Linear regression

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The linear model

For this lecture, we will use the *Advertising* dataset, which can be found here. It shows, for each line, the revenue (`Sales`) depending on the money spent on three different advertising channels: `TV`, `Radio`, `Newspaper`.

Every attribute is continuous, and allows to explain the main R functions used in a plain linear regression task.

Dataset import

Import the dataset (remember to check the current working directory).

```
library(readr)
# here the wd contains a folder 'dataset' which
# in turn contains the file to be read
advertising <- read_csv("./datasets/advertising.csv")
```

```
## Rows: 200 Columns: 4
## -- Column specification ------------------------------------
## Delimiter: ","
## dbl (4): TV, Radio, Newspaper, Sales
##
## i Use `spec()` to retrieve the full column specification for this data.
## i Specify the column types or set `show_col_types = FALSE` to quiet this message.
```

```
advertising
```

```
## # A tibble: 200 x 4
##       TV Radio Newspaper Sales
##    <dbl> <dbl>     <dbl> <dbl>
##  1 230.   37.8      69.2  22.1
##  2  44.5  39.3      45.1  10.4
##  3  17.2  45.9      69.3  12
##  4 152.   41.3      58.5  16.5
##  5 181.   10.8      58.4  17.9
##  6   8.7  48.9      75     7.2
##  7  57.5  32.8      23.5  11.8
##  8 120.   19.6      11.6  13.2
##  9   8.6   2.1       1     4.8
## 10 200.    2.6      21.2  15.6
## # ... with 190 more rows
```

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Basic EDA

Get some statistics with `summary()`

```r
summary(advertising)
```

```
##        TV             Radio          Newspaper     
##  Min.   :  0.70   Min.   : 0.000   Min.   :  0.30  
##  1st Qu.: 74.38   1st Qu.: 9.975   1st Qu.: 12.75  
##  Median :149.75   Median :22.900   Median : 25.75  
##  Mean   :147.04   Mean   :23.264   Mean   : 30.55  
##  3rd Qu.:218.82   3rd Qu.:36.525   3rd Qu.: 45.10  
##  Max.   :296.40   Max.   :49.600   Max.   :114.00  
##      Sales      
##  Min.   : 1.60  
##  1st Qu.:11.00  
##  Median :16.00  
##  Mean   :15.13  
##  3rd Qu.:19.05  
##  Max.   :27.00
```

You can attach the dataset to access columns with less code (not recommended). All the columns of the table are then available in the R environment without prepending the dataset name

```r
attach(advertising)
head(TV)
```

```
## [1] 230.1  44.5  17.2 151.5 180.8   8.7
```

To revert back, detach the dataset

```r
detach(advertising)
```

It is better to use `with()` instead, if really needed. This function basically attach a certain context object (e.g. the dataset namespace), executes the commands inside the curly brackets, and then detaches the context object.

```r
with(advertising, {
  print(head(Sales))
  print(mean(TV))
  print(Radio[Radio < 20])
})
```

```
## [1] 22.1 10.4 12.0 16.5 17.9  7.2
## [1] 147.0425
##  [1] 10.8 19.6  2.1  2.6  5.8  7.6  5.1 15.9 16.9 12.6  3.5
## [12] 16.7 16.0 17.4  1.5  1.4  4.1  8.4  9.9 15.8 11.7  3.1
## [23]  9.6 19.2  2.0 15.5  9.3 14.5 14.3  5.7  1.6  7.7  4.1
## [34] 18.4  4.9  1.5 14.0  3.5  4.3 10.1 17.2 11.0  0.3  0.4
## [45]  8.2 15.4 14.3  0.8 16.0  2.4 11.8  0.0 12.0  2.9 17.0
## [56]  5.7 14.8  1.9  7.3 13.9  8.4 11.6  1.3 18.4 18.1 18.1
## [67] 14.7  3.4  5.2 10.6 11.6  7.1  3.4  7.8  2.3 10.0  2.6
## [78]  5.4  5.7  2.1 13.9 12.1 10.8  4.1  3.7  4.9  9.3  8.6
```

These are some of the functions that might be useful when gathering information about a dataset.

```r
length(advertising) # columns!
```

```
## [1] 4
```

```r
nrow(advertising)
```

```
## [1] 200
```

```r
dim(advertising)
```

```
## [1] 200   4
```

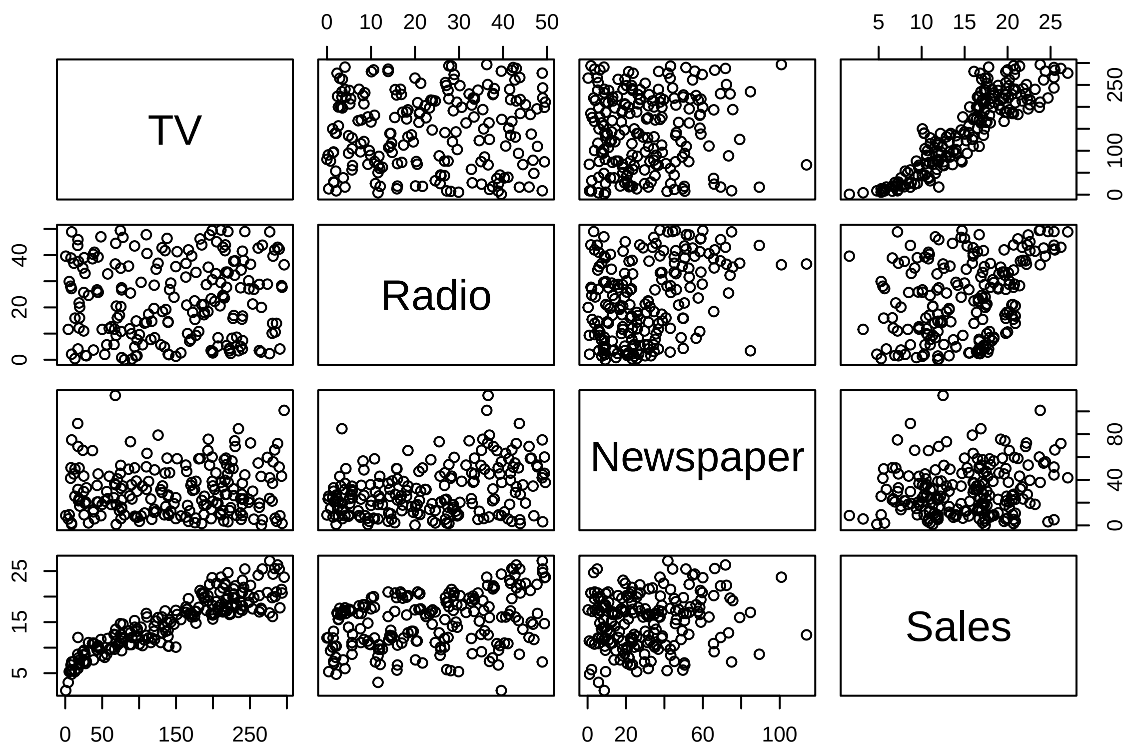
```r
names(advertising)
```

```
## [1] "TV"        "Radio"     "Newspaper" "Sales"
```

