Linear regression (continued)

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Tools for linear regression analysis

We will still use the advertising dataset. Let's load it.

```
library(readr)
advertising <- read_csv("./datasets/advertising.csv")
```

```
## Rows: 200 Columns: 4
## -- Column specification ------------------------------------
## Delimiter: ","
## dbl (4): TV, Radio, Newspaper, Sales
##
## i Use `spec()` to retrieve the full column specification for this data.
## i Specify the column types or set `show_col_types = FALSE` to quiet this message.
```

Plot

The plot function `plot()` provides special plots for a linear model object. See details `?plot.lm`.

```
radio_lm <- lm(Sales ~ Radio, data = advertising)
# residuals vs fitted
plot(radio_lm, which = 1)
```

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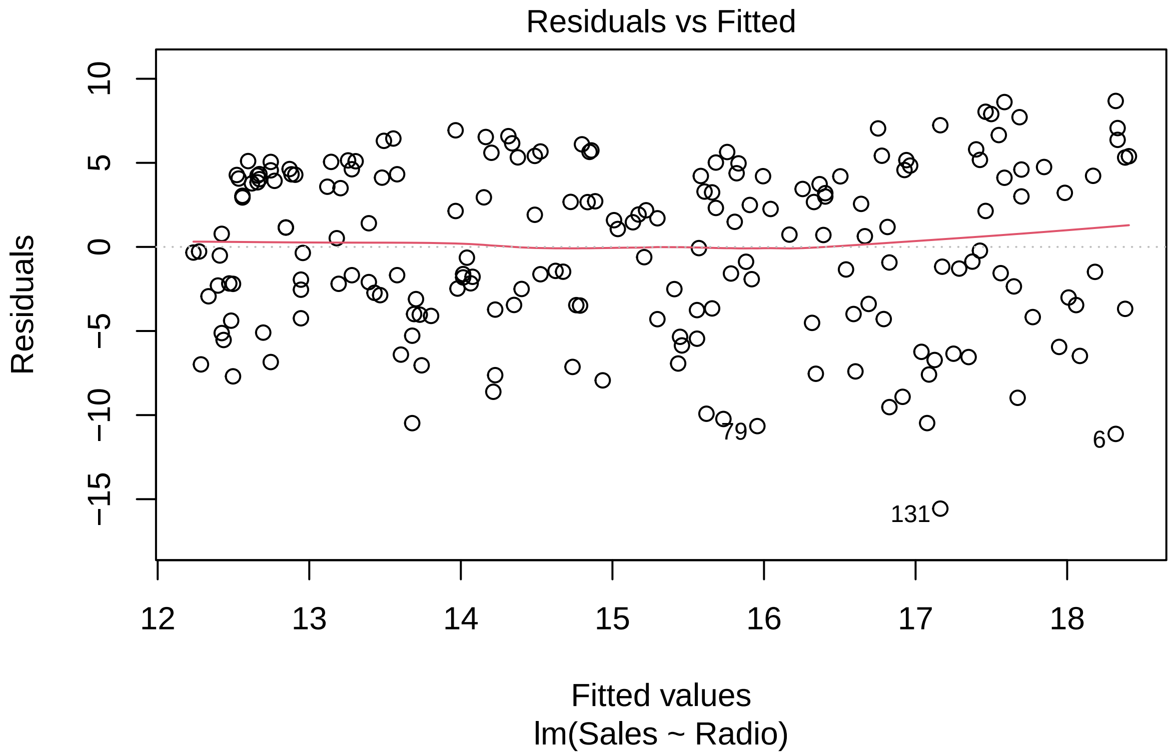
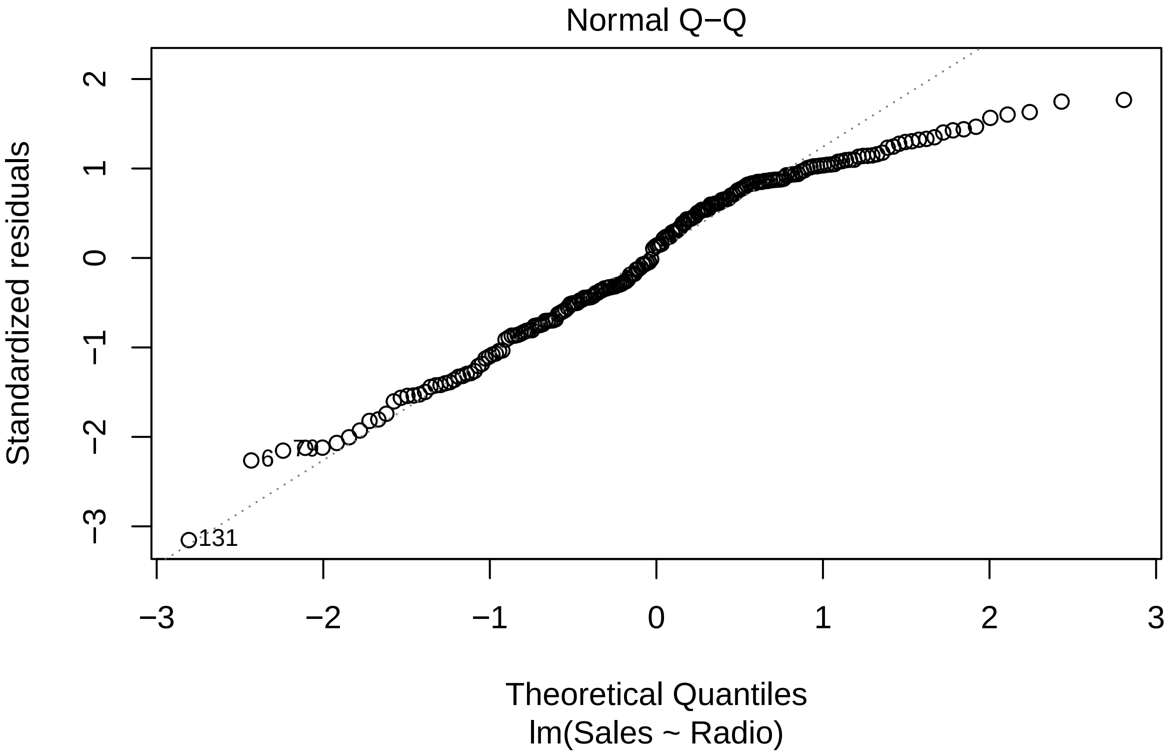


