Background Knowledge for Learning Robot Trajectories subject to Kinematic Joint Constraints

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1 Introduction

This document complements our paper "Learning Robot Trajectories subject to Kinematic Joint Constraints" [1] by providing further information on the implementation and the mathematical background. A preprint of the paper is available at `https://arxiv.org/abs/2011.00563`.

2 System components

The system components required to learn safe movements with a neural network are shown in Figure 1.

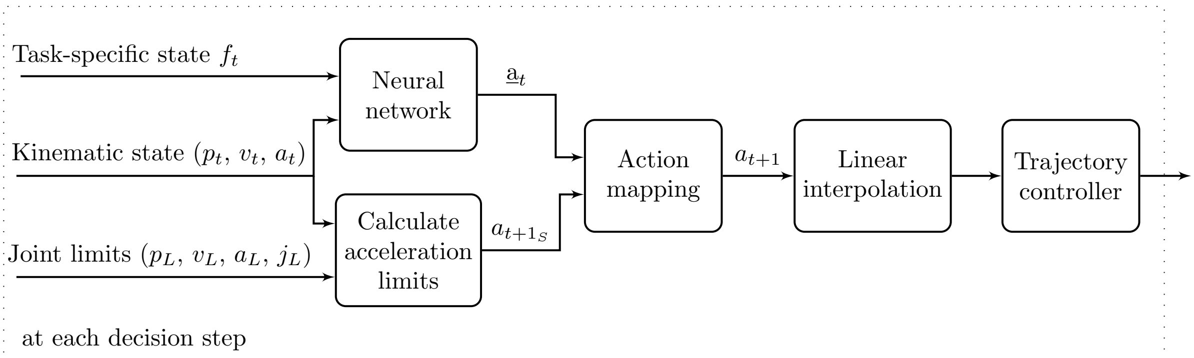

Figure 1: System components for online trajectory generation.

Figure 2 illustrates the steps to derive the range of safe accelerations $a_{t+1_S} = [a_{t+1_{min}}, a_{t+1_{max}}]$. In a first step, each constraint is handled independently. For instance, the maximum accelerations $a_{t+1_{max,j}}$, $a_{t+1_{max,a}}$, $a_{t+1_{max,v}}$ and $a_{t+1_{max,p}}$ are computed to consider the jerk, acceleration, velocity and position constraints, respectively. The desired maximum acceleration $a_{t+1_{max}}$ is set to the smallest of these values. It is important to note that each constraint can nevertheless be influenced by several kinematic limits. For instance, the maximum acceleration complying with position constraints $a_{t+1_{max,p}}$ depends on p_{max}, a_{min} and j_{min}.
In the following, the procedures and equations to compute $a_{t+1_{max,j}}$, $a_{t+1_{max,a}}$, $a_{t+1_{max,v}}$ and $a_{t+1_{max,p}}$ are explained in detail.

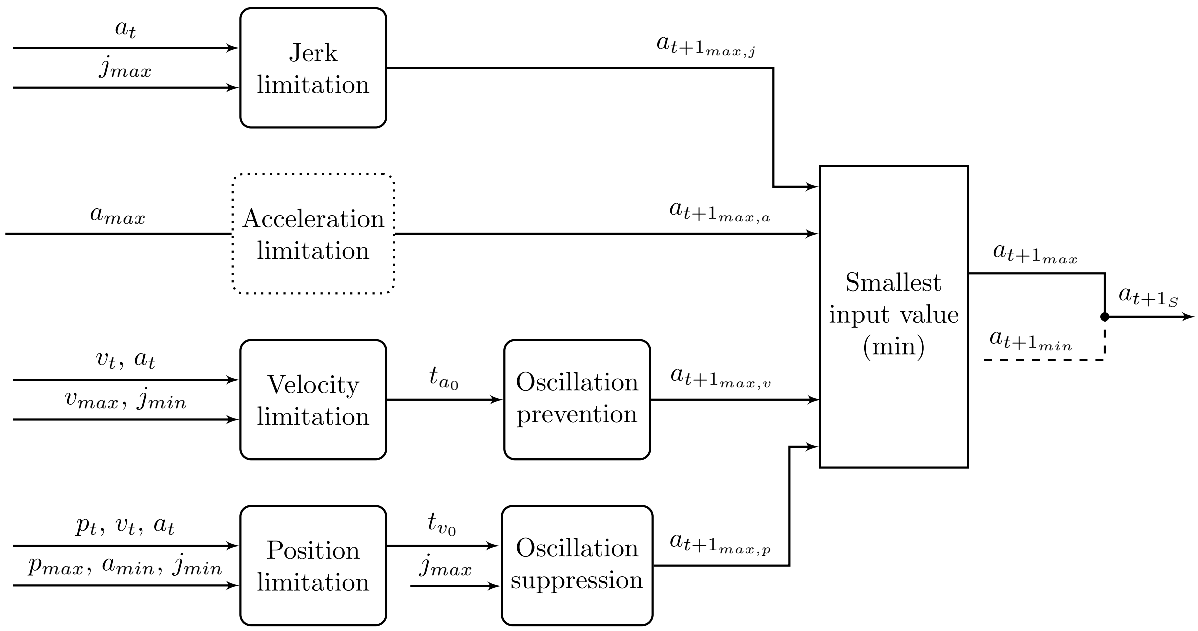

Figure 2: System components to calculate the range of safe accelerations a_{t+1_S}.

3 Exemplary learning tasks

Our method allows to learn robot trajectories without violating kinematic joint constraints. In [1], trajectories that maximize the average absolute joint velocity are generated. In [2], a robot is trained to balance a ball on a plate while following a reference trajectory. In [3], various reaching tasks with up to three robots are learnt.