Simple plots

In exploration, ggplot might be a bit overkill. Faster plots can be drawn with `plot()` and `pairs()`

```r
plot(advertising)
```

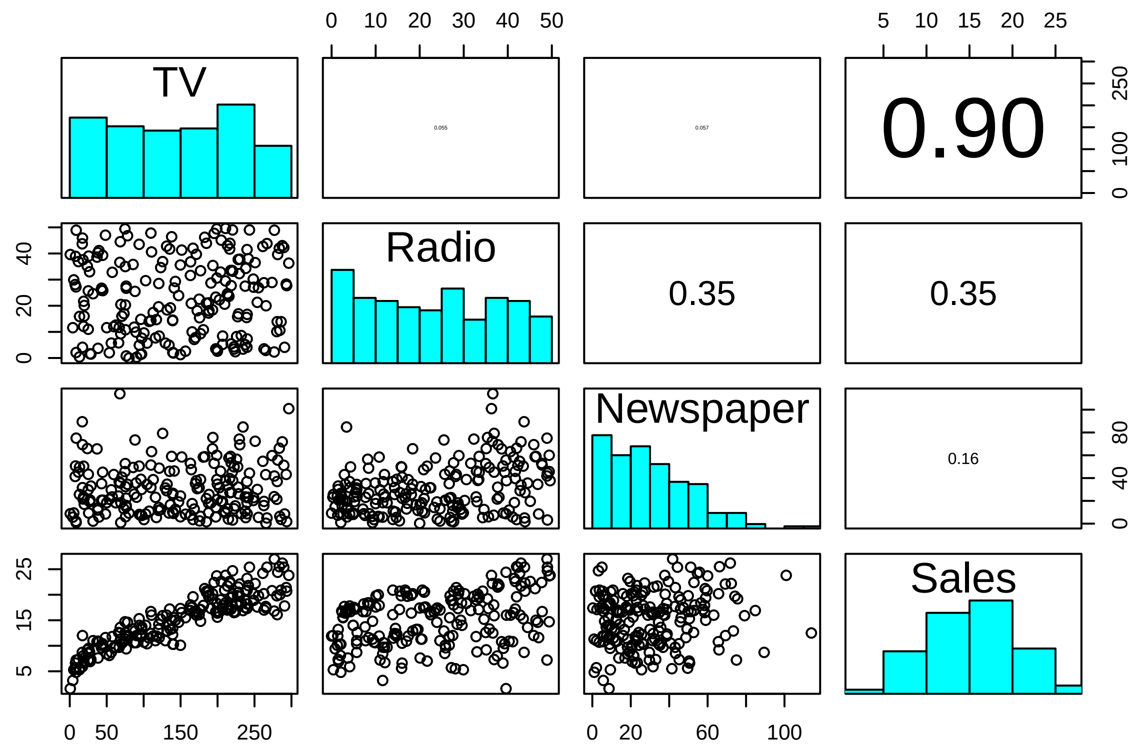


A pair-plot gives a bigger picture of the cross correlations between variables. The base R `pairs()` function can be tweaked (check `?pairs`) in order to draw other kind of information of interest.

```r
## put histograms on the diagonal
panel_hist <- function(x, ...) {
  usr <- par("usr")
  par(usr = c(usr[1:2], 0, 1.5))
  h <- hist(x, plot = FALSE)
  breaks <- h$breaks
  nb <- length(breaks)
  y <- h$counts
```

```
  y <- y / max(y)
  rect(breaks[-nb], 0, breaks[-1], y, col = "cyan", ...)
}
## put (absolute) correlations on the upper panels,
## with size proportional to the correlations.
panel_cor <- function(x, y, digits = 2, prefix = "", cex_cor, ...) {
  par(usr = c(0, 1, 0, 1))
  r <- abs(cor(x, y))
  txt <- format(c(r, 0.123456789), digits = digits)[1]
  txt <- paste0(prefix, txt)
  if (missing(cex_cor)) cex_cor <- 0.8 / strwidth(txt)
  text(0.5, 0.5, txt, cex = cex_cor * r)
}

pairs(advertising,
  upper.panel = panel_cor, diag.panel = panel_hist
)
```



