Relations

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23rd February 2023

Introduction

This chapter contains some material about relations and constructions with them. Notably, it contains:

- A basic discussion and definition of relations (Section 1.1);
- How relations may be viewed as decategorification of profunctors (Remarks 1.1.5 and 1.1.6)
- A discussion of the various kind of categories (a category, a monoidal category, a 2-category, a double category) that relations form (Sections 1.2 to 1.5);
- The various categorical properties of the 2-category of relations, including self-duality, identifications of adjunctions in Rel with functions, of monads in Rel with preorders, of comonads in Rel with subsets, the partial co/completeness of Rel, and its closedness, including how right Kan extensions and right Kan lifts exist in Rel (Section 1.6);
- A discussion of the various kinds of operations involving relations, such as graphs, domains, ranges, unions, intersections, products, inverse relations, composition of relations, and collages (Section 2);
- A discussion of equivalence relations (Section 3) and quotient sets (Section 3.5);
- A lengthy discussion of the adjoint pairs

  $$\begin{aligned} R_* \dashv R_{-1} &\colon \mathcal{P}(A) \rightleftarrows \mathcal{P}(B), \\ R^{-1} \dashv R_! &\colon \mathcal{P}(B) \rightleftarrows \mathcal{P}(A) \end{aligned}$$

  of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R\colon A \nrightarrow B$, along with a discussion of the properties of R_*, R_{-1}, R^{-1}, and $R_!$ (Section 4).

  These two pairs of adjoint functors are the counterpart of the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ induced by a function $f\colon A \to B$ studied in

Constructions With Sets, Section 3, and indeed we have $R_{-1} = R^{-1}$ iff R is total and functional (Item 7 of Proposition 4.2.3). Thus when R comes from a function this pair of adjunctions reduces to the triple adjunction $f_* \dashv f^{-1} \dashv f_!$ from before.

The pairs $R_* \dashv R_{-1}$ and $R^{-1} \dashv R_!$ will later make an appearance in the context of continuous, open, and closed relations between topological spaces (Topological Spaces, Section 5).

- A discussion of spans (Section 5) and their relation to functions (Proposition 5.2.1) and relations (Propositions 5.3.1 and 5.3.3 and Remark 5.3.5);
- A discussion of "hyperpointed sets" (Section 6). I don't know why I wrote this...

Notes to Myself

1. Define $\wedge$ and $\vee$.
2. Write about cospans.

Table of Contents

1 Relations

1.1 Foundations

Let A and B be sets.

Definition 1.1.1 ► Relations

A **relation** $R\colon A \nrightarrow B$ **from** A **to** B[1,2] is a subset R of $A \times B$.[3]

[1] *Further Terminology:* Also called a **multivalued function from** A **to** B, a **relation over** A **and** B, **relation on** A **and** B, a **binary relation over** A **and** B, or a **binary relation on** A **and** B.

[2] *Further Terminology:* When $A = B$, we also call $R \subset A \times A$ a **relation on** A.

[3] *Further Notation:* Given elements $a \in A$ and $b \in B$, we write $a \sim_R b$ to mean $(a, b) \in R$.

Definition 1.1.2 ► The Po/Set of Relations Over Two Sets

Let A and B be sets.

1. The **set of relations from** A **to** B is the set $\mathrm{Rel}(A, B)$ defined by

$$\mathrm{Rel}(A, B) \overset{\mathrm{def}}{=} \{\text{Relations from } A \text{ to } B\}.$$

2. The **poset of relations from** A **to** B is the poset $\mathbf{Rel}(A, B) \overset{\mathrm{def}}{=} (\mathrm{Rel}(A, B), \subset)$ consisting of

   - *The Underlying Set.* The set $\mathrm{Rel}(A, B)$ of Item 1;
   - *The Partial Order.* The partial order

$$\subset\colon \mathrm{Rel}(A, B) \times \mathrm{Rel}(A, B) \longrightarrow \{\mathrm{true}, \mathrm{false}\}$$

   on $\mathrm{Rel}(A, B)$ given by inclusion of relations.

Remark 1.1.3 ► Equivalent Definitions of Relations

A relation from A to B is equivalently:[1]

1. A subset of $A \times B$.
2. A function from $A \times B$ to $\{\mathrm{true}, \mathrm{false}\}$.
3. A function from A to $\mathcal{P}(B)$.
4. A function from B to $\mathcal{P}(A)$.
5. A cocontinuous morphism of posets from $(\mathcal{P}(A), \subset)$ to $(\mathcal{P}(B), \subset)$.

That is: we have bijections of sets

$$\mathrm{Rel}(A, B) \overset{\mathrm{def}}{=} \mathcal{P}(A \times B),$$

$$
\begin{aligned}
&\cong \mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}),\\
&\cong \mathsf{Sets}(A, \mathcal{P}(B)),\\
&\cong \mathsf{Sets}(B, \mathcal{P}(A)),\\
&\cong \mathrm{Hom}^{\mathrm{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A), \mathcal{P}(B)),
\end{aligned}
$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

[1] *Intuition:* In particular, we may think of a relation $R\colon A \to \mathcal{P}(B)$ from A to B as a multivalued function from A to B (including the possibility of a given $a \in A$ having no value at all).

Proof 1.1.4 ▶ Proof of Remark 1.1.3

We claim that Items 1 to 5 are indeed equivalent:

- The equivalence between Items 1 and 2 is a special case of Sets, **??** of **??**.
- The equivalence between Items 2 and 3 is an instance of currying, following from the bijections

$$
\begin{aligned}
\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\
&\cong \mathsf{Sets}(A, \mathcal{P}(B)). \qquad (\text{Sets, ?? of ??})
\end{aligned}
$$

- The equivalence between Items 2 and 4 is also an instance of currying, following from the bijections

$$
\begin{aligned}
\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(B, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\
&\cong \mathsf{Sets}(B, \mathcal{P}(A)). \qquad (\text{Sets, ?? of ??})
\end{aligned}
$$

- The equivalence between Items 2 and 5 follows from the universal property of the powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi_X\colon X \hookrightarrow \mathcal{P}(X)$$

  of X into $\mathcal{P}(X)$ (Sets, **??** of **??**).[1]

This finishes the proof.

[1] In particular, given a relation $f\colon A \to \mathcal{P}(B)$ from A to B, we may extend the domain of f from A to all of $\mathcal{P}(A)$ by taking its left Kan extension along χ_X. This also coincides with the direct image function $f_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$ of Constructions With Sets, Definition 3.3.1.

Remark 1.1.5 ► Relations as Decategorifications of Profunctors I

The notion of a relation is a decategorification of that of a profunctor: while a profunctor from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor

$$\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets},$$

a relation on sets A and B is a function

$$R\colon A \times B \to \{\mathsf{true}, \mathsf{false}\},$$

where we notice that:

- The opposite X^{op} of a set X is itself, as $(-)^{\mathsf{op}}\colon \mathsf{Cats} \to \mathsf{Cats}$ restricts to the identity endofunctor on Sets;
- While
  - A category is enriched over the category

    $$\mathsf{Sets} \overset{\text{def}}{=} \mathsf{Cats}_0$$

    of sets, with profunctors taking values on it;
  - A set is enriched over the set

    $$\{\mathsf{true}, \mathsf{false}\} \overset{\text{def}}{=} \mathsf{Cats}_{-1}$$

    of classical truth values, with relations taking values on it;

Remark 1.1.6 ► Relations as Decategorifications of Profunctors II

Extending Remark 1.1.5, the equivalent definitions of relations in Remark 1.1.3 are also related to the corresponding ones for profunctors (Categories, Remark 3.1.2), which state that a profunctor $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ is equivalently:

1. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$;
2. A functor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$;
3. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$;
4. A colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$.

Indeed:

- The equivalence between Items 1 and 2 (and also that between Items 1 and 3, which is proved analogously) is an instance of currying, both for profunctors as well as for relations, using the isomorphisms

$$\begin{aligned}\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)),\\ \mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{D}, \mathsf{Sets}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Sets}))\\ &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})).\end{aligned}$$

- The equivalence between Items 1 and 3 follows from the universal properties of:
  - The powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding
    $$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$
    of X into $\mathcal{P}(X)$ (Sets, **??** of **??**);
  - The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on a category $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding
    $$よ \colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$
    of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$ (Categories, **??** of Proposition 7.3.2).

Example 1.1.7 ▶ The Trivial Relation

The **trivial relation on A and B** is the relation $\sim_{\mathsf{triv}}$ defined by[1,2,3]

$$\sim_{\mathsf{triv}} \overset{\mathrm{def}}{=} A \times A.$$

[1] This is the unique relation R on A and B such that we have $a \sim_R b$ for all $a \in A$ and all $b \in B$.

[2] As a function from $A \times A$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\mathsf{triv}}$ is the constant function

$$\Delta_{\mathsf{true}} \colon A \times B \to \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking value true.

[3] As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\mathsf{triv}}$ is the function

$$\Delta_{\mathsf{true}} \colon A \to \mathcal{P}(B)$$

defined by

$$\Delta_{\mathsf{true}}(a) \overset{\mathrm{def}}{=} B$$

for each $a \in A$.

Example 1.1.8 ► The Cotrivial Relation

The **cotrivial relation on A and B** is the relation $\sim_{\text{cotriv}}$ defined by[1,2,3]

$$\sim_{\text{cotriv}} \overset{\text{def}}{=} \emptyset.$$

---

[1]This is the unique relation R on A and B such that we have $a \sim_R b$ for no $a \in A$ and no $b \in B$.

[2]As a function from $A \times B$ to $\{\text{true}, \text{false}\}$, the relation $\sim_{\text{cotriv}}$ is the constant function

$$\Delta_{\text{false}} \colon A \times B \to \{\text{true}, \text{false}\}$$

from $A \times B$ to $\{\text{true}, \text{false}\}$ taking value false.

[3]As a function from A to $\mathcal{P}(A)$, the relation $\sim_{\text{cotriv}}$ is the function

$$\Delta_{\text{false}} \colon A \to \mathcal{P}(A)$$

defined by

$$\Delta_{\text{true}}(a) \overset{\text{def}}{=} \emptyset$$

for each $a \in A$.

Example 1.1.9 ► The Characteristic Relation

The characteristic relation on A of Sets, **??** of **??** is another example of a relation. It is in fact the unique relation on A making the following conditions equivalent, for each $a, b \in A$:

1. We have $a \sim_{\text{id}} b$.
2. We have $a = b$.

Example 1.1.10 ► Square Roots

Square roots are examples of relations:

1. *Square Roots in $\mathbb{R}$.* The assignment $x \mapsto \sqrt{x}$ defines a relation

   $$\sqrt{-} \colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$$

   from $\mathbb{R}$ to itself, being explicitly given by

   $$\sqrt{x} \overset{\text{def}}{=} \begin{cases} 0 & \text{if } x = 0, \\ \left\{-\sqrt{|x|}, \sqrt{|x|}\right\} & \text{if } x \neq 0. \end{cases}$$

2. *Square Roots in $\mathbb{Q}$.* Square roots in $\mathbb{Q}$ are similar to square roots in $\mathbb{R}$, though now additionally it may also occur that $\sqrt{-} \colon \mathbb{Q} \to \mathcal{P}(\mathbb{Q})$ sends a rational number x (e.g. 2) to the empty set (since $\sqrt{2} \notin \mathbb{Q}$).

Example 1.1.11 ► Complex Logarithms

The complex logarithm defines a relation

$$\log\colon \mathbb{C} \to \mathcal{P}(\mathbb{C})$$

from $\mathbb{C}$ to itself, where we have

$$\log(a+bi) \overset{\text{def}}{=} \left\{\log\left(\sqrt{a^2+b^2}\right) + i\arg(a+bi) + (2\pi i)k \,\middle|\, k \in \mathbb{Z}\right\}$$

for each $a+bi \in \mathbb{C}$.

Example 1.1.12 ► More Examples of Relations

See [Wik22] for more examples of relations, such as antiderivation, inverse trigonometric functions, and inverse hyperbolic functions.

1.2 The Category of Relations

Definition 1.2.1 ► The Category of Relations

The **category of relations** is the category Rel where

- *Objects.* The objects of Rel are sets;
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathsf{Rel}(A,B) \overset{\text{def}}{=} \mathrm{Rel}(A,B);$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\mathbb{1}^{\mathsf{Rel}}_A \colon \mathrm{pt} \to \mathsf{Rel}(A,A)$$

  of Rel at A is defined by

$$\mathrm{id}^{\mathsf{Rel}}_A \overset{\text{def}}{=} \chi_A(-_1,-_2),$$

  where $\chi_A(-_1,-_2)$ is the characteristic relation of A of Sets, **??** of **??**;
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map

$$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathsf{Rel}(B,C) \times \mathsf{Rel}(A,B) \to \mathsf{Rel}(A,C)$$

of Rel at (A, B, C) is defined by

$$S \circ^{\mathsf{Rel}}_{A,B,C} R \stackrel{\text{def}}{=} S \diamond R$$

for each $(S, R) \in \mathsf{Rel}(B, C) \times \mathsf{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 2.11.1.

1.3 The Closed Symmetric Monoidal Category of Relations

Definition 1.3.1 ► The Closed Symmetric Monoidal Category of Relations

The **closed symmetric monoidal category of relations** is the closed symmetric monoidal category $(\mathsf{Rel}\ , \times, \mathbb{1}_{\mathsf{Rel}}, \alpha^{\mathsf{Rel}}, \lambda^{\mathsf{Rel}}, \rho^{\mathsf{Rel}}, \sigma^{\mathsf{Rel}}, \ \mathbf{Hom}_{\mathsf{Rel}}\ )$ consisting of

- *The Underlying Category.* The category Rel of sets and relations;
- *The Monoidal Product.* The functor

  $$\times \colon \ \mathsf{Rel} \times \mathsf{Rel} \longrightarrow \mathsf{Rel}$$

  where

  - *Action on Objects.* We have

    $$\times(A, B) \stackrel{\text{def}}{=} A \times B,$$

    where $A \times B$ is the Cartesian product of sets of Sets, **??**;
  - *Action on Morphisms.* For each pair of morphisms

    $$\begin{aligned} R &\colon A \nrightarrow B, \\ S &\colon C \nrightarrow D \end{aligned}$$

    of Rel, the image

    $$R \times S \colon \ A \times C \nrightarrow B \times D$$

    of (R, S) by $\times$ is the relation

    $$R \times S \colon \ (A \times C) \times (B \times D) \longrightarrow \{\mathsf{true}, \mathsf{false}\}$$

    of Definition 2.8.1;

- *The Monoidal Unit.* The functor

$$\mathbb{1}_{\mathsf{Rel}}\colon \mathrm{pt} \longrightarrow \mathsf{Rel}$$

picking the punctual set pt;

- *The Associator.* The natural isomorphism

$$\alpha^{\mathsf{Rel}}\colon \times\circ((\times)\times\mathrm{id}) \overset{\cong}{\Longrightarrow} \times\circ(\mathrm{id}\times(\times)), \qquad \begin{tikzcd} \mathsf{Rel}\times\mathsf{Rel}\times\mathsf{Rel} \arrow[r, "\mathrm{id}\times(\times)"] \arrow[d, "(\times)\times\mathrm{id}"'] & \mathsf{Rel}\times\mathsf{Rel} \arrow[d, "\times"] \\ \mathsf{Rel}\times\mathsf{Rel} \arrow[r, "\times"'] \arrow[ur, phantom, "\Nearrow\alpha^{\mathsf{Rel}}"] & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Rel}}_{A,B,C}\colon (A\times B)\times C \mathrel{\rightarrow\mkern-15mu+\mkern6mu} A\times(B\times C)$$

at (A, B, C) is defined by declaring

$$((a,b),c) \sim_{\alpha^{\mathsf{Rel}}_{A,B,C}} (a',(b',c'))$$

iff $a = a'$, $b = b'$, and $c = c'$;

- *The Left Unitor.* The natural isomorphism

$$\lambda^{\mathsf{Rel}}\colon \times\circ\left(\mathbb{1}^{\mathsf{Rel}}\times\mathrm{id}\right) \overset{\cong}{\Longrightarrow} \lambda^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathrm{pt}\times\mathsf{Rel} \arrow[r, "\mathbb{1}^{\mathsf{Rel}}\times\mathrm{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel}\times\mathsf{Rel} \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow\lambda^{\mathsf{Rel}}"] \\ & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Rel}}_{A}\colon \mathbb{1}_{\mathsf{Rel}}\times A \mathrel{\rightarrow\mkern-15mu+\mkern6mu} A$$

at A is defined by declaring

$$(\star, a) \sim_{\lambda^{\mathsf{Rel}}_{A}} b$$

iff $a = b$;

- *The Right Unitor.* The natural isomorphism

$$\rho^{\mathsf{Rel}}\colon \times\circ\left(\mathrm{id}\times\mathbb{1}^{\mathsf{Rel}}\right)\overset{\cong}{\Longrightarrow}\rho^{\mathsf{Cats}_2}_{\mathsf{Rel}},\qquad \begin{tikzcd} \mathsf{Rel}\times\mathrm{pt} \arrow[r, "\mathrm{id}\times\mathbb{1}^{\mathsf{Rel}}"] \arrow[rd, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel}\times\mathsf{Rel} \arrow[d, "\times"] \arrow[ld, phantom, "\overset{\rho^{\mathsf{Rel}}}{\Swarrow}"] \\ & \mathsf{Rel}, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Rel}}_A\colon A\times\mathbb{1}_{\mathsf{Rel}}\nrightarrow A$$

at A is defined by declaring

$$(a,\star)\sim_{\rho^{\mathsf{Rel}}_A} b$$

iff $a=b$;

- *The Symmetry.* The natural isomorphism

$$\sigma^{\mathsf{Rel}}\colon \times\Longrightarrow\times\circ\,\sigma^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}},\qquad \begin{tikzcd} \mathsf{Rel}\times\mathsf{Rel} \arrow[rr, "\times"] \arrow[rd, bend right, "\sigma^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}"'] & \arrow[d, phantom, "\overset{\sigma^{\mathsf{Rel}}}{\Downarrow}"] & \mathsf{Rel}, \\ & \mathsf{Rel}\times\mathsf{Rel} \arrow[ru, bend right, "\times"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Rel}}_{A,B}\colon A\times B\longrightarrow B\times A$$

at (A,B) is defined by declaring

$$(a,b)\sim_{\sigma^{\mathsf{Rel}}_{A,B}}(b',a')$$

iff $a=a'$ and $b=b'$.

- *The Internal* **Hom**. The bifunctor[1]

$$\mathbf{Hom}_{\mathsf{Rel}}\colon\ \mathsf{Rel}^{\mathsf{op}}\times\mathsf{Rel}\longrightarrow\mathsf{Rel}$$

defined by

$$\mathbf{Hom}_{\mathsf{Rel}}(A,B) \overset{\mathrm{def}}{=} A \times B$$

for each $A, B \in \mathrm{Obj}(\mathsf{Rel})$, with its left and right partial functors being adjoint to $\times$, witnessed by bijections of sets[2]

$$\begin{aligned}\mathsf{Rel}(A\times B, C) &\cong \mathsf{Rel}(A, \mathbf{Hom}_{\mathsf{Rel}}(B,C)) \\ &\overset{\mathrm{def}}{=} \mathsf{Rel}(A, B\times C), \\ \mathsf{Rel}(A\times B, C) &\cong \mathsf{Rel}(B, \mathbf{Hom}_{\mathsf{Rel}}(A,C)) \\ &\overset{\mathrm{def}}{=} \mathsf{Rel}(B, A\times C),\end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

[1] More precisely, $\mathbf{Hom}_{\mathsf{Rel}}$ is given by the composition

$$\mathsf{Rel}^{\mathsf{op}} \times \mathsf{Rel} \xrightarrow{\cong} \mathsf{Rel} \times \mathsf{Rel} \xrightarrow{\times} \mathsf{Rel},$$

where the self-duality equivalence $\mathsf{Rel}^{\mathsf{op}} \cong \mathsf{Rel}$ comes from **??** of Proposition 1.6.1.

[2] Indeed, we have

$$\begin{aligned}\mathsf{Rel}(A\times B, C) &\overset{\mathrm{def}}{=} \mathsf{Sets}(A\times B\times C, \{\mathsf{true}, \mathsf{false}\}) \\ &\overset{\mathrm{def}}{=} \mathsf{Rel}(A, B\times C) \\ &\overset{\mathrm{def}}{=} \mathsf{Rel}(A, \mathbf{Hom}_{\mathsf{Rel}}(B,C)),\end{aligned}$$

and similarly for the isomorphism $\mathsf{Rel}(A\times B, C) \cong \mathsf{Rel}(B, \mathbf{Hom}_{\mathsf{Rel}}(A,C))$.

