ST2334 2011/2012 Semester 2

Topic 3
Probability

Experiments

- **An action or process that leads to one of several possible outcomes.**

Experiment	Outcomes
Flip a coin	Heads, Tails
Assembly Time	t > 0 seconds
Course Grades	A+, A, A-, B+, ... , F
Roll a die twice	(1,1), (1,2), ..., (6,5), (6,6)

- **The *sample space* is the set of all possible outcomes**

Sample Space

- Finite vs Infinite sample space
  - Number of elements (points) they contain
  - Finite: card drawn from a deck
  - Infinite: number of die rolls until you roll 6
- Discrete vs Continuous sample space
  - Type of elements (points) they contain
  - Discrete: Finitely many or countably infinity of elements
  - Continuous: all elements part of a continuum
    - Eg. lifetime of a bulb, GPS coordinates

Events

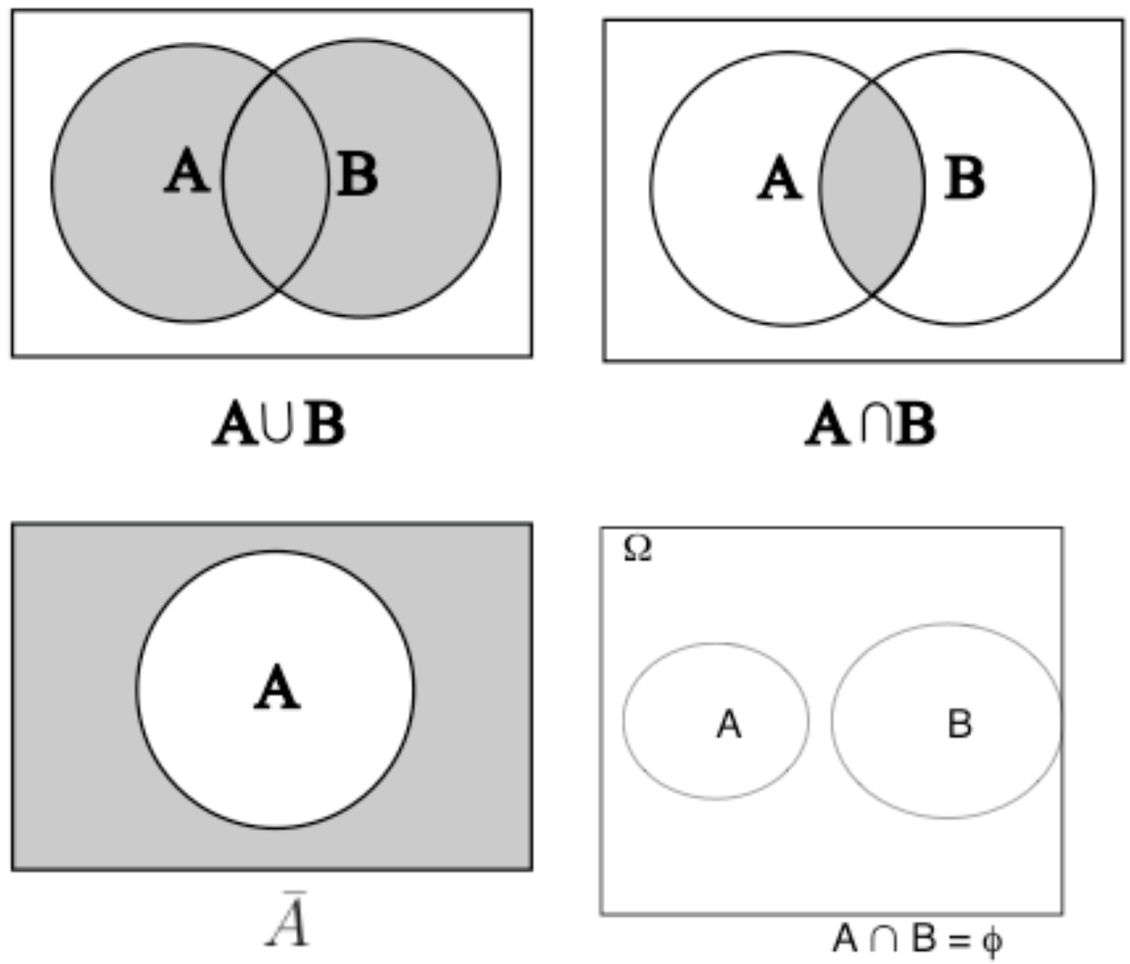
- Any subset of a sample space is called an *event*, denoted by A, B, C, …
  - Including whole set Ω (book uses S)
  - Empty set $\emptyset$.
- Two events are *mutually exclusive*, if they have no elements in common. (also known as *disjoint*)