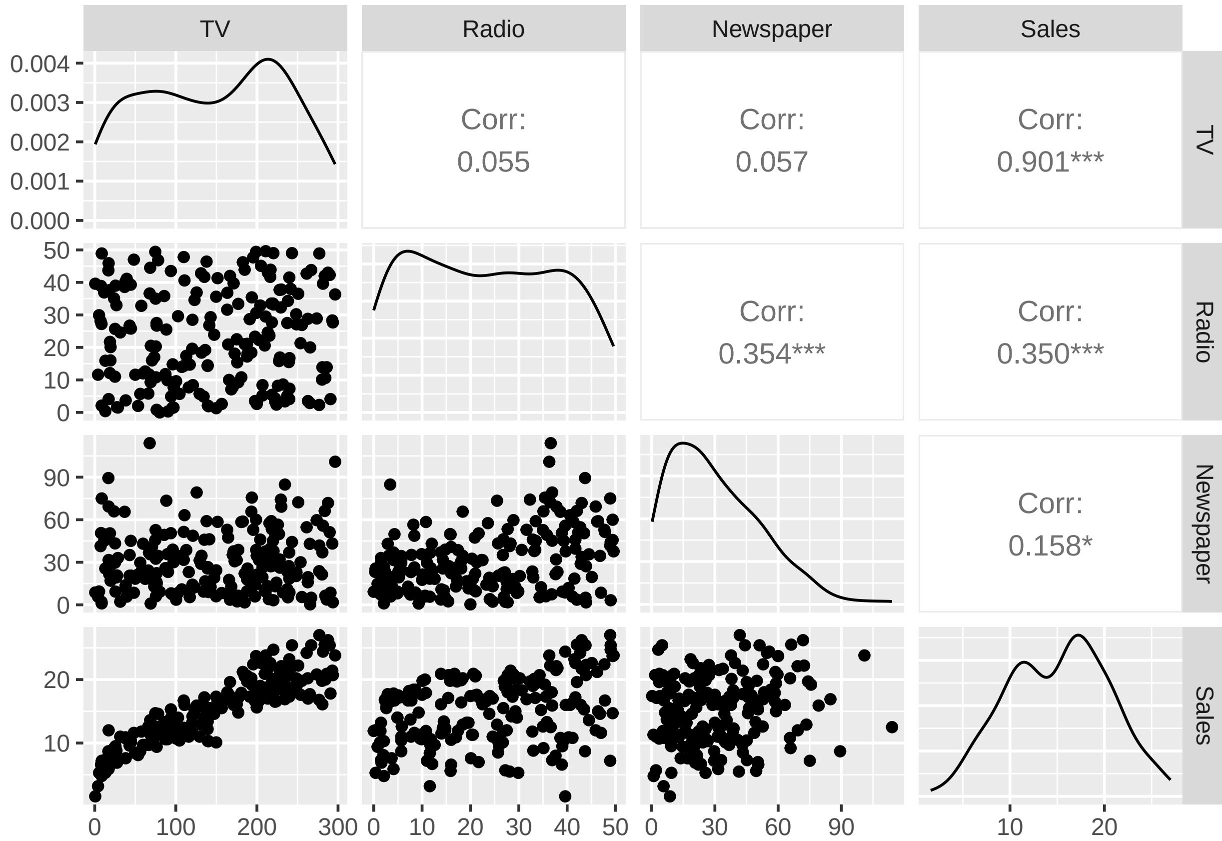
With much less effort, we can obtain a prettier version of the pair-plot.

```
library(GGally)
```

```
## Loading required package: ggplot2
```

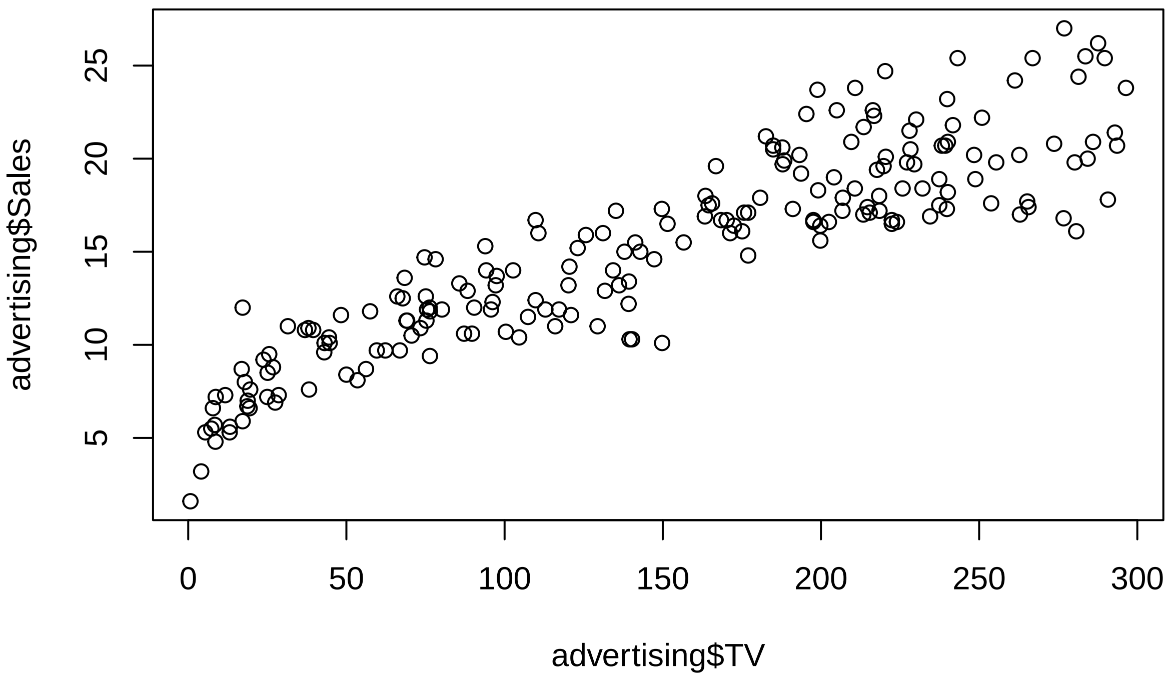
```
## Registered S3 method overwritten by 'GGally':
##   method from
##   +.gg   ggplot2
```

```r
ggpairs(advertising, progress = FALSE)
```