4 Procedure to consider jerk constraints

The jerk between decision steps is constant as the acceleration is linearly interpolated. Consequently, the maximum acceleration $a_{t+1_{max,j}}$ can be computed as follows:

$$a_{t+1_{max,j}} = a_t + j_{max} \cdot \Delta t, \tag{1}$$

with Δt being the time between decision steps.

5 Procedure to consider acceleration constraints

Due to the design of the action space, the desired maximum acceleration $a_{t+1_{max,j}}$ corresponds to the specified maximum acceleration a_{max}.

$$a_{t+1_{max,j}} = a_{max} \tag{2}$$

6 Procedure to consider velocity constraints

The following equations are used to determine the maximum acceleration a_1 at the next decision step t_1 that does not lead to a mandatory violation of the velocity limit v_{max}. The solution corresponds to the desired value of $a_{t+1_{max,v}}$.

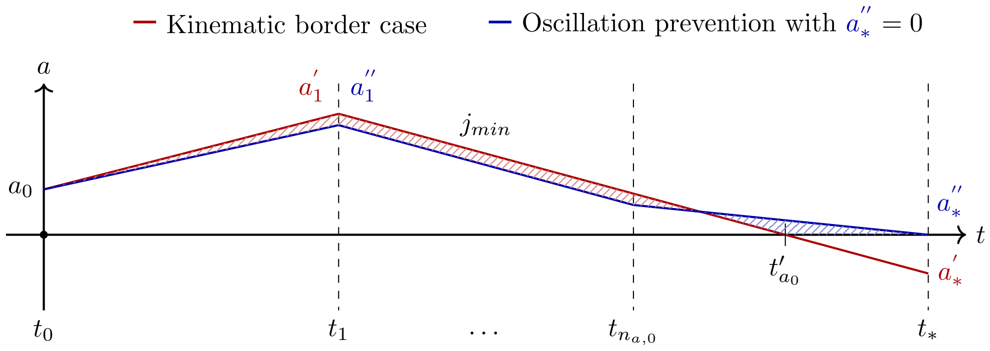

Figure 3: Course of acceleration to comply with velocity constraints.

From a mathematical perspective, the most dynamic behavior is achieved if the maximum velocity v_{max} is reached at an acceleration of zero:

$$v_0 + \int_0^{t'_{a_0}} a(t)\ dt = v_{max} \tag{3}$$

$$a(t'_{a_0}) = 0 \tag{4}$$

Two different cases have to be considered. In the first case, the maximum velocity is reached within the next time step, meaning that $t'_{a_0} \leq t_1$. If so, a'_1 can be determined by solving the system of equation (6). In the second case, a'_1 is greater than zero and followed by a deceleration with the minimum jerk. In that case, a'_1 and t'_{a_0} can be computed by solving (7).
The two cases can be distinguished by the following condition:

$$v_0 + \frac{a_0 \cdot \Delta t}{2} \begin{cases} \geq v_{max} & \text{Case } t'_{a_0} \leq t_1 \\ < v_{max} & \text{Case } t'_{a_0} > t_1 \end{cases} \tag{5}$$

Δt is the time between decision steps. In the second case, the solution for a'_1 received by (7) is prone to oscillations, which are caused by the discrete nature of decision making. These oscillations can be prevented by enforcing zero acceleration at t_*, the discrete decision step that follows t'_{a_0}. As shown in Figure 4, the corresponding acceleration a''_1 can be computed by solving equation (8).

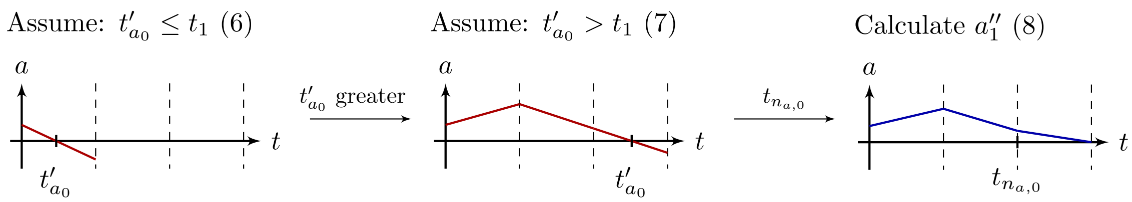

Figure 4: Visualization of the procedure to calculate a''_1.

7 Equations to consider velocity constraints

7.1 Kinematic border case: $t'_{a_0} \leq t_1$

$$v_0 + \int_0^{t'_{a_0}} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t\ dt = v_{max} \tag{6a}$$

$$a_0 + \frac{a_1' - a_0}{\Delta t} \cdot t_{a_0}' = 0 \tag{6b}$$

7.2 Kinematic border case: $t_{a_0}' > t_1$

$$v_0 + \int_0^{t_1} a_0 + \frac{a_1' - a_0}{\Delta t} \cdot t \; dt + \int_{t_1}^{t_{a_0}'} a_1' + j_{min} \cdot (t - t_1) \; dt = v_{max} \tag{7a}$$

$$a_1' + j_{min} \cdot (t_{a_0}' - \Delta t) = 0 \tag{7b}$$

7.3 Oscillation prevention

$$\begin{aligned} &v_0 + \int_0^{t_1} a_0 + \frac{a_1'' - a_0}{\Delta t} \cdot t \; dt + \int_{t_1}^{t_{n_{a,0}}} a_1'' + j_{min} \cdot (t - t_1) \; dt \\ + &\int_{t_{n_{a,0}}}^{t_*} \left(a_1'' + j_{min} \cdot \left(t_{n_{a,0}} - t_1\right)\right) \cdot \left(1 - \frac{t - t_{n_{a,0}}}{\Delta t}\right) \; dt = v_{max}, \end{aligned} \tag{8}$$

with

$$t_{n_{a,0}} = \Delta t \cdot \left\lfloor \frac{t_{a_0}'}{\Delta t} \right\rfloor \tag{9}$$

$$t_* = \Delta t \cdot \left\lceil \frac{t_{a_0}'}{\Delta t} \right\rceil . \tag{10}$$