1.4 The 2-Category of Relations

Definition 1.4.1 ▶ The 2-Category of Relations

The **2-category of relations** is the locally posetal 2-category **Rel** where

- *Objects.* The objects of **Rel** are sets;
- **Hom**-*Posets.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Rel}}(A,B) &\overset{\mathrm{def}}{=} \mathbf{Rel}(A,B) \\ &\overset{\mathrm{def}}{=} (\mathsf{Rel}(A,B), \subset);\end{aligned}$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathbf{Rel})$, the unit map

$$\mathbb{1}^{\mathbf{Rel}}_A \colon \mathrm{pt} \longrightarrow \mathbf{Rel}(A,A)$$

of **Rel** at A is defined by

$$\mathrm{id}^{\mathbf{Rel}}_A \overset{\mathrm{def}}{=} \chi_A(-_1, -_2),$$

where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Sets, **??** of **??**;

- *Composition*. For each $A, B, C \in \mathrm{Obj}(\mathbf{Rel})$, the composition map[1]

$$\circ^{\mathbf{Rel}}_{A,B,C} \colon \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, C)$$

of $\mathbf{Rel}$ at (A, B, C) is defined by

$$S \circ^{\mathbf{Rel}}_{A,B,C} R \stackrel{\text{def}}{=} S \diamond R$$

for each $(S, R) \in \mathbf{Rel}(B, C) \times \mathbf{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 2.11.1.

[1]Note that this is indeed a morphism of posets: given relations $R_1, R_2 \in \mathbf{Rel}(A, B)$ and $S_1, S_2 \in \mathbf{Rel}(B, C)$ such that

$$R_1 \subset R_2,$$
$$S_1 \subset S_2,$$

we have also $S_1 \diamond R_1 \subset S_2 \diamond R_2$.

1.5 The Double Category of Relations

Definition 1.5.1 ► The Double Category of Relations

The **double category of relations** is the locally posetal double category $\mathrm{Rel}^{\mathsf{dbl}}$ where

- *Objects*. The objects of $\mathrm{Rel}^{\mathsf{dbl}}$ are sets;
- *Vertical Morphisms*. The vertical morphisms of $\mathrm{Rel}^{\mathsf{dbl}}$ are maps of sets $f \colon A \to B$;
- *Horizontal Morphisms*. The horizontal morphisms of $\mathrm{Rel}^{\mathsf{dbl}}$ are relations $R \colon A \nrightarrow X$;
- *2-Morphisms*. A 2-cell

$$\begin{tikzcd}
A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\
X \arrow[r, "S"', "\shortmid"{marking}] & Y
\end{tikzcd}$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is either non-existent or an inclusion of relations of the form

$$R \subset S \circ (f \times g), \qquad \begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\ X \times Y \arrow[r, "S"'] & \{\mathrm{true}, \mathrm{false}\}; \end{tikzcd}$$

- *Horizontal Identities.* The horizontal unit functor

$$\nVdash^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_0 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

  of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

  - *Action on Objects.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}_0\right)$, we have

$$\nVdash_A \overset{\mathrm{def}}{=} \chi_A(-_1, -_2);$$

  - *Action on Morphisms.* For each vertical morphism $f \colon A \to B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "\nVdash_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \nVdash_f"] & A \arrow[d, "f"] \\ B \arrow[r, "\nVdash_B"', "\shortmid"{marking}] & B \end{tikzcd}$$

    of f is the inclusion

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f\times f"'] \arrow[dr, phantom, "\subset"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathrm{true}, \mathrm{false}\} \end{tikzcd}$$

    of Sets, Definition 1.2.3;

- *Vertical Identities.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}\right)$, we have

$$\mathrm{id}^{\mathsf{Rel}^{\mathsf{dbl}}}_A \overset{\mathrm{def}}{=} \mathrm{id}_A;$$

- *Identity 2-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the identity 2-morphism

$$
\begin{tikzcd}
A \arrow[r, "R", "|" marking] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\Downarrow \mathrm{id}_R"] & B \arrow[d, "\mathrm{id}_B"] \\
A \arrow[r, "R"', "|" marking] & B
\end{tikzcd}
$$

of R is the identity inclusion

$$
R \subset R, \qquad
\begin{tikzcd}
B \times A \arrow[r, "R"] \arrow[d, "\mathrm{id}_B \times \mathrm{id}_A"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\
B \times A \arrow[r, "R"'] & \{\mathrm{true}, \mathrm{false}\};
\end{tikzcd}
$$

- *Horizontal Composition.* The horizontal composition functor

$$\odot^{\mathsf{Rel}^{\mathsf{dbl}}}\colon \mathsf{Rel}^{\mathsf{dbl}}_1 \underset{\mathsf{Rel}^{\mathsf{dbl}}_0}{\times} \mathsf{Rel}^{\mathsf{dbl}}_1 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

  - *Action on Objects.* For each composable pair $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, we have

$$S \odot R \overset{\mathrm{def}}{=} S \diamond R,$$

  where $S \diamond R$ is the composition of R and S of Definition 2.11.1;

  - *Action on Morphisms.* For each horizontally composable pair

$$
\begin{tikzcd}
A \arrow[r, "R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \alpha"] & B \arrow[d, "g"] \\
X \arrow[r, "T"', "|" marking] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "S", "|" marking] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow \beta"] & C \arrow[d, "h"] \\
Y \arrow[r, "U"', "|" marking] & Z
\end{tikzcd}
$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each pair

$$\begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times C \arrow[r, "S"] \arrow[d, "g \times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ Y \times Z \arrow[r, "U"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of inclusions of relations, the horizontal composition

$$\begin{tikzcd} A \arrow[r, "S \odot R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \beta \odot \alpha"] & C \arrow[d, "h"] \\ X \arrow[r, "U \odot T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of α and β is the inclusion of relations

$$(U \diamond T) \circ (f \times h) \subset (S \diamond R) \qquad \begin{tikzcd} A \times C \arrow[r, "S \diamond R"] \arrow[d, "f \times h"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Z \arrow[r, "U \diamond T"'] & \{\mathsf{true}, \mathsf{false}\}, \end{tikzcd}$$

which is justified by noting that, given $(a, c) \in A \times C$, the statement

- We have $a \sim_{(U \diamond T) \circ (f \times h)} c$, i.e. $f(a) \sim_{U \diamond T} h(c)$, i.e. there exists some $y \in Y$ such that:
  1. We have $f(a) \sim_T y$;
  2. We have $y \sim_U h(c)$;

is implied by the statement

- We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
  1. We have $a \sim_R b$;
  2. We have $b \sim_S c$;

since:

- If $a \sim_R b$, then $f(a) \sim_T g(b)$, as $T \circ (f \times g) \subset R$;
- If $b \sim_S c$, then $g(b) \sim_U h(c)$, as $U \circ (g \times h) \subset S$;

- *Vertical Composition of* 1*-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. maps of sets, we have

$$g \circ^{\mathsf{Rel}^{\mathsf{dbl}}} f \overset{\text{def}}{=} g \circ f;$$

- *Vertical Composition of* 2*-Morphisms.* For each vertically composable pair

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "g"] \\ B \arrow[r, "S"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & Y \arrow[d, "k"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each each pair

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathsf{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times Y \arrow[r, "S"] \arrow[d, "h \times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathsf{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of inclusions of relations, we define the vertical composition

$$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "h \circ f"'] \arrow[dr, phantom, "\Downarrow\beta \circ \alpha"] & X \arrow[d, "k \circ g"] \\ C \arrow[r, "T"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of α and β as the inclusion of relations

$$T \circ [(h \circ f) \times (k \circ g)] \subset R, \qquad \begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "(h \circ f) \times (k \circ g)"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathsf{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

given by the pasting of inclusions

$$\begin{tikzcd}
A \times X \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\subset"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\
B \times Y \arrow[r, "S"] \arrow[d, "h \times k"'] \arrow[dr, phantom, "\subset"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\
C \times Z \arrow[r, "T"'] & \{\mathrm{true}, \mathrm{false}\},
\end{tikzcd}$$

which is justified by noting that, given $(a, x) \in A \times X$, the statement

- We have $h(f(a)) \sim_T k(g(x))$;

is implied by the statement

- We have $a \sim_R x$;

since

- If $a \sim_R x$, then $f(a) \sim_S g(x)$, as $S \circ (f \times g) \subset R$;
- If $b \sim_S y$, then $h(b) \sim_T k(y)$, as $T \circ (h \times k) \subset S$, and thus, in particular:
  - If $f(a) \sim_S g(x)$, then $h(f(a)) \sim_T k(g(x))$;

- *Associators.* For each composable triple $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C \overset{T}{\nrightarrow} D$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R} \colon (T \odot S) \odot R \overset{\cong}{\Longrightarrow} T \odot (S \odot R),
\qquad
\begin{tikzcd}
A \arrow[r, "R", "|"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "S", "|"{marking}] \arrow[d, phantom, "\Downarrow \alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R}"] & C \arrow[r, "T", "|"{marking}] & D \arrow[d, "\mathrm{id}_D"] \\
A \arrow[r, "R"', "|"{marking}] & B \arrow[r, "S"', "|"{marking}] & C \arrow[r, "T"', "|"{marking}] & D
\end{tikzcd}$$

of the associator of $\mathsf{Rel}^{\mathsf{dbl}}$ at (R, S, T) is the identity inclusion

$$(T \diamond S) \diamond R = T \diamond (S \diamond R)
\qquad
\begin{tikzcd}
A \times B \arrow[r, "(T \diamond S) \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "="] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\
A \times B \arrow[r, "T \diamond (S \diamond R)"'] & \{\mathrm{true}, \mathrm{false}\},
\end{tikzcd}$$

justified by Item 2 of Proposition 2.11.5;

- *Left Unitors*. For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_{R}\colon \chi_B \odot R \overset{\cong}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "\chi_B", "|" marking] \arrow[d, phantom, "\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_{R}\ \Downarrow"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "|" marking] & & B \end{tikzcd}$$

of the left unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion

$$R = \chi_B \diamond R, \qquad \begin{tikzcd} A \times B \arrow[r, "\chi_B \diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\not\Rightarrow"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathrm{true}, \mathrm{false}\}, \end{tikzcd}$$

justified by Item 3 of Proposition 2.11.5;

- *Right Unitors*. For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_{R}\colon R \odot \chi_A \overset{\cong}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "\chi_A", "|" marking] \arrow[d, "\mathrm{id}_A"'] & A \arrow[r, "R", "|" marking] \arrow[d, phantom, "\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_{R}\ \Downarrow"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "|" marking] & & B \end{tikzcd}$$

of the right unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion

$$R = R \diamond \chi_A, \qquad \begin{tikzcd} A \times B \arrow[r, "R \diamond \chi_A"] \arrow[d, equal] \arrow[dr, phantom, "\not\Rightarrow"] & \{\mathrm{true}, \mathrm{false}\} \arrow[d, "\mathrm{id}_{\{\mathrm{true},\mathrm{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathrm{true}, \mathrm{false}\}, \end{tikzcd}$$

justified by Item 3 of Proposition 2.11.5.

1.6 Properties of the Category of Relations

Proposition 1.6.1 ▶ Properties of the Category of Relations

Let A and B be sets.

1. *Self-Duality I.* The category Rel is self-dual, i.e. we have an equivalence of categories $\mathrm{Rel}^{\mathrm{op}} \overset{\text{eq.}}{\cong} \mathrm{Rel}$.

2. *Self-Duality II.* The bicategory **Rel** is self-dual, i.e. we have a biequivalence of bicategories $\mathbf{Rel}^{\mathrm{op}} \overset{\text{eq.}}{\cong} \mathbf{Rel}$.

3. *Equivalences and Isomorphisms in* Rel. Let $R\colon A \nrightarrow B$ be a relation from A to B. The following conditions are equivalent:

   (a) The relation $R\colon A \nrightarrow B$ is an equivalence in **Rel**.

   (b) The relation $R\colon A \nrightarrow B$ is an isomorphism in Rel, i.e. there exists a relation $R^{-1}\colon B \nrightarrow A$ from B to A such that we have

   $$R^{-1} \diamond R = \chi_A,$$
   $$R \diamond R^{-1} = \chi_B.$$

   (c) There exists a bijection $f\colon A \xrightarrow{\cong} B$ with $R = \Gamma(f)$.

4. *Adjunctions in **Rel**.* We have a natural bijection

   $$\left\{\begin{matrix}\text{Adjunctions in } \mathbf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\} \cong \left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

5. *Monads in **Rel**.* We have a natural bijection

   $$\left\{\begin{matrix}\text{Monads in} \\ \mathbf{Rel} \text{ on } A\end{matrix}\right\} \cong \{\text{Preorders on } A\}.$$

6. *Comonads in **Rel**.* We have a natural bijection

   $$\left\{\begin{matrix}\text{Comonads in} \\ \mathbf{Rel} \text{ on } A\end{matrix}\right\} \cong \{\text{Subsets of } A\}.$$

7. *As a Kleisli Category.* We have an isomorphism of categories

   $$\mathrm{Rel} \cong \mathrm{FreeAlg}_{\mathcal{P}},$$

   where $\mathcal{P}$ is the powerset monad of Monads, Example 3.11.1.

8. *Co/Completeness (Or Lack Thereof)*. The category Rel is not co/complete, but admits some co/limits:

   (a) *Zero Objects*. The category Rel has a zero object, the empty set $\emptyset$.

   (b) *Co/Products*. The category Rel has co/products, both given by disjoint union of sets.

   (c) *Lack of Co/Equalisers*. The category Rel does not have co/equalisers.

   (d) *Limits of Graphs of Functions*. The category Rel has limits whose arrows are all graphs of functions.

   (e) *Colimits of Graphs of Functions*. The category Rel has colimits whose arrows are all graphs of functions, and these agree with the corresponding limits in Sets.

9. *Closedness*. The bicategory **Rel** is a closed bicategory, where given a relation $R\colon A \nrightarrow B$ and a set X:

   - *Right Kan Extensions*. The right adjoint

     $$\mathrm{Ran}_R\colon \mathrm{Rel}(A, X) \longrightarrow \mathrm{Rel}(B, X)$$

     to the precomposition functor $R^*\colon \mathrm{Rel}(B, X) \longrightarrow \mathrm{Rel}(A, X)$ is given by

     $$\mathrm{Ran}_R(S) \overset{\mathrm{def}}{=} \int_{a\in A} \mathbf{Hom}_{\{\mathrm{true},\mathrm{false}\}}\left(R_a^{-2}, S_a^{-1}\right)$$

     for each $S \in \mathrm{Rel}(A, X)$, so we have $b \sim_{\mathrm{Ran}_R(S)} x$ iff, for each $a \in A$, if $a \sim_R b$, then $a \sim_S x$.

   - *Right Kan Lifts*. The right adjoint to the postcomposition functor

     $$\mathrm{Rift}_R\colon \mathrm{Rel}(X, B) \longrightarrow \mathrm{Rel}(X, A)$$

     to the postcomposition functor $R_*\colon \mathrm{Rel}(X, A) \longrightarrow \mathrm{Rel}(X, B)$ is given by

     $$\mathrm{Rift}_R(S) \overset{\mathrm{def}}{=} \int_{b\in B} \mathbf{Hom}_{\{\mathrm{true},\mathrm{false}\}}\left(R^b_{-_1}, S^b_{-_2}\right)$$

     for each $S \in \mathrm{Rel}(X, B)$, so we have $x \sim_{\mathrm{Rift}_R(S)} a$ iff, for each $b \in B$, if $a \sim_R b$, then $x \sim_S b$.

PROOF 1.6.2 ► PROOF OF PROPOSITION 1.6.1

Item 1: Self-Duality I

Omitted.

Item 2: Self-Duality II

Omitted.

Item 3: Equivalences and Isomorphisms in Rel

Omitted.

Item 4: Adjunctions in Rel

Indeed, an adjunction in Rel from A to B consists of a pair of relations

$$\begin{aligned} R\colon A &\nrightarrow B,\\ S\colon B &\nrightarrow A, \end{aligned}$$

together with inclusions

$$\begin{aligned} \chi_A &\subset R \diamond S,\\ S \diamond R &\subset \chi_B. \end{aligned}$$

These conditions then imply the following statements:

- (★) Given $a \in A$, there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_S a$, and thus R is an entire relation.
- (★) If $a \sim_R b$, then there exists, by the above item, some $b' \in B$ such that $a \sim_R b'$ and $b' \sim_S a$. But since $S \diamond R \subset \chi_B$, we have $b = b'$, and thus R is a functional relation.

Conversely, every function $f\colon A \to B$ gives rise to an adjunction $\Gamma(f) \dashv \Gamma(f)^{\dagger}$ in Rel from A to B.

Item 5: Monads in Rel

Omitted.

Item 6: Comonads in Rel

Omitted.

Item 7: As a Kleisli Category

Omitted.

Item 8: Co/Completeness (Or Lack Thereof)

Omitted.

Item 9: Closedness

Omitted.

2 Operations With Relations

2.1 Graphs of Functions

Let $f\colon A \to B$ be a function.

Definition 2.1.1 ► The Graph of a Function

The **graph of** f is the relation $\Gamma(f)\colon A \nrightarrow B$ defined as follows:

- Viewing relations as subsets of $A \times B$, we define

$$\Gamma(f) \overset{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\};$$

- Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we define

$$\Gamma(f)_{a,b} \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } b = f(a), \\ \text{false} & \text{otherwise} \end{cases}$$

  for each $(a, b) \in A \times B$;

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$[\Gamma(f)](a) \overset{\text{def}}{=} \{f(a)\}$$

  for each $a \in A$, i.e. we define $\Gamma(f)$ as the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_B} \mathcal{P}(B).$$

Proposition 2.1.2 ► Properties of Graphs of Functions

Let $f\colon A \to B$ be a function.

1. *Functoriality.* The assignment $A \mapsto \Gamma(A)$ defines a functor

$$\Gamma\colon \mathsf{Sets} \to \mathsf{Rel}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$\Gamma(A) \stackrel{\text{def}}{=} A;$$
- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets
$$\Gamma_{A,B} \colon \mathsf{Sets}(A, B) \to \underbrace{\mathrm{Rel}(\Gamma(A), \Gamma(B))}_{\stackrel{\text{def}}{=} \mathrm{Rel}(A,B)}$$
of Γ at (A, B) is defined by
$$\Gamma_{A,B}(f) \stackrel{\text{def}}{=} \Gamma(f),$$
where $\Gamma(f)$ is the graph of f as in Definition 2.1.1.

2. *Internal Adjointness.* We have an adjunction
$$\left(\Gamma(f) \dashv \Gamma(f)^{\dagger}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\Gamma(f)", "\shortmid"{marking}] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "\Gamma(f)^{\dagger}", "\shortmid"{marking}] \end{tikzcd}$$
in **Rel**.

3. *Adjointness.* We have an adjunction
$$(\Gamma \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\Gamma"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$
witnessed by a bijection of sets
$$\mathrm{Rel}(\Gamma(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$
natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathrm{Rel})$.

4. *Cocontinuity.* The functor $\Gamma \colon \mathsf{Sets} \to \mathrm{Rel}$ of Item 1 preserves colimits.

5. *Characterisations.* Let $R \colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

(a) There exists a function $f\colon A \to B$ such that $R = \Gamma(f)$.

(b) The relation R is total and functional.

(c) The weak and strong inverse images of R agree, i.e. we have $R^{-1} = R_{-1}$.

(d) The relation R has a right adjoint $R^{\dagger}$ in Rel.

Proof 2.1.3 ▶ Proof of Proposition 2.1.2

Item 1: Functoriality

Omitted.

Item 2: Internal Adjointness

This follows from Item 4.

Item 3: Adjointness

Omitted.

Item 4: Cocontinuity

Omitted.

Item 5: Characterisations

We claim that Items (a) to (d) are indeed equivalent:

- *Item (a)* $\iff$ *Item (b)*. Clear.
- *Item (a)* $\iff$ *Item (c)*. The implication Item (a) $\implies$ Item (b) is clear. Conversely, if $R^{-1} = R_{-1}$, then we have
- *Item (a)* $\implies$ *Item (c)*. Clear.
- *Item (c)* $\implies$ *Item (b)*. We claim that R is indeed total and functional:
  - *Totality.* If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$. But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction. Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.
  - *Functionality.* If $R^{-1} = R_{-1}$, then we have

$$\begin{aligned} \{a\} &= R^{-1}(\{b\}) \\ &= R_{-1}(\{b\}) \end{aligned}$$

for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, and thus we see that R is functional.

- *Item (a)* $\iff$ *Item (d)*. This follows from Item 4 of Proposition 1.6.1.

This finishes the proof. ∎

2.2 Representable Relations

Let A and B be sets.

Definition 2.2.1 ▶ Representable Relations

Let $f\colon A \to B$ and $g\colon B \to A$ be functions.[1]

1. The **representable relation associated to** f is the relation $\chi_f\colon A \nrightarrow B$ defined as the composition

$$A \times B \xrightarrow{f\times\mathrm{id}_B} B \times B \xrightarrow{\chi_B} \{\mathsf{true}, \mathsf{false}\},$$

i.e. by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

2. The **corepresentable relation associated to** g is the relation $\chi^g\colon B \nrightarrow A$ defined as the composition

$$B \times A \xrightarrow{g\times\mathrm{id}_A} A \times A \xrightarrow{\chi_A} \{\mathsf{true}, \mathsf{false}\},$$

i.e. by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

[1] More generally, given functions

$$\begin{aligned} f&\colon A \to C,\\ g&\colon B \to D \end{aligned}$$

and a relation $B \nrightarrow D$, we may consider the composite relation

$$A \times B \xrightarrow{f\times g} C \times D \xrightarrow{R} \{\mathsf{true}, \mathsf{false}\},$$

for which we have $a \sim_{R\circ(f\times g)} b$ iff $f(a) \sim_R g(b)$.

2.3 The Domain and Range of a Relation

Let A and B be sets.