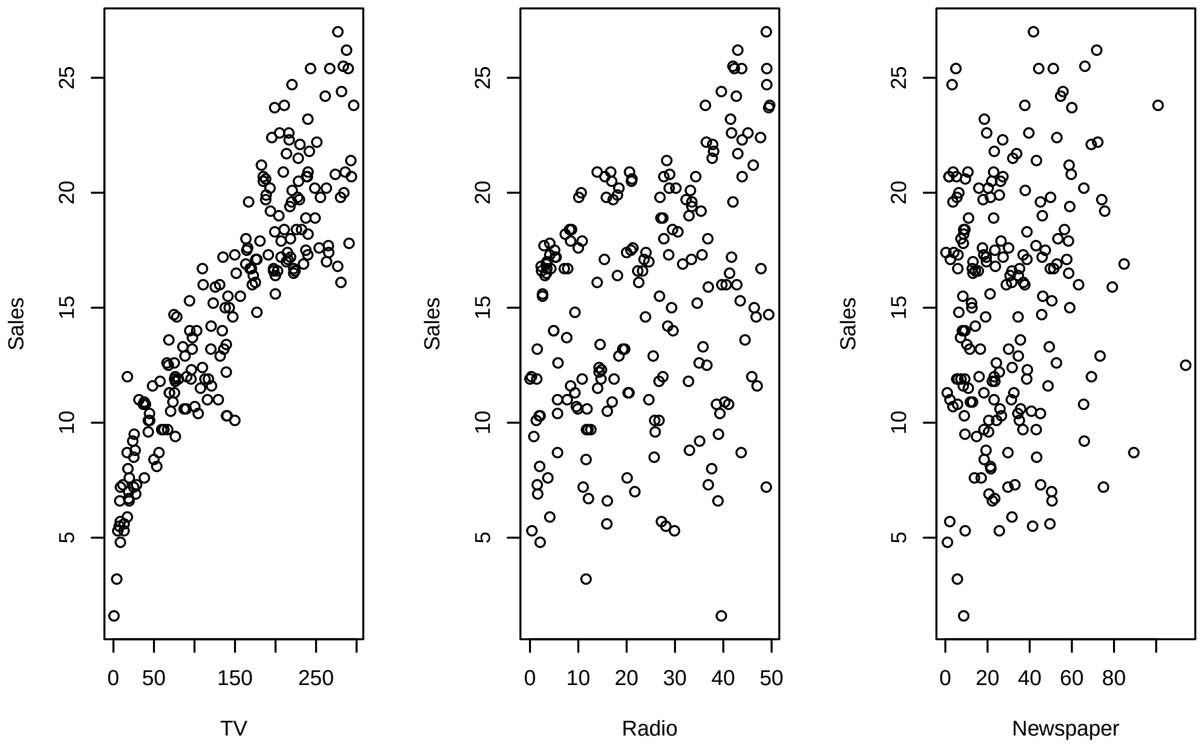
Again, with base R let's plot the predictors against the response. We know how to do that with a single predictor.

```r
plot(advertising$TV, advertising$Sales)
```



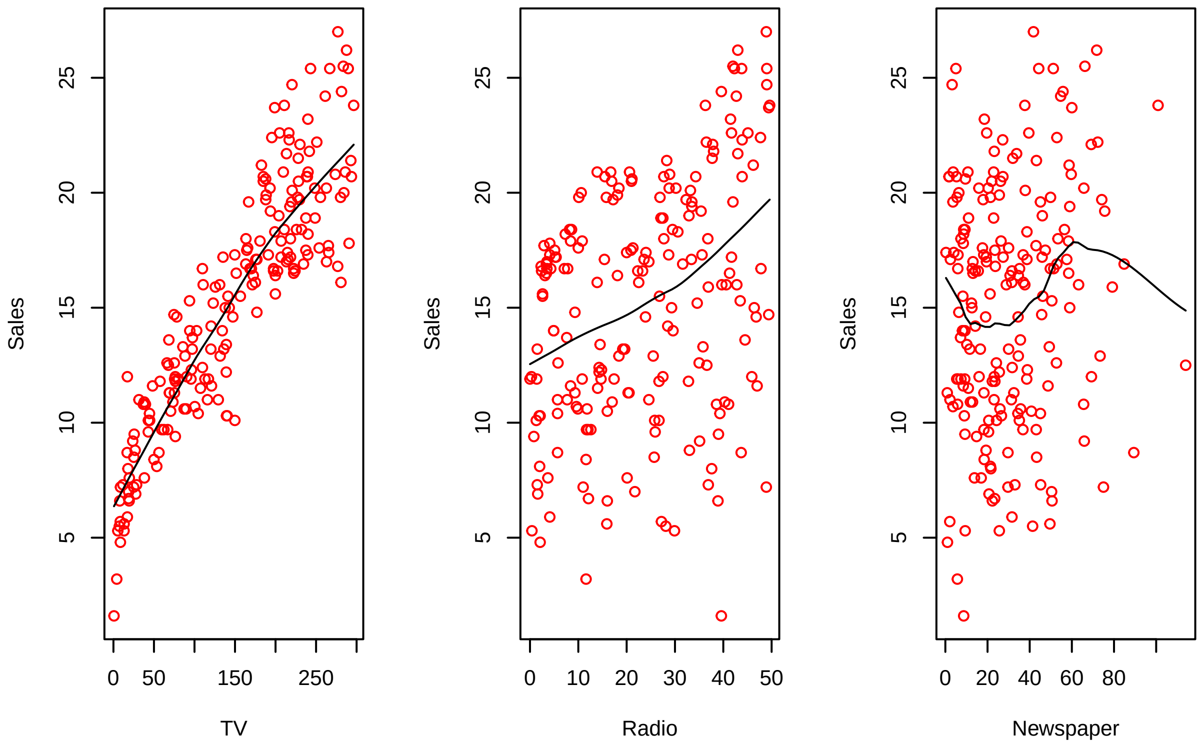
In case we want to add all plots to a single figure, we can do it as follow (base R)

```
# set plot parameters
with(advertising, {
  par(mfrow = c(1, 3))
  plot(TV, Sales)
  plot(Radio, Sales)
  plot(Newspaper, Sales)
})
```