8 Procedure to consider position constraints

In the following, the desired acceleration $a_{t+1_{max,p}}$ corresponds to the largest possible value of a_1. To stay within the specified position limits, the maximum position must be reached at a velocity of zero:

$$v_0 + \int_0^{t_{v_0}} a(t) \; dt = 0 \tag{11}$$

$$p_0 + v_0 \cdot t_{v_0} + \int_0^{t_{v_0}} \int_0^{t} a(t) \; dt \, dt = p_{max}, \tag{12}$$

with t_{v_0} being the continuous time at which the maximum position is reached.
The most dynamic behaviour is achieved if a_1 is followed by the strongest deceleration that complies with the jerk and acceleration limits. This case is denoted as kinematic border case and illustrated by the red line in Figure 5.
The expressions required to solve equation (11) and (12) for a_1 and t_{v_0} depend on the phase (I - VI) that t_{v_0} is within. Note that phase III and V are needed as decisions can be made at discrete time steps only. Figure 7 illustrates how the phase of t_{v_0}' can be determined by pursuing a systematic trial and error strategy. Assuming that t_{v_0}' is within a specified phase, t_{v_0}' and a_1' can be computed by solving the system of equations that is referenced in brackets. In case that t_{v_0}' is in phase I, a_1' is the desired maximum acceleration and no further calculations are required. In all other cases, the value of t_{v_0}' is used for subsequent calculations to suppress oscillations.
The equation systems (18) and (19) can only be solved if $t_{n_{a,min}}$ is known. $t_{n_{a,min}}$ is the time of the discrete decision step at the end of phase II. The duration of phase II can be described as

$$T_{II}(a_1) = \left\lfloor \frac{a_{min} - a_1}{j_{min} \cdot \Delta t} \right\rfloor \cdot \Delta t, \quad \text{with } \Delta t \text{ being the time between decisions steps.} \tag{13}$$

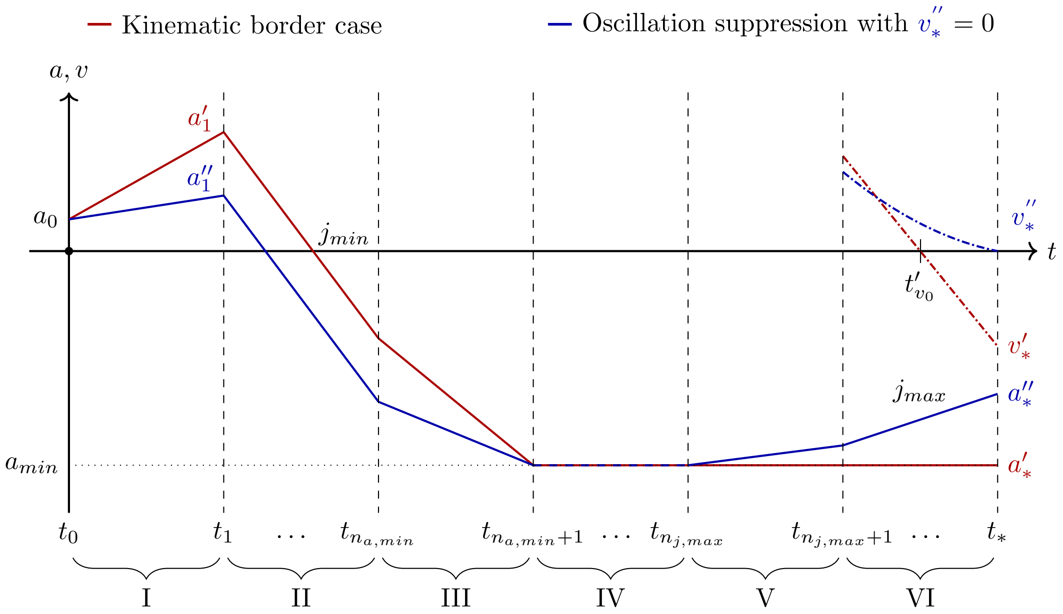

Figure 5: Course of acceleration and velocity to comply with position limits.

However, equation (13) cannot be solved for a_1 as the floor function is not invertible. To circumvent this problem, $t_{n_{a,min}}$ is deduced from additional calculations assuming continuous decision steps. Figure 6 shows how an upper bound $\overline{a'_1}$ and lower bound $\underline{a'_1}$ for a'_1 can be determined.

