Constructions With Categories

Emily

February 27, 2023

Introduction

This chapter contains material about constructions with categories. Notably, it contains:

- A discussion of co/limits, 2-co/limits, some weighted 2-co/limits, pseudo co/limits, lax co/limits, and oplax co/limits of categories, all with very explicit descriptions (Sections 1 to 6);
- A discussion of deloopings of monoids, classifying spaces of categories, opposite categories, categories of pointed objects (i.e. $\mathbb{E}_0$-monoids), joins, arrow categories, the funny tensor product, and the category of simplices of a category (Section 7);
- A discussion of endomorphisms, automorphisms, involutions, idempotent morphisms, and the categories they form (Section 8);
- A discussion of slice categories (Section 9);
- A discussion of coslice categories (Section 10);
- A discussion of quotients of categories (Section 11), where:
    - In Section 11.1 we discuss a notion (I made up) of the quotient of a category by a profunctor (to be thought of as a categorified relation);
    - In Section 11.2 we discuss the usual notion of a quotient of a category by a congruence relation on morphisms;
    - In Section 11.3 we discuss the notion of a quotient of a category by a *generalised* congruence relation, introduced in [BBP99];
    - In Section 11.4 we define generalised congruence relations in a two-step process, first defining the quotient $\mathcal{C}/\simeq$ of a category $\mathcal{C}$ by a congruence relation $\simeq$ on objects, and then defining a generalised congruence relation to be a congruence relation on objects $\simeq$ together with a (classical) congruence relation $\sim$ on $\mathcal{C}/\simeq$.

- A discussion of Gabriel–Zisman localisations (Section 12);
- A discussion of Karoubi envelopes (Section 13);

Notes to Myself

TODO:

1. Classifying space of categories
2. isojoin
3. Adjunction between join and slice

Contents

1 Limits and Colimits of Categories

1.1 Products

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 1.1.1 ▶ Products of Categories

The **product category of** $\mathcal{C}$ **and** $\mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ where

- *Objects.* The objects of $\mathcal{C} \times \mathcal{D}$ are pairs (A, B) with[1]
  - A an object of $\mathcal{C}$;
  - B an object of $\mathcal{D}$;
- *Morphisms.* For each $(A, B), (A', B') \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, we have

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, B), (A', B')) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A, A') \times \mathrm{Hom}_{\mathcal{D}}(B, B');$$

- *Identities.* For each $(A, B) \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, the unit map

$$\mathbb{1}^{\mathcal{C}\times\mathcal{D}}_{(A,B)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, B), (A, B))$$

  of $\mathcal{C} \times \mathcal{D}$ at (A, B) is given by the composition

$$\begin{aligned} \mathrm{pt} &\overset{\sim}{\dashrightarrow} \mathrm{pt} \times \mathrm{pt} \\ &\xrightarrow{\mathbb{1}^{\mathcal{C}}_{A} \times \mathbb{1}^{\mathcal{D}}_{B}} \mathrm{Hom}_{\mathcal{C}}(A, A) \times \mathrm{Hom}_{\mathcal{C}}(B, B) \\ &\stackrel{\text{def}}{=\!=} \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, B), (A, B)) \end{aligned}$$

  in Sets, i.e. we have

$$\mathrm{id}_{(A,B)} \stackrel{\text{def}}{=} (\mathrm{id}_A, \mathrm{id}_B);$$

- *Composition.* For each $\mathbf{X} = (A, B), \mathbf{X}' = (A', B'), \mathbf{X}'' = (A'', B'') \in \mathrm{Obj}(\mathcal{C} \times \mathcal{D})$, the composition morphism

$$\circ^{\mathcal{C}\times\mathcal{D}}_{\mathbf{X},\mathbf{X}',\mathbf{X}''} \colon \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}(\mathbf{X}', \mathbf{X}'') \times \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}(\mathbf{X}, \mathbf{X}') \longrightarrow \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}(\mathbf{X}, \mathbf{X}'')$$

  of $\mathcal{C} \times \mathcal{D}$ at $((A, B), (A', B'), (A'', B''))$ is given by the composition

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A', B'), (A'', B'')) \times \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, B), (A', B')) &\stackrel{\text{def}}{=\!=} (\mathcal{C}(A', A'') \times \mathcal{D}(B', B'')) \times (\mathcal{C}(A, A') \times \mathcal{D}(B, B')) \\ &\overset{\sim}{\dashrightarrow} (\mathcal{C}(A', A'') \times \mathcal{C}(A, A')) \times (\mathcal{D}(B', B'') \times \mathcal{D}(B, B')) \\ &\xrightarrow{\circ^{\mathcal{C}}_{A,A',A''} \times \circ^{\mathcal{D}}_{B,B',B''}} \mathcal{C}(A, A'') \times \mathcal{D}(B, B'') \\ &\stackrel{\text{def}}{=\!=} \mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A, B), (A'', B'')) \end{aligned}$$

in Sets, i.e. for each pair of morphisms

$$\begin{aligned}(f,g)&\colon (A,B) \longrightarrow (A',B'),\\ (h,k)&\colon (A',B') \longrightarrow (A'',B'')\end{aligned}$$

of $\mathcal{C} \times \mathcal{D}$, we have

$$(h,k) \circ (f,g) \stackrel{\text{def}}{=} (h \circ f, k \circ g).$$

[1]That is, we have

$$\mathrm{Obj}(\mathcal{C} \times \mathcal{D}) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D}).$$

1.2 Coproducts

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 1.2.1 ▶ Coproducts of Categories

The **coproduct of $\mathcal{C}$ and $\mathcal{D}$** is the category $\mathcal{C} \coprod \mathcal{D}$ where

- *Objects.* We have

$$\mathrm{Obj}(\mathcal{C} \coprod \mathcal{D}) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathcal{C}) \coprod \mathrm{Obj}(\mathcal{D});$$

- *Morphisms.* For each $A, B \in \mathcal{C} \coprod \mathcal{D}$, with

$$\mathrm{Hom}_{\mathcal{C}\coprod\mathcal{D}}(A,B) \stackrel{\text{def}}{=} \begin{cases} \mathrm{Hom}_{\mathcal{C}}(A,B) & \text{if } A,B \in \mathcal{C},\\ \mathrm{Hom}_{\mathcal{D}}(A,B) & \text{if } A,B \in \mathcal{D},\\ \emptyset & \text{otherwise;}\end{cases}$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathcal{C} \coprod \mathcal{D})$, the unit morphism

$$\mathbb{1}^{\mathcal{C}\coprod\mathcal{D}}_{A} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}\coprod\mathcal{D}}(A,A)$$

of $\mathcal{C} \coprod \mathcal{D}$ at A is defined by

$$\mathbb{1}^{\mathcal{C}\coprod\mathcal{D}}_{A} \stackrel{\text{def}}{=} \begin{cases} \mathbb{1}^{\mathcal{C}}_{A} & \text{if } A \in \mathrm{Obj}(\mathcal{C}),\\ \mathbb{1}^{\mathcal{D}}_{A} & \text{if } A \in \mathrm{Obj}(D);\end{cases}$$

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C} \coprod \mathcal{D})$, the composition morphism

$$\circ^{\mathcal{C}\coprod\mathcal{D}}_{A,B,C} \colon \mathrm{Hom}_{\mathcal{C}\coprod\mathcal{D}}(B,C) \times \mathrm{Hom}_{\mathcal{C}\coprod\mathcal{D}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{C}\coprod\mathcal{D}}(A,C)$$

of $\mathcal{C} \coprod \mathcal{D}$ at (A, B, C) is defined by

$$\circ^{\mathcal{C} \coprod \mathcal{D}}_{A,B,C} \overset{\text{def}}{=} \begin{cases} \circ^{\mathcal{C}}_{A,B,C} & \text{if } A, B, C \in \mathsf{Obj}(\mathcal{C}), \\ \circ^{\mathcal{D}}_{A,B,C} & \text{if } A, B, C \in \mathsf{Obj}(\mathcal{D}), \\ \mathrm{id}_{\emptyset} & \text{if } A, B \in \mathsf{Obj}(\mathcal{C}) \text{ and } C \in \mathsf{Obj}(\mathcal{D}), \\ !_{\mathrm{Hom}_{\mathcal{C}}(A,C)} & \text{if } A, C \in \mathsf{Obj}(\mathcal{C}) \text{ and } B \in \mathsf{Obj}(\mathcal{D}), \\ \mathrm{id}_{\emptyset} & \text{if } B, C \in \mathsf{Obj}(\mathcal{C}) \text{ and } A \in \mathsf{Obj}(\mathcal{D}), \\ \mathrm{id}_{\emptyset} & \text{if } A \in \mathsf{Obj}(\mathcal{C}) \text{ and } B, C \in \mathsf{Obj}(\mathcal{D}), \\ !_{\mathrm{Hom}_{\mathcal{D}}(A,C)} & \text{if } B \in \mathsf{Obj}(\mathcal{C}) \text{ and } A, C \in \mathsf{Obj}(\mathcal{D}), \\ \mathrm{id}_{\emptyset} & \text{if } C \in \mathsf{Obj}(\mathcal{C}) \text{ and } B \in \mathsf{Obj}(\mathcal{D}). \end{cases}$$

1.3 Pullbacks

Let $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$ be functors.

Definition 1.3.1 ▶ Pullbacks of Categories

The **pullback of $\mathcal{C}$ and $\mathcal{D}$ over $\mathcal{E}$ along F and G** is the category $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ where

- *Objects.* The objects of $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ are pairs (A, B) consisting of
  - An object A of $\mathcal{C}$;
  - An object B of $\mathcal{D}$;

  such that $F_A = G_B$ in $\mathcal{E}$;
- *Morphisms.* For each $(A, B), (A', B') \in \mathsf{Obj}(\mathcal{C} \times_{\mathcal{E}} \mathcal{D})$, we have

$$\mathrm{Hom}_{\mathcal{C} \times_{\mathcal{E}} \mathcal{D}}((A, B), (A', B')) \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A, A') \times_{\mathrm{Hom}_{\mathcal{E}}(F_A, F_{A'})} \mathrm{Hom}_{\mathcal{D}}(B, B'),$$

  as in the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C} \times_{\mathcal{E}} \mathcal{D}}((A, B), (A', B')) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}_{\mathcal{D}}(B, B') \arrow[d, "G_{B,B'}"] \\ \mathrm{Hom}_{\mathcal{C}}(A, A') \arrow[r, "F_{A,A'}"'] & \underbrace{\mathrm{Hom}_{\mathcal{E}}(F_A, F_{A'})}_{=\mathrm{Hom}_{\mathcal{E}}(G_B, G_{B'})}; \end{tikzcd}$$

  In other words, a morphism of $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$ from (A, B) to (A', B') is a pair (f, g) consisting of

- A morphism $f\colon A \longrightarrow B$ of C;
- A morphism $g\colon A' \longrightarrow B'$ of $\mathcal{D}$;

such that $F(f) = G(g)$;

- *Identities.* For each $(A, B) \in \mathrm{Obj}(C \times_{\mathcal{E}} \mathcal{D})$, the unit morphism

$$\mathbb{1}^{C\times_{\mathcal{E}}\mathcal{D}}_{(A,B)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{C\times_{\mathcal{E}}\mathcal{D}}((A,B),(A,B))$$

of $C \times_{\mathcal{E}} \mathcal{D}$ at (A, B) is the dashed morphism in the diagram

$$\begin{tikzcd}
\mathrm{pt} \arrow[rrd, bend left, "\mathbb{1}^{\mathcal{E}}_{B}"] \arrow[rd, dashed, "\exists!"] \arrow[rdd, bend right, "\mathbb{1}^{\mathcal{D}}_{A}"'] & & \\
& \mathrm{Hom}_{C\times_{\mathcal{E}}\mathcal{D}}((A,B),(A,B)) \arrow[r, two heads] \arrow[d] \arrow[rd, phantom, "\lrcorner", very near start] & \mathrm{Hom}_{\mathcal{E}}(B,B), \arrow[d, "G_{B,B}"] \\
& \mathrm{Hom}_{\mathcal{D}}(A,A) \arrow[r, "F_{A,A}"'] & \underbrace{\mathrm{Hom}_{C}(F_A,F_A)}_{=\mathrm{Hom}_{C}(G_B,G_B)};
\end{tikzcd}$$

In other words, we have

$$\mathrm{id}^{C\times_{\mathcal{E}}\mathcal{D}}_{(A,B)} \overset{\mathrm{def}}{=} (\mathrm{id}_A, \mathrm{id}_B)$$

for each $(A, B) \in \mathrm{Obj}(C \times_{\mathcal{E}} \mathcal{D})$;

- *Composition.* For each $\mathbf{X} = (A, B), \mathbf{X}' = (A', B'), \mathbf{X}'' = (A'', B'') \in \mathrm{Obj}(C \times_{\mathcal{E}} \mathcal{D})$, the composition morphism

$$\circ^{C\times_{\mathcal{E}}\mathcal{D}}_{\mathbf{X},\mathbf{X}',\mathbf{X}''} \colon \mathrm{Hom}_{C\times_{\mathcal{E}}\mathcal{D}}(\mathbf{X}',\mathbf{X}'') \times \mathrm{Hom}_{C\times_{\mathcal{E}}\mathcal{D}}(\mathbf{X},\mathbf{X}') \longrightarrow \mathrm{Hom}_{C\times_{\mathcal{E}}\mathcal{D}}(\mathbf{X},\mathbf{X}'')$$

of $C \times_{\mathcal{E}} \mathcal{D}$ at $(A, B), (A', B'), (A'', B'')$ is the dashed morphism in the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{C\times_{\mathcal{E}}\mathcal{D}}((A',B'),(A'',B'')) \times \mathrm{Hom}_{C\times_{\mathcal{E}}\mathcal{D}}((A,B),(A',B')) \arrow[rr, two heads] \arrow[dd] \arrow[rd, dashed, "\exists!"] & & \mathrm{Hom}_{\mathcal{E}}(B',B'') \times \mathrm{Hom}_{\mathcal{E}}(B,B') \arrow[d, "\circ^{\mathcal{E}}_{B,B',B''}"] \\
& \mathrm{Hom}_{C\times_{\mathcal{E}}\mathcal{D}}((A,B),(A'',B'')) \arrow[r, two heads] \arrow[d] \arrow[rd, phantom, "\lrcorner", very near start] & \mathrm{Hom}_{\mathcal{E}}(B,B'') \arrow[d, "G_{B,B''}"] \\
\mathrm{Hom}_{\mathcal{D}}(A',A'') \times \mathrm{Hom}_{\mathcal{D}}(A,A') \arrow[r, "\circ^{\mathcal{D}}_{A,A',A''}"'] & \mathrm{Hom}_{\mathcal{D}}(A,A'') \arrow[r, "F_{A,A''}"'] & \underbrace{\mathrm{Hom}_{C}(F_A,F_{A''})}_{=\mathrm{Hom}_{C}(G_B,G_{B''})}.
\end{tikzcd}$$

In other words, we have

$$(f',g')\circ^{\mathcal{C}\times_{\mathcal{E}}\mathcal{D}}_{\mathbf{X},\mathbf{X}',\mathbf{X}''}(f,g)\overset{\text{def}}{=}(f'\circ f,g'\circ g)$$

for each $(f,g)\in\mathrm{Hom}_{\mathcal{C}\times_{\mathcal{E}}\mathcal{D}}(\mathbf{X},\mathbf{X}')$ and each $(f',g')\in\mathrm{Hom}_{\mathcal{C}\times_{\mathcal{E}}\mathcal{D}}(\mathbf{X}',\mathbf{X}'')$.

1.4 Pushouts

Let $\mathcal{C}\xleftarrow{F}\mathcal{E}\xrightarrow{G}\mathcal{D}$ be functors.

Definition 1.4.1 ▶ Pushouts of Categories

The **pushout of $\mathcal{C}$ and $\mathcal{D}$ over $\mathcal{E}$ along F and G** is the category $\mathcal{C}\coprod_{\mathcal{E}}\mathcal{D}$ defined by

$$\mathcal{C}\textstyle\coprod_{\mathcal{E}}\mathcal{D}\overset{\text{def}}{=}(\mathcal{C}\coprod\mathcal{D})/(\simeq_{\mathcal{E}},\sim_{\mathcal{E}}),$$

where $(\simeq_{\mathcal{E}},\sim_{\mathcal{E}})$ is the generalised congruence relation on $\mathcal{C}\coprod\mathcal{D}$ generated by the relations given by declaring $F(C)\simeq G(C)$ and $F(f)\sim G(f)$.

Remark 1.4.2 ▶ Unwinding Definition 1.4.1

In detail, the relation $(\simeq_{\mathcal{E}},\sim_{\mathcal{E}})$ of Definition 1.4.1 is the generalised congruence relation on $\mathcal{C}\coprod\mathcal{D}$ consisting of

- *The Equivalence Relation on Objects.* The relation $\simeq_{\mathcal{E}}$ on $\mathrm{Obj}(\mathcal{C})$ given by declaring $A\simeq_{\mathcal{E}}B$ iff one of the following conditions is satisfied:
  - We have $A,B\in\mathrm{Obj}(\mathcal{C})$ and $A=B$.
  - We have $A,B\in\mathrm{Obj}(\mathcal{D})$ and $A=B$.
  - There exist $X_1,\ldots,X_n\in\mathrm{Obj}(\mathcal{C}\coprod\mathcal{D})$ such that

    $$A\simeq' X_1\simeq'\cdots\simeq' X_n\simeq' B,$$

    where we declare $X\simeq' Y$ if one of the following conditions is satisfied:

    1. There exists $C\in\mathrm{Obj}(\mathcal{E})$ such that $X=F(C)$ and $Y=G(C)$.
    2. There exists $C\in\mathrm{Obj}(\mathcal{E})$ such that $X=G(C)$ and $Y=F(C)$.

    That is: we require the following condition to be satisfied:

    (★) There exist $X_1,\ldots,X_n\in\mathrm{Obj}(\mathcal{C}\coprod\mathcal{D})$ satisfying the following conditions:

1. There exists $C_0 \in \mathrm{Obj}(\mathcal{E})$ satisfying one of the following conditions:
   (a) We have $A = F(C_0)$ and $X_1 = G(C_0)$.
   (b) We have $A = G(C_0)$ and $X_1 = F(C_0)$.
2. For each $1 \le i \le n-1$, there exists $C_i \in \mathrm{Obj}(\mathcal{E})$ satisfying one of the following conditions:
   (a) We have $X_i = F(C_i)$ and $X_{i+1} = G(C_i)$.
   (b) We have $X_i = G(C_i)$ and $X_{i+1} = F(C_i)$.
3. There exists $C_n \in \mathrm{Obj}(\mathcal{E})$ satisfying one of the following conditions:
   (a) We have $X_n = F(C_n)$ and $B = G(C_n)$.
   (b) We have $X_n = G(C_n)$ and $B = F(C_n)$.

- *The Congruence Relation on* $(\mathcal{C} \coprod \mathcal{D})/{\simeq_{\mathcal{E}}}$. The congruence relation $\sim_{\mathcal{E}}$ on $(\mathcal{C} \coprod \mathcal{D})/{\simeq_{\mathcal{E}}}$ defined as follows:

  1. First, for each $[A], [B] \in \mathrm{Obj}(\mathcal{C} \coprod \mathcal{D})/{\simeq_{\mathcal{E}}}$ we define an equivalence relation $\sim'$ on $\mathrm{Hom}_{(\mathcal{C}\coprod\mathcal{D})/\simeq_{\mathcal{E}}}([A], [B])$ by declaring

     $$f_n \mathbin{\square} \cdots \mathbin{\square} f_1 \sim' g_m \mathbin{\square} \cdots \mathbin{\square} g_1$$

     iff one of the following conditions is satisfied:

     - We have $f_n \mathbin{\square} \cdots \mathbin{\square} f_1 = g_m \mathbin{\square} \cdots \mathbin{\square} g_1$;
     - There exist morphisms

       $$\begin{gathered} h_{1,n} \mathbin{\square} \cdots \mathbin{\square} h_{1,1} \\ h_{2,n} \mathbin{\square} \cdots \mathbin{\square} h_{2,1} \\ \vdots \\ h_{k-1,n} \mathbin{\square} \cdots \mathbin{\square} h_{k-1,1} \\ h_{k,n} \mathbin{\square} \cdots \mathbin{\square} h_{k,1} \end{gathered}$$

in $\mathrm{Hom}_{(\mathcal{C}\coprod\mathcal{D})/\simeq_{\mathcal{E}}}([A],[B])$ such that

$$\begin{aligned}
f_n \mathbin{\square} \cdots \mathbin{\square} f_1 &\sim' h_{1,n} \mathbin{\square} \cdots \mathbin{\square} h_{1,1}\\
h_{1,n} \mathbin{\square} \cdots \mathbin{\square} h_{1,1} &\sim' h_{2,n} \mathbin{\square} \cdots \mathbin{\square} h_{2,1}\\
h_{2,n} \mathbin{\square} \cdots \mathbin{\square} h_{2,1} &\sim' h_{3,n} \mathbin{\square} \cdots \mathbin{\square} h_{3,1}\\
&\vdots\\
h_{k-2,n} \mathbin{\square} \cdots \mathbin{\square} h_{k-2,1} &\sim' h_{k-1,n} \mathbin{\square} \cdots \mathbin{\square} h_{k-1,1}\\
h_{k-1,n} \mathbin{\square} \cdots \mathbin{\square} h_{k-1,1} &\sim' h_{k,n} \mathbin{\square} \cdots \mathbin{\square} h_{k,1}\\
h_{k,n} \mathbin{\square} \cdots \mathbin{\square} h_{k,1} &\sim' g_m \mathbin{\square} \cdots \mathbin{\square} g_1,
\end{aligned}$$

where we declare $\phi_n \square \cdots \square \phi_1 \sim' \psi_n \square \cdots \square \psi_1$ if the following condition is satisfied:

(★) For each $1 \leq i \leq n$, there exists $\chi_i \in \mathrm{Mor}(\mathcal{E})$ such that

$$\begin{aligned} \phi_i &= F(\chi_i) \\ \psi_i &= G(\chi_i) \end{aligned} \quad \text{or} \quad \begin{aligned} \phi_i &= G(\chi_i) \\ \psi_i &= F(\chi_i). \end{aligned}$$

That is: we require the following condition to be satisfied:

(★) There exist morphisms

$$\begin{gathered}
h_{1,n} \mathbin{\square} \cdots \mathbin{\square} h_{1,1}\\
h_{2,n} \mathbin{\square} \cdots \mathbin{\square} h_{2,1}\\
\vdots\\
h_{k-1,n} \mathbin{\square} \cdots \mathbin{\square} h_{k-1,1}\\
h_{k,n} \mathbin{\square} \cdots \mathbin{\square} h_{k,1}
\end{gathered}$$

in $\mathrm{Hom}_{(\mathcal{C}\coprod\mathcal{D})/\simeq_{\mathcal{E}}}([A],[B])$ satisfying the following conditions:

(a) There exist $\chi_{0,n}, \ldots, \chi_{0,1} \in \mathrm{Mor}(\mathcal{E})$ such that

$$\begin{aligned} f_i &= F\big(\chi_{0,i}\big) \\ h_{1,i} &= G\big(\chi_{0,i}\big) \end{aligned} \quad \text{or} \quad \begin{aligned} f_i &= G\big(\chi_{0,i}\big) \\ h_{1,i} &= F\big(\chi_{0,i}\big) \end{aligned}$$

for each $1 \leq i \leq n$.

(b) For each $1 \leq i \leq n-1$, there exists $\chi_{i,n}, \ldots, \chi_{i,1} \in \mathrm{Mor}(\mathcal{E})$ such that

$$\begin{aligned} h_{j,i} &= F\big(\chi_{j,i}\big) \\ h_{j+1,i} &= G\big(\chi_{j,i}\big) \end{aligned} \quad \text{or} \quad \begin{aligned} h_{j,i} &= G\big(\chi_{j,i}\big) \\ h_{j+1,i} &= F\big(\chi_{j,i}\big) \end{aligned}$$

for each $1 \leq j \leq k$.

(c) There exist $\chi_{k,n}, \ldots, \chi_{k,1} \in \mathrm{Mor}(\mathcal{E})$ such that

$$\begin{aligned} h_{k,i} &= G(\chi_{k,i}) \\ g_i &= F(\chi_{k,i}) \end{aligned} \quad \text{or} \quad \begin{aligned} h_{k,i} &= F(\chi_{k,i}) \\ g_i &= G(\chi_{k,i}) \end{aligned}$$

for each $1 \leq i \leq n$.

2. We then define $\sim_{\mathcal{E}|[A],[B]}$ as the free congruence on $\sim'$, so that we declare

$$f_n \mathbin{\Box} \cdots \mathbin{\Box} f_1 \sim_{\mathcal{E}|[A],[B]} g_m \mathbin{\Box} \cdots \mathbin{\Box} g_1$$

iff there exist $[A_1], \ldots, [A_n] \in \mathrm{Obj}(\mathcal{C} \coprod \mathcal{D})/\simeq_{\mathcal{E}}$ satisfying the following conditions:

(a) We have $A_1 = A$;

(b) We have $A_n = B$;

(c) For each $1 \leq i \leq k-1$, there exist:

(i) A morphism $f_{i,n_i} \mathbin{\Box} \cdots \mathbin{\Box} f_{i,1}$ in $\mathrm{Hom}_{\mathcal{E}}([A_i], [A_{i+1}])$;

(ii) A morphism $g_{i,m_i} \mathbin{\Box} \cdots \mathbin{\Box} g_{i,1}$ in $\mathrm{Hom}_{\mathcal{E}}([A_i], [A_{i+1}])$;

such that we have

$$f_{i,n_i} \mathbin{\Box} \cdots \mathbin{\Box} f_{i,1} \sim' g_{i,m_i} \mathbin{\Box} \cdots \mathbin{\Box} g_{i,1}.$$

(d) We have

$$f_n \mathbin{\Box} \cdots \mathbin{\Box} f_1 = \mathop{\Large\Box}_{i=1}^{n} f_{i,n_i} \mathbin{\Box} \cdots \mathbin{\Box} f_{i,1}.$$

(e) We have

$$g_m \mathbin{\Box} \cdots \mathbin{\Box} g_1 = \mathop{\Large\Box}_{i=1}^{n} g_{i,n_i} \mathbin{\Box} \cdots \mathbin{\Box} g_{i,1}.$$

1.5 Equalisers

Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 1.5.1 ▶ Equalisers of Categories

The **equaliser of** F **and** G is the category $\mathsf{Eq}(F, G)$ where

- *Objects*. We have

$$\begin{aligned}\mathsf{Obj}(\mathsf{Eq}(F,G)) &\overset{\text{def}}{=} \mathsf{Eq}(F_0, G_0)\\ &\overset{\text{def}}{=} \mathsf{Eq}\left(\mathsf{Obj}(\mathcal{C}) \underset{G_0}{\overset{F_0}{\rightrightarrows}} \mathsf{Obj}(\mathcal{D})\right)\\ &\cong \{A \in \mathsf{Obj}(\mathcal{C}) \mid F_A = G_A\};\end{aligned}$$

- *Morphisms*. For each $A, B \in \mathsf{Obj}(\mathsf{Eq}(F, G))$, we have

$$\begin{aligned}\mathsf{Hom}_{\mathsf{Eq}(F,G)}(A,B) &\overset{\text{def}}{=} \mathsf{Eq}(F_{A,B}, G_{A,B})\\ &\overset{\text{def}}{=} \mathsf{Eq}\left(\mathsf{Hom}_{\mathcal{C}}(A,B) \underset{G_{A,B}}{\overset{F_{A,B}}{\rightrightarrows}} \mathsf{Hom}_{\mathcal{C}}(F_A, F_B)\right)\\ &\cong \left\{f \in \mathsf{Hom}_{\mathsf{Eq}(F,G)}(A,B) \,\middle|\, F_f = G_f\right\};\end{aligned}$$

- *Identities*. For each $A \in \mathsf{Obj}(\mathsf{Eq}(F, G))$, the unit map

$$\mathbb{1}^{\mathsf{Eq}(F,G)}_A : \mathrm{pt} \longrightarrow \mathsf{Hom}_{\mathsf{Eq}(F,G)}(A, A)$$

  of $\mathsf{Eq}(F, G)$ at A is defined by

$$\mathrm{id}^{\mathsf{Eq}(F,G)}_A \overset{\text{def}}{=} \mathrm{id}_A;$$

- *Composition*. For each $A, B, C \in \mathsf{Obj}(\mathsf{Eq}(F, G))$, the unit map

$$\circ^{\mathsf{Eq}(F,G)}_{A,B,C} : \mathsf{Hom}_{\mathsf{Eq}(F,G)}(B,C) \times \mathsf{Hom}_{\mathsf{Eq}(F,G)}(A,B) \longrightarrow \mathsf{Hom}_{\mathsf{Eq}(F,G)}(A,C)$$

  of $\mathsf{Eq}(F, G)$ at (A, B, C) is defined by

$$g \circ^{\mathsf{Eq}(F,G)}_{A,B,C} f \overset{\text{def}}{=} g \circ^{\mathcal{C}}_{A,B,C} f$$

  for each $f \in \mathsf{Hom}_{\mathsf{Eq}(F,G)}(A, B)$ and each $g \in \mathsf{Hom}_{\mathsf{Eq}(F,G)}(B, C)$.

1.6 Coequalisers

Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 1.6.1 ▶ Coequalisers of Categories

The **coequaliser of $\mathcal{C}$ and $\mathcal{D}$ over $\mathcal{E}$** is the category $\mathsf{CoEq}(F, G)$ defined by

$$\mathsf{CoEq}(F, G) \stackrel{\text{def}}{=} \mathcal{C}/\left(\simeq_{F,G}, \sim_{F,G}\right),$$

where $\left(\simeq_{F,G}, \sim_{F,G}\right)$ is the generalised congruence relation on $\mathcal{D}$ generated by the relations given by declaring $F(A) \simeq G(A)$ and $F(f) \sim G(f)$.

Remark 1.6.2 ▶ Unwinding Definition 1.6.1

In detail, the relation $\left(\simeq_{F,G}, \sim_{F,G}\right)$ of Definition 1.6.1 is the generalised congruence relation on $\mathcal{D}$ consisting of

- *The Equivalence Relation on Objects.* The relation $\simeq_{F,G}$ on $\mathrm{Obj}(\mathcal{D})$ given by declaring $A \simeq_{F,G} B$ iff one of the following conditions is satisfied:
  - We have $A = B$.
  - There exist $X_1, \ldots, X_n \in \mathrm{Obj}(\mathcal{D})$ such that

    $$A \simeq' X_1 \simeq' \cdots \simeq' X_n \simeq' B,$$

    where we declare $X \simeq' Y$ if one of the following conditions is satisfied:

    1. There exists $C \in \mathrm{Obj}(\mathcal{C})$ such that $X = F(C)$ and $Y = G(C)$.
    2. There exists $C \in \mathrm{Obj}(\mathcal{C})$ such that $X = G(C)$ and $Y = F(C)$.

    That is: we require the following condition to be satisfied:

    ($\star$) There exist $X_1, \ldots, X_n \in \mathrm{Obj}(\mathcal{D})$ satisfying the following conditions:

      1. There exists $C_0 \in \mathrm{Obj}(\mathcal{C})$ satisfying one of the following conditions:
         (a) We have $A = F(C_0)$ and $X_1 = G(C_0)$.
         (b) We have $A = G(C_0)$ and $X_1 = F(C_0)$.
      2. For each $1 \leq i \leq n-1$, there exists $C_i \in \mathrm{Obj}(\mathcal{C})$ satisfying one of the following conditions:
         (a) We have $X_i = F(C_i)$ and $X_{i+1} = G(C_i)$.
         (b) We have $X_i = G(C_i)$ and $X_{i+1} = F(C_i)$.
      3. There exists $C_n \in \mathrm{Obj}(\mathcal{C})$ satisfying one of the following conditions:

(a) We have $X_n = F(C_n)$ and $B = G(C_n)$.
(b) We have $X_n = G(C_n)$ and $B = F(C_n)$.

- *The Congruence Relation on $\mathcal{D}/{\simeq_{F,G}}$.* The congruence relation $\sim_{F,G}$ on $\mathcal{D}/{\simeq_{F,G}}$ defined as follows:

  1. First, for each $[A], [B] \in \mathrm{Obj}(\mathcal{D})/{\simeq_{F,G}}$ we define an equivalence relation $\sim'$ on $\mathrm{Hom}_{\mathcal{D}/\simeq_{F,G}}([A],[B])$ by declaring

$$f_n \mathbin{\square} \cdots \mathbin{\square} f_1 \sim' g_m \mathbin{\square} \cdots \mathbin{\square} g_1$$

  iff one of the following conditions is satisfied:

  - We have $f_n \mathbin{\square} \cdots \mathbin{\square} f_1 = g_m \mathbin{\square} \cdots \mathbin{\square} g_1$;
  - There exist morphisms

$$\begin{gathered} h_{1,n} \mathbin{\square} \cdots \mathbin{\square} h_{1,1} \\ h_{2,n} \mathbin{\square} \cdots \mathbin{\square} h_{2,1} \\ \vdots \\ h_{k-1,n} \mathbin{\square} \cdots \mathbin{\square} h_{k-1,1} \\ h_{k,n} \mathbin{\square} \cdots \mathbin{\square} h_{k,1} \end{gathered}$$

    in $\mathrm{Hom}_{\mathcal{D}/\simeq_{F,G}}([A],[B])$ such that

$$\begin{aligned} f_n \mathbin{\square} \cdots \mathbin{\square} f_1 &\sim' h_{1,n} \mathbin{\square} \cdots \mathbin{\square} h_{1,1} \\ h_{1,n} \mathbin{\square} \cdots \mathbin{\square} h_{1,1} &\sim' h_{2,n} \mathbin{\square} \cdots \mathbin{\square} h_{2,1} \\ h_{2,n} \mathbin{\square} \cdots \mathbin{\square} h_{2,1} &\sim' h_{3,n} \mathbin{\square} \cdots \mathbin{\square} h_{3,1} \\ &\vdots \\ h_{k-2,n} \mathbin{\square} \cdots \mathbin{\square} h_{k-2,1} &\sim' h_{k-1,n} \mathbin{\square} \cdots \mathbin{\square} h_{k-1,1} \\ h_{k-1,n} \mathbin{\square} \cdots \mathbin{\square} h_{k-1,1} &\sim' h_{k,n} \mathbin{\square} \cdots \mathbin{\square} h_{k,1} \\ h_{k,n} \mathbin{\square} \cdots \mathbin{\square} h_{k,1} &\sim' g_m \mathbin{\square} \cdots \mathbin{\square} g_1, \end{aligned}$$

    where we declare $\phi_n \mathbin{\square} \cdots \mathbin{\square} \phi_1 \sim' \psi_n \mathbin{\square} \cdots \mathbin{\square} \psi_1$ if the following condition is satisfied:

    $(\star)$ For each $1 \leq i \leq n$, there exists $\chi_i \in \mathrm{Mor}(\mathcal{E})$ such that

$$\begin{aligned} \phi_i &= F(\chi_i) \\ \psi_i &= G(\chi_i) \end{aligned} \quad \text{or} \quad \begin{aligned} \phi_i &= G(\chi_i) \\ \psi_i &= F(\chi_i). \end{aligned}$$

    That is: we require the following condition to be satisfied:

(★) There exist morphisms

$$\begin{gathered} h_{1,n} \mathbin{\Box} \cdots \mathbin{\Box} h_{1,1} \\ h_{2,n} \mathbin{\Box} \cdots \mathbin{\Box} h_{2,1} \\ \vdots \\ h_{k-1,n} \mathbin{\Box} \cdots \mathbin{\Box} h_{k-1,1} \\ h_{k,n} \mathbin{\Box} \cdots \mathbin{\Box} h_{k,1} \end{gathered}$$

in $\mathrm{Hom}_{\mathcal{D}/\simeq_{F,G}}([A],[B])$ satisfying the following conditions:

(a) There exist $\chi_{0,n}, \ldots, \chi_{0,1} \in \mathrm{Mor}(\mathcal{E})$ such that

$$\begin{aligned} f_i &= F(\chi_{0,i}) \\ h_{1,i} &= G(\chi_{0,i}) \end{aligned} \quad \text{or} \quad \begin{aligned} f_i &= G(\chi_{0,i}) \\ h_{1,i} &= F(\chi_{0,i}) \end{aligned}$$

for each $1 \leq i \leq n$.

(b) For each $1 \leq j \leq k-1$, there exists $\chi_{j,n}, \ldots, \chi_{j,1} \in \mathrm{Mor}(\mathcal{E})$ such that

$$\begin{aligned} h_{j,i} &= F(\chi_{j,i}) \\ h_{j+1,i} &= G(\chi_{j,i}) \end{aligned} \quad \text{or} \quad \begin{aligned} h_{j,i} &= G(\chi_{j,i}) \\ h_{j+1,i} &= F(\chi_{j,i}) \end{aligned}$$

for each $1 \leq j \leq k$.

(c) There exist $\chi_{k,n}, \ldots, \chi_{k,1} \in \mathrm{Mor}(\mathcal{E})$ such that

$$\begin{aligned} h_{k,i} &= G(\chi_{k,i}) \\ g_i &= F(\chi_{k,i}) \end{aligned} \quad \text{or} \quad \begin{aligned} h_{k,i} &= F(\chi_{k,i}) \\ g_i &= G(\chi_{k,i}) \end{aligned}$$

for each $1 \leq i \leq n$.

2. We then define $\sim_{F,G|[A],[B]}$ as the free congruence on $\sim'$, so that we declare

$$f_n \mathbin{\Box} \cdots \mathbin{\Box} f_1 \sim_{F,G|[A],[B]} g_m \mathbin{\Box} \cdots \mathbin{\Box} g_1$$

iff there exist $[A_1], \ldots, [A_n] \in \mathrm{Obj}(\mathcal{D})/{\simeq_{F,G}}$ satisfying the following conditions:

(a) We have $A_1 = A$;

(b) We have $A_n = B$;

(c) For each $1 \leq i \leq k-1$, there exist:

(i) A morphism $f_{i,n_i} \mathbin{\square} \cdots \mathbin{\square} f_{i,1}$ in $\mathrm{Hom}_{\mathcal{D}/\simeq_{F,G}}([A_i],[A_{i+1}])$;
(ii) A morphism $g_{i,m_i} \mathbin{\square} \cdots \mathbin{\square} g_{i,1}$ in $\mathrm{Hom}_{\mathcal{D}/\simeq_{F,G}}([A_i],[A_{i+1}])$;

such that we have

$$f_{i,n_i} \mathbin{\square} \cdots \mathbin{\square} f_{i,1} \sim' g_{i,m_i} \mathbin{\square} \cdots \mathbin{\square} g_{i,1}.$$

(d) We have

$$f_n \mathbin{\square} \cdots \mathbin{\square} f_1 = \mathop{\Box}_{i=1}^{n} f_{i,n_i} \mathbin{\square} \cdots \mathbin{\square} f_{i,1}.$$

(e) We have

$$g_m \mathbin{\square} \cdots \mathbin{\square} g_1 = \mathop{\Box}_{i=1}^{n} g_{i,n_i} \mathbin{\square} \cdots \mathbin{\square} g_{i,1}.$$

Example 1.6.3 ▶ The Coequaliser of $[0] \rightrightarrows [1]$

The coequaliser of the two inclusions $[0] \rightrightarrows [1]$ is isomorphic to $\mathrm{B}\mathbb{N}$.

1.7 Tensors by Sets

Let $\mathcal{C}$ be a category and let X be a set.

Definition 1.7.1 ▶ Tensors of Categories by Sets

The **tensor of $\mathcal{C}$ by X** is the category $X \odot \mathcal{C}$ given by

$$\begin{aligned} X \odot \mathcal{C} &\cong \coprod_{x \in X} \mathcal{C} \\ &\cong X_{\mathsf{disc}} \times \mathcal{C} \\ &\cong X_{\mathsf{disc}} \mathbin{\square} \mathcal{C}. \end{aligned}$$

1.8 Cotensors by Sets

Let $\mathcal{C}$ be a category and let X be a set.

Definition 1.8.1 ▶ Cotensors of Categories by Sets

The **cotensor of $\mathcal{C}$ by X** is the category $X \pitchfork \mathcal{C}$ given by

$$\begin{aligned} X \pitchfork \mathcal{C} &\cong \prod_{x\in X} \mathcal{C} \\ &\cong \mathsf{Fun}(X_{\mathsf{disc}}, \mathcal{C}) \\ &\cong \mathsf{Fun}^{\mathsf{unnat}}(X_{\mathsf{disc}}, \mathcal{C}). \end{aligned}$$

2 2-Limits and 2-Colimits of Categories

2.1 2-Products

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 2.1.1 ▶ 2-Products

The 2-**product of $\mathcal{C}$ and $\mathcal{D}$** in Cats_2 agrees with their product in Cats, described in Definition 1.1.1.

2.2 2-Coproducts

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 2.2.1 ▶ 2-Coproducts

The 2-**coproduct of $\mathcal{C}$ and $\mathcal{D}$** in Cats_2 agrees with their coproduct in Cats, described in Definition 1.2.1.

2.3 2-Pullbacks

Let $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$ be functors.

Definition 2.3.1 ▶ 2-Pullbacks

The 2-**pullback of $\mathcal{C}$ and $\mathcal{D}$ over $\mathcal{E}$ along F and G** in Cats_2 agrees with their pullback in Cats, described in Definition 1.3.1.

2.4 2-Pushouts

Let $\mathcal{C} \xleftarrow{F} \mathcal{E} \xrightarrow{G} \mathcal{D}$ be functors.

DEFINITION 2.4.1 ▶ 2-PUSHOUTS

The 2**-pushout of** $\mathcal{C}$ **and** $\mathcal{D}$ **over** $\mathcal{E}$ **along** F **and** G in Cats_2 agrees with their pushout in Cats, described in Definition 1.4.1.

2.5 2-Equalisers

Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

DEFINITION 2.5.1 ▶ 2-EQUALISERS

The 2**-equaliser of** $\mathcal{C}$ **and** $\mathcal{D}$ **over** $\mathcal{E}$ in Cats_2 agrees with their equaliser in Cats, described in Definition 1.5.1.

2.6 2-Coequalisers

Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

DEFINITION 2.6.1 ▶ 2-COEQUALISERS

The 2**-coequaliser of** $\mathcal{C}$ **and** $\mathcal{D}$ **over** $\mathcal{E}$ in Cats_2 agrees with their coequaliser in Cats, described in Definition 1.6.1.

2.7 Tensors by Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

DEFINITION 2.7.1 ▶ TENSORS OF CATEGORIES BY CATEGORIES

The **tensor of** $\mathcal{D}$ **by** $\mathcal{C}$ is the category $\mathcal{C} \odot \mathcal{D}$ given by

$$\mathcal{C} \odot \mathcal{D} \cong \mathcal{C} \times \mathcal{D}.$$

2.8 Cotensors by Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 2.8.1 ▶ Cotensors of Categories by Categories

The **cotensor of $\mathcal{D}$ by $\mathcal{C}$** is the category $\mathcal{C} \pitchfork \mathcal{D}$ given by

$$\mathcal{C} \pitchfork \mathcal{D} \cong \mathsf{Fun}(\mathcal{C}, \mathcal{D}).$$

3 Weighted 2-Limits and 2-Colimits of Categories

3.1 Equifiers

Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors and let $\alpha, \beta \colon F \Rrightarrow G$ be natural transformations.

Definition 3.1.1 ▶ Equifiers

The **equifier of α and β** is the category $\mathsf{Eqf}(\alpha, \beta)$ where

- *Objects.* We have

$$\mathsf{Obj}(\mathsf{Eqf}(\alpha, \beta)) \overset{\text{def}}{=} \{A \in \mathsf{Obj}(\mathcal{C}) \mid \alpha_A = \beta_A\};$$

- *Morphisms.* For each $A, B \in \mathsf{Obj}(\mathsf{Eqf}(\alpha, \beta))$, we have

$$\mathsf{Hom}_{\mathsf{Eqf}(\alpha,\beta)}(A, B) \overset{\text{def}}{=} \mathsf{Hom}_{\mathcal{C}}(A, B);$$

- *Identities.* For each $A \in \mathsf{Obj}(\mathsf{Eqf}(\alpha, \beta))$, the unit map

$$\mathbb{1}^{\mathsf{Eqf}(\alpha,\beta)}_A \colon \mathrm{pt} \longrightarrow \mathsf{Hom}_{\mathsf{Eqf}(\alpha,\beta)}(A, A)$$

  of $\mathsf{Eqf}(\alpha, \beta)$ at A is defined by

$$\mathrm{id}^{\mathsf{Eqf}(\alpha,\beta)}_A \overset{\text{def}}{=} \mathrm{id}_A;$$

- *Composition.* For each $A, B, C \in \mathsf{Obj}(\mathsf{Eqf}(\alpha, \beta))$, the composition map

$$\circ^{\mathsf{Eqf}(\alpha,\beta)}_{A,B,C} \colon \mathsf{Hom}_{\mathsf{Eqf}(\alpha,\beta)}(B, C) \times \mathsf{Hom}_{\mathsf{Eqf}(\alpha,\beta)}(A, B) \longrightarrow \mathsf{Hom}_{\mathsf{Eqf}(\alpha,\beta)}(A, C)$$

  of $\mathsf{Eqf}(\alpha, \beta)$ at (A, B, C) is defined by

$$g \circ^{\mathsf{Eqf}(\alpha,\beta)}_A f \overset{\text{def}}{=} g \circ f$$

  for each $f \in \mathsf{Hom}_{\mathsf{Eqf}(\alpha,\beta)}(A, B)$ and each $g \in \mathsf{Hom}_{\mathsf{Eqf}(\alpha,\beta)}(B, C)$.

3.2 Coequifiers

Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors and let $\alpha, \beta \colon F \Rrightarrow G$ be natural transformations.

Definition 3.2.1 ▶ Coequifiers

The **coequifier of** α **and** β is the category $\mathsf{CoEqf}(\alpha, \beta)$ defined by

$$\mathsf{CoEqf}(\alpha, \beta) \overset{\text{def}}{=} \mathcal{D}/\sim_{\alpha,\beta},$$

where $\sim_{\alpha,\beta}$ is the congruence on $\mathcal{D}$ whose component $\sim_{\alpha,\beta|X,Y}$ at $X, Y \in \mathrm{Obj}(\mathcal{D})$ is defined as follows:

- If $X = F_A$ and $Y = G_A$ for some $A \in \mathrm{Obj}(\mathcal{C})$, then $\sim_{\alpha,\beta|X,Y}$ is the equivalence relation generated by $\alpha_A \sim \beta_A$;
- Otherwise, $\sim_{\alpha,\beta|X,Y}$ is the trivial equivalence relation.

Remark 3.2.2 ▶ Unwinding Definition 3.2.1

In detail, the component $\sim_{\alpha,\beta|X,Y}$ at $X, Y \in \mathrm{Obj}(\mathcal{D})$ of the relation $\sim_{\alpha,\beta}$ of Definition 3.2.1 is defined as follows:

- If $X = F_A$ and $Y = G_A$ for some $A \in \mathrm{Obj}(\mathcal{C})$, then we declare $f \sim_{\alpha,\beta|X,Y} g$ iff one of the following conditions is satisfied:
  - We have $f = g$;
  - There exist $\phi_1, \ldots, \phi_n \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ such that

    $$f \sim' \phi_1 \sim' \cdots \sim' \phi_n \sim' g,$$

    where we declare $\phi \sim' \psi$ if one of the following conditions is satisfied:

    1. There exists $A \in \mathrm{Obj}(\mathcal{C})$ such that $\phi = \alpha_A$ and $\psi = \beta_A$.
    2. There exists $A \in \mathrm{Obj}(\mathcal{C})$ such that $\phi = \beta_A$ and $\psi = \alpha_A$.

    That is: we require the following condition to be satisfied:

    ($\star$) There exist $\phi_1, \ldots, \phi_n \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ satisfying the following conditions:

    1. There exists $A_0 \in \mathrm{Obj}(\mathcal{C})$ satisfying one of the following conditions:
       (a) We have $f = \alpha_{A_0}$ and $\phi_1 = \beta_{A_0}$.

(b) We have $f = \beta_{A_0}$ and $\phi_1 = \alpha_{A_0}$.

2. For each $1 \leq i \leq n-1$, there exists $A_i \in \mathrm{Obj}(\mathcal{C})$ satisfying one of the following conditions:
   (a) We have $\phi_i = \alpha_{A_i}$ and $\phi_{i+1} = \beta_{A_i}$.
   (b) We have $\phi_i = \beta_{A_i}$ and $\phi_{i+1} = \alpha_{A_i}$.
3. There exists $A_n \in \mathrm{Obj}(\mathcal{C})$ satisfying one of the following conditions:
   (a) We have $\phi_n = \alpha_{A_n}$ and $g = \beta_{A_n}$.
   (b) We have $\phi_n = \beta_{A_n}$ and $g = \alpha_{A_n}$.

