Bayesian learning and the Hierarchical Gaussian Filter

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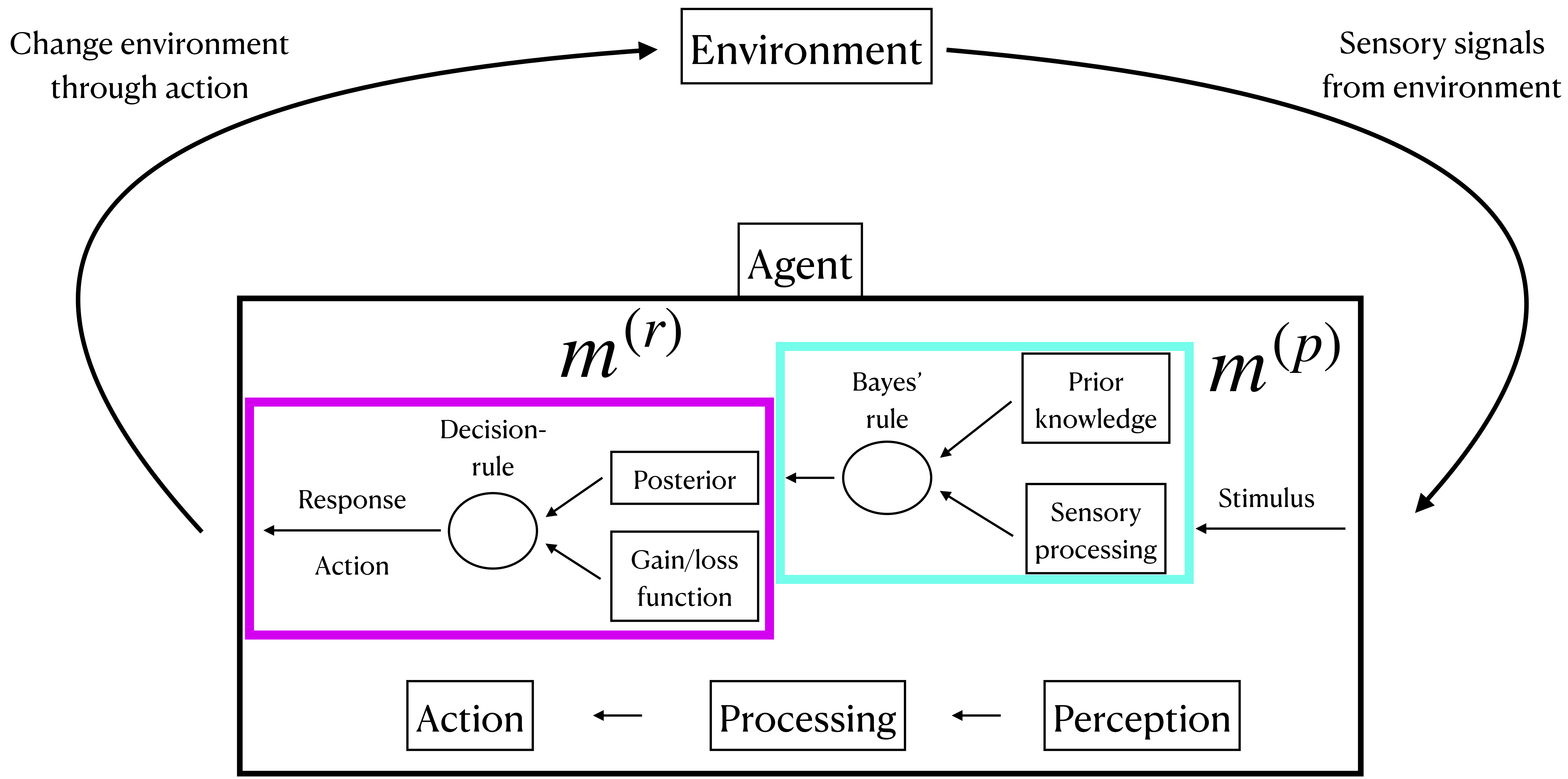
Introduction

- Computational psychiatry is concerned with understanding mental disorders through formalisation and model-building
- Underlying processes can often be described in terms of inference
- And these can be studied through decision-making tasks
- **Inverse Bayesian decision theory** (see Daunizeau et al (2010)) :
  - "a meta-Bayesian procedure which allows for Bayesian inferences about subject's Bayesian inferences"
- The **Hierarchical Gaussian Filter** (HGF) is a way of modelling inferences
  - Remember: to model belief-updating means one needs to choose prior and likelihood terms and derive an equation that describes how their belief changes in response to evidence
    - The HGF represents a particular set of choices for these assumptions
  - It is applicable for tasks where there is a latent variable that is to be inferred by subjects
  - It can be combined with any response model to account for different types of variables such as choices or reaction times
- The model estimates a "belief trajectory" for each subject, which can be correlated with other data such as fMRI or EEG/MEG signals to quantify effects of momentary belief uncertainty or prediction errors
- There is a MatLab toolbox and the new Julia package that can be used for analyses using the model