Definition 2.3.1 ► The Domain and Range of a Relation

Let $R \subset A \times B$ be a relation.[1,2]

1. The **domain of** R is the subset $\mathrm{dom}(R)$ of A defined by

$$\mathrm{dom}(R) \stackrel{\text{def}}{=} \left\{ a \in A \,\middle|\, \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right\}.$$

2. The **range of** R is the subset $\mathrm{range}(R)$ of B defined by

$$\mathrm{range}(R) \stackrel{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{there exists some } a \in A \\ \text{such that } a \sim_R b \end{array} \right\}.$$

[1] Following Categories, Definition 3.3.1, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned} \chi_{\mathrm{dom}(R)}(a) &\cong \operatorname*{colim}_{b \in B}(R^a_b) \qquad (a \in A) \\ &\cong \bigvee_{b \in B} R^a_b, \\ \chi_{\mathrm{range}(R)}(b) &\cong \operatorname*{colim}_{a \in A}(R^a_b) \qquad (b \in B) \\ &\cong \bigvee_{a \in A} R^a_b, \end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\mathrm{true}, \mathrm{false}\}, \preceq)$ of Sets, Definition A.2.5.

[2] Viewing R as a function $R\colon A \to \mathcal{P}(B)$, we have

$$\begin{aligned} \mathrm{dom}(R) &\cong \operatorname*{colim}_{y \in Y}(R(y)) \\ &\cong \bigcup_{y \in Y} R(y), \\ \mathrm{range}(R) &\cong \operatorname*{colim}_{x \in X}(R(x)) \\ &\cong \bigcup_{x \in X} R(x), \end{aligned}$$

2.4 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 2.4.1 ► Binary Unions of Relations

The **union of** R **and** S[1] is the relation $R \cup S$ from A to B defined as their union as sets.

[1] *Further Terminology:* Also called the **binary union of** R **and** S, for emphasis.

Remark 2.4.2 ► Unwinding Definition 2.4.1, I

Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we may define the union of R and S as the relation $R \cup S$ from A to B defined by

$$R \cup S \overset{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$

Remark 2.4.3 ► Unwinding Definition 2.4.1, II

Viewing relations as functions $A \to \mathcal{P}(B)$, we may define the union of R and S as the relation $R \cup S$ from A to B defined by

$$[R \cup S](a) \overset{\text{def}}{=} R(a) \cup S(a)$$

for each $a \in A$.

Proposition 2.4.4 ► Properties of Binary Unions of Relations

Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have

$$(R \cup S)^\dagger = R^\dagger \cup S^\dagger.$$

2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \overset{\text{poss}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

Proof 2.4.5 ► Proof of Proposition 2.4.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:
      (i) $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

or

(i) $a \sim_{R_2} b$ and $b \sim_{S_2} c$;

3. The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:

   (a) There exists some $b \in B$ such that:

   (i) $a \sim_{R_1} b$ or $a \sim_{R_2} b$;

   and

   (i) $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ.

2.5 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

Definition 2.5.1 ▶ The Union of a Family of Relations

The **union of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ from A to B defined as its union as a family of sets.

Remark 2.5.2 ▶ Unwinding Definition 2.5.1, I

Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we may define the union of the family $\{R_i\}_{i \in I}$ as the relation $\bigcup_{i \in I} R_i$ from A to B defined by

$$\bigcup_{i \in I} R_i \overset{\text{def}}{=} \left\{ (a, b) \in (A \times B)^{\times I} \;\middle|\; \begin{array}{l} \text{there exists some } i \in I \\ \text{such that } a \sim_{R_i} b \end{array} \right\}.$$

Remark 2.5.3 ▶ Unwinding Definition 2.5.1, II

Viewing relations as functions $A \to \mathcal{P}(B)$, we may define the union of the family $\{R_i\}_{i \in I}$ as the relation $\bigcup_{i \in I} R_i$ from A to B defined by

$$\left[\bigcup_{i \in I} R_i \right](a) \overset{\text{def}}{=} \bigcup_{i \in I} R_i(a)$$

for each $a \in A$.

Proposition 2.5.4 ► Properties of Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Inverses.* We have

$$\left(\bigcup_{i\in I} R_i\right)^{\dagger} = \bigcup_{i\in I} R_i^{\dagger}.$$

Proof 2.5.5 ► Proof of Proposition 2.5.4

Item 1: Interaction With Inverses

Clear. ∎

2.6 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B.

Definition 2.6.1 ► Binary Intersections of Relations

The **intersection of** R **and** S[1] is the relation $R \cap S$ from A to B defined as their intersection as sets.

[1] *Further Terminology*: Also called the **binary intersection of** R **and** S, for emphasis.

Remark 2.6.2 ► Unwinding Definition 2.6.1, I

Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we may define the intersection of R and S as the relation $R \cup S$ from A to B defined by

$$R \cap S \stackrel{\text{def}}{=} \{(a,b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

Remark 2.6.3 ► Unwinding Definition 2.6.1, II

Viewing relations as functions $A \to \mathcal{P}(B)$, we may define the intersection of R and S as the relation $R \cup S$ from A to B defined by

$$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

for each $a \in A$.

Proposition 2.6.4 ▶ Properties of Binary Intersections of Relations

Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

1. *Interaction With Inverses.* We have
$$(R \cap S)^{\dagger} = R^{\dagger} \cap S^{\dagger}.$$

2. *Interaction With Composition.* We have
$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

Proof 2.6.5 ▶ Proof of Proposition 2.6.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:
      (i) $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
   and
      (i) $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
3. The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:
   (a) There exists some $b \in B$ such that:
      (i) $a \sim_{R_1} b$ and $a \sim_{R_2} b$;
   and
      (i) $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$.

2.7 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

Definition 2.7.1 ► The Intersection of a Family of Relations

The **intersection of the family** $\{R_i\}_{i\in I}$ is the relation $\bigcup_{i\in I} R_i$ defined as its intersection as a family of sets.

Remark 2.7.2 ► Unwinding Definition 2.7.1, I

Viewing relations as functions $A \times B \longrightarrow \{\text{true}, \text{false}\}$, we may define the intersection of the family $\{R_i\}_{i\in I}$ as the relation $\bigcup_{i\in I} R_i$ from A to B defined by

$$\bigcup_{i\in I} R_i \overset{\text{def}}{=} \left\{ (a,b) \in (A\times B)^{\times I} \;\middle|\; \begin{array}{l}\text{for each } i \in I\text{, we}\\ \text{have } a \sim_{R_i} b\end{array} \right\}.$$

Remark 2.7.3 ► Unwinding Definition 2.7.1, II

Viewing relations as functions $A \longrightarrow \mathcal{P}(B)$, we may define the intersection of the family $\{R_i\}_{i\in I}$ as the relation $\bigcap_{i\in I} R_i$ from A to B defined by

$$\left[\bigcap_{i\in I} R_i\right](a) \overset{\text{def}}{=} \bigcap_{i\in I} R_i(a)$$

for each $a \in A$.

Proposition 2.7.4 ► Properties of Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

1. *Interaction With Inverses*. We have

$$\left(\bigcup_{i\in I} R_i\right)^{\dagger} = \bigcup_{i\in I} R_i^{\dagger}.$$

Proof 2.7.5 ► Proof of Proposition 2.7.4

Item 1: Interaction With Inverses

Clear.

2.8 Binary Products of Relations

Let A, B, X, and Y be sets, let $R\colon A \nrightarrow B$ be a relation from A to B, and let $S\colon X \nrightarrow Y$ be a relation from X to Y.

Definition 2.8.1 ▶ Binary Products of Relations

The **product of R and S**[1] is the relation $R \times S$ from $A \times X$ to $B \times Y$ defined as their Cartesian product as sets.

[1] *Further Terminology:* Also called the **binary product of R and S**, for emphasis.

Remark 2.8.2 ▶ Unwinding Definition 2.8.1, I

In detail, the product of R and S is the relation $R \times S$ from $A \times X$ to $B \times Y$ defined by

$$R \times S \stackrel{\text{def}}{=} \{((a, x), (b, y)) \in (A \times X) \times (B \times Y) \mid \text{we have } a \sim_R b \text{ and } x \sim_S y\},$$

i.e. where we declare $(a, x) \sim_{R\times S} (b, y)$ iff $a \sim_R b$ and $x \sim_S y$.

Remark 2.8.3 ▶ Unwinding Definition 2.8.1, II

Viewing relations as functions $A \to \mathcal{P}(B)$, we may define the product of R and S as the relation

$$R \times S \colon A \times X \to \mathcal{P}(B \times Y)$$

from $A \times X$ to $B \times Y$ defined as the composition

$$A \times X \xrightarrow{R\times S} \mathcal{P}(B) \times \mathcal{P}(Y) \overset{\mathcal{P}^{\otimes}_{B,Y}}{\hookrightarrow} \mathcal{P}(B \times Y)$$

in Sets, i.e. by

$$[R \times S](a, x) \stackrel{\text{def}}{=} R(a) \times S(x)$$

for each $(a, x) \in A \times X$.

Proposition 2.8.4 ▶ Properties of Binary Products of Relations

Let A, B, X, and Y be sets.

1. *Interaction With Inverses.* Let

$$\begin{aligned} R &\colon A \nrightarrow A, \\ S &\colon X \nrightarrow X \end{aligned}$$

We have

$$(R \times S)^{\dagger} = R^{\dagger} \times S^{\dagger}.$$

2. *Interaction With Composition.* Let

$$\begin{aligned} R_1 &: A \nrightarrow B, \\ S_1 &: B \nrightarrow C, \\ R_2 &: X \nrightarrow Y, \\ S_2 &: Y \nrightarrow Z \end{aligned}$$

be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

Proof 2.8.5 ► Proof of Proposition 2.4.4

Item 1: Interaction With Inverses

Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(R \times S)^\dagger} (b, y)$ iff:
   - We have $(b, y) \sim_{R \times S} (a, x)$, i.e. iff:
     - We have $b \sim_R a$;
     - We have $y \sim_S x$;
2. We have $(a, x) \sim_{R^\dagger \times S^\dagger} (b, y)$ iff:
   - We have $a \sim_{R^\dagger} b$ and $x \sim_{S^\dagger} y$, i.e. iff:
     - We have $b \sim_R a$;
     - We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.

Item 2: Interaction With Composition

Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$ iff:
   (a) We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:
      (i) There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
      (ii) There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;
2. We have $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$ iff:

(a) There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:

(i) We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;

(ii) We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal.

2.9 Products of Families of Relations

Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets, and let $\{R_i : A_i \nrightarrow B_i\}_{i \in I}$ be a family of relations.

Definition 2.9.1 ▶ The Product of a Family of Relations

The **product of the family** $\{R_i\}_{i \in I}$ is the relation $\prod_{i \in I} R_i$ from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$ defined as its product as a family of sets.

Remark 2.9.2 ▶ Unwinding Definition 2.9.1, I

Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we may define the product of the family $\{R_i\}_{i \in I}$ as the relation $\prod_{i \in I} R_i$ from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$ defined by

$$\prod_{i \in I} R_i \overset{\text{def}}{=} \left\{ (a_i, b_i)_{i \in I} \in \prod_{i \in I} (A_i \times B_i) \;\middle|\; \begin{array}{l} \text{for each } i \in I \text{, we} \\ \text{have } a_i \sim_{R_i} b_i \end{array} \right\}.$$

Remark 2.9.3 ▶ Unwinding Definition 2.9.1, II

Viewing relations as functions $A \to \mathcal{P}(B)$, we may define the product of the family $\{R_i\}_{i \in I}$ as the relation $\prod_{i \in I} R_i$ from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$ defined by

$$\left[\prod_{i \in I} R_i\right]\left((a_i)_{i \in I}\right) \overset{\text{def}}{=} \prod_{i \in I} R_i(a_i)$$

for each $(a_i)_{i \in I} \in \prod_{i \in I} R_i$.

2.10 The Inverse of a Relation

Let A, B, and C be sets and let $R \subset A \times B$ be a relation.

Definition 2.10.1 ▶ The Inverse of a Relation

The **inverse of** R[1] is the relation $R^{\dagger}$ defined by

$$R^{\dagger} \overset{\text{def}}{=} \{(b,a) \in B \times A \mid \text{we have } b \sim_R a\}.$$

[1] *Further Terminology:* Also called the **opposite of** R, the **transpose of** R, or the **converse of** R.

Remark 2.10.2 ▶ Unwinding Definition 2.10.1, I

Viewing relations as functions $A \times B \longrightarrow \{\text{true}, \text{false}\}$, we may define the inverse of R as the relation $R^{\dagger}$ from B to A defined by

$$\left[R^{\dagger}\right]^{b}_{a} \overset{\text{def}}{=} R^{a}_{b}$$

for each $(a,b) \in A \times B$.

Remark 2.10.3 ▶ Unwinding Definition 2.10.1, II

Viewing relations as functions $A \longrightarrow \mathcal{P}(B)$, we may define the inverse of R as the relation $R^{\dagger}$ from B to A defined by

$$\begin{aligned}\left[R^{\dagger}\right](b) &\overset{\text{def}}{=} R^{\dagger}(\{b\})\\ &\overset{\text{def}}{=} \{a \in A \mid b \in R(a)\}\end{aligned}$$

for each $b \in B$, where $R^{\dagger}(\{b\})$ is the fibre of R over $\{b\}$.

Example 2.10.4 ▶ Examples of Inverses of Relations

Here are some examples of inverses of relations.

1. *Less Than Equal Signs.* We have $(\leq)^{\dagger} = \geq$.
2. *Greater Than Equal Signs.* Dually to Item 1, we have $(\geq)^{\dagger} = \leq$.

Proposition 2.10.5 ▶ Properties of Inverses of Relations

Let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

1. *Interaction With Ranges and Domains.* We have

$$\operatorname{dom}\left(R^{\dagger}\right) = \operatorname{range}(R),$$

$$\mathrm{range}\left(R^{\dagger}\right) = \mathrm{dom}(R).$$

2. *Interaction With Composition I.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

3. *Interaction With Composition II.* We have

$$\begin{aligned}\chi_B(-_1,-_2) &\subset R \diamond R^{\dagger},\\ \chi_A(-_1,-_2) &\subset R^{\dagger} \diamond R.\end{aligned}$$

4. *Invertibility.* We have

$$\left(R^{\dagger}\right)^{\dagger} = R.$$

5. *Identity.* We have

$$\chi_A^{\dagger}(-_1,-_2) = \chi_A(-_1,-_2).$$

Proof 2.10.6 ► Proof of Proposition 2.10.5

Item 1: Interaction With Ranges and Domains

Clear.

Item 2: Interaction With Composition I

Clear.

Item 3: Interaction With Composition II

Clear.

Item 4: Invertibility

Clear.

Item 5: Identity

Clear.

2.11 Composition of Relations

Let A, B, and C be sets and let $R \subset A \times B$ and $S \subset B \times C$ be relations.

Definition 2.11.1 ▶ Composition of Relations

The **composition of** R **and** S is the relation $S \diamond R$ defined by

$$S \diamond R \stackrel{\text{def}}{=} \left\{ (a,c) \in A \times C \,\middle|\, \begin{array}{l}\text{there exists some } b \in B \text{ such}\\ \text{that } a \sim_R b \text{ and } b \sim_S c\end{array} \right\}.$$

Remark 2.11.2 ▶ Unwinding Definition 2.11.1, I

Viewing relations as functions $A \times B \to \{\text{true}, \text{false}\}$, we may define the composition of R and S as the relation $S \diamond R$ from A to C defined by

$$\begin{aligned}(S \diamond R)^{-1}_{-2} &\stackrel{\text{def}}{=} \int^{y \in B} S^{-1}_{y} \times R^{y}_{-2}\\ &= \bigvee_{y \in B} S^{-1}_{y} \times R^{y}_{-2},\end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\text{true}, \text{false}\}, \preceq)$ of Sets, Definition A.2.5.

Remark 2.11.3 ▶ Unwinding Definition 2.11.1, II

Viewing relations as functions $A \to \mathcal{P}(B)$, we may define the composition of R and S as the relation $S \diamond R$ from A to C defined by

$$S \diamond R \stackrel{\text{def}}{=} \operatorname{Lan}_{\chi_B}(S) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "S"] \arrow[d, "\chi_B"'] & \mathcal{P}(C), \\ A \arrow[r, "R"'] & \mathcal{P}(B) \arrow[ur, "\operatorname{Lan}_{\chi_B}(S)"'] \arrow[u, phantom, shift left=3, "\Downarrow"] & \end{tikzcd}$$

where $\operatorname{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned}\left[\operatorname{Lan}_{\chi_B}(S)\right](V) &\cong \int^{y \in B} \chi_{\mathcal{P}(B)}(\chi_y, V) \odot S_y\\ &\cong \int^{y \in B} \chi_V(y) \odot S_y\\ &\cong \bigcup_{y \in B} \chi_V(y) \odot S_y\\ &\cong \bigcup_{y \in V} S_y\end{aligned}$$

for each $V \in \mathcal{P}(B)$. Thus, we have[1]

$$\begin{aligned}[S \diamond R](a) &\overset{\text{def}}{=} S(R(a)) \\ &\overset{\text{def}}{=} \bigcup_{x \in R(a)} S(x).\end{aligned}$$

[1] That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i \in I}$ in B, and then the relation S may send the image of each of the b_i's to a number of elements $\{S(b_i)\}_{i \in I} = \left\{\{c_{j_i}\}_{j_i \in J_i}\right\}_{i \in I}$ in C.

Example 2.11.4 ▶ Examples of Composition of Relations

Here are some examples of composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* We have

$$\begin{aligned}\leq \diamond \geq &= \sim_{\text{triv}}, \\ \geq \diamond \leq &= \sim_{\text{triv}} .\end{aligned}$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* We have

$$\begin{aligned}\leq \diamond \leq &= \leq, \\ \geq \diamond \geq &= \geq.\end{aligned}$$

Proposition 2.11.5 ▶ Properties of Composition of Relations

Let $R \colon A \nrightarrow B$, $S \colon B \nrightarrow C$, and $T \colon C \nrightarrow D$ be relations.

1. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\mathrm{dom}(S \diamond R) &\subset \mathrm{dom}(R), \\ \mathrm{range}(S \diamond R) &\subset \mathrm{range}(S).\end{aligned}$$

2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

3. *Unitality.* We have

$$\begin{aligned}\chi_B \diamond R &= R, \\ R \diamond \chi_A &= R.\end{aligned}$$

4. *Interaction With Inverses.* We have
$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

5. *Interaction With Composition.* We have
$$\begin{aligned}\chi_B(-_1, -_2) &\subset R \diamond R^{\dagger},\\ \chi_A(-_1, -_2) &\subset R^{\dagger} \diamond R.\end{aligned}$$

Proof 2.11.6 ▶ Proof of Proposition 2.11.5

Item 1: Interaction With Ranges and Domains

Clear.

Item 2: Associativity

Indeed, we have
$$\begin{aligned}(T \diamond S) \diamond R &\overset{\text{def}}{=} \left(\int^{y \in C} T^{-1}_{x} \times S^{x}_{-2}\right) \diamond R\\ &\overset{\text{def}}{=} \int^{x \in B} \left(\int^{y \in C} T^{-1}_{x} \times S^{x}_{y}\right) \diamond R^{y}_{-2}\\ &= \int^{x \in B} \int^{y \in C} \left(T^{-1}_{x} \times S^{x}_{y}\right) \diamond R^{y}_{-2}\\ &= \int^{y \in C} \int^{x \in B} \left(T^{-1}_{x} \times S^{x}_{y}\right) \diamond R^{y}_{-2}\\ &= \int^{y \in C} \int^{x \in B} T^{-1}_{x} \times \left(S^{x}_{y} \diamond R^{y}_{-2}\right)\\ &= \int^{x \in B} T^{-1}_{x} \times \left(\int^{y \in C} S^{x}_{y} \diamond R^{y}_{-2}\right)\\ &\overset{\text{def}}{=} \int^{x \in B} T^{-1}_{x} \times (S \diamond R)^{x}_{-2}\\ &\overset{\text{def}}{=} T \diamond (S \diamond R).\end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T \diamond S) \diamond R} d$, i.e. there exists some $b \in B$ such that:

   (a) We have $a \sim_R b$;
   (b) We have $b \sim_{T \diamond S} d$, i.e. there exists some $c \in C$ such that:
      (i) We have $b \sim_S c$;
      (ii) We have $c \sim_T d$;

2. We have $a \sim_{T \diamond (S \diamond R)} d$, i.e. there exists some $c \in C$ such that:
   (a) We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
      (i) We have $a \sim_R b$;
      (ii) We have $b \sim_S c$;
   (b) We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3: Unitality

Indeed, we have

$$\begin{aligned}\chi_B \diamond R &\stackrel{\text{def}}{=} \int^{x \in B} (\chi_B)^{-1}_x \times R^x_{-2} \\ &= \bigvee_{x \in B} (\chi_B)^{-1}_x \times R^x_{-2} \\ &= \bigvee_{\substack{x \in B \\ x = -1}} R^x_{-2} \\ &= R^{-1}_{-2},\end{aligned}$$

and

$$\begin{aligned}R \diamond \chi_A &\stackrel{\text{def}}{=} \int^{x \in A} R^{-1}_x \times (\chi_A)^x_{-2} \\ &= \bigvee_{x \in B} R^{-1}_x \times (\chi_A)^x_{-2} \\ &= \bigvee_{\substack{x \in B \\ x = -2}} R^{-1}_x \\ &= R^{-1}_{-2}.\end{aligned}$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality
$$\chi_B \diamond R = R$$
witnesses the equivalence of the following two statements:
  1. We have $a \sim_b B$.
  2. There exists some $b' \in B$ such that:
     (a) We have $a \sim_R b'$
     (b) We have $b' \sim_{\chi_B} b$, i.e. $b' = b$.
- The equality
$$R \diamond \chi_A = R$$
witnesses the equivalence of the following two statements:
  1. There exists some $a' \in A$ such that:
     (a) We have $a \sim_{\chi_B} a'$, i.e. $a = a'$.
     (b) We have $a' \sim_R b$
  2. We have $a \sim_b B$.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

Clear.