- Otherwise, $\sim_{\alpha,\beta|X,Y}$ is the trivial equivalence relation.

3.3 Identifiers

Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor and let $\alpha\colon F \Longrightarrow F$ be a natural transformation.

Definition 3.3.1 ▶ Identifiers

The **identifier of** α is the category $\mathrm{Idf}(\alpha)$ defined as the equifier of α and id_F:

$$\mathrm{Idf}(\alpha) \overset{\text{def}}{=} \mathrm{Eqf}(\alpha, \mathrm{id}_F).$$

Remark 3.3.2 ▶ Unwinding Definition 3.3.1

In detail the **identifier of** α is the category $\mathrm{Idf}(\alpha)$ where

- *Objects.* We have

$$\mathrm{Obj}(\mathrm{Idf}(\alpha)) \overset{\text{def}}{=} \left\{ A \in \mathrm{Obj}(\mathcal{C}) \,\middle|\, \alpha_A = \mathrm{id}_{F_A} \right\};$$

- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathrm{Idf}(\alpha))$, we have

$$\mathrm{Hom}_{\mathrm{Idf}(\alpha)}(A, B) \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A, B);$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathrm{Idf}(\alpha))$, the unit map

$$\mathbb{1}^{\mathrm{Idf}(\alpha)}_A \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathrm{Idf}(\alpha)}(A, A)$$

of $\mathrm{Idf}(\alpha)$ at A is defined by

$$\mathrm{id}^{\mathrm{Idf}(\alpha)}_A \overset{\text{def}}{=} \mathrm{id}_A;$$

- *Composition*. For each $A, B, C \in \mathrm{Obj}(\mathrm{Idf}(\alpha))$, the composition map

$$\circ^{\mathrm{Idf}(\alpha)}_{A,B,C} \colon \mathrm{Hom}_{\mathrm{Idf}(\alpha)}(B, C) \times \mathrm{Hom}_{\mathrm{Idf}(\alpha)}(A, B) \longrightarrow \mathrm{Hom}_{\mathrm{Idf}(\alpha)}(A, C)$$

  of $\mathrm{Idf}(\alpha)$ at (A, B, C) is defined by

$$g \circ^{\mathrm{Idf}(\alpha)}_{A,B,C} f \stackrel{\mathrm{def}}{=} g \circ f$$

  for each $f \in \mathrm{Hom}_{\mathrm{Idf}(\alpha)}(A, B)$ and each $g \in \mathrm{Hom}_{\mathrm{Idf}(\alpha)}(B, C)$.

3.4 Coidentifiers

Let $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor and let $\alpha \colon F \Longrightarrow F$ be a natural transformation.

Definition 3.4.1 ▶ Coidentifiers

The **coidentifier of** α is the category $\mathrm{CoIdf}(\alpha)$ defined as the coequifier of α and id_F:

$$\mathrm{CoIdf}(\alpha) \stackrel{\mathrm{def}}{=} \mathrm{CoEqf}(\alpha, \mathrm{id}_F).$$

Remark 3.4.2 ▶ Unwinding Definition 3.4.1

In detail, the **coidentifier of** α is the category $\mathrm{CoIdf}(\alpha)$ defined by

$$\mathrm{CoIdf}(\alpha) \stackrel{\mathrm{def}}{=} \mathcal{D}/\sim_\alpha,$$

where $\sim_\alpha$ is the congruence on $\mathcal{D}$ whose component $\sim_{\alpha|X,Y}$ at $X, Y \in \mathrm{Obj}(\mathcal{D})$ is defined as follows:

- If $X = Y = F_A$ for some $A \in \mathrm{Obj}(\mathcal{C})$, then $\sim_{\alpha|X,Y}$ is the equivalence relation generated by $\alpha_A \sim \mathrm{id}_{F_A}$.

  That is: we declare $f \sim_{\alpha|X,Y} g$ iff one of the following conditions is satisfied:

  - We have $f = g$;
  - There exist $\phi_1, \ldots, \phi_n \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ such that

  $$f \sim' \phi_1 \sim' \cdots \sim' \phi_n \sim' g,$$

    where we declare $\phi \sim' \psi$ if one of the following conditions is satisfied:

    1. There exists $A \in \mathrm{Obj}(\mathcal{C})$ such that $\phi = \alpha_A$ and $\psi = \mathrm{id}_{F_A}$.

2. There exists $A \in \mathsf{Obj}(\mathcal{C})$ such that $\phi = \mathrm{id}_{F_A}$ and $\psi = \alpha_A$.

That is: we require the following condition to be satisfied:

($\star$) There exist $\phi_1, \ldots, \phi_n \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ satisfying the following conditions:

1. There exists $A_0 \in \mathsf{Obj}(\mathcal{C})$ satisfying one of the following conditions:
   (a) We have $f = \alpha_{A_0}$ and $\phi_1 = \mathrm{id}_{F_{A_0}}$.
   (b) We have $f = \mathrm{id}_{F_{A_0}}$ and $\phi_1 = \alpha_{A_0}$.
2. For each $1 \leq i \leq n-1$, there exists $A_i \in \mathsf{Obj}(\mathcal{C})$ satisfying one of the following conditions:
   (a) We have $\phi_i = \alpha_{A_i}$ and $\phi_{i+1} = \mathrm{id}_{F_{A_i}}$.
   (b) We have $\phi_i = \mathrm{id}_{F_{A_i}}$ and $\phi_{i+1} = \alpha_{A_i}$.
3. There exists $A_n \in \mathsf{Obj}(\mathcal{C})$ satisfying one of the following conditions:
   (a) We have $\phi_n = \alpha_{A_n}$ and $g = \mathrm{id}_{F_{A_n}}$.
   (b) We have $\phi_n = \mathrm{id}_{F_{A_n}}$ and $g = \alpha_{A_n}$.

- Otherwise, $\sim_{\alpha|X,Y}$ is the trivial equivalence relation.

3.5 Inverters

Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors and let $\alpha \colon F \Longrightarrow G$ be a natural transformation.

Definition 3.5.1 ▶ Inverters

The **inverter of** α is the category $\mathsf{Inv}(\alpha)$ where

- *Objects.* We have
$$\mathsf{Obj}(\mathsf{Inv}(\alpha)) \overset{\text{def}}{=} \{A \in \mathsf{Obj}(\mathcal{C}) \mid \alpha_A \text{ is an isomorphism}\};$$
- *Morphisms.* For each $A, B \in \mathsf{Obj}(\mathsf{Inv}(\alpha))$, we have
$$\mathrm{Hom}_{\mathsf{Inv}(\alpha)}(A, B) \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A, B);$$
- *Identities.* For each $A \in \mathsf{Obj}(\mathsf{Inv}(\alpha))$, the unit map
$$\mathbb{1}^{\mathsf{Inv}(\alpha)}_A \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{Inv}(\alpha)}(A, A)$$

of $\mathsf{Inv}(\alpha)$ at A is defined by

$$\mathrm{id}_A^{\mathsf{Inv}(\alpha)} \overset{\mathrm{def}}{=} \mathrm{id}_A;$$

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Inv}(\alpha))$, the composition map

$$\circ_{A,B,C}^{\mathsf{Inv}(\alpha)} \colon \mathrm{Hom}_{\mathsf{Inv}(\alpha)}(B, C) \times \mathrm{Hom}_{\mathsf{Inv}(\alpha)}(A, B) \longrightarrow \mathrm{Hom}_{\mathsf{Inv}(\alpha)}(A, C)$$

  of $\mathsf{Inv}(\alpha)$ at (A, B, C) is defined by

$$g \circ_{A,B,C}^{\mathsf{Inv}(\alpha)} f \overset{\mathrm{def}}{=} g \circ f$$

  for each $f \in \mathrm{Hom}_{\mathsf{Inv}(\alpha)}(A, B)$ and each $g \in \mathrm{Hom}_{\mathsf{Inv}(\alpha)}(B, C)$.

3.6 Coinverters

Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors and let $\alpha \colon F \Longrightarrow G$ be a natural transformation.

Definition 3.6.1 ▶ Coinverters

The **coinverter of** α is the category $\mathsf{CoInv}(\alpha)$ constructed as follows:

1. First we take the coinserter $\mathsf{CoIns}(F, G)$ of F and G, which comes with a functor $\mathrm{coins}(F, G) \colon \mathcal{D} \longrightarrow \mathsf{CoIns}(F, G)$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, "\mathrm{coins}(F,G)"] & \mathsf{CoIns}(F, G) \end{tikzcd}$$

   and a natural transformation $\beta \colon \mathrm{coins}(F, G) \circ F \Longrightarrow \mathrm{coins}(F, G) \circ G$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{coins}(F,G)\circ F"] \arrow[r, bend right, "\mathrm{coins}(F,G)\circ G"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathsf{CoIns}(F, G); \end{tikzcd}$$

2. Then we take the coequifier of the natural transformations

$$\begin{aligned} \left(\mathrm{id}_{\mathrm{coins}(F,G)} \star \alpha\right) \circ \beta \colon \mathrm{coins}(F, G) \circ G &\Longrightarrow \mathrm{coins}(F, G) \circ G, \\ \mathrm{id}_{\mathrm{coins}(F,G)\circ G} \colon \mathrm{coins}(F, G) \circ G &\Longrightarrow \mathrm{coins}(F, G) \circ G. \end{aligned}$$

3. Finally, we take the coequifier of the natural transformations

$$\beta \circ (\mathrm{id}_{\mathrm{coeqf}} \star \mathrm{id}_{\mathrm{coins}(F,G)} \star \alpha) \colon \mathrm{coeqf} \circ \mathrm{coins}(F, G) \circ F \Longrightarrow \mathrm{coeqf} \circ \mathrm{coins}(F, G) \circ F,$$

$$\mathrm{id}_{\mathsf{coeqf}\circ\mathsf{coins}(F,G)\circ F}\colon \mathsf{coeqf}\circ\mathsf{coins}(F,G)\circ F \Longrightarrow \mathsf{coeqf}\circ\mathsf{coins}(F,G)\circ F.$$

Example 3.6.2 ▶ Localisations

Let

- $\mathcal{C}$ be a category;
- W be a subset of $\mathrm{Mor}(\mathcal{C})$;
- $\mathcal{W}$ be the full subcategory of $\mathrm{Arr}(\mathcal{C})$ spanned by those morphisms in W;
- $\mathrm{src}\colon \mathcal{W} \Longrightarrow \mathcal{C}$ be the source functor from $\mathcal{W}$ to $\mathcal{C}$;
- $\mathrm{tgt}\colon \mathcal{W} \Longrightarrow \mathcal{C}$ be the target functor from $\mathcal{W}$ to $\mathcal{C}$;
- $\alpha\colon \mathrm{src} \Longrightarrow \mathrm{tgt}$ be the natural transformation consisting of the collection

  $$\left\{\alpha_f\colon \mathrm{src}(f) \longrightarrow \mathrm{tgt}(f)\right\}_{f\in\mathrm{Obj}(\mathcal{W})}$$

  defined by

  $$\alpha_f \stackrel{\mathrm{def}}{=} f$$

  for each $f \in \mathrm{Obj}(\mathrm{Arr}(\mathcal{C}))$.

We have an equivalence of categories

$$\mathcal{C}\left[W^{-1}\right] \stackrel{\text{eq.}}{\cong} \mathrm{CoInv}(\alpha).$$

3.7 Inserters

Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 3.7.1 ▶ Inserters

The **inserter of F and G** is the category $\mathrm{Ins}(F, G)$ where

- *Objects.* An object of $\mathrm{Ins}(F, G)$ is a pair (A, ϕ) consisting of
  - An object A of $\mathcal{C}$;
  - A morphism $\phi\colon F_A \longrightarrow G_A$ of $\mathcal{D}$;
- *Morphisms.* A morphism of $\mathrm{Ins}(F, G)$ from (A, ϕ) to (B, ψ) is a morphism

$f\colon A \longrightarrow B$ such that the diagram

$$\begin{tikzcd} F_A \arrow[r, "F_f"] \arrow[d, "\phi"'] & F_B \arrow[d, "\psi"] \\ G_A \arrow[r, "G_f"'] & G_B \end{tikzcd}$$

commutes;

- *Identities.* For each $(A, \phi) \in \mathrm{Obj}(\mathsf{Ins}(F, G))$, the unit map

$$\mathbb{1}^{\mathsf{Ins}(F,G)}_{(A,\phi)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{Ins}(F,G)}((A, \phi), (A, \phi))$$

of $\mathsf{Ins}(F, G)$ at (A, ϕ) is defined by

$$\mathrm{id}^{\mathsf{Ins}(F,G)}_{(A,\phi)} \overset{\text{def}}{=} \mathrm{id}_A;$$

- *Composition.* For each $\mathbf{A} = (A, \phi), \mathbf{B} = (B, \psi), \mathbf{C} = (C, \chi) \in \mathrm{Obj}(\mathsf{Ins}(F, G))$, the composition map

$$\circ^{\mathsf{Ins}(F,G)}_{\mathbf{A},\mathbf{B},\mathbf{C}} \colon \mathrm{Hom}_{\mathsf{Ins}(F,G)}(\mathbf{B}, \mathbf{C}) \times \mathrm{Hom}_{\mathsf{Ins}(F,G)}(\mathbf{A}, \mathbf{B}) \longrightarrow \mathrm{Hom}_{\mathsf{Ins}(F,G)}(\mathbf{A}, \mathbf{C})$$

of $\mathsf{Ins}(F, G)$ at $((A, \phi), (B, \psi), (C, \chi))$ is defined by

$$g \circ^{\mathsf{Ins}(F,G)}_{\mathbf{A},\mathbf{B},\mathbf{C}} f \overset{\text{def}}{=} g \circ f$$

for each $(g, f) \in \mathrm{Hom}_{\mathsf{Ins}(F,G)}(\mathbf{B}, \mathbf{C}) \times \mathrm{Hom}_{\mathsf{Ins}(F,G)}(\mathbf{A}, \mathbf{B})$.

3.8 Coinserters

Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 3.8.1 ▶ Coinserters

The **coinserter of** F **and** G is the category $\mathsf{CoIns}(F, G)$ defined by

$$\mathsf{CoIns}(F, G) \overset{\text{def}}{=} \mathcal{D}'/{\sim_{F,G}},$$

where

- $\mathcal{D}'$ is the free category on the underlying directed graph of $\mathcal{D}$ adjoined with morphisms of the form $\alpha_A \colon F_A \longrightarrow G_A$ for each $A \in \mathrm{Obj}(\mathcal{C})$;

- $\sim_{F,G}$ is the congruence on $\mathcal{D}'$ generated by the relation declaring $\phi \sim_{F,G} \psi$ if one of the following conditions is satisfied:
  1. We have $\phi = [\mathrm{id}_A]$ and $\psi = \mathrm{id}_A$.
  2. We have $\phi = [g \circ f]$ and $\psi = g \circ f$.
  3. We have $\phi = \alpha_B \circ \left[F_f\right]$ and $\psi = \left[G_f\right] \circ \alpha_A$.[1]

[1] *Picture:*

$$\begin{tikzcd}
 & F_{C_1} \arrow[r, "\alpha_{C_1}"] \arrow[d, "F_{h_1}"] & G_{C_1} \arrow[d, "G_{h_1}"] \arrow[ddr, "g_1"] & \\
 & F_{C_2} \arrow[r, "\alpha_{C_2}"] \arrow[d, "F_{h_2}"] & G_{C_2} \arrow[d, "G_{h_2}"] \arrow[dr, "g_2"] & \\
A \arrow[uur, "f_1"] \arrow[ur, "f_2"] \arrow[dr, "f_{n-1}"] \arrow[ddr, "f_n"'] & \vdots \arrow[d, "F_{h_{n-1}}"] & \vdots \arrow[d, "G_{h_{n-1}}"] & B. \\
 & F_{C_{n-1}} \arrow[r, "\alpha_{C_{n-1}}"] \arrow[d, "F_{h_n}"] & G_{C_{n-1}} \arrow[d, "G_{h_n}"] \arrow[ur, "g_{n-1}"] & \\
 & F_{C_n} \arrow[r, "\alpha_{C_n}"'] & G_{C_n} \arrow[uur, "g_n"'] &
\end{tikzcd}$$

3.9 Isoinserters

Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 3.9.1 ▶ Isoinserters

The **isoinserter of** F **and** G is the category $\mathrm{IsoIns}(F, G)$ where

- *Objects.* An object of $\mathrm{IsoIns}(F, G)$ is a pair (A, ϕ) consisting of
  - An object A of $\mathcal{C}$;
  - An isomorphism $\phi \colon F_A \xrightarrow{\cong} G_A$ of $\mathcal{D}$;
- *Morphisms.* A morphism of $\mathrm{IsoIns}(F, G)$ from (A, ϕ) to (B, ψ) is a mor-

phism $f\colon A \longrightarrow B$ such that the diagram

$$\begin{tikzcd} F_A \arrow[r, "F_f"] \arrow[d, dashed, "\phi"', "\wr"] & F_B \arrow[d, dashed, "\wr"', "\psi"] \\ G_A \arrow[r, "G_f"'] & G_B \end{tikzcd}$$

commutes;

- *Identities.* For each $(A, \phi) \in \mathrm{Obj}(\mathsf{IsoIns}(F, G))$, the unit map

$$\mathbb{1}^{\mathsf{IsoIns}(F,G)}_{(A,\phi)}\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{IsoIns}(F,G)}((A, \phi), (A, \phi))$$

of $\mathsf{IsoIns}(F, G)$ at (A, ϕ) is defined by

$$\mathrm{id}^{\mathsf{IsoIns}(F,G)}_{(A,\phi)} \overset{\mathrm{def}}{=} \mathrm{id}_A;$$

- *Composition.* For each $\mathbf{A} = (A, \phi), \mathbf{B} = (B, \psi), \mathbf{C} = (C, \chi) \in \mathrm{Obj}(\mathsf{IsoIns}(F, G))$, the composition map

$$\circ^{\mathsf{IsoIns}(F,G)}_{\mathbf{A},\mathbf{B},\mathbf{C}}\colon \mathrm{Hom}_{\mathsf{IsoIns}(F,G)}(\mathbf{B}, \mathbf{C}) \times \mathrm{Hom}_{\mathsf{IsoIns}(F,G)}(\mathbf{A}, \mathbf{B}) \longrightarrow \mathrm{Hom}_{\mathsf{IsoIns}(F,G)}(\mathbf{A}, \mathbf{C})$$

of $\mathsf{IsoIns}(F, G)$ at $((A, \phi), (B, \psi), (C, \chi))$ is defined by

$$g \circ^{\mathsf{IsoIns}(F,G)}_{\mathbf{A},\mathbf{B},\mathbf{C}} f \overset{\mathrm{def}}{=} g \circ f$$

for each $f \in \mathrm{Hom}_{\mathsf{IsoIns}(F,G)}(\mathbf{A}, \mathbf{B})$ and each $g \in \mathrm{Hom}_{\mathsf{IsoIns}(F,G)}(\mathbf{B}, \mathbf{C})$.

3.10 Coisoinserters

Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 3.10.1 ▶ Coisoinserters

The **coisoinserter of** F **and** G is the category $\mathsf{IsoCoIns}(F, G)$ defined by

$$\mathsf{IsoCoIns}(F, G) \overset{\mathrm{def}}{=} \mathcal{D}'/{\sim_{F,G}},$$

where

- $\mathcal{D}'$ is the free category on the underlying directed graph of $\mathcal{D}$ adjoined with morphisms of the form

$$\alpha_A \colon F_A \longrightarrow G_A,$$
$$\alpha_A^{-1} \colon G_A \longrightarrow F_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$;

- $\sim_{F,G}$ is the congruence on $\mathcal{D}'$ generated by the relation declaring $\phi \sim_{F,G} \psi$ if one of the following conditions is satisfied:
  1. We have $\phi = [\mathrm{id}_A]$ and $\psi = \mathrm{id}_A$.
  2. We have $\phi = [g \circ f]$ and $\psi = g \circ f$.
  3. We have $\phi = \alpha_B \circ [F_f]$ and $\psi = [G_f] \circ \alpha_A$.
  4. We have $\phi = \alpha_A^{-1} \circ \alpha_A$ and $\psi = \mathrm{id}_{F_A}$.
  5. We have $\phi = \alpha_A \circ \alpha_A^{-1}$ and $\psi = \mathrm{id}_{G_A}$.

3.11 Comma Categories

Let $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$ be functors.

Definition 3.11.1 ▶ Comma Categories

The **comma category** of F and G is the category[1] $F \downarrow G$ where

1. *Objects.* The objects of $F \downarrow G$ are triples (A, B, ϕ) consisting of
   - An A object of $\mathcal{C}$;
   - An B object of $\mathcal{D}$;
   - A morphism $\phi \colon F_A \longrightarrow G_B$ of $\mathcal{E}$;
2. *Morphisms.* A morphism of $F \downarrow G$ from (A, B, ϕ) to (A', B', ϕ') is a pair (f, g) consisting of
   - A morphism $f \colon A \longrightarrow A'$ of $\mathcal{C}$;
   - A morphism $g \colon B \longrightarrow B'$ of $\mathcal{D}$;

such that the diagram

$$
\begin{tikzcd}
F_A \arrow[r, "F_f"] \arrow[d, "\phi"'] & F_{A'} \arrow[d, "\phi'"] \\
G_B \arrow[r, "G_g"'] & G_{B'}
\end{tikzcd}
$$

commutes.

3. *Identities.* For each $(A, B, \phi) \in \mathrm{Obj}(F \downarrow G)$, the unit map

$$\mathbb{1}^{F\downarrow G}_{(A,B,\phi)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{F\downarrow G}((A, B, \phi), (A, B, \phi))$$

of $F \downarrow G$ at (A, B, ϕ) is defined by

$$\mathrm{id}_{(A,B,\phi)} \stackrel{\text{def}}{=} (\mathrm{id}_A, \mathrm{id}_B);$$

4. *Composition.* For each $\mathbf{X} = (A, B, \phi), \mathbf{X}' = (A', B', \phi'), \mathbf{X}'' = (A'', B'', \phi'') \in \mathrm{Obj}(F \downarrow G)$, the composition map

$$\circ^{F\downarrow G}_{\mathbf{X},\mathbf{X}',\mathbf{X}''} \colon \mathrm{Hom}_{F\downarrow G}(\mathbf{X}', \mathbf{X}'') \times \mathrm{Hom}_{F\downarrow G}(\mathbf{X}, \mathbf{X}') \longrightarrow \mathrm{Hom}_{F\downarrow G}(\mathbf{X}, \mathbf{X}'')$$

of $F \downarrow G$ at $((A, B, \phi), (A', B', \phi'), (A'', B'', \phi''))$ is defined by

$$(f', g') \circ (f, g) \stackrel{\text{def}}{=} (f' \circ f, g' \circ g).$$

[1] *Further Notation:* Also written F/G.

PROPOSITION 3.11.2 ▶ PROPERTIES OF COMMA CATEGORIES

Let $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$ be functors.

1. *Functoriality.* The assignment $\left(\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}\right) \mapsto F \downarrow G$ defines a functor

$$- \downarrow - \colon \mathrm{CoSpan}(\mathsf{Cats}) \longrightarrow \mathsf{Cats}.$$

2. *Duality.* We have an isomorphism of categories

$$(F \downarrow G)^{\mathsf{op}} \cong G^{\mathsf{op}} \downarrow F^{\mathsf{op}},\qquad \begin{tikzcd} (F \downarrow G)^{\mathsf{op}} \arrow[r] \arrow[d] & \mathcal{C}^{\mathsf{op}} \arrow[d, "F^{\mathsf{op}}"] \\ \mathcal{D}^{\mathsf{op}} \arrow[r, "G^{\mathsf{op}}"'] \arrow[ur, Rightarrow, phantom, ""] & \mathcal{E}^{\mathsf{op}}. \end{tikzcd}$$

3. *As a Pullback.* We have an isomorphism of categories

$$F \downarrow G \cong (\mathcal{C} \times \mathcal{D}) \underset{\mathcal{E}\times\mathcal{E}}{\times} \mathsf{Arr}(\mathcal{E}),\qquad \begin{tikzcd} F \downarrow G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Arr}(\mathcal{E}) \arrow[d, "\mathsf{ev}_0 \times \mathsf{ev}_1"] \\ \mathcal{C} \times \mathcal{D} \arrow[r, "F\times G"'] & \mathcal{E} \times \mathcal{E}. \end{tikzcd}$$

4. *As a Weighted 2-Limit.* Let

   - $W \colon \vee \longrightarrow \mathsf{Cats}$ be the functor picking out the cospan $\mathrm{pt} \longrightarrow I \longleftarrow \mathrm{pt}$;
   - $D \colon \vee \longrightarrow \mathsf{Cats}$ be the functor picking out the cospan $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$;

   We have an isomorphism of categories

$$\begin{aligned} F \downarrow G &\cong \mathsf{2lim}^{[W]}(D) \\ &\overset{\text{def}}{=} \mathsf{2lim}^{[\mathrm{pt}\to I \leftarrow \mathrm{pt}]}\left(\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}\right), \end{aligned}\qquad \begin{tikzcd} F \downarrow G \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "G"] \\ \mathcal{C} \arrow[r, "F"'] \arrow[ur, Rightarrow, phantom, ""] & \mathcal{E}. \end{tikzcd}$$

5. *Relation to Co/Slices*[1]. We have isomorphisms of categories

$$\begin{tikzcd} \mathcal{C}_{X/} \arrow[r, "忘"] \arrow[d] & \mathcal{C} \arrow[d, "\mathrm{id}_{\mathcal{C}}"] \\ \mathrm{pt} \arrow[r, "{[X]}"'] & \mathcal{C}, \end{tikzcd}\qquad \begin{aligned} \mathcal{C}_{X/} &\cong [X] \downarrow \mathrm{id}_{\mathcal{C}}, \\ \mathcal{C}_{/X} &\cong \mathrm{id}_{\mathcal{C}} \downarrow [X], \end{aligned}\qquad \begin{tikzcd} \mathcal{C}_{/X} \arrow[r, two heads] \arrow[d, "忘"'] & \mathrm{pt} \arrow[d, "{[X]}"] \\ \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}}"'] & \mathcal{C}. \end{tikzcd}$$

6. *Relation to Natural Transformations.* Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xleftarrow{G} \mathcal{C}$ be functors.

   (a) Let $\alpha \colon F \Longrightarrow G$ be a natural transformation. We have an induced functor $T_\alpha \colon \mathcal{C} \longrightarrow F \downarrow G$ where

   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have
   $$T_\alpha(A) \overset{\text{def}}{=} (A, A, \alpha_A);$$
   - *Action on Morphisms.* For each morphism $f \colon A \longrightarrow B$ of $\mathcal{C}$, we have
   $$T_\alpha(f) \overset{\text{def}}{=} (f, f);$$

   (b) Conversely, given a functor $T \colon \mathcal{C} \longrightarrow F \downarrow G$ such that the diagram[2]

$$\begin{tikzcd}
\mathcal{C} \arrow[drr, bend left, equal] \arrow[ddr, bend right, equal] \arrow[dr, "T"] & & \\
& F \downarrow G \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & \mathcal{C} \arrow[d, "G"] \\
& \mathcal{C} \arrow[r, "F"'] \arrow[ur, phantom, "\Nearrow"] & \mathcal{D}
\end{tikzcd}$$

   commutes, we have an associated natural transformation $\alpha_T \colon F \Longrightarrow G$.

[1] This is a repetition of **??** of **??**.

[2] That is, such that

$$\begin{aligned}
\mathrm{pr}_1 \circ T &= \mathrm{id}_{\mathcal{C}}, \\
\mathrm{pr}_2 \circ T &= \mathrm{id}_{\mathcal{C}}.
\end{aligned}$$

Proof 3.11.3 ▶ Proof of Proposition 3.11.2

Item 1: Functoriality

Omitted.

Item 2: Duality

Omitted.

Item 3: As a Pullback

Omitted.

Item 4: As a Weighted 2-Limit

Omitted.

Item 5: Relation to Co/Slices

This was proved in its repetition, **??** of **??**.

Item 6: Relation to Natural Transformations

Omitted.

3.12 Cocomma Categories

Let $\mathcal{C} \xleftarrow{F} \mathcal{E} \xrightarrow{G} \mathcal{D}$ be functors.

Definition 3.12.1 ▶ Cocomma Categories

The **cocomma category** of F and G is the category $F \uparrow G$ defined by

$$F \uparrow G \overset{\text{def}}{=} (\mathcal{C} \coprod \mathcal{D})' / {\sim_{F,G}},$$

where

- $(\mathcal{C} \coprod \mathcal{D})'$ is the free category on the underlying directed graph of $\mathcal{C} \coprod \mathcal{D}$ adjoined with morphisms of the form $\alpha_A \colon F_A \longrightarrow G_A$ for each $A \in \text{Obj}(\mathcal{C})$;
- $\sim_{F,G}$ is the congruence on $(\mathcal{C} \coprod \mathcal{D})'$ generated by the relation declaring $\phi \sim_{F,G} \psi$ if one of the following conditions is satisfied:
  1. We have $\phi = [\mathrm{id}_A]$ and $\psi = \mathrm{id}_A$.
  2. We have $\phi = [g \circ f]$ and $\psi = g \circ f$.
  3. We have $\phi = \alpha_B \circ [F_f]$ and $\psi = [G_f] \circ \alpha_A$.[1]

[1] *Picture:*

$$
\begin{tikzcd}
 & F_{C_1} \arrow[r, "\alpha_{C_1}"] \arrow[d, dashed, "F_{h_1}"] & G_{C_1} \arrow[d, dashed, "G_{h_1}"] \arrow[ddr, "g_1"] & \\
 & F_{C_2} \arrow[r, dashed, "\alpha_{C_2}"] \arrow[d, dashed, "F_{h_2}"] & G_{C_2} \arrow[d, dashed, "G_{h_2}"] \arrow[dr, dashed, "g_2"] & \\
A \arrow[uur, "f_1"] \arrow[ur, dashed, "f_2"] \arrow[dr, dashed, "f_{n-1}"] \arrow[ddr, "f_n"'] & \vdots \arrow[d, dashed, "F_{h_{n-1}}"] & \vdots \arrow[d, dashed, "G_{h_{n-1}}"] & B. \\
 & F_{C_{n-1}} \arrow[r, dashed, "\alpha_{C_{n-1}}"] \arrow[d, dashed, "F_{h_n}"] & G_{C_{n-1}} \arrow[d, dashed, "G_{h_n}"] \arrow[ur, dashed, "g_{n-1}"] & \\
 & F_{C_n} \arrow[r, "\alpha_{C_n}"'] & G_{C_n} \arrow[uur, "g_n"'] &
\end{tikzcd}
$$

3.13 Isocomma Categories

Let $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$ be functors.

Definition 3.13.1 ▶ Isocomma Categories

The **isocomma category** of F and G is the category $F \mathbin{\rotatebox[origin=c]{-90}{$\dashrightarrow$}} G$ where

1. *Objects.* The objects of $F \mathbin{\rotatebox[origin=c]{-90}{$\dashrightarrow$}} G$ are triples (A, B, ϕ) consisting of
   - An object A of $\mathcal{C}$;
   - An object B of $\mathcal{D}$;
   - An isomorphism $\phi \colon F_A \xrightarrow{\cong} G_B$ of $\mathcal{E}$;
2. *Morphisms.* A morphism of $F \mathbin{\rotatebox[origin=c]{-90}{$\dashrightarrow$}} G$ from (A, B, ϕ) to (A', B', ϕ') is a pair (f, g) consisting of
   - A morphism $f \colon A \longrightarrow A'$ of $\mathcal{C}$;
   - A morphism $g \colon B \longrightarrow B'$ of $\mathcal{D}$;

such that the diagram

$$\begin{tikzcd}
F_A \arrow[r, "F_f"] \arrow[d, dashed, "\phi", "\wr"'] & F_{A'} \arrow[d, dashed, "\phi'"', "\wr"] \\
G_B \arrow[r, "G_g"'] & G_{B'}
\end{tikzcd}$$

commutes.

3. *Identities.* For each $(A, B, \phi) \in \mathrm{Obj}\left(F \downarrow G\right)$, the unit map

$$\mathbb{1}^{F \downarrow G}_{(A,B,\phi)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{F \downarrow G}((A, B, \phi), (A, B, \phi))$$

of $F \downarrow G$ at (A, B, ϕ) is defined by

$$\mathrm{id}_{(A,B,\phi)} \stackrel{\text{def}}{=} (\mathrm{id}_A, \mathrm{id}_B);$$

4. *Composition.* For each $\mathbf{X} = (A, B, \phi)$, $\mathbf{X}' = (A', B', \phi')$, $\mathbf{X}'' = (A'', B'', \phi'') \in \mathrm{Obj}\left(F \downarrow G\right)$, the composition map

$$\circ^{F \downarrow G}_{\mathbf{X},\mathbf{X}',\mathbf{X}''} \colon \mathrm{Hom}_{F \downarrow G}(\mathbf{X}', \mathbf{X}'') \times \mathrm{Hom}_{F \downarrow G}(\mathbf{X}, \mathbf{X}') \longrightarrow \mathrm{Hom}_{F \downarrow G}(\mathbf{X}, \mathbf{X}'')$$

of $F \downarrow G$ at $((A, B, \phi), (A', B', \phi'), (A'', B'', \phi''))$ is defined by

$$(f', g') \circ (f, g) \stackrel{\text{def}}{=} (f' \circ f, g' \circ g).$$

Proposition 3.13.2 ▶ Properties of Comma Categories

Let $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$ be functors.

1. *Functoriality.* The assignment $\left(\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}\right) \mapsto F \downarrow G$ defines a functor

$$- \downarrow - \colon \mathsf{CoSpan}(\mathsf{Cats}) \longrightarrow \mathsf{Cats}.$$

2. *Duality.* We have an isomorphism of categories

$$\left(F \mathbin{\dashdownarrow} G\right)^{\mathsf{op}} \cong G^{\mathsf{op}} \mathbin{\dashdownarrow} F^{\mathsf{op}}, \qquad \begin{tikzcd} \left(F \mathbin{\dashdownarrow} G\right)^{\mathsf{op}} \arrow[r] \arrow[d] & \mathcal{C}^{\mathsf{op}} \arrow[d, "F^{\mathsf{op}}"] \\ \mathcal{D}^{\mathsf{op}} \arrow[r, "G^{\mathsf{op}}"'] \arrow[ur, phantom, "\Nearrow"] & \mathcal{E}^{\mathsf{op}}. \end{tikzcd}$$

3. *As a Pullback.* We have an isomorphism of categories

$$F \mathbin{\dashdownarrow} G \cong (\mathcal{C} \times \mathcal{D}) \underset{\mathcal{E}\times\mathcal{E}}{\times} \mathsf{Iso}(\mathcal{E}), \qquad \begin{tikzcd} F \mathbin{\dashdownarrow} G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Iso}(\mathcal{E}) \arrow[d, "\mathsf{ev}_0 \times \mathsf{ev}_1"] \\ \mathcal{C} \times \mathcal{D} \arrow[r, "F\times G"'] & \mathcal{E} \times \mathcal{E}. \end{tikzcd}$$

4. *Relation to Natural Isomorphisms.* Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xleftarrow{G} \mathcal{C}$ be functors.

   (a) Let $\alpha \colon F \overset{\cong}{\Longrightarrow} G$ be a natural isomorphism. We have an induced functor $T_\alpha \colon \mathcal{C} \longrightarrow F \mathbin{\dashdownarrow} G$ where

   - *Action on Objects.* For each $A \in \mathsf{Obj}(\mathcal{C})$, we have

     $$T_\alpha(A) \overset{\mathrm{def}}{=} (A, A, \alpha_A);$$

   - *Action on Morphisms.* For each morphism $f \colon A \longrightarrow B$ of $\mathcal{C}$, we have

     $$T_\alpha(f) \overset{\mathrm{def}}{=} (f, f);$$

   (b) Conversely, given a functor $T \colon \mathcal{C} \longrightarrow F \mathbin{\dashdownarrow} G$ such that the diagram[1]

$$\begin{tikzcd} \mathcal{C} \arrow[dr, "T"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\ & F \mathbin{\dashdownarrow} G \arrow[r, "\mathsf{pr}_2"] \arrow[d, "\mathsf{pr}_1"] & \mathcal{C} \arrow[d, "G"] \\ & \mathcal{C} \arrow[r, "F"'] \arrow[ur, phantom, "\Nearrow"] & \mathcal{D} \end{tikzcd}$$

commutes, we have an associated natural isomorphism $\alpha_T \colon F \overset{\cong}{\Longrightarrow} G$.

[1]That is, such that

$$\begin{aligned} \mathrm{pr}_1 \circ T &= \mathrm{id}_{\mathcal{C}}, \\ \mathrm{pr}_2 \circ T &= \mathrm{id}_{\mathcal{C}}. \end{aligned}$$

Proof 3.13.3 ▶ Proof of Proposition 3.13.2

Item 1: Functoriality

Omitted.

Item 2: Duality

Omitted.

Item 3: As a Pullback

Omitted.

Item 4: Relation to Natural Isomorphisms

Omitted.

3.14 Isococomma Categories

Let $\mathcal{C} \xleftarrow{F} \mathcal{E} \xrightarrow{G} \mathcal{D}$ be functors.

Definition 3.14.1 ▶ Isococomma Categories

The **isococomma category** of F and G is the category $F \uparrow G$ defined by

$$F \uparrow G \overset{\text{def}}{=} (\mathcal{C} \coprod \mathcal{D})'/{\sim_{F,G}},$$

where

- $(\mathcal{C} \coprod \mathcal{D})'$ is the free category on the underlying directed graph of $\mathcal{C} \coprod \mathcal{D}$ adjoined with isomorphisms of the form $\alpha_A \colon F_A \xrightarrow{\cong} G_A$ for each $A \in \mathrm{Obj}(\mathcal{C})$;
- $\sim_{F,G}$ is the congruence on $(\mathcal{C} \coprod \mathcal{D})'$ generated by the relation declaring $\phi \sim_{F,G} \psi$ if one of the following conditions is satisfied:
  1. We have $\phi = [\mathrm{id}_A]$ and $\psi = \mathrm{id}_A$.
  2. We have $\phi = [g \circ f]$ and $\psi = g \circ f$.

3. We have $\phi = \alpha_B \circ [F_f]$ and $\psi = [G_f] \circ \alpha_A$.[1]

[1] *Picture:*

$$
\begin{tikzcd}
& F_{C_1} \arrow[r, dashed, "\alpha_{C_1}"] \arrow[d, "F_{h_1}"] & G_{C_1} \arrow[d, "G_{h_1}"] \arrow[ddr, "g_1"] & \\
& F_{C_2} \arrow[r, dashed, "\alpha_{C_2}"] \arrow[d, "F_{h_2}"] & G_{C_2} \arrow[d, "G_{h_2}"] \arrow[dr, "g_2"] & \\
A \arrow[uur, "f_1"] \arrow[ur, "f_2"] \arrow[dr, "f_{n-1}"] \arrow[ddr, "f_n"'] & \vdots \arrow[d, "F_{h_{n-1}}"] & \vdots \arrow[d, "G_{h_{n-1}}"] & B. \\
& F_{C_{n-1}} \arrow[r, dashed, "\alpha_{C_{n-1}}"] \arrow[d, "F_{h_n}"] & G_{C_{n-1}} \arrow[d, "G_{h_n}"] \arrow[ur, "g_{n-1}"] & \\
& F_{C_n} \arrow[r, dashed, "\alpha_{C_n}"'] & G_{C_n} \arrow[uur, "g_n"'] &
\end{tikzcd}
$$

4 Pseudolimits and Pseudocolimits of Categories

4.1 Pseudoproducts

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

DEFINITION 4.1.1 ▶ PSEUDOPRODUCTS

The **pseudoproduct of $\mathcal{C}$ and $\mathcal{D}$** in Cats_2 agrees with their product in Cats, described in Definition 1.1.1.

4.2 Pseudocoproducts

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

DEFINITION 4.2.1 ▶ PSEUDOCOPRODUCTS

The **pseudocoproduct of $\mathcal{C}$ and $\mathcal{D}$** in Cats_2 agrees with their coproduct in Cats, described in Definition 1.2.1.

4.3 Pseudopullbacks

Let $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$ be functors.

Definition 4.3.1 ▶ Pseudopullbacks of Categories

The **pseudopullback of $\mathcal{C}$ and $\mathcal{D}$ over $\mathcal{E}$ along F and G** is the category $\mathcal{C} \times^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D}$ where

- *Objects.* The objects of $\mathcal{C} \times^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D}$ are quintuples (A, B, C, ϕ, ψ) consisting of
  - An object A of $\mathcal{C}$;
  - An object B of $\mathcal{D}$;
  - An object C of $\mathcal{E}$;
  - An isomorphism $\phi\colon F_A \xrightarrow{\cong} C$ of $\mathcal{E}$;
  - An isomorphism $\psi\colon G_B \xrightarrow{\cong} C$ of $\mathcal{E}$;
- *Morphisms.* A morphism of $\mathcal{C} \times^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D}$ from (A, B, C, ϕ, ψ) to $(A', B', C', \phi', \psi')$ is a pair (f, g, h) consisting of
  - A morphism $f\colon A \longrightarrow A'$ of $\mathcal{C}$;
  - A morphism $g\colon B \longrightarrow B'$ of $\mathcal{D}$;
  - A morphism $h\colon C \longrightarrow C'$ of $\mathcal{E}$;

  making the diagrams

$$
\begin{tikzcd}
F_A \arrow[r, "F_f"] \arrow[d, dashed, "\phi"', "\wr"] & F_{A'} \arrow[d, dashed, "\phi'", "\wr"'] \\
C \arrow[r, "h"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
G_B \arrow[r, "G_g"] \arrow[d, dashed, "\psi"', "\wr"] & G_{B'} \arrow[d, dashed, "\psi'", "\wr"'] \\
C \arrow[r, "h"'] & C'
\end{tikzcd}
$$

  commute;
- *Identities.* For each $(A, B, C, \phi, \psi) \in \mathrm{Obj}\Big(\mathcal{C} \times^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D}\Big)$, the unit morphism

$$\mathbb{1}^{\mathcal{C}\times^{\mathsf{ps}}_{\mathcal{E}}\mathcal{D}}_{(A,B,C,\phi,\psi)}\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{ps}}_{\mathcal{E}}\mathcal{D}}((A, B, C, \phi, \psi), (A, B, C, \phi, \psi))$$

  of $\mathcal{C} \times^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D}$ at (A, B, C, ϕ, ψ) is defined by

$$\mathrm{id}_{(A,B,C,\phi,\psi)} \overset{\mathrm{def}}{=} (\mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_C);$$

- *Composition.* For each triple of objects

$$\begin{aligned}\mathbf{X} &= (A, B, C, \phi, \psi),\\ \mathbf{X}' &= (A', B', C', \phi', \psi'),\\ \mathbf{X}'' &= (A'', B'', C'', \phi'', \psi'')\end{aligned}$$

  of $\mathcal{C} \times^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D}$, the composition morphism

$$\circ^{\mathcal{C}\times^{\mathsf{ps}}_{\mathcal{E}}\mathcal{D}}_{\mathbf{X},\mathbf{X}',\mathbf{X}''} \colon \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{ps}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}', \mathbf{X}'') \times \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{ps}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}, \mathbf{X}') \longrightarrow \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{ps}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}, \mathbf{X}'')$$

  of $\mathcal{C} \times^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D}$ at $\mathbf{X}, \mathbf{X}', \mathbf{X}''$ is defined by

$$(f', g', h') \circ^{\mathcal{C}\times^{\mathsf{ps}}_{\mathcal{E}}\mathcal{D}}_{\mathbf{X},\mathbf{X}',\mathbf{X}''} (f, g, h) \overset{\text{def}}{=} (f' \circ f, g' \circ g, h' \circ h)$$

  for each $(f, g, h) \in \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{ps}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}, \mathbf{X}')$ and each $(f', g', h') \in \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{ps}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}', \mathbf{X}'')$.

4.4 Pseudopushouts

Let $\mathcal{C} \overset{F}{\longleftarrow} \mathcal{E} \overset{G}{\longrightarrow} \mathcal{D}$ be functors.

DEFINITION 4.4.1 ▶ PSEUDOPUSHOUTS

The **pseudopushout of** $\mathcal{C}$ **and** $\mathcal{D}$ **over** $\mathcal{E}$ **along** F **and** G is the category $\mathcal{C} \coprod^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D}$ defined by

$$\mathcal{C} \textstyle\coprod^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D} \overset{\text{def}}{=} (\mathcal{C} \coprod \mathcal{D})'/{\sim_{F,G}},$$

where

- $(\mathcal{C} \coprod \mathcal{D})'$ is the free category on the underlying directed graph of $\mathcal{C} \coprod \mathcal{D}$ adjoined with morphisms of the form

$$\begin{aligned}\alpha_C &\colon F_C \longrightarrow G_C,\\ \alpha^{-1}_C &\colon G_C \longrightarrow F_C\end{aligned}$$

  for each $C \in \mathrm{Obj}(\mathcal{E})$;
- $\sim_{F,G}$ is the congruence on $(\mathcal{C} \coprod \mathcal{D})'$ generated by the relation declaring $\phi \sim_{F,G} \psi$ if one of the following conditions is satisfied:

  1. We have $\phi = [\mathrm{id}_A]$ and $\psi = \mathrm{id}_A$.

2. We have $\phi = [g \circ f]$ and $\psi = g \circ f$.
3. We have $\phi = \alpha_B \circ [F_f]$ and $\psi = [G_f] \circ \alpha_A$.
4. We have $\phi = \alpha_C^{-1} \circ \alpha_C$ and $\psi = \mathrm{id}_{F_C}$.
5. We have $\phi = \alpha_C \circ \alpha_C^{-1}$ and $\psi = \mathrm{id}_{G_C}$.

4.5 Pseudoequalisers

Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 4.5.1 ▶ Pseudoequalisers of Categories

The **pseudoequaliser of** F **and** G is the category $\mathsf{PsEq}(F, G)$ where

- *Objects.* An object of $\mathsf{PsEq}(F, G)$ is a quadruple (A, B, ϕ, ψ) consisting of
  - An object A of $\mathcal{C}$;
  - An object B of $\mathcal{D}$;
  - An isomorphism $\phi \colon F_A \xrightarrow{\cong} B$ of $\mathcal{D}$;
  - An isomorphism $\psi \colon G_A \xrightarrow{\cong} B$ of $\mathcal{D}$;
- *Morphisms.* A morphism of $\mathsf{PsEq}(F, G)$ from (A, B, ϕ, ψ) to (A', B', ϕ', ψ') is a pair (f, g) consisting of
  - A morphism $f \colon A \longrightarrow A'$ of $\mathcal{C}$;
  - A morphism $g \colon B \longrightarrow B'$ of $\mathcal{D}$;

  making the diagrams

$$\begin{tikzcd} F_A \arrow[r, "F_f"] \arrow[d, dashed, "\phi"', "\wr"] & F_{A'} \arrow[d, dashed, "\phi'", "\wr"'] \\ B \arrow[r, "g"'] & B' \end{tikzcd} \qquad \begin{tikzcd} G_A \arrow[r, "G_f"] \arrow[d, dashed, "\psi"', "\wr"] & G_{A'} \arrow[d, dashed, "\psi'", "\wr"'] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

  commute;
- *Identities.* For each $(A, B, \phi, \psi) \in \mathrm{Obj}(\mathsf{PsEq}(F, G))$, the unit map

$$\mathbb{1}^{\mathsf{PsEq}(F,G)}_{(A,B,\phi,\psi)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{PsEq}(F,G)}((A, B, \phi, \psi), (A, B, \phi, \psi))$$

of $\mathsf{PsEq}(F, G)$ at (A, B, ϕ, ψ) is defined by

$$\mathbb{1}^{\mathsf{PsEq}(F,G)}_{(A,B,\phi,\psi)} \overset{\text{def}}{=} (\mathrm{id}_A, \mathrm{id}_B);$$

- *Composition*. For each $\mathbf{A} = (A, B, \phi, \psi)$, $\mathbf{A}' = (A', B', \phi', \psi')$, $\mathbf{A}'' = (A'', B'', \phi'', \psi'') \in \mathrm{Obj}(\mathsf{PsEq}(F, G))$, the unit map

$$\circ^{\mathsf{PsEq}(F,G)}_{\mathbf{A},\mathbf{A}',\mathbf{A}''} \colon \mathrm{Hom}_{\mathsf{PsEq}(F,G)}(\mathbf{A}', \mathbf{A}'') \times \mathrm{Hom}_{\mathsf{PsEq}(F,G)}(\mathbf{A}, \mathbf{A}') \longrightarrow \mathrm{Hom}_{\mathsf{PsEq}(F,G)}(\mathbf{A}, \mathbf{A}'')$$

of $\mathsf{PsEq}(F, G)$ at $(\mathbf{A}, \mathbf{A}', \mathbf{A}'')$ is defined by

$$\circ^{\mathsf{PsEq}(F,G)}_{\mathbf{A},\mathbf{A}',\mathbf{A}''} \overset{\text{def}}{=} \left(\circ^{\mathcal{C}}_{A,A',A''}, \circ^{\mathcal{C}}_{B,B',B''}\right).$$

4.6 Pseudocoequalisers

Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 4.6.1 ▶ Pseudocoequalisers

The **pseudocoequaliser of** F **and** G in Cats_2 agrees with their coisoinserter in Cats, described in Definition 3.10.1.