Modelling the inference process

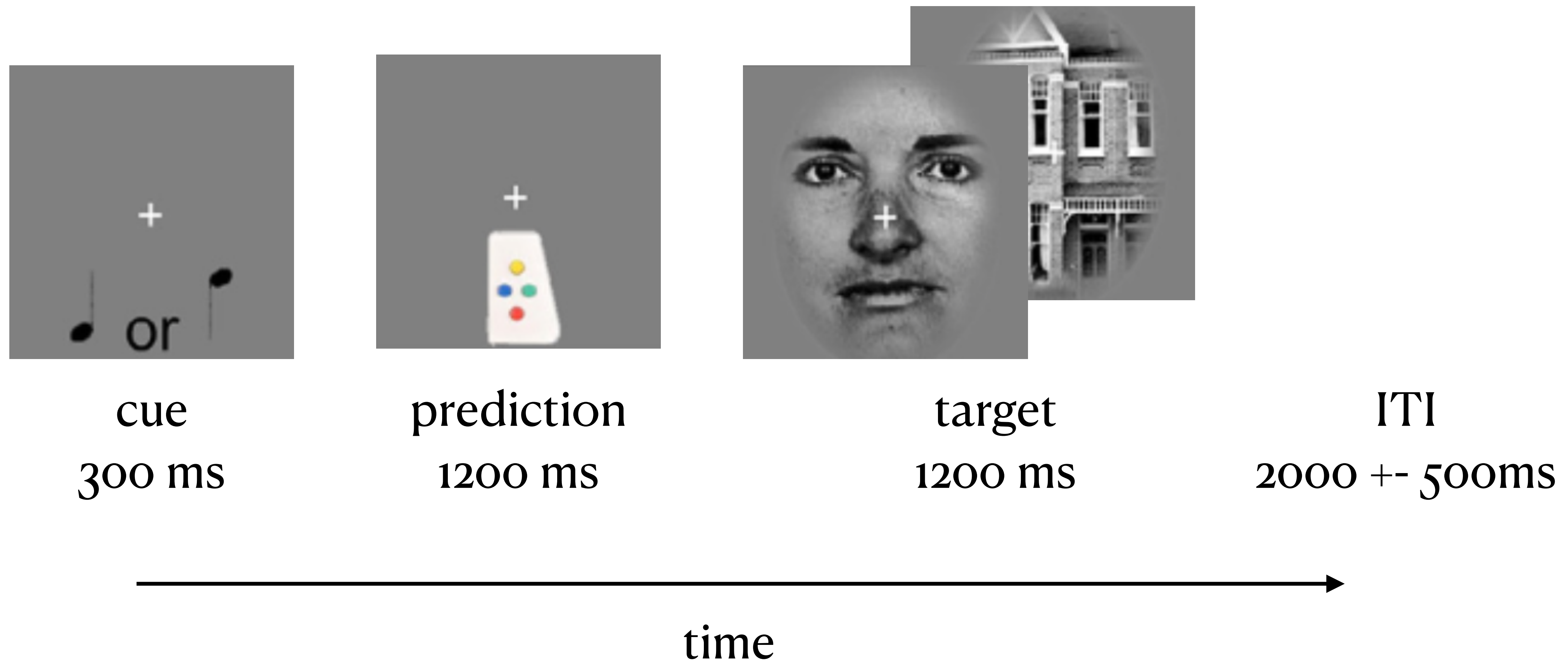


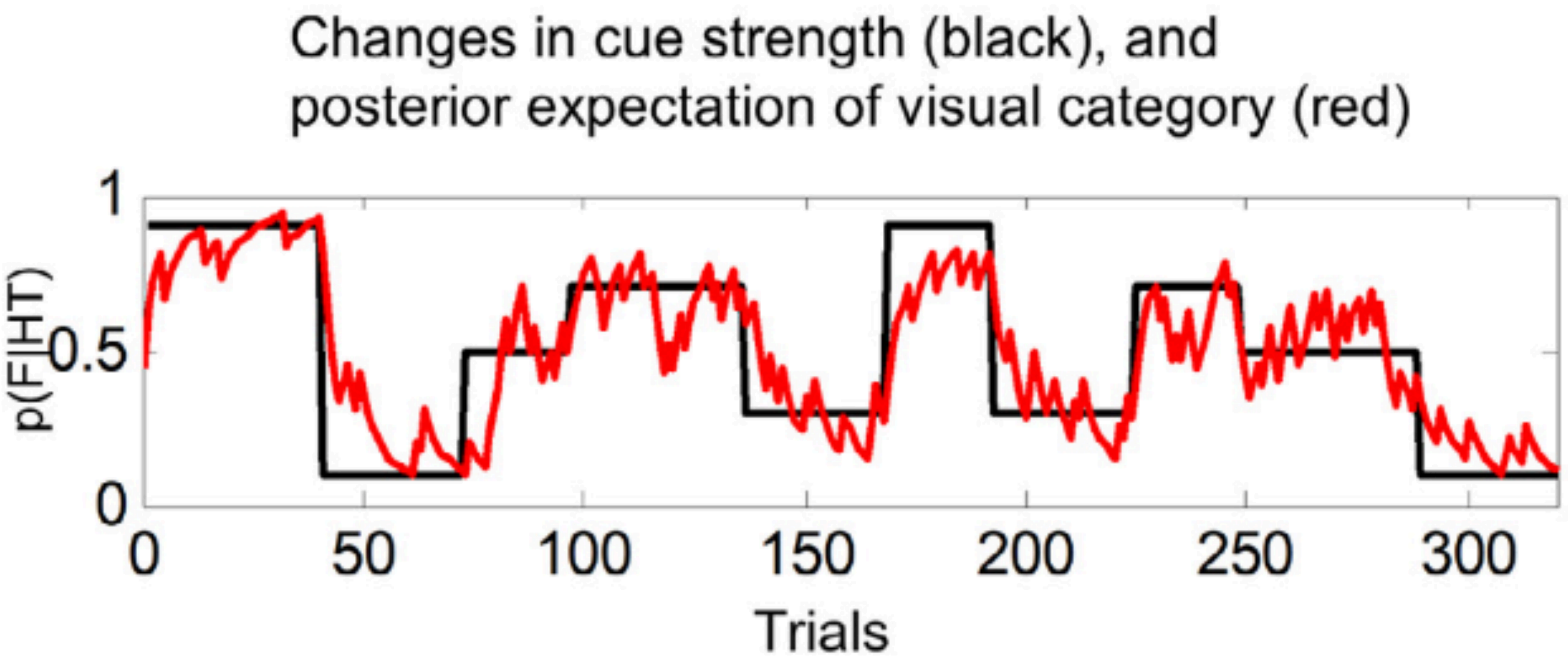
Example:
Reversal learning task

Example: reversal learning task

Link to paper: https://doi.org/10.1016/j.neuroimage.2020.117590

- Implicit inference problem: how reliable is the cue?
  - We will consider this outcome: 1 if high/low tone followed by face/house, 0 otherwise
- Different phases of cue reliability (black line)
- Task same as in earlier paper
- There, the authors found precision-weighted PEs to be associated with fMRI signals in
  - dopaminergic midbrain (for outcome PEs)
  - cholinergic basal forebrain (for contingency PEs)
- Follow-up to test these results under pharamacological interventions (dopaminergic/cholinergic drugs)





If assume this perceptual model:

$$m^{(p)} : \begin{cases} p(u^{(t)}\,|\,x) = Ber(x) & t = 1,\ldots,T \\ p(x) = Beta(1,1) & \end{cases}$$

Also written as:

$$x \sim Beta(a, b)$$
$$u_t \sim Ber(x),\ t = 1,\ldots,T$$

We find the posterior to be:

$$\begin{aligned} p(x\,|\,u^{(1:T)}) &\propto p(u^{(1:T)}\,|\,x) \cdot p(x) \\ &= \left(x^{\sum_t u^{(t)}}(1-x)^{T-\sum_t u^{(t)}}\right) \cdot \left(\frac{1}{B(a,b)}x^{a-1}(1-x)^{1-b}\right) \\ &= \frac{1}{B(a,b)}x^{a+\sum_t u^{(t)}-1}(1-x)^{b+(T-\sum_t u^{(t)})-1} \end{aligned}$$

Which is just the prior with updated parameters!

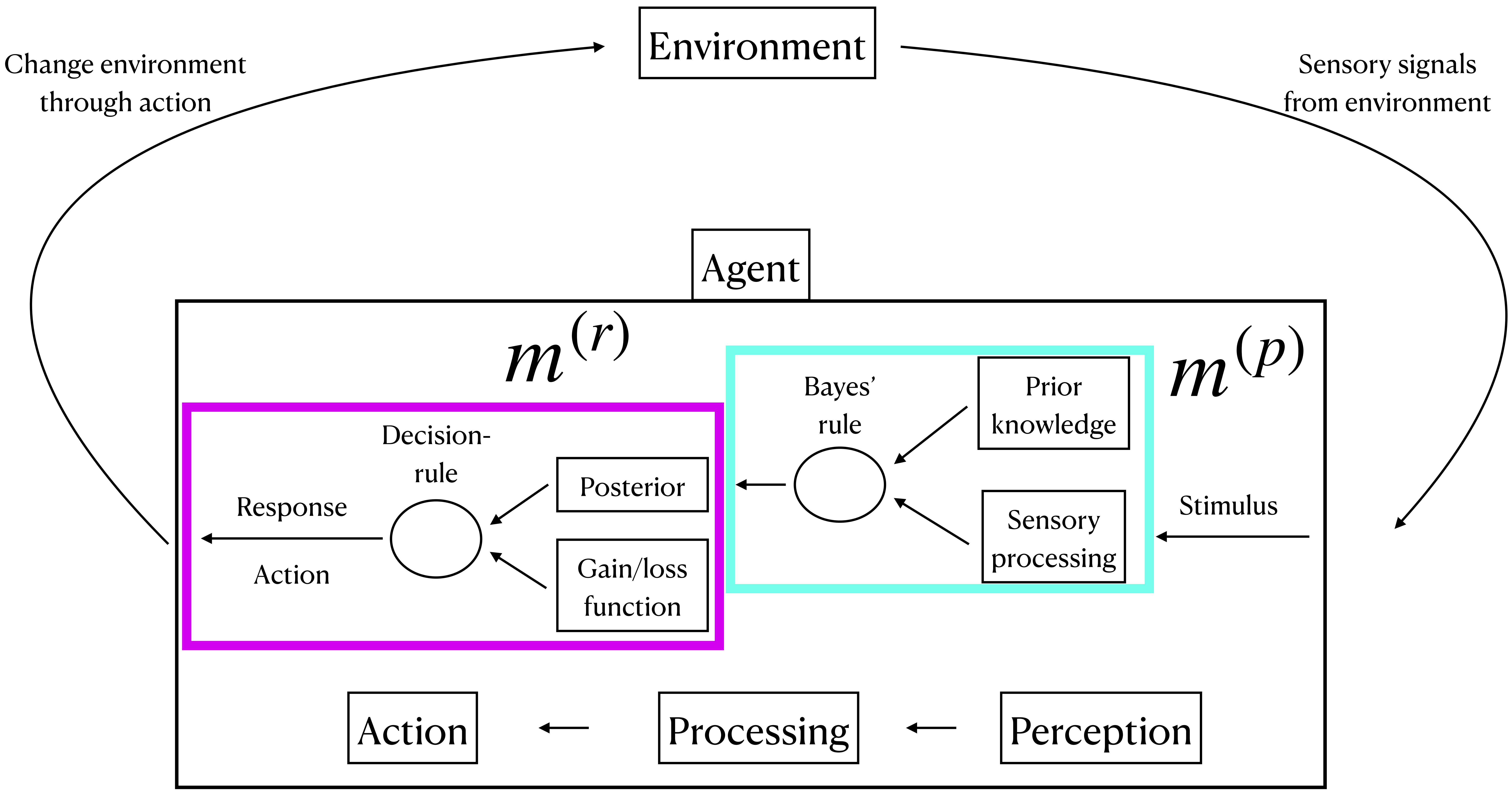
$$p(x\,|\,u^{(1:T)}) = Beta\left(a + \sum_{t=1}^{T} u^{(t)}; b + T - \sum_{t=1}^{T} u^{(t)}\right)$$

This gives the following sequence of parameters:

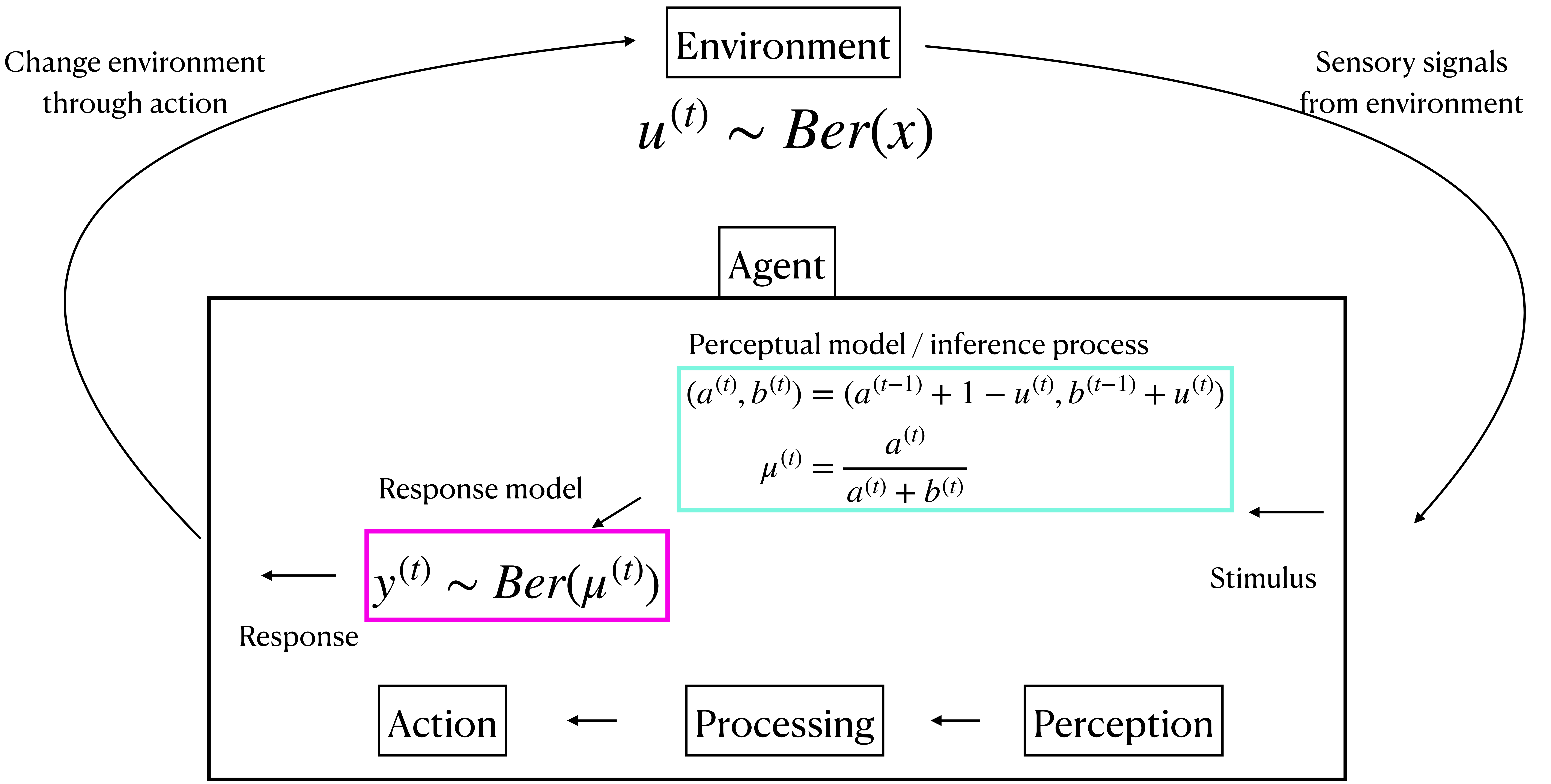
$$(a^{(t)},\, b^{(t)}) = (a^{(t-1)} + u^{(t)},\, b^{(t-1)} + 1 - u^{(t)})$$