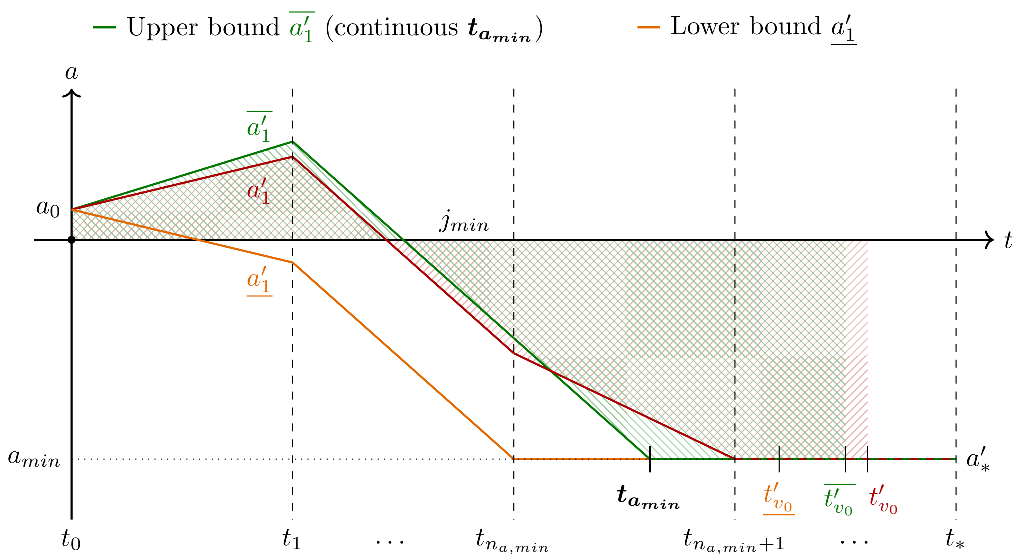

Figure 6: Upper bound $\overline{a'_1}$ and lower bound $\underline{a'_1}$ to determine $t_{n_{a,min}}$ for the kinematic border case. The areas hatched in red and green have to be of equal size in order to ensure a velocity of 0 at t'_{v_0}.

The upper bound is computed by assuming continuity of $t_{a_{min}}$. Once $t_{a_{min}}$ is known, $t_{n_{a,min}}$ can be calculated as follows:

$$t_{n_{a,min}} = \Delta t \cdot \left\lfloor \frac{t_{a_{min}}}{\Delta t} \right\rfloor$$

The value of $t_{n_{a,min}}$ remains valid for the kinematic border case with discrete decision steps.

Proof:
If

$$a_1' \geq \overline{a_1'},$$

the maximum position is exceeded as the velocity is always higher than the velocity of the upper bound.
If

$$a_1' \leq \underline{a_1'},$$

the maximum position is never reached as the velocity is always lower than the velocity of the upper bound. Consequently, the desired solution a_1' is required to be between $\underline{a_1'}$ and $\overline{a_1'}$, meaning that the value of $t_{n_{a,min}}$ does not change.

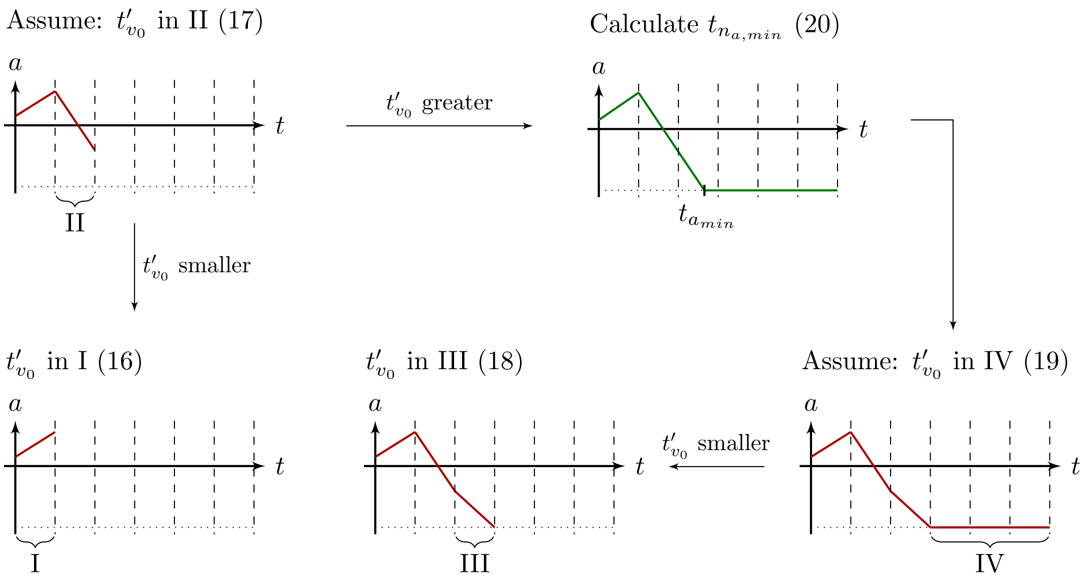

Figure 7: Procedure to determine the value and phase of t'_{v_0}.

The kinematic border case described above leads to two problems: Firstly, oscillations might occur once the maximum position is reached. Secondly, the velocity $v_*^{'}$ at decision step t_* might exceed the minimum velocity v_{min}. Both problems can be mitigated by calculating the maximum value for $a_1^{''}$ while ensuring a final velocity $v_*^{''}$ of zero as indicated by the blue line in Figure 5.
The value of t_* can be derived based on the results from the kinematic border case:

$$t_* = \Delta t \cdot \left\lceil \frac{t'_{v_0}}{\Delta t} \right\rceil \tag{14}$$

As shown in Figure 5, the desired target velocity $v_*^{''}$ can be reached by increasing the acceleration at time $t_{n_{j,max}}$.
Figure 8 illustrates how $t_{n_{j,max}}$ can be determined by computing the continuous switching time $t_{j_{max}}$:

$$t_{n_{j,max}} = \Delta t \cdot \left\lfloor \frac{t_{j_{max}}}{\Delta t} \right\rfloor \tag{15}$$