5 Lax Limits and Lax Colimits of Categories

5.1 Lax Products

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 5.1.1 ▶ Lax Products

The **lax product of** $\mathcal{C}$ **and** $\mathcal{D}$ in Cats_2 agrees with their product in Cats, described in Definition 1.1.1.

5.2 Lax Coproducts

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 5.2.1 ▶ Lax Coproducts

The **lax coproduct of** $\mathcal{C}$ **and** $\mathcal{D}$ in Cats_2 agrees with their coproduct in Cats, described in Definition 1.2.1.

5.3 Lax Pullbacks

Let $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$ be functors.

Definition 5.3.1 ▶ Lax Pullbacks of Categories

The **lax pullback of** $\mathcal{C}$ **and** $\mathcal{D}$ **over** $\mathcal{E}$ **along** F **and** G is the category $\mathcal{C} \times^{\mathsf{lax}}_{\mathcal{E}} \mathcal{D}$ where

- *Objects*. The objects of $\mathcal{C} \times^{\mathsf{lax}}_{\mathcal{E}} \mathcal{D}$ are quintuples (A, B, C, ϕ, ψ) consisting of
  - An object A of $\mathcal{C}$;
  - An object B of $\mathcal{D}$;
  - An object C of $\mathcal{E}$;
  - A morphism $\phi \colon F_A \longrightarrow C$ of $\mathcal{E}$;
  - A morphism $\psi \colon G_B \longrightarrow C$ of $\mathcal{E}$;
- *Morphisms*. A morphism of $\mathcal{C} \times^{\mathsf{lax}}_{\mathcal{E}} \mathcal{D}$ from (A, B, C, ϕ, ψ) to $(A', B', C', \phi', \psi')$ is a pair (f, g, h) consisting of
  - A $f \colon A \longrightarrow A'$ morphism of $\mathcal{C}$;
  - A $g \colon B \longrightarrow B'$ morphism of $\mathcal{D}$;
  - A $h \colon C \longrightarrow C'$ morphism of $\mathcal{E}$;

  making the diagrams

$$
\begin{tikzcd}
F_A \arrow[r, "F_f"] \arrow[d, "\phi"'] & F_{A'} \arrow[d, "\phi'"] \\
C \arrow[r, "h"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
G_B \arrow[r, "G_g"] \arrow[d, "\psi"'] & G_{B'} \arrow[d, "\psi'"] \\
C \arrow[r, "h"'] & C'
\end{tikzcd}
$$

  commute;

- *Identities.* For each $(A, B, C, \phi, \psi) \in \mathrm{Obj}\Big(\mathcal{C} \times^{\mathsf{lax}}_{\mathcal{E}} \mathcal{D}\Big)$, the unit morphism

$$\mathbb{1}^{\mathcal{C}\times^{\mathsf{lax}}_{\mathcal{E}}\mathcal{D}}_{(A,B,C,\phi,\psi)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{lax}}_{\mathcal{E}}\mathcal{D}}((A, B, C, \phi, \psi), (A, B, C, \phi, \psi))$$

  of $\mathcal{C} \times^{\mathsf{lax}}_{\mathcal{E}} \mathcal{D}$ at (A, B, C, ϕ, ψ) is defined by

$$\mathrm{id}_{(A,B,C,\phi,\psi)} \overset{\mathrm{def}}{=} (\mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_C);$$

- *Composition.* For each triple of objects

$$\begin{aligned}\mathbf{X} &= (A, B, C, \phi, \psi),\\ \mathbf{X}' &= (A', B', C', \phi', \psi'),\\ \mathbf{X}'' &= (A'', B'', C'', \phi'', \psi'')\end{aligned}$$

  of $\mathcal{C} \times^{\mathsf{lax}}_{\mathcal{E}} \mathcal{D}$, the composition morphism

$$\circ^{\mathcal{C}\times^{\mathsf{lax}}_{\mathcal{E}}\mathcal{D}}_{\mathbf{X},\mathbf{X}',\mathbf{X}''} \colon \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{lax}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}', \mathbf{X}'') \times \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{lax}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}, \mathbf{X}') \longrightarrow \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{lax}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}, \mathbf{X}'')$$

  of $\mathcal{C} \times^{\mathsf{lax}}_{\mathcal{E}} \mathcal{D}$ at $\mathbf{X}, \mathbf{X}', \mathbf{X}''$ is defined by

$$(f', g', h') \circ^{\mathcal{C}\times^{\mathsf{lax}}_{\mathcal{E}}\mathcal{D}}_{\mathbf{X},\mathbf{X}',\mathbf{X}''} (f, g, h) \overset{\mathrm{def}}{=} (f' \circ f, g' \circ g, h' \circ h)$$

  for each $(f, g, h) \in \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{lax}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}, \mathbf{X}')$ and each $(f', g', h') \in \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{lax}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}', \mathbf{X}'')$.

5.4 Lax Pushouts

5.5 Lax Equalisers

Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 5.5.1 ▶ Lax Equalisers of Categories

The **lax equaliser of** F **and** G is the category $\mathrm{Eq}^{\mathsf{lax}}(F, G)$ where

- *Objects.* An object of $\mathrm{Eq}^{\mathsf{lax}}(F, G)$ is a quadruple (A, B, ϕ, ψ) consisting of
  - An object A of $\mathcal{C}$;

- An object B of $\mathcal{D}$;
- A morphism $\phi\colon F_A \longrightarrow B$ of $\mathcal{D}$;
- A morphism $\psi\colon G_A \longrightarrow B$ of $\mathcal{D}$;

- *Morphisms.* A morphism of $\mathsf{Eq}^{\mathsf{lax}}(F, G)$ from (A, B, ϕ, ψ) to (A', B', ϕ', ψ') is a pair (f, g) consisting of

  - A morphism $f\colon A \longrightarrow A'$ of $\mathcal{C}$;
  - A morphism $g\colon B \longrightarrow B'$ of $\mathcal{D}$;

  making the diagrams

$$\begin{tikzcd} F_A \arrow[r, "F_f"] \arrow[d, "\phi"'] & F_{A'} \arrow[d, "\phi'"] \\ B \arrow[r, "g"'] & B' \end{tikzcd} \qquad \begin{tikzcd} G_A \arrow[r, "G_f"] \arrow[d, "\psi"'] & G_{A'} \arrow[d, "\psi'"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

  commute;

- *Identities.* For each $(A, B, \phi, \psi) \in \mathrm{Obj}\Big(\mathsf{Eq}^{\mathsf{lax}}(F, G)\Big)$, the unit map

$$\mathbb{1}^{\mathsf{Eq}^{\mathsf{lax}}(F,G)}_{(A,B,\phi,\psi)}\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{Eq}^{\mathsf{lax}}(F,G)}((A, B, \phi, \psi), (A, B, \phi, \psi))$$

  of $\mathsf{Eq}^{\mathsf{lax}}(F, G)$ at (A, B, ϕ, ψ) is defined by

$$\mathbb{1}^{\mathsf{Eq}^{\mathsf{lax}}(F,G)}_{(A,B,\phi,\psi)} \overset{\mathrm{def}}{=} (\mathrm{id}_A, \mathrm{id}_B);$$

- *Composition.* For each $\mathbf{A} = (A, B, \phi, \psi), \mathbf{A}' = (A', B', \phi', \psi'), \mathbf{A}'' = (A'', B'', \phi'', \psi'') \in \mathrm{Obj}\Big(\mathsf{Eq}^{\mathsf{lax}}(F, G)\Big)$, the composition map

$$\circ^{\mathsf{Eq}^{\mathsf{lax}}(F,G)}_{\mathbf{A},\mathbf{A}',\mathbf{A}''}\colon \mathrm{Hom}_{\mathsf{Eq}^{\mathsf{lax}}(F,G)}(\mathbf{A}', \mathbf{A}'') \times \mathrm{Hom}_{\mathsf{Eq}^{\mathsf{lax}}(F,G)}(\mathbf{A}, \mathbf{A}') \longrightarrow \mathrm{Hom}_{\mathsf{Eq}^{\mathsf{lax}}(F,G)}(\mathbf{A}, \mathbf{A}'')$$

  of $\mathsf{Eq}^{\mathsf{lax}}(F, G)$ at $(\mathbf{A}, \mathbf{A}', \mathbf{A}'')$ is defined by

$$\circ^{\mathsf{Eq}^{\mathsf{lax}}(F,G)}_{\mathbf{A},\mathbf{A}',\mathbf{A}''} \overset{\mathrm{def}}{=} \Big(\circ^{\mathcal{C}}_{A,A',A''}, \circ^{\mathcal{D}}_{B,B',B''}\Big).$$

5.6 Lax Coequalisers

6 Oplax Limits and Oplax Colimits of Categories

6.1 Oplax Products

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 6.1.1 ▶ Oplax Products

The **oplax product of $\mathcal{C}$ and $\mathcal{D}$** in Cats_2 agrees with their product in Cats, described in Definition 1.1.1.

6.2 Oplax Coproducts

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 6.2.1 ▶ Oplax Coproducts

The **oplax coproduct of $\mathcal{C}$ and $\mathcal{D}$** in Cats_2 agrees with their coproduct in Cats, described in Definition 1.2.1.

6.3 Oplax Pullbacks

Let $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$ be functors.

Definition 6.3.1 ▶ Oplax Pullbacks of Categories

The **oplax pullback of $\mathcal{C}$ and $\mathcal{D}$ over $\mathcal{E}$ along F and G** is the category $\mathcal{C} \times^{\mathrm{oplax}}_{\mathcal{E}} \mathcal{D}$ where

- *Objects.* The objects of $\mathcal{C} \times^{\mathrm{oplax}}_{\mathcal{E}} \mathcal{D}$ are quintuples (A, B, C, ϕ, ψ) consisting of
  - An object A of $\mathcal{C}$;
  - An object B of $\mathcal{D}$;
  - An object C of $\mathcal{E}$;
  - A morphism $\phi\colon C \longrightarrow F_A$ of $\mathcal{E}$;
  - A morphism $\psi\colon C \longrightarrow G_B$ of $\mathcal{E}$;
- *Morphisms.* A morphism of $\mathcal{C} \times^{\mathrm{oplax}}_{\mathcal{E}} \mathcal{D}$ from (A, B, C, ϕ, ψ) to $(A', B', C', \phi', \psi')$ is a pair (f, g, h) consisting of

- A morphism $f\colon A \longrightarrow A'$ of $\mathcal{C}$;
- A morphism $g\colon B \longrightarrow B'$ of $\mathcal{D}$;
- A morphism $h\colon C \longrightarrow C'$ of $\mathcal{E}$;

making the diagrams

$$
\begin{tikzcd}
C \arrow[r, "h"] \arrow[d, "\phi"'] & C \arrow[d, "\phi'"] \\
F_A \arrow[r, "F_f"'] & F_{A'}
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "h"] \arrow[d, "\psi"'] & C' \arrow[d, "\psi'"] \\
G_B \arrow[r, "G_g"'] & G_{B'}
\end{tikzcd}
$$

commute;

- *Identities.* For each $(A, B, C, \phi, \psi) \in \mathrm{Obj}\left(\mathcal{C} \times^{\mathsf{oplax}}_{\mathcal{E}} \mathcal{D}\right)$, the unit morphism

$$\mathbb{1}^{\mathcal{C}\times^{\mathsf{oplax}}_{\mathcal{E}}\mathcal{D}}_{(A,B,C,\phi,\psi)}\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{oplax}}_{\mathcal{E}}\mathcal{D}}((A, B, C, \phi, \psi), (A, B, C, \phi, \psi))$$

of $\mathcal{C} \times^{\mathsf{oplax}}_{\mathcal{E}} \mathcal{D}$ at (A, B, C, ϕ, ψ) is defined by

$$\mathrm{id}_{(A,B,C,\phi,\psi)} \overset{\mathrm{def}}{=} (\mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_C);$$

- *Composition.* For each triple of objects

$$
\begin{aligned}
\mathbf{X} &= (A, B, C, \phi, \psi),\\
\mathbf{X}' &= (A', B', C', \phi', \psi'),\\
\mathbf{X}'' &= (A'', B'', C'', \phi'', \psi'')
\end{aligned}
$$

of $\mathcal{C} \times^{\mathsf{oplax}}_{\mathcal{E}} \mathcal{D}$, the composition morphism

$$\circ^{\mathcal{C}\times^{\mathsf{oplax}}_{\mathcal{E}}\mathcal{D}}_{\mathbf{X},\mathbf{X}',\mathbf{X}''}\colon \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{oplax}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}', \mathbf{X}'') \times \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{oplax}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}, \mathbf{X}') \longrightarrow \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{oplax}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}, \mathbf{X}'')$$

of $\mathcal{C} \times^{\mathsf{oplax}}_{\mathcal{E}} \mathcal{D}$ at $\mathbf{X}, \mathbf{X}', \mathbf{X}''$ is defined by

$$(f', g', h') \circ^{\mathcal{C}\times^{\mathsf{oplax}}_{\mathcal{E}}\mathcal{D}}_{\mathbf{X},\mathbf{X}',\mathbf{X}''} (f, g, h) \overset{\mathrm{def}}{=} (f' \circ f, g' \circ g, h' \circ h)$$

for each $(f, g, h) \in \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{oplax}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}, \mathbf{X}')$ and each $(f', g', h') \in \mathrm{Hom}_{\mathcal{C}\times^{\mathsf{oplax}}_{\mathcal{E}}\mathcal{D}}(\mathbf{X}', \mathbf{X}'')$.

6.4 Oplax Pushouts

6.5 Oplax Equalisers

Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

Definition 6.5.1 ▶ Oplax Equalisers of Categories

The **oplax biequaliser of** F **and** G is the category $\mathsf{Eq}^{\mathsf{oplax}}(F, G)$ where

- *Objects.* An object of $\mathsf{Eq}^{\mathsf{oplax}}(F, G)$ is a quadruple (A, B, ϕ, ψ) consisting of
  - An object A of $\mathcal{C}$;
  - An object B of $\mathcal{D}$;
  - A morphism $\phi\colon B \longrightarrow F_A$ of $\mathcal{D}$;
  - A morphism $\psi\colon B \longrightarrow G_A$ of $\mathcal{D}$;
- *Morphisms.* A morphism of $\mathsf{Eq}^{\mathsf{oplax}}(F, G)$ from (A, B, ϕ, ψ) to (A', B', ϕ', ψ') is a pair (f, g) consisting of
  - A morphism $f\colon A \longrightarrow A'$ of $\mathcal{C}$;
  - A morphism $g\colon B \longrightarrow B'$ of $\mathcal{D}$;

  making the diagrams

$$
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "\phi"'] & B' \arrow[d, "\phi'"] \\
F_A \arrow[r, "F_f"'] & F_{A'}
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "\psi"'] & B' \arrow[d, "\psi'"] \\
G_A \arrow[r, "G_f"'] & G_{A'}
\end{tikzcd}
$$

  commute;
- *Identities.* For each $(A, B, \phi, \psi) \in \mathrm{Obj}\Big(\mathsf{Eq}^{\mathsf{oplax}}(F, G)\Big)$, the unit map

$$\mathbb{1}^{\mathsf{Eq}^{\mathsf{oplax}}(F,G)}_{(A,B,\phi,\psi)}\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{Eq}^{\mathsf{oplax}}(F,G)}((A, B, \phi, \psi), (A, B, \phi, \psi))$$

  of $\mathsf{Eq}^{\mathsf{oplax}}(F, G)$ at (A, B, ϕ, ψ) is defined by

$$\mathbb{1}^{\mathsf{Eq}^{\mathsf{oplax}}(F,G)}_{(A,B,\phi,\psi)} \overset{\text{def}}{=} (\mathrm{id}_A, \mathrm{id}_B);$$

- *Composition*. For each $\mathbf{A} = (A, B, \phi, \psi), \mathbf{A}' = (A', B', \phi', \psi'), \mathbf{A}'' = (A'', B'', \phi'', \psi'') \in \mathrm{Obj}\left(\mathsf{Eq}^{\mathsf{oplax}}(F, G)\right)$, the composition map

$$\circ^{\mathsf{Eq}^{\mathsf{oplax}}(F,G)}_{\mathbf{A},\mathbf{A}',\mathbf{A}''} \colon \mathrm{Hom}_{\mathsf{Eq}^{\mathsf{oplax}}(F,G)}(\mathbf{A}', \mathbf{A}'') \times \mathrm{Hom}_{\mathsf{Eq}^{\mathsf{oplax}}(F,G)}(\mathbf{A}, \mathbf{A}') \longrightarrow \mathrm{Hom}_{\mathsf{Eq}^{\mathsf{oplax}}(F,G)}(\mathbf{A}, \mathbf{A}'')$$

  of $\mathsf{Eq}^{\mathsf{oplax}}(F, G)$ at $(\mathbf{A}, \mathbf{A}', \mathbf{A}'')$ is defined by

$$\circ^{\mathsf{Eq}^{\mathsf{oplax}}(F,G)}_{\mathbf{A},\mathbf{A}',\mathbf{A}''} \overset{\text{def}}{=} \left(\circ^{\mathcal{C}}_{A,A',A''}, \circ^{\mathcal{D}}_{B,B',B''}\right).$$

6.6 Oplax Coequalisers

7 More Constructions with Categories

7.1 Deloopings

Let A be a monoid.

Definition 7.1.1 ▶ The Delooping of a Monoid

The **delooping of** A is the category with a distinguished object $(\mathrm{B}A, \star)$ consisting of

- *The Category*. The category $\mathrm{B}A$ where
  - *Objects*. We have

$$\mathrm{Obj}(\mathrm{B}A) \overset{\text{def}}{=} \mathrm{pt};$$

  - *Morphisms*. We have

$$\mathrm{Hom}_{\mathrm{B}A}(\star, \star) \overset{\text{def}}{=} A;$$

  - *Identities*. The identity map

$$\mathbb{1}^{\mathrm{B}A}_{\star} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathrm{B}A}(\star, \star)$$

    of $\mathrm{B}A$ at $\star$ is defined by

$$\mathbb{1}^{\mathrm{B}A}_{\star} \overset{\text{def}}{=} \eta_A;$$

- *Composition*. The composition map

$$\circ^{\mathsf{B}A}_{\star,\star,\star}\colon \underbrace{\mathrm{Hom}_{\mathsf{B}A}(\star,\star)\times\mathrm{Hom}_{\mathsf{B}A}(\star,\star)}_{\overset{\text{def}}{=}A\times A}\longrightarrow\underbrace{\mathrm{Hom}_{\mathsf{B}A}(\star,\star)}_{\overset{\text{def}}{=}A}$$

of $\mathsf{B}A$ at $(\star,\star,\star)$ is defined by

$$\circ^{\mathsf{B}A}_{\star,\star,\star}\overset{\text{def}}{=}\mu_A.$$

- *The Distinguished Object*. The object $\star$ of $\mathsf{B}A$.

Proposition 7.1.2 ▶ Properties of Deloopings of Monoids

Let A be a monoid.

1. *Functoriality*. The assignments $A\mapsto\mathsf{B}A$, $(\mathsf{B}A,\star)$ defines functors

$$\mathsf{B}\colon \mathsf{Mon}\longrightarrow\mathsf{Cats},$$
$$\mathsf{B}\colon \mathsf{Mon}\longrightarrow\mathsf{Cats}_*.$$

2. *Fully Faithfulness*. The functors of Item 1 are fully faithful, determining isomorphisms of categories[1]

$$\begin{tikzcd} \mathsf{Mon} \arrow[r, "\mathsf{B}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cats} \arrow[d, "\mathrm{Obj}"] \\ \mathrm{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets}, \end{tikzcd} \qquad \begin{aligned} \mathsf{Mon}&\cong\mathrm{pt}\times_{\mathsf{Sets}}\mathsf{Cats},\\ \mathsf{Mon}&\cong\mathrm{pt}\times_{\mathsf{Sets}}\mathsf{Cats}_*. \end{aligned} \qquad \begin{tikzcd} \mathsf{Mon} \arrow[r, "\mathsf{B}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Cats}_* \arrow[d, "\mathrm{Obj}"] \\ \mathrm{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets}. \end{tikzcd}$$

3. *Adjointness I*. We have an adjunction

$$\left(\mathsf{B}^\dagger\dashv\mathsf{B}\right)\colon\quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathsf{B}^\dagger"] \arrow[r, phantom, "\perp"] & \mathsf{Mon}, \arrow[l, bend left, "\mathsf{B}"] \end{tikzcd}$$

where $\mathsf{B}^\dagger\colon\mathsf{Cats}\longrightarrow\mathsf{Mon}$ is the functor defined on objects by

$$\mathsf{B}^\dagger(\mathcal{C})\overset{\text{def}}{=}\mathrm{Mor}(\mathcal{C})/\sim$$

for each $\mathcal{C}\in\mathrm{Obj}(\mathsf{Cats})$, where $\sim$ is the relation on $\mathrm{Mor}(\mathcal{C})$ obtained by declaring

$$[\mathrm{id}_A]\sim 1_{\mathsf{B}^\dagger(\mathcal{C})},$$

$$[g \circ f] \sim [g][f]$$

for each $A \in \mathrm{Obj}(\mathcal{C})$ and each composable pair $(f, g) \in \mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{C})$.

4. *Adjointness II.* We have an adjunction

$$(\mathsf{B} \dashv \mathsf{End})\colon \quad \begin{tikzcd} \mathsf{Mon} \arrow[r, bend left, "\mathsf{B}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}_* \arrow[l, bend left, "\mathsf{End}"] \end{tikzcd},$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Cats}_*}((\mathsf{B}A, \star), (\mathcal{C}, X)) \cong \mathrm{Hom}_{\mathsf{Mon}}(A, \mathrm{End}_{\mathcal{C}}(X)),$$

natural in $A \in \mathrm{Obj}(\mathsf{Mon})$ and $(\mathcal{C}, X) \in \mathrm{Obj}(\mathsf{Cats}_*)$.

5. *Preservation of Limits.* The functor $\mathsf{B}\colon \mathsf{Mon} \longrightarrow \mathsf{Cats}$ of Item 1 preserves limits. In particular, we have isomorphisms of categories

$$\begin{aligned} \mathsf{B}(A \times B) &\cong \mathsf{B}A \times \mathsf{B}B, \\ \mathsf{B}(A \times_C B) &\cong \mathsf{B}A \times_{\mathsf{B}C} \mathsf{B}B, \\ \mathsf{B}\,\mathrm{Eq}(f, g) &\cong \mathrm{Eq}(\mathsf{B}f, \mathsf{B}g), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Mon})$ and parallel $f, g \in \mathrm{Mor}(\mathsf{Mon})$.

6. *Interaction With Adjunctions, Equivalences, and Isomorphisms.* Let $\mathsf{B}f\colon \mathsf{B}A \longrightarrow \mathsf{B}B$ and $\mathsf{B}g\colon \mathsf{B}B \longrightarrow \mathsf{B}A$ be functors. The following conditions are equivalent:[2]

   (a) The pair $(\mathsf{B}f, \mathsf{B}g)$ determines an equivalence of categories $\mathsf{B}A \overset{\text{eq.}}{\cong} \mathsf{B}B$.

   (b) The pair $(\mathsf{B}f, \mathsf{B}g)$ determines an isomorphism of categories $\mathsf{B}A \cong \mathsf{B}B$.

   (c) The pair (f, g) determines an isomorphism of monoids $A \cong B$.

---

[1]Here $\mathrm{pt} \times_{\mathsf{Sets}} \mathsf{Cats}$ (resp. $\mathrm{pt} \times_{\mathsf{Sets}} \mathsf{Cats}_*$) is the full subcategory of Cats (resp. Cats_*) spanned by the one-object categories (resp. the one-object categories with a distinguished object).

[2] *Warning.* The following condition *is not* equivalent to the conditions in Items (a) to (c):

- The pair $(\mathsf{B}f, \mathsf{B}g)$ determines an adjunction

$$(\mathsf{B}f \dashv \mathsf{B}g)\colon \quad \begin{tikzcd} \mathsf{B}A \arrow[r, bend left, "\mathsf{B}f"] \arrow[r, phantom, "\perp"] & \mathsf{B}B \arrow[l, bend left, "\mathsf{B}g"] \end{tikzcd}.$$

In detail, it means that we have a bijection *of sets*

$$\phi\colon \underbrace{\mathrm{Hom}_{\mathsf{B}B}(\mathsf{B}f(\star),\star)}_{\substack{\overset{\text{def}}{=}\mathrm{Hom}_{\mathsf{B}B}(\star,\star)\\ \overset{\text{def}}{=}B}} \xrightarrow{\cong} \underbrace{\mathrm{Hom}_{\mathsf{B}A}(\star,\mathsf{B}g(\star))}_{\substack{\overset{\text{def}}{=}\mathrm{Hom}_{\mathsf{B}A}(\star,\star)\\ \overset{\text{def}}{=}A}}$$

which is moreover natural in that, for each morphism $a \in A$ of $\mathsf{B}A$ or each morphism $b \in B$ of $\mathsf{B}B$, either (or, equivalently, both) of the diagrams

$$\begin{tikzcd} B \arrow[r, dashed, "\phi", "\sim"'] \arrow[d, "\cdot f(a)"'] & A \arrow[d, "\cdot a"] \\ B \arrow[r, dashed, "\sim", "\phi"'] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, dashed, "\phi", "\sim"'] \arrow[d, "\cdot b"'] & A \arrow[d, "\cdot g(b)"] \\ B \arrow[r, dashed, "\sim", "\phi"'] & A \end{tikzcd}$$

commute, i.e. such that, for each $a, a' \in A$ and each $b, b' \in B$, we have

$$\phi(f(a)b) = a\phi(b),$$
$$\phi(bb') = g(b)\phi(b').$$

A counterexample is given in [Mol05]. See also [Cun20].

PROOF 7.1.3 ▶ PROOF OF PROPOSITION 7.1.2

Item 1: Functoriality

Omitted.

Item 2: Fully Faithfulness

Omitted.

Item 3: Adjointness I

Omitted.

Item 4: Adjointness II

Omitted.

Item 5: Preservation of Limits

This follows from Item 3 and Categories, Item 4 of Proposition 6.1.3.

Item 6: Interaction With Adjunctions, Equivalences, and Isomorphisms

Omitted. □

7.2 The Classifying Space of a Category

Proposition 7.2.1 ▶ Properties of the Classifying Space of a Category

Let $\mathcal{C}$ be a category.

1. *Contractibility Criteria*. Suppose that one of the following conditions is satisfied:

   (a) The category $\mathcal{C}$ has an initial object.

   (b) The category $\mathcal{C}$ has a terminal object.

   Then $\mathsf{B}\mathcal{C}$ is contractible.

Proof 7.2.2 ▶ Proof of Proposition 7.2.1

Item 1: Contractibility Criteria

Omitted.

7.3 Opposite Categories

Let $\mathcal{C}$ be a category.

Definition 7.3.1 ▶ Opposite Categories

The **opposite category** of $\mathcal{C}$ is the category $\mathcal{C}^{\mathsf{op}}$ where

1. *Objects*. We have
$$\mathrm{Obj}(\mathcal{C}^{\mathsf{op}}) \overset{\mathrm{def}}{=} \mathrm{Obj}(\mathcal{C});$$

2. *Morphisms*. For each $A, B \in \mathrm{Obj}(\mathcal{C})$, we have
$$\mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(A, B) \overset{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{C}}(B, A);$$

3. *Identities*. For each $A \in \mathrm{Obj}(\mathcal{C})$, the unit map
$$\mathbb{1}^{\mathcal{C}^{\mathsf{op}}}_A : \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(A, A)$$
of $\mathcal{C}^{\mathsf{op}}$ at A is given by
$$\mathbb{1}^{\mathsf{op}}_A = \mathbb{1}_A;$$

4. *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the composition map

$$\circ^{\mathcal{C}^{\mathsf{op}}}_{A,B,C} \colon \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(B, C) \times \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(A, C)$$

of $\mathcal{C}^{\mathsf{op}}$ at (A, B, C) is given by the composition

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(B, C) \times \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(A, B) &\overset{\text{def}}{=\!=} \mathrm{Hom}_{\mathcal{C}}(C, B) \times \mathrm{Hom}_{\mathcal{C}}(B, A) \\ &\overset{\sim}{\dashrightarrow} \mathrm{Hom}_{\mathcal{C}}(B, A) \times \mathrm{Hom}_{\mathcal{C}}(C, B) \\ &\xrightarrow{\circ^{\mathcal{C}}_{C,B,A}} \mathrm{Hom}_{\mathcal{C}}(C, A) \\ &\overset{\text{def}}{=\!=} \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(A, C)\end{aligned}$$

Proposition 7.3.2 ▶ Properties of Opposite Categories

The following statements are true:

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathcal{C}^{\mathsf{op}}$ defines a functor

$$(-)^{\mathsf{op}} \colon \mathsf{Cats} \longrightarrow \mathsf{Cats}.$$

2. *Interaction With Undercategories and Overcategories.* Let $A \in \mathrm{Obj}(\mathcal{C})$. We have equivalences of categories

$$\begin{aligned}\mathcal{C}_{A/} &\overset{\text{eq.}}{\cong} \left(\mathcal{C}^{\mathsf{op}}_{/A}\right)^{\mathsf{op}}, \\ \mathcal{C}_{/A} &\overset{\text{eq.}}{\cong} \left(\mathcal{C}^{\mathsf{op}}_{A/}\right)^{\mathsf{op}}.\end{aligned}$$

Proof 7.3.3 ▶ Proof of Proposition 7.3.2

Item 1: Functoriality

Clear.

Item 2: Interaction With Undercategories and Overcategories

Omitted. □

7.4 Categories of Pointed Objects

Let $(\mathcal{C}, \not\Bbbk_{\mathcal{C}})$ be a pointed category.

Definition 7.4.1 ▶ Pointed Objects in a Pointed Category

A **pointed object in** $(\mathcal{C}, \mathbb{1}_{\mathcal{C}})$ is a pair $X = (X, x_0)$ consisting of

- *The Underlying Object.* An object X of $\mathcal{C}$;
- *The Basepoint.* A morphism

$$x_0 \colon \mathbb{1}_{\mathcal{C}} \longrightarrow X$$

  of $\mathcal{C}$, called the **basepoint of** X.

Definition 7.4.2 ▶ Morphisms of Pointed Objects in a Pointed Category

A **morphism of pointed objects in** $(\mathcal{C}, \mathbb{1}_{\mathcal{C}})$ **from** (X, x_0) **to** (Y, y_0) is a morphism

$$f \colon X \longrightarrow Y$$

of $\mathcal{C}$ making the diagram

$$\begin{tikzcd}
 & \mathbb{1}_{\mathcal{C}} \arrow[dl, "x_0"'] \arrow[dr, "y_0"] & \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

commute.

Definition 7.4.3 ▶ The Category of Pointed Objects in a Pointed Category

The **category of pointed objects in** $(\mathcal{C}, \mathbb{1}_{\mathcal{C}})$ is the category $\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$[1] defined as the coslice category of $\mathcal{C}$ by $\mathbb{1}_{\mathcal{C}}$.

[1] *Further Notation:* Also written $\mathcal{C}_*$ when $\mathbb{1}_{\mathcal{C}}$ is the terminal object of $\mathcal{C}$.

Remark 7.4.4 ▶ Unwinding Definition 7.4.3

In detail, **category of pointed objects in** $(\mathcal{C}, \mathbb{1}_{\mathcal{C}})$ is the category $\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$ where

- *Objects.* The objects of $\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$ are pointed objects in $\mathcal{C}$;
- *Morphisms.* The morphisms of $\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$ are morphisms of pointed objects in $\mathcal{C}$;

- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/})$, the unit map

$$\mathbb{1}^{\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}}_{(X,x_0)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}}((X, x_0), (X, x_0))$$

  of $\mathcal{C}$ at (X, x_0) is the map of sets picking the morphism id_X of $\mathrm{Hom}_{\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}}((X, x_0), (X, x_0))$;

- *Composition.* For each $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/})$, the composition map

$$\circ^{\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathrm{Hom}_{\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}}((Y, y_0), (Z, z_0)) \times \mathrm{Hom}_{\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}}((X, x_0), (Y, y_0)) \longrightarrow \mathrm{Hom}_{\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}}((X, x_0), (Z, z_0))$$

  of $\mathcal{C}$ at $((X, x_0), (Y, y_0), (Z, z_0))$ is the map of sets defined by

$$\circ^{\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}}_{(X,x_0),(Y,y_0),(Z,z_0)} \overset{\mathrm{def}}{=} \circ^{\mathcal{C}}_{X,Y,Z}\Big|_{\mathrm{Hom}_{\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}}((Y,y_0),(Z,z_0)) \times \mathrm{Hom}_{\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}}((X,x_0),(Y,y_0))}.$$

Remark 7.4.5 ▶ Forgetful Functor

We have a natural forgetful functor $忘\colon \mathcal{C}_{\mathbb{1}_{\mathcal{C}}/} \twoheadrightarrow \mathcal{C}$ where

- *Action on Objects.* For each $(X, x_0) \in \mathrm{Obj}(\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/})$, we have

$$忘(X, x_0) \overset{\mathrm{def}}{=} X;$$

- *Action on Morphisms.* For each $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/})$, the action on Hom-sets

$$忘_{(X,x_0),(Y,y_0)} \colon \mathrm{Hom}_{\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}}((X, x_0), (Y, y_0)) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

  of 忘 at $((X, x_0), (Y, y_0))$ is defined by

$$忘(f) \overset{\mathrm{def}}{=} f$$

  for each $f \in \mathrm{Hom}_{\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}}((X, x_0), (Y, y_0))$.

Proposition 7.4.6 ▶ Properties of Pointed Objects in a Pointed Category

Let $(\mathcal{C}, \mathbb{1}_{\mathcal{C}})$ be a pointed category.

1. *Functoriality.* The assignments $\mathcal{C} \mapsto \mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$, $(\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}, 忘)$ define functors

$$\begin{aligned}(-)_{\mathbb{1}_{\mathcal{C}}/} &\colon \mathsf{Cats}_{*} \longrightarrow \mathsf{Cats},\\ (-)_{\mathbb{1}_{\mathcal{C}}/} &\colon \mathsf{Cats}_{*} \longrightarrow \mathsf{DFib}.\end{aligned}$$

2. *2-Functoriality.* The assignments $\mathcal{C} \mapsto \mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$, $(\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}, 忘)$ define 2-functors

$$\begin{aligned}(-)_{\mathbb{1}_{\mathcal{C}}/} &\colon \mathsf{Cats}_{*,2} \longrightarrow \mathsf{Cats}_{2},\\ (-)_{\mathbb{1}_{\mathcal{C}}/} &\colon \mathsf{Cats}_{*,2} \longrightarrow \mathsf{DFib}_{2}.\end{aligned}$$

3. *Adjointness.* If $\mathcal{C}$ has coproducts, then we have an adjunction

$$\Big((-)^{+} \dashv 忘\Big)\colon \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "(-)^{+}"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{\mathbb{1}_{\mathcal{C}}/} \arrow[l, bend left, "忘"] \end{tikzcd},$$

   where

$$(-)^{+}\colon \mathcal{C} \longrightarrow \mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$$

   is the functor where

   - *Action on Objects.* For each $X \in \mathrm{Obj}(\mathcal{C})$, we have

   $$X^{+} \overset{\mathrm{def}}{=} X \coprod \mathbb{1}_{\mathcal{C}};$$

   - *Action on Morphisms.* For each morphism $f\colon X \longrightarrow Y$ of $\mathcal{C}$, the image

   $$f^{+}\colon X^{+} \longrightarrow Y^{+}$$

   of f by $(-)^{+}$ is defined by

   $$f^{+} \overset{\mathrm{def}}{=} f \coprod \mathrm{id}_{\mathbb{1}_{\mathcal{C}}}.$$

4. *Initial and Zero Objects.* Let $(\mathcal{C}, \mathbb{1}_{\mathcal{C}})$ be a pointed category.

   (a) The object $(\mathbb{1}_{\mathcal{C}}, \mathrm{id}_{\mathbb{1}_{\mathcal{C}}})$ is initial in $\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$.

   (b) If $\mathbb{1}_{\mathcal{C}}$ is terminal in $\mathcal{C}$, then $(\mathbb{1}_{\mathcal{C}}, \mathrm{id}_{\mathbb{1}_{\mathcal{C}}})$ is a zero object of $\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$.

5. *Symmetric Closed Monoidality.* Let $\mathcal{C}$ be a category. If:

   (a) The category $\mathcal{C}$ has a terminal object $\mathbb{1}_{\mathcal{C}}$;

(b) The category $\mathcal{C}$ is finitely bicomplete;

(c) The category $\mathcal{C}$ is Cartesian closed;

then the quadruple $(\mathcal{C}_*, \wedge, S^0, \mathbf{Hom}_{\mathcal{C}_*})$ consisting of

- *The Underlying Category.* The category $\mathcal{C}_*$ of pointed objects in $\mathcal{C}$;
- *The Monoidal Product.* The functor

$$\wedge \colon \mathcal{C}_* \times \mathcal{C}_* \longrightarrow \mathcal{C}_*,$$

called the **smash product of** $\mathcal{C}$, defined on objects by

$$X \wedge Y \overset{\text{def}}{=} \mathbb{1}_{\mathcal{C}} \coprod_{X \coprod Y} (X \times Y), \qquad \begin{tikzcd} X \wedge Y & X \times Y \arrow[l] \\ \mathbb{1}_{\mathcal{C}} \arrow[u] \arrow[ur, phantom, "\ulcorner", very near end] & X \coprod Y; \arrow[l] \arrow[u] \end{tikzcd}$$

- *The Monoidal Unit.* The object S^0 of $\mathcal{C}$ defined by

$$S^0 \overset{\text{def}}{=} \mathbb{1}_{\mathcal{C}} \coprod \mathbb{1}_{\mathcal{C}};$$

- *The Internal* **Hom**. For each $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathcal{C}_*)$, the pointed object $\mathbf{Hom}_{\mathcal{C}_*}(X, Y)$ in $\mathcal{C}$ consisting of
  - *The Underlying Object.* The object $\mathbf{Hom}_{\mathcal{C}_*}(X, Y)$ of $\mathcal{C}$ defined by

$$\mathbf{Hom}_{\mathcal{C}_*}(X, Y) \overset{\text{def}}{=} \mathbb{1}_{\mathcal{C}}^{\mathbb{1}_{\mathcal{C}}} \times_{Y^{\mathbb{1}_{\mathcal{C}}}} Y^X, \qquad \begin{tikzcd} \mathbf{Hom}_{\mathcal{C}_*}(X, Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^X \arrow[d, "x_0^*"] \\ \mathbb{1}_{\mathcal{C}}^{\mathbb{1}_{\mathcal{C}}} \arrow[r, "(y_0)_*"'] & Y^{\mathbb{1}_{\mathcal{C}}}; \end{tikzcd}$$

  - *The Basepoint.* The morphism

$$\Delta_{y_0} \colon \mathbb{1}_{\mathcal{C}} \longrightarrow \mathbf{Hom}_{\mathcal{C}_*}(X, Y)$$

of $\mathcal{C}$ given by the dashed morphism in the diagram

$$
\begin{tikzcd}
\mathbb{1}_{\mathcal{C}} \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathbf{Hom}_{\mathcal{C}_*}(X, Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^X \arrow[d, "x_0^*"] \\
& \mathbb{1}_{\mathcal{C}}^{\mathbb{1}_{\mathcal{C}}} \arrow[r, "(y_0)_*"'] & Y^{\mathbb{1}_{\mathcal{C}}},
\end{tikzcd}
$$

where

- The morphism $\mathbb{1}_{\mathcal{C}} \longrightarrow \mathbb{1}_{\mathcal{C}}^{\mathbb{1}_{\mathcal{C}}}$ is the adjunct of the isomorphism

  $$\mathbb{1}_{\mathcal{C}} \times \mathbb{1}_{\mathcal{C}} \xrightarrow{\cong} \mathbb{1}_{\mathcal{C}}$$

  in $\mathcal{C}$ under the adjunction $\mathbb{1}_{\mathcal{C}} \times - \dashv (-)^{\mathbb{1}_{\mathcal{C}}}$;
- The morphism $\mathbb{1}_{\mathcal{C}} \longrightarrow Y^X$ is the adjunct of the composition

  $$
  \begin{aligned}
  X \times \mathbb{1}_{\mathcal{C}} &\xrightarrow{!_X \times \mathrm{id}_{\mathbb{1}_{\mathcal{C}}}} \mathbb{1}_{\mathcal{C}} \times \mathbb{1}_{\mathcal{C}} \\
  &\overset{\sim}{\dashrightarrow} \mathbb{1}_{\mathcal{C}} \\
  &\xrightarrow{y_0} Y
  \end{aligned}
  $$

  in $\mathcal{C}$ under the adjunction $X \times - \dashv (-)^X$;

is a symmetric closed monoidal category.

6. *Co/Completeness.* If $\mathcal{C}$ is co/complete, then so is $\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$.[1]

7. *Symmetric Strong Monoidality of Free Pointed Objects With Respect to Coproducts.* If $\mathcal{C}$ has binary coproducts and an initial object $\varnothing_{\mathcal{C}}$, then the functor $(-)^+$ of Item 3 has a symmetric strong monoidal structure

   $$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathcal{C}, \coprod, \varnothing_{\mathcal{C}}) \longrightarrow \left(\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}, \vee, (\mathbb{1}_{\mathcal{C}}, \mathrm{id}_{\mathbb{1}_{\mathcal{C}}})\right)$$

   being equipped with isomorphisms

   $$
   \begin{gathered}
   (-)^{+,\coprod}_{X,Y} \colon X^+ \vee Y^+ \xrightarrow{\cong} (X \textstyle\coprod Y)^+, \\
   (-)^{+,\coprod}_{\mathbb{1}} \colon \mathbb{1}_{\mathcal{C}} \xrightarrow{\cong} \varnothing_{\mathcal{C}}^+,
   \end{gathered}
   $$

natural in $X, Y \in \mathrm{Obj}(\mathcal{C})$.

8. *Symmetric Strong Monoidality of Free Pointed Objects With Respect to Products.* If $\mathcal{C}$ has binary co/products, and $\mathbb{1}_{\mathcal{C}}$ is terminal in $\mathcal{C}$, then the functor $(-)^+$ of Item 3 has a symmetric strong monoidal structure

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathcal{C}, \times, \mathbb{1}_{\mathcal{C}}) \longrightarrow \left(\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}, \wedge, S^0\right)$$

being equipped with isomorphisms

$$(-)^{+,\times}_{X,Y} \colon X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+,$$
$$(-)^{+,\times}_{\mathbb{1}} \colon S^0 \xrightarrow{\cong} \mathbb{1}^+_{\mathcal{C}},$$

natural in $X, Y \in \mathrm{Obj}(\mathcal{C})$.

9. *Universal Property I.* Suppose that $\mathcal{C}$ has binary coproducts and that $\mathbb{1}_{\mathcal{C}}$ is terminal in $\mathcal{C}$. The symmetric monoidal category structure on $\mathcal{C}_*$ of Item 5 is uniquely determined by the following requirements:

   (a) *Two-Sided Preservation of Colimits.* The tensor product

   $$\otimes_{\mathcal{C}_*} \colon \mathcal{C}_* \times \mathcal{C}_* \longrightarrow \mathcal{C}_*$$

   of $\mathcal{C}_*$ preserves colimits separately in each variable.

   (b) *The Unit Object Is S^0.* We have $\mathbb{1}_{\mathcal{C}_*} = S^0 \stackrel{\text{def}}{=} \mathbb{1}_{\mathcal{C}} \coprod \mathbb{1}_{\mathcal{C}}$.

10. *Universal Property II.* Suppose that $\mathcal{C}$ has binary coproducts and that $\mathbb{1}_{\mathcal{C}}$ is terminal in $\mathcal{C}$. The symmetric monoidal structure of Item 5 is the unique symmetric monoidal structure on $\mathcal{C}_*$ such that the free functor

$$(-)^+ \colon \mathcal{C} \longrightarrow \mathcal{C}_*$$

admits a symmetric monoidal structure.

11. *Distributivity of Smash Products Over Wedge Sums.* If $\mathcal{C}$ has binary co/products and a terminal object, then we have isomorphisms

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z),$$
$$(X \vee Y) \wedge Z \cong (X \wedge Z) \vee (Y \wedge Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathcal{C}_*)$.

12. *Comonoids in* $\mathcal{C}_*$. Suppose that $\mathcal{C}$ has coproducts and a terminal object $\mathbb{1}_{\mathcal{C}}$. The symmetric monoidal functor

$$\left((-)^+, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathcal{C}, \times, \mathbb{1}_{\mathcal{C}}) \longrightarrow \left(\mathcal{C}_*, \wedge, S^0\right)$$

of Item 8 lifts to an equivalence of categories

$$\mathsf{CoMon}\left(\mathcal{C}_*, \wedge, S^0\right) \overset{\text{eq.}}{\cong} \mathsf{CoMon}(\mathcal{C}, \times, \mathbb{1}_{\mathcal{C}}).$$

[1] In particular, $\mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$ has binary coproducts, called **wedge sums**, and given by the pushout

$$X \vee Y \overset{\text{def}}{=} X \coprod_{\mathbb{1}_{\mathcal{C}}} Y, \qquad \begin{tikzcd} X \vee Y \arrow[dr, phantom, very near start, "\ulcorner"] & Y \arrow[l] \\ X \arrow[u] & \mathbb{1}_{\mathcal{C}} \arrow[l, "x_0"] \arrow[u, "y_0"'] \end{tikzcd}$$

in $\mathcal{C}$.

Proof 7.4.7 ▶ Proof of Proposition 7.4.6

Item 1: Functoriality

This follows from Item 1 of Proposition 10.1.2.

Item 2: 2-Functoriality

Omitted.

Item 3: Adjointness

Omitted.

Item 4: Initial and Zero Objects

Omitted.

Item 5: Symmetric Closed Monoidality

See [Rie14, Lemma 3.3.16].

Item 6: Co/Completeness

Omitted.

Item 7: Symmetric Strong Monoidality of Free Pointed Objects With Respect to Coproducts

Omitted.

Item 8: Symmetric Strong Monoidality of Free Pointed Objects With Respect to Products

Omitted.

Item 9: Universal Property I

Omitted.

Item 10: Universal Property II

Omitted.

Item 11: Distributivity of Smash Products Over Wedge Sums

Omitted.

Item 12: Comonoids in $\mathcal{C}_*$

Omitted.

7.5 Joins of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

DEFINITION 7.5.1 ▶ JOINS OF CATEGORIES ([LUR20, TAG 0161])

The **join of $\mathcal{C}$ and $\mathcal{D}$** is the category $\mathcal{C} \star \mathcal{D}$ where[1]

- *Objects.* We have

$$\mathrm{Obj}(\mathcal{C} \star \mathcal{D}) \overset{\mathrm{def}}{=} \mathrm{Obj}(\mathcal{C}) \coprod \mathrm{Obj}(\mathcal{D});$$

- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C} \star \mathcal{D})$, we have

$$\mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(A, B) \overset{\mathrm{def}}{=} \begin{cases} \mathrm{Hom}_{\mathcal{C}}(A, B) & \text{if } A, B \in \mathrm{Obj}(\mathcal{C}), \\ \mathrm{Hom}_{\mathcal{D}}(A, B) & \text{if } A, B \in \mathrm{Obj}(\mathcal{D}), \\ \mathrm{pt} & \text{if } A \in \mathrm{Obj}(\mathcal{C}) \text{ and } B \in \mathrm{Obj}(\mathcal{D}), \\ \emptyset & \text{if } A \in \mathrm{Obj}(\mathcal{D}) \text{ and } B \in \mathrm{Obj}(\mathcal{C}). \end{cases}$$

- *Identities.* For each $X \in \mathrm{Obj}(\mathcal{C} \star \mathcal{D})$, the unit map

$$\mathbb{1}^{\mathcal{C}\star\mathcal{D}}_X \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(X, X)$$

of $\mathcal{C} \star \mathcal{D}$ at X is defined by

$$\mathbb{1}^{\mathcal{C}\star\mathcal{D}}_X \overset{\mathrm{def}}{=} \begin{cases} \mathbb{1}^{\mathcal{C}}_X & \text{if } X \in \mathrm{Obj}(\mathcal{C}), \\ \mathbb{1}^{\mathcal{D}}_X & \text{if } X \in \mathrm{Obj}(\mathcal{D}); \end{cases}$$

- *Composition*. For each $X, Y, Z \in \mathrm{Obj}(\mathcal{C} \star \mathcal{D})$, the composition map

  $$\circ^{\mathcal{C}\star\mathcal{D}}_{X,Y,Z} \colon \mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(X,Z)$$

  of $\mathcal{C} \star \mathcal{D}$ at (X, Y, Z) is defined as follows:

  - If $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, then we have

    $$g \circ^{\mathcal{C}\star\mathcal{D}}_{X,Y,Z} f \overset{\text{def}}{=} g \circ^{\mathcal{C}}_{X,Y,Z} f$$

    for each $f \in \mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(X,Y)$ and each $g \in \mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(Y,Z)$.
  - If $X, Y, Z \in \mathrm{Obj}(\mathcal{D})$, then we have

    $$g \circ^{\mathcal{C}\star\mathcal{D}}_{X,Y,Z} f \overset{\text{def}}{=} g \circ^{\mathcal{C}}_{X,Y,Z} f$$

    for each $f \in \mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(X,Y)$ and each $g \in \mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(Y,Z)$.
  - If $X \in \mathrm{Obj}(\mathcal{C})$ and $Z \in \mathrm{Obj}(\mathcal{D})$, then the composition map

    $$\circ^{\mathcal{C}\star\mathcal{D}}_{X,Y,Z} \colon \underbrace{\mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(B,C)}_{\overset{\text{def}}{=}\mathrm{pt}} \times \underbrace{\mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(A,B)}_{\overset{\text{def}}{=}\mathrm{pt}} \longrightarrow \underbrace{\mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(A,C)}_{\overset{\text{def}}{=}\mathrm{pt}}$$

    is the terminal map.