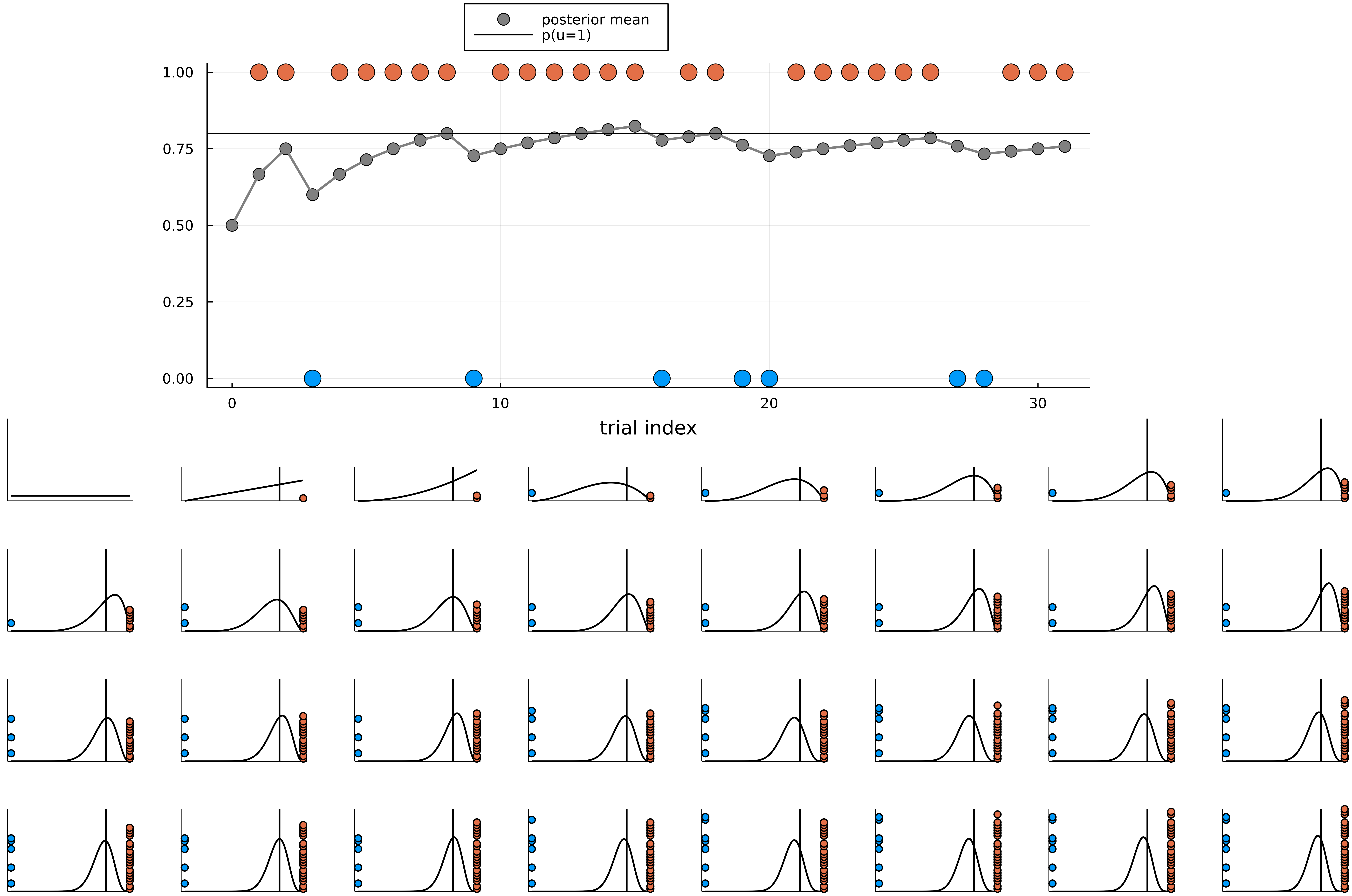
And these expectations: $\mu^{(t)} = \dfrac{a^{(t)}}{a^{(t)} + b^{(t)}}$

