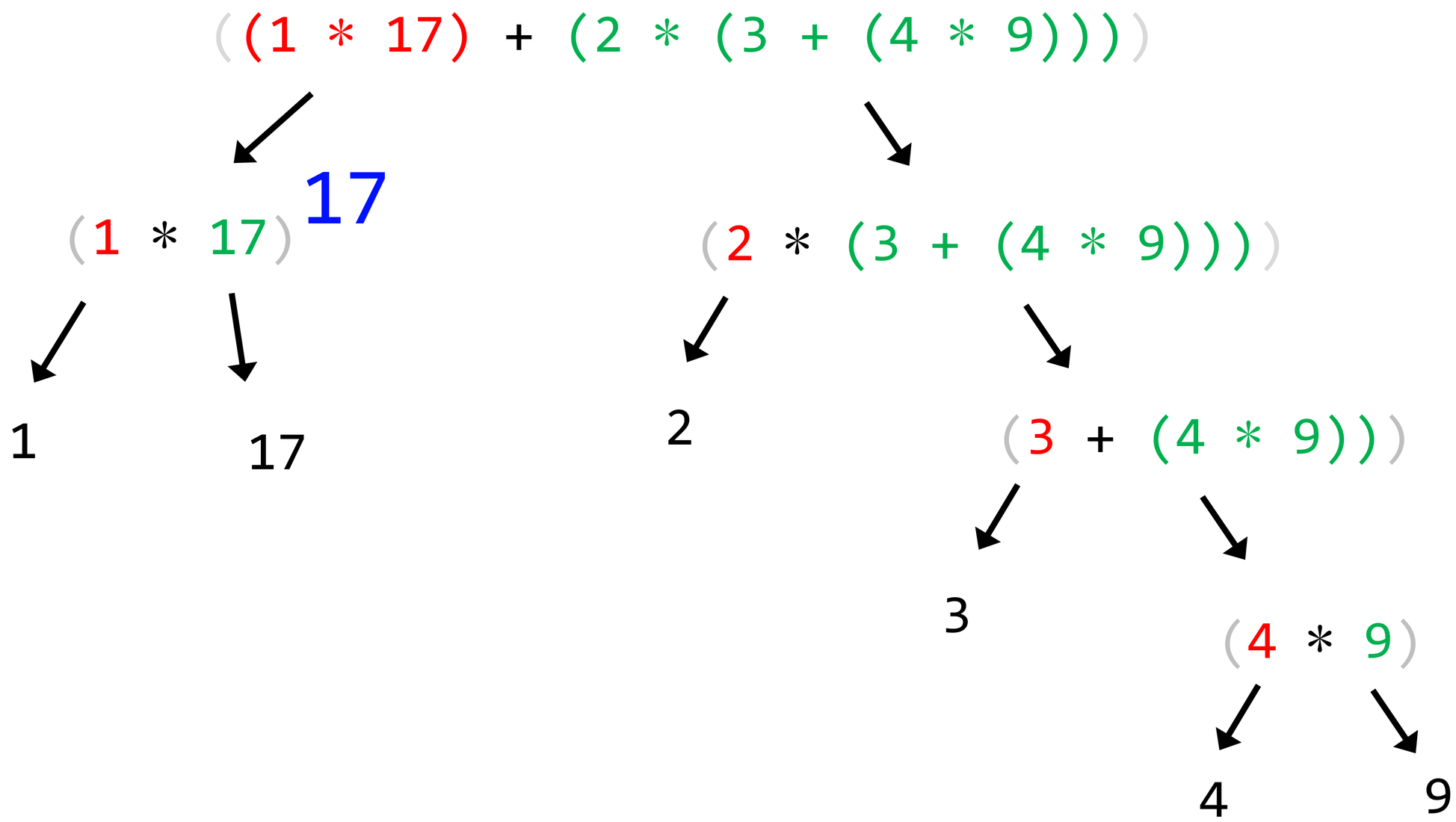
`int evaluate(expression):`

- If *expression* is a number, return *expression*
- Otherwise, break up *expression* by its joining *operator*:
  - *leftResult* = `evaluate(`*leftExpression*`)`