[1]*Slogan:* The join $\mathcal{C} \star \mathcal{D}$ of $\mathcal{C}$ and $\mathcal{D}$ is the disjoint union of $\mathcal{C}$ and $\mathcal{D}$ with a unique morphism from each object of $\mathcal{C}$ to each object of $\mathcal{D}$.

Example 7.5.2 ▶ Examples of Joins of Categories

Here are some examples of joins of categories.

1. We have an isomorphism of categories

   $$\mathsf{n+1} \cong \mathsf{n} \star \mathsf{0},$$

   so that e.g.:

   $$\begin{aligned} \mathsf{1} &\cong \mathsf{0} \star \mathsf{0}, \\ \mathsf{2} &\cong \mathsf{1} \star \mathsf{0} \\ &\cong (\mathsf{0} \star \mathsf{0}) \star \mathsf{0}, \\ \mathsf{3} &\cong \mathsf{2} \star \mathsf{0} \\ &\cong (\mathsf{1} \star \mathsf{0}) \star \mathsf{0} \end{aligned}$$

$$\begin{aligned} &\cong ((0 \star 0) \star 0) \star 0, \\ 4 &\cong 3 \star 0 \\ &\cong (2 \star 0) \star 0 \\ &\cong ((1 \star 0) \star 0) \star 0 \\ &\cong (((0 \star 0) \star 0) \star 0) \star 0, \end{aligned}$$

More generally, we have an isomorphism of categories

$$n \star m \cong n+m+1.$$

Proposition 7.5.3 ▶ Properties of Joins of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. *Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathcal{C} \star \mathcal{D}$ define functors

$$\begin{aligned} \mathcal{C} \star -\colon \mathsf{Cats} &\longrightarrow \mathsf{Cats}_{\mathcal{C}/}, \\ - \star \mathcal{D}\colon \mathsf{Cats} &\longrightarrow \mathsf{Cats}_{\mathcal{D}/}, \\ -_1 \star -_2\colon \mathsf{Cats} \times \mathsf{Cats} &\longrightarrow \mathsf{Cats}. \end{aligned}$$

2. *Adjointness.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, we have adjunctions

$$(\mathcal{C} \star - \dashv -_{(-)/})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathcal{C}\star-"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}_{\mathcal{C}/}, \arrow[l, bend left, "-_{(-)/}"] \end{tikzcd}$$

$$(- \star \mathcal{D} \dashv -_{/(-)})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "-\star\mathcal{D}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}_{\mathcal{D}/}, \arrow[l, bend left, "-_{/(-)}"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathsf{Fun}_{\mathcal{C}/}(\mathcal{C} \star \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}(\mathcal{D}, \mathcal{E}_{F/}), \\ \mathsf{Fun}_{/\mathcal{D}}(\mathcal{C} \star \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}(\mathcal{C}, \mathcal{E}_{G/}), \end{aligned}$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$, in $(F, \mathcal{E}) \in \mathrm{Obj}(\mathsf{Cats}_{/\mathcal{C}})$, and in $(G, \mathcal{E}) \in \mathrm{Obj}(\mathsf{Cats}_{/\mathcal{D}})$, where

$$\begin{aligned} -_{/(-)}\colon \mathsf{Cats}_{\mathcal{C}/} &\longrightarrow \mathsf{Cats}, \\ -_{/(-)}\colon \mathsf{Cats}_{\mathcal{D}/} &\longrightarrow \mathsf{Cats} \end{aligned}$$

are the functors given by

$$(F\colon \mathcal{C} \longrightarrow \mathcal{E}) \mapsto \mathcal{E}_{F/},$$
$$(G\colon \mathcal{D} \longrightarrow \mathcal{E}) \mapsto \mathcal{E}_{/G},$$

respectively.

3. *Monoidality.* The triple $(\mathsf{Cats}, \star, \emptyset_{\mathsf{cat}})$ is a monoidal category.[1]
4. *Interaction With Opposites.* We have a natural isomorphism

$$(\mathcal{C} \star \mathcal{D})^{\mathsf{op}} \cong \mathcal{D}^{\mathsf{op}} \star \mathcal{C}^{\mathsf{op}}.$$

5. *As a Pushout.* The diagram

$$\begin{tikzcd}
\mathcal{C} \star \mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{C} \times \mathbb{I} \times \mathcal{D} \arrow[d] \\
(\mathcal{C} \times \{0\}) \coprod (\{1\} \times \mathcal{D}) \arrow[r] & (\mathcal{C} \times \{0\} \times \mathcal{D}) \coprod (\mathcal{C} \times \{1\} \times \mathcal{D})
\end{tikzcd}$$

is a pushout square in Cats.

[1] In particular, we have isomorphisms

$$\mathcal{C} \star \emptyset_{\mathsf{cat}} \cong \mathcal{C} \cong \emptyset_{\mathsf{cat}} \star \mathcal{C},$$
$$(\mathcal{C} \star \mathcal{D}) \star \mathcal{E} \cong \mathcal{C} \star (\mathcal{D} \star \mathcal{E}).$$

Proof 7.5.4 ▶ Proof of Proposition 7.5.3

Item 1: Functoriality

See [Lur20, Tag 0163].

Item 2: Adjointness

See [Lur20, Tag 016H].

Item 3: Monoidality

See [Lur20, Tag 0167].

Item 4: Interaction With Opposites

See [Lur20, Tag 0168].

Item 5: As a Pushout

See [Lur20, Tag 016E].

7.6 Arrow Categories

7.6.1 The Walking Arrow

Definition 7.6.1 ▶ The Walking Arrow Category

The **walking arrow category**[1] is the category $\mathbb{1}$ where

- *Objects.* We have
$$\mathrm{Obj}(\mathbb{1}) = \{[0], [1]\};$$
- *Morphisms.* We have
$$\begin{aligned}\mathrm{Hom}_{\mathbb{1}}([0], [0]) &= \left\{\mathrm{id}_{[0]}\right\},\\ \mathrm{Hom}_{\mathbb{1}}([1], [1]) &= \left\{\mathrm{id}_{[1]}\right\},\\ \mathrm{Hom}_{\mathbb{1}}([0], [1]) &= \mathrm{pt},\\ \mathrm{Hom}_{\mathbb{1}}([1], [0]) &= \emptyset;\end{aligned}$$
- *Identities.* The unit maps
$$\begin{aligned}\mathbb{1}^{\mathbb{1}}_{[0]} &\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathbb{1}}([0], [0]),\\ \mathbb{1}^{\mathbb{1}}_{[1]} &\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathbb{1}}([1], [1])\end{aligned}$$
of $\mathbb{1}$ are the unique ones;
- *Composition.* The composition maps of $\mathbb{1}$ are also the unique ones possible.

[1] *Further Terminology:* Also called the **interval category**.

7.6.2 Arrow Categories

Let $\mathcal{C}$ be a category.

Definition 7.6.2 ▶ Arrow Categories

The **arrow category of $\mathcal{C}$**[1] is the category $\mathrm{Arr}(\mathcal{C})$[2] defined by
$$\mathrm{Arr}(\mathcal{C}) \stackrel{\text{def}}{=} \mathrm{Fun}(\mathbb{1}, \mathcal{C}).$$

[1] *Further Terminology:* Also called the **category of morphisms of $\mathcal{C}$**.
[2] *Further Notation:* Also written $\mathcal{C}^{\rightarrow}$.

Remark 7.6.3 ▶ Unwinding Definition 7.6.2

In detail, $\mathsf{Arr}(\mathcal{C})$ is the category where

- *Objects.* The objects of $\mathsf{Arr}(\mathcal{C})$ are the morphisms of $\mathcal{C}$;
- *Morphisms.* A morphism of $\mathsf{Arr}(\mathcal{C})$ from $f \colon A \longrightarrow B$ to $g \colon A' \longrightarrow B'$ is a pair (ϕ, ψ) consisting of
  - A morphism $\phi \colon A \longrightarrow A'$ of $\mathcal{C}$;
  - A morphism $\psi \colon B \longrightarrow B'$ of $\mathcal{C}$;

  commutative square of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

- *Identities.* For each $f \in \mathrm{Obj}(\mathsf{Arr}(\mathcal{C}))$, the unit map

$$\mathbb{1}^{\mathsf{Arr}(\mathcal{C})}_{f} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{Arr}(\mathcal{C})}(f, f)$$

  of $\mathsf{Arr}(\mathcal{C})$ at f is defined by

$$\mathrm{id}_f \stackrel{\text{def}}{=} \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[r, "f"'] & B; \end{tikzcd}$$

- *Composition.* For each $f, g, h \in \mathrm{Obj}(\mathsf{Arr}(\mathcal{C}))$, the composition map

$$\circ^{\mathsf{Arr}(\mathcal{C})}_{f,g,h} \colon \mathrm{Hom}_{\mathsf{Arr}(\mathcal{C})}(g, h) \times \mathrm{Hom}_{\mathsf{Arr}(\mathcal{C})}(f, g) \longrightarrow \mathrm{Hom}_{\mathsf{Arr}(\mathcal{C})}(f, h)$$

of $\mathrm{Arr}(\mathcal{C})$ at (f, g, h) is defined by

$$\begin{pmatrix}\begin{tikzcd} A' \arrow[r, "g"] \arrow[d, "\phi_2"'] & B' \arrow[d, "\psi_2"] \\ A'' \arrow[r, "h"'] & B'' \end{tikzcd}\end{pmatrix} \circ \begin{pmatrix}\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi_1"'] & B \arrow[d, "\psi_1"] \\ A' \arrow[r, "g"'] & B' \end{tikzcd}\end{pmatrix} = \begin{pmatrix}\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi_1"'] & B \arrow[d, "\psi_1"] \\ A' \arrow[r, "g"] \arrow[d, "\phi_2"'] & B' \arrow[d, "\psi_2"] \\ A'' \arrow[r, "h"'] & B'' \end{tikzcd}\end{pmatrix}.$$

Proposition 7.6.4 ▶ Properties of Arrow Categories

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{Arr}(\mathcal{C})$ defines a functor
$$\mathrm{Arr}\colon \mathsf{Cats} \longrightarrow \mathsf{Cats}.$$
2. 2*-Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{Arr}(\mathcal{C})$ defines a 2-functor
$$\mathrm{Arr}\colon \mathsf{Cats}_2 \longrightarrow \mathsf{Cats}_2.$$
3. *Adjointness.* We have an adjunction
$$(- \times \mathbb{1} \dashv \mathrm{Arr})\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "- \times \mathbb{1}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathrm{Arr}"] \end{tikzcd}$$
witnessed by a bijection
$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathbb{1}, \mathcal{D}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathrm{Arr}(\mathcal{D})),$$
natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.
4. 2*-Adjointness.* We have a 2-adjunction
$$(- \times \mathbb{1} \dashv \mathrm{Arr})\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "- \times \mathbb{1}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathrm{Arr}"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Fun}(\mathcal{C} \times \mathbb{I}, \mathcal{D}) \cong \mathsf{Fun}(\mathcal{C}, \mathsf{Arr}(\mathcal{D})),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}_2)$.

5. *As a Comma Category.* We have an isomorphism of categories

$$\mathsf{Arr}(\mathcal{C}) \cong \mathrm{id}_{\mathcal{C}} \downarrow \mathrm{id}_{\mathcal{C}}.$$

Proof 7.6.5 ▶ Proof of Proposition 7.6.4

Item 1: Functoriality

This is a special case of Categories, Item 1 of Proposition 2.3.2.

Item 2: 2-Functoriality

This is a special case of Categories, Item 2 of Proposition 2.3.2.

Item 3: Adjointness

This is a special case of Categories, Item 4 of Proposition 2.3.2.

Item 4: 2-Adjointness

This is a special case of Categories, Item 3 of Proposition 2.3.2.

Item 5: As a Comma Category

Omitted. □

7.7 The Funny Tensor Product

7.7.1 Separately Functorial Bifunctors

Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

Definition 7.7.1 ▶ Separately Functorial Bifunctors

A **separately functorial bifunctor** $F\colon \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$ **from** $\mathcal{C} \times \mathcal{D}$ **to** $\mathcal{E}$ consists of

1. *Action on Objects.* A map of sets

$$F\colon \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D}) \longrightarrow \mathrm{Obj}(\mathcal{E}),$$

called the **action on objects of** F;

2. *Left Action on* Hom*-sets.* For each $B \in \mathrm{Obj}(\mathcal{D})$ and each $A, A' \in \mathrm{Obj}(\mathcal{C})$, a map

$$F^{\mathsf{L}}_{A,A'|B} \colon \mathrm{Hom}_{\mathcal{C}}(A, A') \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(A, B), F(A', B)),$$

called the **left action on** Hom**-sets of** F **at** (A, A');

3. *Right Action on* Hom*-sets.* For each $A \in \mathrm{Obj}(\mathcal{C})$ and each $B, B' \in \mathrm{Obj}(\mathcal{D})$, a map

$$F^{\mathsf{R}}_{A|B,B'} \colon \mathrm{Hom}_{\mathcal{D}}(B, B') \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(A, B), F(A, B')),$$

called the **right action on** Hom**-sets of** F **at** (B, B');

satisfying the following conditions:

1. *Left Preservation of Composition.* For each $A, A', A'' \in \mathrm{Obj}(\mathcal{C})$ and each $B \in \mathrm{Obj}(\mathcal{D})$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A', A'') \times \mathrm{Hom}_{\mathcal{C}}(A, A') \arrow[r, "\circ^{\mathcal{C}}_{A,A',A''}"] \arrow[d, "F^{\mathsf{L}}_{A',A''|B} \times F^{\mathsf{L}}_{A,A'|B}"'] & \mathrm{Hom}_{\mathcal{C}}(A, A'') \arrow[d, "F^{\mathsf{L}}_{A,A''|B}"] \\
\mathrm{Hom}_{\mathcal{E}}(F_{A',B}, F_{A'',B}) \times \mathrm{Hom}_{\mathcal{E}}(F_{A,B}, F_{A',B}) \arrow[r, "\circ^{\mathcal{E}}_{F_{A,B},F_{A',B},F_{A'',B}}"'] & \mathrm{Hom}_{\mathcal{E}}(F_{A,B}, F_{A'',B})
\end{tikzcd}$$

commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$F(g \circ f, \mathrm{id}_B) = F(g, \mathrm{id}_B) \circ F(f, \mathrm{id}_B).$$

2. *Right Preservation of Composition.* For each $A \in \mathrm{Obj}(\mathcal{C})$ and each $B, B', B'' \in \mathrm{Obj}(\mathcal{D})$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(B', B'') \times \mathrm{Hom}_{\mathcal{C}}(B, B') \arrow[r, "\circ^{\mathcal{D}}_{B,B',B''}"] \arrow[d, "F^{\mathsf{R}}_{A|B',B''} \times F^{\mathsf{R}}_{A|B,B'}"'] & \mathrm{Hom}_{\mathcal{D}}(B, B'') \arrow[d, "F^{\mathsf{R}}_{A|B,B''}"] \\
\mathrm{Hom}_{\mathcal{E}}(F_{A,B'}, F_{A,B''}) \times \mathrm{Hom}_{\mathcal{E}}(F_{A,B}, F_{A,B'}) \arrow[r, "\circ^{\mathcal{E}}_{F_{A,B},F_{A,B'},F_{A,B''}}"'] & \mathrm{Hom}_{\mathcal{E}}(F_{A,B}, F_{A,B''})
\end{tikzcd}$$

commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{D}$, we have

$$F(\mathrm{id}_A, g \circ f) = F(\mathrm{id}_A, g) \circ F(\mathrm{id}_A, f).$$

3. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$ and each $B \in \mathrm{Obj}(\mathcal{D})$, the diagrams

$$\begin{tikzcd} \mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, "\mathbb{1}^{\mathcal{E}}_{F_{A,B}}"] & \\ \mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "F^{\mathsf{L}}_{A,A|B}"'] & \mathrm{Hom}_{\mathcal{E}}(F_{A,B},F_{A,B}) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{D}}_{B}"'] \arrow[dr, "\mathbb{1}^{\mathcal{E}}_{F_{A,B}}"] & \\ \mathrm{Hom}_{\mathcal{D}}(B,B) \arrow[r, "F^{\mathsf{R}}_{A|B,B}"'] & \mathrm{Hom}_{\mathcal{E}}(F_{A,B},F_{A,B}) \end{tikzcd}$$

commute, i.e. we have

$$F(\mathrm{id}_A, \mathrm{id}_B) = \mathrm{id}_{F_{A,B}}.$$

Definition 7.7.2 ▶ Categories of Separately Bilinear Bifunctors

The **category of separately bifunctorial bifunctors from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$** is the category $\mathrm{Bil}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ where

- *Objects.* The objects of $\mathrm{Bil}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ are separately bifunctorial bifunctors from $\mathcal{C}$ to $\mathcal{D}$;
- *Morphisms.* The morphisms of $\mathrm{Bil}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ are unnatural transformations;
- *Identities.* The identities of $\mathrm{Bil}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ are given by the identity unnatural transformations;
- *Composition.* The composition maps of $\mathrm{Bil}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ are given by composition of unnatural transformations.

7.7.2 The Funny Tensor Product

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 7.7.3 ▶ The Funny Tensor Product

The **funny tensor product of $\mathcal{C}$ and $\mathcal{D}$** is the category $\mathcal{C} \square \mathcal{D}$ such that we have an isomorphism of categories

$$\mathrm{Fun}(\mathcal{C} \square \mathcal{D}, \mathcal{E}) \cong \mathrm{Bil}(\mathcal{C} \times \mathcal{D}, \mathcal{E}),$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

Construction 7.7.4 ▶ Construction of the Funny Tensor Product

Explicitly, the **funny tensor product of** $\mathcal{C}$ **and** $\mathcal{D}$ is the category $\mathcal{C} \,\square\, \mathcal{D}$ given by

$$\mathcal{C} \,\square\, \mathcal{D} \overset{\text{def}}{=} (\mathcal{C}_0 \times \mathcal{D}) \coprod_{\mathcal{C}_0 \times \mathcal{D}_0} (\mathcal{C} \times \mathcal{D}_0), \qquad \begin{tikzcd} \mathcal{C} \,\square\, \mathcal{D} \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{C} \times \mathcal{D}_0 \arrow[l] \\ \mathcal{C}_0 \times \mathcal{D} \arrow[u] & \mathcal{C}_0 \times \mathcal{D}_0, \arrow[l] \arrow[u] \end{tikzcd}$$

where $\mathcal{C}_0$ and $\mathcal{D}_0$ are the discrete categories on $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Obj}(\mathcal{D})$.

Remark 7.7.5 ▶ Unwinding Construction 7.7.4

In detail, the **funny tensor product of** $\mathcal{C}$ **and** $\mathcal{D}$ is the category $\mathcal{C} \,\square\, \mathcal{D}$ where

- *Objects*. We have[1]

$$\mathrm{Obj}(\mathcal{C} \,\square\, \mathcal{D}) \overset{\text{def}}{=} \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D});$$

- *Morphisms*. For each $A \,\square\, B, A' \,\square\, B' \in \mathrm{Obj}(\mathcal{C} \,\square\, \mathcal{D})$, the morphisms of $\mathcal{C} \,\square\, \mathcal{D}$ from $A \,\square\, B$ to $A' \,\square\, B'$ are freely generated under composition by pairs of the form[2]

$$\begin{aligned} f \,\square\, B &\colon A \,\square\, B \longrightarrow A' \,\square\, B, \\ A \,\square\, g &\colon A \,\square\, B \longrightarrow A \,\square\, B' \end{aligned}$$

  consisting of

  - An object A of $\mathcal{C}$;
  - An object B of $\mathcal{D}$;
  - A morphism $f \colon A \longrightarrow A'$ of $\mathcal{C}$;
  - A morphism $g \colon B \longrightarrow B'$ of $\mathcal{D}$;

  subject to the following relations:

  1. *Identities*. For each $A \in \mathrm{Obj}(\mathcal{C})$ and each $B \in \mathrm{Obj}(\mathcal{D})$, we have

$$\begin{aligned} \mathrm{id}_A \,\square\, B &= \mathrm{id}_{A \square B}, \\ A \,\square\, \mathrm{id}_B &= \mathrm{id}_{A \square B}. \end{aligned}$$

2. *Left Composition.* For each composable pair

$$\begin{aligned} f &\colon A \longrightarrow A', \\ f' &\colon A' \longrightarrow A'' \end{aligned}$$

of morphisms of $\mathcal{C}$ and each $B \in \mathrm{Obj}(\mathcal{D})$, we have

$$(f' \mathbin{\square} B) \circ (f \mathbin{\square} B) = (f' \circ f) \mathbin{\square} B.$$

3. *Right Composition.* For each composable pair

$$\begin{aligned} g &\colon B \longrightarrow B', \\ g' &\colon B' \longrightarrow B'' \end{aligned}$$

of morphisms of $\mathcal{D}$ and each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$(A \mathbin{\square} g') \circ (A \mathbin{\square} g) = A \mathbin{\square} (g' \circ g).$$

- *Identities.* For each $A \mathbin{\square} B \in \mathrm{Obj}(\mathcal{C} \mathbin{\square} \mathcal{D})$, the unit map

$$\mathbb{1}^{\mathcal{C} \square \mathcal{D}}_{A \square B} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C} \square \mathcal{D}}(A \mathbin{\square} B, A \mathbin{\square} B)$$

of $\mathcal{C} \mathbin{\square} \mathcal{D}$ at $A \mathbin{\square} B$ is defined by

$$\mathrm{id}^{\mathcal{C} \square \mathcal{D}}_{A \square B} \overset{\mathrm{def}}{=} \mathrm{id}_{A \square B};$$

- *Composition.* For each $\mathbf{X} = A \mathbin{\square} B, \mathbf{X}' = A' \mathbin{\square} B', \mathbf{X}'' = A'' \mathbin{\square} B'' \in \mathrm{Obj}(\mathcal{C} \mathbin{\square} \mathcal{D})$, the composition map

$$\circ^{\mathcal{C} \square \mathcal{D}}_{\mathbf{X},\mathbf{X}',\mathbf{X}''} \colon \mathrm{Hom}_{\mathcal{C} \square \mathcal{D}}(\mathbf{X}', \mathbf{X}'') \times \mathrm{Hom}_{\mathcal{C} \square \mathcal{D}}(\mathbf{X}, \mathbf{X}') \longrightarrow \mathrm{Hom}_{\mathcal{C} \square \mathcal{D}}(\mathbf{X}, \mathbf{X}'')$$

of $\mathcal{C} \mathbin{\square} \mathcal{D}$ at $(A \mathbin{\square} B, A' \mathbin{\square} B', A'' \mathbin{\square} B'')$ is defined by

$$\psi \circ^{\mathcal{C} \square \mathcal{D}}_{\mathbf{X},\mathbf{X}',\mathbf{X}''} \phi \overset{\mathrm{def}}{=} [((f_1' \circ f_1) \mathbin{\square} B) \circ (A \mathbin{\square} (g_1' \circ g_1)) \circ \cdots \circ ((f_1' \circ f_1) \mathbin{\square} B') \circ (A' \mathbin{\square} (g_1' \circ g_1))],$$

for each

$$\begin{aligned} \phi &= \left[(f_1 \mathbin{\square} B) \circ (A \mathbin{\square} g_1) \circ \cdots \circ (f_1 \mathbin{\square} B') \circ (A' \mathbin{\square} g_1)\right] \in \mathrm{Hom}_{\mathcal{C} \square \mathcal{D}}(\mathbf{X}, \mathbf{X}'), \\ \psi &= \left[(f_1' \mathbin{\square} B) \circ (A \mathbin{\square} g_1') \circ \cdots \circ (f_1' \mathbin{\square} B') \circ (A' \mathbin{\square} g_1')\right] \in \mathrm{Hom}_{\mathcal{C} \square \mathcal{D}}(\mathbf{X}, \mathbf{X}''). \end{aligned}$$

[1] *Further Notation:* We write $A \mathbin{\square} B$ for a pair $(A, B) \in \mathrm{Obj}(\mathcal{C} \mathbin{\square} \mathcal{D})$.

[2] *Further Terminology:* The morphisms of $\mathcal{C} \mathbin{\square} \mathcal{D}$ of the form $f \mathbin{\square} B$ and $A \mathbin{\square} g$ are called the **basic morphisms of $\mathcal{C} \mathbin{\square} \mathcal{D}$**.

Example 7.7.6 ▶ Tensoring With Sets

Given a set X and a category $\mathcal{C}$, we have isomorphisms

$$\begin{aligned} X \odot \mathcal{C} &\cong X_{\mathsf{disc}} \times \mathcal{C} \\ &\cong X_{\mathsf{disc}} \mathbin{\square} \mathcal{C}. \end{aligned}$$

Proposition 7.7.7 ▶ Properties of the Funny Tensor Product

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. *Functoriality.* The assignments $\mathcal{C}, \mathcal{D}, (\mathcal{C}, \mathcal{D}) \mapsto \mathcal{C} \mathbin{\square} \mathcal{D}$ define functors

$$\begin{aligned} \mathcal{C} \mathbin{\square} - \colon \mathsf{Cats} &\longrightarrow \mathsf{Cats}, \\ - \mathbin{\square} \mathcal{D} \colon \mathsf{Cats} &\longrightarrow \mathsf{Cats}, \\ -_1 \mathbin{\square} -_2 \colon \mathsf{Cats} \times \mathsf{Cats} &\longrightarrow \mathsf{Cats}. \end{aligned}$$

2. *Adjointness.* We have adjunctions

$$\left(\mathcal{C} \mathbin{\square} - \dashv \mathsf{Fun}^{\mathsf{unnat}}(\mathcal{C}, -)\right) \colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathcal{C} \mathbin{\square} -"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}^{\mathsf{unnat}}(\mathcal{C}, -)"] \end{tikzcd}$$

$$\left(- \mathbin{\square} \mathcal{D} \dashv \mathsf{Fun}^{\mathsf{unnat}}(\mathcal{D}, -)\right) \colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "- \mathbin{\square} \mathcal{D}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}^{\mathsf{unnat}}(\mathcal{D}, -)"] \end{tikzcd}$$

where $\mathsf{Fun}^{\mathsf{unnat}}(\mathcal{C}, \mathcal{D})$ is the category of functors and unnatural transformations from $\mathcal{C}$ to $\mathcal{D}$, witnessed by isomorphisms of categories

$$\begin{aligned} \mathsf{Fun}(\mathcal{C} \mathbin{\square} \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}\left(\mathcal{D}, \mathsf{Fun}^{\mathsf{unnat}}(\mathcal{C}, \mathcal{E})\right), \\ \mathsf{Fun}(\mathcal{C} \mathbin{\square} \mathcal{D}, \mathcal{E}) &\cong \mathsf{Fun}\left(\mathcal{C}, \mathsf{Fun}^{\mathsf{unnat}}(\mathcal{D}, \mathcal{E})\right), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof 7.7.8 ▶ Proof of Proposition 7.7.7

Item 1: Functoriality

Omitted.

Item 2: Adjointness

Omitted.

7.8 The Category of Simplices of a Category

Let $\mathcal{C}$ be a category.

Definition 7.8.1 ▶ The Category of Simplices of a Category

The **category of simplices of** $\mathcal{C}$ is the category $\int^{\Delta}\mathcal{C}$ defined by

$$\int^{\Delta}\mathcal{C} \overset{\text{def}}{=} \int^{\Delta}\mathrm{N}_{\bullet}(\mathcal{C}),$$

where $\int^{\Delta}\mathrm{N}_{\bullet}(\mathcal{C})$ is the category of simplices of $\mathrm{N}_{\bullet}(\mathcal{C})$ of Simplicial Objects, **??**.

Remark 7.8.2 ▶ Unwinding Definition 7.8.1

In detail, the **category of simplices of** $\mathcal{C}$ is the category $\int^{\Delta}\mathcal{C}$ where

- *Objects.* The objects of $\int^{\Delta}\mathcal{C}$ are pairs $([n], F)$ consisting of
  - An object $[n]$ of Δ;
  - A functor $F\colon \mathsf{n} \longrightarrow \mathcal{C}$;
- *Morphisms.* A morphism of $\int^{\Delta}\mathcal{C}$ from $([n], F)$ to $([m], G)$ is a morphism $\phi\colon [n] \longrightarrow [m]$ of Δ making the diagram

$$\begin{tikzcd}
\mathsf{n} \arrow[rr, "\phi"] \arrow[dr, "F"'] & & \mathsf{m} \arrow[dl, "G"] \\
& \mathcal{C} &
\end{tikzcd}$$

  commute;
- *Identities.* For each $([n], F) \in \mathrm{Obj}\left(\int^{\Delta}\mathcal{C}\right)$, the unit map

$$\mathbb{1}^{\int^{\Delta}\mathcal{C}}_{([n],F)}\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\int^{\Delta}\mathcal{C}}(([n], F), ([n], F))$$

  of $\int^{\Delta}\mathcal{C}$ at $([n], F)$ is defined by

$$\mathrm{id}^{\int^{\Delta}\mathcal{C}}_{([n],F)} \overset{\text{def}}{=} \mathrm{id}_{[n]};$$

- *Composition.* For each $\mathbf{A} = ([n], F), \mathbf{B} = ([m], G), \mathbf{C} = ([p], H) \in \mathrm{Obj}\left(\int^{\Delta} \mathcal{C}\right)$, the composition map
$$\circ^{\int^{\Delta}\mathcal{C}}_{\mathbf{A},\mathbf{B},\mathbf{C}} \colon \mathrm{Hom}_{\int^{\Delta}\mathcal{C}}(\mathbf{B},\mathbf{C}) \times \mathrm{Hom}_{\int^{\Delta}\mathcal{C}}(\mathbf{A},\mathbf{B}) \longrightarrow \mathrm{Hom}_{\int^{\Delta}\mathcal{C}}(\mathbf{A},\mathbf{C})$$
of $\int^{\Delta} \mathcal{C}$ at $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is defined by
$$\psi \circ^{\int^{\Delta}\mathcal{C}}_{\mathbf{A},\mathbf{B},\mathbf{C}} \phi \overset{\text{def}}{=} \psi \circ^{\Delta}_{[n],[m],[p]} \phi,$$
as in the diagram
$$\begin{tikzcd} n \arrow[r, "\phi"] \arrow[dr, "F"'] & m \arrow[d, "G"] \arrow[r, "\psi"] & p \arrow[dl, "H"] \\ & \mathcal{C}. & \end{tikzcd}$$

Definition 7.8.3 ▶ The Forgetful Functor From the Category of Simplices

The **the forgetful functor from the category of simplices of** $\mathcal{C}$ is the functor
$$q \colon \int^{\Delta} \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$$
where

- *Action on Objects.* For each $([n], F) \in \mathrm{Obj}\left(\int^{\Delta} \mathcal{C}\right)$, we have
$$q([n], F) \overset{\text{def}}{=} (F_0, F_n);$$
- *Action on Morphisms.* For each morphism $\phi \colon [n] \longrightarrow [m]$ of Δ, the image
$$q(\phi) \colon \underbrace{q([n], F)}_{(F_0, F_n)} \longrightarrow \underbrace{q([m], G)}_{(G_0, G_m)}$$
of ϕ by q is defined by
$$q(\phi) \overset{\text{def}}{=} \left(q_{\phi,0}, q_{\phi,n}\right)$$
where

- $q_{\phi,0} \colon F_0 \longrightarrow G_0$ is the composition

$$G_0 \xrightarrow{G_{i_{0,\phi_0}}} G_{\phi_0} = F_0$$

  in C^{op}, where
  - i_{0,ϕ_0} is the unique morphism of m from 0 to ϕ_0;
  - We have $G_{\phi_0} = F_0$ since ϕ is a morphism of $\int^{\Delta} C$;
- $q_{\phi,n} \colon F_n \longrightarrow G_m$ is the composition

$$F_n = G_{\phi_n} \xrightarrow{G_{i_{\phi_n,m}}} \stackrel{\text{def}}{=} G_m$$

  in C^{op}, where
  - We have $F_n = G_{\phi_n}$ since ϕ is a morphism of $\int^{\Delta} C$;
  - $i_{\phi_n,m}$ is the unique morphism of m from ϕ_n to m.

8 Endomorphisms, Automorphisms, Involutions, and Idempotents

8.1 Endomorphisms in Categories

8.1.1 Foundations

Let C be a category.

Definition 8.1.1 ▶ Endomorphisms in Categories

An **endomorphism in** C is a functor $\phi \colon \mathrm{B}\mathbb{N} \longrightarrow C$.

Remark 8.1.2 ▶ Unwinding Definition 8.1.1

In detail, an **endomorphism in** C is a pair (A, ϕ) consisting of

- *The Underlying Object.* An object A of C;
- *The Endomorphism.* A morphism $\phi \colon A \longrightarrow A$ of C.

Proof 8.1.3 ▶ Proof of Remark 8.1.2

Indeed, a functor $\phi\colon \mathrm{B}\mathbb{N} \longrightarrow \mathcal{C}$ consists of

- *Action on Objects.* A map of sets

$$\phi_0\colon \underbrace{\mathrm{Obj}(\mathrm{B}\mathbb{N})}_{\stackrel{\mathrm{def}}{=}\mathrm{pt}} \longrightarrow \mathrm{Obj}(\mathcal{C})$$

  picking an object A of $\mathcal{C}$;

- *Action on Morphisms.* A map of sets

$$\phi_{\star,\star}\colon \underbrace{\mathrm{Hom}_{\mathrm{B}\mathbb{N}}(\star,\star)}_{\stackrel{\mathrm{def}}{=}\mathbb{N}} \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,A);$$

preserving composition and identities. This makes $\phi_{\star,\star}$ into a morphism of monoids

$$\phi_{\star,\star}\colon \underbrace{\left(\mathrm{Hom}_{\mathrm{B}\mathbb{N}}(\star,\star), \circ^{\mathrm{B}\mathbb{N}}_{\star,\star,\star}, \mathbb{1}^{\mathrm{B}\mathbb{N}}_{\star}\right)}_{\stackrel{\mathrm{def}}{=}(\mathbb{N},+,0)} \longrightarrow (\mathrm{Hom}_{\mathcal{C}}(A,A), \circ, \mathrm{id}_A),$$

determining and being determined by, via Monoids, Item 2 of Proposition 1.1.10, an element $\phi\colon A \longrightarrow A$ of $\mathrm{Hom}_{\mathcal{C}}(A,A)$. ∎

Definition 8.1.4 ▶ Morphisms of Endomorphisms in Categories

A **morphism of endomorphisms in** $\mathcal{C}$ from ϕ to ψ is a natural transformation $\alpha\colon \phi \Longrightarrow \psi$ of functors from $\mathrm{B}\mathbb{N}$ to $\mathcal{C}$.

Remark 8.1.5 ▶ Unwinding Definition 8.1.4

In detail, a **morphism of endomorphisms in** $\mathcal{C}$ from (A, ϕ) to (B, ψ) is a morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

Definition 8.1.6 ▶ The Category of Endomorphisms in a Category

The **category of endomorphisms in** $\mathcal{C}$ is the category $\mathrm{End}(\mathcal{C})$[1,2] defined by

$$\mathrm{End}(\mathcal{C}) \overset{\text{def}}{=} \mathrm{Fun}(\mathrm{B}\mathbb{N}, \mathcal{C}).$$

[1] *Further Notation:* Also written $\mathcal{C}^{\circlearrowleft}$.

[2] Since $\mathrm{B}\mathbb{N}$ may be thought of as a categorical realisation of the "directed circle", we also write $\mathcal{L}^{\mathrm{dir}}(\mathcal{C})$ for $\mathrm{End}(\mathcal{C})$, which we may view as a "**categorical free directed loop space**" **of** $\mathcal{C}$.

Homotopy-theoretic information about $\mathcal{L}^{\mathrm{dir}}(\mathcal{C})$ is often not of much interest, however, as many categories commonly appearing in practice tend to be contractible for reasons which also hold true for categories of functors into them (as is the case of $\mathcal{L}^{\mathrm{dir}}(\mathcal{C}) \overset{\text{def}}{=} \mathrm{Fun}(\mathrm{B}\mathbb{N}, \mathcal{C})$), such as admitting initial/final objects or binary co/products.

Remark 8.1.7 ▶ Unwinding Definition 8.1.6

In detail, the **category of endomorphisms in** $\mathcal{C}$ is the category $\mathrm{End}(\mathcal{C})$ where

- *Objects.* The objects of $\mathrm{End}(\mathcal{C})$ are endomorphisms in $\mathcal{C}$;
- *Morphisms.* The morphisms of $\mathrm{End}(\mathcal{C})$ are morphisms of endomorphisms in $\mathcal{C}$;
- *Identities.* For each $(A, \phi) \in \mathrm{Obj}(\mathrm{End}(\mathcal{C}))$, the unit map

$$\mathbb{1}^{\mathrm{End}(\mathcal{C})}_{(A,\phi)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathrm{End}(\mathcal{C})}((A, \phi), (A, \phi))$$

of $\mathrm{End}(\mathcal{C})$ at (A, ϕ) is defined by

$$\mathrm{id}^{\mathrm{End}(\mathcal{C})}_{(A,\phi)} \overset{\text{def}}{=} \mathrm{id}_A;$$

- *Composition*. For each $(A, \phi), (B, \psi), (C, \chi) \in \mathrm{Obj}(\mathsf{End}(\mathcal{C}))$, the composition map

$$\circ^{\mathsf{End}(\mathcal{C})}_{\phi,\psi,\chi} \colon \mathrm{Hom}_{\mathsf{End}(\mathcal{C})}(\psi, \chi) \times \mathrm{Hom}_{\mathsf{End}(\mathcal{C})}(\phi, \psi) \longrightarrow \mathrm{Hom}_{\mathsf{End}(\mathcal{C})}(\phi, \chi)$$

of $\mathsf{End}(\mathcal{C})$ at $(A, \phi), (B, \psi), (C, \chi)$ is defined by

$$g \circ^{\mathsf{End}(\mathcal{C})}_{\phi,\psi,\chi} f \stackrel{\text{def}}{=} g \circ f.$$

Proposition 8.1.8 ▶ Properties of Categories of Endomorphisms

Let $\mathcal{C}$ be a category.

1. *Functoriality*. The assignment $\mathcal{C} \mapsto \mathsf{End}(\mathcal{C})$ defines a functor

$$\mathsf{End} \colon \mathsf{Cats} \longrightarrow \mathsf{Cats}.$$

2. 2*-Functoriality*. The assignment $\mathcal{C} \mapsto \mathsf{End}(\mathcal{C})$ defines a 2-functor

$$\mathsf{End} \colon \mathsf{Cats}_2 \longrightarrow \mathsf{Cats}_2.$$

3. *Adjointness I*. If $\mathcal{C}$ has products and coproducts, then we have a triple adjunction[1]

$$\left(\mathbb{N} \odot (-) \dashv 忘 \dashv \mathbb{N} \pitchfork (-)\right)\colon \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left=45, "\mathbb{N}\odot(-)"] \arrow[r, bend right=45, "\mathbb{N}\pitchfork(-)"'] & \mathsf{End}(\mathcal{C}) \arrow[l, "忘"'] \end{tikzcd},$$

where[2]

- $\mathbb{N} \odot (-) \colon \mathcal{C} \longrightarrow \mathsf{End}(\mathcal{C})$ is the functor defined on objects by

$$\begin{aligned} \mathbb{N} \odot (A) &\stackrel{\text{def}}{=} (\mathbb{N} \odot A, \mathbb{N} \odot \mathrm{id}_A) \\ &\cong \left(A^{\coprod \mathbb{N}}, \mathrm{id}_A^{\coprod \mathbb{N}}\right); \end{aligned}$$

(Weighted Category Theory, Construction 1.2.2)

- $忘 \colon \mathsf{End}(\mathcal{C}) \longrightarrow \mathcal{C}$ is the **forgetful functor from** $\mathsf{End}(\mathcal{C})$ **to** $\mathcal{C}$, defined on objects by

$$忘(A, \phi) \stackrel{\text{def}}{=} A;$$

- $\mathbb{N} \pitchfork (-) \colon \mathcal{C} \longrightarrow \mathsf{End}(\mathcal{C})$ is the functor defined on objects by

$$\begin{aligned}\mathbb{N} \pitchfork (A) &\overset{\text{def}}{=} (\mathbb{N} \pitchfork A, \mathbb{N} \pitchfork \mathrm{id}_A)\\ &\cong \left(A^{\times \mathbb{N}}, \mathrm{id}_A^{\times \mathbb{N}}\right).\end{aligned}$$

(Weighted Category Theory, Construction 1.2.2)

4. *Adjointness II.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction

$$(\mathrm{colim}^{\circlearrowleft} \dashv \iota \dashv \lim^{\circlearrowleft})\colon \quad \begin{tikzcd} \mathsf{End}(\mathcal{C}) \arrow[r, bend left=45, "\mathrm{colim}^{\circlearrowleft}"] \arrow[r, bend right=45, "\lim^{\circlearrowleft}"'] & \mathcal{C}, \arrow[l, "\iota"' description] \end{tikzcd}$$

where[3,4]

- $\mathrm{colim}^{\circlearrowleft} \colon \mathsf{End}(\mathcal{C}) \longrightarrow \mathcal{C}$ is the functor defined on objects by

$$\begin{aligned}\mathrm{colim}^{\circlearrowleft}(A, \phi) &\overset{\text{def}}{=} \mathrm{colim}\left(\mathsf{B}\mathbb{N} \xrightarrow{(A,\phi)} \mathcal{C}\right)\\ &\overset{\text{def}}{=} \mathrm{colim}(A \circlearrowleft \phi);\end{aligned}$$

- $\iota \colon \mathcal{C} \hookrightarrow \mathsf{End}(\mathcal{C})$ is the functor defined on objects by[5]

$$\iota(A) \overset{\text{def}}{=} (A, \mathrm{id}_A);$$

- $\lim^{\circlearrowleft} \colon \mathsf{End}(\mathcal{C}) \longrightarrow \mathcal{C}$ is the functor defined on objects by

$$\begin{aligned}\lim^{\circlearrowleft}(A, \phi) &\overset{\text{def}}{=} \lim\left(\mathsf{B}\mathbb{N} \xrightarrow{(A,\phi)} \mathcal{C}\right)\\ &\overset{\text{def}}{=} \lim(A \circlearrowleft \phi).\end{aligned}$$

5. 2-*Adjointness.* We have a 2-adjunction

$$(\mathsf{B}\mathbb{N} \times - \dashv \mathsf{End})\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathsf{B}\mathbb{N}\times -"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2. \arrow[l, bend left, "\mathsf{End}"] \end{tikzcd}$$

---

[1] Here $\mathcal{C} \cong \mathsf{Fun}(\mathrm{pt}, \mathcal{C})$, which we may think of as the "**category of identities of** $\mathcal{C}$".

[2] In a sense, $(\mathbb{N} \odot A, \mathbb{N} \odot \mathrm{id}_A)$ and $(\mathbb{N} \pitchfork A, \mathbb{N} \pitchfork \mathrm{id}_A)$ are the co/universal ways of producing an endomorphism starting with an identity.

[3] In a sense, $\mathrm{colim}^{\circlearrowleft}(A, \phi)$ and $\lim^{\circlearrowleft}(A, \phi)$ are the co/universal ways of producing an identity starting with an endomorphism.

[4]*Example:* Let $\mathcal{C} = \mathsf{Sets}$, let X be a set, and let $\phi\colon X \longrightarrow X$ be a morphism of sets. Then

$$\begin{aligned}\operatorname{colim}^{\circlearrowleft}(X,\phi) &\cong X/\sim,\\ \lim^{\circlearrowleft}(X,\phi) &\cong \{x \in X \mid \phi(x) = x\},\end{aligned}$$

where $\sim$ is the equivalence relation on X generated by declaring $x \sim y$ iff $\phi(x) = y$ for each $x, y \in X$.

[5]Viewing $\mathcal{C} \cong \mathsf{Fun}(\mathrm{pt}, \mathcal{C})$ as the "category of identities of $\mathcal{C}$", we see that the functor ι is just the inclusion of categories from the category of identities of $\mathcal{C}$ to the category of endomorphisms of $\mathcal{C}$.

PROOF 8.1.9 ▶ PROOF OF PROPOSITION 8.1.8

Item 1: Functoriality

Omitted.

Item 2: 2-Functoriality

Omitted.

Item 3: Adjointness I

We give two proofs, one via Kan extensions and the other by directly verifying that the functors form an adjunction.

Indeed, applying Kan Extensions, ?? of Proposition 1.1.6 to the functor $[\star]\colon \mathrm{pt} \twoheadrightarrow \mathsf{B}\mathbb{N}$, we obtain a triple adjunction

$$\left(\mathsf{Lan}_{[\star]} \dashv [\star]^* \dashv \mathsf{Ran}_{[\star]}\right):\quad \begin{tikzcd} \mathsf{Fun}(\mathrm{pt}, \mathcal{C}) \arrow[r, bend left=50, "\mathsf{Lan}_{[\star]}"] \arrow[r, bend right=50, "\mathsf{Ran}_{[\star]}"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C}) \arrow[l, "{[\star]^*}" description] \end{tikzcd}.$$

Here $\mathsf{Fun}(\mathrm{pt}, \mathcal{C}) \cong \mathcal{C}$ via ?? of ?? and $\mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C}) \overset{\text{def}}{=} \mathsf{End}(\mathcal{C})$ by definition. We claim that $\mathsf{Lan}_{[\star]} \cong \mathbb{N} \odot -$, $[\star]^* \cong$ 忘, and $\mathsf{Ran}_{[\star]} \cong \mathbb{N} \pitchfork (-)$:

- *Computing* $\mathsf{Lan}_{[\star]}$. Let A be an object of $\mathcal{C}$. By Kan Extensions, Item 4 of Proposition 1.1.6, we have

  $$\mathsf{Lan}_{[\star]}(A) \cong \operatorname{colim}\left([\star] \downarrow \underline{\star} \twoheadrightarrow \mathrm{pt} \xrightarrow{A} \mathcal{C}\right).$$

  Unwinding the description of $[\star] \downarrow \underline{\star}$ given in ??, we see that it is the category having the form

  $$\begin{tikzcd} \star \arrow[r, "1"] \arrow[d, "0"'] & \star \arrow[r, "1"] \arrow[d, "1"] & \star \arrow[r, "1"] \arrow[d, "2"] & \star \arrow[r, "1"] \arrow[d, "3"] & \star \arrow[r, "1"] \arrow[d, "4"] & \star \arrow[r, "1"] \arrow[d, "5"] & \cdots \\ \star \arrow[r, equal] & \star \arrow[r, equal] & \star \arrow[r, equal] & \star \arrow[r, equal] & \star \arrow[r, equal] & \star \arrow[r, equal] & \cdots. \end{tikzcd}$$

Moreover, the composition $[\star] \downarrow \underline{\star} \twoheadrightarrow \mathrm{pt} \xrightarrow{A} \mathcal{C}$ is given by the diagram in $\mathcal{C}$ having $\mathbb{N}$ factors of A, and thus its colimit is given by $A^{\coprod \mathbb{N}}$. Similarly, one sees that the endomorphism this object carries is $\mathrm{id}_A^{\coprod \mathbb{N}}$.

Alternatively, we may use Kan Extensions, Item 5 of Proposition 1.1.6 and directly compute $\mathrm{Lan}_{[\star]}(A)$:

$$\begin{aligned}
\mathrm{Lan}_{[\star]}(A) &\cong \int^{\star \in \mathrm{pt}} \mathrm{Hom}_{\mathsf{B}\mathbb{N}}(\star, \star) \odot A, \\
&\cong \int^{\star \in \mathrm{pt}} \mathbb{N} \odot A, \\
&\cong \mathbb{N} \odot A.
\end{aligned}$$

- *Computing* $[\star]^*$. Let (A, ϕ) be an object of $\mathrm{End}(\mathcal{C})$, viewed as a functor $\phi \colon \mathsf{B}\mathbb{N} \longrightarrow \mathcal{C}$. Then the composition

$$\mathrm{pt} \xrightarrow{[\star]} \mathsf{B}\mathbb{N} \xrightarrow{(A,\phi)} \mathcal{C}$$

corresponds precisely to A, and we see that $[\star]^* \cong \text{忘}$.