Modelling the inference process



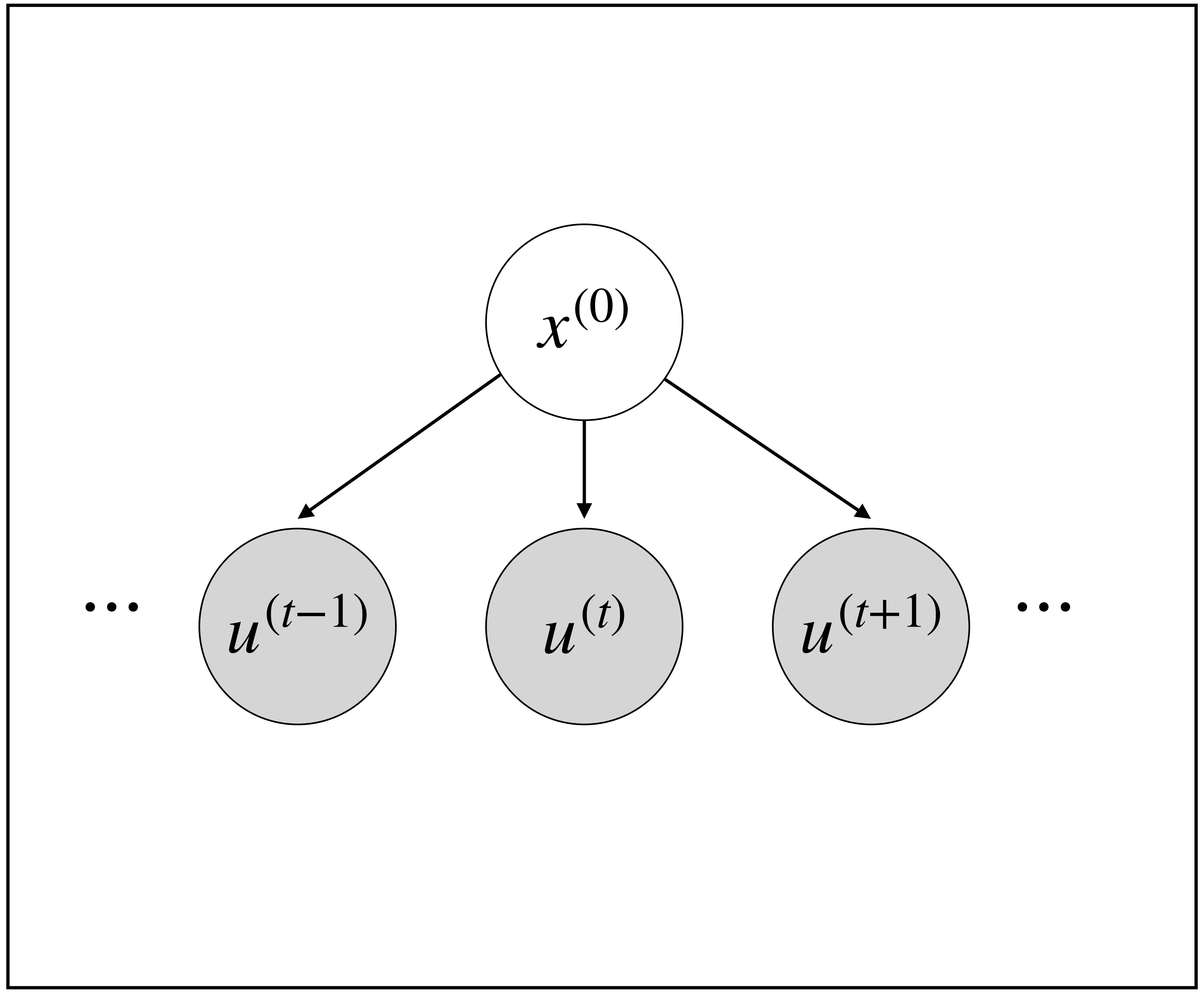
Example inference process

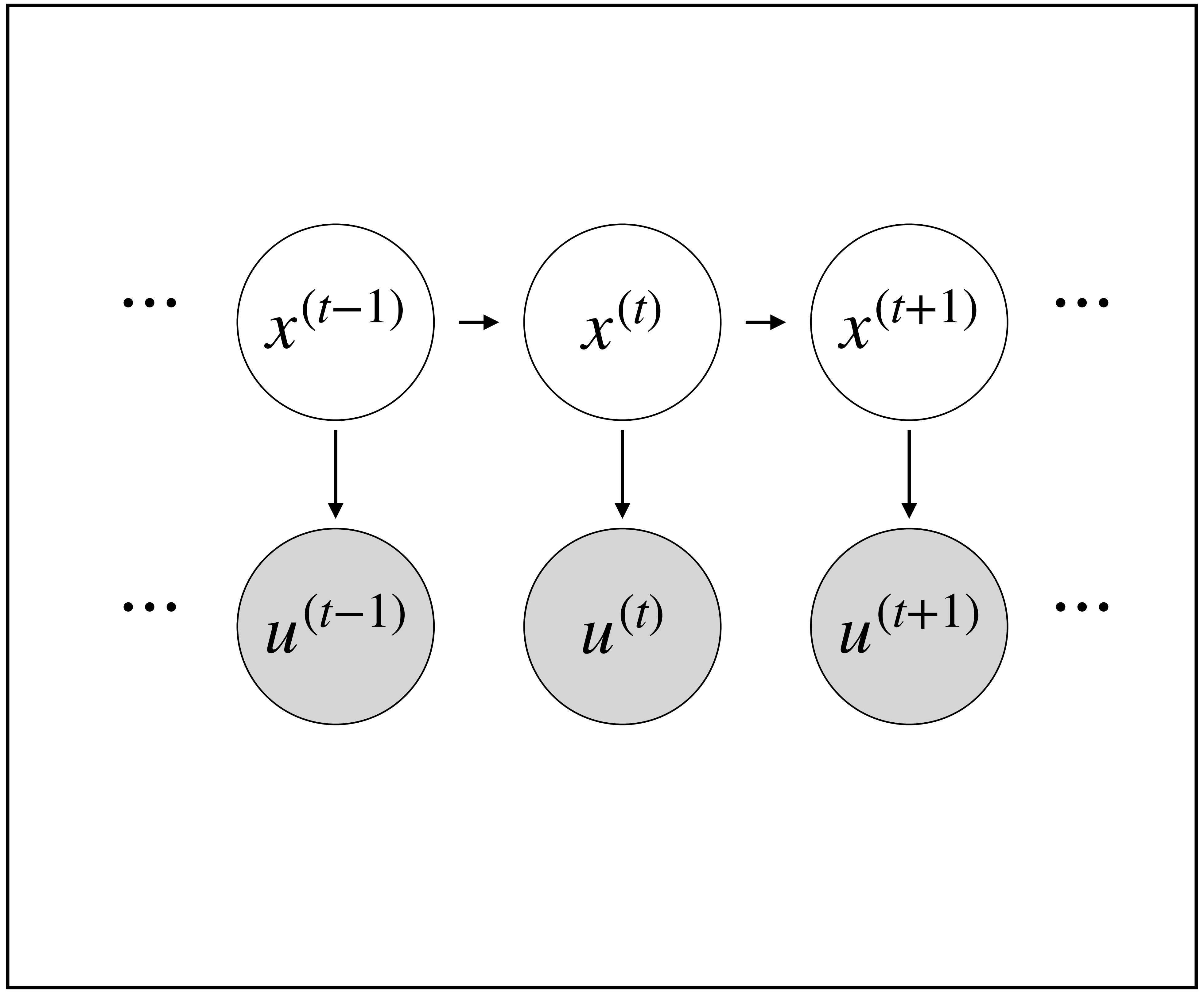



posterior mean
p(u=1)
1.00
0.75
0.50
0.25
0.00
0
10
20
30
trial index

Dynamic inference problems and different types of uncertainty

Static vs. dynamic generative models

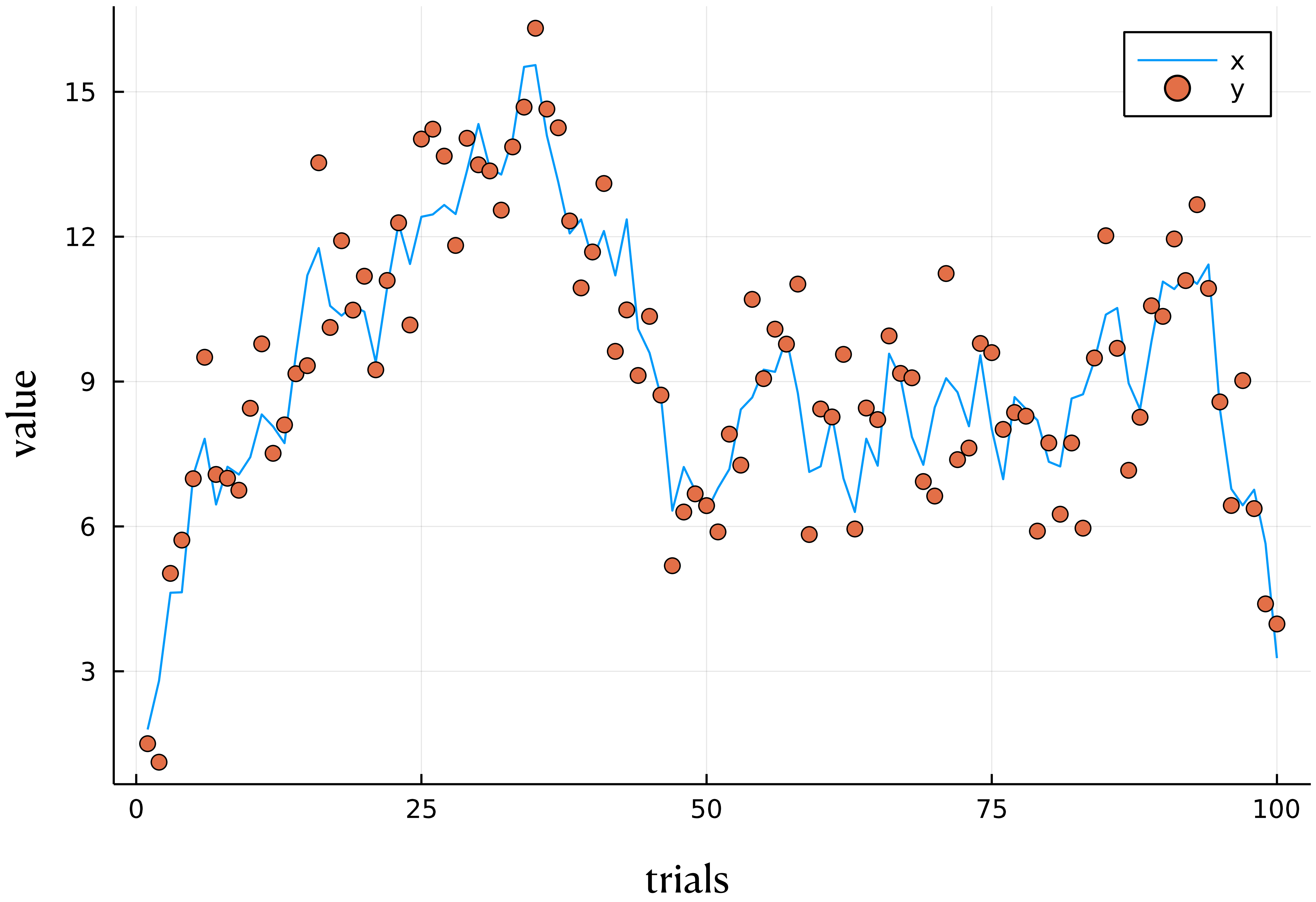




For changing environmental statistics (here: success probabilities / optimal choice)
we need to be forgetting about some of the old data.
This means computing a running average, where old data is down-weighted.

Generative model

$$x^{(0)} \sim \mathcal{N}(\mu^{(0)}, {\sigma^2}^{(0)})$$
$$x^{(t)} \sim \mathcal{N}(x^{(t-1)}, \sigma_x^2)$$
$$y^{(t)} \sim \mathcal{N}(x^{(t)}, \sigma_\epsilon^2)$$



Batch Bayesian estimation

Basic example: fit a normal distribution:

1. Take a batch of data
2. Compute the mean and the variance
3. Done

$$p(\boldsymbol{\theta} \mid \mathbf{y}_{1:T}) = \frac{1}{Z} p(\boldsymbol{\theta}) \prod_{k=1}^{T} p(\mathbf{y}_k \mid \boldsymbol{\theta})$$

Recursive Bayesian estimation

Here we recursively update a distribution representing our current belief, starting from the prior $p(\theta)$.

$$\begin{aligned}
p(\boldsymbol{\theta} \mid \mathbf{y}_1) &= \frac{1}{Z_1} p(\mathbf{y}_1 \mid \boldsymbol{\theta}) p(\boldsymbol{\theta}), \\
p(\boldsymbol{\theta} \mid \mathbf{y}_{1:2}) &= \frac{1}{Z_2} p(\mathbf{y}_2 \mid \boldsymbol{\theta}) p(\boldsymbol{\theta} \mid \mathbf{y}_1), \\
p(\boldsymbol{\theta} \mid \mathbf{y}_{1:3}) &= \frac{1}{Z_3} p(\mathbf{y}_3 \mid \boldsymbol{\theta}) p(\boldsymbol{\theta} \mid \mathbf{y}_{1:2}), \\
&\vdots \\
p(\boldsymbol{\theta} \mid \mathbf{y}_{1:T}) &= \frac{1}{Z_T} p(\mathbf{y}_T \mid \boldsymbol{\theta}) p(\boldsymbol{\theta} \mid \mathbf{y}_{1:T-1})
\end{aligned}$$