adding a smoothed line would look like this

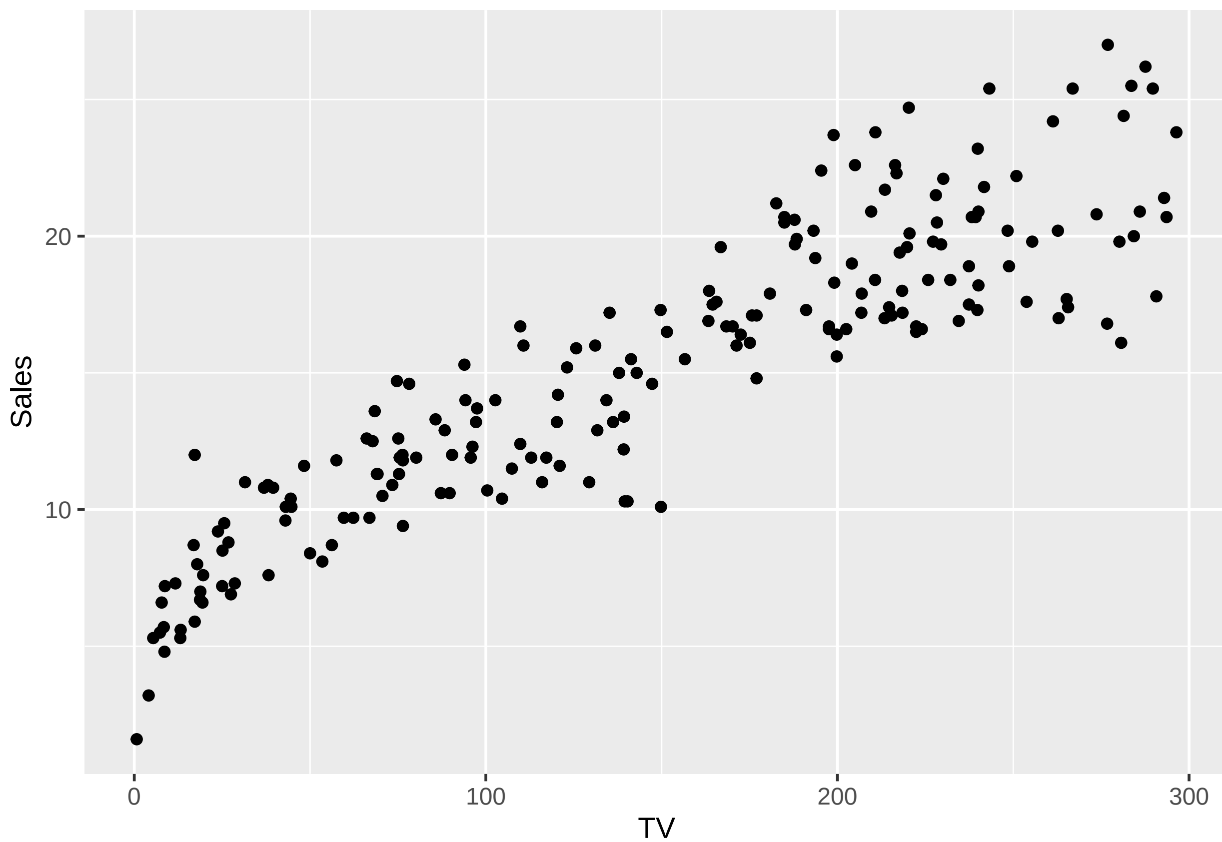
```
# set plot parameters
with(advertising, {
  par(mfrow = c(1, 3))
  scatter.smooth(TV, Sales, col = "red")
  scatter.smooth(Radio, Sales, col = "red")
  scatter.smooth(Newspaper, Sales, col = "red", span = .3) # tweak with span
})
```



The same pictures can also be plotted with `ggplot`, if preferred (in two ways).

```
library(ggplot2)

ggplot(advertising) +
  geom_point(aes(TV, Sales))
```