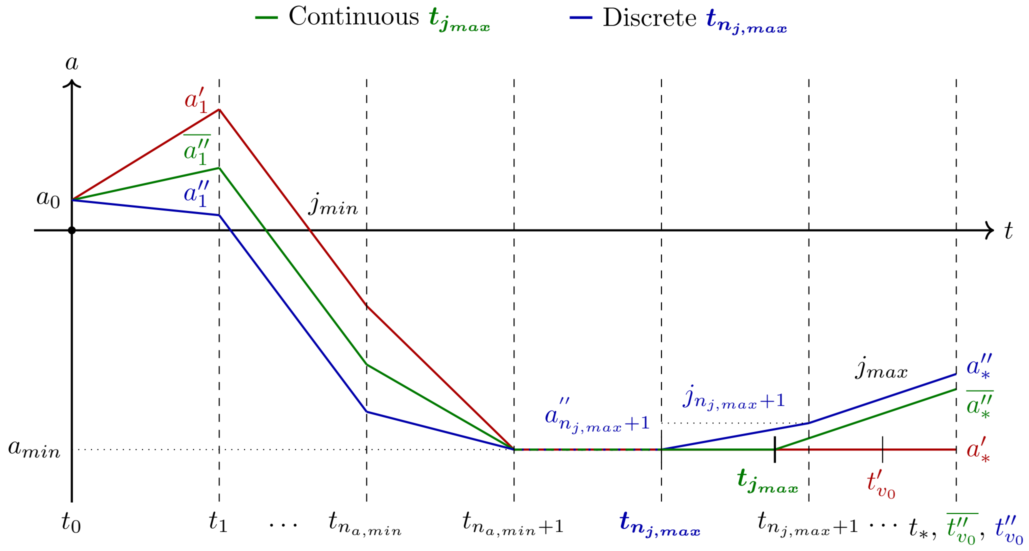

Figure 8: The discrete switching time $t_{n_{j,max}}$ for oscillation suppression is derived based on the results of the kinematic border case and an additional calculation assuming continuity of the switching time $t_{j_{max}}$.

Based on the phase of $t_{j_{max}}$, two different cases have to be analyzed to calculate a''_1. Figure 9 illustrates the procedure and refers to the underlying systems of equations.

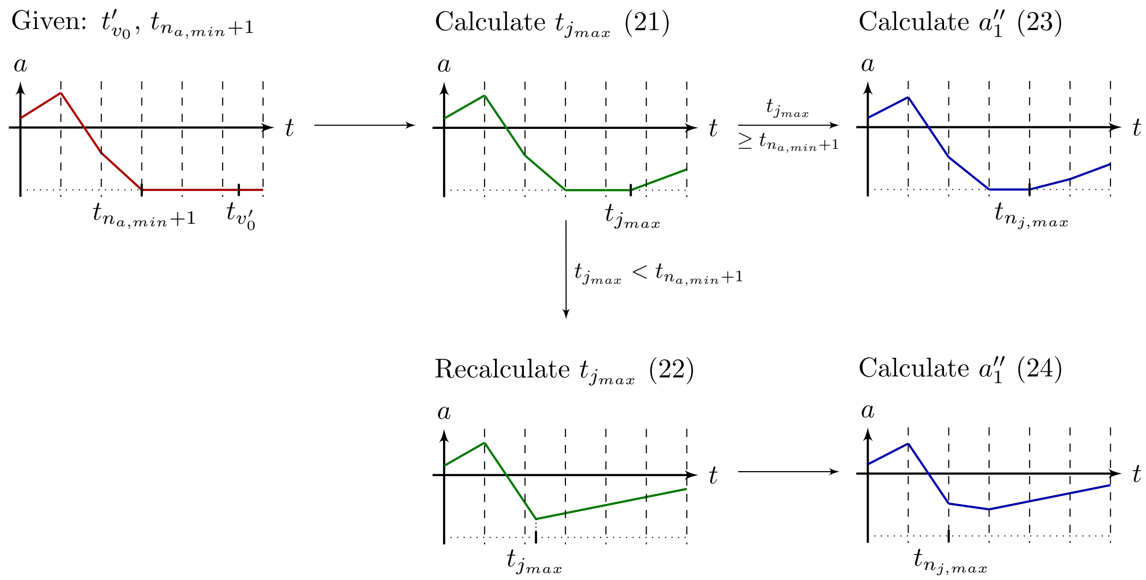

Figure 9: Procedure to suppress oscillations by calculating a''_1.

9 Equations to consider position constraints

9.1 Kinematic border case: Phase I

$$v(t'_{v_0}) = v_0 + \int_0^{t'_{v_0}} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \; dt = 0 \tag{16a}$$

$$p(t'_{v_0}) = p_0 + \int\limits_0^{t'_{v_0}} \int\limits_0^{t} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \; dt\, dt + t'_{v_0} \cdot v_0 = p_{max} \tag{16b}$$

9.2 Kinematic border case: Phase II

$$v(t'_{v_0}) = v_0 + \int_0^{t_1} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \; dt + \int_{t_1}^{t'_{v_0}} a'_1 + j_{min} \cdot (t - t_1) \; dt = 0 \tag{17a}$$

$$\begin{aligned} p(t'_{v_0}) = p_0 + \int\limits_0^{t_1} \int\limits_0^{t} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \; dt\, dt + \int\limits_{t_1}^{t'_{v_0}} \int\limits_{t_1}^{t} a'_1 + j_{min} \cdot (t - t_1) \; dt\, dt \\ + t'_{v_0} \cdot v_0 + \left(t'_{v_0} - t_1\right) \cdot \int\limits_0^{t_1} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \; dt = p_{max} \end{aligned} \tag{17b}$$