Batch

Recursive

With drift

$$x \sim \mathcal{N}(\mu, \sigma^2)$$
$$y^{(t)} \sim \mathcal{N}(x, \sigma_\epsilon^2),\ t = 1,2,\ldots,T$$

$$x^{(0)} \sim \mathcal{N}(\mu^{(0)}, \sigma^{2(0)})$$
$$x^{(t)} \sim \mathcal{N}(x^{(t-1)}, \sigma_x^2)$$
$$y^{(t)} \sim \mathcal{N}(x^{(t)}, \sigma_\epsilon^2)$$

The posterior is a closed form expression. We update the prior parameters with summed up data statistics

$$\boxed{p\left(x \mid y^{(1:T)}\right)} \propto p_{\mathrm{N}}\left(x \mid \mu^{(0)}, \sigma^{(0)}\right) \prod^{T} p_{\mathrm{N}}\left(y^{(t)} \mid x, \sigma_\epsilon\right)$$
$$= \boxed{p_{\mathrm{N}}\left(x \mid \mu^{(T)}, \sigma^{(T)}\right)}$$

$$\mu^{(T)} = \left(\frac{1}{\sigma^{(0)}} + \frac{T}{\sigma_\epsilon}\right)^{-1} \cdot \left(\frac{T}{\sigma_\epsilon}\frac{\sum_{t=1}^{T} y^{(t)}}{T} + \frac{1}{\sigma^{(0)}}\mu^{(0)}\right)$$
$$\sigma^{(T)} = \left(\frac{1}{\sigma^{(0)}} + \frac{T}{\sigma_\epsilon}\right)^{-1}$$

We compute the same posterior, just incrementally updating the sufficient statistics

$$p(x \mid y^{(1:t)}) \propto p(y^{(t)} \mid x, \sigma_\epsilon) p(x \mid y^{(1:t-1)})$$
$$\propto \boxed{p_{\mathrm{N}}(x \mid \mu^{(t)}, \sigma^{(t)})}$$

$$\mu^{(t)} = \left(\frac{1}{\sigma^{(t-1)}} + \frac{1}{\sigma_\epsilon}\right)^{-1} \cdot \left(\frac{1}{\sigma_\epsilon} y^{(t)} + \frac{1}{\sigma^{(t-1)}}\mu^{(t-1)}\right)$$
$$= \frac{\sigma^{(t-1)}}{\sigma^{(t-1)} + \sigma_\epsilon} \cdot y^{(t)} + \frac{\sigma_\epsilon}{\sigma^{(t-1)} + \sigma_\epsilon} \cdot \mu^{(t-1)}$$
$$\sigma^{(t)} = \left(\frac{1}{\sigma^{(t-1)}} + \frac{1}{\sigma_\epsilon}\right)^{-1}$$

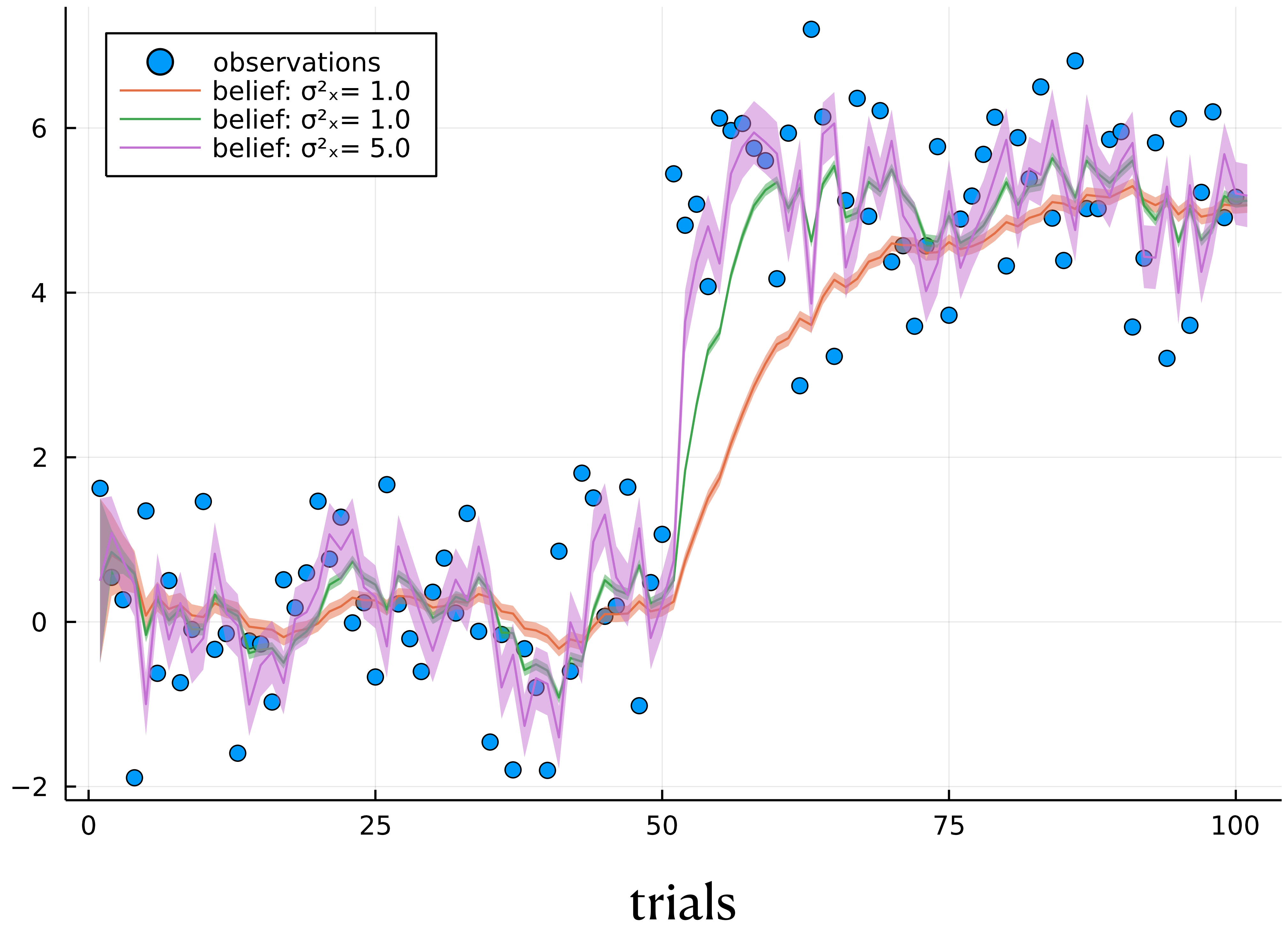
Here, because of the drift, we update with a different prior

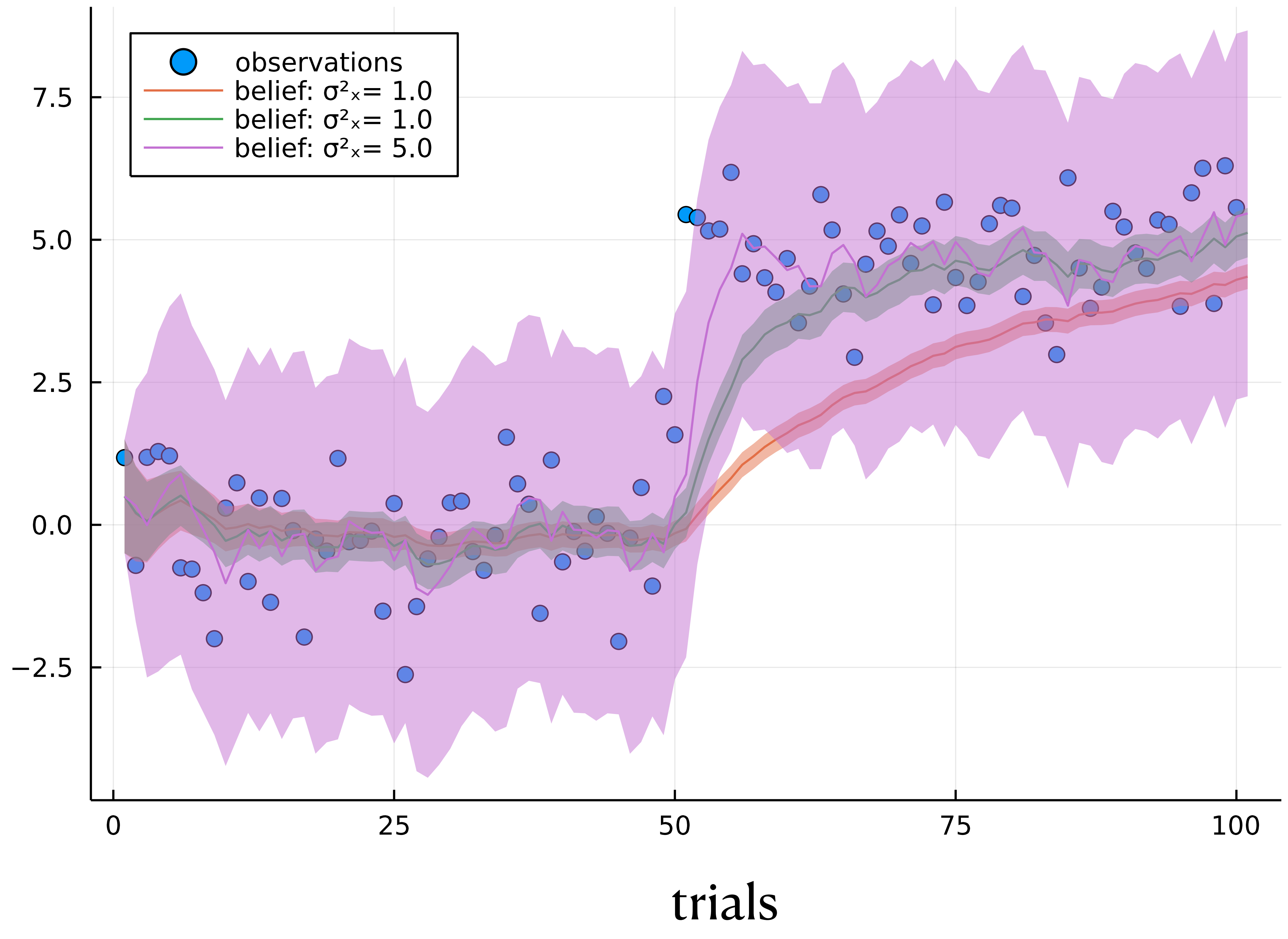
$$\hat{\mu}^{(t)} = \mu^{(t-1)}$$
$$\hat{\sigma}^{(t)} = \left(\frac{1}{\sigma^{(t-1)}} + \frac{1}{\sigma_x}\right)^{-1}$$