This shows how the residuals e are distributed against the fitted values (or predictions) $\hat{y}$. Ideally, there should be no correlation, i.e. randomly spread around 0. If that is not the case and the plot shows a trend, the assumption of a linear regression model might not be appropriate.

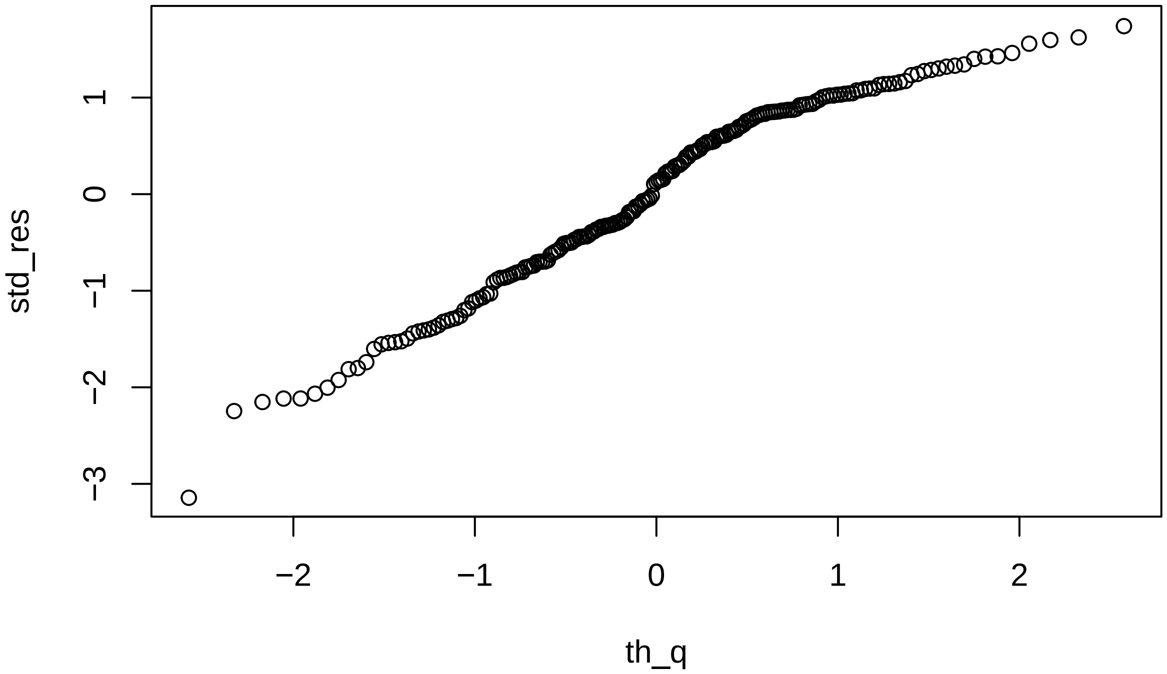
```
# normal q-q plot
plot(radio_lm, which = 2)
```



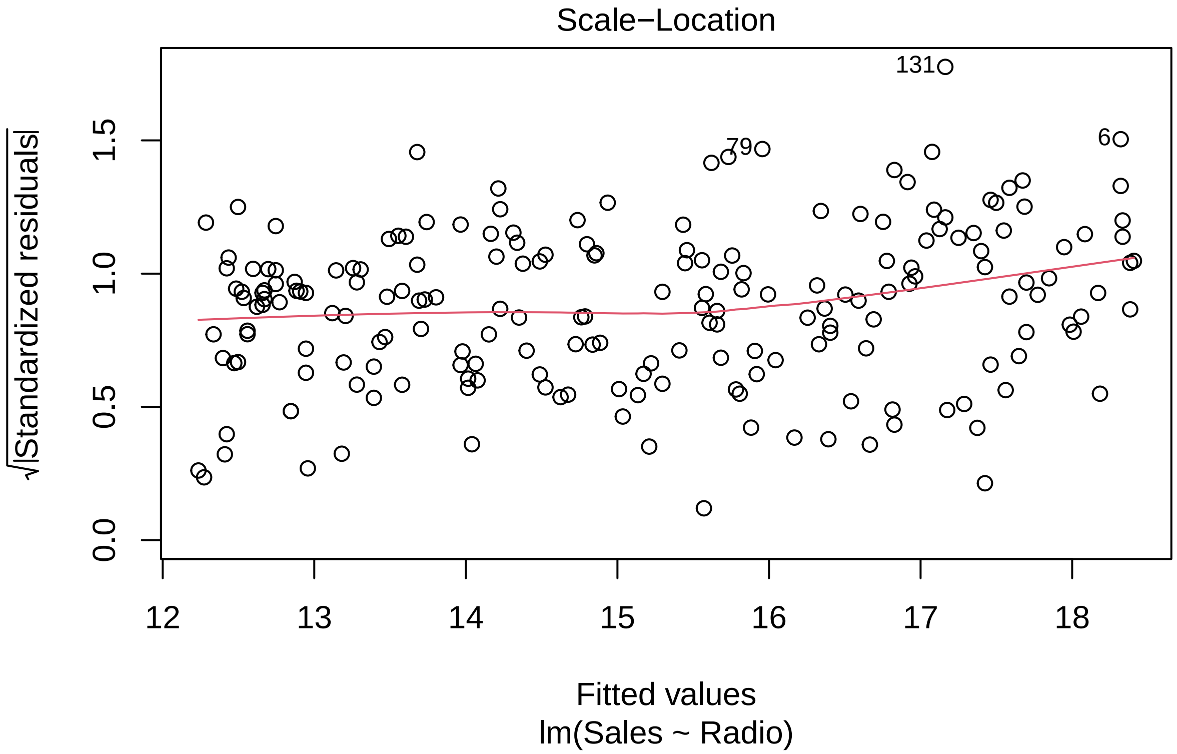
The Q-Q plot compares two distributions (or one theoretical distribution with an empirical one). Given some data, the question it helps answering to is: *is one distribution close to the other one?* In particular, the above plot is a Normal Q-Q plot of e, where its (standardized) distribution is compared to the standard normal distribution. More precisely, once the residuals are standardized (i.e. minus the mean, divided by the standard deviation) and sorted in ascending order, they are plotted against the theoretical quantiles of the standard Normal distributions. Example: the theoretical quantile of the k^{th} residual (in ascending order) is q_α where $\alpha = P(Z <= q_\alpha) = k/n$