Operations on Events

- The *union* of two sets *A* and *B* is denoted by $A \cup B$
  - Consist of all elements in *A* or *B* (or both).
- The *intersection* of two sets *A* and *B* is denoted by $A \cap B$
  - Consist of all elements in both *A* and *B*.
- The *complement* of set *A* is denoted by $\bar{A}$ or A^c
  - Consist of all elements not in *A*.

Venn Diagrams

- Graphical representation of sets, useful for depicting relationships and operations of events.



Simple rules

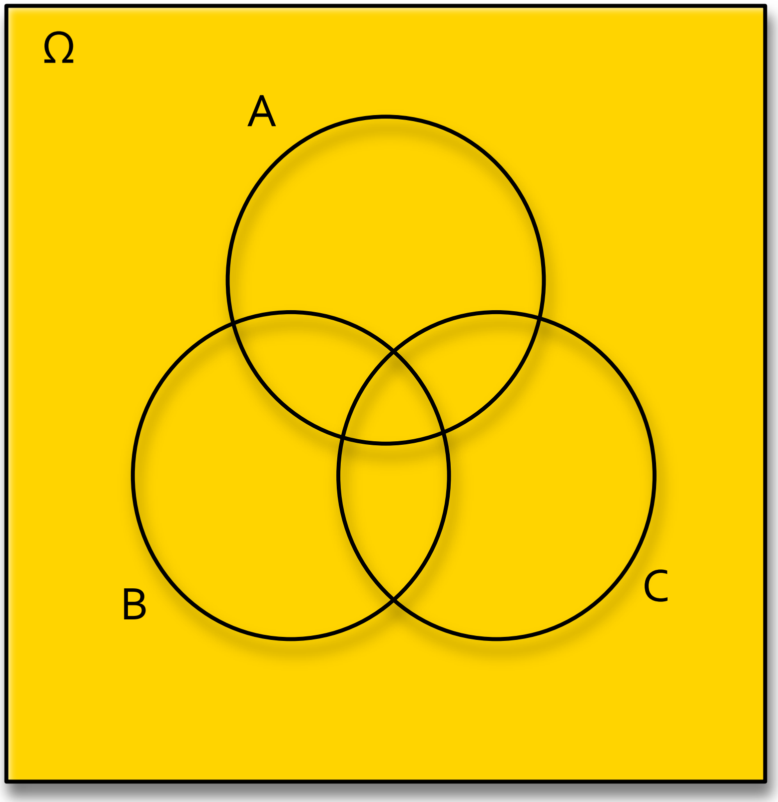
- Can use Venn diagram to show/remind you of some simple rules.
  - You can distribute "complement" eg.

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

  - Can also distribute union and intersection. E.g.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



Some useful Counting tools

- Factorial
  - Number of ways to arrange *n* objects
- Permutations
  - Number of ways to arrange *r* objects out of *n*
- Combinations
  - Number of ways to choose *r* objects out of *n*

Factorial

- If there are *n* objects, the number of ways of arranging them is

$$n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$$

- Example: Suppose that you have a deck of cards and you take all the hearts (13 of them) and shuffle them. How many different possible orderings can there be?

Permutation

- The number of ways to arrange *r* out of *n* objects is

$$P_n^r = \frac{n!}{(n-r)!}$$

  - order matters!

- Example: If you have 4 persons and 2 chairs, there are 4!/(4-2)! = 12 ways to fill the chairs.
- Example: You deal 5 hearts out of your 13 hearts. How many possible arrangements are there?