$$\boxed{p(x^{(t)} \mid y^{(1:t)}) = p_{\mathrm{N}}(x^{(t)} \mid \mu^{(t)}, \sigma^{(t)})}$$

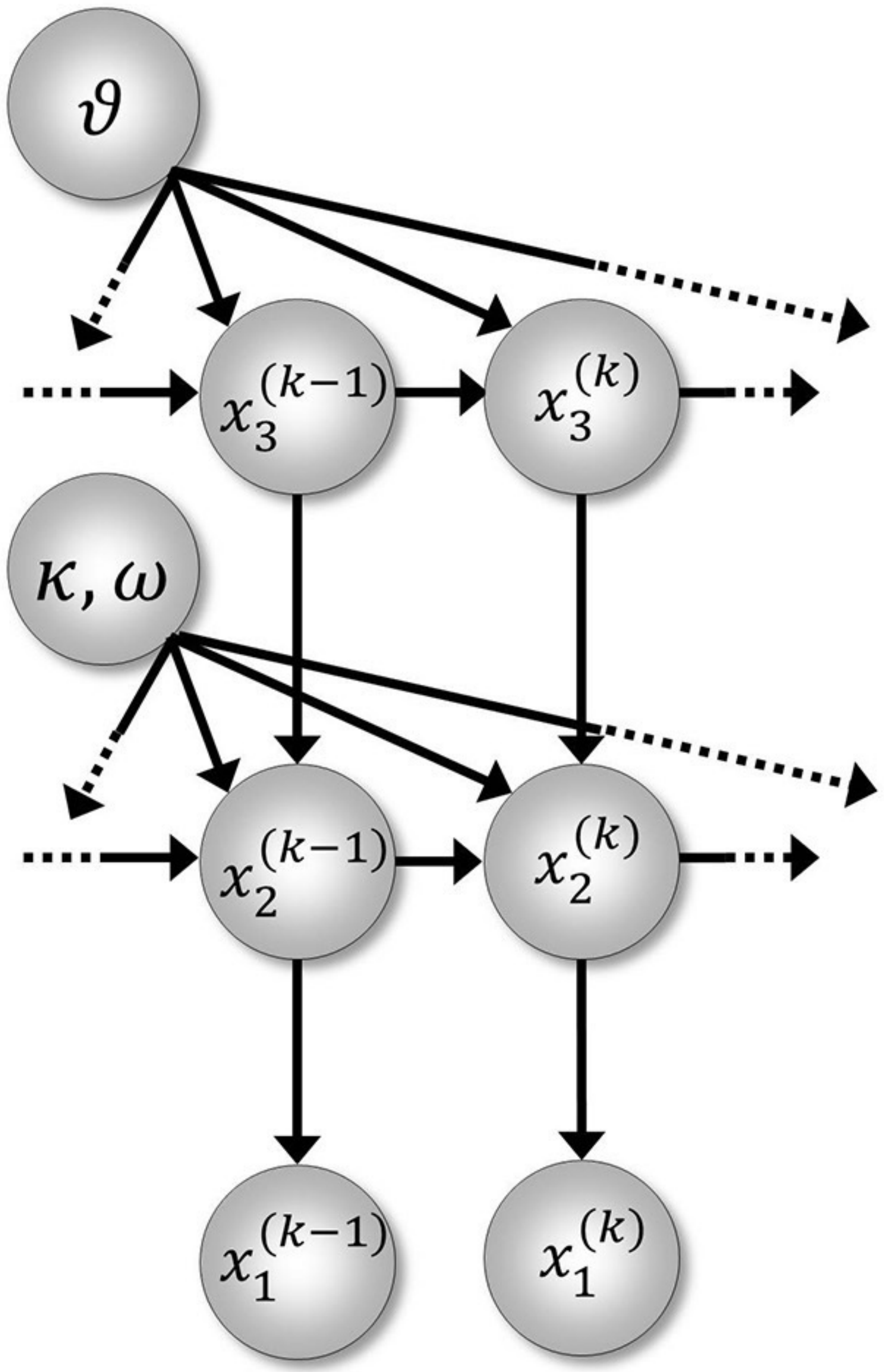
$$\mu^{(t)} = \hat{\mu}^{(t-1)} + \frac{\hat{\sigma}^{(t)}}{\hat{\sigma}^{(t)} + \sigma_\epsilon} \cdot (y^{(t)} - \hat{\mu}^{(t)})$$
$$\sigma^{(t)} = \hat{\sigma}^{(t)} - \frac{\hat{\sigma}^{(t)2}}{\hat{\sigma}^{(t)} + \sigma_\epsilon}$$
$$= \left(1 - \frac{\hat{\sigma}^{(t)}}{\hat{\sigma}^{(t)} + \sigma_\epsilon}\right)\hat{\sigma}^{(t)}$$

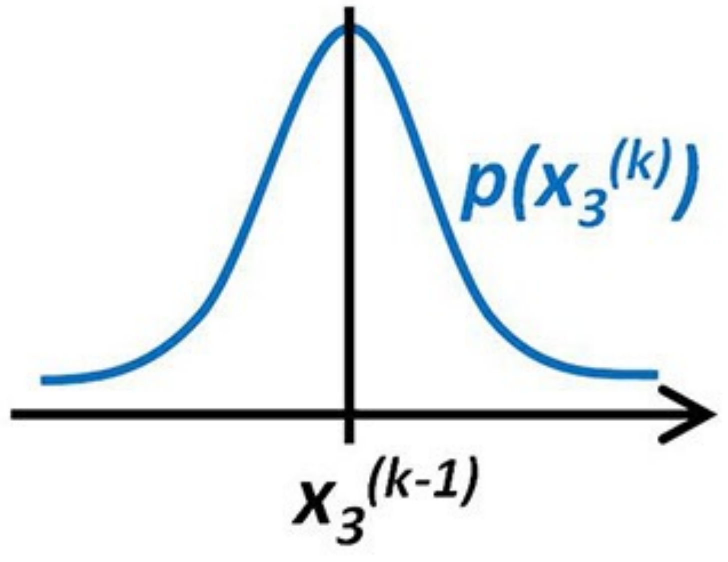
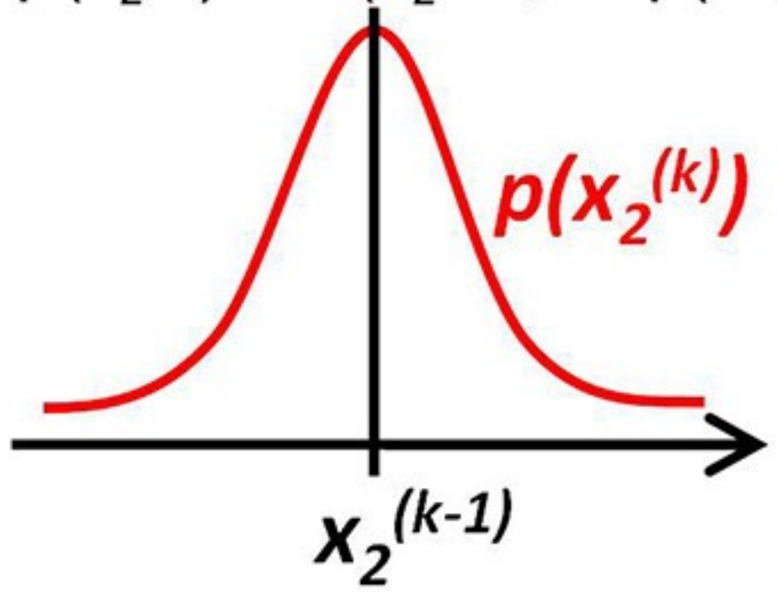
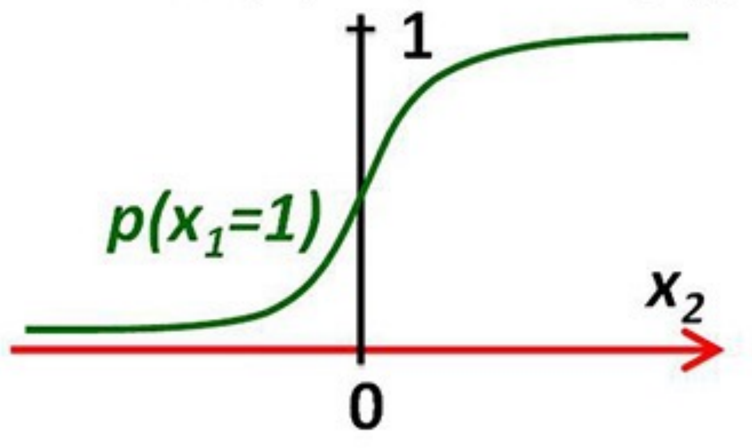
Let’s simulate belief trajectories for a given sequence of observations:





