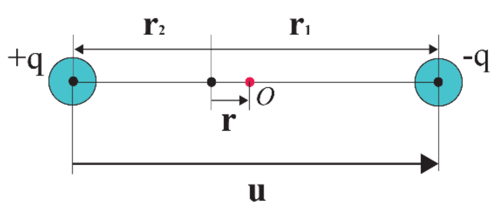


$$\mathbf{r}_1 = \mathbf{r} + \mathbf{u}/2$$
$$\mathbf{r}_2 = \mathbf{r} - \mathbf{u}/2$$

$$\boldsymbol{\tau} = -\mathbf{r}_1 \times \left( q\mathbf{E} + q\frac{d\mathbf{r}_1}{dt} \times \mathbf{B} \right) + \mathbf{r}_2 \times \left( q\mathbf{E} + q\frac{d\mathbf{r}_2}{dt} \times \mathbf{B} \right) =$$

$$q\left\{ -(\mathbf{r}+\mathbf{u}/2)\times\mathbf{E} + (\mathbf{r}-\mathbf{u}/2)\times\mathbf{E} - (\mathbf{r}+\mathbf{u}/2)\times\left(\frac{d\mathbf{r}}{dt}\times\mathbf{B} + \frac{1}{2}\frac{d\mathbf{u}}{dt}\times\mathbf{B}\right) + (\mathbf{r}-\mathbf{u}/2)\times\left(\frac{d\mathbf{r}}{dt}\times\mathbf{B} - \frac{1}{2}\frac{d\mathbf{u}}{dt}\times\mathbf{B}\right) \right\} =$$

$$q\left\{ -(\mathbf{r}+\mathbf{u}/2)\times\mathbf{E} + (\mathbf{r}-\mathbf{u}/2)\times\mathbf{E} - (\mathbf{r}+\mathbf{u}/2)\times\left(\frac{d\mathbf{r}}{dt}\times\mathbf{B} + \frac{1}{2}\frac{d\mathbf{u}}{dt}\times\mathbf{B}\right) + (\mathbf{r}-\mathbf{u}/2)\times\left(\frac{d\mathbf{r}}{dt}\times\mathbf{B} - \frac{1}{2}\frac{d\mathbf{u}}{dt}\times\mathbf{B}\right) \right\} =$$

$$q\left\{ \underbrace{-\mathbf{u}}_{q(-\mathbf{u})=\mathbf{p}} \times\mathbf{E} - \mathbf{r}\times\left(\frac{d\mathbf{u}}{dt}\times\mathbf{B}\right) - \mathbf{u}\times\left(\underbrace{\frac{d\mathbf{r}}{dt}}_{=0}\times\mathbf{B}\right) \right\} = \mathbf{p}\times\mathbf{E} + \mathbf{r}\times\underbrace{\left(\frac{d\mathbf{p}}{dt}\times\mathbf{B}\right)}_{2nd\ order\ small}$$

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$$\mathbf{L} = m\mathbf{r}_1 \times \frac{d\mathbf{r}_1}{dt} + m\mathbf{r}_2 \times \frac{d\mathbf{r}_2}{dt} =$$

$$m\left\{ (\mathbf{r}+\mathbf{u}/2)\times\left(\frac{d\mathbf{r}}{dt} + \frac{1}{2}\frac{d\mathbf{u}}{dt}\right) + (\mathbf{r}-\mathbf{u}/2)\times\left(\frac{d\mathbf{r}}{dt} - \frac{1}{2}\frac{d\mathbf{u}}{dt}\right) \right\} =$$

$$\frac{m}{q^2}\left\{ 2\mathbf{r}\times\underbrace{\frac{d\mathbf{r}}{dt}}_{=0} + \frac{1}{2}\mathbf{p}\times\frac{d\mathbf{p}}{dt} \right\} = \frac{m}{2q^2}\mathbf{p}\times\frac{d\mathbf{p}}{dt}$$

$$\Downarrow$$

$$\frac{d\mathbf{L}}{dt} = \frac{m}{2q^2}\frac{d}{dt}\left(\mathbf{p}\times\frac{d\mathbf{p}}{dt}\right)$$

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$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} \quad \Rightarrow \quad \frac{d}{dt}\left(\mathbf{p}\times\frac{d\mathbf{p}}{dt}\right) = \frac{2q^2}{m}\mathbf{p}\times\mathbf{E}$$