Exercise: draw a Q-Q plot "by hand":

```r
n <- nrow(advertising)
std_res <- with(radio_lm, {
  sort((residuals - mean(residuals)) / sd(residuals),
    decreasing = FALSE
  )
})
th_q <- qnorm(1:n / n)
plot(th_q, std_res)
```

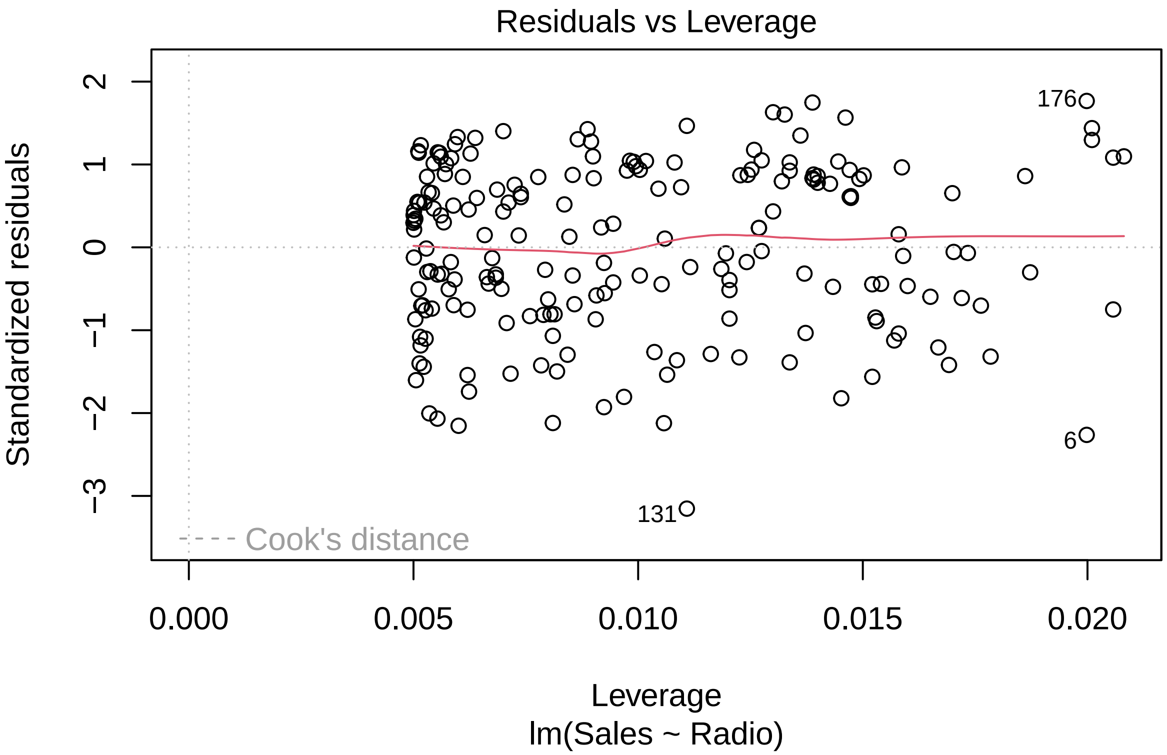


```
# scale-location
plot(radio_lm, which = 3)
```