The generative model assumed for HGF (3-level for binary outcomes)



State of the world	Model	
Log-volatility x_3 of tendency	Gaussian random walk with constant step size ϑ	$p(x_3^{(k)}) \sim N(x_3^{(k-1)}, \vartheta)$ 
Tendency x_2 towards category "1"	Gaussian random walk with step size $\exp(\kappa x_3+\omega)$	$p(x_2^{(k)}) \sim N(x_2^{(k-1)}, \exp(\kappa x_3+\omega))$ 
Stimulus category x_1 ("0" or "1")	Sigmoid trans-formation of x_2	$p(x_1=1) = s(x_2)$ $p(x_1=0) = 1-s(x_2)$ 

Precision weights and types of uncertainty

The learners observations are generated by:

$$u^{(t)} \sim \mathrm{Ber}\left(x_1^{(t)}\right)$$

which leads to these updates for the belief about the latent process:

$$\mu_2^{(t)} = \mu_2^{(t-1)} + \frac{\hat{\pi}_u}{\pi_2^{(t)}}\delta_1^{(t)} \qquad \hat{\mu}_1^{(t)} = s\left(\mu_2^{(t)}\right)$$

The precision weight can be decomposed into factors corresponding to different kinds of uncertainty:

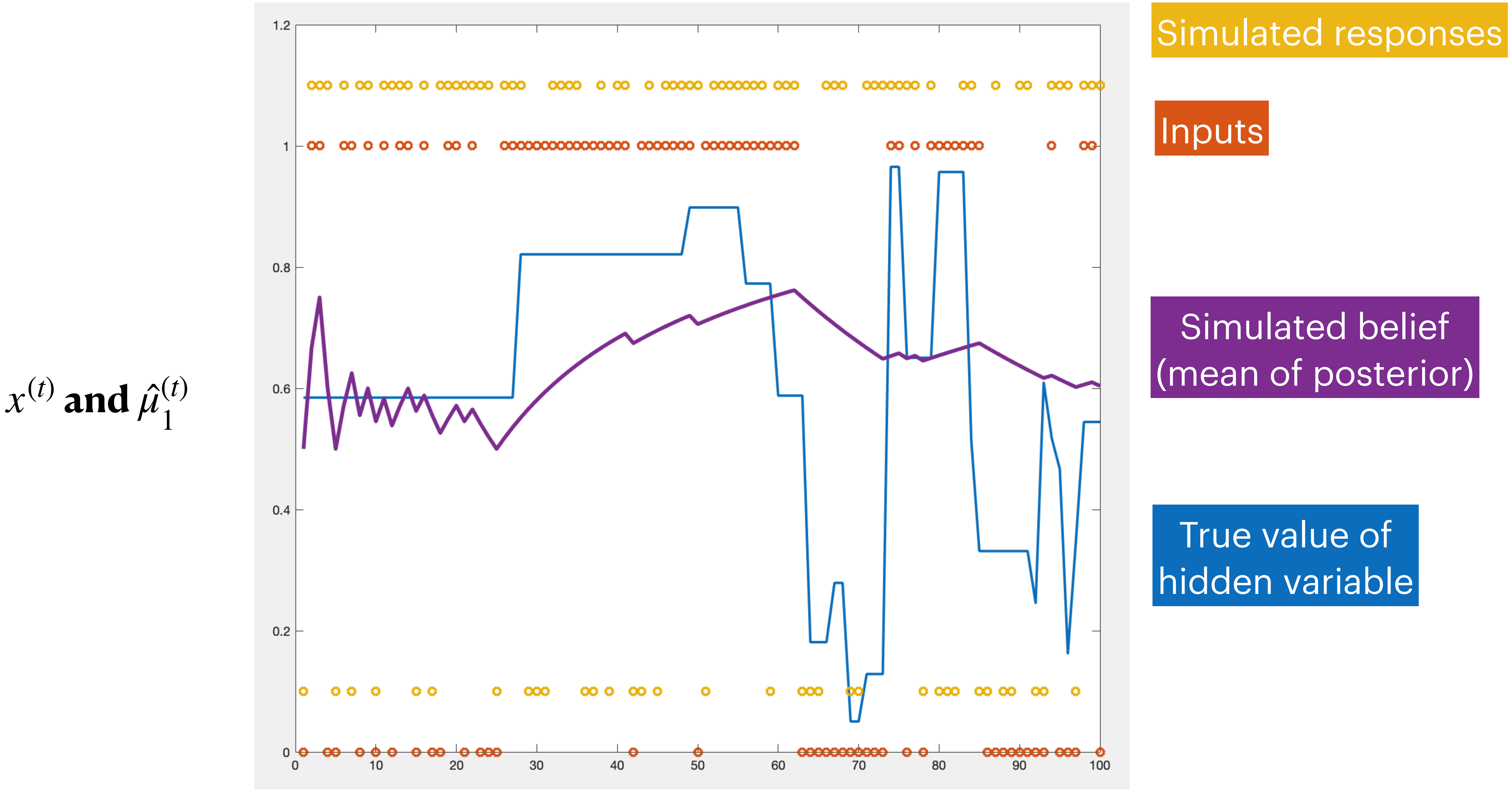
$$\frac{\hat{\pi}_u}{\pi_2^{(t)}} = \frac{\hat{\pi}_u}{\hat{\pi}^{(t-1)} + \hat{\pi}_u} = \frac{\hat{\pi}_u}{\frac{1}{\sigma_2^{(t-1)} \exp(\kappa\mu_3^{(t-1)} + \omega)} + \hat{\pi}_u}$$

Estimation uncertainty

Estimated volatility of the environment

Irreducible uncertainty about the outcome

Simulation of Beta-Bernoulli model

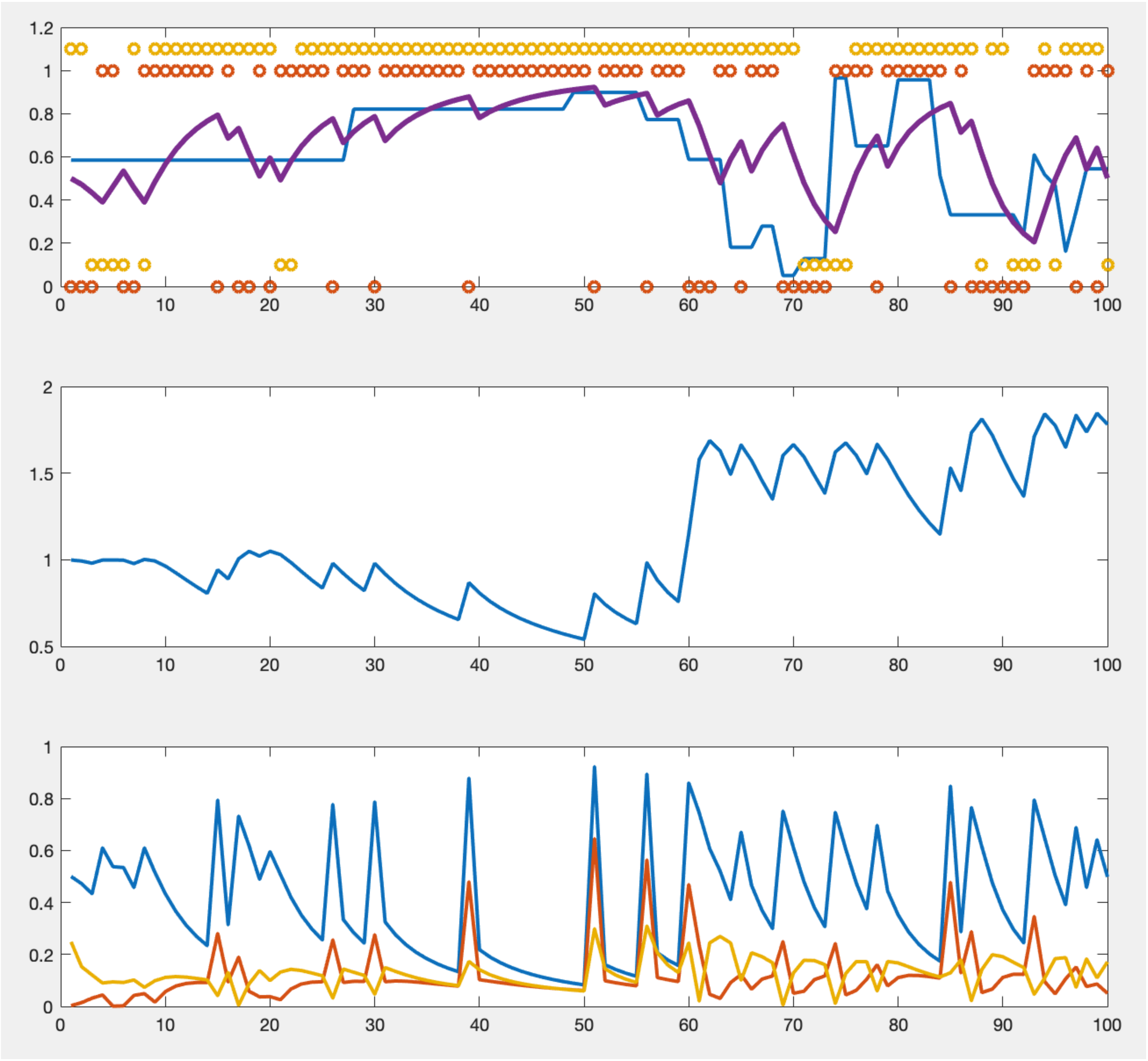


Simulation of HGF

$x^{(t)}$ **and** $\hat{\mu}_1^{(t)}$

Belief about volatility, $\hat{\mu}_3^{(t)}$

Prediction errors, $\delta_{1:3}^{(t)}$



Simulated responses

Inputs

Simulated belief (mean of posterior)