2.12 The Collage of a Relation

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

Definition 2.12.1 ▶ The Collage of a Relation

The **collage of** R[1] is the poset $\mathbf{Coll}(R) \overset{\text{def}}{=} \left(\mathrm{Coll}(R), \preceq_{\mathbf{Coll}(R)}\right)$ consisting of

- *The Underlying Set.* The set $\mathrm{Coll}(R)$ defined by
$$\mathrm{Coll}(R) \overset{\text{def}}{=} A \coprod B.$$
- *The Partial Order.* The partial order
$$\preceq_{\mathbf{Coll}(R)} \colon \mathrm{Coll}(R) \times \mathrm{Coll}(R) \to \{\mathrm{true}, \mathrm{false}\}$$

on Coll(R) defined by

$$\preceq (a,b) \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \text{false} & \text{otherwise.} \end{cases}$$

[1] *Further Terminology:* Also called the **cograph of** R.

PROPOSITION 2.12.2 ▶ PROPERTIES OF COLLAGES OF RELATIONS

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

1. *Functoriality.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor[1]

$$\mathbf{Coll}\colon \ \mathbf{Rel}(A,B) \to \mathsf{Pos}_{/\Delta^1}(A,B)$$

where

- *Action on Objects.* For each $R \in \mathrm{Obj}(\mathbf{Rel}(A,B))$, we have

$$[\mathbf{Coll}](R) \overset{\text{def}}{=} \mathbf{Coll}(R)$$

for each $R \in \mathbf{Rel}(A,B)$, where $\mathbf{Coll}(R)$ is the collage of R of Definition 2.12.1;

- *Action on Morphisms.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$, the action on Hom-sets

$$\mathbf{Coll}_{R,S}\colon \ \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \to \mathrm{Hom}_{\mathsf{Pos}_{/\Delta^1}}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

of **Coll** at (R,S) is given by sending an inclusion

$$\iota\colon \ R \subset S$$

to the morphism

$$\mathbf{Coll}(\iota)\colon \ \mathbf{Coll}(R) \to \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \overset{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.[2]

2. *Equivalence.* The functor of Item 1 is an equivalence of categories.

[1]Here $\mathsf{Pos}_{/\Delta^1}(A, B)$ is the category defined as the pullback

$$\mathsf{Pos}_{/\Delta^1}(A, B) \overset{\text{def}}{=} \mathrm{pt} \underset{[A],\mathsf{Pos},\mathrm{fib}_0}{\times} \mathsf{Pos}_{/\Delta^1} \underset{\mathrm{fib}_1,\mathsf{Pos},[B]}{\times} \mathrm{pt},$$

as in the diagram

$$\begin{tikzcd}
 & & \mathbf{Pos}_{/\Delta^1}(A, B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & \mathbf{Pos}_{/\Delta^1} \underset{\mathbf{Pos}}{\times} \mathrm{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \mathrm{pt} \underset{\mathbf{Pos}}{\times} \mathbf{Pos}_{/\Delta^1} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
\mathrm{pt} \arrow[dr, "{[A]}"'] & & \mathbf{Pos}_{/\Delta^1} \arrow[dl, "\mathrm{fib}_{[0]}"] \arrow[dr, "\mathrm{fib}_{[1]}"'] & & \mathrm{pt}. \arrow[dl, "{[B]}"] \\
 & \mathbf{Pos} & & \mathbf{Pos} &
\end{tikzcd}$$

[2]Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$, and thus $x \preceq_{\mathbf{Coll}(S)} y$.

Proof 2.12.3 ► Proof of Proposition 2.12.2

Item 1: Functoriality

Omitted.

Item 2: Equivalence

Omitted.

3 Equivalence Relations

3.1 Reflexive Relations

3.1.1 Foundations

Let A be a set.

Definition 3.1.1 ► Reflexive Relations

A **reflexive relation** is equivalently:[1]

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \chi_A)$;
- A pointed object in $(\mathbf{Rel}(A, A), \chi_A)$.

[1] Note that since $\mathbf{Rel}(A, A)$ is posetal, reflexivity is a property of a relation, instead of a structure.

Remark 3.1.2 ► Unwinding Definition 3.1.1

In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R \colon \chi_A \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a \in A$, we have $a \sim_R a$.

Definition 3.1.3 ► The Po/Set of Reflexive Relations on a Set

Let A be a set.

1. The **set of reflexive relations on** A is the subset $\mathrm{Rel}^{\mathrm{refl}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the reflexive relations.
2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{refl}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the reflexive relations.

Proposition 3.1.4 ► Properties of Reflexive Relations

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is reflexive, then so is $R^{\dagger}$.
2. *Interaction With Composition.* If R and S are reflexive, then so is $S \diamond R$.

Proof 3.1.5 ► Proof of Proposition 3.1.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear. □

3.1.2 The Reflexive Closure of a Relation

Let R be a relation on A.

Definition 3.1.6 ► The Reflexive Closure of a Relation

The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{refl}}$[1] satisfying the following universal property:[2]

(UP) Given another reflexive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{refl}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{refl}.
[2] *Slogan:* The reflexive closure of R is the smallest reflexive relation containing R.

Construction 3.1.7 ► The Reflexive Closure of a Relation

Concretely, $\sim_R^{\mathrm{refl}}$ is the free pointed object on R in $(\mathbf{Rel}(A,A), \chi_A)$[1], being given by

$$\begin{aligned} R^{\mathrm{refl}} &\stackrel{\mathrm{def}}{=} R \coprod^{\mathbf{Rel}(A,A)} \Delta_A \\ &= R \cup \Delta_A \\ &= \{(a,b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } a = b\}. \end{aligned}$$

[1] Or, equivalently, the free $\mathbb{E}_0$-monoid on R in $(\mathsf{N}_\bullet(\mathbf{Rel}(A,A)), \chi_A)$.

Proof 3.1.8 ► Proof of Construction 3.1.7

Clear. ∎

Proposition 3.1.9 ► Properties of the Reflexive Closure of a Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{refl}} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{refl}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{refl}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{refl}}\left(\sim_R^{\mathrm{refl}}, \sim_S\right) \cong \mathbf{Rel}(\sim_R, \sim_S),$$

   natural in $\sim_R \in \mathrm{Obj}\left(\mathbf{Rel}^{\mathrm{refl}}(A,A)\right)$ and $\sim_S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\mathrm{refl}} = R$.

3. *Idempotency.* We have

$$\left(R^{\mathrm{refl}}\right)^{\mathrm{refl}} = R^{\mathrm{refl}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{refl}} = \left(R^{\mathrm{refl}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathsf{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"] \arrow[d, "(-)^{\dagger}"'] & \mathsf{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathsf{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"'] & \mathsf{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{refl}} = S^{\mathrm{refl}} \diamond R^{\mathrm{refl}}, \qquad \begin{tikzcd} \mathsf{Rel}(A,A) \times \mathsf{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{refl}} \times (-)^{\mathrm{refl}}"'] & \mathsf{Rel}(A,A) \arrow[d, "(-)^{\mathrm{refl}}"] \\ \mathsf{Rel}(A,A) \times \mathsf{Rel}(A,A) \arrow[r, "\diamond"'] & \mathsf{Rel}(A,A). \end{tikzcd}$$

Proof 3.1.10 ► Proof of Proposition 3.1.9

Item 1: Adjointness

This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 3.1.6.

Item 2: The Reflexive Closure of a Reflexive Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 3.1.4. □

3.2 Symmetric Relations

3.2.1 Foundations

Let A be a set.

Definition 3.2.1 ▶ Symmetric Relations

A relation R on A is **symmetric** if, for each $a, b \in A$, the following conditions are equivalent:[1]

1. We have $a \sim_R b$.
2. We have $b \sim_R a$.

[1]That is, R is symmetric if $R^\dagger = R$.

Definition 3.2.2 ▶ The Po/Set of Symmetric Relations on a Set

Let A be a set.

1. The **set of symmetric relations on** A is the subset $\mathrm{Rel}^{\mathrm{symm}}(A, A)$ of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.
2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathrm{symm}}(A, A)$ of $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

Proposition 3.2.3 ▶ Properties of Symmetric Relations

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is symmetric, then so is $R^\dagger$.
2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

Proof 3.2.4 ▶ Proof of Proposition 3.2.3

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

Clear. □

3.2.2 The Symmetric Closure of a Relation

Let R be a relation on A.

Definition 3.2.5 ▶ The Symmetric Closure of a Relation

The **symmetric closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{symm}}$[1] satisfying the following universal property:[2]

(UP) Given another symmetric relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{symm}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{symm}.
[2] *Slogan:* The symmetric closure of R is the smallest symmetric relation containing R.

Construction 3.2.6 ▶ The Symmetric Closure of a Relation

Concretely, $\sim_R^{\mathrm{symm}}$ is the symmetric relation on A defined by

$$\begin{aligned} R^{\mathrm{symm}} &\stackrel{\mathrm{def}}{=} R \cup R^{\dagger} \\ &= \{(a,b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

Proof 3.2.7 ▶ Proof of Construction 3.2.6

Clear. ∎

Proposition 3.2.8 ▶ Properties of the Symmetric Closure of a Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\Big((-)^{\mathrm{symm}} \dashv 忘\Big)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{symm}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{symm}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{symm}}\left(\sim_R^{\mathrm{symm}}, \sim_S\right) \cong \mathbf{Rel}(\sim_R, \sim_S),$$

natural in $\sim_R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{symm}}(A,A))$ and $\sim_S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\mathrm{symm}} = R$.

3. *Idempotency.* We have

$$(R^{\mathrm{symm}})^{\mathrm{symm}} = R^{\mathrm{symm}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathsf{symm}} = \left(R^{\mathsf{symm}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathsf{Rel}(A, A) \arrow[r, "(-)^{\mathsf{symm}}"] \arrow[d, "(-)^{\dagger}"'] & \mathsf{Rel}(A, A) \arrow[d, "(-)^{\dagger}"] \\ \mathsf{Rel}(A, A) \arrow[r, "(-)^{\mathsf{symm}}"'] & \mathsf{Rel}(A, A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathsf{symm}} = S^{\mathsf{symm}} \diamond R^{\mathsf{symm}}, \qquad \begin{tikzcd} \mathsf{Rel}(A, A) \times \mathsf{Rel}(A, A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathsf{symm}} \times (-)^{\mathsf{symm}}"'] & \mathsf{Rel}(A, A) \arrow[d, "(-)^{\mathsf{symm}}"] \\ \mathsf{Rel}(A, A) \times \mathsf{Rel}(A, A) \arrow[r, "\diamond"'] & \mathsf{Rel}(A, A). \end{tikzcd}$$

Proof 3.2.9 ► Proof of Proposition 3.2.8

Item 1: Adjointness

This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 3.2.5.

Item 2: The Symmetric Closure of a Symmetric Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

Clear.

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 3.2.3. □

3.3 Transitive Relations

3.3.1 Foundations

Let A be a set.

Definition 3.3.1 ▶ Transitive Relations

A **transitive relation** is equivalently:[1]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$;
- A non-unital monoid in $(\mathbf{Rel}(A, A), \diamond)$.

[1] Note that since $\mathbf{Rel}(A, A)$ is posetal, transitivity is a property of a relation, instead of a structure.

Remark 3.3.2 ▶ Unwinding Definition 3.3.1

In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R : R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, we have:

($\star$) If $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

Definition 3.3.3 ▶ The Po/Set of Transitive Relations on a Set

Let A be a set.

1. The **set of transitive relations from** A **to** B is the subset $\mathrm{Rel}^{\mathrm{trans}}(A)$ of $\mathrm{Rel}(A, A)$ spanned by the transitive relations.
2. The **poset of relations from** A **to** B is is the subposet $\mathbf{Rel}^{\mathrm{trans}}(A)$ of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

Proposition 3.3.4 ▶ Properties of Transitive Relations

Let R and S be relations on A.

1. *Interaction With Inverses.* If R is transitive, then so is $R^\dagger$.
2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ ***may fail to be transitive***.

Proof 3.3.5 ▶ Proof of Proposition 3.3.4

Item 1: Interaction With Inverses

Clear.

Item 2: Interaction With Composition

See [MSE 2096272].[1]

[1] *Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

1. If $a \sim_{S \diamond R} c$ and $c \sim_{S \diamond r} e$, then:
   (a) There is some $b \in A$ such that:
      (i) $a \sim_R b$;
      (ii) $b \sim_S c$;
   (b) There is some $d \in A$ such that:
      (i) $c \sim_R d$;
      (ii) $d \sim_S e$.

3.3.2 The Transitive Closure of a Relation

Let R be a relation on A.

Definition 3.3.6 ► The Transitive Closure of a Relation

The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{trans}}$[1] satisfying the following universal property:[2]

(UP) Given another transitive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{trans}} \subset \sim_S$.

[1] *Further Notation:* Also written R^{trans}.
[2] *Slogan:* The transitive closure of R is the smallest transitive relation containing R.

Construction 3.3.7 ► The Transitive Closure of a Relation

Concretely, $\sim_R^{\mathrm{trans}}$ is the free non-unital monoid on R in $(\mathbf{Rel}(A, A), \diamond)$[1], being given by

$$\begin{aligned} R^{\mathrm{trans}} &\overset{\mathrm{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \left\{ (a, b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exist } (x_1, \ldots, x_n) \in R^{\times n} \text{ such} \\ \text{that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b \end{array} \right\}. \end{aligned}$$

[1] Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on R in $(\mathsf{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.

Proof 3.3.8 ► Proof of Construction 3.3.7

Clear. □

Proposition 3.3.9 ► Properties of the Transitive Closure of a Relation

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{trans}} \dashv 忘\right)\colon\quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{trans}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{trans}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{trans}}\left(\sim_R^{\mathrm{trans}}, \sim_S\right) \cong \mathbf{Rel}(\sim_R, \sim_S),$$

   natural in $\sim_R \in \mathrm{Obj}\left(\mathbf{Rel}^{\mathrm{trans}}(A,A)\right)$ and $\sim_S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\mathrm{trans}} = R$.

3. *Idempotency.* We have

$$\left(R^{\mathrm{trans}}\right)^{\mathrm{trans}} = R^{\mathrm{trans}}.$$

4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{trans}} = \left(R^{\mathrm{trans}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{trans}} \overset{\mathrm{poss}}{\neq} S^{\mathrm{trans}} \diamond R^{\mathrm{trans}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{trans}} \times (-)^{\mathrm{trans}}"'] \arrow[dr, phantom, "\times"] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{trans}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof 3.3.10 ► Proof of Proposition 3.3.9

Item 1: Adjointness

This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 3.3.6.

Item 2: The Transitive Closure of a Transitive Relation

Clear.

Item 3: Idempotency

This follows from Item 2.

Item 4: Interaction With Inverses

We have

$$\begin{aligned}
\left(R^{\dagger}\right)^{\mathrm{trans}} &= \bigcup_{n=1}^{\infty}\left(R^{\dagger}\right)^{\diamond n} && \text{(Construction 3.3.7)}\\
&= \bigcup_{n=1}^{\infty}\left(R^{\diamond n}\right)^{\dagger} && \text{(Item 4 of Proposition 2.11.5)}\\
&= \left(\bigcup_{n=1}^{\infty} R^{\diamond n}\right)^{\dagger} && \text{(Item 1 of Proposition 2.5.4)}\\
&= \left(R^{\mathrm{trans}}\right)^{\dagger}. && \text{(Construction 3.3.7)}
\end{aligned}$$

Item 5: Interaction With Composition

This follows from Item 2 of Proposition 3.3.4. ∎

3.4 Equivalence Relations

3.4.1 Foundations

Let A be a set.

Definition 3.4.1 ► Equivalence Relations

A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.[1]

[1] *Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

Example 3.4.2 ▶ The Kernel of a Function

The **kernel of a function** $f\colon A \to B$ is the equivalence $\sim_{\mathrm{Ker}(f)}$ on A obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[1]

[1]The kernel $\mathrm{Ker}(f)\colon A \nrightarrow A$ of f is the induced monad of the adjunction $\Gamma(f) \dashv \Gamma(f)^{\dagger}\colon A \rightleftarrows B$ in **Rel**.

Definition 3.4.3 ▶ The Po/Set of Equivalence Relations on a Set

Let A and B be sets.

1. The **set of equivalence relations from A to B** is the subset $\mathrm{Rel}^{\mathrm{eq}}(A, B)$ of $\mathrm{Rel}(A, B)$ spanned by the equivalence relations.
2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A, B)$ of $\mathbf{Rel}(A, B)$ spanned by the equivalence relations.

3.4.2 The Equivalence Closure of a Relation

Let R be a relation on A.

Definition 3.4.4 ▶ The Equivalence Closure of a Relation

The **equivalence closure**[1] of $\sim_R$ is the relation $\sim^{\mathrm{eq}}_R$[2] satisfying the following universal property:[3]

(UP) Given another equivalence relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim^{\mathrm{eq}}_R \subset \sim_S$.

[1]*Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.
[2]*Further Notation:* Also written R^{eq}.
[3]*Slogan:* The equivalence closure of R is the smallest equivalence relation containing R.

Construction 3.4.5 ▶ The Equivalence Closure of a Relation

Concretely, $\sim^{\mathrm{eq}}_R$ is the equivalence relation on A defined by

$$\begin{aligned} R^{\mathrm{eq}} &\overset{\mathrm{def}}{=} \left(\left(R^{\mathrm{refl}}\right)^{\mathrm{symm}}\right)^{\mathrm{trans}} \\ &= \left(\left(R^{\mathrm{symm}}\right)^{\mathrm{trans}}\right)^{\mathrm{refl}} \end{aligned}$$

$$= \left\{ (a,b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exist } (x_1, \ldots, x_n) \in R^{\times n} \text{ satisfying at least one} \\ \text{of the following conditions:} \\ \quad 1.\ \text{The following conditions are satisfied:} \\ \qquad \text{(a) We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \qquad \text{(b) We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \text{ for} \\ \qquad\quad \text{each } 1 \leq i \leq n-1; \\ \qquad \text{(c) We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \quad 2.\ \text{We have } a = b. \end{array} \right\}.$$

PROOF 3.4.6 ► PROOF OF CONSTRUCTION 3.4.5

From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 3.1.6, 3.2.5 and 3.3.6), we see that it suffices to prove that:

1. The symmetric closure of a reflexive relation is still reflexive;
2. The transitive closure of a symmetric relation is still symmetric;

which are both clear. ∎

PROPOSITION 3.4.7 ► PROPERTIES OF EQUIVALENCE RELATIONS

Let R be a relation on A.

1. *Adjointness.* We have an adjunction

$$((-)^{\mathrm{eq}} \dashv 忘): \quad \begin{tikzcd} \mathbf{Rel}(A,B) \arrow[r, bend left, "(-)^{\mathrm{eq}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{eq}}(A,B), \arrow[l, bend left, "忘"] \end{tikzcd}$$

   witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{eq}}\left(\sim_R^{\mathrm{eq}}, \sim_S\right) \cong \mathbf{Rel}(\sim_R, \sim_S),$$

   natural in $\sim_R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{eq}}(A,B))$ and $\sim_S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\mathrm{eq}} = R$.

3. *Idempotency.* We have

$$(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.$$

Proof 3.4.8 ▶ Proof of Proposition 3.4.7

Item 1: Adjointness

This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 3.4.4.

Item 2: The Equivalence Closure of an Equivalence Relation

Clear.

Item 3: Idempotency

This follows from Item 2. □

3.5 Quotients by Equivalence Relations

3.5.1 Equivalence Classes

Let A be a set, let R be a relation on A, and let $a \in A$.

Definition 3.5.1 ▶ Equivalence Classes

The **equivalence class associated to** a is the set $[a]$ defined by[1,2]

$$\begin{aligned} [a] &\overset{\mathrm{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. \qquad \text{(since R is symmetric)} \end{aligned}$$

[1] Note that since R is symmetric, we have $a \in [a]$.

[2] Note that since R is transitive and symmetric, if $x, y \in [a]$, then $x \sim_R y$. As a consequence, if $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.

3.5.2 Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A.

Definition 3.5.2 ▶ Quotients of Sets by Equivalence Relations

The **quotient of** X **by** R is the set $X/{\sim_R}$ defined by

$$X/{\sim_R} \overset{\mathrm{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

Remark 3.5.3 ▶ Why "Equivalence" Relations for Quotient Sets

The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.
- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by

  $$[a] \stackrel{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

  but we could equally well define

  $$[a]' \stackrel{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

  instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.[1]
- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \nsim_R b$, and equal otherwise.

[1] When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to presheaves and copresheaves; see Constructions With Categories, Definition 11.1.1.