- *Computing* $\mathrm{Ran}_{[\star]}$. Let A be an object of $\mathcal{C}$. By Kan Extensions, Item 4 of Proposition 1.1.6, we have

$$\mathrm{Ran}_{[\star]}(A) \cong \lim\left(\underline{\star} \downarrow [\star] \twoheadrightarrow \mathrm{pt} \xrightarrow{A} \mathcal{C}\right).$$

Unwinding the description of $\underline{\star} \downarrow [\star]$ given in ??, we see that it is the category having the form

$$\begin{tikzcd}
\star \arrow[r, equal] \arrow[d, "0"'] & \star \arrow[r, equal] \arrow[d, dashed, "1"] & \star \arrow[r, equal] \arrow[d, dashed, "2"] & \star \arrow[r, equal] \arrow[d, dashed, "3"] & \star \arrow[r, equal] \arrow[d, dashed, "4"] & \star \arrow[r, equal] \arrow[d, dashed, "5"] & \cdots \\
\star \arrow[r, "1"'] & \star \arrow[r, "1"'] & \star \arrow[r, "1"'] & \star \arrow[r, "1"'] & \star \arrow[r, "1"'] & \star \arrow[r, "1"'] & \cdots .
\end{tikzcd}$$

Moreover, the composition $\underline{\star} \downarrow [\star] \twoheadrightarrow \mathrm{pt} \xrightarrow{A} \mathcal{C}$ is given by the diagram in $\mathcal{C}$ having $\mathbb{N}$ factors of A, and thus its limit is given by $A^{\times \mathbb{N}}$. Similarly, one sees that the endomorphism this object carries is $\mathrm{id}_A^{\times \mathbb{N}}$.

Alternatively, we may use Kan Extensions, Item 5 of Proposition 1.1.6 and directly compute $\mathrm{Ran}_{[\star]}(A)$:

$$\mathrm{Ran}_{[\star]}(A) \cong \int_{\star \in \mathrm{pt}} \mathrm{Hom}_{\mathsf{B}\mathbb{N}}(\star, \star) \pitchfork A,$$

$$\begin{aligned}
&\cong \int_{\star\in\mathrm{pt}} \mathbb{N} \pitchfork A, \\
&\cong \mathbb{N} \pitchfork A.
\end{aligned}$$

We may also just explicitly verify that the stated adjunction holds (we give a partial proof, not verifying naturality):

- *The Adjunction* $\mathbb{N} \odot (-) \dashv$ 忘. Given $A \in \mathrm{Obj}(C)$ and $(B, \phi) \in \mathrm{Obj}(\mathrm{End}(C))$, we have a bijection

$$\mathrm{Hom}_{\mathrm{End}(C)}((\mathbb{N} \odot A, \mathbb{N} \odot \mathrm{id}_A), (B, \phi)) \cong \mathrm{Hom}_C(A, B).$$

  Indeed, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{End}(C)}((\mathbb{N} \odot A, \mathbb{N} \odot \mathrm{id}_A), (B, \phi)) &\cong \mathrm{Hom}_{\mathrm{End}(C)}\left(\left(A^{\coprod\mathbb{N}}, \mathrm{id}_A^{\coprod\mathbb{N}}\right), (B, \phi)\right) \\
&\cong \mathrm{Hom}_{\mathrm{End}(C)}((A, \mathrm{id}_A), (B, \phi))^{\times\mathbb{N}},
\end{aligned}$$

  and hence a morphism $(\mathbb{N} \odot A, \mathbb{N} \odot \mathrm{id}_A) \longrightarrow (B, \phi)$ of $\mathrm{End}(C)$ is equivalently given by an $\mathbb{N}$-indexed collection

$$\{f_n \colon A \longrightarrow B\}_{n\in\mathbb{N}}$$

  of morphisms of C such that, for each $n \in \mathbb{N}$, the diagram

$$\begin{tikzcd}
A \arrow[r, "f_n"] \arrow[d, equal] & B \arrow[d, "\phi"] \\
A \arrow[r, "f_{n+1}"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "f_n"] \arrow[dr, "f_{n+1}"'] & B \arrow[d, "\phi"] \\
 & B
\end{tikzcd}$$

  commutes. Now, given a morphism $f \colon A \longrightarrow B$ of C, we have a corresponding morphism

$$
\begin{tikzcd}
A \arrow[r, "f_0"] \arrow[d, equal] & B \arrow[d, "\phi"] \\
A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[d, "\phi"] \\
A \arrow[r, "f_2"] \arrow[d, equal] & B \arrow[d, "\phi"] \\
A \arrow[r, "f_3"] \arrow[d, equal] & B \arrow[d, "\phi"] \\
\vdots & \vdots
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "f_0"] \arrow[dr, "f_1"] \arrow[ddr, "f_2"] \arrow[dddr, "f_3"] \arrow[ddddr, "f_4"] & B \arrow[d, "\phi"] \\
& B \arrow[d, "\phi"] \\
\cdots & B \arrow[d, "\phi"] \\
& B \arrow[d, "\phi"] \\
& \vdots
\end{tikzcd}
$$

of $\mathsf{End}(\mathcal{C})$, and conversely every such morphism comes uniquely from a morphism of $\mathcal{C}$.

- *The Adjunction* 忘 $\dashv \mathbb{N} \pitchfork (-)$. Given $(A, \phi) \in \mathrm{Obj}(\mathsf{End}(\mathcal{C}))$ and $B \in \mathrm{Obj}(\mathcal{C})$, we have a bijection

$$\mathrm{Hom}_{\mathsf{End}(\mathcal{C})}((A, \phi), (\mathbb{N} \pitchfork B, \mathbb{N} \pitchfork \mathrm{id}_B)) \cong \mathrm{Hom}_{\mathcal{C}}(A, B).$$

Indeed, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{End}(\mathcal{C})}((A, \phi), (\mathbb{N} \pitchfork B, \mathbb{N} \pitchfork \mathrm{id}_B)) &\cong \mathrm{Hom}_{\mathsf{End}(\mathcal{C})}\left((A, \phi), \left(B^{\times \mathbb{N}}, \mathrm{id}_B^{\times \mathbb{N}}\right)\right) \\
&\cong \mathrm{Hom}_{\mathsf{End}(\mathcal{C})}((A, \phi), (B, \mathrm{id}_B))^{\times \mathbb{N}},
\end{aligned}$$

and hence a morphism $(A, \phi) \longrightarrow (\mathbb{N} \pitchfork B, \mathbb{N} \pitchfork \mathrm{id}_B)$ of $\mathsf{End}(\mathcal{C})$ is equivalently given by an $\mathbb{N}$-indexed collection

$$\{f_n \colon A \longrightarrow B\}_{n \in \mathbb{N}}$$

of morphisms of $\mathcal{C}$ such that, for each $n \in \mathbb{N}$, the diagram

$$\begin{tikzcd} A \arrow[r, "f_n"] \arrow[d, "\phi"'] & B \arrow[d, equal] \\ A \arrow[r, "f_{n+1}"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f_n"] \arrow[d, "\phi"'] & B \\ A \arrow[ur, "f_{n+1}"'] & \end{tikzcd}$$

commutes. Now, given a morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$, we have a corresponding morphism

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "\phi"'] & B \arrow[d, equal] \\ A \arrow[r, "f_1"] \arrow[d, "\phi"'] & B \arrow[d, equal] \\ A \arrow[r, "f_2"] \arrow[d, "\phi"'] & B \arrow[d, equal] \\ A \arrow[r, "f_3"] \arrow[d, "\phi"'] & B \arrow[d, equal] \\ \vdots & \vdots \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "\phi"'] & B \\ A \arrow[ur, "f_1"] \arrow[d, "\phi"'] & \\ A \arrow[uur, "f_2"] \arrow[d, "\phi"'] & \ddots \\ A \arrow[uuur, "f_3"] \arrow[d, "\phi"'] & \\ \vdots \arrow[uuuur, "f_4"] & \end{tikzcd}$$

of $\mathsf{End}(\mathcal{C})$, and conversely every such morphism comes uniquely from a morphism of $\mathcal{C}$.

Item 4: Adjointness II

Indeed, applying Kan Extensions, ?? of Proposition 1.1.6 to the terminal functor $!\colon \mathsf{B}\mathbb{N} \twoheadrightarrow \mathrm{pt}$ from $\mathsf{B}\mathbb{N}$, we obtain a triple adjunction

$$(\mathrm{Lan}_! \dashv {!^*} \dashv \mathrm{Ran}_!)\colon \quad \begin{tikzcd} \mathsf{Fun}(\mathsf{B}\mathbb{N}, \mathcal{C}) \arrow[r, bend left=40, "\mathrm{Lan}_!"] \arrow[r, bend right=40, "\mathrm{Ran}_!"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{Fun}(\mathrm{pt}, \mathcal{C}). \arrow[l, "!^*"'] \end{tikzcd}$$

Here $\mathsf{Fun}(\mathsf{B}\mathbb{N}, C) \stackrel{\text{def}}{=} \mathsf{End}(C)$ by definition and $\mathsf{Fun}(\mathrm{pt}, C) \cong C$ via **??** of **??**. We claim that $\mathrm{Lan}_! \cong \mathrm{colim}^{\circlearrowleft}(\phi)$, $!^* \cong \iota$, and $\mathrm{Ran}_! \cong \lim^{\circlearrowleft}(\phi)$:

- *Computing* $\mathrm{Lan}_!$. Let (A, ϕ) be an object of $\mathsf{End}(C)$. By Kan Extensions, Item 4 of Proposition 1.1.6, we have

$$\mathrm{Lan}_!(A, \phi) \cong \mathrm{colim}\left(! \downarrow \underline{\star} \twoheadrightarrow \mathsf{B}\mathbb{N} \xrightarrow{(A,\phi)} C\right).$$

  Unwinding the description of $! \downarrow \underline{\star}$ given in **??**, we see that it is isomorphic to $\mathsf{B}\mathbb{N}$ via the functor $! \downarrow \underline{\star} \longrightarrow \mathsf{B}\mathbb{N}$. Thus $\mathrm{Lan}_! \cong \mathrm{colim}^{\circlearrowleft}$.

- *Computing* $!^*$. Let A be an object of C, viewed as a functor $[A]\colon \mathrm{pt} \longrightarrow C$. Then the composition

$$\mathsf{B}\mathbb{N} \xrightarrow{!} \mathrm{pt} \xrightarrow{A} C$$

  corresponds precisely to (A, id_A), and we see that $!^* \cong \iota$.

- *Computing* $\mathrm{Ran}_!$. Let (A, ϕ) be an object of $\mathsf{End}(C)$. By Kan Extensions, Item 4 of Proposition 1.1.6, we have

$$\mathrm{Ran}_!(A, \phi) \cong \lim\left(\underline{\star} \downarrow ! \twoheadrightarrow \mathsf{B}\mathbb{N} \xrightarrow{(A,\phi)} C\right).$$

  Unwinding the description of $\underline{\star} \downarrow !$ given in **??**, we see that it is isomorphic to $\mathsf{B}\mathbb{N}$ via the functor $\underline{\star} \downarrow ! \longrightarrow \mathsf{B}\mathbb{N}$. Thus $\mathrm{Ran}_! \cong \lim^{\circlearrowleft}$.

Item 5: 2-Adjointness

This is a special case of **??** of **??**.

8.1.2 The Endomorphism Monoid of an Object of a Category

Let C be a category, let $X \in \mathrm{Obj}(C)$, and let (C, X) be a category with a distinguished object.

Definition 8.1.10 ▶ The Endomorphism Monoid of an Object

The **endomorphism monoid of** X **in** C is the monoid $\mathrm{End}_C(X)$ consisting of

- *The Underlying Set.* The set $\mathrm{End}_{\mathcal{C}}(X)$ defined by

$$\mathrm{End}_{\mathcal{C}}(X) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(X, X);$$

- *The Multiplication Map.* The map of sets

$$\mu_{\mathrm{End}_{\mathcal{C}}(X)} \colon \underbrace{\mathrm{End}_{\mathcal{C}}(X) \times \mathrm{End}_{\mathcal{C}}(X)}_{\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(X,X) \times \mathrm{Hom}_{\mathcal{C}}(X,X)} \longrightarrow \underbrace{\mathrm{End}_{\mathcal{C}}(X)}_{\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(X,X)}$$

  defined by

$$\mu_{\mathrm{End}_{\mathcal{C}}(X)} \stackrel{\text{def}}{=} \circ^{\mathcal{C}}_{X,X,X};$$

- *The Unit Map.* The map of sets

$$\eta_{\mathrm{End}_{\mathcal{C}}(X)} \colon \mathrm{pt} \longrightarrow \underbrace{\mathrm{End}_{\mathcal{C}}(X)}_{\stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(X,X)}$$

  defined by

$$\eta_{\mathrm{End}_{\mathcal{C}}(X)} \stackrel{\text{def}}{=} \mathbb{k}^{\mathcal{C}}_{X}.$$

DEFINITION 8.1.11 ▶ THE ENDOMORPHISM MONOID OF A POINTED CATEGORY

The **endomorphism monoid of** $(\mathcal{C}, X)$ is the endomorphism monoid $\mathrm{End}_{\mathcal{C}}(X)$ of X in $\mathcal{C}$.

PROPOSITION 8.1.12 ▶ PROPERTIES OF ENDOMORPHISM MONOIDS

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $(\mathcal{C}, X) \mapsto \mathrm{End}_{\mathcal{C}}(X)$ defines a functor

$$\mathrm{End} \colon \mathsf{Cats}_* \longrightarrow \mathsf{Mon},$$

   where

   - *Action on Objects.* For each $(\mathcal{C}, X) \in \mathrm{Obj}(\mathsf{Cats}_*)$, we have

$$\mathrm{End}(\mathcal{C}, X) \stackrel{\text{def}}{=} \mathrm{End}_{\mathcal{C}}(X);$$

   - *Action on Morphisms.* For each morphism $F \colon (\mathcal{C}, X) \longrightarrow (\mathcal{D}, Y)$ of Cats_*, the image

$$\mathrm{End}(F) \colon \mathrm{End}_{\mathcal{C}}(X) \longrightarrow \mathrm{End}_{\mathcal{D}}(Y)$$

of F by End is defined by

$$\mathsf{End}(F) \overset{\text{def}}{=} F_{X,X}.$$

2. *Adjointness.* We have an adjunction

$$(\mathsf{B} \dashv \mathsf{End})\colon \quad \begin{tikzcd} \mathsf{Mon} \arrow[r, bend left, "\mathsf{B}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}_{*}, \arrow[l, bend left, "\mathsf{End}"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Cats}_{*}((\mathsf{B}A, \star), (\mathcal{C}, X)) \cong \mathsf{Mon}(A, \mathsf{End}_{\mathcal{C}}(X)),$$

natural in $A \in \mathrm{Obj}(\mathsf{Mon})$ and $(\mathcal{C}, X) \in \mathrm{Obj}(\mathsf{Cats}_{*})$.

3. *Interaction With Groupoids I: Functoriality.* The functor of Item 1 restricts to a functor

$$\mathsf{Aut}\colon \mathsf{Grpd}_{*} \longrightarrow \mathsf{Grp}.$$

4. *Interaction With Groupoids II: Adjointness.* The adjunction of Item 2 restricts to an adjunction

$$(\mathsf{B} \dashv \mathsf{Aut})\colon \quad \begin{tikzcd} \mathsf{Grp} \arrow[r, bend left, "\mathsf{B}"] \arrow[r, phantom, "\perp"] & \mathsf{Grpd}_{*}, \arrow[l, bend left, "\mathsf{Aut}"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Grpd}_{*}((\mathsf{B}G, \star), (\mathcal{C}, X)) \cong \mathsf{Grpd}(G, \mathsf{Aut}_{\mathcal{C}}(X)),$$

natural in $G \in \mathrm{Obj}(\mathsf{Grp})$ and $(\mathcal{C}, X) \in \mathrm{Obj}(\mathsf{Cats}_{*})$.

5. *Preservation of Limits.* The functor $\mathsf{End}\colon \mathsf{Cats}_{*} \longrightarrow \mathsf{Mon}$ of Item 1 preserves limits. In particular, we have isomorphisms of categories

$$\begin{aligned} \mathsf{End}_{\mathcal{C}\wedge\mathcal{D}}(*_{\mathcal{C}\wedge\mathcal{D}}) &\cong \mathsf{End}_{\mathcal{C}}(*_{\mathcal{C}}) \times \mathsf{End}_{\mathcal{D}}(*_{\mathcal{D}}), \\ \mathsf{End}_{\mathrm{Eq}(F,G)}(*_{\mathcal{C}}) &\cong \mathrm{Eq}(\mathsf{End}(F), \mathsf{End}(G)), \end{aligned}$$

natural in $(\mathcal{C}, *_{\mathcal{C}}), (\mathcal{D}, *_{\mathcal{D}}) \in \mathrm{Obj}(\mathsf{Cats}_{*})$ and parallel $F, G \in \mathrm{Mor}(\mathsf{Cats}_{*})$.

Proof 8.1.13 ▶ Proof of Proposition 8.1.12

Item 1: Functoriality

Clear.

Item 2: Adjointness

Omitted.

Item 3: Interaction With Groupoids I: Functoriality

Clear.

Item 4: Interaction With Groupoids II: Adjointness

Clear.

Item 5: Preservation of Limits

This follows from Item 2 and **??** of **??**.

8.2 Automorphisms in Categories

8.2.1 Foundations

Let $\mathcal{C}$ be a category.

Definition 8.2.1 ▶ Automorphisms in Categories

An **automorphism in** $\mathcal{C}$ is a functor $\phi\colon \mathsf{B}\mathbb{Z} \longrightarrow \mathcal{C}$.

Remark 8.2.2 ▶ Unwinding Definition 8.2.1

In detail, an **automorphism in** $\mathcal{C}$ is a pair (A, ϕ) consisting of[1]

- *The Underlying Object.* An object A of $\mathcal{C}$;
- *The Automorphism.* An isomorphism $\phi\colon A \xrightarrow{\cong} A$ in $\mathcal{C}$.

[1] In other words, an **automorphism in** $\mathcal{C}$ is an endomorphism of $\mathcal{C}$ which is additionally an isomorphism in $\mathcal{C}$.

Proof 8.2.3 ▶ Proof of Remark 8.2.2

Indeed, a functor $\phi\colon \mathsf{B}\mathbb{Z} \longrightarrow \mathcal{C}$ consists of

- *Action on Objects.* A map of sets

$$\phi_0 \colon \underbrace{\mathrm{Obj}(\mathrm{B}\mathbb{Z})}_{\stackrel{\mathrm{def}}{=}\mathrm{pt}} \longrightarrow \mathrm{Obj}(C)$$

  picking an object A of C;

- *Action on Morphisms.* A map of sets

$$\phi_{\star,\star} \colon \underbrace{\mathrm{Hom}_{\mathrm{B}\mathbb{Z}}(\star,\star)}_{\stackrel{\mathrm{def}}{=}\mathbb{Z}} \longrightarrow \mathrm{Hom}_C(A,A);$$

preserving composition and identities. This makes $\phi_{\star,\star}$ into a morphism of monoids

$$\phi_{\star,\star} \colon \underbrace{\left(\mathrm{Hom}_{\mathrm{B}\mathbb{Z}}(\star,\star), \circ^{\mathrm{B}\mathbb{Z}}_{\star,\star,\star}, \mathbb{1}^{\mathrm{B}\mathbb{Z}}_{\star}\right)}_{\stackrel{\mathrm{def}}{=}(\mathbb{Z},+,0)} \longrightarrow (\mathrm{Hom}_C(A,A), \circ, \mathrm{id}_A),$$

determining and being determined by, via Monoids, **??** of **??**, an invertible element $\phi \colon A \xrightarrow{\cong} A$ of $\mathrm{Hom}_C(A,A)$, i.e. an isomorphism in C from A to itself.

Definition 8.2.4 ▶ Morphisms of Automorphisms in Categories

A **morphism of automorphisms in** C from ϕ to ψ is a natural transformation $\alpha \colon \phi \Longrightarrow \psi$ of functors from $\mathrm{B}\mathbb{Z}$ to C.

Remark 8.2.5 ▶ Unwinding Definition 8.2.4

In detail, a **morphism of automorphisms in** C from (A,ϕ) to (B,ψ) is a morphism $f \colon A \longrightarrow B$ of C such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & B \arrow[d, "\psi"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

Definition 8.2.6 ▶ The Category of Automorphisms in a Category

The **category of automorphisms in** $\mathcal{C}$ is the category $\mathsf{Aut}(\mathcal{C})$[1] defined by

$$\mathsf{Aut}(\mathcal{C}) \overset{\text{def}}{=} \mathsf{Fun}(\mathrm{B}\mathbb{Z}, \mathcal{C}).$$

[1]Since $\mathrm{B}\mathbb{Z}$ may be thought of as a categorical realisation of the circle (as $|\mathrm{N}_\bullet(\mathrm{B}\mathbb{Z})| \simeq S^1$), we also write $\mathcal{L}(\mathcal{C})$ for $\mathsf{Aut}(\mathcal{C})$, which we may view as the **categorical free loop space of** $\mathcal{C}$.

Homotopy-theoretic information about $\mathcal{L}(\mathcal{C})$ is often not of much interest, however, as many categories commonly appearing in practice tend to be contractible for reasons which also hold true for categories of functors into them (as is the case of $\mathcal{L}(\mathcal{C}) \overset{\text{def}}{=} \mathsf{Fun}(\mathrm{B}\mathbb{Z}, \mathcal{C})$), such as admitting initial/final objects or binary co/products.

Remark 8.2.7 ▶ Unwinding Definition 8.2.6

In detail, the **category of automorphisms in** $\mathcal{C}$ is the category $\mathsf{Aut}(\mathcal{C})$ where

- *Objects*. The objects of $\mathsf{Aut}(\mathcal{C})$ are automorphisms in $\mathcal{C}$;
- *Morphisms*. The morphisms of $\mathsf{Aut}(\mathcal{C})$ are morphisms of automorphisms in $\mathcal{C}$;
- *Identities*. For each $(A, \phi) \in \mathrm{Obj}(\mathsf{Aut}(\mathcal{C}))$, the unit map

$$\mathbb{1}^{\mathsf{Aut}(\mathcal{C})}_{(A,\phi)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{Aut}(\mathcal{C})}((A, \phi), (A, \phi))$$

  of $\mathsf{Aut}(\mathcal{C})$ at (A, ϕ) is defined by

$$\mathrm{id}^{\mathsf{Aut}(\mathcal{C})}_{(A,\phi)} \overset{\text{def}}{=} \mathrm{id}_A;$$

- *Composition*. For each $(A, \phi), (B, \psi), (C, \chi) \in \mathrm{Obj}(\mathsf{Aut}(\mathcal{C}))$, the composition map

$$\circ^{\mathsf{Aut}(\mathcal{C})}_{\phi,\psi,\chi} \colon \mathrm{Hom}_{\mathsf{Aut}(\mathcal{C})}(\psi, \chi) \times \mathrm{Hom}_{\mathsf{Aut}(\mathcal{C})}(\phi, \psi) \longrightarrow \mathrm{Hom}_{\mathsf{Aut}(\mathcal{C})}(\phi, \chi)$$

  of $\mathsf{Aut}(\mathcal{C})$ at $(A, \phi), (B, \psi), (C, \chi)$ is defined by

$$g \circ^{\mathsf{Aut}(\mathcal{C})}_{\phi,\psi,\chi} f \overset{\text{def}}{=} g \circ f.$$

Proposition 8.2.8 ▶ Properties of Categories of Automorphisms

Let $\mathcal{C}$ be a category.[1]

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Aut}(\mathcal{C})$ defines a functor
$$\mathsf{Aut}\colon \mathsf{Cats} \longrightarrow \mathsf{Cats}.$$
2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Aut}(\mathcal{C})$ defines a 2-functor
$$\mathsf{Aut}\colon \mathsf{Cats}_2 \longrightarrow \mathsf{Cats}_2.$$
3. *Adjointness I.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction
$$\left(\chi^{\mathsf{L}} \dashv \iota \dashv \chi^{\mathsf{R}}\right)\colon \quad \begin{tikzcd} \mathsf{End}(\mathcal{C}) \arrow[r, bend left, "\chi^{\mathsf{L}}"] \arrow[r, bend right, "\chi^{\mathsf{R}}"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathsf{Aut}(\mathcal{C}), \arrow[l, "\iota"'] \end{tikzcd}$$
where[2,3]
   - $\chi^{\mathsf{L}}\colon \mathsf{End}(\mathcal{C}) \longrightarrow \mathsf{Aut}(\mathcal{C})$ is the functor defined on objects by
   $$\chi^{\mathsf{L}}(A,\phi) \overset{\text{def}}{=} \left(\chi^{\mathsf{L}}_{\phi}(A), \chi^{\mathsf{L}}(\phi)\right),$$
   where
      - $\chi^{\mathsf{L}}_{\phi}(A)$ is the object of $\mathcal{C}$ defined by
      $$\begin{aligned}\chi^{\mathsf{L}}_{\phi}(A) &\overset{\text{def}}{=} \mathrm{colim}\left(\cdots \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} \cdots\right)\\ &\cong \mathrm{colim}\left(A \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} \cdots\right);\end{aligned}$$
      - $\chi^{\mathsf{L}}(\phi)\colon \chi^{\mathsf{L}}_{\phi}(A) \longrightarrow \chi^{\mathsf{L}}_{\phi}(A)$ is the automorphism of $\chi^{\mathsf{L}}_{\phi}(A)$ obtained by applying functoriality of colimits (Limits and Colimits, Item 3 of Proposition 1.6.4) to the natural transformation of diagrams
      $$\begin{tikzcd} \cdots \arrow[r, "\phi"] & A \arrow[r, "\phi"] \arrow[d, "\phi"] & A \arrow[r, "\phi"] \arrow[d, "\phi"] & A \arrow[r, "\phi"] \arrow[d, "\phi"] & \cdots \\ \cdots \arrow[r, "\phi"'] & A \arrow[r, "\phi"'] & A \arrow[r, "\phi"'] & A \arrow[r, "\phi"'] & \cdots; \end{tikzcd}$$

- $\iota\colon \mathrm{Aut}(\mathcal{C}) \longrightarrow \mathrm{End}(\mathcal{C})$ is the fully faithful inclusion of categories defined on objects by

$$\iota(A,\phi) \stackrel{\text{def}}{=} (A,\phi);$$

- $\chi^{\mathsf{R}}\colon \mathrm{End}(\mathcal{C}) \longrightarrow \mathrm{Aut}(\mathcal{C})$ is the functor defined on objects by

$$\chi^{\mathsf{R}}(A,\phi) \stackrel{\text{def}}{=} \left(\chi^{\mathsf{R}}_{\phi}(A), \chi^{\mathsf{R}}(\phi)\right),$$

where

- $\chi^{\mathsf{R}}_{\phi}(A)$ is the object of $\mathcal{C}$ defined by

$$\begin{aligned}\chi^{\mathsf{R}}_{\phi}(A) &\stackrel{\text{def}}{=} \lim\left(\cdots \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} \cdots\right)\\ &\cong \lim\left(\cdots \xrightarrow{\phi} A \xrightarrow{\phi} A \xrightarrow{\phi} A\right);\end{aligned}$$

- $\chi^{\mathsf{R}}(\phi)\colon \chi^{\mathsf{R}}_{\phi}(A) \longrightarrow \chi^{\mathsf{R}}_{\phi}(A)$ is the automorphism of $\chi^{\mathsf{R}}_{\phi}(A)$ obtained by applying functoriality of limits (Limits and Colimits, Item 3 of Proposition 1.6.4) to the natural transformation of diagrams

$$\begin{tikzcd}\cdots \arrow[r, "\phi"] & A \arrow[r, "\phi"] \arrow[d, "\phi"] & A \arrow[r, "\phi"] \arrow[d, "\phi"] & A \arrow[r, "\phi"] \arrow[d, "\phi"] & \cdots\\ \cdots \arrow[r, "\phi"'] & A \arrow[r, "\phi"'] & A \arrow[r, "\phi"'] & A \arrow[r, "\phi"'] & \cdots;\end{tikzcd}$$

4. *Adjointness II.* If $\mathcal{C}$ has products and coproducts, then we have a triple adjunction

$$\left(\mathbb{Z}\odot(-) \dashv 忘 \dashv \mathbb{Z}\pitchfork(-)\right)\colon\quad \begin{tikzcd}\mathcal{C} \arrow[r, bend left, "\mathbb{Z}\odot(-)"] \arrow[r, bend right, "\mathbb{Z}\pitchfork(-)"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathrm{Aut}(\mathcal{C}) \arrow[l, "忘"']\end{tikzcd},$$

where[4]

- $\mathbb{Z}\odot(-)\colon \mathcal{C} \longrightarrow \mathrm{Aut}(\mathcal{C})$ is the functor defined on objects by

$$\begin{aligned}\mathbb{Z}\odot(A) &\stackrel{\text{def}}{=} (\mathbb{Z}\odot A, \mathbb{Z}\odot \mathrm{id}_A)\\ &\cong \left(A^{\coprod\mathbb{Z}}, \mathrm{id}_A^{\coprod\mathbb{Z}}\right);\end{aligned}$$ (Weighted Category Theory, Construction 1.2.2)

- 忘: $\mathrm{Aut}(C) \longrightarrow C$ is the **forgetful functor from** $\mathrm{Aut}(C)$ **to** C, defined on objects by

$$忘(A, \phi) \overset{\text{def}}{=} A;$$

- $\mathbb{Z} \pitchfork (-)\colon C \longrightarrow \mathrm{Aut}(C)$ is the functor defined on objects by

$$\begin{aligned}\mathbb{Z} \pitchfork (A) &\overset{\text{def}}{=} (\mathbb{Z} \pitchfork A, \mathbb{Z} \pitchfork \mathrm{id}_A) \\ &\cong \left(A^{\times \mathbb{Z}}, \mathrm{id}_A^{\times \mathbb{Z}}\right).\end{aligned}$$

(Weighted Category Theory, Construction 1.2.2)

5. *Adjointness III.* If C is bicomplete, then we have a triple adjunction

$$(\mathrm{colim}^{\circlearrowleft} \dashv \iota \dashv \lim^{\circlearrowleft})\colon \quad \begin{tikzcd} \mathrm{Aut}(C) \arrow[r, bend left=45, "\mathrm{colim}^{\circlearrowleft}"] \arrow[r, bend right=45, "\lim^{\circlearrowleft}"'] & C \arrow[l, "\iota" description] \end{tikzcd},$$

where[5]

- $\mathrm{colim}^{\circlearrowleft}\colon \mathrm{Aut}(C) \longrightarrow C$ is the functor defined on objects by

$$\begin{aligned}\mathrm{colim}^{\circlearrowleft}(A, \phi) &\overset{\text{def}}{=} \mathrm{colim}\left(\mathrm{B}\mathbb{Z} \xrightarrow{(A,\phi)} C\right) \\ &\overset{\text{def}}{=} \mathrm{colim}(A \circlearrowleft \phi);\end{aligned}$$

- $\iota\colon C \hookrightarrow \mathrm{Aut}(C)$ is the functor defined on objects by[6]

$$\iota(A) \overset{\text{def}}{=} (A, \mathrm{id}_A);$$

- $\lim^{\circlearrowleft}\colon \mathrm{Aut}(C) \longrightarrow C$ is the functor defined on objects by

$$\begin{aligned}\lim^{\circlearrowleft}(A, \phi) &\overset{\text{def}}{=} \lim\left(\mathrm{B}\mathbb{Z} \xrightarrow{(A,\phi)} C\right) \\ &\overset{\text{def}}{=} \lim(A \circlearrowleft \phi).\end{aligned}$$

6. *2-Adjointness.* We have a 2-adjunction

$$(\mathrm{B}\mathbb{Z} \times - \dashv \mathrm{Aut})\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathrm{B}\mathbb{Z}\times -"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2 \arrow[l, bend left, "\mathrm{Aut}"] \end{tikzcd}.$$

[1]There are two other natural triple adjunctions not included here:

- The first is the adjunction between $\mathsf{End}(\mathcal{C})$ and $\mathsf{Aut}(\mathcal{C})$ induced by taking left and right Kan extensions along the functor $\mathsf{B}\mathbb{Z} \longrightarrow \mathsf{B}\mathbb{N}$ corresponding to the morphism of monoids $0\colon \mathbb{Z} \longrightarrow \mathbb{N}$. One of the functors involved is the functor
$$0^*\colon \mathsf{End}(\mathcal{C}) \longrightarrow \mathsf{Aut}(\mathcal{C})$$
defined by
$$0^*(A,\phi) \stackrel{\text{def}}{=} (A, \mathrm{id}_A);$$
- The second is the family of adjunctions between $\mathsf{End}(\mathcal{C})$ and $\mathsf{Aut}(\mathcal{C})$ induced by taking left and right Kan extensions along the functor $\mathsf{B}\mathbb{N} \longrightarrow \mathsf{B}\mathbb{Z}$ corresponding to the morphism of monoids $k\colon \mathbb{N} \longrightarrow \mathbb{Z}$ picking $k \in \mathbb{Z}$. One of the functors involved is the functor
$$k^*\colon \mathsf{Aut}(\mathcal{C}) \longrightarrow \mathsf{End}(\mathcal{C})$$
defined by
$$k^*(A,\phi) \stackrel{\text{def}}{=} \left(A, \phi^{\circ k}\right).$$

[2]In a sense, χ^{L} and χ^{R} are the co/universal ways of producing an automorphism starting with an endomorphism.

[3]*Examples:* Examples of χ^{L} include the following:

(a) The localisation $A\left[a^{-1}\right]$ of a monoid A by a single element $a \in A$ (Monoids, **??**);

(b) The localisation $A\left[a^{-1}\right]$ of a monoid with zero $(A, 0_A)$ by a single element $a \in A$ (Monoids With Zero, **??**);

(c) The localisation $M\left[r^{-1}\right]$ of an R-module M by a single element $r \in R$ (Modules, Definition 3.6.5);

(d) The coperfection of a characteristic p ring of ().

Similarly, an example of χ^{R} is given by the perfection of a characteristic p ring of ().

[4]In a sense, $(\mathbb{Z} \odot A, \mathbb{Z} \odot \mathrm{id}_A)$ and $(\mathbb{Z} \pitchfork A, \mathbb{Z} \pitchfork \mathrm{id}_A)$ are the co/universal ways of producing an automorphism starting with an identity.

[5]In a sense, $\mathrm{colim}^{\circlearrowleft}(A,\phi)$ and $\lim^{\circlearrowleft}(A,\phi)$ are the co/universal ways of producing an identity starting with an automorphism.

[6]Viewing $\mathcal{C} \cong \mathsf{Fun}(\mathsf{pt}, \mathcal{C})$ as the "category of identities of $\mathcal{C}$", we see that the functor ι is just the inclusion of categories from the category of identities of $\mathcal{C}$ to the category of automorphisms of $\mathcal{C}$.

Proof 8.2.9 ▶ Proof of Proposition 8.2.8

Item 1: Functoriality

Omitted.

Item 2: 2-Functoriality

Omitted.

Item 3: Adjointness I

Omitted.

Item 4: Adjointness II

Omitted.

Item 5: Adjointness III

Omitted.

Item 6: 2-Adjointness

This is a special case of **??** of **??**.

8.2.2 The Automorphism Group of an Object of a Category

Let $\mathcal{C}$ be a category, let $X \in \mathsf{Obj}(\mathcal{C})$, and let $(\mathcal{C}, X)$ be a category with a distinguished object.

Definition 8.2.10 ▶ The Automorphism Group of an Object

The **automorphism group** of an object A of $\mathcal{C}$ is the group $\mathsf{Aut}_{\mathcal{C}}(A)$ consisting of

- *The Underlying Set.* The set $\mathsf{Aut}_{\mathcal{C}}(A)$ defined by
$$\mathsf{Aut}_{\mathcal{C}}(A) \stackrel{\text{def}}{=} \{f \in \mathsf{End}_{\mathcal{C}}(A) \mid f \text{ is an isomorphism}\};$$
- *The Multiplication Map.* The map of sets
$$\mu_{\mathsf{Aut}_{\mathcal{C}}(A)} \colon \mathsf{Aut}_{\mathcal{C}}(A) \times \mathsf{Aut}_{\mathcal{C}}(A) \longrightarrow \mathsf{Aut}_{\mathcal{C}}(A)$$
defined by
$$\mu_{\mathsf{Aut}_{\mathcal{C}}(A)} \stackrel{\text{def}}{=} \circ^{\mathcal{C}}_{A,A,A}\Big|_{\mathsf{Aut}_{\mathcal{C}}(A)};$$
- *The Unit Map.* The map of sets
$$\eta_{\mathsf{Aut}_{\mathcal{C}}(A)} \colon \mathrm{pt} \longrightarrow \mathsf{Aut}_{\mathcal{C}}(A)$$
defined by
$$\eta_{\mathsf{Aut}_{\mathcal{C}}(A)} \stackrel{\text{def}}{=} \mathbb{1}^{\mathcal{C}}_{A};$$
- *The Antipode.* The map of sets
$$\chi_{\mathsf{Aut}_{\mathcal{C}}(A)} \colon \mathsf{Aut}_{\mathcal{C}}(A) \longrightarrow \mathsf{Aut}_{\mathcal{C}}(A)$$
defined by
$$\chi_{\mathsf{Aut}_{\mathcal{C}}(A)}(f) \stackrel{\text{def}}{=} f^{-1}$$
for each $f \in \mathsf{Aut}_{\mathcal{C}}(A)$.

Definition 8.2.11 ▶ The Automorphism Group of a Pointed Category

The **automorphism group of** $(\mathcal{C}, X)$ is the automorphism group $\mathrm{Aut}_{\mathcal{C}}(X)$ of X in $\mathcal{C}$.[1]

[1] *Warning:* The assignment $(\mathcal{C}, X) \mapsto \mathrm{Aut}_{\mathcal{C}}(X)$ *does not* define a functor $\mathrm{Aut}\colon \mathsf{Cats}_* \longrightarrow \mathsf{Grp}$; see [MSE 570202].

8.3 Involutions in Categories

Let $\mathcal{C}$ be a category.

Definition 8.3.1 ▶ Involutions in Categories

An **involution in** $\mathcal{C}$ is a functor $\sigma\colon \mathrm{B}\mathbb{Z}_{/2} \longrightarrow \mathcal{C}$.

Remark 8.3.2 ▶ Unwinding Definition 8.3.1

In detail, an **involution in** $\mathcal{C}$ is a pair (A, σ) consisting of[1,2]

- *The Underlying Object.* An object A of $\mathcal{C}$;
- *The Involution.* An automorphism $\sigma\colon A \xrightarrow{\cong} A$ of $\mathcal{C}$ such that we have

$$\sigma^2 = \mathrm{id}_A, \qquad \begin{tikzcd} A \arrow[r, "\sigma"] \arrow[dr, "\mathrm{id}_A"'] & A \arrow[d, "\sigma"] \\ & A. \end{tikzcd}$$

[1] In other words, an **involution in** $\mathcal{C}$ is an involutory element of $\mathrm{End}_{\mathcal{C}}(A)$.
[2] In yet other words, an **involution in** $\mathcal{C}$ is an order 2 automorphism of A in $\mathcal{C}$.

Proof 8.3.3 ▶ Proof of Remark 8.3.2

Indeed, a functor $\sigma\colon \mathrm{B}\mathbb{Z}_{/2} \longrightarrow \mathcal{C}$ consists of

- *Action on Objects.* A map of sets

$$\sigma_0\colon \underbrace{\mathrm{Obj}(\mathrm{B}\mathbb{Z}_{/2})}_{\overset{\mathrm{def}}{=}\mathrm{pt}} \longrightarrow \mathrm{Obj}(\mathcal{C})$$

picking an object A of $\mathcal{C}$;

- *Action on Morphisms.* A map of sets

$$\sigma_{\star,\star}\colon \underbrace{\mathrm{Hom}_{\mathrm{B}\mathbb{Z}_{/2}}(\star,\star)}_{\overset{\text{def}}{=}\mathbb{Z}_{/2}} \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A,A);$$

preserving composition and identities. This makes $\sigma_{\star,\star}$ into a morphism of monoids

$$\sigma_{\star,\star}\colon \underbrace{\left(\mathrm{Hom}_{\mathrm{B}\mathbb{Z}_{/2}}(\star,\star), \circ^{\mathrm{B}\mathbb{Z}_{/2}}_{\star,\star,\star}, \mathbb{1}^{\mathrm{B}\mathbb{Z}_{/2}}_{\star}\right)}_{\overset{\text{def}}{=}(\mathbb{Z}_{/2},+,0)} \longrightarrow (\mathrm{Hom}_{\mathcal{C}}(A,A), \circ, \mathrm{id}_A),$$

determining and being determined by, via Monoids, **??** of **??**, an involutory element $\sigma\colon A \xrightarrow{\cong} A$ of $\mathrm{Hom}_{\mathcal{C}}(A,A)$, satisfying $\sigma^2 = \mathrm{id}_A$, i.e. an involution of A.

Definition 8.3.4 ▶ Morphisms of Involutions in Categories

A **morphism of involutions in** $\mathcal{C}$ from σ to τ is a natural transformation $\alpha\colon \sigma \Longrightarrow \tau$ of functors from $\mathrm{B}\mathbb{Z}_{/2}$ to $\mathcal{C}$.

Remark 8.3.5 ▶ Unwinding Definition 8.3.4

In detail, a **morphism of involutions in** $\mathcal{C}$ from (A,σ) to (B,τ) is a morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\sigma"'] & B \arrow[d, "\tau"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

Definition 8.3.6 ▶ The Category of Involutions in a Category

The **category of involutions in** $\mathcal{C}$ is the category $\mathrm{Inv}(\mathcal{C})$ defined by

$$\mathrm{Inv}(\mathcal{C}) \overset{\text{def}}{=} \mathrm{Fun}(\mathrm{B}\mathbb{Z}_{/2}, \mathcal{C}).$$

Remark 8.3.7 ▶ Unwinding Definition 8.3.6

In detail, the **category of involutions in $\mathcal{C}$** is the category $\mathsf{Inv}(\mathcal{C})$ where

- *Objects.* The objects of $\mathsf{Inv}(\mathcal{C})$ are involutions in $\mathcal{C}$;
- *Morphisms.* The morphisms of $\mathsf{Inv}(\mathcal{C})$ are morphisms of involutions in $\mathcal{C}$;
- *Identities.* For each $(A,\sigma)\in\mathrm{Obj}(\mathsf{Inv}(\mathcal{C}))$, the unit map
$$\mathbb{1}^{\mathsf{Inv}(\mathcal{C})}_{(A,\sigma)}\colon \mathrm{pt}\longrightarrow \mathrm{Hom}_{\mathsf{Inv}(\mathcal{C})}((A,\sigma),(A,\sigma))$$
of $\mathsf{Inv}(\mathcal{C})$ at (A,σ) is defined by
$$\mathrm{id}^{\mathsf{Inv}(\mathcal{C})}_{(A,\sigma)}\overset{\mathrm{def}}{=}\mathrm{id}_A;$$
- *Composition.* For each $(A,\sigma),(B,\rho),(C,\tau)\in\mathrm{Obj}(\mathsf{Inv}(\mathcal{C}))$, the composition map
$$\circ^{\mathsf{Inv}(\mathcal{C})}_{\sigma,\rho,\tau}\colon \mathrm{Hom}_{\mathsf{Inv}(\mathcal{C})}(\rho,\tau)\times\mathrm{Hom}_{\mathsf{Inv}(\mathcal{C})}(\sigma,\rho)\longrightarrow\mathrm{Hom}_{\mathsf{Inv}(\mathcal{C})}(\sigma,\tau)$$
of $\mathsf{Inv}(\mathcal{C})$ at $(A,\sigma),(B,\rho),(C,\tau)$ is defined by
$$g\circ^{\mathsf{Inv}(\mathcal{C})}_{\sigma,\rho,\tau}f\overset{\mathrm{def}}{=}g\circ f.$$

Proposition 8.3.8 ▶ Properties of Categories of Involutions

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C}\mapsto\mathsf{Inv}(\mathcal{C})$ defines a functor
$$\mathsf{Inv}\colon \mathsf{Cats}\longrightarrow\mathsf{Cats}.$$
2. *2-Functoriality.* The assignment $\mathcal{C}\mapsto\mathsf{Inv}(\mathcal{C})$ defines a 2-functor
$$\mathsf{Inv}\colon \mathsf{Cats}_2\longrightarrow\mathsf{Cats}_2.$$
3. *Adjointness I.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction
$$(\mathrm{L}\dashv\iota\dashv\mathrm{R})\colon\quad \begin{tikzcd} \mathsf{Aut}(\mathcal{C}) \arrow[r, bend left, "\mathrm{L}"] \arrow[r, bend right, "\mathrm{R}"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{Inv}(\mathcal{C}), \arrow[l, "\iota"'] \end{tikzcd}$$

obtained via precomposition and Kan extensions along the delooping $\mathsf{B}(\mathrm{mod}\,2)\colon \mathsf{B}\mathbb{Z} \longrightarrow \mathsf{B}\mathbb{Z}_{/2}$ of the parity map, where

- $\mathsf{L}\colon \mathsf{Aut}(\mathcal{C}) \longrightarrow \mathsf{Inv}(\mathcal{C})$ is the functor defined on objects by

$$\mathsf{L}(A,\phi) \overset{\text{def}}{=} (\mathsf{L}(A), \mathsf{L}(\phi)),$$

  where $\mathsf{L}(A)$ is the colimit

$$\mathsf{L}(A) \overset{\text{def}}{=} \operatorname{colim}\left(\begin{tikzcd} A \arrow[loop left, "\mathrm{id}_A"] \arrow[loop above, "\phi^{-2}"] \arrow[loop below, "\phi^{2}"] \arrow[r, shift left=7, "\phi^{-3}"] \arrow[r, shift left=5, "\phi^{3}"] \arrow[r, shift left=3, "\phi^{-1}"] \arrow[r, shift left=1, "\phi"] & A \arrow[l, shift left=1, "\phi"] \arrow[l, shift left=3, "\phi^{-1}"] \arrow[l, shift left=5, "\phi^{3}"] \arrow[l, shift left=7, "\phi^{-3}"] \arrow[loop right, "\mathrm{id}_A"] \arrow[loop above, "\phi^{2}"] \arrow[loop below, "\phi^{-2}"] \end{tikzcd}\right)$$

  in $\mathcal{C}$;
- $\iota\colon \mathsf{Inv}(\mathcal{C}) \hookrightarrow \mathsf{Aut}(\mathcal{C})$ is the natural inclusion of categories of $\mathsf{Inv}(\mathcal{C})$ into $\mathsf{Aut}(\mathcal{C})$;
- $\mathsf{R}\colon \mathsf{Aut}(\mathcal{C}) \longrightarrow \mathsf{Inv}(\mathcal{C})$ is the functor defined on objects by

$$\mathsf{R}(A,\phi) \overset{\text{def}}{=} (\mathsf{R}(A), \mathsf{R}(\phi)),$$

  where $\mathsf{R}(A)$ is the limit

$$\mathsf{R}(A) \overset{\text{def}}{=} \lim\left(\begin{tikzcd} A \arrow[loop left, "\mathrm{id}_A"] \arrow[loop above, "\phi^{-2}"] \arrow[loop below, "\phi^{2}"] \arrow[r, shift left=7, "\phi^{-3}"] \arrow[r, shift left=5, "\phi^{3}"] \arrow[r, shift left=3, "\phi^{-1}"] \arrow[r, shift left=1, "\phi"] & A \arrow[l, shift left=1, "\phi"] \arrow[l, shift left=3, "\phi^{-1}"] \arrow[l, shift left=5, "\phi^{3}"] \arrow[l, shift left=7, "\phi^{-3}"] \arrow[loop right, "\mathrm{id}_A"] \arrow[loop above, "\phi^{2}"] \arrow[loop below, "\phi^{-2}"] \end{tikzcd}\right)$$

  in $\mathcal{C}$.

