

# Background Knowledge for Learning Robot Trajectories subject to Kinematic Joint Constraints

## 1 Introduction

This document complements our paper “Learning Robot Trajectories subject to Kinematic Joint Constraints” by providing further information on the implementation and the mathematical background. A preprint of the paper is available at <https://arxiv.org/abs/2011.00563>.

## 2 System components

The system components required to learn safe movements with a neural network are shown in Figure 1.

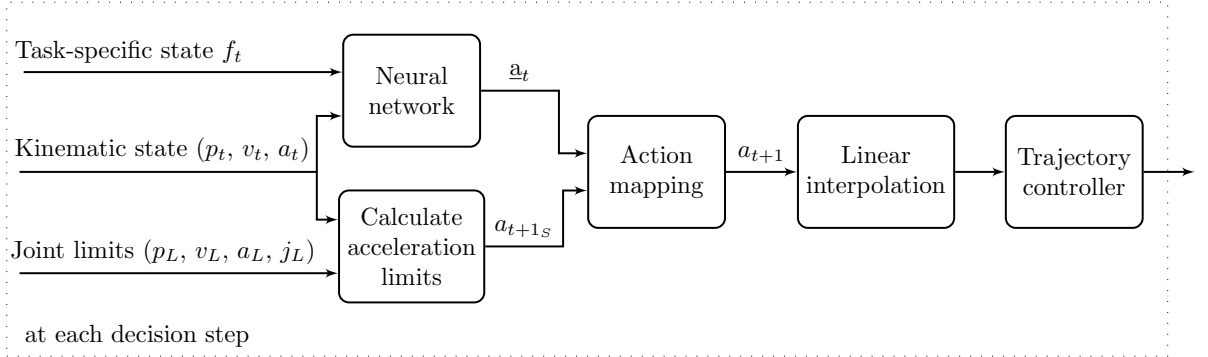


Figure 1: System components for online trajectory generation.

Figure 2 illustrates the steps to derive the range of safe accelerations  $a_{t+1s} = [a_{t+1min}, a_{t+1max}]$ . In a first step, each constraint is handled independently. For instance, the maximum accelerations  $a_{t+1max,j}$ ,  $a_{t+1max,a}$ ,  $a_{t+1max,v}$  and  $a_{t+1max,p}$  are computed to consider the jerk, acceleration, velocity and position constraints, respectively. The desired maximum acceleration  $a_{t+1max}$  is set to the smallest of these values. It is important to note that each constraint can nevertheless be influenced by several kinematic limits. For instance, the maximum acceleration complying with position constraints  $a_{t+1max,p}$  depends on  $p_{max}$ ,  $a_{min}$  and  $j_{min}$ . In the following, the procedures and equations to compute  $a_{t+1max,j}$ ,  $a_{t+1max,a}$ ,  $a_{t+1max,v}$  and  $a_{t+1max,p}$  are explained in detail.

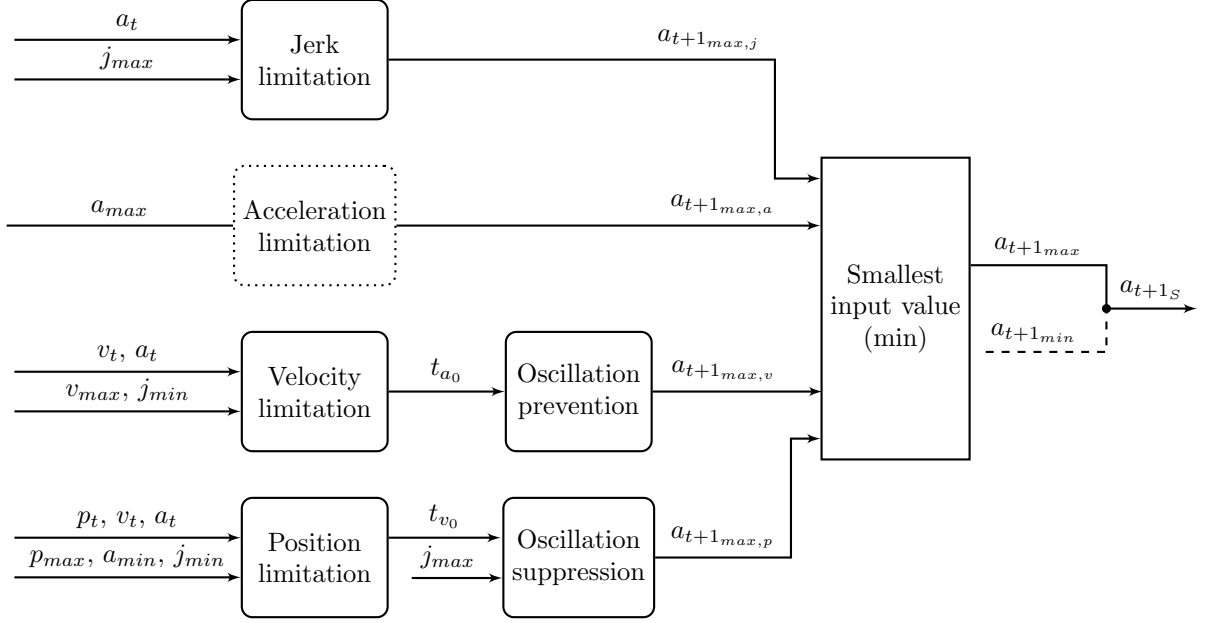


Figure 2: System components to calculate the range of safe accelerations  $a_{t+1s}$ .