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$$\frac{d}{dt}\left(\mathbf{p}\times\frac{d\mathbf{p}}{dt}\right)=\frac{2q^2}{m}\mathbf{p}\times\mathbf{E}$$

$$\mathbf{p}(t)=\mathbf{p}_0+\mathbf{p}_1(t)$$

$$\frac{d\mathbf{p}_0}{dt}=0;\quad \mathbf{p}_1(t)=\mathbf{p}_0\times\delta\boldsymbol{\alpha}(t)\quad (\delta\boldsymbol{\alpha}\parallel\mathbf{z},\quad \mathbf{p}_0\in[xoy])$$

$$\frac{d}{dt}\left(\mathbf{p}_0\times\frac{d\mathbf{p}_1}{dt}\right)=\frac{2q^2}{m}\mathbf{p}_0\times\mathbf{E}$$

$\Downarrow$

$$\frac{d^2}{dt^2}\left(\mathbf{p}_0\times\mathbf{p}_1\right)=\frac{2q^2}{m}\mathbf{p}_0\times\mathbf{E}$$

$\Downarrow$

$$\frac{d^2}{dt^2}\left(\mathbf{p}_0\times\mathbf{p}_1\right)=\frac{d^2}{dt^2}\left\{\mathbf{p}_0\times\left[\mathbf{p}_0\times\delta\boldsymbol{\alpha}(t)\right]\right\}=\frac{d^2}{dt^2}\left\{\left(\underbrace{\mathbf{p}_0\cdot\delta\boldsymbol{\alpha}}_{=0}\right)\cdot\mathbf{p}_0-\left(\mathbf{p}_0\cdot\mathbf{p}_0\right)\cdot\delta\boldsymbol{\alpha}\right\}=$$

$$-\mathbf{z}\left|p_0\right|^2\frac{d^2(\delta\alpha)}{dt^2}=\mathbf{z}\omega^2\left(p_{x0}^2+p_{y0}^2\right)\cdot\delta\alpha$$

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$$\mathbf{p}_0\times\mathbf{E}=(\mathbf{x}p_{xo}+\mathbf{y}p_{yo})\times(\mathbf{x}E_x+\mathbf{y}E_y)=\mathbf{z}\left(p_{x0}E_y-p_{y0}E_x\right)$$

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$$\omega^2\left(p_{x0}^2+p_{y0}^2\right)\cdot\delta\alpha=\frac{2q^2}{m}\left(p_{x0}E_y-p_{y0}E_x\right)$$

$\Downarrow$

$$\delta\alpha=\frac{2q^2}{\omega^2 m}\frac{p_{x0}E_y-p_{y0}E_x}{p_{x0}^2+p_{y0}^2}=A\cdot E_y-B\cdot E_x\qquad\left\{A=\frac{2q^2}{\omega^2 m}\frac{p_{x0}}{p_{x0}^2+p_{y0}^2},\quad B=\frac{2q^2}{\omega^2 m}\frac{p_{y0}}{p_{x0}^2+p_{y0}^2}\right\}$$

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$$\mathbf{p}_1(t) = \left(\mathbf{x}p_{x0} + \mathbf{y}p_{y0}\right)\times\mathbf{z}\left(A\cdot E_y - B\cdot E_x\right) = -\mathbf{y}p_{x0}\left(A\cdot E_y - B\cdot E_x\right) + \mathbf{x}p_{y0}\left(A\cdot E_y - B\cdot E_x\right)$$

$\Downarrow$

$$p_{1x} = p_{y0}\left(A\cdot E_y - B\cdot E_x\right) = -\frac{2q^2}{\omega^2 m}\frac{(p_{y0})^2}{p_{x0}^2 + p_{y0}^2}\cdot E_x + \frac{2q^2}{\omega^2 m}\frac{p_{y0}\cdot p_{x0}}{p_{x0}^2 + p_{y0}^2}\cdot E_y$$

$$p_{1y} = -p_{x0}\left(A\cdot E_y - B\cdot E_x\right) = \frac{2q^2}{\omega^2 m}\frac{p_{x0}\cdot p_{y0}}{p_{x0}^2 + p_{y0}^2}\cdot E_x - \frac{2q^2}{\omega^2 m}\frac{(p_{x0})^2}{p_{x0}^2 + p_{y0}^2}\cdot E_y$$

$\Downarrow$

$$\begin{pmatrix} p_{1x} \\ p_{1y} \end{pmatrix} = \begin{pmatrix} -\frac{2q^2}{\omega^2 m\varepsilon_0}\frac{(p_{y0})^2}{p_{x0}^2 + p_{y0}^2} & \frac{2q^2}{\omega^2 m\varepsilon_0}\frac{p_{y0}\cdot p_{x0}}{p_{x0}^2 + p_{y0}^2} \\ \frac{2q^2}{\omega^2 m\varepsilon_0}\frac{p_{x0}\cdot p_{y0}}{p_{x0}^2 + p_{y0}^2} & -\frac{2q^2}{\omega^2 m\varepsilon_0}\frac{(p_{x0})^2}{p_{x0}^2 + p_{y0}^2} \end{pmatrix}\cdot\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \varepsilon_0\overline{\overline{\varepsilon}}\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$\Downarrow$

$$\begin{pmatrix} p_{1x} \\ p_{1y} \end{pmatrix} = \begin{pmatrix} -\varepsilon_1 & \varepsilon_2 \\ \varepsilon_2 & -\varepsilon_3 \end{pmatrix}\cdot\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \varepsilon_0\overline{\overline{\varepsilon}}\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