Proposition 3.5.4 ▶ Properties of Quotient Sets

Let $f : X \to Y$ be a function and let R be a relation on X.

1. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[1,2]

   $$X/{\sim_{\mathrm{Ker}(f)}} \cong \mathrm{Im}(f).$$

2. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X. The following conditions are equivalent:

   (a) There exists a map

   $$\bar{f} : X/{\sim_R} \to Y$$

making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R} \arrow[ur, dotted, "\exists", "\bar{f}"'] & \end{tikzcd}$$

commute.

(b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

3. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X. If the conditions of Item 2 hold, then $\bar{f}$ is the *unique* map making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R} \arrow[ur, dashed, "\exists!", "\bar{f}"'] & \end{tikzcd}$$

commute.

4. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X. If the conditions of Item 2 hold, then the following conditions are equivalent:

   (a) The map $\bar{f}$ is an injection.

   (b) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

5. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X. If the conditions of Item 2 hold, then the following conditions are equivalent:

   (a) The map $f \colon X \to Y$ is surjective.

   (b) The map $\bar{f} \colon X/{\sim_R} \to Y$ is surjective.

6. *Descending Functions to Quotient Sets, V.* Let R be a relation on X and let $\sim_R^{\mathrm{eq}}$ be the equivalence relation associated to R. The following conditions are equivalent:

   (a) The map f satisfies the equivalent conditions of Item 2:

- There exists a map

$$\bar{f}\colon X/{\sim}_R^{\mathrm{eq}} \to Y$$

making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim}_R^{\mathrm{eq}} \arrow[ur, dotted, "\exists"', "\bar{f}"] & \end{tikzcd}$$

commute.

- For each $x, y \in X$, if $x \sim_R^{\mathrm{eq}} y$, then $f(x) = f(y)$.

(b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

[1] *Further Terminology*: The set $X/{\sim}_{\mathrm{Ker}(f)}$ is often called the **coimage of** f, and denoted by $\mathrm{Coim}(f)$.

[2] In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f:

(a) The kernel $\mathrm{Ker}(f)\colon X \nrightarrow X$ of f is the induced monad of the adjunction $\Gamma(f) \dashv \Gamma(f)^\dagger\colon X \rightleftarrows Y$ in **Rel**;

(b) The image $\mathrm{Im}(f) \subset Y$ of f is the induced comonad of the adjunction $\Gamma(f) \dashv \Gamma(f)^\dagger\colon X \rightleftarrows Y$ in **Rel**.

Proof 3.5.5 ▶ Proof of Proposition 3.5.4

Item 1: The First Isomorphism Theorem for Sets

Clear.

Item 2: Descending Functions to Quotient Sets, I

See [Pro23c].

Item 3: Descending Functions to Quotient Sets, II

See [Pro23d].

Item 4: Descending Functions to Quotient Sets, III

See [Pro23a].

Item 5: Descending Functions to Quotient Sets, IV

See [Pro23b].

Item 6: Descending Functions to Quotient Sets, V

The implication Item (a) $\implies$ Item (b) is clear.

Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R, we see that the condition $x \sim_R^{\mathrm{eq}} y$ unwinds to the following:

($\star$) There exist $(x_1, \ldots, x_n) \in R^{\times n}$ satisfying at least one of the following conditions:

1. The following conditions are satisfied:
   (a) We have $x \sim_R x_1$ or $x_1 \sim_R x$;
   (b) We have $x_i \sim_R x_{i+1}$ or $x_{i+1} \sim_R x_i$ for each $1 \leq i \leq n-1$;
   (c) We have $y \sim_R x_n$ or $x_n \sim_R y$;
2. We have $x = y$.

Now, if $x = y$, then $f(x) = f(y)$ trivially; otherwise, we have

$$\begin{aligned} f(x) &= f(x_1), \\ f(x_1) &= f(x_2), \\ &\vdots \\ f(x_{n-1}) &= f(x_n), \\ f(x_n) &= f(y), \end{aligned}$$

and $f(x) = f(y)$, as we wanted to show.

4 Functoriality of Powersets

4.1 Direct Images

Let A and B be sets and let $R \colon A \nrightarrow B$ be a relation.

Definition 4.1.1 ▶ Direct Images

The **direct image function associated to** R is the function[1]

$$R_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[2,3]

$$R_*(U) \stackrel{\text{def}}{=} R(U)$$

$$\begin{aligned}&\stackrel{\text{def}}{=} \bigcup_{a\in U} R(a)\\&= \left\{ b \in B \;\middle|\; \begin{array}{l}\text{there exists some } a \in\\ U \text{ such that } b \in R(a)\end{array} \right\}\end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Notation:* Also written $\exists_R\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_R(U)$.
- There exists some $a \in U$ such that $b \in f(a)$.

[2] *Further Terminology:* The set $R(U)$ is called the **direct image of U by R**.

[3] We also have

$$R_*(U) = B \setminus R_!(A \setminus U);$$

see Item 7 of Proposition 4.1.3.

Remark 4.1.2 ► Unwinding Definition 4.1.1

Identifying subsets of A with relations from pt to A via Constructions With Sets, Item 7 of Proposition 3.2.3, we see that the direct image function associated to R is equivalently the function

$$R_*\colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt},A)} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt},B)}$$

defined by

$$R_*(U) \stackrel{\text{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(A)$, where $R \diamond U$ is the composition

$$\mathrm{pt} \overset{U}{\nrightarrow} A \overset{R}{\nrightarrow} B.$$

Proposition 4.1.3 ► Properties of Direct Image Functions

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

$$R_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_*](U) \overset{\text{def}}{=} R_*(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_*(U) \subset R_*(V)$.

2. *Adjointness.* We have an adjunction

$$
(R_* \dashv R_{-1}) \colon \quad
\begin{tikzcd}
\mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"]
\end{tikzcd}
$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
   (a) We have $R_*(U) \subset V$;
   (b) We have $U \subset R_{-1}(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_*(U) \cup R_*(V) = R_*(U \cup V),$$
$$R_*(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_*\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$R_*(U \cap V) \subset R_*(U) \cap R_*(V),$$
$$R_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$R_{*|U,V}^{\otimes} \colon R_*(U) \cup R_*(V) \xrightarrow{=} R_*(U \cup V),$$
$$R_{*|\mathbb{1}}^{\otimes} \colon \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$R_{*|U,V}^{\otimes} \colon R_*(U \cap V) \subset R_*(U) \cap R_*(V),$$
$$R_{*|\mathbb{1}}^{\otimes} \colon R_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images With Compact Support.* We have

$$R_*(U) = B \setminus R_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof 4.1.4 ► Proof of Proposition 4.1.3

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from Kan Extensions, Item 2 of Proposition 1.1.6.

Item 3: Preservation of Colimits

This follows from ?? and Categories, ?? of Proposition 6.1.3.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from ??.

Item 7: Relation to Direct Images With Compact Support

The proof proceeds in the same way as in the case of functions (Constructions With Sets, Item 7 of Proposition 3.3.3): applying Item 7 of Proposition 4.4.3 to $A \setminus U$, we have

$$\begin{aligned} R_!(A \setminus U) &= B \setminus R_*(A \setminus (A \setminus U)) \\ &= B \setminus R_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} R_*(U) &= B \setminus (B \setminus R_*(U)), \\ &= B \setminus R_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

PROPOSITION 4.1.5 ▶ PROPERTIES OF THE DIRECT IMAGE FUNCTION OPERATION

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*\colon \mathsf{Rel}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*\colon \mathsf{Rel}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[1]

$$(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition*. For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[2]

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathcal{P}(B) \arrow[d, "S_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

[1]That is, the postcomposition

$$(\chi_A)_*\colon \mathrm{Rel}(\mathrm{pt}, A) \to \mathrm{Rel}(\mathrm{pt}, A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},A)}$.

[2]That is, we have

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathrm{Rel}(\mathrm{pt}, B) \arrow[d, "S_*"] \\ & \mathrm{Rel}(\mathrm{pt}, C). \end{tikzcd}$$

Proof 4.1.6 ► Proof of Proposition 4.1.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned} (\chi_A)_*(U) &\stackrel{\text{def}}{=} \bigcup_{a \in U} \chi_A(a) \\ &\stackrel{\text{def}}{=} \bigcup_{a \in U} \{a\} \\ &= U \\ &\stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{P}(A)}(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$
\begin{aligned}
(S \diamond R)_*(U) &\overset{\text{def}}{=} \bigcup_{a\in U} [S \diamond R](a) \\
&\overset{\text{def}}{=} \bigcup_{a\in U} S(R(a)) \\
&\overset{\text{def}}{=} \bigcup_{a\in U} S_*(R(a)) \\
&= S_*\left(\bigcup_{a\in U} R(a)\right) \\
&\overset{\text{def}}{=} S_*(R_*(U)) \\
&\overset{\text{def}}{=} [S_* \circ R_*](U)
\end{aligned}
$$

for each $U \in \mathcal{P}(A)$, where we used Item 3 of Proposition 4.1.3. Thus $(S \diamond R)_* = S_* \circ R_*$.

4.2 Strong Inverse Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 4.2.1 ▶ Strong Inverse Images

The **strong inverse image function associated to** R is the function

$$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[1]

$$R_{-1}(V) \overset{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Terminology:* The set $R_{-1}(V)$ is called the **strong inverse image of** V **by** R.

Remark 4.2.2 ▶ Unwinding Definition 4.2.1

Identifying subsets of B with relations from pt to B via Constructions With Sets, Item 7 of Proposition 3.2.3, we see that the inverse image function associated to

R is equivalently the function

$$R_{-1}\colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt},B)} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt},A)}$$

defined by

$$R_{-1}(V) \stackrel{\text{def}}{=} \mathrm{Rift}_R(V), \qquad \begin{tikzcd} & A \arrow[d, "R", "\shortmid" marking] \\ \mathrm{pt} \arrow[r, "V"', "\shortmid" marking] \arrow[ur, dashed, "\mathrm{Rift}_R(V)"] & B, \end{tikzcd}$$

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\stackrel{\text{def}}{=} \mathrm{Rift}_R(V) \\ &\cong \int_{x\in B} \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R^{x}_{-_1}, V^{x}_{-_2}\right). \end{aligned}$$

Thus, we have

$$\begin{aligned} R_{-1}(V) &\cong \left\{a \in A \;\middle|\; \int_{x\in B} \mathrm{Hom}_{\{\mathrm{t},\mathrm{f}\}}\left(R^{x}_{a}, V^{x}_{\star}\right) = \mathrm{true}\right\} \\ &= \left\{a \in A \;\middle|\; \begin{array}{l} \text{for each } x \in B\text{, at least one of the follow-} \\ \text{ing conditions hold:} \\ \quad 1.\ \text{We have } R^{x}_{a} = \mathrm{false}; \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } R^{x}_{a} = \mathrm{true}; \\ \qquad \text{(b) We have } V^{x}_{\star} = \mathrm{true}; \end{array} \right\} \\ &= \left\{a \in A \;\middle|\; \begin{array}{l} \text{for each } x \in B\text{, at least one of the follow-} \\ \text{ing conditions hold:} \\ \quad 1.\ \text{We have } x \notin R(a); \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } x \in R(a); \\ \qquad \text{(b) We have } x \in V; \end{array} \right\} \end{aligned}$$

$$= \{a \in A \mid \text{for each } x \in R(a), \text{we have } x \in V\}$$
$$= \{a \in A \mid R(a) \subset V\}.$$

PROPOSITION 4.2.3 ▶ PROPERTIES OF STRONG INVERSE IMAGES

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$[R_{-1}](V) \stackrel{\text{def}}{=} R_{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R_{-1}(U) \subset R_{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1})\colon\quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:

(a) We have $R_*(U) \subset V$;

(b) We have $U \subset R_{-1}(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_{-1}(U_i) \subset R_{-1}\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{gathered} R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V), \\ \emptyset \subset R_{-1}(\emptyset), \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} R_{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{gathered} R_{-1}(U \cap V) = R_{-1}(U) \cap R_{-1}(V), \\ R_{-1}(B) = B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_{-1}, R^{\otimes}_{-1}, R^{\otimes}_{-1|\mathbb{1}}\right) : (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$\begin{gathered} R^{\otimes}_{-1|U,V} : R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V), \\ R^{\otimes}_{-1|\mathbb{1}} : \emptyset \subset R_{-1}(\emptyset), \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_{-1}, R^{\otimes}_{-1}, R^{\otimes}_{-1|\mathbb{1}}\right) : (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$\begin{gathered} R^{\otimes}_{-1|U,V} : R_{-1}(U \cap V) \xrightarrow{=} R_{-1}(U) \cap R_{-1}(V), \\ R^{\otimes}_{-1|\mathbb{1}} : R_{-1}(A) \xrightarrow{=} B, \end{gathered}$$

natural in $U, V \in \mathcal{P}(B)$.

7. *Interaction With Weak Inverse Images.* Let $R\colon A \nrightarrow B$ be a relation from A to B.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V) \subset R^{-1}(V)$$

   natural in $V \in \mathcal{P}(B)$. We also have equalities

   $$\begin{aligned} R^{-1}(B \setminus V) &= A \setminus R_{-1}(V), \\ R_{-1}(B \setminus V) &= A \setminus R^{-1}(V) \end{aligned}$$

   for each $V \in \mathcal{P}(B)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof 4.2.4 ► Proof of Proposition 4.2.3

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from Kan Extensions, Item 2 of Proposition 1.1.6.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from Item 2 and Categories, **??** of Proposition 6.1.3.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from **??**.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Interaction With Weak Inverse Images

The first part of **??** is clear, while the second follows by noting that

$$A \setminus R_{-1}(V) = \{a \in A \mid R(a) \not\subset V\},$$

$$
\begin{aligned}
R^{-1}(B \setminus V) &= \{a \in A \mid R(a) \setminus V \neq \emptyset\}, \\
R_{-1}(B \setminus V) &= \{a \in A \mid R(a) \subset B \setminus V\}, \\
A \setminus R^{-1}(V) &= \{a \in A \mid R(a) \cap V = \emptyset\}.
\end{aligned}
$$

???? follow from Item 5 of Proposition 2.1.2.

Proposition 4.2.5 ▶ Properties of the Strong Inverse Image Function Operation

Let $R \colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1} \colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable relations $R \colon A \nrightarrow B$ and $S \colon B \nrightarrow C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S_{-1}"] \arrow[dr, "(S \diamond R)_{-1}"'] & \mathcal{P}(B) \arrow[d, "R_{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof 4.2.6 ▶ Proof of Proposition 4.2.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$\begin{aligned}(\chi_A)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid \chi_A(a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$\begin{aligned}(S \diamond R)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid [S \diamond R](a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid S(R(a)) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid S_*(R(a)) \subset U\} \\ &= \{a \in A \mid R(a) \subset S_{-1}(U)\} \\ &\stackrel{\text{def}}{=} R_{-1}(S_{-1}(U)) \\ &\stackrel{\text{def}}{=} [R_{-1} \circ S_{-1}](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 4.2.3, which implies that the conditions

- We have $S_*(R(a)) \subset U$;
- We have $R(a) \subset S_{-1}(U)$;

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$. □

4.3 Weak Inverse Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 4.3.1 ► Weak Inverse Images

The **weak inverse image function associated to** R[1] is the function

$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[2]

$$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(B)$.

[1] *Further Terminology:* Also called simply the **inverse image function associated to** R.

[2] *Further Terminology:* The set $R^{-1}(V)$ is called the **weak inverse image of** V **by** R or simply the **inverse image of** V **by** R.

Remark 4.3.2 ► Unwinding Definition 4.3.1

Identifying subsets of B with relations from B to pt via Constructions With Sets, Item 7 of Proposition 3.2.3, we see that the weak inverse image function associated to R is equivalently the function

$$R^{-1}\colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})}$$

defined by

$$R^{-1}(V) \overset{\text{def}}{=} V \diamond R$$

for each $V \in \mathcal{P}(A)$, where $R \diamond V$ is the composition

$$A \overset{R}{\nrightarrow} B \overset{V}{\nrightarrow} \mathrm{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\overset{\text{def}}{=} V \diamond R \\ &\overset{\text{def}}{=} \int^{x \in B} V_x^{-1} \times R_{-2}^x, \end{aligned}$$

and thus $R^{-1}(V)$ is the subset of A given by

$$\begin{aligned} R^{-1}(V) &\cong \left\{ a \in A \,\middle|\, \int^{x \in B} V_x^{\star} \times R_a^x = \text{true} \right\} \\ &= \left\{ a \in A \,\middle|\, \begin{array}{l} \text{there exists } x \in B \text{ such that the follow-} \\ \text{ing conditions hold:} \\ \quad 1. \ \text{We have } V_x^{\star} = \text{true}; \\ \quad 2. \ \text{We have } R_a^x = \text{true}; \end{array} \right\} \end{aligned}$$

$$= \left\{ a \in A \,\middle|\, \begin{array}{l} \text{there exists } x \in B \text{ such that the follow-} \\ \text{ing conditions hold:} \\ \quad 1.\ \text{We have } x \in V; \\ \quad 2.\ \text{We have } x \in R(a); \end{array} \right\}$$

$$= \{a \in A \mid \text{there exists } x \in V \text{ such that } x \in R(a)\}$$

$$= \{a \in A \mid R(a) \cap V \neq \emptyset\}.$$

Proposition 4.3.3 ▶ Properties of Weak Inverse Image Functions

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

$$R^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$\left[R^{-1}\right](V) \stackrel{\text{def}}{=} R^{-1}(V);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:

(a) We have $R^{-1}(U) \subset V$;

(b) We have $U \subset R_!(V)$.

3. *Preservation of Colimits.* We have an equality of sets

$$R^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} R^{-1}(U) \cup R^{-1}(V) &= R^{-1}(U \cup V), \\ R^{-1}(\emptyset) &= \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R^{-1}\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$\begin{aligned} R^{-1}(U \cap V) &\subset R^{-1}(U) \cap R^{-1}(V), \\ R^{-1}(A) &\subset B, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} R^{-1,\otimes}_{U,V} \colon R^{-1}(U) \cup R^{-1}(V) &\xrightarrow{=} R^{-1}(U \cup V), \\ R^{-1,\otimes}_{\mathbb{1}} \colon \emptyset &\xrightarrow{=} \emptyset, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$R^{-1,\otimes}_{U,V}\colon R^{-1}(U\cap V)\subset R^{-1}(U)\cap R^{-1}(V),$$
$$R^{-1,\otimes}_{\mathbb{1}}\colon R^{-1}(A)\subset B,$$

natural in $U,V\in\mathcal{P}(B)$.

7. *Interaction With Strong Inverse Images.* Let $R\colon A \nrightarrow B$ be a relation from A to B.

   (a) If R is a total relation, then we have an inclusion of sets

   $$R_{-1}(V)\subset R^{-1}(V)$$

   natural in $V\in\mathcal{P}(B)$. We also have equalities

   $$R^{-1}(B\setminus V)=A\setminus R_{-1}(V),$$
   $$R_{-1}(B\setminus V)=A\setminus R^{-1}(V)$$

   for each $V\in\mathcal{P}(B)$.

   (b) If R is total and functional, then the above inclusion is in fact an equality.

   (c) Conversely, if we have $R_{-1}=R^{-1}$, then R is total and functional.

Proof 4.3.4 ▶ Proof of Proposition 4.3.3

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from Kan Extensions, Item 2 of Proposition 1.1.6.

Item 3: Preservation of Colimits

This follows from **??** and Categories, **??** of Proposition 6.1.3.

Item 4: Oplax Preservation of Limits

Omitted.

Item 5: Symmetric Strict Monoidality With Respect to Unions

This follows from Item 3.

Item 6: Symmetric Oplax Monoidality With Respect to Intersections

This follows from **??**.

Item 7: Interaction With Strong Inverse Images

This was proved in Item 7 of Item 7.

Proposition 4.3.5 ▶ Properties of the Weak Inverse Image Function Operation

Let $R\colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function
$$(-)^{-1}\colon \mathsf{Rel}(A,B) \to \mathsf{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$
2. *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function
$$(-)^{-1}\colon \mathsf{Rel}(A,B) \to \mathsf{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$
3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[1]
$$(\chi_A)^{-1} = \mathrm{id}_{\mathcal{P}(A)};$$
4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[2]
$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathcal{P}(B) \arrow[d, "R^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

---

[1] That is, the postcomposition
$$(\chi_A)^{-1}\colon \mathsf{Rel}(\mathrm{pt},A) \to \mathsf{Rel}(\mathrm{pt},A)$$
is equal to $\mathrm{id}_{\mathsf{Rel}(\mathrm{pt},A)}$.

[2] That is, we have
$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathsf{Rel}(\mathrm{pt},C) \arrow[r, "R^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathsf{Rel}(\mathrm{pt},B) \arrow[d, "S^{-1}"] \\ & \mathsf{Rel}(\mathrm{pt},A). \end{tikzcd}$$

Proof 4.3.6 ► Proof of Proposition 4.3.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

This follows from Categories, Item 5 of Proposition 1.4.3.

Item 4: Interaction With Composition

This follows from Categories, Item 2 of Proposition 1.4.3.

4.4 Direct Images With Compact Support

Let A and B be sets and let $R\colon A \rightarrow B$ be a relation.