  - *rightResult* = `evaluate(`*rightExpression*`)`
  - `Return` *leftResult operator rightResult*

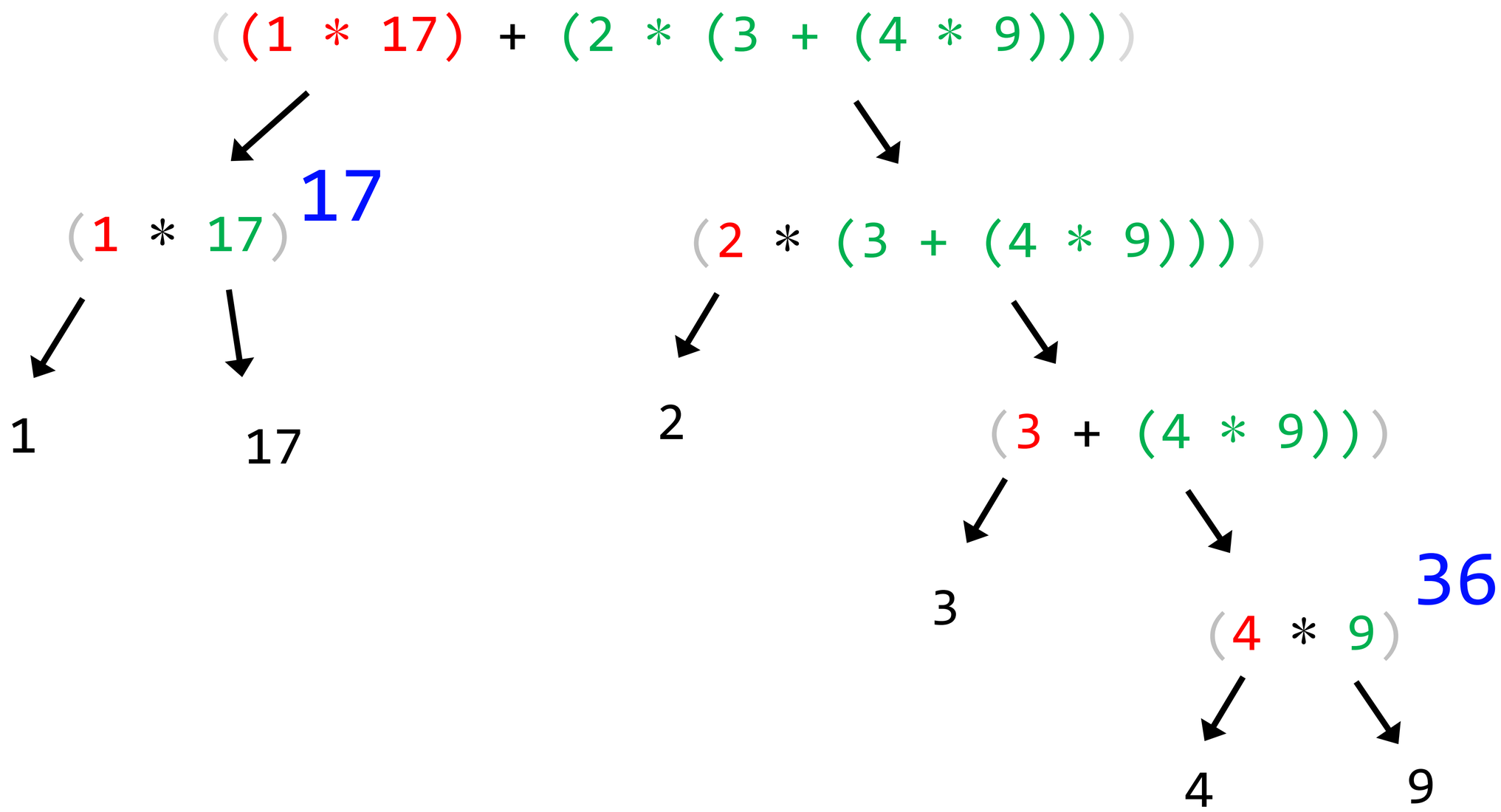
How do we evaluate ((1*17)+(2*(3+(4*9))))?