4. *Adjointness II.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction

$$(\mathsf{L} \dashv \iota \dashv \mathsf{R})\colon\quad \begin{tikzcd} \mathsf{End}(\mathcal{C}) \arrow[r, bend left=45, "\mathsf{L}"] \arrow[r, bend right=45, "\mathsf{R}"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{Inv}(\mathcal{C}), \arrow[l, "\iota"] \end{tikzcd}$$

obtained by either

- Combining the triple adjunctions in Item 3 of Proposition 8.2.8 and Item 3, or;
- Via precomposition and Kan extensions along the delooping $\mathrm{B}(\mathrm{mod}\,2)\colon \mathrm{B}\mathbb{N} \hookrightarrow \mathrm{B}\mathbb{Z}_{/2}$ of the parity map;

where

- $\mathsf{L}\colon \mathsf{End}(\mathcal{C}) \longrightarrow \mathsf{Inv}(\mathcal{C})$ is the functor defined on objects by
$$\mathsf{L}(A,\phi) \overset{\text{def}}{=} (\mathsf{L}(A), \mathsf{L}(\phi)),$$
where $\mathsf{L}(A)$ is the colimit
$$\mathsf{L}(A) \overset{\text{def}}{=} \operatorname{colim}\left(\begin{tikzcd} A \arrow[loop below, "\mathrm{id}_A"] \arrow[loop left, "\phi^2"] \arrow[loop left, "\phi^4"] \arrow[loop above, "\phi^6"] \arrow[r, shift left=1, "\phi"] \arrow[r, shift left=2, "\phi^3"] \arrow[r, shift left=3, "\phi^5"] \arrow[r, shift left=4, "\phi^7"] & A \arrow[loop below, "\mathrm{id}_A"] \arrow[loop right, "\phi^2"] \arrow[loop right, "\phi^4"] \arrow[loop above, "\phi^6"] \arrow[l, shift left=1, "\phi"] \arrow[l, shift left=2, "\phi^3"] \arrow[l, shift left=3, "\phi^5"] \arrow[l, shift left=4, "\phi^7"] \end{tikzcd}\right)$$
in $\mathcal{C}$;
- $\iota\colon \mathsf{Inv}(\mathcal{C}) \hookrightarrow \mathsf{End}(\mathcal{C})$ is the natural inclusion of categories of $\mathsf{Inv}(\mathcal{C})$ into $\mathsf{End}(\mathcal{C})$;
- $\mathsf{R}\colon \mathsf{End}(\mathcal{C}) \longrightarrow \mathsf{Inv}(\mathcal{C})$ is the functor defined on objects by
$$\mathsf{R}(A,\phi) \overset{\text{def}}{=} (\mathsf{R}(A), \mathsf{R}(\phi)),$$
where $\mathsf{R}(A)$ is the limit
$$\mathsf{R}(A) \overset{\text{def}}{=} \lim\left(\begin{tikzcd} A \arrow[loop below, "\mathrm{id}_A"] \arrow[loop left, "\phi^2"] \arrow[loop left, "\phi^4"] \arrow[loop above, "\phi^6"] \arrow[r, shift left=1, "\phi"] \arrow[r, shift left=2, "\phi^3"] \arrow[r, shift left=3, "\phi^5"] \arrow[r, shift left=4, "\phi^7"] & A \arrow[loop below, "\mathrm{id}_A"] \arrow[loop right, "\phi^2"] \arrow[loop right, "\phi^4"] \arrow[loop above, "\phi^6"] \arrow[l, shift left=1, "\phi"] \arrow[l, shift left=2, "\phi^3"] \arrow[l, shift left=3, "\phi^5"] \arrow[l, shift left=4, "\phi^7"] \end{tikzcd}\right)$$
in $\mathcal{C}$.

5. *Adjointness III.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction
$$(\mathbb{Z}_{/2} \odot (-) \dashv \iota \dashv \mathbb{Z}_{/2} \pitchfork (-))\colon \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left=40, "\mathbb{Z}_{/2}\odot(-)"] \arrow[r, bend right=40, "\mathbb{Z}_{/2}\pitchfork(-)"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{Inv}(\mathcal{C}), \arrow[l, "\iota"] \end{tikzcd}$$

obtained by either

- Combining the triple adjunctions in Item 3 of Proposition 8.1.8, Item 3 of Proposition 8.2.8 and Item 3, or;
- Via precomposition and Kan extensions along the delooping $\mathrm{B}\{\star\} \twoheadrightarrow \mathrm{B}\mathbb{Z}_{/2}$ of the initial map from $\{\star\}$ to $\mathbb{Z}_{/2}$;

where

- $\mathbb{Z}_{/2} \odot (-) \colon \mathcal{C} \longrightarrow \mathsf{Inv}(\mathcal{C})$ is defined on objects by
$$\mathbb{Z}_{/2} \odot A \stackrel{\text{def}}{=} \left(A \coprod A, \beta^{\mathcal{C},\coprod}_{A,A}\right),$$
where $\beta^{\mathcal{C},\coprod}_{A,A} \colon A \coprod A \longrightarrow A \coprod A$ is the morphism swapping the two factors of A in $A \coprod A$;
- $\iota \colon \mathsf{Inv}(\mathcal{C}) \longrightarrow \mathcal{C}$ is the forgetful functor defined on objects by
$$\iota(A, \sigma) \stackrel{\text{def}}{=} A;$$
- $\mathbb{Z}_{/2} \pitchfork (-) \colon \mathcal{C} \longrightarrow \mathsf{Inv}(\mathcal{C})$ is defined on objects by
$$\mathbb{Z}_{/2} \pitchfork A \stackrel{\text{def}}{=} \left(A \times A, \beta^{\mathcal{C},\times}_{A,A}\right),$$
where $\beta^{\mathcal{C},\times}_{A,A} \colon A \times A \longrightarrow A \times A$ is the morphism swapping the two factors of A in $A \times A$.

6. *Adjointness IV.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction
$$(\mathsf{L} \dashv \iota \dashv \mathsf{R}) \colon \quad \begin{tikzcd} \mathsf{Inv}(\mathcal{C}) \arrow[r, bend left=45, "\mathsf{L}"] \arrow[r, bend right=45, "\mathsf{R}"'] & \mathcal{C}, \arrow[l, "\iota" description] \end{tikzcd}$$
obtained via precomposition and Kan extensions along the delooping $\mathrm{B}\mathbb{Z}_{/2} \twoheadrightarrow \mathrm{B}\{\star\}$ of the terminal map from $\mathbb{Z}_{/2}$ to $\{\star\}$, where

- $\mathrm{colim}^{\circlearrowleft} \colon \mathsf{Inv}(\mathcal{C}) \longrightarrow \mathcal{C}$ is the restriction to $\mathsf{Inv}(\mathcal{C})$ of the functor $\mathrm{colim}^{\circlearrowleft}$ of Item 4 of Proposition 8.1.8, being defined on objects by
$$\begin{aligned} \mathrm{colim}^{\circlearrowleft}(A, \sigma) &\stackrel{\text{def}}{=} \mathrm{colim}\left(\mathrm{B}\mathbb{Z}_{/2} \xrightarrow{(A,\sigma)} \mathcal{C}\right) \\ &\stackrel{\text{def}}{=} \mathrm{colim}(A \circlearrowleft \sigma); \end{aligned}$$

- $\iota\colon \mathcal{C} \hookrightarrow \mathsf{End}(\mathcal{C})$ is the functor defined on objects by[1]

$$\iota(A) \overset{\text{def}}{=} (A, \mathrm{id}_A);$$

- $\lim^{\circlearrowleft}\colon \mathsf{Inv}(\mathcal{C}) \longrightarrow \mathcal{C}$ is the restriction to $\mathsf{Inv}(\mathcal{C})$ of the functor $\lim^{\circlearrowleft}$ of Item 4 of Proposition 8.1.8, being defined on objects by

$$\begin{aligned}\lim^{\circlearrowleft}(A, \sigma) &\overset{\text{def}}{=} \lim\left(\mathrm{B}\mathbb{Z}_{/2} \xrightarrow{(A,\sigma)} \mathcal{C}\right)\\ &\overset{\text{def}}{=} \lim(A \circlearrowleft \sigma).\end{aligned}$$

7. *2-Adjointness.* We have a 2-adjunction

$$\left(\mathrm{B}\mathbb{Z}_{/2} \times - \dashv \mathsf{Inv}\right)\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathrm{B}\mathbb{Z}_{/2}\times -"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2. \arrow[l, bend left, "\mathsf{Inv}"] \end{tikzcd}$$

[1] Viewing $\mathcal{C} \cong \mathsf{Fun}(\mathrm{pt}, \mathcal{C})$ as the "category of identities of $\mathcal{C}$", we see that the functor ι is just the inclusion of categories from the category of identities of $\mathcal{C}$ to the category of endomorphisms of $\mathcal{C}$.

Proof 8.3.9 ▶ Proof of Proposition 8.3.8

Item 1: Functoriality

Omitted.

Item 2: 2-Functoriality

Omitted.

Item 3: Adjointness I

Omitted.

Item 4: Adjointness II

Omitted.

Item 5: Adjointness III

Omitted.

Item 6: Adjointness IV

Omitted.

Item 7: 2-Adjointness

This is a special case of **??** of **??**. ∎

8.4 Idempotent Morphisms in Categories

Let $\mathcal{C}$ be a category.

Definition 8.4.1 ▶ Idempotent Morphisms

An **idempotent morphism in** $\mathcal{C}$ is a functor $\sigma\colon \mathsf{B}\mathbb{B} \longrightarrow \mathcal{C}$.

Remark 8.4.2 ▶ Unwinding Definition 8.4.1

In detail, an **idempotent morphism in** $\mathcal{C}$ is a pair (A, σ) consisting of[1]

- *The Underlying Object.* An object A of $\mathcal{C}$;
- *The Idempotent Morphism.* A morphism $\sigma\colon A \xrightarrow{\cong} A$ of $\mathcal{C}$ such that we have

$$\sigma^2 = \sigma, \qquad \begin{tikzcd} A \arrow[r, "\sigma"] \arrow[dr, "\sigma"'] & A \arrow[d, "\sigma"] \\ & A. \end{tikzcd}$$

[1] In other words, an **idempotent morphism in** $\mathcal{C}$ is an idempotent element of $\mathrm{End}_{\mathcal{C}}(A)$.

Proof 8.4.3 ▶ Proof of Remark 8.4.2

Indeed, a functor $\sigma\colon \mathsf{B}\mathbb{B} \longrightarrow \mathcal{C}$ consists of

- *Action on Objects.* A map of sets
$$\sigma_0\colon \underbrace{\mathrm{Obj}(\mathsf{B}\mathbb{B})}_{\overset{\text{def}}{=}\mathrm{pt}} \longrightarrow \mathrm{Obj}(\mathcal{C})$$
picking an object A of $\mathcal{C}$;
- *Action on Morphisms.* A map of sets
$$\sigma_{\star,\star}\colon \underbrace{\mathrm{Hom}_{\mathsf{B}\mathbb{B}}(\star, \star)}_{\overset{\text{def}}{=}\mathbb{B}} \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, A);$$

preserving composition and identities. This makes $\sigma_{\star,\star}$ into a morphism of monoids

$$\sigma_{\star,\star}\colon \underbrace{\left(\mathrm{Hom}_{\mathsf{B}\mathbb{B}}(\star,\star), \circ^{\mathsf{B}\mathbb{B}}_{\star,\star,\star}, \mathbb{1}^{\mathsf{B}\mathbb{B}}_{\star}\right)}_{\overset{\text{def}}{=}(\mathbb{B},+,0)} \longrightarrow (\mathrm{Hom}_{C}(A,A), \circ, \mathrm{id}_A),$$

determining and being determined by, via Monoids, **??** of **??**, an idempotent element $\sigma\colon A \longrightarrow A$ of $\mathrm{End}_C(A,A)$, satisfying $\sigma^2 = \sigma$, i.e. an idempotent morphism in C from A to itself.

Definition 8.4.4 ▶ Morphisms of Idempotent Morphisms

A **morphism of idempotent morphisms in** C from σ to τ is a natural transformation $\alpha\colon \sigma \Longrightarrow \tau$ of functors from $\mathsf{B}\mathbb{B}$ to C.

Remark 8.4.5 ▶ Unwinding Definition 8.4.4

In detail, a **morphism of idempotent morphisms in** C from (A,σ) to (B,τ) is a morphism $f\colon A \longrightarrow B$ of C such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\sigma"'] & B \arrow[d, "\tau"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

Definition 8.4.6 ▶ The Category of Idempotent Morphisms of a Category

The **category of idempotent morphisms of** C is the category $\mathrm{Idem}(C)$ defined by

$$\mathrm{Idem}(C) \overset{\text{def}}{=} \mathrm{Fun}(\mathsf{B}\mathbb{B}, C).$$

Remark 8.4.7 ▶ Unwinding Definition 8.4.6

In detail, the **category of idempotent morphisms in** C is the category $\mathrm{Idem}(C)$ where

- *Objects.* The objects of $\mathsf{Idem}(\mathcal{C})$ are idempotent morphisms in $\mathcal{C}$;
- *Morphisms.* The morphisms of $\mathsf{Idem}(\mathcal{C})$ are morphisms of idempotent morphisms in $\mathcal{C}$;
- *Identities.* For each $(A, \sigma) \in \mathrm{Obj}(\mathsf{Idem}(\mathcal{C}))$, the unit map

$$\mathbb{1}^{\mathsf{Idem}(\mathcal{C})}_{(A,\sigma)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathsf{Idem}(\mathcal{C})}((A, \sigma), (A, \sigma))$$

of $\mathsf{Idem}(\mathcal{C})$ at (A, σ) is defined by

$$\mathrm{id}^{\mathsf{Idem}(\mathcal{C})}_{(A,\sigma)} \overset{\text{def}}{=} \mathrm{id}_A;$$

- *Composition.* For each $(A, \sigma), (B, \rho), (C, \tau) \in \mathrm{Obj}(\mathsf{Idem}(\mathcal{C}))$, the composition map

$$\circ^{\mathsf{Idem}(\mathcal{C})}_{\sigma,\rho,\tau} \colon \mathrm{Hom}_{\mathsf{Idem}(\mathcal{C})}(\rho, \tau) \times \mathrm{Hom}_{\mathsf{Idem}(\mathcal{C})}(\sigma, \rho) \longrightarrow \mathrm{Hom}_{\mathsf{Idem}(\mathcal{C})}(\sigma, \tau)$$

of $\mathsf{Idem}(\mathcal{C})$ at $((A, \sigma), (B, \rho), (C, \tau))$ is defined by

$$g \circ^{\mathsf{Idem}(\mathcal{C})}_{\sigma,\rho,\tau} f \overset{\text{def}}{=} g \circ f.$$

Proposition 8.4.8 ▶ Properties of Categories of Idempotent Morphisms

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Idem}(\mathcal{C})$ defines a functor

$$\mathsf{Idem} \colon \mathsf{Cats} \longrightarrow \mathsf{Cats}.$$

2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Idem}(\mathcal{C})$ defines a 2-functor

$$\mathsf{Idem} \colon \mathsf{Cats}_2 \longrightarrow \mathsf{Cats}_2.$$

3. *Adjointness I.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction

$$(\mathrm{L} \dashv \iota \dashv \mathrm{R})\colon\quad \begin{tikzcd} \mathsf{End}(\mathcal{C}) \arrow[r, bend left=50, "\mathrm{L}"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, bend right=50, "\mathrm{R}"'] & \mathsf{Idem}(\mathcal{C}), \arrow[l, "\iota" description] \end{tikzcd}$$

obtained via precomposition and Kan extensions along the delooping $\mathrm{B}\mathbb{N} \longrightarrow \mathrm{B}\mathbb{B}$ of the map picking $1 \in \mathbb{B}$ via Monoids, Item 2 of Proposition 1.1.10, where

- $\mathsf{L}\colon \mathsf{End}(\mathcal{C}) \longrightarrow \mathsf{Idem}(\mathcal{C})$ is the functor defined on objects by

  $$\mathsf{L}(A, \phi) \overset{\text{def}}{=} (\mathsf{L}(A), \mathsf{L}(\phi)),$$

  where $\mathsf{L}(A)$ is the coequaliser

  $$\begin{aligned}\mathsf{L}(A) &\cong \mathrm{CoEq}\left(\coprod_{n\in\mathbb{N}} \mathbb{B} \odot A \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \mathbb{B} \odot A\right)\\ &\cong \mathrm{CoEq}\left(\coprod_{n\in\mathbb{N}} A \textstyle\coprod A \underset{\rho}{\overset{\lambda}{\rightrightarrows}} A \textstyle\coprod A\right)\end{aligned}$$

  in $\mathcal{C}$, where

  $$\begin{aligned}\lambda &\overset{\text{def}}{=} \mathrm{id}_{A\coprod A} \textstyle\coprod \displaystyle\coprod_{n=1}^{\infty} (\mathrm{inj}_2 \textstyle\coprod \mathrm{inj}_2),\\ \rho &\overset{\text{def}}{=} \mathrm{id}_{A\coprod A} \textstyle\coprod \displaystyle\coprod_{n=1}^{\infty} (\phi^n \textstyle\coprod \phi^n);\end{aligned}$$

- $\iota\colon \mathsf{Idem}(\mathcal{C}) \hookrightarrow \mathsf{End}(\mathcal{C})$ is the natural inclusion of categories of $\mathsf{Idem}(\mathcal{C})$ into $\mathsf{End}(\mathcal{C})$;
- $\mathsf{R}\colon \mathsf{End}(\mathcal{C}) \longrightarrow \mathsf{Idem}(\mathcal{C})$ is the functor defined on objects by

  $$\mathsf{R}(A, \phi) \overset{\text{def}}{=} (\mathsf{R}(A), \mathsf{R}(\phi)),$$

  where $\mathsf{R}(A)$ is the equaliser

  $$\begin{aligned}\mathsf{R}(A) &\cong \mathrm{Eq}\left(\mathbb{B} \pitchfork A \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \prod_{n\in\mathbb{N}} \mathbb{B} \pitchfork A\right)\\ &\cong \mathrm{Eq}\left(A \times A \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \prod_{n\in\mathbb{N}} A \times A\right)\end{aligned}$$

  in $\mathcal{C}$, where

  $$\lambda \overset{\text{def}}{=} \mathrm{id}_{A\times A} \times \prod_{n=1}^{\infty} (\mathrm{pr}_2 \times \mathrm{pr}_2),$$

$$\rho \overset{\text{def}}{=} \mathrm{id}_{A\times A} \times \prod_{n=1}^{\infty}(\phi^n \times \phi^n).$$

4. *Adjointness II.* If $\mathcal{C}$ is bicomplete, then we have a triple adjunction

$$(\mathbb{B} \odot (-) \dashv \iota \dashv \mathbb{B} \pitchfork (-)):\quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "\mathbb{B}\odot(-)"] \arrow[r, bend right=50, "\mathbb{B}\pitchfork(-)"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathsf{Idem}(\mathcal{C}) \arrow[l, "\iota"'] \end{tikzcd},$$

obtained by either

- Combining the triple adjunctions in Item 3 of Proposition 8.1.8 and Item 3, or;
- Via precomposition and Kan extensions along the delooping $\mathrm{B}\{\star\} \twoheadrightarrow \mathrm{B}\mathbb{B}$ of the initial map from $\{\star\}$ to $\mathbb{B}$;

where

- $\mathbb{B} \odot (-) \colon \mathcal{C} \longrightarrow \mathsf{Idem}(\mathcal{C})$ is defined on objects by
$$\mathbb{B} \odot A \overset{\text{def}}{=} (A \textstyle\coprod A, \sigma_{A,A}),$$
where $\sigma_{A,A} \colon A \coprod A \longrightarrow A \coprod A$ is the morphism defined by[1,2]
$$\sigma_{A,A} \overset{\text{def}}{=} \mathrm{inj}_2 \textstyle\coprod \mathrm{inj}_2;$$
- $\iota \colon \mathsf{Idem}(\mathcal{C}) \longrightarrow \mathcal{C}$ is the forgetful functor defined on objects by
$$\iota(A, \sigma) \overset{\text{def}}{=} A;$$
- $\mathbb{B} \pitchfork (-) \colon \mathcal{C} \longrightarrow \mathsf{Idem}(\mathcal{C})$ is defined on objects by
$$\mathbb{B} \pitchfork A \overset{\text{def}}{=} (A \times A, \sigma_{A,A}),$$
where $\sigma_{A,A} \colon A \times A \longrightarrow A \times A$ is the morphism defined by[3,4]
$$\sigma_{A,A} \overset{\text{def}}{=} \mathrm{pr}_2 \times \mathrm{pr}_2\,.$$

5. *2-Adjointness.* We have a 2-adjunction

$$(\mathrm{B}\mathbb{B} \times - \dashv \mathsf{Idem}):\quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathrm{B}\mathbb{B}\times-"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2 \arrow[l, bend left, "\mathsf{Idem}"] \end{tikzcd}.$$

[1]For $C = \mathsf{Sets}$, the map $\sigma_{A,A}$ is explicitly given by sending each $x \in A \coprod A$ in either factor of A in $A \coprod A$ to the copy of x in the second factor of A in $A \coprod A$.

[2]When C has an initial object $\varnothing_C$, the map $\sigma_{A,A}$ is the same as the composition

$$A \coprod A \xrightarrow{\nabla_A} A \xrightarrow{\cong} \varnothing_C \coprod A \hookrightarrow A \coprod A$$

where $\nabla_A \colon A \coprod A \longrightarrow A$ is the fold map of A.

[3]For $C = \mathsf{Sets}$, the map $\sigma_{A,A}$ is explicitly given by

$$\sigma_{A,A}(x, y) \overset{\text{def}}{=} (y, y)$$

for each $(x, y) \in A \times A$.

[4]When C has a terminal object $\varnothing_C$, the map $\sigma_{A,A}$ is the same as the composition

$$A \times A \twoheadrightarrow \mathrm{pt} \times A \xrightarrow{\cong} A \xrightarrow{\delta_A} A \times A$$

where $\Delta_A \colon A \longrightarrow A \times A$ is the diagonal map of A.

Proof 8.4.9 ▶ Proof of Proposition 8.4.8

Item 1: Functoriality

Omitted.

Item 2: 2-Functoriality

Omitted.

Item 3: Adjointness I

Omitted.

Item 4: Adjointness II

Omitted.

Item 5: 2-Adjointness

This is a special case of **??** of **??**.

9 Slice Categories

9.1 Slice Categories

Let C be a category.

Definition 9.1.1 ▶ Slice Categories

The **slice category of $\mathcal{C}$ by X**[1] is the category $\mathcal{C}_{/X}$ where

- *Objects.* The objects of $\mathcal{C}_{/X}$ are pairs (A, ϕ) consisting of
  - An object A of $\mathcal{C}$;
  - A morphism $\phi \colon A \longrightarrow X$ of $\mathcal{C}$;
- *Morphisms.* A morphism of $\mathcal{C}_{/X}$ from (A, ϕ) to (B, ψ) is a morphism $f \colon A \longrightarrow B$ of $\mathcal{C}$ making the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\phi"'] & & B \arrow[dl, "\psi"] \\ & X & \end{tikzcd}$$

  commute;
- *Identities.* For each $(A, \phi) \in \mathrm{Obj}(\mathcal{C}_{/X})$, the unit map

$$\mathbb{1}^{\mathcal{C}_{/X}}_{(A,\phi)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}_{/X}}((A, \phi), (A, \phi))$$

  of $\mathcal{C}_{/X}$ at (A, ϕ) is given by

$$\mathrm{id}^{\mathcal{C}_{/X}}_{(A,\phi)} \overset{\mathrm{def}}{=} \mathrm{id}_A,$$

  as witnessed by the commutativity of the diagram

$$\begin{tikzcd} A \arrow[rr, equal, "\mathrm{id}_A"] \arrow[dr, "\phi"'] & & A \arrow[dl, "\phi"] \\ & X & \end{tikzcd}$$

  in $\mathcal{C}$;
- *Composition.* For each $(A, \phi), (B, \psi), (C, \chi) \in \mathrm{Obj}(\mathcal{C}_{/X})$, the composition map

$$\circ^{\mathcal{C}_{/X}}_{(A,\phi),(B,\psi),(C,\chi)} \colon \mathrm{Hom}_{\mathcal{C}_{/X}}((B, \psi), (C, \chi)) \times \mathrm{Hom}_{\mathcal{C}_{/X}}((A, \phi), (B, \psi)) \longrightarrow \mathrm{Hom}_{\mathcal{C}_{/X}}((A, \phi), (C, \chi))$$

  of $\mathcal{C}_{/X}$ at $((A, \phi), (B, \psi), (C, \chi))$ is defined by

$$\circ^{\mathcal{C}_{/X}}_{(A,\phi),(B,\psi),(C,\chi)} \overset{\mathrm{def}}{=} \circ^{\mathcal{C}}_{A,B,C},$$

as witnessed by the commutativity of the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\phi"'] & B \arrow[r, "g"] \arrow[d, "\psi"] & C \arrow[dl, "\chi"] \\ & X & \end{tikzcd}$$

in $\mathcal{C}$.

[1] *Further Terminology:* Also called the **category of objects of $\mathcal{C}$ over X**.

EXAMPLE 9.1.2 ▶ SLICE CATEGORIES OF Sets [LUR20, TAG 015Q]

Let S be a set. We have an equivalence of categories

$$\mathsf{Sets}_{/S} \overset{\text{eq.}}{\cong} \prod_{s\in S} \mathsf{Sets}$$

given on objects by $\left(X \xrightarrow{f} S\right) \mapsto \left\{f^{-1}(\{s\})\right\}_{s\in S}$.

PROPOSITION 9.1.3 ▶ PROPERTIES OF SLICE CATEGORIES

Let $\phi\colon X \longrightarrow Y$ be a morphism of $\mathcal{C}$.

1. *Functoriality.* The assignment $X \mapsto \mathcal{C}_{/X}$ defines functors

$$\mathcal{C}_{/(-)}\colon \mathcal{C} \longrightarrow \mathsf{Cats},$$
$$\left(\mathcal{C}_{/(-)}, 忘^{\mathrm{cov},\mathcal{C}_{/(-)}}\right)\colon \mathcal{C} \longrightarrow \mathsf{Cats}_{/\mathcal{C}}.$$

2. *Base Change I.* We have a functor

$$\mathcal{C}_{/\phi}\colon \mathcal{C}_{/X} \longrightarrow \mathcal{C}_{/Y},$$

also written $\phi_*\colon \mathcal{C}_{/X} \longrightarrow \mathcal{C}_{/Y}$.

3. *Base Change II.* If $\mathcal{C}$ has pullbacks, then we have a functor

$$\phi^*\colon \mathcal{C}_{/Y} \longrightarrow \mathcal{C}_{/X}$$

where

- *Action on Objects.* For each $(A, \theta) \in \mathrm{Obj}(\mathcal{C}_{/Y})$, we have

$$\phi^*(A, \theta) \overset{\text{def}}{=} (A \times_Y X, \mathrm{pr}_2),$$

where $A \times_Y X$ is the pullback

$$\begin{tikzcd}
A \times_Y X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\phi"] \\
A \arrow[r, "\theta"'] & Y;
\end{tikzcd}$$

- *Action on Morphisms.* For each morphism $f\colon (A, \theta) \longrightarrow (B, \theta')$ of $\mathcal{C}_{/Y}$, the image

$$\phi^*(f)\colon \phi^*(A, \theta) \longrightarrow \phi^*(B, \theta')$$

of f by ϕ^* is the dashed morphism in the diagram

$$\begin{tikzcd}
A \times_Y X \arrow[rr, "\mathrm{pr}_2"] \arrow[dd, "\mathrm{pr}_1"'] \arrow[dr, dashed, "\exists!"] \arrow[dr, phantom, "\lrcorner", very near start] & & X \arrow[dd, "f"] \arrow[dr, equal] & \\
& B \times_Y X \arrow[rr, "\mathrm{pr}_2"] \arrow[dd, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & & X \arrow[dd, "\phi"] \\
A \arrow[rr, "\theta"] \arrow[dr, "\phi"'] & & Y \arrow[dr, equal] & \\
& B \arrow[rr, "\theta'"'] & & Y,
\end{tikzcd}$$

obtained via Limits and Colimits, Item 3 of Proposition 1.6.4.

4. *Base Change III.* If $\mathcal{C}$ has pullbacks, then we have an adjunction

$$(\phi^* \dashv \phi_*)\colon \quad \begin{tikzcd} \mathcal{C}_{/Y} \arrow[r, bend left, "\phi^*"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{/X}. \arrow[l, bend left, "\phi_*"] \end{tikzcd}$$

5. *Base Change IV.* If $\mathcal{C}$ is locally Cartesian closed, then we have a functor

$$\phi_! \colon \mathcal{C}_{/X} \longrightarrow \mathcal{C}_{/Y},$$

assembling into a triple adjunction

$$(\phi_! \dashv \phi^* \dashv \phi_*)\colon \quad \begin{tikzcd} \mathcal{C}_{/X} \arrow[r, bend left=45, "\phi_!"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, bend right=45, "\phi_*"'] & \mathcal{C}_{/Y} \arrow[l, "\phi^*" description] \end{tikzcd}$$

between $\mathcal{C}_{/X}$ and $\mathcal{C}_{/Y}$.

6. *Relation to the Grothendieck Construction*[1]. We have an isomorphisms of categories

$$\mathcal{C}_{/X} \cong \int^{\mathcal{C}} h_X.$$

7. *Duality.* We have an isomorphism of categories

$$(\mathcal{C}_{/X})^{\mathsf{op}} \cong (\mathcal{C}^{\mathsf{op}})_{X/}.$$

8. *As Comma Categories.* We have an isomorphisms of categories

$$\mathcal{C}_{/X} \cong \mathrm{id}_{\mathcal{C}} \downarrow [X], \qquad \begin{tikzcd} \mathcal{C}_{/X} \arrow[r, two heads] \arrow[d, "忘^{\mathrm{cov}}"'] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow"] & \mathrm{pt} \arrow[d, "{[X]}"] \\ \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}}"'] & \mathcal{C}. \end{tikzcd}$$

9. *As Pullbacks.* We have an isomorphism of categories

$$\mathcal{C}_{/X} \cong \{X\} \times_{\iota_X, \mathcal{C}, \mathrm{ev}_1} \mathrm{Arr}(\mathcal{C}), \qquad \begin{tikzcd} \mathcal{C}_{/X} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Arr}(\mathcal{C}) \arrow[d, "\mathrm{ev}_1"] \\ \{X\} \arrow[r, "\iota_X"'] & \mathcal{C}. \end{tikzcd}$$

10. *Slices of Presheaf Categories.* We have an equivalence of categories

$$\mathsf{PSh}(\mathcal{C}_{/X}) \overset{\text{eq.}}{\cong} \mathsf{PSh}(\mathcal{C})_{/h_X}.$$

More generally, given a preshehaf $\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$ on $\mathcal{C}$, we have an equivalence of categories

$$\mathsf{PSh}\left(\int^{\mathcal{C}} \mathcal{F}\right) \overset{\text{eq.}}{\cong} \mathsf{PSh}(\mathcal{C})_{/\mathcal{F}}.$$

[1]This is a repetition of Fibred Categories, Example 9.3.1.

Proof 9.1.4 ▶ Proof of Proposition 9.1.3

Item 1: Functoriality

Omitted.

Item 2: Base Change I

Indeed, define $\mathcal{C}_{/f}\colon \mathcal{C}_{/X} \longrightarrow \mathcal{C}_{/Y}$ as the functor sending

- *Action on Objects.* An object $(A, \phi) \overset{\text{def}}{=} \left(A, A \xrightarrow{\phi} X\right)$ of $\mathcal{C}_{/X}$ to the object $(A, f \circ \phi)$ of $\mathcal{C}_{/Y}$;
- *Action on Morphisms.* A morphism

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "\phi"'] & & B \arrow[dl, "\psi"] \\ & X & \end{tikzcd}$$

of $\mathcal{C}_{/X}$ to the morphism

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "\phi"'] & & B \arrow[dl, "\psi"] \\ & X \arrow[d, "f"] & \\ & Y & \end{tikzcd}$$

of $\mathcal{C}_{/Y}$.

That $\mathcal{C}_{/f}$ preserves identities and composition is clear.

Item 3: Base Change II

Omitted.

Item 4: Base Change III

Omitted.

Item 5: Base Change IV

Omitted.

Item 6: Relation to the Grothendieck Construction

This was proved in its repetition, Fibred Categories, Example 9.3.1.

Item 7: Duality

Clear.[1]

Item 8: As Comma Categories

Clear.[2]

Item 9: As Pullbacks

Clear.[3]

Item 10: Slices of Presheaf Categories

See [nLab23c, Proposition 3.2].

[1] *Reference*: [Lur20, Tag 015T].
[2] *Reference*: [Lur20, Tag 015U].
[3] *Reference*: [Lur20, Tag 015U].

9.2 Slice Categories of Morphisms

Let C be a category and let $f\colon X \longrightarrow Y$ be a morphism of C.

Definition 9.2.1 ▶ Slice Categories of Morphisms

The **slice category of C by f**[1] is the category $C_{/f}$ defined by[2]

$$\begin{aligned} C_{/f} &\overset{\text{def}}{=} C_{/X} \times_{C_{/Y}} C_{/Y} \\ &\cong C_{/X}, \end{aligned} \qquad \begin{tikzcd} C_{/f} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_{/Y} \arrow[d, "\mathrm{id}_{C_{/Y}}"] \\ C_{/X} \arrow[r, "C_{/f}"'] & C_{/Y}. \end{tikzcd}$$

[1] *Further Terminology*: Also called the **category of objects of C over f**.

[2]We also have an isomorphism of categories

$$
C_{/f} \cong \mathrm{id}_- \downarrow [f], \qquad
\begin{tikzcd}
C_{/f} \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d, "{[f]}"] \\
C \arrow[r, hook] \arrow[ur, phantom, "\Nearrow"] & \mathrm{Arr}(C).
\end{tikzcd}
$$

Remark 9.2.2 ▶ Unwinding Definition 9.2.1

In detail, $C_{/f}$ is the category where

- *Objects.* The objects of $C_{/f}$ are pairs (A, ϕ, ψ) consisting of
  - An object A of C;
  - A morphism $\phi\colon X \longrightarrow A$ of C;
  - A morphism $\psi\colon X \longrightarrow B$ of C;

  such that the diagram

$$
\begin{tikzcd}
 & X \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
A \arrow[rr, "f"'] & & B
\end{tikzcd}
$$

- *Morphisms.* A morphism of $C_{/f}$ from (A, ϕ_A, ψ_A) to (B, ϕ_B, ψ_B) is a morphism $\theta\colon A \longrightarrow B$ of C such that the diagram

$$
\begin{tikzcd}
 & A \arrow[dl, "\phi_A"'] \arrow[dr, "\psi_A"] \arrow[rrr, "\theta"] & & & & \\
X \arrow[rr, "f"] \arrow[drrr, "\mathrm{id}_A"'] & & Y \arrow[drrr, dashed, "\mathrm{id}_B"] & & B \arrow[dl, "\phi_B"'] \arrow[dr, "\psi_B"] & \\
 & & & X \arrow[rr, "f"'] & & Y
\end{tikzcd}
$$

  commutes, i.e. such that the diagrams

$$
\begin{tikzcd}
A \arrow[rr, "\theta"] \arrow[dr, "\phi_A"'] & & B \arrow[dl, "\phi_B"] \\
 & X &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "\theta"] \arrow[dr, "\psi_A"'] & & B \arrow[dl, "\psi_B"] \\
 & Y &
\end{tikzcd}
$$

commute.

- *Identities*. For each $(A, \phi, \psi) \in \mathrm{Obj}(\mathcal{C}_{/f})$, the unit map

$$\mathbb{1}^{\mathcal{C}_{/f}}_{(A,\phi,\psi)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}_{/f}}((A, \phi, \psi), (A, \phi, \psi))$$

of $\mathcal{C}_{f/}$ at (A, ϕ, ψ) is defined by

$$\mathrm{id}^{\mathcal{C}_{/f}}_{(A,\phi,\psi)} \stackrel{\text{def}}{=} (A, \mathrm{id}_A, f);$$

- *Composition*. For each

$$\begin{aligned}\mathbf{A} &= (A, \phi_A, \psi_A),\\ \mathbf{B} &= (B, \phi_B, \psi_B),\\ \mathbf{C} &= (C, \phi_C, \psi_C)\end{aligned}$$

in $\mathrm{Obj}(\mathcal{C}_{/f})$, the composition map

$$\circ^{\mathcal{C}_{/f}}_{\mathbf{A},\mathbf{B},\mathbf{C}} \colon \mathrm{Hom}_{\mathcal{C}_{/f}}(\mathbf{B}, \mathbf{C}) \times \mathrm{Hom}_{\mathcal{C}_{/f}}(\mathbf{A}, \mathbf{B}) \longrightarrow \mathrm{Hom}_{\mathcal{C}_{/f}}(\mathbf{A}, \mathbf{C})$$

of $\mathcal{C}_{/f}$ at $((A, \phi_A, \psi_A), (B, \phi_B, \psi_B), (C, \phi_C, \psi_C))$ is defined by

$$\theta' \circ^{\mathcal{C}_{/f}}_{\mathbf{A},\mathbf{B},\mathbf{C}} \theta \stackrel{\text{def}}{=} \theta' \circ \theta.$$

9.3 Slice Categories Over Diagrams

Let $\mathcal{C}$ be a category and let $D \colon \mathcal{K} \longrightarrow \mathcal{C}$ be a functor.

Definition 9.3.1 ▶ Slice Categories Over Diagrams

[1]The **slice category of $\mathcal{C}$ over D** is the category $\mathcal{C}_{/D}$ defined by

$$\mathcal{C}_{/D} \stackrel{\text{def}}{=} \mathcal{C} \times_{\mathsf{Fun}(\mathcal{K},\mathcal{C})} \mathsf{Fun}(\mathcal{K}, \mathcal{C})_{/D}, \qquad \begin{tikzcd} \mathcal{C}_{/D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Fun}(\mathcal{K},\mathcal{C})_{/D} \arrow[d, "忘"] \\ \mathcal{C} \arrow[r, "\Delta_{(-)}"'] & \mathsf{Fun}(\mathcal{K},\mathcal{C}). \end{tikzcd}$$

[1] *Reference*: [Lur20, Tag 015V].

Remark 9.3.2 ▶ Unwinding Definition 9.3.1

In detail, $\mathcal{C}_{/D}$ is the category where

- *Objects.* The objects of $\mathcal{C}_{/D}$ are pairs (A, α) consisting of
  - An object A of $\mathcal{C}$;
  - A natural transformation $\alpha \colon \Delta_A \Longrightarrow D$ from Δ_A to D;
- *Morphisms.* A morphism of $\mathcal{C}_{/D}$ from (A, α) to (B, β) is a morphism $f \colon A \longrightarrow B$ of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
\Delta_A \arrow[rr, Rightarrow, "\Delta_f"] \arrow[dr, Rightarrow, "\alpha"'] & & \Delta_B \arrow[dl, Rightarrow, "\beta"] \\
& X &
\end{tikzcd}$$

  commutes;
- *Identities.* For each $(A, \alpha) \in \mathrm{Obj}(\mathcal{C}_{/D})$, the unit map

$$\mathbb{1}^{\mathcal{C}_{/D}}_{(A,\alpha)} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}_{/D}}((A, \alpha), (A, \alpha))$$

  of $\mathcal{C}_{/D}$ at (A, α) is given by

$$\mathrm{id}^{\mathcal{C}_{/D}}_{(A,\alpha)} \overset{\text{def}}{=} \mathrm{id}_A,$$

  as witnessed by the commutativity of the diagram

$$\begin{tikzcd}
\Delta_A \arrow[rr, equal, "\Delta_{\mathrm{id}_A}"] \arrow[dr, Rightarrow, "\alpha"'] & & \Delta_A \arrow[dl, Rightarrow, "\alpha"] \\
& D &
\end{tikzcd}$$

  in $\mathcal{C}$;
- *Composition.* For each $(A, \alpha), (B, \beta), (C, \gamma) \in \mathrm{Obj}(\mathcal{C}_{/D})$, the composition map

$$\circ^{\mathcal{C}_{/D}}_{(A,\alpha),(B,\beta),(C,\gamma)} \colon \mathrm{Hom}_{\mathcal{C}_{/D}}((B, \beta), (C, \gamma)) \times \mathrm{Hom}_{\mathcal{C}_{/D}}((A, \alpha), (B, \beta)) \longrightarrow \mathrm{Hom}_{\mathcal{C}_{/D}}((A, \alpha), (C, \gamma))$$

  of $\mathcal{C}_{/D}$ at $((A, \alpha), (B, \beta), (C, \gamma))$ is defined by

$$\circ^{\mathcal{C}_{/D}}_{(A,\alpha),(B,\beta),(C,\gamma)} \overset{\text{def}}{=} \circ^{\mathcal{C}}_{A,B,C},$$

as witnessed by the commutativity of the diagram

$$\begin{tikzcd}
\Delta_A \arrow[r, Rightarrow, "\Delta_f"] \arrow[dr, Rightarrow, "\alpha"'] & \Delta_B \arrow[r, Rightarrow, "\Delta_g"] \arrow[d, Rightarrow, "\beta"] & \Delta_C \arrow[dl, Rightarrow, "\gamma"] \\
& D &
\end{tikzcd}$$

in $\mathcal{C}$.

Proposition 9.3.3 ▶ Properties of Slice Categories Over Diagrams

Let $D\colon \mathcal{K} \longrightarrow \mathcal{C}$ be a functor.

1. *Functoriality.* The assignments $D \mapsto \mathcal{C}_{/D}, (\mathcal{C}_{/D}, 忘)$ define functors

$$\begin{aligned} \mathcal{C}_{/(-)}\colon \mathsf{Fun}(\mathcal{K},\mathcal{C}) &\longrightarrow \mathsf{Cats}, \\ (\mathcal{C}_{/(-)}, 忘)\colon \mathsf{Fun}(\mathcal{K},\mathcal{C}) &\longrightarrow \mathsf{Cats}_{/\mathcal{C}}. \end{aligned}$$

2. *Relation to Overcategories.* Let $X \in \mathrm{Obj}(\mathcal{C})$. We have an isomorphism of categories

$$\mathcal{C}_{/[X]} \cong \mathcal{C}_{/X},$$

where we pick $\mathcal{K} = \mathrm{pt}$ and where $[X]\colon \mathcal{C} \xrightarrow{\cong} \mathsf{Fun}(\mathrm{pt}, \mathcal{C})$ is the functor from pt to $\mathcal{C}$ picking X.

3. *Interaction With Opposites.* We have isomorphisms of categories

$$\begin{aligned} (\mathcal{C}_{/D})^{\mathsf{op}} &\cong (\mathcal{C}^{\mathsf{op}})_{D^{\mathsf{op}}/}, \\ (\mathcal{C}_{D/})^{\mathsf{op}} &\cong (\mathcal{C}^{\mathsf{op}})_{/D^{\mathsf{op}}}. \end{aligned}$$

Proof 9.3.4 ▶ Proof of Proposition 9.3.3

Item 1: Functoriality

Omitted.

Item 2: Relation to Overcategories

See [Lur20, Tag 015X].

Item 3: Interaction With Opposites

See [Lur20, Tag 015W].

10 Coslice Categories

10.1 Coslice Categories

Let $\mathcal{C}$ be a category.

Definition 10.1.1 ▶ Coslice Categories

The **coslice category of $\mathcal{C}$ by X**[1] is the category $\mathcal{C}_{X/}$ where

- *Objects.* The objects of $\mathcal{C}_{X/}$ are pairs (A, ϕ) consisting of
  - An object A of $\mathcal{C}$;
  - A morphism $\phi\colon X \longrightarrow A$ of $\mathcal{C}$;
- *Morphisms.* A morphism of $\mathcal{C}_{X/}$ from (A, ϕ) to (B, ψ) is a morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd}$$

  commutes.
- *Identities.* For each $(A, \phi) \in \mathrm{Obj}(\mathcal{C}_{X/})$, the unit map

$$\mathbb{1}^{\mathcal{C}_{X/}}_{(A,\phi)}\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}_{X/}}((A, \phi), (A, \phi))$$

  of $\mathcal{C}_{X/}$ at (A, ϕ) is given by

$$\mathrm{id}^{\mathcal{C}_{X/}}_{(A,\phi)} \overset{\mathrm{def}}{=} \mathrm{id}_A,$$

  as witnessed by the commutativity of the diagram

$$\begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[dr, "\phi"] & \\ A \arrow[rr, equal, "\mathrm{id}_A"'] & & A \end{tikzcd}$$

  in $\mathcal{C}$;

- *Composition.* For each $(A,\phi),(B,\psi),(C,\chi)\in\mathrm{Obj}(\mathcal{C}_{X/})$, the composition map

$$\circ^{\mathcal{C}_{X/}}_{(A,\phi),(B,\psi),(C,\chi)}\colon \mathrm{Hom}_{\mathcal{C}_{X/}}((B,\psi),(C,\chi))\times\mathrm{Hom}_{\mathcal{C}_{X/}}((A,\phi),(B,\psi))\longrightarrow\mathrm{Hom}_{\mathcal{C}_{X/}}((A,\phi),(C,\chi))$$

of $\mathcal{C}_{X/}$ at $((A,\phi),(B,\psi),(C,\chi))$ is defined by

$$\circ^{\mathcal{C}_{X/}}_{(A,\phi),(B,\psi),(C,\chi)}\stackrel{\text{def}}{=}\circ^{\mathcal{C}}_{A,B,C},$$

as witnessed by the commutativity of the diagram

$$\begin{tikzcd}
 & X \arrow[dl, "\phi"'] \arrow[d, "\psi"] \arrow[dr, "\chi"] & \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

in $\mathcal{C}$.

[1] *Further Terminology:* Also called the **category of objects of $\mathcal{C}$ under X**.

Proposition 10.1.2 ▶ Properties of Coslice Categories

Let $\phi\colon X\longrightarrow Y$ be a morphism of $\mathcal{C}$.

1. *Functoriality.* The assignment $X\mapsto\mathcal{C}_{/X}$ defines functors

$$\mathcal{C}_{(-)/}\colon\mathcal{C}^{\mathrm{op}}\longrightarrow\mathsf{Cats},$$
$$\left(\mathcal{C}_{(-)/},\text{忘}^{\mathcal{C}_{(-)/}}\right)\colon\mathcal{C}^{\mathrm{op}}\longrightarrow\mathsf{Cats}_{/\mathcal{C}}.$$

2. *Cobase Change I.* We have a functor

$$\mathcal{C}_{\phi/}\colon\mathcal{C}_{Y/}\longrightarrow\mathcal{C}_{X/},$$

also written $\phi_*\colon\mathcal{C}_{Y/}\longrightarrow\mathcal{C}_{X/}$.

3. *Cobase Change II.* If $\mathcal{C}$ has pushouts, then we have a functor

$$\phi^*\colon\mathcal{C}_{X/}\longrightarrow\mathcal{C}_{Y/}$$

where

- *Action on Objects.* For each $(A,\theta)\in\mathrm{Obj}(\mathcal{C}_{X/})$, we have

$$\phi^*(A,\theta)\overset{\mathrm{def}}{=}\left(A\textstyle\coprod_X Y,\mathrm{inj}_2\right),$$

where $A\coprod_X Y$ is the pushout

$$\begin{tikzcd}
A\coprod_X Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l, "\mathrm{inj}_2"'] \\
A \arrow[u, "\mathrm{inj}_1"] & X; \arrow[l, "\theta"] \arrow[u, "\phi"']
\end{tikzcd}$$

- *Action on Morphisms.* For each morphism $f\colon (A,\theta)\longrightarrow(B,\theta')$ of $\mathcal{C}_{X/}$, the image

$$\phi^*(f)\colon \phi^*(A,\theta)\longrightarrow\phi^*(B,\theta')$$

of f by ϕ^* is the dashed morphism in the diagram

$$\begin{tikzcd}
A\coprod_X Y \arrow[dr, dashed, "\exists!"] & & X \arrow[ll, "\mathrm{inj}_2"'] \arrow[dr, equal] & \\
& B\coprod_Y X & & X \arrow[ll, "\mathrm{inj}_2"'] \\
A \arrow[uu, "\mathrm{inj}_1"] \arrow[dr, "\phi"'] & & Y \arrow[ll, "\theta"'] \arrow[uu, "f"] \arrow[dr, equal] & \\
& B \arrow[uu, "\mathrm{inj}_1"] & & Y, \arrow[ll, "\theta'"] \arrow[uu, "\phi"']
\end{tikzcd}$$

obtained via Limits and Colimits, Item 3 of Proposition 1.6.4.