This plot only reduces the skewness of the standardized residuals by taking its square root.

```
# residuals vs leverage
plot(radio_lm, which = 5)
```



In order to understand this plot we have to introduced the so-called *hat matrix*

$$H = X(X'X)^{-1}X'$$

which is the projection matrix that maps Y to $\hat{Y}$. Its diagonal elements are particularly interesting and are called leverages of the observations. Note that $h_{ii} \in [0, 1]$ and $\sum_i h_{ii} = p$, where p is the number of coefficients in the regression model. The leverage of an observation h_{ii} measures how much an observation y_i has an impact on the fitted value $\hat{y}_i$.[1]

Multiple predictors

Predictors in R linear models are defined in the formula.

What is a formula?

In R, `y ~ x + b` is a formula.

- `y` is the dependent variable(s) (e.g. response, label)
- `x + b` are the independent variables (e.g. predictors/features)
- can also be one-sided e.g. `~ x`
- `+` (plus) sign is not a sum, but a *join* operator
- when formulae are written, variables are not evaluated (symbolic model)
- you can exclude some terms explicitly by using `-` (e.g. `y ~ x - 1` will only estimate the `x` coefficient and not the intercept, which is implicitly added by `lm`)

More on formulae later (see Interactions in the next lesson).

[1] Explanation taken from https://it.mathworks.com/help/stats/hat-matrix-and-leverage.html

Examples:

```
f <- y ~ x + b
class(f)
```

```
## [1] "formula"
```

```
f[[1]] # formula symbol
```

```
## `~`
```

```
f[[2]] # dep vars
```

```
## y
```

```
f[[3]] # indep vars
```

```
## x + b
```

Generally, a linear model formula is composed by multiple predictors.

```
complete_lm <- lm(Sales ~ TV + Radio + Newspaper, data = advertising)
# or
complete_lm <- lm(Sales ~ ., data = advertising)
```

The dot stands for all the possible predictors in the dataset.

```
summary(complete_lm)
```

```
##
## Call:
## lm(formula = Sales ~ ., data = advertising)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -7.3034 -0.8244 -0.0008  0.8976  3.7473
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 4.6251241  0.3075012  15.041   <2e-16 ***
## TV          0.0544458  0.0013752  39.592   <2e-16 ***
## Radio       0.1070012  0.0084896  12.604   <2e-16 ***
## Newspaper   0.0003357  0.0057881   0.058    0.954
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.662 on 196 degrees of freedom
## Multiple R-squared:  0.9026, Adjusted R-squared:  0.9011
## F-statistic: 605.4 on 3 and 196 DF,  p-value: < 2.2e-16
```

Confidence regions

The hypothesis system setup (in the general case, multiple regression) is:

$$\begin{aligned} H_0 &: \quad C\beta = \theta \\ H_1 &: \quad C\beta \neq \theta \end{aligned}$$

The confidence region for β is given by the theorem by which:

$$\frac{(C\beta - C\hat{\beta})'(C(X'X)^{-1}C')^{-1}(C\beta - C\hat{\beta})}{q\text{MSR}} \sim F(q, n-p)$$

where $\text{rank}(C) = q \leq p$.