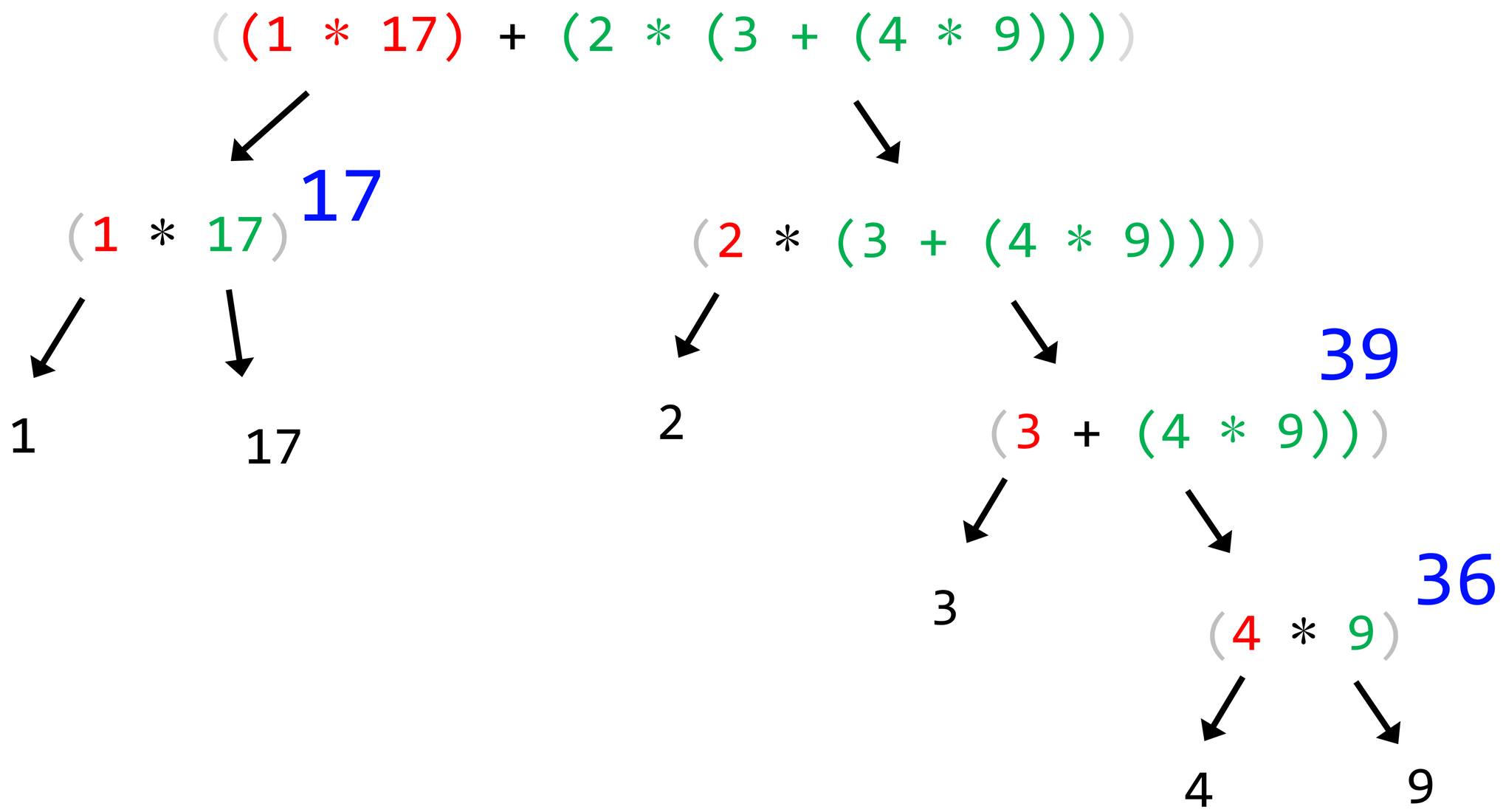
How do we evaluate ((1*17)+(2*(3+(4*9))))?