9.3 Kinematic border case: Phase III

$$\begin{aligned} v(t'_{v_0}) = v_0 + \int_0^{t_1} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \; dt + \int_{t_1}^{t_{n_{a,min}}} a'_1 + j_{min} \cdot (t - t_1) \; dt \\ + \int_{t_{n_{a,min}}}^{t'_{v_0}} a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot \left(t - t_{n_{a,min}}\right) \; dt = 0 \end{aligned} \tag{18a}$$

$$\begin{aligned} p(t'_{v_0}) = p_0 + \int\limits_0^{t_1} \int\limits_0^{t} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \; dt\, dt + \int\limits_{t_1}^{t_{n_{a,min}}} \int\limits_{t_1}^{t} a'_1 + j_{min} \cdot (t - t_1) \; dt\, dt \\ + \int\limits_{t_{n_{a,min}}}^{t'_{v_0}} \int\limits_{t_{n_{a,min}}}^{t} a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot \left(t - t_{n_{a,min}}\right) \; dt\, dt \\ + t'_{v_0} \cdot v_0 + \left(t'_{v_0} - t_1\right) \cdot \int\limits_0^{t_1} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \; dt \\ + \left(t'_{v_0} - t_{n_{a,min}}\right) \cdot \int\limits_{t_1}^{t_{n_{a,min}}} a'_1 + j_{min} \cdot (t - t_1) \; dt = p_{max} \end{aligned} \tag{18b}$$

9.4 Kinematic border case: Phase IV

$$
\begin{aligned}
v(t'_{v_0}) = v_0 &+ \int_0^{t_1} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \; dt + \int_{t_1}^{t_{n_{a,min}}} a'_1 + j_{min} \cdot (t - t_1) \; dt \\
&+ \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}} a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot \left(t - t_{n_{a,min}}\right) \; dt \\
&+ \int_{t_{n_{a,min}+1}}^{t'_{v_0}} a_{min} \; dt = 0
\end{aligned}
\tag{19a}
$$

$$
\begin{aligned}
p(t'_{v_0}) = p_0 &+ \int_0^{t_1}\int_0^{t} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \; dt\,dt + \int_{t_1}^{t_{n_{a,min}}}\int_{t_1}^{t} a'_1 + j_{min} \cdot (t - t_1) \; dt\,dt \\
&+ \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}}\int_{t_{n_{a,min}}}^{t} a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot \left(t - t_{n_{a,min}}\right) \; dt\,dt \\
&+ \int_{t_{n_{a,min}+1}}^{t'_{v_0}}\int_{t_{n_{a,min}+1}}^{t} a_{min} \; dt\,dt + t'_{v_0} \cdot v_0 + \left(t'_{v_0} - t_1\right) \cdot \int_0^{t_1} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \; dt \\
&+ \left(t'_{v_0} - t_{n_{a,min}}\right) \cdot \int_{t_1}^{t_{n_{a,min}}} a'_1 + j_{min} \cdot (t - t_1) \; dt + \left(t'_{v_0} - t_{n_{a,min}+1}\right) \\
&\cdot \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}} a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot \left(t - t_{n_{a,min}}\right) \; dt \\
&= p_{max}
\end{aligned}
\tag{19b}
$$

For the sake of clarity, $t_{n_{a,min}+1}$ is defined as a shortened form of $t_{n_{a,min}} + \Delta t$.

9.5 Kinematic border case: Calculation of $t_{n_{a,min}}$

$$
\begin{aligned}
v(\overline{t'_{v_0}}) = v_0 &+ \int_0^{t_1} a_0 + \frac{\overline{a'_1} - a_0}{\Delta t} \cdot t \; dt + \int_{t_1}^{t_{a_{min}}} \overline{a'_1} + j_{min} \cdot (t - t_1) \; dt \\
&+ \int_{t_{a_{min}}}^{\overline{t'_{v_0}}} a_{min} \; dt = 0
\end{aligned}
\tag{20a}
$$

$$
\begin{aligned}
p(\overline{t'_{v_0}}) = p_0 &+ \int_0^{t_1}\int_0^{t} a_0 + \frac{\overline{a'_1} - a_0}{\Delta t} \cdot t \; dt\,dt + \int_{t_1}^{t_{a_{min}}}\int_{t_1}^{t} \overline{a'_1} + j_{min} \cdot (t - t_1) \; dt\,dt \\
&+ \int_{t_{a_{min}}}^{\overline{t'_{v_0}}}\int_{t_{a_{min}}}^{t} a_{min} \; dt\,dt + \overline{t'_{v_0}} \cdot v_0 + \left(\overline{t'_{v_0}} - t_1\right) \cdot \int_0^{t_1} a_0 + \frac{\overline{a'_1} - a_0}{\Delta t} \cdot t \; dt \\
&+ \left(\overline{t'_{v_0}} - t_{a_{min}}\right) \cdot \int_{t_1}^{t_{a_{min}}} \overline{a'_1} + j_{min} \cdot (t - t_1) \; dt = p_{max}
\end{aligned}
\tag{20b}
$$

$$t_{a_{min}} = \Delta t \cdot \left(1 + \frac{a_{min} - \overline{a'_1}}{j_{min} \cdot \Delta t}\right) \tag{20c}$$