True value of hidden variable

$\delta_1^{(t)}$

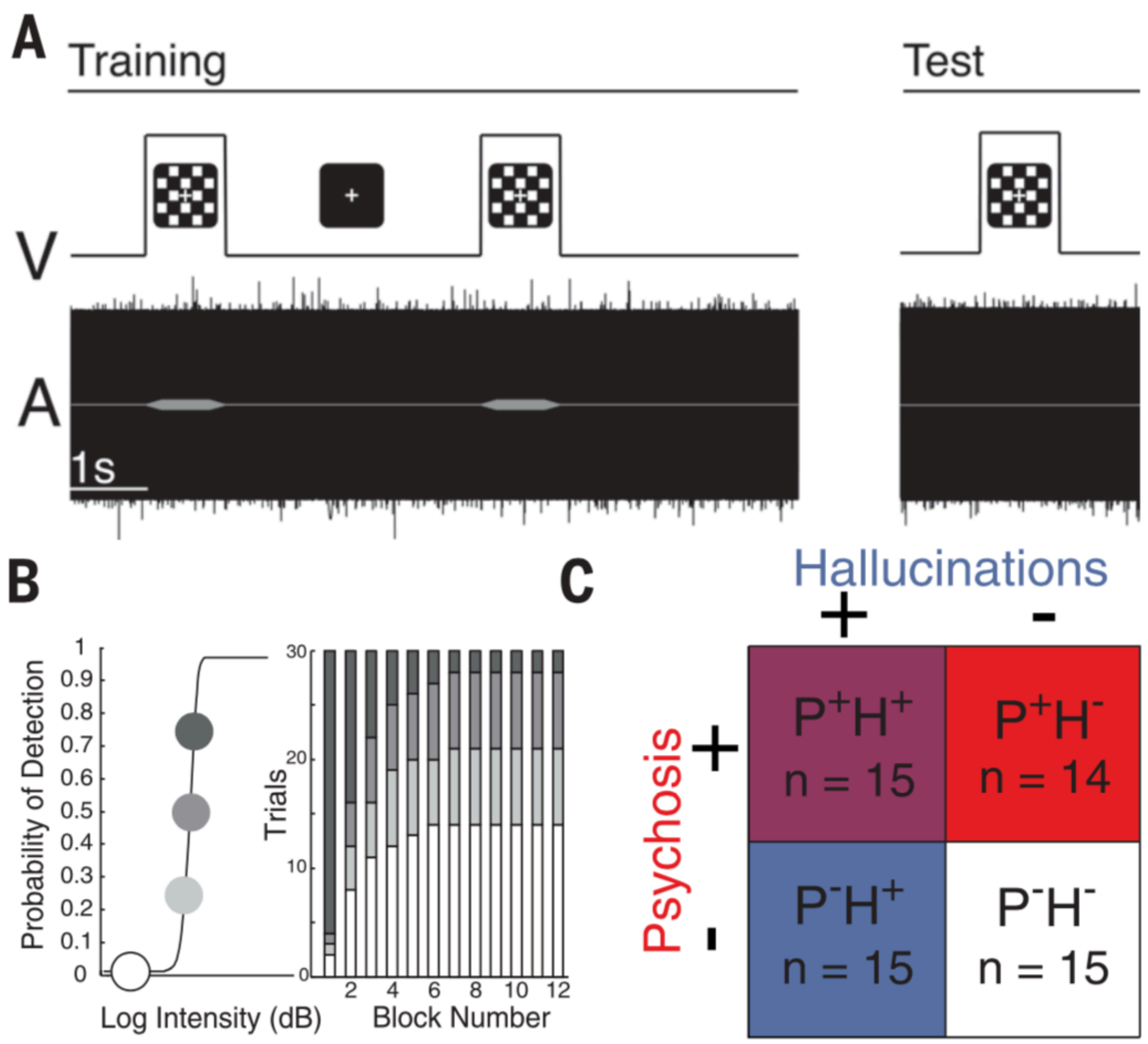
$\delta_2^{(t)}$

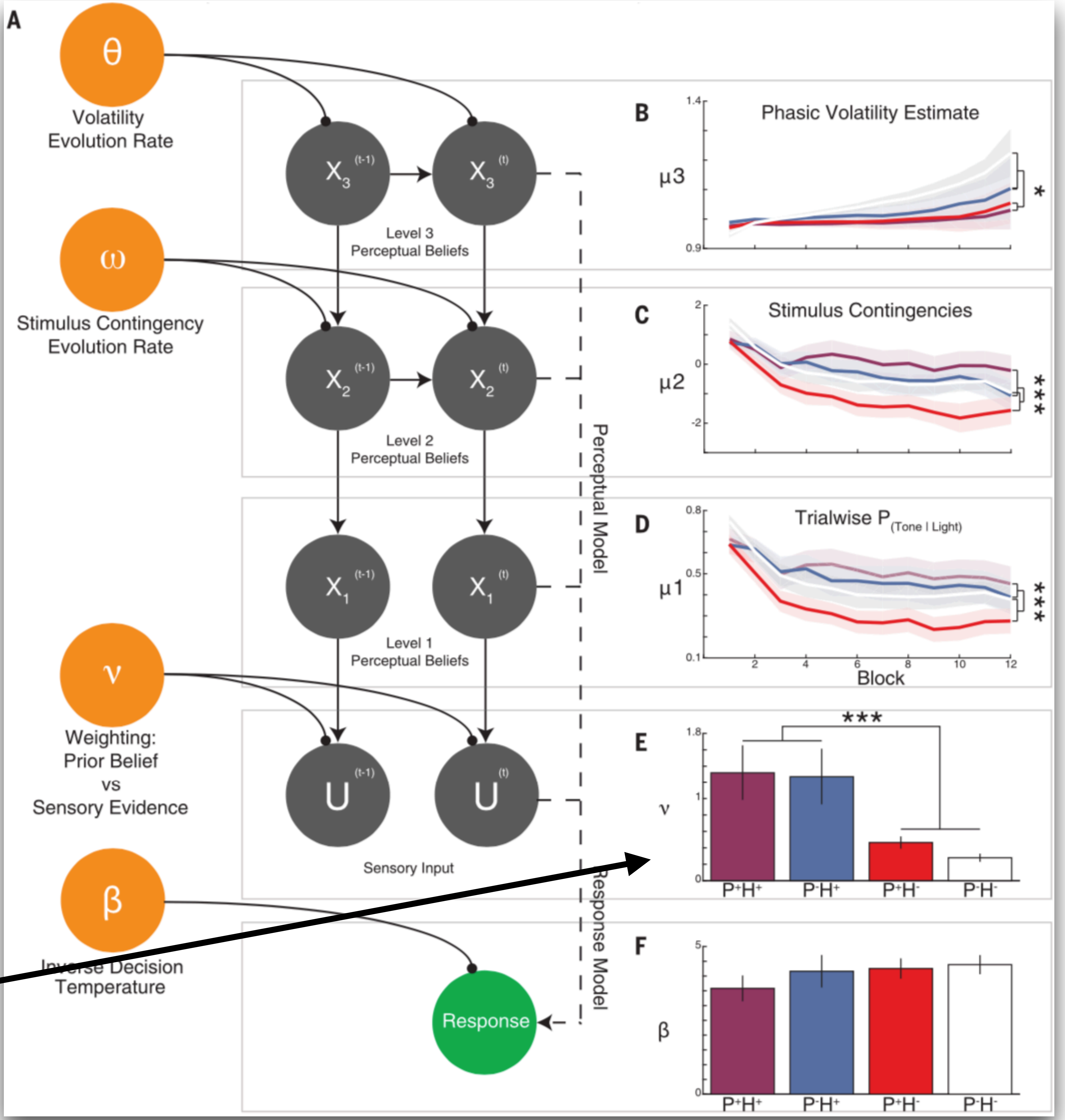
$\delta_3^{(t)}$

References / further reading

- Theory
  - “A reading list on Bayesian methods": http://cocosci.princeton.edu/tom/bayes.html
  - Mathys et al. (2011): “A Bayesian foundation for individual learning under uncertainty”
  - Mathys et al. (2014): “Uncertainty in perception and the Hierarchical Gaussian Filter”
  - Daunizeau et al. (2010): “Observing the Observer (I): Meta-Bayesian Models of Learning and Decision-Making”
  - Maia and Frank (2011): “From Reinforcement Learning Models to Psychiatric and Neurological Disorders”
- Applications
  - Iglesias et al. (2013): “Hierarchical Prediction Errors in Midbrain and Basal Forebrain during Sensory Learning”
  - de Berker et al. (2015): “Computations of uncertainty mediate acute stress responses in humans”
  - Powers et al. (2017): “Pavlovian conditioning–induced hallucinations result from overweighting of perceptual priors”
- General modelling
  - Wilson and Collins (2019): “Ten simple rules for the computational modeling of behavioral data”
  - Palminteri et al. (2017): “The Importance of Falsification in Computational Cognitive Modeling”

Conditioned hallucinations





Subjects with hallucinations show higher estimates for weights on prior beliefs

Powers et al. (2017): "Pavlovian conditioning–induced hallucinations result from overweighting of perceptual priors"