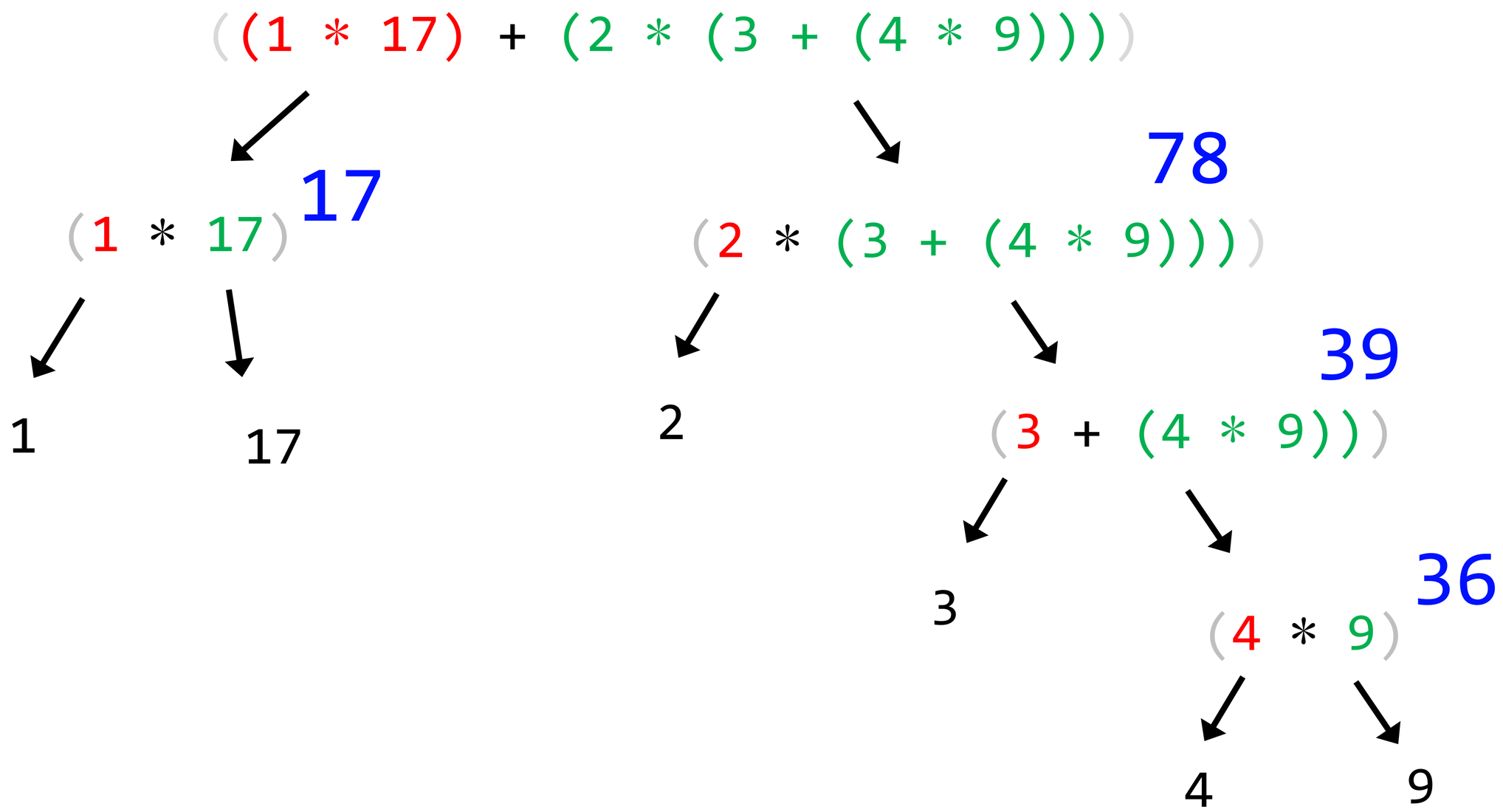
How do we evaluate ((1*17)+(2*(3+(4*9))))?