9.6 Oscillation suppression: Calculation of $t_{n_{j,max}}$

9.6.1 Case analysis: $t_{j_{max}} \geq t_{n_{a,min}+1}$

$$\begin{aligned} v(t_*) &= v_0 + \int_0^{t_1} a_0 + \frac{\overline{a''_1} - a_0}{\Delta t} \cdot t \; dt + \int_{t_1}^{t_{n_{a,min}}} \overline{a''_1} + j_{min} \cdot (t - t_1) \; dt \\ &+ \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}} \overline{a''_1} + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (\overline{a''_1} + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot \left(t - t_{n_{a,min}}\right) \; dt \\ &+ \int_{t_{n_{a,min}+1}}^{t_{j_{max}}} a_{min} \; dt + \int_{t_{j_{max}}}^{t_*} a_{min} + j_{max} \cdot (t - t_{j_{max}}) \; dt = 0 \end{aligned} \tag{21a}$$

$$\begin{aligned} p(t_*) &= p_0 + \int_0^{t_1}\int_0^{t} a_0 + \frac{\overline{a''_1} - a_0}{\Delta t} \cdot t \; dt\,dt + \int_{t_1}^{t_{n_{a,min}}}\int_{t_1}^{t} \overline{a''_1} + j_{min} \cdot (t - t_1) \; dt\,dt \\ &+ \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}}\int_{t_{n_{a,min}}}^{t} \overline{a''_1} + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (\overline{a''_1} + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot \left(t - t_{n_{a,min}}\right) \; dt\,dt \\ &+ \int_{t_{n_{a,min}+1}}^{t_{j_{max}}}\int_{t_{n_{a,min}+1}}^{t} a_{min} \; dt\,dt + \int_{t_{j_{max}}}^{t_*}\int_{t_{j_{max}}}^{t} a_{min} + j_{max} \cdot (t - t_{j_{max}}) \; dt\,dt \\ &+ t_* \cdot v_0 + (t_* - t_1) \cdot \int_0^{t_1} a_0 + \frac{\overline{a''_1} - a_0}{\Delta t} \cdot t \; dt \\ &+ \left(t_* - t_{n_{a,min}}\right) \cdot \int_{t_1}^{t_{n_{a,min}}} \overline{a''_1} + j_{min} \cdot (t - t_1) \; dt \\ &+ \left(t_* - t_{n_{a,min}+1}\right) \\ &\cdot \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}} \overline{a''_1} + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (\overline{a''_1} + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot \left(t - t_{n_{a,min}}\right) \; dt \\ &+ (t_* - t_{j_{max}}) \cdot \int_{t_{n_{a,min}+1}}^{t_{j_{max}}} a_{min} \; dt \\ &= p_{max} \end{aligned} \tag{21b}$$

9.6.2 Case analysis: $t_{j_{max}} < t_{n_{a,min}+1}$

Note: These equations also apply if t'_{v_0} is within phase II.

$$\begin{aligned} v(t_*) &= v_0 + \int_0^{t_1} a_0 + \frac{\overline{a''_1} - a_0}{\Delta t} \cdot t \; dt + \int_{t_1}^{t_{j_{max}}} \overline{a''_1} + j_{min} \cdot (t - t_1) \; dt \\ &+ \int_{t_{j_{max}}}^{t_*} \overline{a''_1} + j_{min} \cdot (t_{j_{max}} - t_1) + j_{max} \cdot (t - t_{j_{max}}) \; dt = 0 \end{aligned} \tag{22a}$$

$$
\begin{aligned}
p(t_*) = p_0 &+ \int_0^{t_1}\int_0^{t} a_0 + \frac{\overline{a_1''} - a_0}{\Delta t} \cdot t \; dt\, dt + \int_{t_1}^{t_{j_{max}}}\int_{t_1}^{t} \overline{a_1''} + j_{min} \cdot (t - t_1) \; dt\, dt \\
&+ \int_{t_{j_{max}}}^{t_*}\int_{t_{j_{max}}}^{t} \overline{a_1''} + j_{min} \cdot (t_{j_{max}} - t_1) + j_{max} \cdot (t - t_{j_{max}}) \; dt\, dt \\
&+ t_* \cdot v_0 + (t_* - t_1) \cdot \int_0^{t_1} a_0 + \frac{\overline{a_1''} - a_0}{\Delta t} \cdot t \; dt \\
&+ (t_* - t_{j_{max}}) \cdot \int_{t_1}^{t_{j_{max}}} \overline{a_1''} + j_{min} \cdot (t - t_1) \; dt \\
&= p_{max}
\end{aligned}
\tag{22b}
$$

9.7 Oscillation suppression: Calculation of a_1''

Given $t_{n_{j,max}}$ from (9.6), the following systems of equations can be solved for a_1'' and $j_{n_{j,max}+1}$. For the sake of clarity, $t_{n_{j,max}+1}$ is defined as a shortened form of $t_{n_{j,max}} + \Delta t$.

9.7.1 Case analysis: $t_{j_{max}} \geq t_{n_{a,min}+1}$

$$
\begin{aligned}
v(t_*) = v_0 &+ \int_0^{t_1} a_0 + \frac{a_1'' - a_0}{\Delta t} \cdot t \; dt + \int_{t_1}^{t_{n_{a,min}}} a_1'' + j_{min} \cdot (t - t_1) \; dt \\
&+ \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}} a_1'' + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a_1'' + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot \left(t - t_{n_{a,min}}\right) \; dt \\
&+ \int_{t_{n_{a,min}+1}}^{t_{n_{j,max}}} a_{min} \; dt + \int_{t_{n_{j,max}}}^{t_{n_{j,max}+1}} a_{min} + j_{n_{j,max}+1} \cdot (t - t_{n_{j,max}}) \; dt \\
&+ \int_{t_{n_{j,max}+1}}^{t_*} a_{min} + j_{n_{j,max}+1} \cdot \Delta t + j_{max} \cdot (t - t_{n_{j,max}+1}) \; dt = v_{*,min}''
\end{aligned}
\tag{23a}
$$