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Maxwell's curl equations for the grating layer:

$$\nabla\times\mathbf{H}(x,y) = -i\omega\varepsilon_0\tilde{\varepsilon}\,\mathbf{E}(x,y) \qquad \|\,\mathbf{D}(x,y) = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0(\mathbf{E} + \overline{\overline{\varepsilon}}\mathbf{P})$$
$$\nabla\times\mathbf{E}(x,y) = i\omega\mu_0\mathbf{H}(x,y)$$

where

$$\tilde{\varepsilon} = \begin{pmatrix} 1-\varepsilon_1 & \varepsilon_2 \\ \varepsilon_2 & 1-\varepsilon_3 \end{pmatrix} = \begin{pmatrix} 1-\frac{2q^2}{\omega^2 m\varepsilon_0}\sin^2\varphi & \frac{2q^2}{\omega^2 m\varepsilon_0}\sin\varphi\cdot\cos\varphi \\ \frac{2q^2}{\omega^2 m\varepsilon_0}\sin\varphi\cdot\cos\varphi & 1-\frac{2q^2}{\omega^2 m\varepsilon_0}\cos^2\varphi \end{pmatrix} \equiv \begin{pmatrix} \tilde{\varepsilon}_1 & \tilde{\varepsilon}_2 \\ \tilde{\varepsilon}_2 & \tilde{\varepsilon}_3 \end{pmatrix}$$

$$\left\{\mathbf{E}(x,y),\mathbf{H}(x,y)\right\} \sim e^{-i\omega t + ik_x x \pm ik_y y}$$
$$\frac{\partial}{\partial z}\left\{\mathbf{E}(x,y),\mathbf{H}(x,y)\right\} = 0$$

(No field variations along the z-direction). We get the following expressions for Maxwell equation:

$$\frac{\partial H_z}{\partial y} = -i\omega\varepsilon_0(\tilde{\varepsilon}_1 E_x + \tilde{\varepsilon}_2 E_y)$$
$$-\frac{\partial H_z}{\partial x} = -i\omega\varepsilon_0(\tilde{\varepsilon}_2 E_x + \tilde{\varepsilon}_3 E_y)$$
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega\mu_0 H_z$$

The components of the electromagnetic fields are periodic functions of x and can be approximately expanded in truncated generalized Fourier series:

$$\left\{E_x(x,y),H_z(x,y)\right\}=\sum_{n=-\infty}^{\infty}\left\{E_{x,n}(y),H_{z,n}(y)\right\}e^{i(k_x+\frac{2\pi}{h}n)x}$$

where $h$ is a period of the grating. The relation among the Fourier coefficients can be written as:

$$\frac{\partial \mathbf{h}_z(y)}{\partial y}=-i\omega\varepsilon_0\left\{[\![\tilde{\varepsilon}_1]\!]\mathbf{e}_x(y)+[\![\tilde{\varepsilon}_2]\!]\mathbf{e}_y(y)\right\}$$

$$-i\mathbf{K}\mathbf{h}_z(y)=-i\omega\varepsilon_0\left\{[\![\tilde{\varepsilon}_2]\!]\mathbf{e}_x(y)+[\![\tilde{\varepsilon}_3]\!]\mathbf{e}_y(y)\right\}$$

$$i\mathbf{K}\mathbf{e}_y(y)-\frac{\partial \mathbf{e}_x(y)}{\partial y}=i\omega\mu_0\mathbf{h}_z(y)$$

with

$\mathbf{K}=[k_{xn}\delta_{nm}]$ - diagonal matrix

$$\left[\mathbf{e}_x(y)\right]_n=E_{x,n}(y)$$

$$\left[\mathbf{e}_y(y)\right]_n=E_{y,n}(y)$$

$[\![\tilde{\varepsilon}_p(x)]\!]_{n,m}=\frac{1}{h}\int_0^h dx\tilde{\varepsilon}_p(x)e^{-i(n-m)\frac{2\pi}{h}x}$ - (Toeplitz matrix), $p$=1, 2, 3