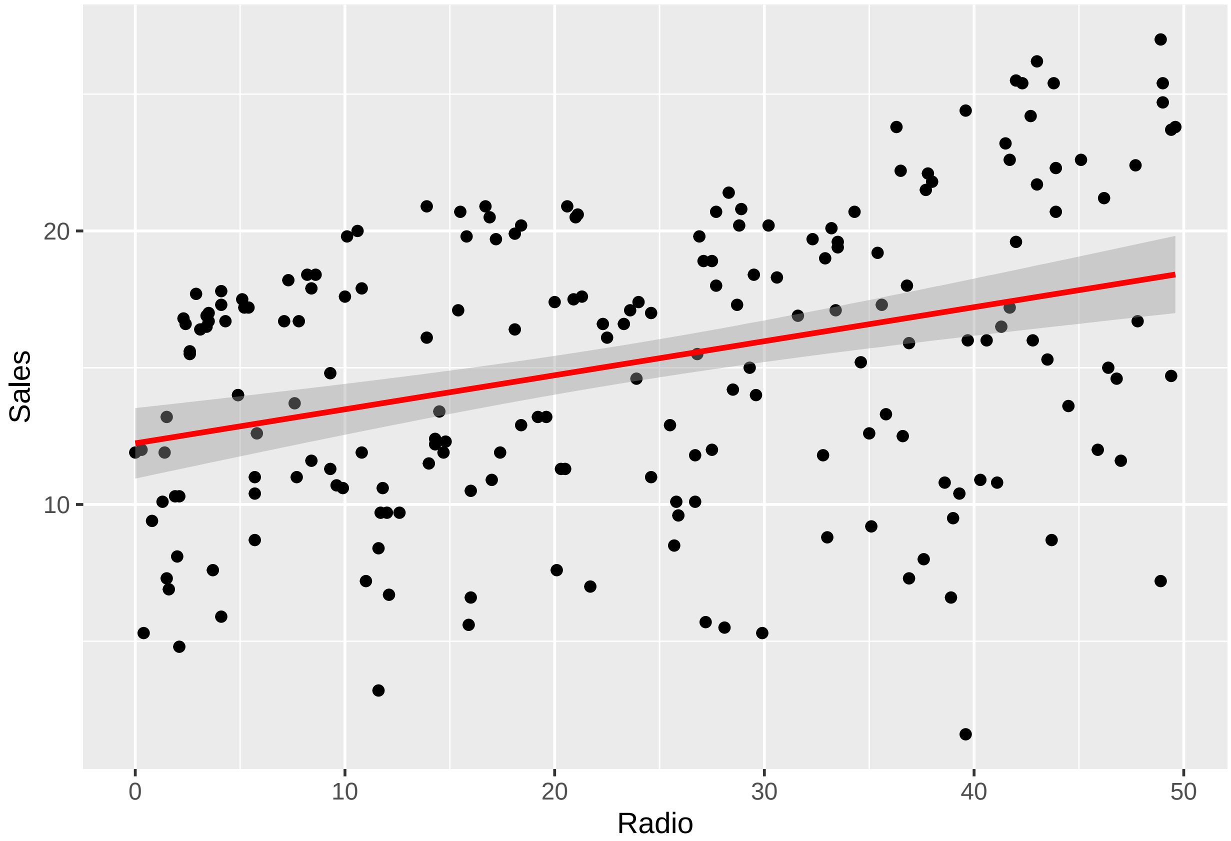
In case of $C = I$, the $(1-\alpha)$-confidence region is an ellipsoid in $\mathbb{R}^p$.

$$(\hat{\beta} - \beta)X'X(\hat{\beta} - \beta) \leq F_\alpha(p, n-p)p\text{MSR}$$

As already seen before, some plotting functions also draw a confidence region for the regression line.

```
library(ggplot2)
ggplot(radio_lm, mapping = aes(Radio, Sales)) +
  geom_point() +
  geom_smooth(method = "lm", color = "red")
```

```
## `geom_smooth()` using formula = 'y ~ x'
```



This confidence region is drawn according to that formula.