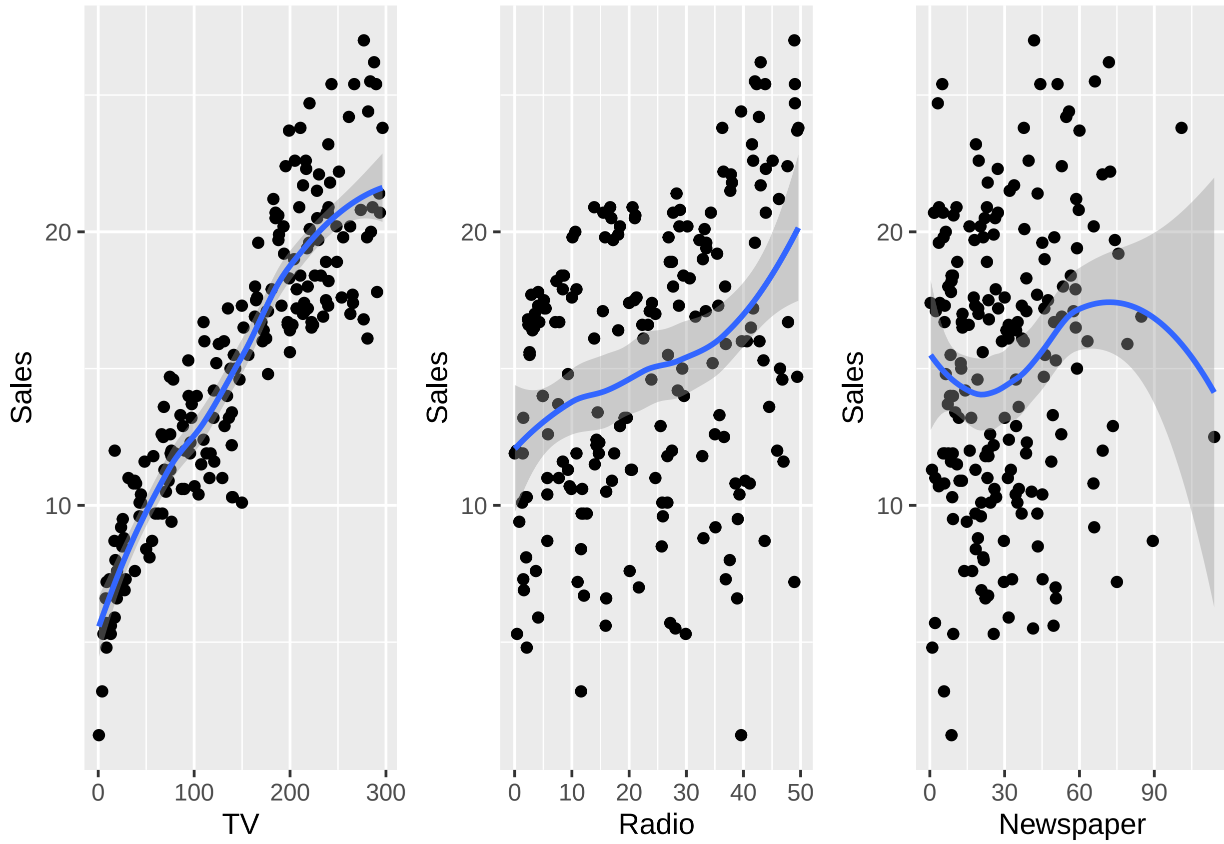


First method: draw three separate plots and arrange them together with `ggarrange()`

```
library(ggpubr)
tvplt <- ggplot(advertising, mapping = aes(TV, Sales)) +
  geom_point() +
  geom_smooth()
radplt <- ggplot(advertising, mapping = aes(Radio, Sales)) +
  geom_point() +
  geom_smooth()
nwsplt <- ggplot(advertising, mapping = aes(Newspaper, Sales)) +
  geom_point() +
  geom_smooth()

ggarrange(tvplt, radplt, nwsplt,
  ncol = 3
)
```

```
## `geom_smooth()` using method = 'loess' and formula = 'y ~
## x'
## `geom_smooth()` using method = 'loess' and formula = 'y ~
## x'
## `geom_smooth()` using method = 'loess' and formula = 'y ~
## x'
```



Second method: rearrange the dataset first, and feed everything to ggplot.

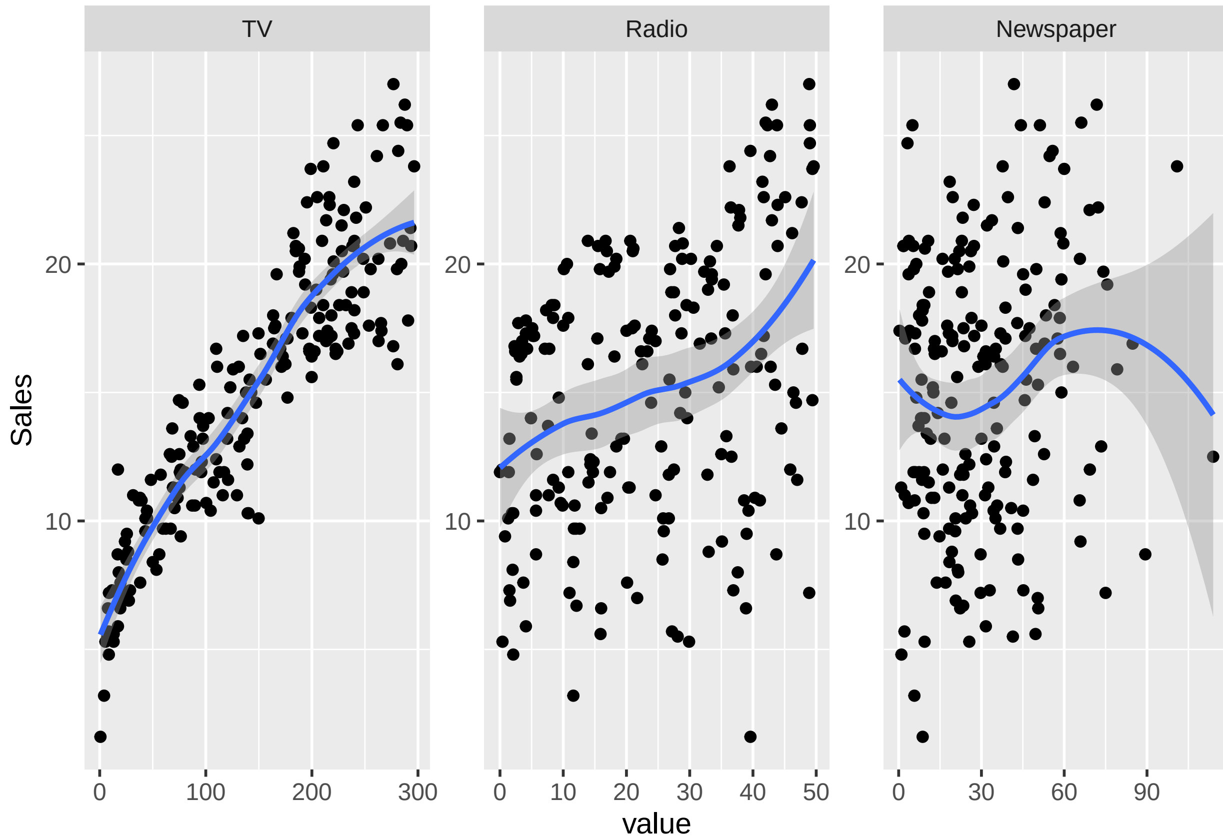
```
library(reshape2)

wide_adv <- melt(advertising, id.vars = "Sales")
head(wide_adv)
```

```
##   Sales variable value
## 1  22.1       TV 230.1
## 2  10.4       TV  44.5
## 3  12.0       TV  17.2
## 4  16.5       TV 151.5
## 5  17.9       TV 180.8
## 6   7.2       TV   8.7
```

```
ggplot(wide_adv, aes(value, Sales)) +
  geom_point() +
  geom_smooth() +
  facet_wrap(~variable, scales = "free")
```

```
## `geom_smooth()` using method = 'loess' and formula = 'y ~
## x'
```



If you're not sure why we need to use the `melt()` function, check the first lab (Iris dataset), or simply read the `?melt` helper.