4. *Cobase Change III.* If $\mathcal{C}$ has pushouts, then we have an adjunction

$$(\phi_*\dashv\phi^*)\colon\quad \begin{tikzcd}
\mathcal{C}_{Y/} \arrow[r, bend left, "\phi_*"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{X/}. \arrow[l, bend left, "\phi^*"]
\end{tikzcd}$$

5. *Cobase Change IV.* If $\mathcal{C}$ is locally coCartesian coclosed, then we have a functor
$$\phi_! \colon \mathcal{C}_{Y/} \longrightarrow \mathcal{C}_{X/},$$
assembling into a triple adjunction
$$(\phi_* \dashv \phi^* \dashv \phi_!)\colon \begin{tikzcd} \mathcal{C}_{Y/} \arrow[r, bend left=45, "\phi_*"] \arrow[r, phantom, shift left=2ex, "\perp"] \arrow[r, phantom, shift right=2ex, "\perp"] \arrow[r, bend right=45, "\phi_!"'] & \mathcal{C}_{X/} \arrow[l, "\phi^*"] \end{tikzcd}$$
between $\mathcal{C}_{Y/}$ and $\mathcal{C}_{X/}$.
6. *Relation to the Grothendieck Construction*[1]. We have an isomorphisms of categories
$$\mathcal{C}_{X/} \cong \int_{\mathcal{C}} h^X.$$
7. *Duality.* We have an isomorphism of categories
$$(\mathcal{C}_{X/})^{\mathsf{op}} \cong (\mathcal{C}^{\mathsf{op}})_{/X}.$$
8. *As Comma Categories.* We have an isomorphisms of categories
$$\mathcal{C}_{X/} \cong [X] \downarrow \mathrm{id}_{\mathcal{C}}, \qquad \begin{tikzcd} \mathcal{C}_{X/} \arrow[r, "忘"] \arrow[d] & \mathcal{C} \arrow[d, "\mathrm{id}_{\mathcal{C}}"] \\ \mathrm{pt} \arrow[r, "{[X]}"'] \arrow[ur, phantom, "\nearrow"] & \mathcal{C}. \end{tikzcd}$$
9. *As Pullbacks.* We have an isomorphism of categories
$$\mathcal{C}_{X/} \cong \{X\} \times_{\iota_X, \mathcal{C}, \mathrm{ev}_0} \mathsf{Arr}(\mathcal{C}), \qquad \begin{tikzcd} \mathcal{C}_{X/} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{Arr}(\mathcal{C}) \arrow[d, "\mathrm{ev}_0"] \\ \{X\} \arrow[r, "\iota_X"'] & \mathcal{C}. \end{tikzcd}$$
10. *Coslices of Copresheaf Categories.* We have an equivalence of categories
$$\mathsf{CoPSh}(\mathcal{C}_{X/}) \overset{\text{eq.}}{\cong} \mathsf{CoPSh}(\mathcal{C})_{/h^X}.$$

More generally, given a copresheaf $F: C \longrightarrow \mathsf{Sets}$ on C, we have an equivalence of categories

$$\mathsf{CoPSh}\left(\int_C F\right) \stackrel{\text{eq.}}{\cong} \mathsf{CoPSh}(C)_{/F}.$$

[1] This is a repetition of Fibred Categories, Example 9.3.1.

Proof 10.1.3 ▶ Proof of Proposition 10.1.2

Item 1: Functoriality

This is dual to Item 1 of Proposition 9.1.3.

Item 2: Cobase Change I

Omitted.

Item 3: Cobase Change II

Omitted.

Item 4: Cobase Change III

Omitted.

Item 5: Cobase Change IV

Omitted.

Item 6: Relation to the Grothendieck Construction

Omitted.

Item 7: Duality

Clear.[1]

Item 8: As Comma Categories

Clear.[2]

Item 9: As Pullbacks

Clear.[3]

Item 10: Coslices of Copresheaf Categories

See [nLab23a, Section 2].

[1] *Reference:* [Lur20, Tag 015T].
[2] *Reference:* [Lur20, Tag 015U].
[3] *Reference:* [Lur20, Tag 015U].

10.2 Coslice Categories of Morphisms

Let C be a category and let $f: X \longrightarrow Y$ be a morphism of C.

Definition 10.2.1 ▶ Coslice Categories of Morphisms

The **coslice category of $\mathcal{C}$ by f**[1] is the category $\mathcal{C}_{f/}$ defined by[2]

$$\begin{aligned}\mathcal{C}_{f/} &\overset{\text{def}}{=} \mathcal{C}_{Y/} \times_{\mathcal{C}_{X/}} \mathcal{C}_{X/}\\ &\cong \mathcal{C}_{Y/},\end{aligned} \qquad \begin{tikzcd} \mathcal{C}_{f/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C}_{X/} \arrow[d, "\mathrm{id}_{\mathcal{C}_{X/}}"] \\ \mathcal{C}_{Y/} \arrow[r, "\mathcal{C}_{/f}"'] & \mathcal{C}_{X/}. \end{tikzcd}$$

[1] *Further Terminology:* Also called the **category of objects of $\mathcal{C}$ under f**.
[2] We also have an isomorphism of categories

$$\mathcal{C}_{f/} \cong [f] \downarrow \mathrm{id}_{-}, \qquad \begin{tikzcd} \mathcal{C}_{f/} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, hook] \\ \mathrm{pt} \arrow[r, "{[f]}"'] \arrow[ur, Rightarrow] & \mathrm{Arr}(\mathcal{C}). \end{tikzcd}$$

Remark 10.2.2 ▶ Unwinding Definition 10.2.1

In detail, $\mathcal{C}_{f/}$ is the category where

- *Objects.* The objects of $\mathcal{C}_{f/}$ are pairs (A, ϕ, ψ) consisting of
  - An object A of $\mathcal{C}$;
  - A morphism $\phi \colon A \longrightarrow X$ of $\mathcal{C}$;
  - A morphism $\psi \colon B \longrightarrow X$ of $\mathcal{C}$;

  such that the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\phi"'] & & B \arrow[dl, "\psi"] \\ & X & \end{tikzcd}$$

- *Morphisms.* A morphism of $\mathcal{C}_{/f}$ from (A, ϕ_A, ψ_A) to (B, ϕ_B, ψ_B) is a morphism $\theta\colon A \longrightarrow B$ of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi_A"'] \arrow[drr, "\mathrm{id}_A"] & Y \arrow[dl, "\psi_A"] \arrow[drr, "\mathrm{id}_B"] & & \\
A \arrow[drr, "\theta"'] & & X \arrow[r, "f"] \arrow[d, "\phi_B"] & Y \arrow[dl, "\psi_B"] \\
& & B &
\end{tikzcd}$$

commutes, i.e. such that the diagrams

$$\begin{tikzcd}
& X \arrow[dl, "\phi_B"'] \arrow[dr, "\phi_A"] & \\
A \arrow[rr, "\theta"'] & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& Y \arrow[dl, "\psi_B"'] \arrow[dr, "\psi_A"] & \\
A \arrow[rr, "\theta"'] & & B
\end{tikzcd}$$

commute.

- *Identities*. For each $(A, \phi, \psi) \in \mathrm{Obj}(\mathcal{C}_{f/})$, the unit map

$$\mathbb{1}^{\mathcal{C}_{f/}}_{(A,\phi,\psi)}\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}_{f/}}((A, \phi, \psi), (A, \phi, \psi))$$

of $\mathcal{C}_{f/}$ at (A, ϕ, ψ) is defined by

$$\mathrm{id}^{\mathcal{C}_{f/}}_{(A,\phi,\psi)} \stackrel{\text{def}}{=} (A, \mathrm{id}_A, f);$$

- *Composition*. For each

$$\begin{aligned}
\mathbf{A} &= (A, \phi_A, \psi_A),\\
\mathbf{B} &= (B, \phi_B, \psi_B),\\
\mathbf{C} &= (C, \phi_C, \psi_C)
\end{aligned}$$

in $\mathrm{Obj}(\mathcal{C}_{f/})$, the composition map

$$\circ^{\mathcal{C}_{f/}}_{\mathbf{A},\mathbf{B},\mathbf{C}}\colon \mathrm{Hom}_{\mathcal{C}_{f/}}(\mathbf{B}, \mathbf{C}) \times \mathrm{Hom}_{\mathcal{C}_{f/}}(\mathbf{A}, \mathbf{B}) \longrightarrow \mathrm{Hom}_{\mathcal{C}_{f/}}(\mathbf{A}, \mathbf{C})$$

of $\mathcal{C}_{f/}$ at (A, ϕ_A, ψ_A), (B, ϕ_B, ψ_B), (C, ϕ_C, ψ_C) is defined by

$$\theta' \circ^{\mathcal{C}_{f/}}_{\mathbf{A},\mathbf{B},\mathbf{C}} \theta \stackrel{\text{def}}{=} \theta' \circ \theta.$$

10.3 Coslice Categories Under Diagrams

Let $\mathcal{C}$ be a category and let $D\colon \mathcal{K} \longrightarrow \mathcal{C}$ be a functor.

Definition 10.3.1 ▶ Coslice Categories Under Diagrams

[1]The **coslice category of $\mathcal{C}$ under D** is the category $\mathcal{C}_{D/}$ defined by

$$\mathcal{C}_{D/} \overset{\text{def}}{=} \mathcal{C} \times_{\mathsf{Fun}(\mathcal{K},\mathcal{C})} \mathsf{Fun}(\mathcal{K},\mathcal{C})_{D/}, \qquad \begin{tikzcd} \mathcal{C}_{D/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Fun}(\mathcal{K},\mathcal{C})_{D/} \arrow[d, "忘"] \\ \mathcal{C} \arrow[r, "\Delta_{(-)}"'] & \mathsf{Fun}(\mathcal{K},\mathcal{C}). \end{tikzcd}$$

[1] *Reference:* [Lur20, Tag 015V].

Remark 10.3.2 ▶ Unwinding Definition 10.3.1

In detail, $\mathcal{C}_{D/}$ is the category where

- *Objects.* The objects of $\mathcal{C}_{D/}$ are pairs (A, α) consisting of
  - An object A of $\mathcal{C}$;
  - A natural transformation $\alpha\colon D \Longrightarrow \Delta_A$ from D to Δ_A;
- *Morphisms.* A morphism of $\mathcal{C}_{D/}$ from (A, α) to (B, β) is a morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} & D \arrow[dl, Rightarrow, "\alpha"'] \arrow[dr, Rightarrow, "\beta"] & \\ \Delta_A \arrow[rr, Rightarrow, "\Delta_f"'] & & \Delta_B \end{tikzcd}$$

  commutes;
- *Identities.* For each $(A, \alpha) \in \mathrm{Obj}(\mathcal{C}_{D/})$, the unit map

$$\mathbb{1}^{\mathcal{C}_{D/}}_{(A,\alpha)}\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}_{D/}}((A, \alpha), (A, \alpha))$$

  of $\mathcal{C}_{D/}$ at (A, α) is given by

$$\mathrm{id}^{\mathcal{C}_{D/}}_{(A,\alpha)} \overset{\text{def}}{=} \mathrm{id}_A,$$

as witnessed by the commutativity of the diagram

$$\begin{tikzcd} & D \arrow[dl, Rightarrow, "\alpha"'] \arrow[dr, Rightarrow, "\alpha"] & \\ A \arrow[rr, equal, "\mathrm{id}_A"'] & & A \end{tikzcd}$$

in $\mathcal{C}$;

- *Composition*. For each $(A, \alpha), (B, \beta), (C, \gamma) \in \mathrm{Obj}(\mathcal{C}_{D/})$, the composition map

$$\circ^{\mathcal{C}_{D/}}_{(A,\alpha),(B,\beta),(C,\gamma)} \colon \mathrm{Hom}_{\mathcal{C}_{D/}}((B,\beta),(C,\gamma)) \times \mathrm{Hom}_{\mathcal{C}_{D/}}((A,\alpha),(B,\beta)) \longrightarrow \mathrm{Hom}_{\mathcal{C}_{D/}}((A,\alpha),(C,\gamma))$$

of $\mathcal{C}_{D/}$ at $((A,\alpha),(B,\beta),(C,\gamma))$ is defined by

$$\circ^{\mathcal{C}_{D/}}_{(A,\alpha),(B,\beta),(C,\gamma)} \overset{\text{def}}{=} \circ^{\mathcal{C}}_{A,B,C},$$

as witnessed by the commutativity of the diagram

$$\begin{tikzcd} & D \arrow[dl, Rightarrow, "\alpha"'] \arrow[d, equal] \arrow[dr, Rightarrow, "\gamma"] & \\ & D \arrow[d, Rightarrow, "\beta"] & \\ \Delta_A \arrow[r, Rightarrow, "\Delta_f"'] & \Delta_B \arrow[r, Rightarrow, "\Delta_g"'] & \Delta_C \end{tikzcd}$$

in $\mathcal{C}$.

Proposition 10.3.3 ▶ Properties of Coslice Categories Under Diagrams

Let $D \colon \mathcal{K} \longrightarrow \mathcal{C}$ be a functor.

1. *Functoriality*. The assignments $D \mapsto \mathcal{C}_{D/}, (\mathcal{C}_{D/}, 忘)$ define functors

$$\begin{aligned} \mathcal{C}_{(-)/} \colon \mathrm{Fun}(\mathcal{K}, \mathcal{C}) &\longrightarrow \mathsf{Cats}, \\ (\mathcal{C}_{(-)/}, 忘) \colon \mathrm{Fun}(\mathcal{K}, \mathcal{C}) &\longrightarrow \mathsf{Cats}_{/\mathcal{C}}. \end{aligned}$$

2. *Relation to Overcategories*. Let $X \in \mathrm{Obj}(\mathcal{C})$. We have an isomorphism of categories

$$\mathcal{C}_{[X]/} \cong \mathcal{C}_{X/},$$

where we pick $\mathcal{K} = \mathrm{pt}$ and where $[X]\colon \mathcal{C} \xrightarrow{\cong} \mathsf{Fun}(\mathrm{pt}, \mathcal{C})$ is the functor from pt to $\mathcal{C}$ picking X.

3. *Interaction With Opposites.* We have isomorphisms of categories

$$(\mathcal{C}_{/D})^{\mathsf{op}} \cong (\mathcal{C}^{\mathsf{op}})_{D^{\mathsf{op}}/},$$
$$(\mathcal{C}_{D/})^{\mathsf{op}} \cong (\mathcal{C}^{\mathsf{op}})_{/D^{\mathsf{op}}}.$$

Proof 10.3.4 ▶ Proof of Proposition 10.3.3

Item 1: Functoriality

Omitted.

Item 2: Relation to Overcategories

See [Lur20, Tag 015X].

Item 3: Interaction With Opposites

See [Lur20, Tag 015W].

11 Quotients of Categories

11.1 Quotients of Categories by Profunctors

Definition 11.1.1 ▶ Equivalence Classes of Objects by Profunctors

Let $R\colon \mathcal{C} \nrightarrow \mathcal{C}$ be a profunctor on $\mathcal{C}$ and let $A \in \mathrm{Obj}(\mathcal{C})$.[1,2]

1. The **left equivalence class of A by R** is the presheaf $[A]^{\mathsf{L}}\colon \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Sets}$ on $\mathcal{C}$ defined as the composition

$$\mathcal{C}^{\mathsf{op}} \xrightarrow{\cong} \mathcal{C}^{\mathsf{op}} \times \mathrm{pt} \xrightarrow{\mathrm{id}\times[A]} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{R} \mathsf{Sets}.$$

2. The **right equivalence class of A by R** is the copresheaf $[A]^{\mathsf{R}}\colon \mathcal{C} \longrightarrow \mathsf{Sets}$ on $\mathcal{C}$ defined as the composition

$$\mathcal{C} \xrightarrow{\cong} \mathrm{pt} \times \mathcal{C} \xrightarrow{[A]\times\mathrm{id}} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{R} \mathsf{Sets}.$$

[1] On objects, we have

$$[A]^{\mathsf{L}}(X) \stackrel{\text{def}}{=} R^X_A,$$

$$[A]^{\mathsf{R}}(X) \overset{\text{def}}{=} R^A_X$$

for each $X \in \mathrm{Obj}(\mathcal{C})$.

[2]Viewing R as a functor $R\colon \mathcal{C} \longrightarrow \mathsf{PSh}(\mathcal{C})$ or as a functor $R\colon \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{CoPSh}(\mathcal{C})$, we have more simply

$$[A]^{\mathsf{L}} \overset{\text{def}}{=} R(A),$$
$$[A]^{\mathsf{R}} \overset{\text{def}}{=} R(A)$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

Definition 11.1.2 ▶ Quotients of Categories by Profunctors

Let $R\colon \mathcal{C} \nrightarrow \mathcal{C}$ be a profunctor on $\mathcal{C}$.

1. The **left quotient of** $\mathcal{C}$ **by** R is the full subcategory $\mathcal{C} \setminus R$ of $\mathsf{PSh}(\mathcal{C})$ spanned by the presheaves of the form $[A]^{\mathsf{L}}$ with $A \in \mathrm{Obj}(\mathcal{C})$.

2. The **right quotient of** $\mathcal{C}$ **by** R is the full subcategory $\mathcal{C}/R$ of $\mathsf{CoPSh}(\mathcal{C})$ spanned by the copresheaves of the form $[A]^{\mathsf{R}}$ with $A \in \mathrm{Obj}(\mathcal{C})$.

Definition 11.1.3 ▶ The Quotient Projection Functor

Let $R\colon \mathcal{C} \nrightarrow \mathcal{C}$ be a profunctor on $\mathcal{C}$.

1. The **left quotient projection functor from** $\mathcal{C}$ **to** $\mathcal{C} \setminus R$ is the functor

$$\pi^{\mathsf{L}}_{\mathcal{C}}\colon \mathcal{C}^{\mathsf{op}} \twoheadrightarrow \mathcal{C} \setminus R$$

   where

   - *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\pi^{\mathsf{L}}_{\mathcal{C}}(A) \overset{\text{def}}{=} [A]^{\mathsf{L}};$$

   - *Action on Morphisms.* For each morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$, the image

$$\pi^{\mathsf{L}}_{\mathcal{C}}(f)\colon \pi^{\mathsf{L}}_{\mathcal{C}}(B) \Longrightarrow \pi^{\mathsf{L}}_{\mathcal{C}}(A)$$

     of f by $\pi^{\mathsf{L}}_{\mathcal{C}}$ is the natural transformation defined by

$$\begin{aligned}\pi^{\mathsf{L}}_{\mathcal{C}}(f) &\overset{\text{def}}{=} \left\{[f]^{\mathsf{L}}_X\colon [B]^{\mathsf{L}}_X \longrightarrow [A]^{\mathsf{L}}_X\right\}_{X \in \mathrm{Obj}(\mathcal{C})} \\ &\overset{\text{def}}{=} \left\{R^f_X\colon R^B_X \longrightarrow R^A_X\right\}_{X \in \mathrm{Obj}(\mathcal{C})}.\end{aligned}$$

2. The **right quotient projection functor from $\mathcal{C}$ to $\mathcal{C}/R$** is the functor

$$\pi^{\mathsf{R}}_{\mathcal{C}}\colon \mathcal{C} \twoheadrightarrow \mathcal{C}/R$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\pi^{\mathsf{R}}_{\mathcal{C}}(A) \overset{\text{def}}{=} [A]^{\mathsf{R}};$$

- *Action on Morphisms.* For each morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$, the image

$$\pi^{\mathsf{R}}_{\mathcal{C}}(f)\colon \pi^{\mathsf{R}}_{\mathcal{C}}(A) \Longrightarrow \pi^{\mathsf{R}}_{\mathcal{C}}(B)$$

of f by $\pi^{\mathsf{R}}_{\mathcal{C}}$ is the natural transformation defined by

$$\begin{aligned}\pi^{\mathsf{R}}_{\mathcal{C}}(f) &\overset{\text{def}}{=} \left\{[f]^{\mathsf{R}}_X\colon [A]^{\mathsf{R}}_X \longrightarrow [B]^{\mathsf{R}}_X\right\}_{X\in\mathrm{Obj}(\mathcal{C})}\\ &\overset{\text{def}}{=} \left\{R^X_f\colon R^X_A \longrightarrow R^X_B\right\}_{X\in\mathrm{Obj}(\mathcal{C})}.\end{aligned}$$

PROOF 11.1.4 ▶ PROOF OF DEFINITION 11.1.3

$\pi^{\mathsf{L}}_{\mathcal{C}}(f)$ Is Indeed a Natural Transformation

Naturality for $\pi^{\mathsf{L}}_{\mathcal{C}}(f)$ corresponds to the condition that, for each morphism $g\colon X \longrightarrow Y$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
{[B]^{\mathsf{L}}_X} \arrow[r, "{[B]^{\mathsf{L}}_g}"] \arrow[d, "{[f]^{\mathsf{L}}_X}"'] & {[B]^{\mathsf{L}}_Y} \arrow[d, "{[f]^{\mathsf{L}}_Y}"] \\
{[A]^{\mathsf{L}}_X} \arrow[r, "{[A]^{\mathsf{L}}_g}"'] & {[A]^{\mathsf{L}}_Y}
\end{tikzcd}$$

whose entries are defined by

$$
\begin{tikzcd}
R^B_X \arrow[r, "R^{\mathrm{id}_B}_g"] \arrow[d, "R^f_{\mathrm{id}_X}"'] & R^B_Y \arrow[d, "R^f_{\mathrm{id}_Y}"] \\
R^A_X \arrow[r, "R^{\mathrm{id}_A}_g"'] & R^A_Y
\end{tikzcd}
$$

commutes, i.e. that we have

$$R^f_{\mathrm{id}_Y} \circ R^{\mathrm{id}_B}_g = R^{\mathrm{id}_A}_g \circ R^f_{\mathrm{id}_X}.$$

And indeed, by the functoriality of R, we have

$$
\begin{aligned}
R^f_{\mathrm{id}_Y} \circ R^{\mathrm{id}_B}_g &= R^{\mathrm{id}_B \circ f}_{\mathrm{id}_Y \circ g} \\
&= R^f_g \\
&= R^{f \circ \mathrm{id}_A}_{g \circ \mathrm{id}_X} \\
&= R^{\mathrm{id}_A}_g \circ R^f_{\mathrm{id}_X}.
\end{aligned}
$$

π^{L}_C Indeed Preserves Composition

We claim that, given morphisms $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ of C, we have

$$\pi^{\mathsf{L}}_C(g \circ f) = \pi^{\mathsf{L}}_C(f) \circ \pi^{\mathsf{L}}_C(g).$$

Indeed, we have

$$
\begin{aligned}
\pi^{\mathsf{L}}_C(g \circ f) &\overset{\text{def}}{=} \left\{R^{g \circ f}_X \colon R^C_X \longrightarrow R^A_X\right\}_{X \in \mathrm{Obj}(C)} \\
&= \left\{R^f_X \circ R^g_X \colon R^C_X \longrightarrow R^A_X\right\}_{X \in \mathrm{Obj}(C)} \\
&\overset{\text{def}}{=} \left\{R^f_X \colon R^C_X \longrightarrow R^B_X\right\}_{X \in \mathrm{Obj}(C)} \circ \left\{R^g_X \colon R^B_X \longrightarrow R^A_X\right\}_{X \in \mathrm{Obj}(C)} \\
&\overset{\text{def}}{=} \pi^{\mathsf{L}}_C(f) \circ \pi^{\mathsf{L}}_C(g).
\end{aligned}
$$

π^{L}_C Indeed Preserves Identities

We claim that, given an object $A \in \mathrm{Obj}(C)$, we have

$$\pi^{\mathsf{L}}_C(\mathrm{id}_A) = \mathrm{id}_{\pi^{\mathsf{L}}_C(A)}.$$

Indeed, we have

$$\begin{aligned}\pi^{\mathsf{L}}_{C}(\mathrm{id}_A) &\stackrel{\text{def}}{=} \left\{R^{\mathrm{id}_A}_X \colon R^A_X \longrightarrow R^A_X\right\}_{X\in\mathrm{Obj}(C)}\\ &= \left\{\mathrm{id}_{R^A_X} \colon R^A_X \longrightarrow R^A_X\right\}_{X\in\mathrm{Obj}(C)}\\ &\stackrel{\text{def}}{=} \mathrm{id}_{\pi^{\mathsf{L}}_{C}(A)}.\end{aligned}$$

This finishes the proof that π^{L}_{C} is indeed a functor. The proof for π^{R}_{C} is similar, and therefore omitted. ∎

Example 11.1.5 ▶ Examples of Quotients of Categories by Profunctors

Here are some examples of quotients of categories by profunctors.

1. *The Trivial Quotient.* If $R = \Delta_{\mathrm{pt}}$ is the trivial profunctor, then

$$\begin{aligned}[A]^{\mathsf{L}} &= \Delta_{\mathrm{pt}},\\ [A]^{\mathsf{R}} &= \Delta_{\mathrm{pt}}\end{aligned}$$

for all $A \in \mathrm{Obj}(C)$, and thus we have equivalences of categories

$$\begin{aligned}C\backslash R &\stackrel{\text{eq.}}{\cong} \mathrm{pt},\\ C/R &\stackrel{\text{eq.}}{\cong} \mathrm{pt},\end{aligned}$$

and the left and right quotient projection functors

$$\begin{aligned}\pi^{\mathsf{L}}_{C} &\colon C \twoheadrightarrow C\backslash R,\\ \pi^{\mathsf{R}}_{C} &\colon C \twoheadrightarrow C/R,\end{aligned}$$

are both given by the terminal functor from C to pt.

This corresponds to the trivial relation $\sim_{\mathrm{triv}} \stackrel{\text{def}}{=} \Delta_{\mathrm{true}}$ on a set X of Relations, Example 1.1.7, for which we have $X/{\sim_{\mathrm{triv}}} \cong \mathrm{pt}$, and whose projection map $X \longrightarrow X/{\sim_{\mathrm{triv}}}$ is given by the terminal map from X to pt.

2. *The Identity Quotient.* If $R = \mathrm{Hom}_{C}(-_1, -_2)$ is the identity profunctor of C, then

$$\begin{aligned}[A]^{\mathsf{L}} &= h_A,\\ [A]^{\mathsf{R}} &= h^A\end{aligned}$$

for all $A \in \mathrm{Obj}(\mathcal{C})$, and thus we have equivalences of categories

$$\mathcal{C} \backslash R \overset{\text{eq.}}{\cong} \mathcal{C},$$
$$\mathcal{C} / R \overset{\text{eq.}}{\cong} \mathcal{C}.$$

and the left and right quotient projection functors

$$\pi^{\mathsf{L}}_{\mathcal{C}} \colon \mathcal{C} \twoheadrightarrow \mathcal{C} \backslash R,$$
$$\pi^{\mathsf{R}}_{\mathcal{C}} \colon \mathcal{C} \twoheadrightarrow \mathcal{C} / R,$$

are both given by the identity functor from $\mathcal{C}$ to itself.

This corresponds to the characteristic relation $\chi_X(-_1, -_2) \overset{\text{def}}{=} \sim_{\text{id}} \overset{\text{def}}{=} \Delta_X$ on a set X of Relations, Example 1.1.9, for which we have $X/{\sim_{\text{id}}} \cong X$, and whose projection map $X \longrightarrow X/{\sim_{\text{id}}}$ is given by the identity map from X to itself.

3. *Quotients by Congruences.* The notion of a congruence relation on a category and its associated quotient is a special case of quotients by profunctors; see Proposition 11.2.6.[1]

4. *Categories With the Same Object Class.* Given a category $\mathcal{D}$ and a bijection $\mathrm{Obj}(\mathcal{D}) \cong \mathrm{Obj}(\mathcal{C})$, there exists a profunctor $R_{\mathcal{D}}$ such that we have

$$\mathcal{D} \cong \mathcal{C} \backslash R_{\mathcal{D}},$$
$$\mathcal{D}^{\mathsf{op}} \cong \mathcal{C} / R_{\mathcal{D}}.$$

Moreover, $R_{\mathcal{D}}$ in this case is a promonad, coming with natural transformations

$$\mu_R \colon R_{\mathcal{D}} \diamond R_{\mathcal{D}} \Longrightarrow R_{\mathcal{D}},$$
$$\eta_R \colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow R_{\mathcal{D}}$$

such that the diagrams

$$
\begin{tikzcd}
 & R_{\mathcal{D}} \diamond (R_{\mathcal{D}} \diamond R_{\mathcal{D}}) \arrow[dr, "\mathrm{id}_R \star \mu_R", "|" marking] & \\
(R_{\mathcal{D}} \diamond R_{\mathcal{D}}) \diamond R_{\mathcal{D}} \arrow[ur, "\alpha^{\mathsf{Prof}}_{A,A,A}", "|" marking] \arrow[d, "\mu_R \star \mathrm{id}_R"', "|" marking] & & R_{\mathcal{D}} \diamond R_{\mathcal{D}} \arrow[d, "\mu_R", "|" marking] \\
R_{\mathcal{D}} \diamond R_{\mathcal{D}} \arrow[rr, "\mu_R"', "|" marking] & & R_{\mathcal{D}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}} \diamond R_{\mathcal{D}} \arrow[r, "\eta_A \star R_{\mathcal{D}}", "|" marking] \arrow[dr, "\lambda^{\mathsf{Prof}}_{R_{\mathcal{D}}}"', "|" marking] & R_{\mathcal{D}} \diamond R_{\mathcal{D}} \arrow[d, "\mu_R", "|" marking] \\
 & R_{\mathcal{D}}
\end{tikzcd}
\qquad
\begin{tikzcd}
R_{\mathcal{D}} \diamond \mathrm{Hom}_{\mathcal{C}} \arrow[r, "R_{\mathcal{D}} \star \eta_A", "|" marking] \arrow[dr, "\rho^{\mathsf{Prof}}_{R_{\mathcal{D}}}"', "|" marking] & R_{\mathcal{D}} \diamond R_{\mathcal{D}} \arrow[d, "\mu_R", "|" marking] \\
 & R_{\mathcal{D}}
\end{tikzcd}
$$

commute.[2]

[1] As such, the examples in Example 11.2.4 are also examples of quotients of categories by profunctors.

[2] Promonads are the profunctor equivalent of reflexive and transitive relations.

11.2 Quotients of Categories by Congruence Relations

11.2.1 Foundations

Let $(\mathcal{C}, \circ^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$ be a category.

Definition 11.2.1 ▶ Congruence Relations on Categories

A **congruence relation $\sim$ on $\mathcal{C}$** is a collection[1]

$$\left\{\sim_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A,B) \nrightarrow \mathrm{Hom}_{\mathcal{C}}(A,B)\right\}_{A,B \in \mathrm{Obj}(\mathcal{C})}$$

of equivalence relations such that for each:

- Triple of objects (A, B, C) of $\mathcal{C}$:
- Pair of parallel morphisms $f_1, f_2 \colon A \rightrightarrows B$ from A to B;
- Pair of parallel morphisms $g_1, g_2 \colon B \rightrightarrows C$ from B to C;

if:

(★) We have $f_1 \sim_{A,B} f_2$ and $g_1 \sim_{B,C} g_2$;

then:

(★) We have $g_1 \circ f_1 \sim_{A,C} g_2 \circ f_2$.

[1] *Further Terminology*: The equivalence relation $\sim_{A,B}$ is called the **component of** $\sim$ **at** (A, B).

Definition 11.2.2 ▶ The Quotient of a Category by a Congruence Relation

The **quotient of** $\mathcal{C}$ **by a congruence relation** $\sim$ **on** $\mathcal{C}$ is the category $\mathcal{C}/\sim$ where

- *Objects*. We have

$$\begin{aligned}\mathrm{Obj}(\mathcal{C}/\sim) &\overset{\text{def}}{=} \{[A] \mid A \in \mathrm{Obj}(\mathcal{C})\}\\ &\cong \mathrm{Obj}(\mathcal{C});\end{aligned}$$

- *Morphisms*. For each $[A], [B] \in \mathrm{Obj}(\mathcal{C}/\sim)$, we have

$$\mathrm{Hom}_{\mathcal{C}/\sim}([A], [B]) \overset{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A, B)/\sim_{A,B};$$

- *Identities*. For each $[A] \in \mathrm{Obj}(\mathcal{C}/\sim)$, the unit map

$$\mathbb{k}^{\mathcal{C}/\sim}_{A} : \mathrm{pt} \longrightarrow \underbrace{\mathrm{Hom}_{\mathcal{C}/\sim}([A], [A])}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,A)/\sim_{A,A}}$$

of $\mathcal{C}/\sim$ at $[A]$ is given by the composition

$$\mathrm{pt} \xrightarrow{\mathbb{k}^{\mathcal{C}}_{A}} \mathrm{Hom}_{\mathcal{C}}(A, A) \twoheadrightarrow \mathrm{Hom}_{\mathcal{C}}(A, A)/\sim_{A,A};$$

- *Composition*. For each $[A], [B], [C] \in \mathrm{Obj}(\mathcal{C}/\sim)$, the composition map

$$\circ^{\mathcal{C}/\sim}_{[A],[B],[C]} : \underbrace{\mathrm{Hom}_{\mathcal{C}/\sim}([B], [C])}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(B,C)/\sim_{B,C}} \times \underbrace{\mathrm{Hom}_{\mathcal{C}/\sim}([A], [B])}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,B)/\sim_{A,B}} \longrightarrow \underbrace{\mathrm{Hom}_{\mathcal{C}/\sim}([A], [C])}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,C)/\sim_{A,C}}$$

of $\mathcal{C}/\sim$ at $[A], [B], [C]$ is defined by[1]

$$[g] \circ^{\mathcal{C}/\sim}_{A,B,C} [f] \overset{\text{def}}{=} [g \circ f].$$

[1] Note that $\circ^{\mathcal{C}/\sim}_{[A],[B],[C]}$ is well-defined since $\sim$ is a congruence relation: if $[f] = [f']$ and $[g] = [g']$, then $[g \circ f] = [g' \circ f']$.

Definition 11.2.3 ▶ The Quotient Functor From $\mathcal{C}$ to $\mathcal{C}/{\sim}$

The **quotient functor from $\mathcal{C}$ to $\mathcal{C}/{\sim}$** is the functor

$$\pi_{\mathcal{C}} \colon \mathcal{C} \twoheadrightarrow \mathcal{C}/{\sim}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\pi_{\mathcal{C}}(A) \overset{\text{def}}{=} [A];$$

- *Action on Morphisms.* For each $(A, B) \in \mathrm{Obj}(\mathcal{C})$, the action on Hom-sets

$$\pi^{\mathcal{C}}_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \underbrace{\mathrm{Hom}_{\mathcal{C}/\sim}([A], [B])}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,B)/\sim_{A,B}}$$

  of $\pi^{\mathcal{C}}$ at (A, B) is defined by

$$\pi_{\mathcal{C}}(f) \overset{\text{def}}{=} [f]$$

  for each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Example 11.2.4 ▶ Examples of Quotients of Categories by Congruence Relations

Here are some examples of quotients of categories by congruence relations.

1. *Commutative Monoids.* Let $\mathcal{C} = \mathsf{Mon}$ be the category of monoids.
   - The relation declaring $f \sim g$ iff $f^{\mathrm{ab}} = g^{\mathrm{ab}}$ defines a congruence relation on Mon.
   - The quotient category $\mathsf{Mon}/{\sim}$ is equivalent to CMon.
   - The composition of the quotient functor

$$\pi_{\mathsf{Mon}} \colon \mathsf{Mon} \longrightarrow \mathsf{Mon}/{\sim}$$

     of Mon by $\sim$ with the equivalence $\mathsf{Mon}/{\sim} \cong \mathsf{CMon}$ is given by the abelianisation functor $(-)^{\mathrm{ab}} \colon \mathsf{Mon} \longrightarrow \mathsf{CMon}$.
2. *Groups.* Let $\mathcal{C} = \mathsf{Mon}$ be the category of monoids.

- The relation declaring $f \sim g$ iff $f^{\mathrm{grp}} = g^{\mathrm{grp}}$ defines a congruence relation on Mon.
- The quotient category $\mathsf{Mon}/{\sim}$ is equivalent to Grp.
- The composition of the quotient functor

$$\pi_{\mathsf{Mon}} \colon \mathsf{Mon} \longrightarrow \mathsf{Mon}/{\sim}$$

of Mon by $\sim$ with the equivalence $\mathsf{Mon}/{\sim} \cong \mathsf{Grp}$ is given by the group completion functor $(-)^{\mathrm{grp}} \colon \mathsf{Mon} \longrightarrow \mathsf{Grp}$.

3. *Quotienting by Homotopies.* Let $\mathcal{C} = \mathsf{Spc}$ be a convenient category of spaces.

- The relation declaring $f \sim g$ if f is homotopic to g defines a congruence relation on Spc.
- The quotient category $\mathsf{Spc}/{\sim}$ is then the homotopy category of spaces $\mathsf{Ho}(\mathsf{Spc})$.
- The quotient functor

$$\pi_{\mathsf{Spc}} \colon \mathsf{Spc} \longrightarrow \mathsf{Ho}(\mathsf{Spc})$$

is then the localisation functor with respect to homotopy, being the identity on objects and sending a map of spaces f to its homotopy class $[f]$.

11.2.2 As a Special Case of Quotients of Categories by Profunctors

Let $\sim$ be a congruence relation on $\mathcal{C}$.

Definition 11.2.5 ▶ The Profunctor Associated to a Congruence Relation

The **profunctor associated to** $\sim$ is the profunctor $R \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathsf{Sets}$ where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathcal{C})$, we have

$$\begin{aligned} R^A_B &\overset{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{C}/\sim}([A], [B]) \\ &\overset{\mathrm{def}}{=} \mathrm{Hom}_{\mathcal{C}}(A, B)/{\sim_{A,B}}; \end{aligned}$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C})$, the action on **Hom**-sets

$$R_{(A,B),(X,Y)} \colon \underbrace{\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}((A,B),(X,Y))}_{\substack{\overset{\mathrm{def}}{=}\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A,X)\times\mathrm{Hom}_{\mathcal{C}}(B,Y)\\ \overset{\mathrm{def}}{=}\mathrm{Hom}_{\mathcal{C}}(X,A)\times\mathrm{Hom}_{\mathcal{C}}(B,Y)}} \longrightarrow \mathrm{Hom}_{\mathsf{Sets}}\left(R^B_X, R^A_Y\right)$$

of R at $((A,B),(X,Y))$ is defined by[1]

$$\begin{aligned} R^f_g &\overset{\mathrm{def}}{=} [f]^* \circ [g]_* \\ &= [g]_* \circ [f]^* \end{aligned}$$

for each $(f,g) \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}((A,B),(X,Y))$.

[1] Here R^f_g sits as the bottom arrow in the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,X) \arrow[r, "f^*\circ g_* = g_*\circ f^*"] \arrow[d, two heads, "\pi^{\mathcal{C}}_{B,X}"'] & \mathrm{Hom}_{\mathcal{C}}(A,Y) \arrow[d, two heads, "\pi^{\mathcal{C}}_{A,Y}"] \\ \mathrm{Hom}_{\mathcal{C}/\sim}([B],[X]) \arrow[r, "{[f]^*\circ[g]_* = [g]_*\circ[f]^*}"'] & \mathrm{Hom}_{\mathcal{C}/\sim}([A],[Y]), \end{tikzcd}$$

defined by

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,X) \arrow[r, "f^*\circ g_* = g_*\circ f^*"] \arrow[d, two heads, "\pi^{\mathcal{C}}_{B,X}"'] & \mathrm{Hom}_{\mathcal{C}}(A,Y) \arrow[d, two heads, "\pi^{\mathcal{C}}_{A,Y}"] \\ \mathrm{Hom}_{\mathcal{C}}(B,X)/{\sim_{B,X}} \arrow[r, "{[f]^*\circ[g]_* = [g]_*\circ[f]^*}"'] & \mathrm{Hom}_{\mathcal{C}}(A,Y)/{\sim_{A,Y}}. \end{tikzcd}$$

Proposition 11.2.6 ▶ Quotients by Congruences as Quotients by Profunctors

We have isomorphisms of categories

$$\begin{aligned} \mathcal{C}/{\sim} &\cong \mathcal{C}\backslash R, \\ (\mathcal{C}/{\sim})^{\mathrm{op}} &\cong \mathcal{C}/R. \end{aligned}$$

Proof 11.2.7 ▶ Proof of Proposition 11.2.6

Unwinding the definitions, we see that $C\setminus R$ can be identified with the full subcategory of $\mathsf{PSh}(C/{\sim})$ spanned by the representable presheaves:

- An object of $C\setminus R$ is a presheaf on C of the form $[A]^{\mathsf{L}}$, which is defined by
$$\begin{aligned}[A]^{\mathsf{L}} &\overset{\text{def}}{=} R^{-}_{A}\\ &\overset{\text{def}}{=} \mathrm{Hom}_{C/\sim}(-,[A]).\end{aligned}$$

- The morphisms of $C\setminus R$ are of the form
$$\begin{aligned}\mathrm{Nat}\Big([A]^{\mathsf{L}},[B]^{\mathsf{L}}\Big) &\overset{\text{def}}{=} \mathrm{Nat}(R^{-}_{A},R^{-}_{B})\\ &\overset{\text{def}}{=} \mathrm{Nat}(\mathrm{Hom}_{C/\sim}(-,[A]),\mathrm{Hom}_{C/\sim}(-,[B]))\\ &\cong \mathrm{Nat}_{\mathsf{PSh}(C/\sim)}\big(h_{[A]},h_{[B]}\big)\\ &\cong \mathrm{Hom}_{C/\sim}([A],[B]).\end{aligned}$$

More precisely, we have an isomorphism of categories F from the full subcategory of $\mathsf{PSh}(C/{\sim})$ spanned by the representable presheaves on $C/{\sim}$ to the category $C\setminus R$, given on objects by
$$F\big(h_{[A]}\big) \overset{\text{def}}{=} [A]^{\mathsf{L}}$$
and on morphisms by
$$F\big(h_{[f]}\big) \overset{\text{def}}{=} [f]^{\mathsf{L}}.$$
Precomposing F with the Yoneda embedding of $C/{\sim}$ then gives an isomorphism between $C/{\sim}$ and $C\setminus R$.

The proof that $(C/{\sim})^{\mathsf{op}} \cong C/R$ is dual to the above one, and is hence omitted.

11.2.3 The First Equivalence Theorem for Categories

Let $F\colon C \longrightarrow \mathcal{D}$ be a functor.

Definition 11.2.8 ▶ The Congruence Relation Associated to a Functor

The **congruence relation on C associated to F** is the congruence relation $\sim_F$ defined by declaring $f \sim_F g$ iff $F(f) = F(g)$.[1]

[1] The profunctor associated to $\sim_F$ is given by the representable profunctor $h_{F_{-}}$ associated to F.

Proposition 11.2.9 ▶ The First Equivalence Theorem for Categories

We have an equivalence of categories

$$\mathrm{Im}(F) \overset{\text{eq.}}{\cong} \mathcal{C}/{\sim_F},$$

between the essential image of F and $\mathcal{C}/{\sim_F}$.[1]

[1] In particular, F factors uniquely through $\pi_{\mathcal{C}}$, so that there exists a unique functor $\mathcal{C}/{\sim_F} \xrightarrow{\exists!} \mathcal{D}$ making the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "\pi_{\mathcal{C}}"'] & & \mathcal{D} \\
& \mathcal{C}/{\sim_F} \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

commute.

Proof 11.2.10 ▶ Proof of Proposition 11.2.9

Let $E\colon \mathcal{C}/{\sim_F} \longrightarrow \mathrm{Im}(F)$ be the functor where

- *Action on Objects.* For each $[A] \in \mathrm{Obj}(\mathcal{C}/{\sim_F})$, we have

$$E([A]) \overset{\text{def}}{=} F(A);$$

- *Action on Morphisms.* For each $[A], [B] \in \mathrm{Obj}(\mathcal{C}/{\sim_F})$, the action on Hom-sets

$$E_{[A],[B]}\colon \underbrace{\mathrm{Hom}_{\mathcal{C}/{\sim_F}}([A],[B])}_{\overset{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,B)/{\sim_F}} \longrightarrow \mathrm{Hom}_{\mathrm{Im}(F)}(F(A),F(B))$$

  of E at $([A],[B])$ is given by

$$E([f]) \overset{\text{def}}{=} F(f).$$

  for each $[f] \in \mathrm{Hom}_{\mathcal{C}/{\sim_F}}([A],[B])$.

The map $E_{[A],[B]}$ is indeed well-defined since if $[f] = [g]$, then $E([f]) \overset{\text{def}}{=} F(f) = F(g) \overset{\text{def}}{=} E([g])$.

Moreover, to prove that E is an equivalence of categories, it suffices (by Categories, Item (a) of Proposition 2.4.2) to show that E is essentially surjective and fully faithful.

That E is essentially surjective follows from the definition of the essential image of F and of the action on objects of E, while that E is fully faithful follows from the definition of the essential image of F and of the action on Hom-sets of E.

Thus E is an equivalence, and we indeed have $\mathrm{Im}(F) \overset{\text{eq.}}{\cong} \mathcal{C}/{\sim_F}$. This finishes the proof. ∎

11.3 Quotients of Categories by Generalised Congruence Relations, I

11.3.1 Generalised Congruence Relations on Categories, I

Let $\mathcal{C}$ be a category.

Definition 11.3.1 ▶ Generalised Congruence Relations on Categories, I

A **generalised congruence relation** $(\sim, \sim)$ **on** $\mathcal{C}$ consists of

- *The Equivalence Relation on Objects.* An equivalence relation $\simeq$ on $\mathrm{Obj}(\mathcal{C})$;
- *The Partial Equivalence Relation on Morphisms.* A partial equivalence relation $\sim$ on $\coprod_{n=1}^{\infty} \mathrm{Mor}(\mathcal{C})^{\times n}$;

satisfying the following conditions:[1,2]

1. *Interaction With Domains and Codomains.* For each $\phi, \psi \in \mathrm{Mor}(\mathcal{C})$, if $\phi \sim \psi$, then $\mathrm{dom}(\phi) \simeq \mathrm{dom}(\psi)$ and $\mathrm{cod}(\phi) \simeq \mathrm{cod}(\psi)$.
2. *Interaction With Identities.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, if $A \simeq B$, then $\mathrm{id}_A \sim \mathrm{id}_B$.
3. *Interaction With $\simeq$-Composition.* For each $f, g, h \in \mathrm{Mor}(\mathcal{C})$, if $(g, f) \sim h$, then $\mathrm{dom}(g) \simeq \mathrm{cod}(f)$.
4. *Interaction With Composition I.* For each composable pair of morphisms f and g of $\mathcal{C}$, we have $(g, f) \sim g \circ f$.
5. *Interaction With Composition II.* For each $f, f', g, g' \in \mathrm{Mor}(\mathcal{C})$, if:
   (a) We have $f \sim f'$;
   (b) We have $g \sim g'$;
   (c) The morphism f is composable with g;
   (d) The morphism f' is composable with g';

   then $g \circ f \sim g' \circ f'$.

[1] *Further Terminology:* Two sequences $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_m)$ of morphisms of $\mathcal{C}$ satisfying $\mathrm{dom}(g_1) \simeq \mathrm{cod}(f_n)$ are called **$\simeq$-composable**.

[2] *Further Terminology:* A sequence $(f_1, \ldots, f_n)$ of morphisms of $\mathcal{C}$ such that $\mathrm{dom}(f_{i+1}) \simeq \mathrm{cod}(f_i)$ for each $1 \leq i \leq n-1$ is called a **$\simeq$-path in $\mathcal{C}$**.

The conditions in Items 1 and 3 to 5 and Definition 11.4.3 ensure that we have $f \sim f$ for any morphism f of $\mathcal{C}$ and also that $(f_1, \ldots, f_n) \sim (f_1, \ldots, f_n)$ holds precisely when $(f_1, \ldots, f_n)$ is a $\simeq$-path; see [BBP99, p. 5].

Definition 11.3.2 ▶ Quotients by Generalised Congruence Relations

The **quotient of $\mathcal{C}$ by a generalised congruence relation** $(\simeq, \sim)$ **on $\mathcal{C}$** is the category $\mathcal{C}/(\simeq, \sim)$ where

- *Objects.* We have
$$\mathrm{Obj}(\mathcal{C}/(\simeq,\sim)) \overset{\mathrm{def}}{=} \mathrm{Obj}(\mathcal{C})/{\simeq};$$

- *Morphisms.* For each $[A], [B] \in \mathrm{Obj}(\mathcal{C}/(\simeq,\sim))$, we have
$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}/(\simeq,\sim)}([A],[B]) &\overset{\mathrm{def}}{=} \{\simeq\text{-paths in } \mathcal{C} \text{ from } A \text{ to } B\}\\ &\overset{\mathrm{def}}{=} \left\{\phi = (f_1, \ldots, f_n) \in \coprod_{n=0}^{\infty} \mathrm{Mor}(\mathcal{C})^{\times n} \;\middle|\; \phi \sim \phi\right\};\end{aligned}$$

- *Identities.* For each $[A] \in \mathrm{Obj}(\mathcal{C}/(\simeq,\sim))$, the unit map
$$\mathbb{1}^{\mathcal{C}/(\simeq,\sim)}_{A} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}/(\simeq,\sim)}([A],[A])$$
of $\mathcal{C}/(\simeq,\sim)$ at $[A]$ is defined by
$$\mathrm{id}_{[A]} \overset{\mathrm{def}}{=} [\mathrm{id}_A];$$

- *Composition.* For each $[A], [B], [C] \in \mathrm{Obj}(\mathcal{C}/(\simeq,\sim))$, the composition map
$$\circ^{\mathcal{C}/(\simeq,\sim)}_{[A],[B],[C]} \colon \mathrm{Hom}_{\mathcal{C}/(\simeq,\sim)}([B],[C]) \times \mathrm{Hom}_{\mathcal{C}/(\simeq,\sim)}([A],[B]) \longrightarrow \mathrm{Hom}_{\mathcal{C}/(\simeq,\sim)}([A],[C])$$
of $\mathcal{C}/(\simeq,\sim)$ at $[A], [B], [C]$ is defined by
$$\psi \circ^{\mathcal{C}/(\simeq,\sim)}_{A,B,C} \phi \overset{\mathrm{def}}{=} [\psi \circ \phi].$$

Definition 11.3.3 ▶ The Quotient Functor From $\mathcal{C}$ to $\mathcal{C}/(\simeq,\sim)$

The **quotient functor from $\mathcal{C}$ to $\mathcal{C}/(\simeq,\sim)$** is the functor

$$\pi_{\mathcal{C}}\colon \mathcal{C} \twoheadrightarrow \mathcal{C}/(\simeq,\sim)$$

where

- *Action on Objects.* For each $A \in \text{Obj}(\mathcal{C})$, we have

$$\pi_{\mathcal{C}}(A) \overset{\text{def}}{=} [A];$$

- *Action on Morphisms.* For each $(A, B) \in \text{Obj}(\mathcal{C})$, the action on Hom-sets

$$\pi^{\mathcal{C}}_{A,B}\colon \text{Hom}_{\mathcal{C}}(A, B) \longrightarrow \text{Hom}_{\mathcal{C}/(\simeq,\sim)}([A], [B])$$

  of $\pi^{\mathcal{C}}$ at (A, B) is defined by

$$\pi_{\mathcal{C}}(f) \overset{\text{def}}{=} [f]$$

  for each $f \in \text{Hom}_{\mathcal{C}}(A, B)$.