### 3 Procedure to consider jerk constraints

The jerk between decision steps is constant as the acceleration is linearly interpolated. Consequently, the maximum acceleration  $a_{t+1max,j}$  can be computed as follows:

$$a_{t+1max,j} = a_t + j_{max} \cdot \Delta t, \quad (1)$$

with  $\Delta t$  being the time between decision steps.

### 4 Procedure to consider acceleration constraints

Due to the design of the action space, the desired maximum acceleration  $a_{t+1max,j}$  corresponds to the specified maximum acceleration  $a_{max}$ .

$$a_{t+1max,j} = a_{max} \quad (2)$$

### 5 Procedure to consider velocity constraints

The following equations are used to determine the maximum acceleration  $a_1$  at the next decision step  $t_1$  that does not lead to a mandatory violation of the velocity limit  $v_{max}$ . The solution corresponds to the desired value of  $a_{t+1max,v}$ .

From a mathematical perspective, the most dynamic behavior is achieved if the maximum velocity  $v_{max}$  is reached at an acceleration of zero:

$$v_0 + \int_0^{t'_{a0}} a(t) dt = v_{max} \quad (3)$$

$$a(t'_{a0}) = 0 \quad (4)$$

Two different cases have to be considered. In the first case, the maximum velocity is reached within the next time step, meaning that  $t'_{a0} \leq t_1$ . If so,  $a'_1$  can be determined by solving the system of

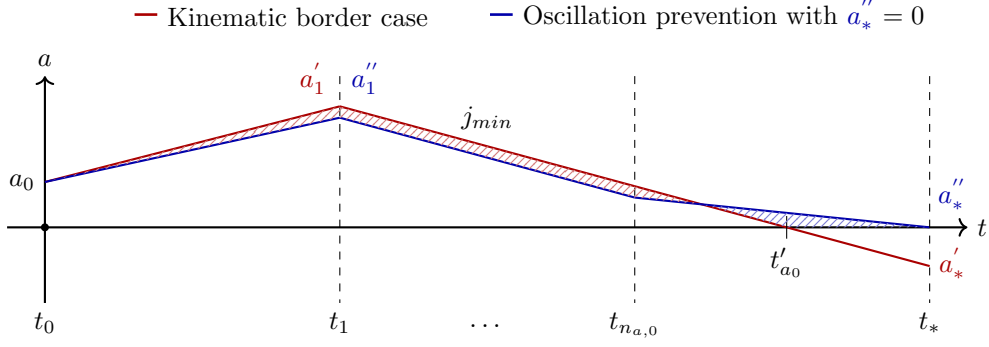


Figure 3: Course of acceleration to comply with velocity constraints.

equation (6). In the second case,  $a'_1$  is greater than zero and followed by a deceleration with the minimum jerk. In that case,  $a'_1$  and  $t'_{a_0}$  can be computed by solving (7). The two cases can be distinguished by the following condition:

$$v_0 + \frac{a_0 \cdot \Delta t}{2} \begin{cases} \geq v_{max} & \text{Case } t'_{a_0} \leq t_1 \\ < v_{max} & \text{Case } t'_{a_0} > t_1 \end{cases} \quad (5)$$

$\Delta t$  is the time between decision steps. In the second case, the solution for  $a'_1$  received by (7) is prone to oscillations, which are caused by the discrete nature of decision making. These oscillations can be prevented by enforcing zero acceleration at  $t_*$ , the discrete decision step that follows  $t'_{a_0}$ . As shown in Figure 4, the corresponding acceleration  $a''_1$  can be computed by solving equation (8).

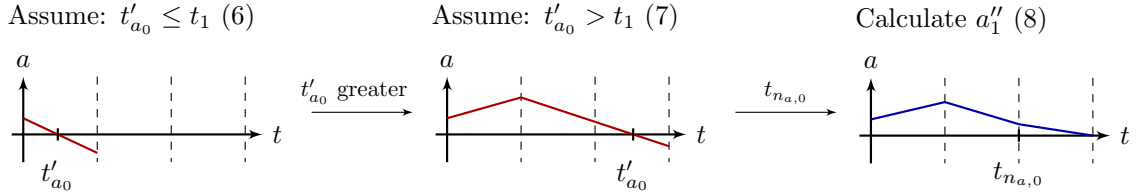


Figure 4: Visualization of the procedure to calculate  $a''_1$ .