Definition 4.4.1 ► Direct Images With Compact Support

The **direct image with compact support function associated to** R is the function[1]

$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[2,3]

$$\begin{aligned} R_!(U) &\stackrel{\text{def}}{=} \left\{ b \in B \,\middle|\, \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \,\middle|\, R^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

[1] *Further Notation:* Also written $\forall_R\colon \mathcal{P}(A) \to \mathcal{P}(B)$. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_R(U)$.
- For each $a \in A$, if $b \in R(a)$, then $a \in U$.

[2] *Further Terminology:* The set $R_!(U)$ is called the **direct image with compact support of** U **by** R.

[3] We also have

$$R_!(U) = B \setminus R_*(A \setminus U);$$

see Item 7 of Proposition 4.4.3.

Remark 4.4.2 ► Unwinding Definition 4.4.1

Identifying subsets of B with relations from pt to B via Constructions With Sets, Item 7 of Proposition 3.2.3, we see that the direct image with compact support function associated to R is equivalently the function

$$R_! \colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})}$$

defined by

$$R_!(U) \stackrel{\text{def}}{=} \mathrm{Ran}_R(U), \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(U)"] \\ A \arrow[ur, "R"] \arrow[r, "U"'] & \mathrm{pt}, \end{tikzcd}$$

being explicitly computed by

$$\begin{aligned} R^*(U) &\stackrel{\text{def}}{=} \mathrm{Ran}_R(U) \\ &\cong \int_{a \in A} \mathrm{Hom}_{\{\mathrm{t,f}\}}\left(R_a^{-2}, U_a^{-1}\right). \end{aligned}$$

Thus, we have

$$\begin{aligned} R^{-1}(U) &\cong \left\{ b \in B \;\middle|\; \int_{a \in A} \mathrm{Hom}_{\{\mathrm{t,f}\}}\left(R_a^b, U_a^{\star}\right) = \mathrm{true} \right\} \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A \text{, at least one of the follow-} \\ \text{ing conditions hold:} \\ \quad 1.\ \text{We have } R_a^b = \mathrm{false}; \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } R_a^b = \mathrm{true}; \\ \qquad \text{(b) We have } U_a^{\star} = \mathrm{true}; \end{array} \right\} \end{aligned}$$

$$= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A, \text{ at least one of the follow-} \\ \text{ing conditions hold:} \\ \quad 1.\ \text{We have } b \notin R(a); \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } b \in R(a); \\ \qquad \text{(b) We have } a \in U; \end{array} \right\}$$

$$= \{ b \in B \mid \text{for each } a \in A, \text{ if } b \in R(a), \text{ then } a \in U. \}$$

$$= \left\{ b \in B \;\middle|\; R^{-1}(b) \subset U \right\}.$$

Proposition 4.4.3 ► Properties of Direct Images With Compact Support

Let $R \colon A \nrightarrow B$ be a relation.

1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_! \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_!](U) \stackrel{\text{def}}{=} R_!(U);$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_!(U) \subset R_!(V)$.

2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

(★) The following conditions are equivalent:
   (a) We have $R^{-1}(U) \subset V$;
   (b) We have $U \subset R_!(V)$.

3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i\in I} R_!(U_i) \subset R_!\left(\bigcup_{i\in I} U_i\right),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$\emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

4. *Preservation of Limits.* We have an equality of sets

$$R_!\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} R_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_!(U \cap V) = R_!(U) \cap R_!(V),$$
$$R_!(A) = B,$$

natural in $U, V \in \mathcal{P}(A)$.

5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \longrightarrow (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R_{!|U,V}^{\otimes} \colon R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$R_{!|\mathbb{1}}^{\otimes} \colon \emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R_{!|U,V}^{\otimes} : R_!(U \cap V) \xrightarrow{=} R_!(U) \cap R_!(V),$$
$$R_{!|\mathbb{1}}^{\otimes} : R_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

7. *Relation to Direct Images.* We have

$$R_!(U) = B \setminus R_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof 4.4.4 ▶ Proof of Proposition 4.4.3

Item 1: Functoriality

Clear.

Item 2: Adjointness

This follows from Kan Extensions, Item 2 of Proposition 1.1.6.

Item 3: Lax Preservation of Colimits

Omitted.

Item 4: Preservation of Limits

This follows from Item 2 and Categories, **??** of Proposition 6.1.3.

Item 5: Symmetric Lax Monoidality With Respect to Unions

This follows from **??**.

Item 6: Symmetric Strict Monoidality With Respect to Intersections

This follows from Item 4.

Item 7: Relation to Direct Images

As with Item 7 of Proposition 4.1.3, the proof proceeds in the same way as in the case of functions (Constructions With Sets, Item 7 of Proposition 3.5.5): We claim that $R_!(U) = B \setminus R_*(A \setminus U)$.

- *The First Implication.* We claim that

$$R_!(U) \subset B \setminus R_*(A \setminus U).$$

  Let $b \in R_!(U)$. We need to show that $b \notin R_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $b \in R(a)$.

  This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_!(U)$).

  Thus $b \in B \setminus R_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus R_*(A \setminus U) \subset R_!(U).$$

  Let $b \in B \setminus R_*(A \setminus U)$. We need to show that $b \in R_!(U)$, i.e. that $R^{-1}(b) \subset U$.

  Since $b \notin R_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

  Thus $b \in R_!(U)$.

This finishes the proof. ∎

Proposition 4.4.5 ▶ Properties of the Direct Image With Compact Support Function Operation

Let $R \colon A \nrightarrow B$ be a relation.

1. *Functionality I.* The assignment $R \mapsto R_!$ defines a function

$$(-)_! \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

2. *Functionality II.* The assignment $R \mapsto R_!$ defines a function

$$(-)_! \colon \mathsf{Sets}(A, B) \to \mathrm{Hom}_{\mathsf{Pos}}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)};$$

4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have

$$
(S \diamond R)_! = S_! \circ R_!,
\qquad
\begin{tikzcd}
\mathcal{P}(A) \arrow[r, "R_!"] \arrow[dr, "(S\diamond R)_!"'] & \mathcal{P}(B) \arrow[d, "S_!"] \\
& \mathcal{P}(C).
\end{tikzcd}
$$

Proof 4.4.6 ▶ Proof of Proposition 4.4.5

Item 1: Functionality I

Clear.

Item 2: Functionality II

Clear.

Item 3: Interaction With Identities

Indeed, we have

$$
\begin{aligned}
(\chi_A)_!(U) &\overset{\text{def}}{=} \left\{a \in A \,\middle|\, \chi_A^{-1}(a) \subset U\right\} \\
&\overset{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\
&= U
\end{aligned}
$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_! = \mathrm{id}_{\mathcal{P}(A)}$.

Item 4: Interaction With Composition

Indeed, we have

$$
\begin{aligned}
(S \diamond R)_!(U) &\overset{\text{def}}{=} \left\{c \in C \,\middle|\, [S \diamond R]^{-1}(c) \subset U\right\} \\
&\overset{\text{def}}{=} \left\{c \in C \,\middle|\, S^{-1}\left(R^{-1}(c)\right) \subset U\right\} \\
&= \left\{c \in C \,\middle|\, R^{-1}(c) \subset S_!(U)\right\} \\
&\overset{\text{def}}{=} R_!(S_!(U)) \\
&\overset{\text{def}}{=} [R_! \circ S_!](U)
\end{aligned}
$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 4.4.3, which implies that the conditions

- We have $S^{-1}\left(R^{-1}(c)\right) \subset U$;

- We have $R^{-1}(c) \subset S_!(U)$;

are equivalent. Thus $(S \diamond R)_! = S_! \circ R_!$.

4.5 Functoriality of Powersets

Proposition 4.5.1 ▶ Functoriality of Powersets I

The assignment $X \mapsto \mathcal{P}(X)$ defines functors[1]

$$\begin{aligned} \mathcal{P}_*\colon \mathsf{Rel} &\to \mathsf{Sets}, \\ \mathcal{P}_{-1}\colon \mathsf{Rel}^{\mathsf{op}} &\to \mathsf{Sets}, \\ \mathcal{P}^{-1}\colon \mathsf{Rel}^{\mathsf{op}} &\to \mathsf{Sets}, \\ \mathcal{P}_!\colon \mathsf{Rel} &\to \mathsf{Sets} \end{aligned}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$\begin{aligned} \mathcal{P}_*(A) &\overset{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_{-1}(A) &\overset{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}^{-1}(A) &\overset{\text{def}}{=} \mathcal{P}(A), \\ \mathcal{P}_!(A) &\overset{\text{def}}{=} \mathcal{P}(A); \end{aligned}$$

- *Action on Morphisms.* For each morphism $R\colon A \nrightarrow B$ of Rel, the images

$$\begin{aligned} \mathcal{P}_*(R)\colon \mathcal{P}(A) &\to \mathcal{P}(B), \\ \mathcal{P}_{-1}(R)\colon \mathcal{P}(B) &\to \mathcal{P}(A), \\ \mathcal{P}^{-1}(R)\colon \mathcal{P}(B) &\to \mathcal{P}(A), \\ \mathcal{P}_!(R)\colon \mathcal{P}(A) &\to \mathcal{P}(B) \end{aligned}$$

  of R by $\mathcal{P}_*, \mathcal{P}_{-1}, \mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned} \mathcal{P}_*(R) &\overset{\text{def}}{=} R_*, \\ \mathcal{P}_{-1}(R) &\overset{\text{def}}{=} R_{-1}, \\ \mathcal{P}^{-1}(R) &\overset{\text{def}}{=} R^{-1}, \\ \mathcal{P}_!(R) &\overset{\text{def}}{=} R_!, \end{aligned}$$

  as in Definitions 4.1.1, 4.2.1, 4.3.1 and 4.4.1.

[1] The functor $\mathcal{P}_*\colon \mathsf{Rel} \to \mathsf{Sets}$ admits a left adjoint; see Item 3 of Proposition 2.1.2.

Proof 4.5.2 ► Proof of Proposition 4.5.1

This follows from Items 3 and 4 of Proposition 4.1.5, Items 3 and 4 of Proposition 4.2.5, Items 3 and 4 of Proposition 4.3.5, and Items 3 and 4 of Proposition 4.4.5.

4.6 Functoriality of Powersets: Relations on Powersets

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

Definition 4.6.1 ► The Relation on Powersets Associated to a Relation

The **relation on powersets associated to** R is the relation

$$\mathcal{P}(R)\colon \mathcal{P}(A) \nrightarrow \mathcal{P}(B)$$

defined by[1]

$$\mathcal{P}(R)^{V}_{U} \overset{\text{def}}{=} \mathbf{Rel}\big(\chi_{\mathrm{pt}}, V \diamond R \diamond U\big)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

[1] Illustration:

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "\shortmid"{marking}, "U"'] \arrow[rrr, bend left, "\shortmid"{marking}, "\chi_{\mathrm{pt}}"] & A \arrow[r, "\shortmid"{marking}, "R"'] & B \arrow[r, "\shortmid"{marking}, "V"'] & \mathrm{pt}. \end{tikzcd}$$

Remark 4.6.2 ► Unwinding Definition 4.6.1

In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff:

- We have $\chi_{\mathrm{pt}} \subset V \diamond R \diamond U$, i.e. iff:
- We have $(V \diamond R \diamond U)^{\star}_{\star} = \mathrm{true}$, i.e. iff we have

  $$\int^{a\in A}\int^{b\in B} V^{\star}_{b} \times R^{b}_{a} \times U^{a}_{\star} = \mathrm{true},$$

  i.e. iff:
- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $U^{a}_{\star} = \mathrm{true}$;
  - We have $R^{b}_{a} = \mathrm{true}$;
  - We have $V^{\star}_{b} = \mathrm{true}$;

i.e. iff:

- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $a \in U$;
  - We have $a \sim_R b$;
  - We have $b \in V$.

Proposition 4.6.3 ► Functoriality of Powersets II

The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P}\colon \ \mathsf{Rel} \longrightarrow \mathsf{Rel}\,.$$

Proof 4.6.4 ► Proof of Proposition 4.6.3

Omitted. □

5 Spans

5.1 Foundations

Let A and B be sets.

Definition 5.1.1 ► Spans

A **span from A to B**[1] is a functor $F\colon \Lambda \longrightarrow \mathsf{Sets}$ such that

$$\begin{aligned} F([-1]) &= A, \\ F([1]) &= B. \end{aligned}$$

[1] *Further Terminology:* Also called a **roof from A to B** or a **correspondence from A to B**.

Remark 5.1.2 ► Unwinding Definition 5.1.1

In detail, a **span from A to B** is a triple (S, f, g) consisting of[1,2]

- *The Underlying Set.* A set S, called the **underlying set of** (S, f, g);
- *The Legs.* A pair of functions $f\colon S \longrightarrow A$ and $g\colon S \longrightarrow B$.

[1] *Picture:*

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B. \end{tikzcd}$$

[2] We may think of a span (S, f, g) from A to B as a multivalued map from A to B, sending an element $a \in A$ to the set $g\left(f^{-1}(a)\right)$ of elements of B.

Definition 5.1.3 ► Morphisms of Spans

A **morphism of spans** (R, f_1, g_1) **to** (S, f_2, g_2)[1] is a natural transformation $(R, f_1, g_1) \Longrightarrow (S, f_2, g_2)$.

[1] *Further Terminology:* Also called a **morphism of roofs from** (R, f_1, g_1) **to** (S, f_2, g_2) or a **morphism of correspondences from** (R, f_1, g_1) **to** (S, f_2, g_2).

Remark 5.1.4 ► Unwinding Definition 5.1.3

In detail, a **morphism of spans from** (R, f_1, g_1) **to** (S, f_2, g_2) is a function $\phi\colon R \to S$ making the diagram[1]

$$\begin{tikzcd} & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[rrr, "\phi"] & & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\ A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A & & B. \end{tikzcd}$$

commute.

[1] *Alternative Picture:*

$$\begin{tikzcd} & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[dd, "\phi"] & \\ A & & B. \\ & S \arrow[ul, "f_2"] \arrow[ur, "g_2"'] & \end{tikzcd}$$

Definition 5.1.5 ► The Category of Spans From A to B

The **category of spans from A to B** is the category $\mathsf{Span}(A,B)$ defined by

$$\mathsf{Span}(A,B) \overset{\text{def}}{=} \mathsf{Fun}(\Lambda,\mathsf{Sets}) \underset{\mathrm{ev}_{[-1]},\mathsf{Sets},[A]}{\times} \mathrm{pt} \underset{[B],\mathsf{Sets},\mathrm{ev}_{[1]}}{\times} \mathsf{Fun}(\Lambda,\mathsf{Sets}),$$

as in the diagram

$$\begin{tikzcd}
 & & & \mathsf{Span}(A,B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner"{rotate=-45}, very near start] & & & \\
 & & \mathsf{Fun}(\Lambda,\mathsf{Sets}) \underset{\mathsf{Sets}}{\times} \mathrm{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner"{rotate=-45}, very near start] & & \mathrm{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Fun}(\Lambda,\mathsf{Sets}) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner"{rotate=-45}, very near start] & & \\
 & \mathsf{Fun}(\Lambda,\mathsf{Sets}) \arrow[dr, "\mathrm{ev}_{[-1]}"'] & & \mathrm{pt} \arrow[dl, "{[A]}"] \arrow[dr, "{[B]}"'] & & \mathsf{Fun}(\Lambda,\mathsf{Sets}). \arrow[dl, "\mathrm{ev}_{[1]}"] & \\
 & & \mathsf{Sets} & & \mathsf{Sets} & &
\end{tikzcd}$$

Remark 5.1.6 ► Unwinding Definition 5.1.5

In detail, the **category of spans from A to B** is the category $\mathsf{Span}(A,B)$ where

- *Objects.* The objects of $\mathsf{Span}(A,B)$ are spans from A to B;
- *Morphisms.* The morphism of $\mathsf{Span}(A,B)$ are morphisms of spans;
- *Identities.* The unit map

  $$\mathbb{1}^{\mathsf{Span}(A,B)}_{(S,f,g)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{Span}_C(A,B)}((S,f,g),(S,f,g))$$

  of $\mathsf{Span}(A,B)$ at (S,f,g) is defined by[1]

  $$\mathrm{id}^{\mathsf{Span}(A,B)}_{(S,f,g)} \overset{\text{def}}{=} \mathrm{id}_S;$$

- *Composition.* The composition map

  $$\circ^{\mathsf{Span}(A,B)}_{R,S,T} \colon \mathrm{Hom}_{\mathsf{Span}_C(A,B)}(S,T) \times \mathrm{Hom}_{\mathsf{Span}_C(A,B)}(R,S) \longrightarrow \mathrm{Hom}_{\mathsf{Span}_C(A,B)}(R,T)$$

  of $\mathsf{Span}(A,B)$ at $((R,f_1,g_1),(S,f_2,g_2),(T,f_3,g_3))$ is defined by[2]

  $$\psi \circ^{\mathsf{Span}(A,B)}_{R,S,T} \phi \overset{\text{def}}{=} \psi \circ \phi.$$

---

[1] *Picture:*

$$
\begin{tikzcd}
 & S \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[rrr, "\mathrm{id}_S"] & & & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A & & B.
\end{tikzcd}
$$

[2] *Picture:*

$$
\begin{tikzcd}
 & R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] \arrow[rrr, "\phi"] & & & S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] \arrow[rrr, "\psi"] & & & T \arrow[dl, "f_3"'] \arrow[dr, "g_3"] & \\
A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A \arrow[rrr, equal] & & B \arrow[rrr, equal] & A & & B.
\end{tikzcd}
$$

Definition 5.1.7 ► The Bicategory of Spans

The **bicategory of spans in C** is the bicategory Span where

- *Objects.* The objects of Span are sets;
- Hom-*Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, we have

$$\mathrm{Hom}_{\mathsf{Span}}(A, B) \stackrel{\text{def}}{=} \mathsf{Span}(A, B);$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, the unit functor

$$\mathbb{1}^{\mathsf{Span}}_A \colon \mathrm{pt} \longrightarrow \mathsf{Span}(A, A)$$

  of Span at A is the functor picking the span $(A, \mathrm{id}_A, \mathrm{id}_A)$:

$$
\begin{tikzcd}
 & A \arrow[dl, "\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_A"] & \\
A & & A.
\end{tikzcd}
$$

- *Composition.* The composition bifunctor

$$\circ^{\mathsf{Span}}_{A,B,C} \colon \mathsf{Span}(B,C) \times \mathsf{Span}(A,B) \to \mathsf{Span}(A,C)$$

of Span at (A,B,C) is the bifunctor where

  - *Action on Objects.* The composition of two spans

$$\begin{tikzcd}
& R \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & \\
A & & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& S \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
A & & B
\end{tikzcd}$$

    in $\mathcal{C}$ is the span $(R \times_B S, f_1 \circ \mathrm{pr}_1, g_2 \circ \mathrm{pr}_2)$, constructed as in the diagram

$$\begin{tikzcd}
& & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] \arrow[ddll, bend right, "f_1 \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "g_2 \circ \mathrm{pr}_2"] & & \\
& R \arrow[dl, "f_1"'] \arrow[dr, "g_2"] & & S \arrow[dl, "f_1"'] \arrow[dr, "g_2"] & \\
A & & B & & C.
\end{tikzcd}$$

  - *Action on Morphisms.* The horizontal composition of 2-morphisms is defined via functoriality of pullbacks: given morphisms of spans

$$\begin{tikzcd}
& R \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\phi"] & \\
A & & B \\
& R' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& S \arrow[dl, "h"'] \arrow[dr, "k"] \arrow[dd, "\psi"] & \\
B & & C, \\
& S' \arrow[ul, "h'"] \arrow[ur, "k'"'] &
\end{tikzcd}$$

their horizontal composition is the morphism of spans

$$
\begin{tikzcd}
& R \times_B S \arrow[dl, "f \circ \mathrm{pr}_1"'] \arrow[dr, "k \circ \mathrm{pr}_2"] \arrow[dd, dashed, "\exists!"] & \\
A & & C, \\
& R' \times_B S' \arrow[ul, "h' \circ \mathrm{pr}'_1"] \arrow[ur, "k' \circ \mathrm{pr}'_2"'] &
\end{tikzcd}
$$

constructed as in the diagram

$$
\begin{tikzcd}
& & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[ddll, bend right, "f \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "k \circ \mathrm{pr}_2"] \arrow[dddd, dashed, bend right, "\exists!"] & & \\
& R \arrow[dl, "f"'] \arrow[dr, "g"] & & S \arrow[dl, "h"'] \arrow[dr, "k"] & \\
A & & B & & C; \\
& R' \arrow[ul, "f'"] \arrow[ur, "g'"'] & & S' \arrow[ul, "h'"] \arrow[ur, "k"'] & \\
& & R' \times_B S' \arrow[ul, "\mathrm{pr}'_1"] \arrow[ur, "\mathrm{pr}'_2"'] \arrow[uull, bend left, "f' \circ \mathrm{pr}'_1"] \arrow[uurr, bend right, "k' \circ \mathrm{pr}'_2"'] & &
\end{tikzcd}
$$

- *Associators and Unitors.* The associator and unitors are defined using the universal property of the pullback.