$$
\begin{aligned}
p(t_*) = p_0 &+ \int_0^{t_1}\int_0^{t} a_0 + \frac{a_1'' - a_0}{\Delta t} \cdot t \; dt\, dt + \int_{t_1}^{t_{n_{a,min}}}\int_{t_1}^{t} a_1'' + j_{min} \cdot (t - t_1) \; dt\, dt \\
&+ \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}} \int_{t_{n_{a,min}}}^{t} a_1'' + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a_1'' + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot \left(t - t_{n_{a,min}}\right) \; dt\, dt \\
&+ \int_{t_{n_{a,min}+1}}^{t_{n_{j,max}}} \int_{t_{n_{a,min}+1}}^{t} a_{min} \; dt\, dt + \int_{t_{n_{j,max}}}^{t_{n_{j,max}+1}} \int_{t_{n_{j,max}}}^{t} a_{min} + j_{n_{j,max}+1} \cdot (t - t_{n_{j,max}}) \; dt\, dt \\
&+ \int_{t_{n_{j,max}+1}}^{t_*} \int_{t_{n_{j,max}+1}}^{t} a_{min} + j_{n_{j,max}+1} \cdot \Delta t + j_{max} \cdot (t - t_{n_{j,max}+1}) \; dt\, dt \\
&+ t_* \cdot v_0 + (t_* - t_1) \cdot \int_0^{t_1} a_0 + \frac{a_1'' - a_0}{\Delta t} \cdot t \; dt \\
&+ \left(t_* - t_{n_{a,min}}\right) \cdot \int_{t_1}^{t_{n_{a,min}}} a_1'' + j_{min} \cdot (t - t_1) \; dt \\
&+ \left(t_* - t_{n_{a,min}+1}\right) \\
&\cdot \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}} a_1'' + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a_1'' + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot \left(t - t_{n_{a,min}}\right) \; dt \\
&+ \left(t_* - t_{n_{j,max}}\right) \cdot \int_{t_{n_{a,min}+1}}^{t_{n_{j,max}}} a_{min} \; dt + \left(t_* - t_{n_{j,max}+1}\right) \cdot \int_{t_{n_{j,max}}}^{t_{n_{j,max}+1}} a_{min} + j_{n_{j,max}+1} \cdot (t - t_{n_{j,max}}) \; dt \\
&= p_{max}
\end{aligned}
\tag{23b}
$$

9.7.2 Case analysis: $t_{j_{max}} < t_{n_{a,min}+1}$

Note: These equations also apply if t'_{v_0} is within phase II.

$$
\begin{aligned}
v(t_*) = v_0 &+ \int_0^{t_1} a_0 + \frac{a_1'' - a_0}{\Delta t} \cdot t \; dt + \int_{t_1}^{t_{n_{j,max}}} a_1'' + j_{min} \cdot (t - t_1) \; dt \\
&+ \int_{t_{n_{j,max}}}^{t_{n_{j,max}+1}} a_1'' + j_{min} \cdot (t_{n_{j,max}} - t_1) + j_{n_{j,max}+1} \cdot (t - t_{n_{j,max}}) \; dt \\
&+ \int_{t_{n_{j,max}+1}}^{t_*} a_1'' + j_{min} \cdot (t_{n_{j,max}} - t_1) + j_{n_{j,max}+1} \cdot \Delta t + j_{max} \cdot (t - t_{n_{j,max}+1}) \; dt = v''_{*,min}
\end{aligned}
\tag{24a}
$$

$$
\begin{aligned}
p(t_*) = p_0 &+ \int_0^{t_1}\int_0^{t} a_0 + \frac{a_1'' - a_0}{\Delta t} \cdot t \; dt\,dt + \int_{t_1}^{t_{n_{j,max}}}\int_{t_1}^{t} a_1'' + j_{min} \cdot (t - t_1) \; dt\,dt \\
&+ \int_{t_{n_{j,max}}}^{t_{n_{j,max}+1}} \int_{t_{n_{j,max}}}^{t} a_1'' + j_{min} \cdot (t_{n_{j,max}} - t_1) + j_{n_{j,max}+1} \cdot (t - t_{n_{j,max}}) \; dt\,dt \\
&+ \int_{t_{n_{j,max}+1}}^{t_*} \int_{t_{n_{j,max}+1}}^{t} a_1'' + j_{min} \cdot (t_{n_{j,max}} - t_1) + j_{n_{j,max}+1} \cdot \Delta t + j_{max} \cdot (t - t_{n_{j,max}+1}) \; dt\,dt \\
&+ t_* \cdot v_0 + (t_* - t_1) \cdot \int_0^{t_1} a_0 + \frac{a_1'' - a_0}{\Delta t} \cdot t \; dt \\
&+ \left(t_* - t_{n_{j,max}}\right) \cdot \int_{t_1}^{t_{n_{j,max}}} a_1'' + j_{min} \cdot (t - t_1) \; dt \\
&+ \left(t_* - t_{n_{j,max}+1}\right) \cdot \int_{t_{n_{j,max}}}^{t_{n_{j,max}+1}} a_1'' + j_{min} \cdot (t_{n_{j,max}} - t_1) + j_{n_{j,max}+1} \cdot (t - t_{n_{j,max}}) \; dt \\
&= p_{max}
\end{aligned}
\tag{24b}
$$

References

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