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From the 2nd expression we get:

$$[\![\tilde{\varepsilon}_3]\!]\mathbf{e}_y=\frac{\mathbf{K}}{\omega\varepsilon_0}\mathbf{h}_z-[\![\tilde{\varepsilon}_2]\!]\mathbf{e}_x$$

$$\Downarrow$$

$$\mathbf{e}_y=[\![\tilde{\varepsilon}_3]\!]^{-1}\frac{\mathbf{K}}{\omega\varepsilon_0}\mathbf{h}_z-[\![\tilde{\varepsilon}_3]\!]^{-1}[\![\tilde{\varepsilon}_2]\!]\mathbf{e}_x$$

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$$\frac{\partial \mathbf{h}_z(y)}{\partial y} = -i\omega\varepsilon_0 [\![\tilde{\varepsilon}_1]\!]\mathbf{e}_x - i\omega\varepsilon_0 [\![\tilde{\varepsilon}_2]\!]\left\{[\![\tilde{\varepsilon}_3]\!]^{-1}\frac{\mathbf{K}}{\omega\varepsilon_0}\mathbf{h}_z - [\![\tilde{\varepsilon}_3]\!]^{-1}[\![\tilde{\varepsilon}_2]\!]\mathbf{e}_x\right\} =$$

$$-i\omega\varepsilon_0\left([\![\tilde{\varepsilon}_1]\!] - [\![\tilde{\varepsilon}_2]\!][\![\tilde{\varepsilon}_3]\!]^{-1}[\![\tilde{\varepsilon}_2]\!]\right)\mathbf{e}_x - i[\![\tilde{\varepsilon}_2]\!][\![\tilde{\varepsilon}_3]\!]^{-1}\mathbf{K}\mathbf{h}_z$$

$\Downarrow$

$$\frac{\partial \tilde{\mathbf{h}}_z(y)}{\partial y} = -ik_0\left([\![\tilde{\varepsilon}_1]\!] - [\![\tilde{\varepsilon}_2]\!][\![\tilde{\varepsilon}_3]\!]^{-1}[\![\tilde{\varepsilon}_2]\!]\right)\mathbf{e}_x - ik_0[\![\tilde{\varepsilon}_2]\!][\![\tilde{\varepsilon}_3]\!]^{-1}\frac{\mathbf{K}}{k_0}\tilde{\mathbf{h}}_z$$

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$$\frac{\partial \mathbf{e}_x(y)}{\partial y} = i\mathbf{K}\mathbf{e}_y(y) - i\omega\mu_0\mathbf{h}_z(y) =$$

$$i\mathbf{K}\left\{[\![\tilde{\varepsilon}_3]\!]^{-1}\frac{\mathbf{K}}{\omega\varepsilon_0}\mathbf{h}_z - [\![\tilde{\varepsilon}_3]\!]^{-1}[\![\tilde{\varepsilon}_2]\!]\mathbf{e}_x\right\} - i\omega\mu_0\mathbf{h}_z =$$

$$i\left\{[\![\tilde{\varepsilon}_3]\!]^{-1}\frac{\mathbf{K}^2}{\omega\varepsilon_0} - \omega\mu_0\right\}\mathbf{h}_z - i\mathbf{K}[\![\tilde{\varepsilon}_3]\!]^{-1}[\![\tilde{\varepsilon}_2]\!]\mathbf{e}_x$$

$\Downarrow$

$$\frac{\partial \mathbf{e}_x(y)}{\partial y} = ik_0\left\{[\![\tilde{\varepsilon}_3]\!]^{-1}\frac{\mathbf{K}^2}{k_0^2} - \mathbf{I}\right\}\tilde{\mathbf{h}}_z - ik_0\frac{\mathbf{K}}{k_0}[\![\tilde{\varepsilon}_3]\!]^{-1}[\![\tilde{\varepsilon}_2]\!]\mathbf{e}_x$$

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$$\frac{\partial}{\partial y}\begin{pmatrix}\mathbf{e}_x(y)\\ \tilde{\mathbf{h}}_z(y)\end{pmatrix} = ik_0\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}\mathbf{e}_x(y)\\ \tilde{\mathbf{h}}_z(y)\end{pmatrix}$$

$$M_{11} = -\frac{\mathbf{K}}{k_0}[\![\tilde{\varepsilon}_3]\!]^{-1}[\![\tilde{\varepsilon}_2]\!], \quad M_{12} = [\![\tilde{\varepsilon}_3]\!]^{-1}\frac{\mathbf{K}^2}{k_0^2} - \mathbf{I}$$

$$M_{21} = -\left([\![\tilde{\varepsilon}_1]\!] - [\![\tilde{\varepsilon}_2]\!][\![\tilde{\varepsilon}_3]\!]^{-1}[\![\tilde{\varepsilon}_2]\!]\right), \quad M_{22} = -[\![\tilde{\varepsilon}_2]\!][\![\tilde{\varepsilon}_3]\!]^{-1}\frac{\mathbf{K}}{k_0}$$

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