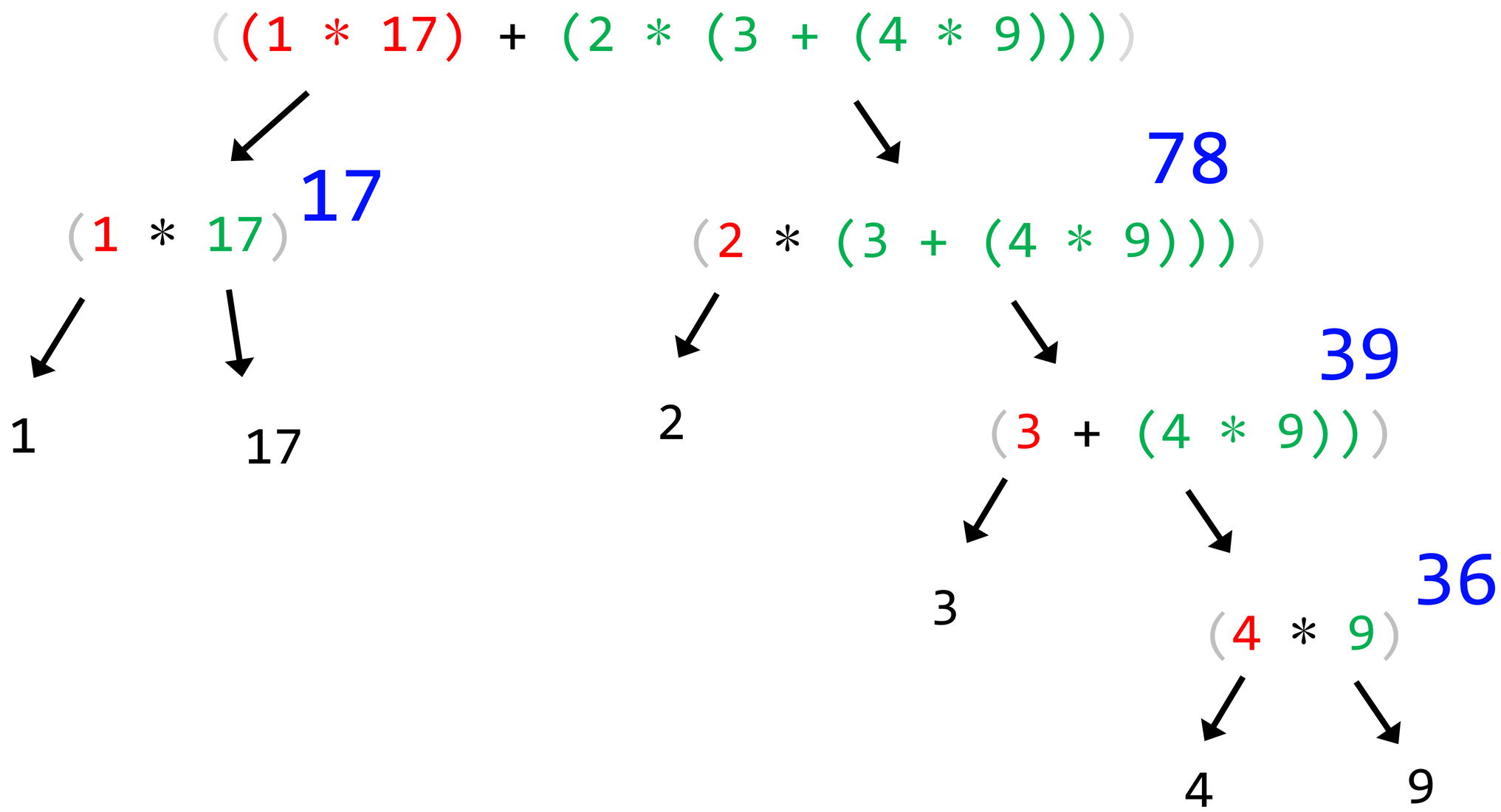
How do we evaluate ((1*17)+(2*(3+(4*9))))?



How do we evaluate ((1*17)+(2*(3+(4*9))))?



How do we evaluate ((1*17)+(2*(3+(4*9))))? 95



```
int evaluate(const string& expression) {
    if (stringIsInteger(expression)) {
        return stringToInteger(expression);
    }
    string unwrappedExpression = expression.substr(1, expression.length() - 2);
    int opIndex = indexOfNextOperator(unwrappedExpression);

    string left = unwrappedExpression.substr(0, opIndex);
    string right = unwrappedExpression.substr(opIndex + 1);
    int leftResult = evaluate(left);
    int rightResult = evaluate(right);

    char op = unwrappedExpression[opIndex];
    if (op == '+') {
        return leftResult + rightResult;
    } else if (op == '*') {
        return leftResult * rightResult;
    }
    throw "invalid expression string";
}
```