Combinations

- The number of ways to choose *r* objects from *n* objects is

$$\binom{n}{r} = C_n^r = \frac{n!}{r!(n-r)!}$$

  - Order does not matter

- Example: If you have 4 persons and 2 chairs, there are 4!/2!2! = 6 ways to choose who gets to sit.

- Example: You deal 5 hearts out of your 13 hearts. How many different possible combinations are there?

Probability Interpretations

- Equally-likely outcomes
- Frequency
- Subjectivity
- Note that these are interpretations, not assumptions.
  - Probability theory is built on rigorous mathematics which provides a framework that is consistent with these classical interpretations.

Equally-likely Outcomes

- If there are *m* equally likely possibilities, of which one must occur and *s* are regarded as a success, then the probability of a success is given by *s/m*.
- Example: Probability of drawing an Ace from a deck of cards is 4/52 = 1/13.
- What if you can't break it down into equally-likely outcomes?
  - Probability of rain tomorrow?

Frequency Interpretation

- The probability of an event is the proportion of times the event will occur in a long run of repeated experiments.
- Example: Probability of a jet from New York to Boston arriving on time is 78%. We mean that flights like this, over the long run will arrive on time 78% of the time.
- But what if it is not a repeatable event?
  - What is the probability of an earth-destroying earthquake occurring next year?

Subjective probabilities

- Subjective probabilities express the strength of one's belief with regard to uncertainties that are involved.
- Example: I think there's a 95% probability that she will say yes if I propose.
- Sometimes thought of as willingness to place a bet

Example: Roll 2 Dice

- Sum of spots on the 2 dice.
  - Sample space = {2, 3, …, 12}
  - Equally likely outcomes?
- Rolling 2 dice vs rolling 1 die twice?

$$
\begin{array}{rlllllll}
\Omega & = & \{(1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\
 & & (2,1) & (2,2), & (2,3), & (2,4), & (2,5), & (2,6), \\
 & & (3,1) & (3,2) & (3,3), & (3,4), & (3,5), & (3,6), \\
 & & (4,1) & (4,2) & (4,3) & (4,4), & (4,5), & (4,6), \\
 & & (5,1) & (5,2) & (5,3) & (5,4) & (5,5), & (5,6), \\
 & & (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\}
\end{array}
$$

Example: Roll 2 Dice

P(2) = 1/36

P(7) = 6/36

P(10) = 3/36

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Example: Birthday Paradox

- Party trick: walk into a room or bar with at least 50 people. Boldly claim that you sense two people sharing the same birthday. Act awesome afterwards.
  - How often are you right?
- We can cast this as a probability question.
  - For *n* randomly chosen people, what is the probability that there is at least two of them with the same birthday?
  - For simplicity, let's assume each of 365 days are equally likely

Example: Birthday Paradox

- Number of possible outcomes = 365^n
  - Equally likely.
- Number of outcomes with no one having the same birthday as anyone else = number of ways you can arrange n distinct birthdays = ${}_{365}P_n$ = $365!/(365-n)!$
- Probability of all birthdays different = $(365!/(365-n)!)/365^n$
- Probability of at least 2 people sharing a birthday = $1 - (365!/(365-n)!)/365^n$

Example: Birthday Paradox

n	P(at least 2 persons have the same birthdate)
10	0.117
20	0.411
21	0.444
22	0.476
23	0.507
24	0.538
30	0.706
50	0.970
57	0.990
100	0.9999997

Mathematical Construct

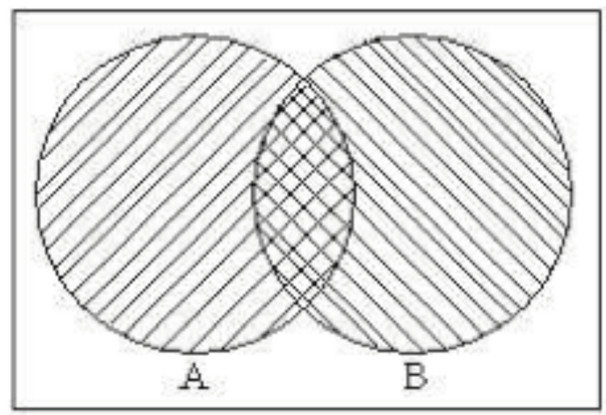
- Any function of events P(.) is a probability (function) if it satisfies the following axioms:
  - $0 \leq P(A) \leq 1$, for any event A
  - $P(\Omega) = 1$
  - If A and B are mutually exclusive events (in Ω), then $P(A \cup B) = P(A) + P(B)$
- Check that our interpretations satisfies the axioms.