Definition 5.1.8 ► The Double Category of Spans

The **double category of spans** is the double category $\mathsf{Span}^{\mathrm{dbl}}$ where

- *Objects.* The objects of $\mathsf{Span}^{\mathrm{dbl}}$ are sets;

- *Vertical Morphisms.* The vertical morphisms of $\mathsf{Span}^{\mathsf{dbl}}$ are functions $f\colon A \to B$;
- *Horizontal Morphisms.* The horizontal morphisms of $\mathsf{Span}^{\mathsf{dbl}}$ are spans $(S, \phi, \psi)\colon A \nrightarrow X$;
- *2-Morphisms.* A 2-cell

$$\begin{tikzcd}
A \arrow[r, "{(R,\phi_R,\psi_R)}", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\
X \arrow[r, "{(S,\phi_S,\psi_S)}"', "\shortmid"{marking}] & Y
\end{tikzcd}$$

of $\mathsf{Span}^{\mathsf{dbl}}$ is a morphism of spans from the span

$$\begin{tikzcd}
 & & R \arrow[dll, "\phi_R"'] \arrow[dr, "\psi_R"] & & \\
 & & & B \arrow[dr, "g"] & \\
A & & & & Y
\end{tikzcd}$$

to the span

$$\begin{tikzcd}
 & & A \times_X S \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "f"] & & S \arrow[dl, "\phi_S"'] \arrow[dr, "\psi_S"] & \\
X & & X & & Y;
\end{tikzcd}$$

- *Horizontal Identities.* The horizontal unit functor

$$\mathbb{1}^{\mathsf{Span}^{\mathsf{dbl}}}\colon \left(\mathsf{Span}^{\mathsf{dbl}}\right)_0 \to \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1$$

of $\mathsf{Span}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each $A \in \mathrm{Obj}\Big(\big(\mathsf{Span}^{\mathsf{dbl}}\big)_0\Big)$, we have

$$\mathbb{1}_A \overset{\text{def}}{=} (A, \mathrm{id}_A, \mathrm{id}_A),$$

as in the diagram

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & \\ A & & A; \end{tikzcd}$$

- *Action on Morphisms.* For each vertical morphism $f\colon A \longrightarrow B$ of $\mathsf{Span}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \mathbb{1}_f"] & A \arrow[d, "f"] \\ B \arrow[r, "\mathbb{1}_B"', "\shortmid"{marking}] & B \end{tikzcd}$$

of f is the morphism of spans from

$$\begin{tikzcd} & & A \arrow[dll, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & & \\ & & & A \arrow[dr, "f"] & \\ A & & & & B \end{tikzcd}$$

to

$$\begin{tikzcd} & & A \times_B B \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & A \arrow[dl, "f"'] \arrow[dr, "f"] & & B \arrow[dl, equal, "\mathrm{id}_B"'] \arrow[dr, equal, "\mathrm{id}_B"] & \\ B & & B & & B \end{tikzcd}$$

given by the isomorphism $A \xrightarrow{\cong} A \times_B B$;

- *Vertical Identities.* For each $A \in \mathrm{Obj}\left(\mathsf{Span}^{\mathsf{dbl}}\right)$, we have

$$\mathrm{id}_A^{\mathsf{Span}^{\mathsf{dbl}}} \stackrel{\mathrm{def}}{=} \mathrm{id}_A;$$

- *Identity 2-Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Span}^{\mathsf{dbl}}$, the identity 2-morphism

$$\begin{tikzcd}
A \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\Downarrow \mathrm{id}_S"] & B \arrow[d, "\mathrm{id}_B"] \\
A \arrow[r, "S"', "\shortmid"{marking}] & B
\end{tikzcd}$$

of R is the morphism of spans from

$$\begin{tikzcd}
 & & S \arrow[dll, "\phi_S"'] \arrow[dr, "\psi_S"] & & \\
 & & & B \arrow[dr, equal, "\mathrm{id}_B"] & \\
A & & & & B
\end{tikzcd}$$

to

$$\begin{tikzcd}
 & & A \times_A S \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] & & S \arrow[dl, "\phi_S"'] \arrow[dr, "\psi_S"] & \\
A & & A & & B
\end{tikzcd}$$

given by the isomorphism $S \xrightarrow{\cong} A \times_A S$;

- *Horizontal Composition.* The horizontal composition functor

$$\odot^{\mathsf{Span}^{\mathsf{dbl}}} \colon \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1 \times_{\left(\mathsf{Span}^{\mathsf{dbl}}\right)_0} \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1 \to \left(\mathsf{Span}^{\mathsf{dbl}}\right)_1$$

of $\mathsf{Span}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each composable pair

$$A \overset{(R,\phi_R,\psi_R)}{\mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}}} B \overset{(S,\phi_S,\psi_S)}{\mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}}} C$$

of horizontal morphisms of $\mathsf{Span}^{\mathsf{dbl}}$, we have

$$(S, \phi_S, \psi_S) \odot (R, \phi_R, \psi_R) \overset{\text{def}}{=} S \circ^{\mathsf{Span}}_{A,B,C} R,$$

where $S \circ^{\mathsf{Span}}_{A,B,C} R$ is the composition of (R, ϕ_R, ψ_R) and (S, ϕ_S, ψ_S) defined as in Definition 5.1.7;

- *Action on Morphisms.* For each horizontally composable pair

$$\begin{tikzcd} A \arrow[r, "(R,\phi_R,\psi_R)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "(T,\phi_T,\psi_T)"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "(S,\phi_S,\psi_S)", "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\ Y \arrow[r, "(U,\phi_U,\psi_U)"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Span}^{\mathsf{dbl}}$, [...];

- *Vertical Composition of 1-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Span}^{\mathsf{dbl}}$, i.e. maps of sets, we have

$$g \circ^{\mathsf{Span}^{\mathsf{dbl}}} f \overset{\text{def}}{=} g \circ f;$$

- *Vertical Composition of 2-Morphisms.* For each vertically composable pair

$$\begin{tikzcd} A \arrow[r, "(R,\phi_R,\psi_R)", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "g"] \\ B \arrow[r, "(S,\phi_S,\psi_S)"', "\shortmid"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "(S,\phi_S,\psi_S)", "\shortmid"{marking}] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & Y \arrow[d, "k"] \\ C \arrow[r, "(T,\phi_T,\psi_T)"', "\shortmid"{marking}] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Span}^{\mathsf{dbl}}$, [...];

- *Associators and Unitors.* The associator and unitors of $\mathsf{Span}^{\mathsf{dbl}}$ are defined using the universal property of the pullback.

5.2 Comparison to Functions

Proposition 5.2.1 ▶ Comparison of Spans to Functions

We have a pseudofunctor

$$\iota\colon \mathsf{Sets}_{\mathsf{bidisc}} \to \mathsf{Span}$$

from $\mathsf{Sets}_{\mathsf{bidisc}}$ to Span where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, we have

$$\iota(A) \stackrel{\mathrm{def}}{=} A;$$

- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets}_{\mathsf{bidisc}})$, the action on Hom-categories

$$\iota_{A,B}\colon \mathsf{Sets}(A, B)_{\mathsf{disc}} \to \mathsf{Span}(A, B)$$

of ι at (A, B) is the functor defined on objects by sending a function $f\colon A \to B$ to the span

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, "f"] & \\ A & & B \end{tikzcd}$$

from A to B.

Proof 5.2.2 ▶ Proof of Proposition 5.2.1

Clear. □

5.3 Comparison to Relations

Proposition 5.3.1 ▶ Comparison of Spans to Relations I

We have a pseudofunctor

$$\iota\colon \mathsf{Span} \to \mathbf{Rel}$$

from Span to $\mathbf{Rel}$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, we have

$$\iota(A) \stackrel{\mathrm{def}}{=} A;$$

- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, the action on Hom-categories
$$\iota_{A,B} \colon \mathsf{Span}(A, B) \longrightarrow \mathbf{Rel}(A, B)$$
of ι at (A, B) is the functor where
  - *Action on Objects.* Given a span
$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A & & B \end{tikzcd}$$
from A to B, we define a relation
$$\iota_{A,B}(S) \colon A \nrightarrow B$$
from A to B as follows:
    - Viewing relations as functions $A \times B \longrightarrow \{\text{true}, \text{false}\}$, we define
$$\iota_{A,B}(S)^a_b \overset{\text{def}}{=} \begin{cases} \text{true} & \text{if there exists } x \in S \text{ such that } a = f(x) \text{ and } b = g(x), \\ \text{false} & \text{otherwise} \end{cases}$$
for each $(a, b) \in A \times B$;
    - Viewing relations as functions $A \longrightarrow \mathcal{P}(B)$, we define
$$\left[\iota_{A,B}(S)\right](a) \overset{\text{def}}{=} g\left(f^{-1}(a)\right)$$
for each $a \in A$;
    - Viewing relations as subsets of $A \times B$, we define
$$\iota_{A,B}(S) \overset{\text{def}}{=} \{(f(x), g(x)) \mid x \in S\}.$$
  - *Action on Morphisms.* Given a morphism of spans
$$\begin{tikzcd} & R \arrow[dl, "f_R"'] \arrow[dr, "g_R"] \arrow[dd, "\phi"] & \\ A & & B, \\ & S \arrow[ul, "f_S"] \arrow[ur, "g_S"'] & \end{tikzcd}$$

we have a corresponding inclusion of relations

$$\iota_{A,B}(\phi)\colon \iota_{A,B}(R) \subset \iota_{A,B}(S),$$

since we have $a \sim_{\iota_{A,B}(R)} b$ iff there exists $x \in R$ such that $a = f_R(x)$ and $b = g_R(x)$, in which case we then have

$$\begin{aligned} a &= f_R(x) \\ &= f_S(\phi(x)), \\ b &= g_R(x) \\ &= g_S(\phi(x)), \end{aligned}$$

so that $a \sim_{\iota_{A,B}(S)} b$, and thus $\iota_{A,B}(R) \subset \iota_{A,B}(S)$.

Proof 5.3.2 ► Proof of Proposition 5.3.1

Omitted. ∎

Proposition 5.3.3 ► Comparison of Spans to Relations II

We have a lax functor

$$\left(\iota, \iota^2, \iota^0\right)\colon\ \mathbf{Rel} \longrightarrow \mathsf{Span}$$

from **Rel** to Span where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Span})$, we have

  $$\iota(A) \stackrel{\text{def}}{=} A;$$

- *Action on* Hom*-Categories.* For each $A, B \in \mathrm{Obj}(\mathsf{Span})$, the action on Hom-categories

  $$\iota_{A,B}\colon\ \mathbf{Rel}(A,B) \longrightarrow \mathsf{Span}(A,B)$$

  of ι at (A,B) is the functor where

  - *Action on Objects.* Given a relation $R\colon\ A \nrightarrow B$ from A to B, we define a span

    $$\iota_{A,B}(R)\colon\ A \nrightarrow B$$

    from A to B by

    $$\iota_{A,B}(R) \stackrel{\text{def}}{=} (R, \mathrm{pr}_1|_R, \mathrm{pr}_2|_R),$$

where $R \subset A \times B$ and $\mathrm{pr}_1|_R$ and $\mathrm{pr}_2|_R$ are the restriction of the projections

$$\begin{aligned}\mathrm{pr}_1 \colon A \times B &\to A,\\ \mathrm{pr}_2 \colon A \times B &\to B\end{aligned}$$

to R;

- *Action on Morphisms.* Given an inclusion $\phi \colon R \subset S$ of relations, we have a corresponding morphism of spans

$$\iota_{A,B}(\phi) \colon \iota_{A,B}(R) \to \iota_{A,B}(S)$$

as in the diagram

$$\begin{tikzcd} & R \arrow[dl, "\mathrm{pr}_1|_R"'] \arrow[dr, "\mathrm{pr}_2|_R"] \arrow[dd, hook] & \\ A & & B. \\ & S \arrow[ul, "\mathrm{pr}_1|_S"] \arrow[ur, "\mathrm{pr}_2|_S"'] & \end{tikzcd}$$

- *The Lax Functoriality Constraints.* The lax functoriality constraint

$$\iota^2_{R,S} \colon \iota(S) \circ \iota(R) \Longrightarrow \iota(S \diamond R)$$

of ι at (R, S) is given by the morphism of spans from

$$\begin{tikzcd} & & R \times_B S \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "\mathrm{pr}_1|_R \circ \mathrm{pr}_1"'] \arrow[ddrr, bend left, "\mathrm{pr}_2|_S \circ \mathrm{pr}_2"] & & \\ & R \arrow[dl, "\mathrm{pr}_1|_R"'] \arrow[dr, "\mathrm{pr}_2|_R"] & & S \arrow[dl, "\mathrm{pr}_1|_S"'] \arrow[dr, "\mathrm{pr}_2|_S"] & \\ A & & B & & C \end{tikzcd}$$

to

$$\begin{tikzcd} & S \diamond R \arrow[dl, "\mathrm{pr}_1|_{S \diamond R}"'] \arrow[dr, "\mathrm{pr}_2|_{S \diamond R}"] & \\ A & & C \end{tikzcd}$$

given by the natural inclusion $R \times_B S \hookrightarrow S \diamond R$, since we have

$$R \times_B S = \{((a_R, b_R), (b_S, c_S)) \in R \times S \mid b_R = b_S\};$$
$$S \diamond R = \left\{(a,c) \in A \times C \,\middle|\, \begin{array}{l}\text{there exists some } b \in B \text{ such} \\ \text{that } (a,b) \in R \text{ and } (b,c) \in S\end{array}\right\};$$

- *The Lax Unity Constraints.* The lax unity constraint[1]

$$\iota^0_A \colon \underbrace{\mathrm{id}_{\iota(A)}}_{(A,\mathrm{id}_A,\mathrm{id}_A)} \Longrightarrow \underbrace{\iota(\chi_A)}_{\left(\Delta_A, \mathrm{pr}_1|_{\Delta_A}, \mathrm{pr}_2|_{\Delta_A}\right)}$$

of ι at A is given by the diagonal morphism of A, as in the diagram

$$\begin{tikzcd} & A \arrow[dl, equal, "\mathrm{id}_A"'] \arrow[dr, equal, "\mathrm{id}_A"] \arrow[d, "\delta_A"] & \\ A & \Delta_A \arrow[l, "\mathrm{pr}_1|_{\Delta_A}"] \arrow[r, "\mathrm{pr}_2|_{\Delta_A}"'] & A. \end{tikzcd}$$

[1]Which is in fact strong, as δ_A is an isomorphism.

Proof 5.3.4 ► Proof of Proposition 5.3.1

Omitted. ∎

Remark 5.3.5 ► Interaction With Multirelations

The pseudofunctor of Proposition 5.3.1 and the lax functor of Proposition 5.3.3 fail to be equivalences of bicategories. This happens essentially because a span $(S, f, g) \colon A \nrightarrow B$ from A to B may relate elements $a \in A$ and $b \in B$ by more than one element, e.g. there could be $s \neq s' \in S$ such that $a = f(s) = f(s')$ and $b = g(s) = g(s')$.

Thus, in a sense, spans may be thought of as "relations with multiplicity". And indeed, if instead of considering relations from A to B, i.e. functions

$$R \colon A \times B \longrightarrow \{\mathrm{true}, \mathrm{false}\}$$

from $A \times B$ to $\{\mathrm{true}, \mathrm{false}\} \cong \{0, 1\}$, we consider functions

$$R \colon A \times B \longrightarrow \mathbb{N} \cup \{\infty\}$$

from $A \times B$ to $\mathbb{N} \cup \{\infty\}$, then we obtain the notion of a **multirelation from A to B**, and these turn out to assemble together with sets into a bicategory MRel that is biequivalent to Span; see [BG03, Propositions 2.5 and 2.6].

Remark 5.3.6 ► Interaction With Double Categories and Adjointness

There are double functors between the double categories $\mathrm{Rel}^{\mathrm{dbl}}$ and $\mathrm{Span}^{\mathrm{dbl}}$ analogous to the functors of Propositions 5.3.1 and 5.3.3, assembling moreover into a strict-lax adjunction of double functors; see [Gra20, Section 4.5.3].

6 Hyperpointed Sets

6.1 Foundations

Definition 6.1.1 ► Hyperpointed Sets

A **hyperpointed set**[1] is equivalently:

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathrm{Rel}), \mathrm{pt})$;
- A pointed object in $(\mathrm{Rel}, \mathrm{pt})$;
- A pointed object in $(\mathbf{Rel}, \mathrm{pt})$.

[1] *Further Terminology:* Also called a **multipointed set** or an **$\mathbb{F}_1$-hypermodule**.

Remark 6.1.2 ► Unwinding Definition 6.1.1, I

Viewing relations $A \nrightarrow B$ as functions $A \times B \to \{\mathrm{true}, \mathrm{false}\}$ via Remark 1.1.3, we see that hyperpointed sets may also be described as follows:

A **hyperpointed set** is a pair (X, x_0) consisting of

- *The Underlying Set.* A set X, called the **underlying set of (X, x_0)**;
- *The Hyperbasepoint.* A morphism

$$J \colon \mathrm{pt} \nrightarrow X$$

  in Rel from pt to X, i.e. a relation

$$J \colon \mathrm{pt} \times X \nrightarrow \{\mathrm{true}, \mathrm{false}\}$$

  from pt to X, called the **hyperbasepoint of X**.

Remark 6.1.3 ▶ Unwinding Definition 6.1.1, II

Viewing relations $A \nrightarrow B$ as functions $A \to \mathcal{P}(B)$ via Remark 1.1.3, we see that hyperpointed sets may also be described as follows:

A **hyperpointed set** is a pair (X, x_0) consisting of

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0);
- *The Hyperbasepoint.* A morphism

$$[x_0]\colon \mathrm{pt} \nrightarrow X$$

  in Rel from pt to X, i.e. a relation

$$[x_0]\colon \mathrm{pt} \to \mathcal{P}(X)$$

  from pt to X, determining a subset x_0 of X, called the **hyperbasepoint of** X.

Example 6.1.4 ▶ The Empty Hyperpointed Set

The **empty hyperpointed set** is the hyperpointed set $(\emptyset, \emptyset)$ consisting of

- *The Underlying Set.* The empty set $\emptyset$;
- *The Hyperbasepoint.* The subset $\emptyset$ of pt.

Example 6.1.5 ▶ The Trivial Hyperpointed Set

The **trivial hyperpointed set** is the hyperpointed set $(\mathrm{pt}, \star)$ consisting of

- *The Underlying Set.* The punctual set $\mathrm{pt} \overset{\text{def}}{=} \{\star\}$;
- *The Hyperbasepoint.* The subset $\{\star\}$ of pt.

Example 6.1.6 ▶ Representable Hyperpointed Sets

The **representable hyperpointed set associated to a pointed set** (X, x_0) is the hyperpointed set $(X, \{x_0\})$ consisting of

- *The Underlying Set.* The set X;
- *The Hyperbasepoint.* The subset $\{x_0\}$ of X.

6.2 Hyperpointed Functions

6.2.1 Lax Hyperpointed Functions

Let (X, x_0) and (Y, y_0) be hyperpointed sets.

Definition 6.2.1 ► Lax Hyperpointed Functions

A **lax hyperpointed function from** (X, x_0) **to** (Y, y_0)[1] is a pair (f, f^0) consisting of

- *The Underlying Function.* A function $f\colon X \to Y$, called the **underlying function of** (f, f^0);
- *The Hyperbasepoint Preservation Constraint.* A natural transformation

$$f^0\colon [y_0] \Longrightarrow f_* \circ [x_0], \qquad \begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] \arrow[d, phantom, "\overset{f^0}{\Swarrow}"] & \\ \mathcal{P}(X) \arrow[rr, "f_*"'] & & \mathcal{P}(Y), \end{tikzcd}$$

called the **lax hyperbasepoint preservation constraint of** (f, f^0), i.e. an inclusion of sets

$$y_0 \subset f(x_0).$$

[1] *Further Terminology:* Also called a **lax multipointed function**, a **lax morphism of hyperpointed sets**, a **lax morphism of multipointed sets**, or a **lax morphism of** $\mathbb{F}_1$**-hypermodules**.

6.2.2 Oplax Hyperpointed Functions

Let (X, x_0) and (Y, y_0) be hyperpointed sets.

Definition 6.2.2 ► Oplax Hyperpointed Functions

A **oplax hyperpointed function from** (X, x_0) **to** (Y, y_0)[1] is a pair (f, f^0) consisting of

- *The Underlying Function.* A function $f\colon X \to Y$, called the **underlying function of** (f, f^0);

- *The Hyperbasepoint Preservation Constraint.* A natural transformation

$$f^0\colon [y_0] \Longrightarrow f_* \circ [x_0], \qquad \begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ \mathcal{P}(X) \arrow[rr, "f_*"'] \arrow[ur, phantom, "\overset{f^0}{\Longrightarrow}"] & & \mathcal{P}(Y), \end{tikzcd}$$

called the **oplax hyperbasepoint preservation constraint of** (f, f^0), i.e. an inclusion of sets

$$f(x_0) \subset y_0.$$

[1] *Further Terminology:* Also called a **oplax multipointed function**, a **oplax morphism of hyperpointed sets**, a **oplax morphism of multipointed sets**, or a **oplax morphism of $\mathbb{F}_1$-hypermodules**.

6.2.3 Strong Hyperpointed Functions

Let (X, x_0) and (Y, y_0) be hyperpointed sets.

Definition 6.2.3 ► Strong Hyperpointed Functions

A **strong hyperpointed function from** (X, x_0) **to** (Y, y_0)[1] is an op/lax hyperpointed function (f, f^0) whose hyperbasepoint preservation constraint is an isomorphism.