11.3.2 The First Isomorphism Theorem for Categories, I

Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.

Definition 11.3.4 ▶ The Generalised Congruence Relation Associated to a Functor

The **congruence relation on $\mathcal{C}$ associated to F** is the generalised congruence relation $(\simeq_F, \sim_F)$ consisting of

- *The Equivalence Relation on Objects.* The equivalence relation $\simeq_F$ on $\text{Obj}(\mathcal{C})$ defined by defined by declaring $A \simeq_F B$ iff $F(A) = F(B)$.
- *The Partial Equivalence Relation on Morphisms.* The partial equivalence relation $\sim_F$ on $\coprod_{n=1}^{\infty} \text{Mor}(\mathcal{C})^{\times n}$ defined by declaring $\phi \sim_F \psi$ iff the following conditions are satisfied:

  1. We have $\phi = (f_n, \dots, f_1)$.
  2. We have $\psi = (g_m, \dots, g_1)$.
  3. The composition $F(f_n) \circ \dots \circ F(f_1)$ is well-defined in $\mathcal{D}$.

4. The composition $F(g_m) \circ \cdots \circ F(g_1)$ is well-defined in $\mathcal{D}$.
5. We have $F(f_n) \circ \cdots \circ F(f_1) = F(g_m) \circ \cdots \circ F(g_1)$.

Example 11.3.5 ▶ The Generalised Congruence Associated to a Quotient Functor

The generalised congruence associated to the quotient functor $\pi_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathcal{C}/(\simeq, \sim)$ of a category $\mathcal{C}$ by a generalised congruence $(\simeq, \sim)$ is precisely $(\simeq, \sim)$.

Proposition 11.3.6 ▶ The First Isomorphism Theorem for Categories

We have an isomorphism of categories

$$\mathrm{Im}(F) \cong \mathcal{C}/(\simeq_F, \sim_F),$$

between the image of F and $\mathcal{C}/(\simeq_F, \sim_F)$.[1]

[1] In particular, F factors uniquely through $\pi_{\mathcal{C}}$, so that there exists a unique functor $\mathcal{C}/(\simeq_F, \sim_F) \xrightarrow{\exists!} \mathcal{D}$ making the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "\pi_{\mathcal{C}}"'] & & \mathcal{D} \\
& \mathcal{C}/(\simeq_F, \sim_F) \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

commute.

Proof 11.3.7 ▶ Proof of Proposition 11.4.8

Omitted. □

11.4 Quotients of Categories by Generalised Congruence Relations, II

11.4.1 Quotients of Categories by Equivalence Relations on Objects

Let $\mathcal{C}$ be a category and let $\simeq$ be an equivalence relation on $\mathrm{Obj}(\mathcal{C})$.

Definition 11.4.1 ▶ The Quotient of a Category by an Equivalence Relation on Objects

The **quotient of $\mathcal{C}$ by** $\simeq$ is the category $\mathcal{C}/\simeq$ where

- *Objects.* The objects of $\mathcal{C}/\simeq$ are the $\simeq$-equivalence classes of the objects of $\mathcal{C}$; i.e. we have
$$\mathrm{Obj}(\mathcal{C}/\simeq) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathcal{C})/\simeq;$$
- *Morphisms.* For each $[A], [B] \in \mathrm{Obj}(\mathcal{C}/\simeq)$, we have[1]
$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}/\simeq}([A],[B]) &\stackrel{\text{def}}{=} \mathrm{Hom}'_{\mathcal{C}/\simeq}([A],[B])/\sim_{[A],[B]}\\ &\stackrel{\text{def}}{=} \left(\coprod_{n=1}^{\infty} \mathrm{Hom}'^{,n}_{\mathcal{C}/\simeq}([A],[B])\right)\Bigg/\sim_{[A],[B]},\end{aligned}$$
where[2]
$$\mathrm{Hom}'^{,n}_{\mathcal{C}/\simeq}([A],[B]) \stackrel{\text{def}}{=} \coprod_{\substack{X_1\in[A_1]\\ X_2,Y_2\in[A_2]\\ \vdots\\ X_{n-1},Y_{n-1}\in[A_{n-1}]\\ Y_n\in[A_n]}} \prod_{i=1}^{n-1} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_{i+1}),$$
and where $\sim_{[A],[B]}$ is the equivalence relation generated by the relation $\sim$ on $\mathrm{Hom}'_{\mathcal{C}/\simeq}([A],[B])$ defined as follows:
  - We say that $\phi \sim \psi$ if one of the following conditions is satisfied:
    1. *Gluing Compositions.* We have $\phi = (f_n, \ldots, f_1)$, the morphisms $f_n, \ldots, f_1$ are composable in $\mathcal{C}$, and $\psi = f_n \circ \cdots \circ f_1$.
    2. *Gluing Identities.* We have $[A] = [B]$ and
    $$\begin{aligned}\phi, \psi &\in \mathrm{Hom}'^{,1}_{\mathcal{C}}([A],[A])\\ &\stackrel{\text{def}}{=} \coprod_{X,Y\in[A]} \mathrm{Hom}_{\mathcal{C}}(X,Y)\end{aligned}$$
    are of the form
    $$\begin{aligned}\phi &= \mathrm{id}_X,\\ \psi &= \mathrm{id}_Y\end{aligned}$$
    with $X, Y \in [A]$.

- *Identities*. For each $[A] \in \mathrm{Obj}(\mathcal{C}/\simeq)$, the unit map

$$\mathbb{1}^{\mathcal{C}/\simeq}_{[A]} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\mathcal{C}/\simeq}([A],[A])$$

  of $\mathcal{C}/\simeq$ at $[A]$ is defined by

$$\mathrm{id}_{[A]} \overset{\mathrm{def}}{=} [\mathrm{id}_X]$$

  with $X \in [A]$;

- *Composition*. For each $[A],[B],[C] \in \mathrm{Obj}(\mathcal{C}/\simeq)$, the composition map

$$\circ^{\mathcal{C}/\simeq}_{[A],[B],[C]} \colon \mathrm{Hom}_{\mathcal{C}/\simeq}([B],[C]) \times \mathrm{Hom}_{\mathcal{C}/\simeq}([A],[B]) \longrightarrow \mathrm{Hom}_{\mathcal{C}/\simeq}([A],[C])$$

  of $\mathcal{C}/\simeq$ at $([A],[B],[C])$ is defined by

$$(g_n \mathbin{\Box} \cdots \mathbin{\Box} g_1) \circ^{\mathcal{C}/\simeq}_{[A],[B],[C]} (f_m \mathbin{\Box} \cdots \mathbin{\Box} f_1) \overset{\mathrm{def}}{=} g_n \mathbin{\Box} \cdots \mathbin{\Box} g_1 \mathbin{\Box} f_m \mathbin{\Box} \cdots \mathbin{\Box} f_1.$$

---

[1] *Further Notation*: We also write $f_n \mathbin{\Box} \cdots \mathbin{\Box} f_1$ for $[(f_n,\ldots,f_1)] \in \mathrm{Hom}_{\mathcal{C}/\simeq}([A],[B])$.

[2] For small n, we have

$$\mathrm{Hom}'^{,1}_{\mathcal{C}/\simeq}([A],[B]) \overset{\mathrm{def}}{=} \coprod_{\substack{X\in[A]\\ Y\in[B]}} \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

$$\mathrm{Hom}'^{,2}_{\mathcal{C}/\simeq}([A],[B]) \overset{\mathrm{def}}{=} \coprod_{\substack{X\in[A]\\ Y,Y'\in[B]\\ Z\in[C]}} \mathrm{Hom}_{\mathcal{C}}(Y',Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y).$$

Definition 11.4.2 ▶ The Quotient Functor From $\mathcal{C}$ to $\mathcal{C}/\simeq$

The **quotient functor from $\mathcal{C}$ to $\mathcal{C}/\simeq$** is the functor

$$\pi^{\simeq}_{\mathcal{C}} \colon \mathcal{C} \twoheadrightarrow \mathcal{C}/\simeq$$

where

- *Action on Objects*. For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\pi^{\simeq}_{\mathcal{C}}(A) \overset{\mathrm{def}}{=} [A];$$

- *Action on Morphisms*. For each $(A,B) \in \mathrm{Obj}(\mathcal{C})$, the action on Hom-sets

$$(\pi^{\simeq}_{\mathcal{C}})_{A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{C}/\simeq}([A],[B])$$

of $\pi_{\mathcal{C}}^{\simeq}$ at (A, B) is defined by

$$\pi_{\mathcal{C}}^{\simeq}(f) \overset{\text{def}}{=} f$$

for each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

11.4.2 Generalised Congruence Relations on Categories, II

Definition 11.4.3 ▶ Generalised Congruence Relations on Categories, II

A **generalised congruence relation** $(\simeq, \sim)$ **on** $\mathcal{C}$ consists of

- *The Equivalence Relation on Objects.* An equivalence relation $\simeq$ on $\mathrm{Obj}(\mathcal{C})$;
- *The Congruence Relation on $\mathcal{C}/\simeq$.* A congruence relation on $\mathcal{C}/\simeq$.

Definition 11.4.4 ▶ Quotients by Generalised Congruence Relations, II

The **quotient of** $\mathcal{C}$ **by a generalised congruence relation** $(\simeq, \sim)$ **on** $\mathcal{C}$ is the category $\mathcal{C}/(\simeq, \sim)$ defined as the quotient of $\mathcal{C}/\simeq$ by $\sim$.

Definition 11.4.5 ▶ The Quotient Functor From $\mathcal{C}$ to $\mathcal{C}/(\simeq, \sim)$

The **quotient functor from** $\mathcal{C}$ **to** $\mathcal{C}/(\simeq, \sim)$ is the functor $\pi_{\mathcal{C}}$ given by the composition

$$\mathcal{C} \xrightarrow{\pi_{\mathcal{C}}^{\simeq}} \mathcal{C}/\simeq \xrightarrow{\pi_{\mathcal{C}/\simeq}} \mathcal{C}/(\simeq, \sim),$$

where $\pi_{\mathcal{C}}^{\simeq}$ is the functor of Definition 11.4.2 and $\pi_{\mathcal{C}/\simeq}$ is the quotient functor from $\mathcal{C}/\simeq$ to $\mathcal{C}/(\simeq, \sim) \overset{\text{def}}{=} (\mathcal{C}/\simeq)/\sim$.

11.4.3 The First Isomorphism Theorem for Categories, II

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor.

Definition 11.4.6 ▶ The Generalised Congruence Relation Associated to a Functor

The **congruence relation on** $\mathcal{C}$ **associated to** F is the generalised congruence relation $(\simeq_F, \sim_F)$ consisting of

- *The Equivalence Relation on Objects.* The equivalence relation $\simeq_F$ on $\mathrm{Obj}(\mathcal{C})$ defined by defined by declaring $A \simeq_F B$ iff $F(A) = F(B)$.
- *The Congruence Relation on $\mathcal{C}/\simeq$.* The congruence relation on $\mathcal{C}/\simeq$ consisting of the collection

$$\left\{\sim_{F|[A],[B]} \colon \mathrm{Hom}_{\mathcal{C}/\simeq_F}([A],[B]) \nrightarrow \mathrm{Hom}_{\mathcal{C}/\simeq_F}([A],[B])\right\}_{[A],[B]\in\mathcal{C}/\simeq_F}$$

of equivalence relations where we declare $f_n \,\square \cdots \square\, f_1 \sim_{F|[A],[B]} g_m \,\square \cdots \square\, g_1$ iff the following conditions are satisfied:

1. The composition $F(f_n) \circ \cdots \circ F(f_1)$ is well-defined in $\mathcal{D}$.
2. The composition $F(g_m) \circ \cdots \circ F(g_1)$ is well-defined in $\mathcal{D}$.
3. We have $F(f_n) \circ \cdots \circ F(f_1) = F(g_m) \circ \cdots \circ F(g_1)$.

Example 11.4.7 ▶ The Generalised Congruence Associated to a Quotient Functor

The generalised congruence associated to the quotient functor $\pi_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathcal{C}/(\simeq,\sim)$ of a category $\mathcal{C}$ by a generalised congruence $(\simeq,\sim)$ is precisely $(\simeq,\sim)$.

Proposition 11.4.8 ▶ The First Isomorphism Theorem for Categories

We have an isomorphism of categories

$$\mathrm{Im}(F) \cong \mathcal{C}/(\simeq_F,\sim_F),$$

between the image of F and $\mathcal{C}/(\simeq_F,\sim_F)$.[1]

[1]In particular, F factors uniquely through $\pi_{\mathcal{C}}$, so that there exists a unique functor $\mathcal{C}/(\simeq_F,\sim_F) \xrightarrow{\exists!} \mathcal{D}$ making the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "\pi_{\mathcal{C}}"'] & & \mathcal{D} \\
& \mathcal{C}/(\simeq_F,\sim_F) \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

commute.

Proof 11.4.9 ▶ Proof of Proposition 11.4.8

Let $E\colon \mathcal{C}/(\simeq_F, \sim_F) \longrightarrow \mathrm{Im}(F)$ be the functor where

- *Action on Objects.* For each $[A] \in \mathrm{Obj}(\mathcal{C}/(\simeq_F, \sim_F))$, we have

$$E([A]) \stackrel{\text{def}}{=} F(A);$$

- *Action on Morphisms.* For each $[A], [B] \in \mathrm{Obj}(\mathcal{C}/(\simeq_F, \sim_F))$, the action on Hom-sets

$$E_{[A],[B]}\colon \underbrace{\mathrm{Hom}_{\mathcal{C}/(\simeq_F,\sim_F)}([A],[B])}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}/\simeq_F}([A],[B])/\sim_{F|[A],[B]}} \longrightarrow \mathrm{Hom}_{\mathrm{Im}(F)}(F(A), F(B))$$

  of E at $([A], [B])$ is given by

$$E([f_n \mathbin{\square} \cdots \mathbin{\square} f_1]) \stackrel{\text{def}}{=} F(f_n) \circ \cdots \circ F(f_1).$$

  for each $[f_n \mathbin{\square} \cdots \mathbin{\square} f_1] \in \mathrm{Hom}_{\mathcal{C}/(\simeq_F,\sim_F)}([A],[B])$.

The map $E_{[A],[B]}$ is indeed well-defined since if $[f_n \mathbin{\square} \cdots \mathbin{\square} f_1] = [g_m \mathbin{\square} \cdots \mathbin{\square} g_1]$, then (by definition) $F(f_n) \circ \cdots \circ F(f_1) = F(g_m) \circ \cdots \circ F(g_1)$, and thus

$$\begin{aligned}
E([f_n \mathbin{\square} \cdots \mathbin{\square} f_1]) &\stackrel{\text{def}}{=} F(f_n) \circ \cdots \circ F(f_1)\\
&= F(g_m) \circ \cdots \circ F(g_1)\\
&\stackrel{\text{def}}{=} E([g_m \mathbin{\square} \cdots \mathbin{\square} g_1]).
\end{aligned}$$

Moreover, to prove that E is an isomorphism of categories, it suffices (by Categories, Item 1 of Proposition 2.4.6) to show that E is surjective on objects and fully faithful.

That E is surjective on objects follows from the definition of the image of F and of the action on objects of E, while that E is fully faithful follows from the definition of the image of F and of the action on Hom-sets of E.

Thus E is an isomorphism, and we indeed have $\mathrm{Im}(F) \cong \mathcal{C}/(\simeq_F, \sim_F)$. This finishes the proof.

12 Gabriel–Zisman Localisations

12.1 Left Calculus of Fractions

Let $\mathcal{C}$ be a category.

Definition 12.1.1 ▶ Left Multiplicative Systems of Morphisms of Categories

A **left multiplicative system of morphisms of** $\mathcal{C}$ is a subset S of $\mathrm{Mor}(\mathcal{C})$ satisfying the following conditions:

1. *Identity.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have $\mathrm{id}_A \in S$.
2. *Composition.* For each composable pair (f, g) of $\mathcal{C}$, if $f, g \in S$, then $g \circ f \in S$.
3. *Lower-Right-Corner Square Completion.* Every diagram as below-left may be completed to a square as below-right:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "t \in S"'] & {} \\ X & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "t \in S"'] & B \arrow[d, dotted, "s \in S"] \\ X \arrow[r, dotted, "g"'] & Y \end{tikzcd}$$

4. *The S-Equalising-S-Coequalising Condition.* For each parallel pair $f, g \colon A \rightrightarrows B$ of morphisms of $\mathcal{C}$ and each $t \in S$ such that

   (a) We have $\mathrm{tgt}(t) = A$;

   (b) The diagram

   $$X \xrightarrow{t \in S} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

   commutes (i.e. $f \circ t = g \circ t$);

   there exists some $s \in S$ such that

   (a) We have $\mathrm{src}(s) = B$;

   (b) The diagram

   $$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{s \in S} Y$$

   commutes (i.e. $s \circ f = s \circ g$);

Proposition-Definition 12.1.2 ▶ Categories of Left Fractions

The **category of left fractions of C by a left multiplicative system S of morphisms of C** is the pair $(S^{-1}C, \gamma)$ with[1]

- $S^{-1}C$ the category where
  - *Objects.* We have
  $$\mathrm{Obj}\left(S^{-1}C\right) \stackrel{\mathrm{def}}{=} \mathrm{Obj}(C);$$
  - *Morphisms.* For each $A, B \in \mathrm{Obj}(S^{-1}C)$, we have
  $$\mathrm{Hom}_{S^{-1}C}(A, B) \stackrel{\mathrm{def}}{=} \left\{ (f, s) \in \prod_{X \in \mathrm{Obj}(C)} \mathrm{Hom}_C(A, X) \times \mathrm{Hom}_C(B, X) \;\middle|\; s \in S \right\} \Big/ \sim,$$
  where $\sim$ is the equivalence relation on $\prod_{X \in \mathrm{Obj}(C)} \mathrm{Hom}_C(A, X) \times \mathrm{Hom}_C(B, X)$ obtained by declaring $(f_1, s_1) \sim (f_2, s_2)$ iff there exists a triple $((f_3, s_3), u, v)$ consisting of
    - An element (f_3, s_3) of $\displaystyle\prod_{X \in \mathrm{Obj}(C)} \mathrm{Hom}_C(A, X) \times \mathrm{Hom}_C(B, X)$;
    - A morphism $u \colon X_1 \longrightarrow X_3$ of C;
    - A morphism $v \colon X_2 \longrightarrow X_3$ of C;

  making the diagram
  $$\begin{tikzcd} & X_1 \arrow[d, "u"] & \\ A \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & X_3 & B \arrow[ul, "s_1"'] \arrow[l, "s_3"'] \arrow[dl, "s_2"] \\ & X_1 \arrow[u, "v"'] & \end{tikzcd}$$
  commute;
  - *Identities.* For each $A \in \mathrm{Obj}(S^{-1}C)$, the unit map
  $$\mathbb{1}^{S^{-1}C}_A \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{S^{-1}C}(A, A)$$
  of $S^{-1}C$ at A is defined by
  $$\mathrm{id}^{S^{-1}C}_A \stackrel{\mathrm{def}}{=} [(\mathrm{id}_A, \mathrm{id}_A)];$$

- *Composition*. For each $A, B, C \in \mathrm{Obj}(S^{-1}C)$, the composition map

$$\circ^{S^{-1}C}_{A,B,C}\colon \mathrm{Hom}_{S^{-1}C}(B,C) \times \mathrm{Hom}_{S^{-1}C}(A,B) \longrightarrow \mathrm{Hom}_{S^{-1}C}(A,C)$$

  of $S^{-1}C$ at (A,B,C) is the map defined by

$$[(g,t)] \circ [(f,s)] \overset{\mathrm{def}}{=} [(h \circ f, u \circ t)]$$

  for each
  - Element $[(g,t)] = \left[\left(B \xrightarrow{g} Y, C \xrightarrow{t} Y\right)\right]$ in $\mathrm{Hom}_{S^{-1}C}(B,C)$;
  - Element $[(f,s)] = \left[\left(A \xrightarrow{g} X, B \xrightarrow{s} X\right)\right]$ in $\mathrm{Hom}_{S^{-1}C}(A,B)$;

  and where h and $u \in S$ are the morphisms filling the square

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "s\in S"'] & Y \arrow[d, dotted, "u\in S"] \\ X \arrow[r, dotted, "h"'] & Z. \end{tikzcd}$$

- $\gamma\colon C \longrightarrow S^{-1}C$ the functor where
  - *Action on Objects*. For each $A \in \mathrm{Obj}(C)$, we have

$$\gamma_A \overset{\mathrm{def}}{=} A;$$

  - *Action on Morphisms*. For each morphism $f\colon A \longrightarrow B$ of C, the action on morphisms

$$\gamma_{A,B}\colon \mathrm{Hom}_C(A,B) \longrightarrow \underbrace{\mathrm{Hom}_{S^{-1}C}(\gamma_A,\gamma_B)}_{\overset{\mathrm{def}}{=}\mathrm{Hom}_{S^{-1}C}(A,B)}$$

    of γ at (A,B) is defined by

$$\gamma_f \overset{\mathrm{def}}{=} [(f, \mathrm{id}_B)].$$

[1] *Further Notation:* We write $s^{-1}f$ for the equivalence class $[(f,s)]$ of (f,s).

Proof 12.1.3 ▶ Proof of Proposition-Definition 12.1.2

See [deJ20, Tag 04VD].

12.2 Right Calculus of Fractions

Let $\mathcal{C}$ be a category.

Definition 12.2.1 ▶ Right Multiplicative Systems of Morphisms of Categories

A **right multiplicative system of morphisms of** $\mathcal{C}$ is a subset S of $\mathrm{Mor}(\mathcal{C})$ satisfying the following conditions:

1. *Identity*. For each $A \in \mathrm{Obj}(\mathcal{C})$, we have $\mathrm{id}_A \in S$.
2. *Composition*. For each composable pair (f, g) of $\mathcal{C}$, if $f, g \in S$, then $g \circ f \in s$.
3. *Upper-Left-Corner Square Completion*. Every diagram as below-left may be completed to a square as below-right:

$$\begin{tikzcd} & B \arrow[d, "s\in S"] \\ X \arrow[r, "g"'] & Y \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r, dotted, "f"] \arrow[d, dotted, "t\in S"'] & B \arrow[d, "s\in S"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

4. *The S-Coequalising-S-Equalising Condition*. For each parallel pair $f, g \colon A \rightrightarrows B$ of morphisms of $\mathcal{C}$ and each $s \in S$ such that

   (a) $\mathrm{src}(s) = B$;

   (b) The diagram

   $$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "s\in S"] & Y \end{tikzcd}$$

   commutes (i.e. $s \circ f = s \circ g$);

   there exists some $t \in S$ such that

   (a) $\mathrm{tgt}(t) = A$;

(b) The diagram

$$\begin{tikzcd} X \arrow[r, "t \in S"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

commutes (i.e. $f \circ t = g \circ t$);

PROPOSITION-DEFINITION 12.2.2 ▶ CATEGORIES OF RIGHT FRACTIONS

The **category of right fractions of $\mathcal{C}$ by a right multiplicative system S of morphisms of $\mathcal{C}$** is the pair $(S^{-1}\mathcal{C}, \gamma)$ with[1]

- $S^{-1}\mathcal{C}$ the category where
  - *Objects.* We have
  $$\mathrm{Obj}\left(S^{-1}\mathcal{C}\right) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathcal{C});$$
  - *Morphisms.* For each $A, B \in \mathrm{Obj}(S^{-1}\mathcal{C})$, we have

$$\mathrm{Hom}_{S^{-1}\mathcal{C}}(A, B) \stackrel{\text{def}}{=} \left\{ (f, s) \in \prod_{X \in \mathrm{Obj}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X, A) \times \mathrm{Hom}_{\mathcal{C}}(X, B) \,\middle|\, s \in S \right\} \Big/ \sim,$$

where $\sim$ is the equivalence relation on $\prod_{X \in \mathrm{Obj}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X, A) \times \mathrm{Hom}_{\mathcal{C}}(X, B)$ obtained by declaring $(f_1, s_1) \sim (f_2, s_2)$ iff there exists a triple $((f_3, s_3), u, v)$ consisting of

- An element (f_3, s_3) of $\displaystyle\prod_{X \in \mathrm{Obj}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(X, A) \times \mathrm{Hom}_{\mathcal{C}}(X, B)$;
- A morphism $u \colon X_3 \longrightarrow X_1$ of $\mathcal{C}$;
- A morphism $v \colon X_3 \longrightarrow X_2$ of $\mathcal{C}$;

making the diagram

$$\begin{tikzcd} & X_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ A & X_3 \arrow[l, "s_3"] \arrow[r, "f_3"'] \arrow[u, "u"] \arrow[d, "v"'] & B \\ & X_1 \arrow[ul, "s_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

commute;

- *Identities.* For each $A \in \mathrm{Obj}\left(S^{-1}C\right)$, the unit map

$$\mathbb{1}^{S^{-1}C}_{A} \colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{S^{-1}C}(A, A)$$

of $S^{-1}C$ at A is defined by

$$\mathrm{id}^{S^{-1}C}_{A} \overset{\mathrm{def}}{=} [(\mathrm{id}_A, \mathrm{id}_A)];$$

- *Composition.* For each $A, B, C \in \mathrm{Obj}\left(S^{-1}C\right)$, the composition map

$$\circ^{S^{-1}C}_{A,B,C} \colon \mathrm{Hom}_{S^{-1}C}(B, C) \times \mathrm{Hom}_{S^{-1}C}(A, B) \longrightarrow \mathrm{Hom}_{S^{-1}C}(A, C)$$

of $S^{-1}C$ at (A, B, C) is the map defined by

$$[(g, t)] \circ [(f, s)] \overset{\mathrm{def}}{=} [(g \circ h, s \circ u)]$$

for each

- Element $[(g, t)] = \left[\left(Y \xrightarrow{g} B, Y \xrightarrow{t} C\right)\right]$ in $\mathrm{Hom}_{S^{-1}C}(B, C)$;
- Element $[(f, s)] = \left[\left(X \xrightarrow{g} A, X \xrightarrow{s} B\right)\right]$ in $\mathrm{Hom}_{S^{-1}C}(A, B)$;

and where h and $u \in S$ are the morphisms filling the square

$$\begin{tikzcd}
Z \arrow[r, dotted, "h"] \arrow[d, dotted, "u \in S"'] & Y \arrow[d, "t \in S"] \\
X \arrow[r, "f"'] & A.
\end{tikzcd}$$

- $\gamma \colon C \longrightarrow S^{-1}C$ the functor where
  - *Action on Objects.* For each $A \in \mathrm{Obj}(C)$, we have

$$\gamma_A \overset{\mathrm{def}}{=} A;$$

  - *Action on Morphisms.* For each morphism $f \colon A \longrightarrow B$ of C, the action on morphisms

$$\gamma_{A,B} \colon \mathrm{Hom}_C(A, B) \longrightarrow \underbrace{\mathrm{Hom}_{S^{-1}C}(\gamma_A, \gamma_B)}_{\overset{\mathrm{def}}{=} \mathrm{Hom}_{S^{-1}C}(A,B)}$$

of γ at (A, B) is defined by

$$\gamma_f \overset{\mathrm{def}}{=} [(f, \mathrm{id}_B)].$$

[1] *Further Notation:* We write $s^{-1}f$ for the equivalence class $[(f, s)]$ of (f, s).

Proof 12.2.3 ▶ Proof of Proposition-Definition 12.1.2

See [deJ20, Tag 04VH].

12.3 Two-Sided Calculus of Fractions

Let $\mathcal{C}$ be a category.

Definition 12.3.1 ▶ Multiplicative Systems of Morphisms of Categories

A **multiplicative system of morphisms of** $\mathcal{C}$ is a subset S of $\mathrm{Mor}(\mathcal{C})$ which is both a left and a right multiplicative system of morphisms of $\mathcal{C}$.

Definition 12.3.2 ▶ Saturated Multiplicative Systems

A multiplicative system S of morphisms of $\mathcal{C}$ is **saturated** if, for each composable triple (f, g, h) of morphisms of $\mathcal{C}$, if $g \circ f \in S$ and $h \circ g \in S$, then $g \in S$:

$$\begin{tikzcd} & C \arrow[ddr, "h"] & \\ & B \arrow[u, "g"] \arrow[dr, "h \circ g"] & \\ A \arrow[ur, "f"] \arrow[uur, "g \circ f"] \arrow[rr, "(h \circ g) \circ f"'] & & D \end{tikzcd} \rightsquigarrow \begin{tikzcd} & C \arrow[ddr, "h"] & \\ & B \arrow[u, "g"] \arrow[dr, "h \circ g"] & \\ A \arrow[ur, "f"] \arrow[uur, "g \circ f"] \arrow[rr, "(h \circ g) \circ f"'] & & D. \end{tikzcd}$$

Proposition 12.3.3 ▶ Properties of Two-Sided Multiplication Systems

Let S be a subclass of $\mathrm{Mor}(\mathcal{C})$.

1. *Categories of Two-Sided Fractions.* If S is a two-sided multiplicative system of morphisms of $\mathcal{C}$, then the categories of left and right fractions of $\mathcal{C}$ of Proposition-Definitions 12.1.2 and 12.2.2 are canonically isomorphic.

2. *Saturation and Isomorphisms.* We have an equality of sets

$$\underbrace{\{f \in \mathrm{Mor}(\mathcal{C}) \mid \gamma(f) \text{ is an isomorphism}\}}_{\overset{\text{def}}{=} \widehat{S}} = \underbrace{\{g \in \mathrm{Mor}(\mathcal{C}) \mid \text{there exist } g, h \in \mathrm{Mor}(\mathcal{C}) \text{ such that } g \circ f, h \circ g \in S\}}_{\overset{\text{def}}{=} S'}.$$

Moreover, $\widehat{S} = S'$ is the smallest saturated multiplicative system containing S.[1]

[1] As such, if S is saturated, then $\widehat{S} = S$.

Proof 12.3.4 ▶ Proof of Proposition 12.3.3

Item 1: Categories of Two-Sided Fractions

See [deJ20, Tag 04VL].

Item 2: Saturation and Isomorphisms

See [deJ20, Tag 05Q9].

12.4 Gabriel–Zisman Localisations

Let $\mathcal{C}$ be a category and S be a subclass of $\mathrm{Mor}(\mathcal{C})$.

Definition 12.4.1 ▶ Gabriel–Zisman Localisations

The **Gabriel–Zisman localisation of $\mathcal{C}$ at S**[1,2] is the pair $(S^{-1}\mathcal{C}, \gamma)$ with

- $S^{-1}\mathcal{C}$[3] a category;
- $\gamma \colon \mathcal{C} \longrightarrow S^{-1}\mathcal{C}$ a functor such that, for each $f \in S$, the morphism $\gamma(f)$ is an isomorphism;

satisfying the following universal property:

(UP) Given another such[4] pair $(\mathcal{D}, \delta)$, there exists a unique morphism $S^{-1}\mathcal{C} \xrightarrow{\exists!} \mathcal{D}$ making the diagram

$$\begin{tikzcd} & S^{-1}\mathcal{C} \arrow[d, dashed, "\exists!"] \\ \mathcal{C} \arrow[ur, "\gamma"] \arrow[r, "\delta"'] & \mathcal{D} \end{tikzcd}$$

commute.

[1] Or simply the **localisation of $\mathcal{C}$ at S**.

[2] *Further Terminology:* The class S is called **a class of weak equivalences of $\mathcal{C}$**, and the elements of S are called the **weak equivalences of $\mathcal{C}$**.

[3] *Further Notation:* Also written $\mathcal{C}[S^{-1}]$.

[4] That is, such that, for each $f \in s$, the morphism $\delta(f)$ is an isomorphism in $\mathcal{D}$.

Construction 12.4.2 ▶ Construction of Gabriel–Zisman Localisations

If S is a left (resp. right) multiplicative system, then the pair $(S^{-1}C, \gamma)$ of Proposition-Definition 12.1.2 (resp. Proposition-Definition 12.1.2) satisfies the universal property of the Gabriel–Zisman localisation of C at S.

Proof 12.4.3 ▶ Proof of Construction 12.4.2

See [de]20, 04VG and 04VK].

Example 12.4.4 ▶ Examples of Gabriel–Zisman Localisations

Here are some examples of Gabriel–Zisman localisations.

1. *Abelian Groups*[1]. The localisation of the category of groups with respect to the class
$$S = \left\{\phi\colon G \longrightarrow H \,\middle|\, \phi \text{ induces an isomorphism } G^{\mathrm{ab}} \xrightarrow{\cong} H^{\mathrm{ab}}\right\}$$
is equivalent to Ab.

2. *The Homotopy Category of Spaces*[2]. The localisation of Spc at the weak homotopy equivalences recovers the homotopy category of spaces $\mathrm{Ho}(\mathsf{Spc})$.

3. *Derived Categories*[3]. The localisation of the category $\mathsf{Ch}_{\geq 0}(\mathcal{A})$ of non-negatively graded chain complexes of objects in an additive category $\mathcal{A}$ at the quasi-isomorphisms recovers the bounded below derived category $\mathcal{D}_{\geq 0}(\mathcal{A})$ of $\mathsf{Ch}_{\geq 0}(\mathcal{A})$.

[1] This is [Ber18, Example 1.2.5].
[2] This is [Ber18, Example 1.2.6].
[3] This is [Ber18, Example 1.2.7].

Proposition 12.4.5 ▶ Properties of Gabriel–Zisman Localisations

Let C be a category and S be a subclass of $\mathrm{Mor}(C)$.

1. *Interaction With Isomorphisms.* If $s \in S$, then γ_s is an isomorphism.

2. *The Filtered Colimit Formula for* Hom*-Sets of Gabriel–Zisman Localisations.* Let $A, B \in \mathrm{Obj}(S^{-1}C)$.

   (a) If S is a left multiplicative system of morphisms of C, then we have a bijection of sets

$$\mathrm{Hom}_{S^{-1}\mathcal{C}}(A,B) \cong \operatorname*{colim}_{(s\colon B\longrightarrow B')\in B/S}(\mathrm{Hom}_{\mathcal{C}}(A,B')),$$

where B/S is the category defined as in [deJ20, Tag 05Q0].

(b) If S is a right multiplicative system of morphisms of $\mathcal{C}$, then we have a bijection of sets

$$\mathrm{Hom}_{S^{-1}\mathcal{C}}(A,B) \cong \operatorname*{colim}_{(s\colon A'\longrightarrow A)\in S/A}(\mathrm{Hom}_{\mathcal{C}}(A',B)),$$

where S/A is the category defined as in [deJ20, Tag 05Q4].

3. *When Localisation Commutes With Finite Colimits.* If S is a left multiplicative system of morphisms of $\mathcal{C}$, then $\gamma\colon \mathcal{C} \longrightarrow S^{-1}\mathcal{C}$ commutes with finite colimits.

4. *When Localisation Commutes With Finite Limits.* If S is a right multiplicative system of morphisms of $\mathcal{C}$, then $\gamma\colon \mathcal{C} \longrightarrow S^{-1}\mathcal{C}$ commutes with finite limits.

Proof 12.4.6 ▶ Proof of Proposition 12.4.5

Item 1: Interaction With Isomorphisms

See [deJ20, Tag 04VG].

Item 2: The Filtered Colimit Formula for Hom-Sets of Gabriel–Zisman Localisations

See [deJ20, Tags 05Q0 and 05Q4].

Item 3: When Localisation Commutes With Finite Colimits

See [deJ20, Tag 05Q2].

Item 4: When Localisation Commutes With Finite Limits

See [deJ20, Tag 05Q6].

Remark 12.4.7 ▶ On Size Issues ([Ber18, p. 15])

Even when $\mathcal{C}$ is locally small, $S^{-1}\mathcal{C}$ may fail to be so. Ways of solving this problem include:

1. Using universes;

2. Requiring S to be a multiplicative system;
3. Switching to model categories.

13 The Karoubi Envelope of a Category

13.1 Split Idempotents

Let $\mathcal{C}$ be a category.

Definition 13.1.1 ▶ Split Idempotents in a Category

An idempotent morphism $e\colon A \longrightarrow A$ of $\mathcal{C}$ is **split** if there exist

- An object B of $\mathcal{C}$;
- A morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$;
- A morphism $g\colon B \longrightarrow A$ of $\mathcal{C}$;

such that

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "e"'] & B \arrow[d, "g"] \\
& A
\end{tikzcd}
\qquad
\begin{aligned}
g \circ f &= e,\\
f \circ g &= \mathrm{id}_B,
\end{aligned}
\qquad
\begin{tikzcd}
B \arrow[r, "g"] \arrow[dr, equal, "\mathrm{id}_B"'] & A \arrow[d, "f"] \\
& B.
\end{tikzcd}
$$

Definition 13.1.2 ▶ Idempotent Complete Category

A category $\mathcal{C}$ is **idempotent complete** if all idempotents in $\mathcal{C}$ are split.

13.2 The Karoubi Envelope of a Category

Let $\mathcal{C}$ be a category.

Definition 13.2.1 ▶ The Karoubi Envelope of a Category

The **Karoubi envelope of $\mathcal{C}$**[1] is the category $\overline{\mathcal{C}}$[2] where

- *Objects.* The objects of $\overline{\mathcal{C}}$ are pairs (A, e) consisting of

- *The Underlying Object.* An object A of $\mathcal{C}$;
- *The Idempotent Morphism.* An idempotent morphism $e\colon A \longrightarrow A$ of $\mathcal{C}$;

- *Morphisms.* A morphism of $\overline{\mathcal{C}}$ from (A, e) to (B, e') is a morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$ making the diagram[3]

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "e"'] & B \arrow[d, "e'"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commute;

- *Identities.* For each $(A, e) \in \mathrm{Obj}(\overline{\mathcal{C}})$, the unit map

$$\mathbb{1}^{\overline{\mathcal{C}}}_{(A,e)}\colon \mathrm{pt} \longrightarrow \mathrm{Hom}_{\overline{\mathcal{C}}}((A, e), (A, e))$$

of $\overline{\mathcal{C}}$ at (A, e) is defined by

$$\mathrm{id}^{\overline{\mathcal{C}}}_{(A,e)} \stackrel{\text{def}}{=} \mathrm{id}_A;$$

- *Composition.* For each $\mathbf{A} = (A, e)$, $\mathbf{B} = (B, e')$, $\mathbf{C} = (C, e'') \in \mathrm{Obj}(\overline{\mathcal{C}})$, the unit map

$$\circ^{\overline{\mathcal{C}}}_{\mathbf{A},\mathbf{B},\mathbf{C}}\colon \mathrm{Hom}_{\overline{\mathcal{C}}}(\mathbf{B}, \mathbf{C}) \times \mathrm{Hom}_{\overline{\mathcal{C}}}(\mathbf{A}, \mathbf{B}) \longrightarrow \mathrm{Hom}_{\overline{\mathcal{C}}}(\mathbf{A}, \mathbf{C})$$

of $\overline{\mathcal{C}}$ at $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is defined by

$$g \circ^{\overline{\mathcal{C}}}{}_{(A,e)} f \stackrel{\text{def}}{=} g \circ f$$

for each $f \in \mathrm{Hom}_{\overline{\mathcal{C}}}(\mathbf{A}, \mathbf{B})$ and each $g \in \mathrm{Hom}_{\overline{\mathcal{C}}}(\mathbf{B}, \mathbf{C})$.

[1] *Further Terminology:* Also called the **Cauchy completion of** $\mathcal{C}$ or the **idempotent completion of** $\mathcal{C}$.
[2] *Further Notation:* Also written $\mathrm{Kar}(\mathcal{C})$ or $\mathrm{Split}(\mathcal{C})$.
[3] This is equivalent to the commutativity of the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "e"'] & B \\ A \arrow[r, "f"'] & B \arrow[u, "e'"'] \end{tikzcd}$$

Definition 13.2.2 ▶ The Embedding Into The Karoubi Envelope

The **embedding of $\mathcal{C}$ into its Karoubi envelope** $\overline{\mathcal{C}}$ is the functor

$$\iota^{\mathsf{Kar}}_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \overline{\mathcal{C}}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\iota^{\mathsf{Kar}}_{\mathcal{C}}(A) \stackrel{\text{def}}{=} (A, \mathrm{id}_A);$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action

$$\iota^{\mathsf{Kar}}_{\mathcal{C}|A,B} \colon \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\overline{\mathcal{C}}}\Big(\iota^{\mathsf{Kar}}_{\mathcal{C}}(A), \iota^{\mathsf{Kar}}_{\mathcal{C}}(B)\Big)$$

  of $\iota^{\mathsf{Kar}}_{\mathcal{C}}$ at (A, B) is defined by

$$\iota^{\mathsf{Kar}}_{\mathcal{C}|A,B}(f) \stackrel{\text{def}}{=} f$$

  for each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Example 13.2.3 ▶ Examples of Karoubi Completions

Here are some examples of Karoubi completions.

1. *Smooth Manifolds.* The embedding $\iota \colon \mathsf{Open} \longrightarrow \mathsf{Man}$ of the full subcategory Open of Man spanned by the open subspacecs of finite-dimensional Cartesian spaces into Man exhibits Man as the Karoubi envelope of Open.[1]
2. *Projective Modules.* The category of projective modules over a ring R is the Karoubi envelope of the category of free modules over R.
3. *Vector Bundles.* The category of vector bundles over a paracompact space X is the Karoubi envelope of the category of trivial bundles over X.

[1] *Reference:* [nLab23b, Theorem 4.1].

Proposition 13.2.4 ▶ Properties of the Karoubi Envelope of a Category

Let $\mathcal{C}$ be a category.

1. *Functoriality.* The assignment $\mathcal{C} \mapsto \overline{\mathcal{C}}$ defines a functor

$$\overline{(-)}\colon \mathsf{Cats} \longrightarrow \mathsf{Cats}.$$

2. *Adjointness.* We have a triple adjunction

$$\left(\mathrm{Free} \dashv 忘 \dashv \overline{(-)}\right)\colon \quad \begin{tikzcd} \mathsf{SemiCats} \arrow[rr, bend left, "\mathrm{Free}"] \arrow[rr, bend right, "\overline{(-)}"'] & & \mathsf{Cats}, \arrow[ll, "忘"'] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathrm{Fun}(\mathrm{Free}(\mathcal{C}), \mathcal{D}) &\cong \mathrm{SemiFun}(\mathcal{C}, \mathcal{D}),\\ \mathrm{SemiFun}(\mathcal{D}, \mathcal{E}) &\cong \mathrm{Fun}(\mathcal{D}, \overline{\mathcal{E}}) \end{aligned}$$

for each $\mathcal{C}, \mathcal{E} \in \mathrm{Obj}(\mathsf{SemiCats})$ and each $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, where

- $\mathrm{Free}\colon \mathsf{SemiCats} \longrightarrow \mathsf{Cats}$ is the functor freely adjoining identities to a semicategory;
- $忘\colon \mathsf{Cats} \longrightarrow \mathsf{SemiCats}$ is the forgetful functor from Cats to $\mathsf{SemiCats}$;
- $\overline{(-)}\colon \mathsf{SemiCats} \longrightarrow \mathsf{Cats}$ is the natural extension of the Karoubi envelope functor of Item 1 from Cats to $\mathsf{SemiCats}$.

3. *Universal Property.* The pair $(\overline{\mathcal{C}}, \iota)$ consisting of

- The Karoubi completion $\overline{\mathcal{C}}$ of $\mathcal{C}$;
- The embedding $\iota^{\mathrm{Kar}}_{\mathcal{C}}\colon \mathcal{C} \hookrightarrow \overline{\mathcal{C}}$ of Definition 13.2.2;

satisfies the following universal property:

(UP) Given another pair $(\mathcal{E}, i)$ consisting of

- A category $\mathcal{E}$;
- A functor $i\colon \mathcal{C} \longrightarrow \mathcal{E}$;

if

(a) The category $\mathcal{E}$ is idempotent complete;

(b) The functor i is fully faithful;

(c) Every object of $\mathcal{E}$ is the retract of an object of the form $i(A)$ with $A \in \mathrm{Obj}(\mathcal{C})\mathcal{C}$;

then we have an equivalence of categories

$$\mathcal{E} \overset{\text{eq.}}{\cong} \overline{\mathcal{C}}.$$

4. *Idempotent Completeness.* The category $\overline{\mathcal{C}}$ is idempotent complete.
5. *Via Retracts.* We have an equivalence of categories

$$\overline{\mathcal{C}} \cong \mathsf{PSh}^{\text{retr.-rep.}}(\mathcal{C}),$$

where $\mathsf{PSh}^{\text{retr.-rep.}}(\mathcal{C})$ is the full subcategory of $\mathsf{PSh}(\mathcal{C})$ spanned by the presheaves on $\mathcal{C}$ which are retracts of representable presheaves.

Proof 13.2.5 ▶ Proof of Proposition 13.2.4

Item 1: Functoriality

Omitted.

Item 2: Adjointness

See [nLab23d, Proposition 3.6].

Item 3: Universal Property

Omitted.

Item 4: Idempotent Completeness

Omitted.

Item 5: Via Retracts

Omitted.

Appendices

A Other Chapters

Category Theory

Categorical Hochschild Co/Homology

Monoidal Categories

Categorical Algebra

Enriched Category Theory

Internal Category Theory

Homological Algebra

Categorical Logic

Sites, Sheaves, and Stacks

Complements on Sheaves

Bicategories

References

[MSE 570202] Servaes. *Taking the automorphism group of a group is not functorial*. Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/570202 (cit. on p. 99).

[BBP99] Marek A. Bednarczyk, Andrzej M. Borzyszkowski, and Wieslaw Pawlowski. "Generalized Congruences—Epimorphisms in *Cat*". In: *Theory Appl. Categ.* 5 (1999), No. 11, 266–280 (cit. on pp. 1, 145).

[Ber18] Julia E. Bergner. *The Homotopy Theory of $(\infty, 1)$-Categories*. Vol. 90. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2018, pp. xiv+273. ISBN: 978-1-107-49902-7; 978-1-107-10136-4 (cit. on pp. 161, 162).

[Cun20] Oscar Cunningham. *Adjunctions between monoids*. 2020. URL: https://oscarcunningham.com/314/adjunctions-between-monoids (cit. on p. 53).

[deJ20] Aise Johan de Jong et al. *The Stacks Project*. 2020. URL: https://stacks.math.columbia.edu (cit. on pp. 156, 159–162).

[Lur20] Jacob Lurie. *Kerodon*. 2020. URL: https://kerodon.net (cit. on pp. 63, 66, 67, 113, 117, 119, 121, 122, 126, 129, 131).

[Mol05] Vladimir Molotkov. *Adjunction in Monoids*. 2005. arXiv: math/0504060 (cit. on p. 53).

[nLab23a] The nLab Authors. *Functors and Comma Categories*. 2023. URL: https://ncatlab.org/nlab/show/functors+and+comma+categories (cit. on p. 126).

[nLab23b] The nLab Authors. *Karoubi Envelope*. 2023. URL: https://ncatlab.org/nlab/show/Karoubi+envelope (cit. on p. 165).

[nLab23c] The nLab Authors. *Over Category*. 2023. URL: https://ncatlab.org/nlab/show/over+category (cit. on p. 117).

[nLab23d] The nLab Authors. *Semicategory*. 2023. URL: https://ncatlab.org/nlab/show/semicategory (cit. on p. 167).

[Rie14] Emily Riehl. *Categorical Homotopy Theory*. Vol. 24. New Mathematical Monographs. Cambridge University Press, Cambridge, 2014, pp. xviii+352. ISBN: 978-1-107-04845-4. DOI: 10.1017/CBO9781107261457. URL: https://doi.org/10.1017/CBO9781107261457 (cit. on p. 62).