To get the confidence intervals values, just use `confint()`

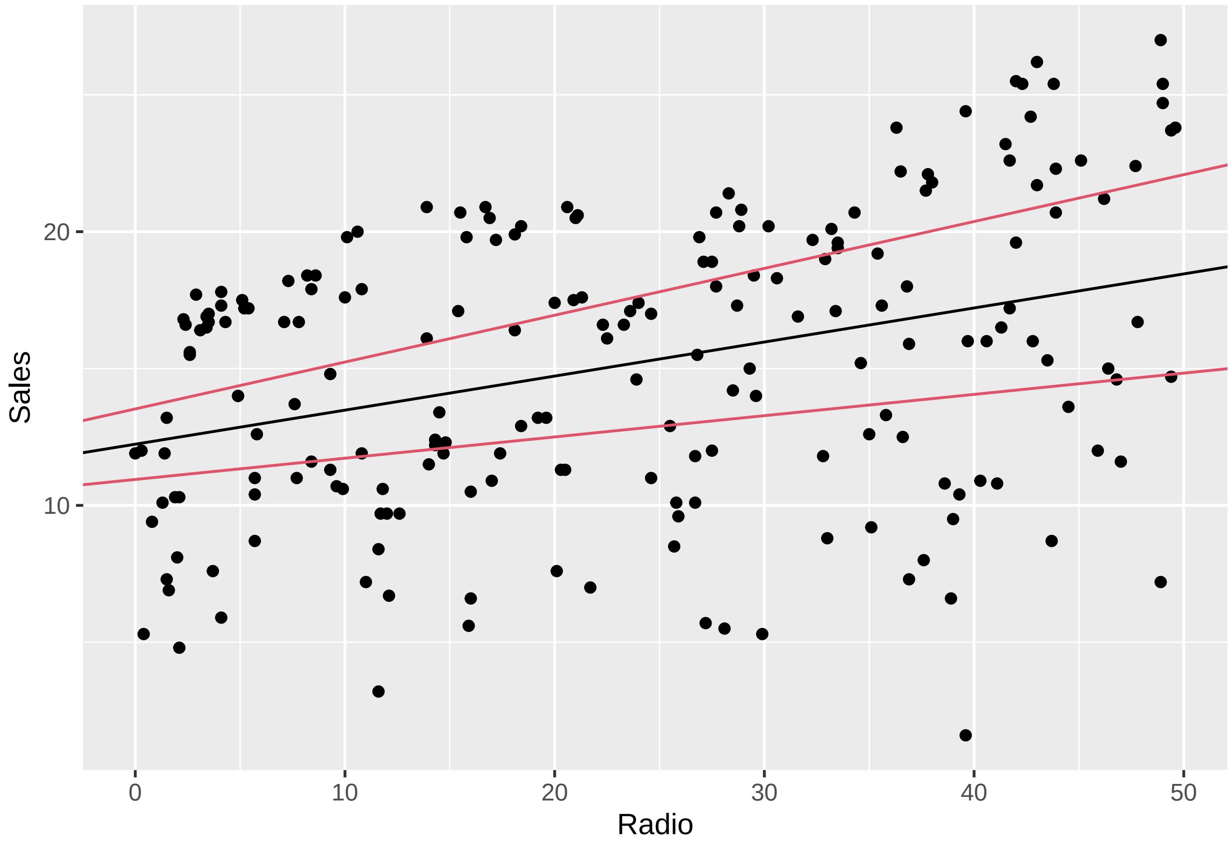
```
confint(radio_lm)
```

```
##                   2.5 %     97.5 %
## (Intercept) 10.94703557 13.5244084
## Radio        0.07770266  0.1711606
```

```
lb <- confint(radio_lm)[, 1]
ub <- confint(radio_lm)[, 2]
```

To plot it, e.g. using `ggplot`:

```
# prevision region
ggplot(radio_lm) +
  geom_point(mapping = aes(Radio, Sales)) +
  geom_abline(intercept = radio_lm$coefficients[[1]],
              slope = radio_lm$coefficients[[2]], color = 1) +
  geom_abline(intercept = lb[1], slope = lb[2], color = 2) +
  geom_abline(intercept = ub[1], slope = ub[2], color = 2)
```



The region between the lines is different (larger) from the one generated by `geom_smooth`. Without entering too much in detail, this happens because the confidence intervals in the second case are computed independently one coefficients from the other, while in ggplot's function the confidence region is jointly computed, therefore it's more accurate.