Some Elementary Theorems

- If A is a subset of B, then $P(A) \leq P(B)$.
- For any event A, $P(A) = 1 - P(A^c)$
  - Hence, $P(\varnothing) = 0$.
- If $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$.
- For any events A, B,
  $P(A \cup B) = P(A) + P(B) - P(A \wedge B)$

Venn Diagrams (again!)

- We can prove the theorems rigorously. (Try it!)
- Venn diagrams is useful in visualizing these theorems.
- Example: we subtract out the intersection because we're double counting.

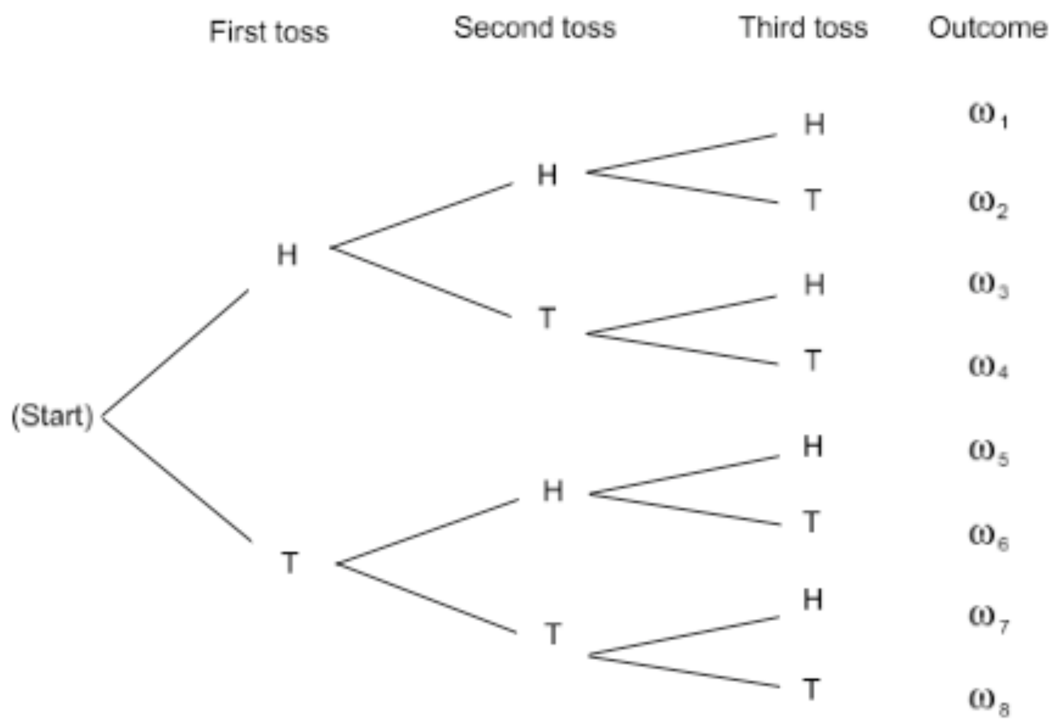


Example: Toss a coin 3 times

- Toss a fair coin 3 times.
  - Any of the 8 outcomes are equally likely.
- Let A be the event that first toss is heads.
  - $P(A) = 0.5$
- Let B be the event that second toss is tails.
  - $P(B) = 0.5$
- What is $P(A \cup B)$?
  - We can list them…

Graphical Representation

- Using tree diagrams is a convenient way of representing outcomes and keeping track of probabilities.
- 6 outcomes
  so 6/8 = ¾



Example: Toss a coin 3 times

- We can also make things a little easier by using our theorem.
  - $A \wedge B = \{\omega_3, \omega_4\}$
  - $P(A \cup B) = P(A) + P(B) – P(A \wedge B) = ½ + ½ - ¼ = ¾$

Errata

- Slide 19: ${}_{365}P_n = 365!/(365-n)!$