Simple regression

Check the `lm()` function: first argument is the *formula*.

Let's fit a simple single predictor linear model

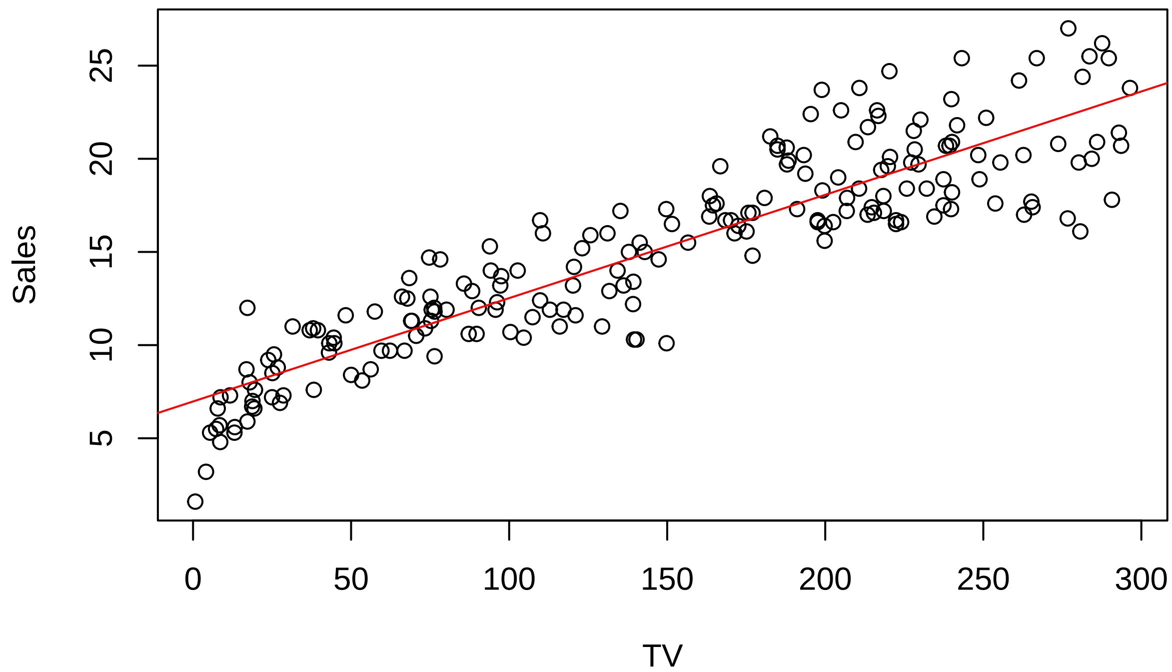
```
simple_reg <- lm(Sales ~ TV, data = advertising)
```

Now that we fitted the model, we have access to the coefficient estimates and we can plot the regression line.

Plot

Here two ways of drawing the regression line, first in base R

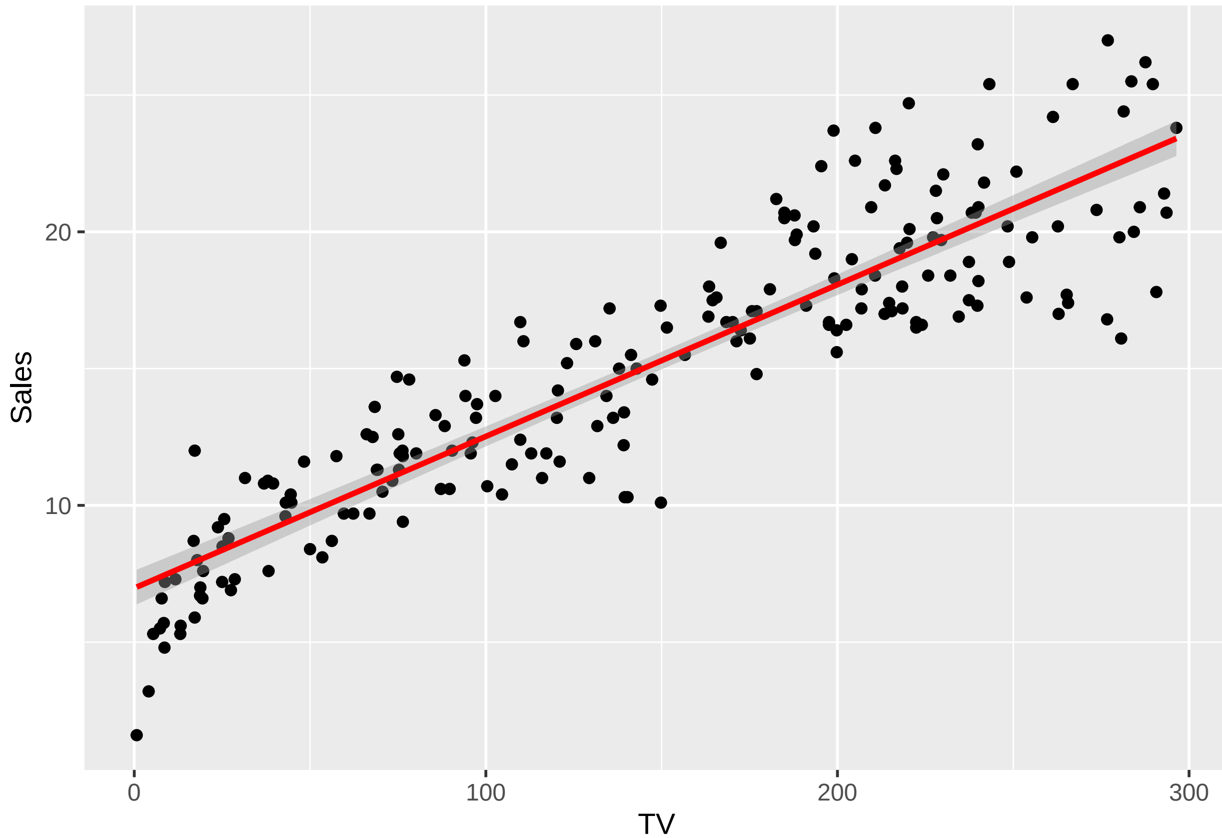
```
with(data = advertising, {
  plot(x = TV, y = Sales)
  # abline draws a line from intercept and slope
  abline(
    simple_reg,
    col = "red"
  )
})
```