[1] *Further Terminology:* Also called simply a **hyperpointed function**, a **strict hyperpointed function**, a **strong/strict multipointed function**, a **strong/strict morphism of hyperpointed sets**, a **strong/strict morphism of multipointed sets**, or a **strong/strict morphism of $\mathbb{F}_1$-hypermodules**.

Remark 6.2.4 ► Unwinding Definition 6.2.3

In detail, a **strong hyperpointed function from** (X, J_X) **to** (Y, J_Y) is a function $f\colon X \to Y$ such that we have an equality of sets

$$f(x_0) = y_0.$$

6.3 Hyperpointed Relations

6.3.1 Lax Hyperpointed Relations

Let (X, J_X) and (Y, J_Y) be hyperpointed sets.

Definition 6.3.1 ▶ Lax Hyperpointed Relations

A **lax hyperpointed relation**[1] is a lax morphism of pointed objects in $(\mathbf{Rel}, \mathrm{pt})$.

[1] *Further Terminology:* Also called a **lax hypermorphism of hyperpointed sets**, or a **lax hypermorphism of $\mathbb{F}_1$-hypermodules**.

Remark 6.3.2 ▶ Unwinding Definition 6.3.1, I

Viewing relations $A \nrightarrow B$ as functions $A \times B \to \{\text{true}, \text{false}\}$ via Remark 1.1.3, we see that lax hyperpointed relations may be described as follows:

A **lax hyperpointed relation from** (X, J_X) **to** (Y, J_Y) is a pair (f, f^0) consisting of

- *The Underlying Relation.* A relation

$$f \colon X \times Y \to \{\text{true}, \text{false}\}$$

from X to Y, called the **underlying relation of** (f, f^0);

- *The Hyperbasepoint Preservation Constraint.* A natural transformation

$$f^0 \colon J_Y \Longrightarrow f \diamond J_X, \qquad \begin{tikzcd} & \mathrm{pt} \arrow[dl, "J_X"', "\shortmid"{marking}] \arrow[dr, "J_Y", "\shortmid"{marking}] \arrow[d, phantom, "\overset{f^0}{\Leftarrow}"] & \\ X \arrow[rr, "f"', "\shortmid"{marking}] & & Y, \end{tikzcd}$$

called the **lax hyperbasepoint preservation constraint of** (f, f^0), with components

$$[f^0]^a \colon [J_Y]^a \to \int^{x \in X} f_x^- \times [J_X]^x$$

in $\{\text{true}, \text{false}\}$, for $a \in X$.

Remark 6.3.3 ▶ Unwinding Definition 6.3.1, II

Viewing relations $A \nrightarrow B$ as functions $A \to \mathcal{P}(B)$ via Remark 1.1.3, we see that lax hyperpointed relations may also be described as follows:

A **lax hyperpointed relation from** (X, x_0) **to** (Y, y_0) is a pair (f, f^0) consisting of

- *The Underlying Relation.* A relation

$$f\colon X\times Y\to\{\text{true},\text{false}\}$$

from X to Y, called the **underlying relation of** (f,f^0);

- *The Hyperbasepoint Preservation Constraint.* A natural transformation

$$f^0\colon [y_0]\Longrightarrow f\diamond[x_0],\qquad \begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"', "\shortmid" marking] \arrow[dr, "{[y_0]}", "\shortmid" marking] & \\ X \arrow[rr, "f"', "\shortmid" marking] \arrow[ur, phantom, "\Leftarrow f^0"] & & Y, \end{tikzcd}$$

called the **lax hyperbasepoint preservation constraint of** (f,f^0), i.e. an inclusion of sets

$$y_0\subset f(x_0),$$

i.e.:

$$y_0\subset\bigcup_{x\in x_0}f(x).$$

6.3.2 Oplax Hyperpointed Relations

Definition 6.3.4 ▶ Oplax Hyperpointed Relations

An **oplax hyperpointed relation**[1] is an oplax morphism of pointed objects in $(\mathbf{Rel},\mathrm{pt})$.

[1] *Further Terminology:* Also called an **oplax hypermorphism of hyperpointed sets** or an **oplax hypermorphism of $\mathbb{F}_1$-hypermodules**.

Remark 6.3.5 ▶ Unwinding Definition 6.3.4, I

Viewing relations $A\nrightarrow B$ as functions $A\times B\to\{\text{true},\text{false}\}$ via Remark 1.1.3, we see that oplax hyperpointed relations may be described as follows:

An **oplax hyperpointed relation from** (X,J_X) **to** (Y,J_Y) is a pair (f,f^0) consisting of

- *The Underlying Relation.* A relation

$$f\colon X\times Y\to\{\text{true},\text{false}\}$$

from X to Y, called the **underlying relation of** (f, f^0);

- *The Hyperbasepoint Preservation Constraint.* A natural transformation

$$f^0 \colon J_Y \Longrightarrow f \diamond J_X, \qquad \begin{tikzcd} & \mathrm{pt} \arrow[dl, "J_X"', "\times" marking] \arrow[dr, "J_Y", "\times" marking] & \\ X \arrow[rr, "f"', "+" marking] \arrow[rr, phantom, shift left=4, "\overset{f^0}{\Rightarrow}"] & & Y, \end{tikzcd}$$

called the **oplax hyperbasepoint preservation constraint of** (f, f^0), with components

$$[f^0]^a \colon \int^{x \in X} f^-_x \times [J_X]^x \to [J_Y]^a$$

in $\{\mathrm{true}, \mathrm{false}\}$, for $a \in X$.

Remark 6.3.6 ▶ Unwinding Definition 6.3.4, II

Viewing relations $A \nrightarrow B$ as functions $A \to \mathcal{P}(B)$ via Remark 1.1.3, we see that oplax hyperpointed relations may also be described as follows:

An **oplax hyperpointed relation from** (X, x_0) **to** (Y, y_0) is a pair (f, f^0) consisting of

- *The Underlying Relation.* A relation

$$f \colon X \times Y \to \{\mathrm{true}, \mathrm{false}\}$$

from X to Y, called the **underlying relation of** (f, f^0);

- *The Hyperbasepoint Preservation Constraint.* A natural transformation

$$f^0 \colon [y_0] \Longrightarrow f \diamond [x_0], \qquad \begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"', "\times" marking] \arrow[dr, "{[y_0]}", "\times" marking] & \\ X \arrow[rr, "f"', "+" marking] \arrow[rr, phantom, shift left=4, "\overset{f^0}{\Rightarrow}"] & & Y, \end{tikzcd}$$

called the **oplax hyperbasepoint preservation constraint of** (f, f^0), i.e. an inclusion of sets

$$f(x_0) \subset y_0,$$

i.e.:

$$\bigcup_{x \in x_0} f(x) \subset y_0.$$

6.3.3 Strong Hyperpointed Relations

Let (X, x_0) and (Y, y_0) be hyperpointed sets.

Definition 6.3.7 ▶ Strong Hyperpointed Relations

A **strong hyperpointed relation from** (X, x_0) **to** (Y, y_0)[1] is equivalently:

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathrm{Rel}), \mathrm{pt})$;
- A morphism of pointed objects in $(\mathrm{Rel}, \mathrm{pt})$;
- A strong morphism of pointed objects in $(\mathbf{Rel}, \mathrm{pt})$;
- A strict morphism of pointed objects in $(\mathbf{Rel}, \mathrm{pt})$.

[1] *Further Terminology:* Also called simply a **hyperpointed relation**, a **strict hyperpointed relation**, a **strong/strict multipointed relation**, a **strong/strict hypermorphism of hyperpointed sets**, a **strong/strict hypermorphism of multipointed sets**, or a **strong/strict hypermorphism of** $\mathbb{F}_1$**-hypermodules**.

Remark 6.3.8 ▶ Unwinding Definition 6.3.7, I

Viewing relations $A \nrightarrow B$ as functions $A \times B \to \{\mathrm{true}, \mathrm{false}\}$ via Remark 1.1.3, we see that strong hyperpointed relations may also be described as follows:

In detail, a **strong hyperpointed relation from** (X, J_X) **to** (Y, J_Y) is an op/lax hyperpointed relation (f, f^0) whose hyperbasepoint preservation constraint is an isomorphism.

Remark 6.3.9 ▶ Unwinding Definition 6.3.7, II

Viewing relations $A \nrightarrow B$ as functions $A \to \mathcal{P}(X)$ via Remark 1.1.3, we see that strong hyperpointed relations may also be described as follows:

A **strong hyperpointed relation from** (X, J_X) **to** (Y, J_Y) is a relation $f \colon X \nrightarrow Y$ such that we have an equality of relations

$$\int^{x \in X} f_x^{-} \times [J_X]^x = J_Y.$$

Remark 6.3.10 ▶ Unwinding Definition 6.3.7, III

Viewing relations $A \nrightarrow B$ as functions $A \times B \to \{\mathrm{true}, \mathrm{false}\}$ via Remark 1.1.3, we see that strong hyperpointed relations may also be described as follows:

A **strong hyperpointed relation from** (X, x_0) **to** (Y, y_0) is a relation $f \colon X \nrightarrow Y$ such that we have an equality of sets

$$f(x_0) = y_0,$$

i.e.:

$$\bigcup_{x \in x_0} f(x) = y_0.$$

6.4 Categories of Hyperpointed Sets

Definition 6.4.1 ▶ Categories of Hyperpointed Sets

Hyperpointed sets and hyperpointed functions/relations assemble into the following (2-)categories:

- The **category** $\mathsf{Sets}_*^{\mathsf{hyp,lax}}$ **of hyperpointed sets and lax hyperpointed morphisms between them**;
- The **category** $\mathsf{Sets}_*^{\mathsf{hyp,oplax}}$ **of hyperpointed sets and oplax hyperpointed morphisms between them**;
- The **category** $\mathsf{Sets}_*^{\mathsf{hyp}}$ **of hyperpointed sets and strong hyperpointed morphisms between them**;
- The **category** $\mathrm{Rel}_*^{\mathsf{hyp,lax}}$ **of hyperpointed sets and lax hyperpointed relations between them**;
- The **category** $\mathrm{Rel}_*^{\mathsf{hyp,oplax}}$ **of hyperpointed sets and oplax hyperpointed relations between them**;
- The **category** $\mathrm{Rel}_*^{\mathsf{hyp}}$ **of hyperpointed sets and strong hyperpointed relations between them**;
- The 2-**category** $\mathbf{Rel}_*^{\mathsf{hyp,lax}}$ **of hyperpointed sets and lax hyperpointed relations between them**;
- The 2-**category** $\mathbf{Rel}_*^{\mathsf{hyp,oplax}}$ **of hyperpointed sets and oplax hyperpointed relations between them**;

- The 2-**category** $\mathsf{Rel}_*^{\mathsf{hyp}}$ **of hyperpointed sets and strong hyperpointed relations between them**.

Proposition 6.4.2 ▶ Relation to Pointed Sets

The assignment $(X, x_0) \mapsto (X, \{x_0\})$ sending a pointed set to its representable hyperpointed set defines a fully faithful functor

$$\mathsf{Sets}_* \hookrightarrow \mathsf{Sets}_*^{\mathsf{hyp}}.$$

Proof 6.4.3 ▶ Proof of Proposition 6.4.2

Omitted. ∎

6.5 Free Hyperpointed Sets

Let X be a set.

Definition 6.5.1 ▶ Free Hyperpointed Sets

The **free hyperpointed set on** X is the hyperpointed set X^+ consisting of

- *The Underlying Set.* The set X^+ defined by

$$X^+ \overset{\text{def}}{=} X \coprod \mathrm{pt};$$

- *The Basepoint.* The element $\star$ of X^+.

Proposition 6.5.2 ▶ Properties of Free Hyperpointed Sets

Let X be a set.

1. *Functoriality I.* The assignment $X \mapsto X^+$ defines functors

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*^{\mathsf{hyp,lax}},$$
$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*^{\mathsf{hyp,oplax}},$$
$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*^{\mathsf{hyp}},$$

   where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$[(-)^{+}](X) \overset{\text{def}}{=} X_{+},$$

where X_{+} is the hyperpointed set of Definition 6.5.1;

- *Action on Morphisms.* For each morphism $f\colon X \to Y$ of Sets, the image

$$f_{+}\colon X_{+} \to Y_{+}$$

of f by $(-)^{+}$ is the hyperpointed function defined by

$$f^{+}(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star & \text{if } x = \star. \end{cases}$$

2. *Functoriality II.* The assignment $X \mapsto X^{+}$ defines functors

$$\begin{aligned} (-)^{+}\colon \ \mathsf{Rel} &\to \mathsf{Rel}_{*}^{\mathsf{hyp,lax}}, \\ (-)^{+}\colon \ \mathsf{Rel} &\to \mathsf{Rel}_{*}^{\mathsf{hyp,oplax}}, \\ (-)^{+}\colon \ \mathsf{Rel} &\to \mathsf{Rel}_{*}^{\mathsf{hyp}}, \end{aligned}$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$[(-)^{+}](X) \overset{\text{def}}{=} X_{+},$$

where X_{+} is the hyperpointed set of Definition 6.5.1;

- *Action on Morphisms.* For each morphism $f\colon X \nrightarrow Y$ of Rel, the image

$$f_{+}\colon X_{+} \nrightarrow Y_{+}$$

of f by $(-)^{+}$ is the hyperpointed relation defined by

$$f^{+}(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \{\star\} & \text{if } x = \star. \end{cases}$$

3. *Adjointness I.* We have an adjunction[1]

$$\left((-)^{+} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^{+}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{*}^{\mathsf{hyp,lax}} \arrow[l, bend left, "忘"] \end{tikzcd},$$

witnessed by a bijection of sets

$$\mathsf{Sets}_*^{\mathsf{hyp,lax}}((X_+, \{\star\}), (Y, y_0)) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}\left(\mathsf{Sets}_*^{\mathsf{hyp,lax}}\right)$.

4. *Adjointness II.* We have adjunctions

$$\left((-)^+ \dashv 忘\right)\colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}_*^{\mathsf{hyp,lax}} \arrow[l, bend left, "忘"] \end{tikzcd},$$

$$\left((-)^+ \dashv 忘\right)\colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}_*^{\mathsf{hyp,oplax}} \arrow[l, bend left, "忘"] \end{tikzcd},$$

$$\left((-)^+ \dashv 忘\right)\colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}_*^{\mathsf{hyp}} \arrow[l, bend left, "忘"] \end{tikzcd},$$

witnessed by bijections of sets

$$\begin{aligned} \mathsf{Rel}_*^{\mathsf{hyp,lax}}((X_+, \{\star\}), (Y, y_0)) &\cong \mathsf{Rel}(X, Y), \\ \mathsf{Rel}_*^{\mathsf{hyp,lax}}((X_+, \{\star\}), (Y, y_0)) &\cong \mathsf{Rel}(X, Y), \\ \mathsf{Rel}_*^{\mathsf{hyp,lax}}((X_+, \{\star\}), (Y, y_0)) &\cong \mathsf{Rel}(X, Y), \end{aligned}$$

natural in $X \in \mathrm{Obj}(\mathsf{Rel})$ and $(Y, y_0) \in \mathrm{Obj}\left(\mathsf{Rel}_*^{\mathsf{hyp,lax}}\right)$, resp. $(Y, y_0) \in \mathrm{Obj}\left(\mathsf{Rel}_*^{\mathsf{hyp,oplax}}\right)$ and $(Y, y_0) \in \mathrm{Obj}\left(\mathsf{Rel}_*^{\mathsf{hyp}}\right)$.

5. *Symmetric Strong Monoidality With Respect to Wedge Sums I.* The free hyperpointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\right)\colon (\mathsf{Sets}, \textstyle\coprod, \emptyset) \to \left(\mathsf{Sets}_*^{\mathsf{hyp,lax}}, \vee, \mathrm{pt}\right),$$

being equipped with isomorphisms of hyperpointed sets

$$\begin{gathered} (-)^{+,\coprod}_{X,Y}\colon X^+ \vee Y^+ \xrightarrow{\cong} (X \textstyle\coprod Y)^+, \\ (-)^{+,\coprod}_{\mathbb{1}}\colon \mathrm{pt} \xrightarrow{\cong} \emptyset^+, \end{gathered}$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

6. *Symmetric Strong Monoidality With Respect to Wedge Sums II.* The free hyperpointed set functors of Item 2 have symmetric strong monoidal structures

$$\left((-)^{+},(-)^{+,\coprod},(-)^{+,\coprod}_{\mathbb{1}}\right)\colon(\mathsf{Rel},\coprod,\emptyset)\to\left(\mathsf{Rel}^{\mathsf{hyp,lax}}_{*},\vee,\mathrm{pt}\right),$$
$$\left((-)^{+},(-)^{+,\coprod},(-)^{+,\coprod}_{\mathbb{1}}\right)\colon(\mathsf{Rel},\coprod,\emptyset)\to\left(\mathsf{Rel}^{\mathsf{hyp,oplax}}_{*},\vee,\mathrm{pt}\right),$$
$$\left((-)^{+},(-)^{+,\coprod},(-)^{+,\coprod}_{\mathbb{1}}\right)\colon(\mathsf{Rel},\coprod,\emptyset)\to\left(\mathsf{Rel}^{\mathsf{hyp,lax}}_{*},\vee,\mathrm{pt}\right),$$

being equipped with isomorphisms of hyperpointed sets

$$(-)^{+,\coprod}_{X,Y}\colon X^{+}\vee Y^{+}\overset{\cong}{\nrightarrow}(X\coprod Y)^{+},$$
$$(-)^{+,\coprod}_{\mathbb{1}}\colon\mathrm{pt}\overset{\cong}{\nrightarrow}\emptyset^{+},$$

natural in $X,Y\in\mathrm{Obj}(\mathsf{Rel})$.

7. *Symmetric Strong Monoidality With Respect to Smash Products I.* The free hyperpointed set functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)^{+},(-)^{+,\times},(-)^{+,\times}_{\mathbb{1}}\right)\colon(\mathsf{Sets},\times,\mathrm{pt})\to\left(\mathsf{Sets}^{\mathsf{hyp,lax}}_{*},\wedge,S^{0}\right),$$

being equipped with isomorphisms of hyperpointed sets

$$(-)^{+,\times}_{X,Y}\colon X^{+}\wedge Y^{+}\xrightarrow{\cong}(X\times Y)^{+},$$
$$(-)^{+,\times}_{\mathbb{1}}\colon S^{0}\xrightarrow{\cong}\mathrm{pt}^{+},$$

natural in $X,Y\in\mathrm{Obj}(\mathsf{Sets})$.

8. *Symmetric Strong Monoidality With Respect to Smash Products II.* The free hyperpointed set functors of Item 2 have symmetric strong monoidal structures

$$\left((-)^{+},(-)^{+,\times},(-)^{+,\times}_{\mathbb{1}}\right)\colon(\mathsf{Rel},\times,\mathrm{pt})\to\left(\mathsf{Rel}^{\mathsf{hyp,lax}}_{*},\wedge,S^{0}\right),$$
$$\left((-)^{+},(-)^{+,\times},(-)^{+,\times}_{\mathbb{1}}\right)\colon(\mathsf{Rel},\times,\mathrm{pt})\to\left(\mathsf{Rel}^{\mathsf{hyp,oplax}}_{*},\wedge,S^{0}\right),$$
$$\left((-)^{+},(-)^{+,\times},(-)^{+,\times}_{\mathbb{1}}\right)\colon(\mathsf{Rel},\times,\mathrm{pt})\to\left(\mathsf{Rel}^{\mathsf{hyp,lax}}_{*},\wedge,S^{0}\right),$$

being equipped with isomorphisms of hyperpointed sets

$$(-)^{+,\times}_{X,Y}\colon X^{+}\wedge Y^{+}\overset{\cong}{\nrightarrow}(X\times Y)^{+},$$
$$(-)^{+,\times}_{\mathbb{1}}\colon S^{0}\overset{\cong}{\nrightarrow}\mathrm{pt}^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Rel})$.

[1] *Warning:* This does not work if we replace $\mathsf{Sets}_*^{\mathsf{hyp,lax}}$ by $\mathsf{Sets}_*^{\mathsf{hyp,oplax}}$ or $\mathsf{Sets}_*^{\mathsf{hyp}}$.

Proof 6.5.3 ► Proof of Proposition 6.5.2

Item 1: Functoriality I

Clear.

Item 2: Functoriality II

Clear.

Item 3: Adjointness I

Clear.

Item 4: Adjointness II

Clear.

Item 6: Symmetric Strong Monoidality With Respect to Wedge Sums I

Omitted.

Item 6: Symmetric Strong Monoidality With Respect to Wedge Sums II

Omitted.

Item 7: Symmetric Strong Monoidality With Respect to Smash Products I

Omitted.

Item 8: Symmetric Strong Monoidality With Respect to Smash Products II

Omitted.

Appendices

A Other Chapters

Logic and Model Theory

1. Logic
2. Model Theory

Type Theory

3. Type Theory
4. Homotopy Type Theory

Set Theory

5. Sets
6. Constructions With Sets
7. Indexed and Fibred Sets
8. Relations
9. Posets

Category Theory

Topics in Scheme Theory

Fundamental Groups of Schemes

Cohomology of Schemes

Group Schemes

Deformation Theory

Étale Cohomology

Crystalline Cohomology

Algebraic K-Theory

Intersection Theory

Monodromy Groups in Algebraic Geometry

Algebraic Spaces

Deligne–Mumford Stacks

Algebraic Stacks

Moduli Theory

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