and in ggplot

```
ggplot(simple_reg, mapping = aes(TV, Sales)) +
  geom_point() +
  geom_smooth(method = "lm", color = "red")
```

```
## `geom_smooth()` using formula = 'y ~ x'
```



Actually, `lm()` does a lot more than just compute the OLS coefficients.

```
summary(simple_reg)
```

```
##
## Call:
## lm(formula = Sales ~ TV, data = advertising)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -6.4438 -1.4857  0.0218  1.5042  5.6932
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 6.974821   0.322553   21.62   <2e-16 ***
## TV          0.055465   0.001896   29.26   <2e-16 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.296 on 198 degrees of freedom
## Multiple R-squared:  0.8122, Adjusted R-squared:  0.8112
## F-statistic: 856.2 on 1 and 198 DF,  p-value: < 2.2e-16
```

We will analyze in detail this output in the next subsection.

Summary

What's in `summary(lm)`?

- residuals: $Y - X\hat{\beta}$
- estimate: $\hat{\beta}$
- std. error: errors on the coefficients $S\sqrt{(X'X)^{-1}_{i+1,i+1}}$ because $\frac{\hat{\beta}_i - \beta_i}{S\sqrt{(X'X)^{-1}}} \sim t(n-p)$
- t-value: value of the beta t-statistic under the null hypothesis $(H_0 : \beta_{i+1} = 0)$
- p-value: probability of the test statistic being beyond t-val
- residual std error: $S = \sqrt{\frac{e'e}{n-p}}$
- multiple R squared: *later*
- adj R squared: *later*

Let's compute them to make sure we understood the concepts

Residuals Easy to retrieve them (actually, they are the realization of the residuals)

$$e = y - x\hat{\beta}$$

```
library(tibble)
y <- advertising$Sales
x <- cbind(1, advertising$TV)
e <- y - x %*% simple_reg$coefficients
head(tibble(
  lm_res = simple_reg$residuals,
  manual_res = as.vector(e)
))
```

```
## # A tibble: 6 x 2
##   lm_res manual_res
##    <dbl>      <dbl>
## 1  2.36       2.36
## 2  0.957      0.957
## 3  4.07       4.07
## 4  1.12       1.12
## 5  0.897      0.897
## 6 -0.257     -0.257
```

Estimate Estimates are just the $\hat{\beta}$ values for each predictor (plus intercept). They are computed with the closed form max likelihood formula.

$$\hat{\beta} = (X'X)^{-1}X'Y$$

```
beta_hat <- solve(t(x) %*% x) %*% t(x) %*% y
head(tibble(
  lm_coeff = simple_reg$coefficients,
  manual_coeff = as.vector(beta_hat)
))
```

```
## # A tibble: 2 x 2
##   lm_coeff manual_coeff
##      <dbl>        <dbl>
## 1   6.97         6.97
## 2   0.0555       0.0555
```

Standard Error For each predictor, this represent the variation in the beta estimator. The lower the standard error is, the higher is the accuracy of that particular coefficient. For predictor i, it's computed as

$$SE_i = S\sqrt{(X'X)^{-1}_{i+1,i+1}}$$

where S is the *residuals standard error* (with S^2 being the RMS - residual mean squares), we get the $i+1^{\text{th}}$ element because of the 1 column for the intercept._

```r
n <- nrow(x)
p <- ncol(x)
rms <- t(e) %*% e / (n - p)

# SE for TV
tv_se <- sqrt(rms * solve(t(x) %*% x)[2, 2])
tv_se
```

```
##             [,1]
## [1,] 0.001895551
```

T-value and p-value These two are related to each other. The first is the test statistics value, under the null hypothesis $H_0 : \beta_i = 0$, for the variable

$$\frac{\hat{\beta}_i - \beta_i}{S\sqrt{(X'X)^{-1}}}$$

which is student-T distributed with $n-2$ degrees of freedom.

```r
# for TV
t_val <- simple_reg$coefficients[2] / tv_se
t_val
```

```
##         [,1]
## [1,] 29.2605
```

And finally, the p-value is simply the probability on a $t(n-2)$ distribution of the statistic to be beyond that critical value. Remember that with *beyond* we mean on both sides of the distribution, since the alternative hypothesis $H_1 : \beta \neq 0$ is two-sided.

```r
# multiply by two because the alternative hyp is two-sided
p_val <- 2 * pt(t_val, n - 2, lower.tail = FALSE)
p_val
```

```
##              [,1]
## [1,] 7.927912e-74
```

Here we cannot appreciate the manual computation since the p-value is very low (meaning that we can reject the null, favoring the alternative).

Exercise: You can try and compute this value manually on another simple regression model where we use `Radio` as predictor.

```r
radio_simple_reg <- lm(Sales ~ Radio, data = advertising)
summary(radio_simple_reg)
```

```
##
## Call:
## lm(formula = Sales ~ Radio, data = advertising)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -15.5632  -3.5293   0.6714   4.2504   8.6796
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  12.2357     0.6535  18.724  < 2e-16 ***
## Radio         0.1244     0.0237   5.251 3.88e-07 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 4.963 on 198 degrees of freedom
## Multiple R-squared:  0.1222, Adjusted R-squared:  0.1178
## F-statistic: 27.57 on 1 and 198 DF,  p-value: 3.883e-07
```