## 6 Equations to consider velocity constraints

### 6.1 Kinematic border case: $t'_{a_0} \leq t_1$

$$v_0 + \int_0^{t'_{a_0}} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \, dt = v_{max} \quad (6a)$$

$$a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t'_{a_0} = 0 \quad (6b)$$

### 6.2 Kinematic border case: $t'_{a_0} > t_1$

$$v_0 + \int_0^{t_1} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \, dt + \int_{t_1}^{t'_{a_0}} a'_1 + j_{min} \cdot (t - t_1) \, dt = v_{max} \quad (7a)$$

$$a'_1 + j_{min} \cdot (t'_{a_0} - \Delta t) = 0 \quad (7b)$$

### 6.3 Oscillation prevention

$$v_0 + \int_0^{t_1} a_0 + \frac{a_1'' - a_0}{\Delta t} \cdot t \, dt + \int_{t_1}^{t_{n_{a,0}}} a_1'' + j_{min} \cdot (t - t_1) \, dt + \int_{t_{n_{a,0}}}^{t_*} (a_1'' + j_{min} \cdot (t_{n_{a,0}} - t_1)) \cdot \left(1 - \frac{t - t_{n_{a,0}}}{\Delta t}\right) dt = v_{max}, \quad (8)$$

with

$$t_{n_{a,0}} = \Delta t \cdot \left\lfloor \frac{t_{a_0}'}{\Delta t} \right\rfloor \quad (9)$$

$$t_* = \Delta t \cdot \left\lceil \frac{t_{a_0}'}{\Delta t} \right\rceil. \quad (10)$$

## 7 Procedure to consider position constraints

In the following, the desired acceleration  $a_{t+1_{max,p}}$  corresponds to the greatest possible value of  $a_1$ . To stay within the specified position limits, the maximum position must be reached at a velocity of 0:

$$v_0 + \int_0^{t_{v_0}} a(t) \, dt = 0 \quad (11)$$

$$p_0 + v_0 \cdot t_{v_0} + \int_0^{t_{v_0}} \int_0^t a(t) \, dt \, dt = p_{max}, \quad (12)$$

with  $t_{v_0}$  being the continuous time at which the maximum position is reached.

The most dynamic behaviour is achieved if  $a_1$  is followed by the strongest deceleration that complies with the jerk and acceleration limits. This case is denoted as kinematic border case and illustrated by the red line in Figure 5.

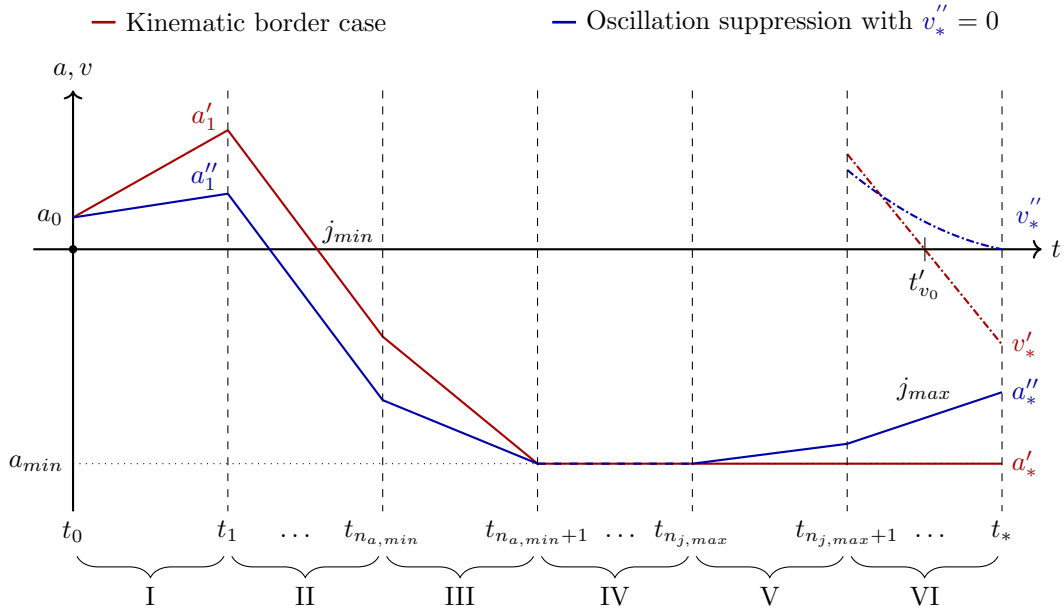


Figure 5: Course of acceleration and velocity to comply with position limits.

The expressions required to solve equation (11) and (12) for  $a_1$  and  $t_{v_0}$  depend on the phase (I - VI) that  $t_{v_0}$  is within. Note that phase III and V are needed as decisions can be made at discrete time

steps only. Figure 6 illustrates how the phase of  $t'_{v_0}$  can be determined by pursuing a systematic trial and error strategy. Assuming that  $t'_{v_0}$  is within a specified phase,  $t'_{v_0}$  and  $a'_1$  can be computed by solving the system of equations that is referenced in brackets. In case that  $t'_{v_0}$  is in phase I,  $a'_1$  is the desired maximum acceleration and no further calculations are required. In all other cases, the value of  $t'_{v_0}$  is used for subsequent calculations to suppress oscillations.

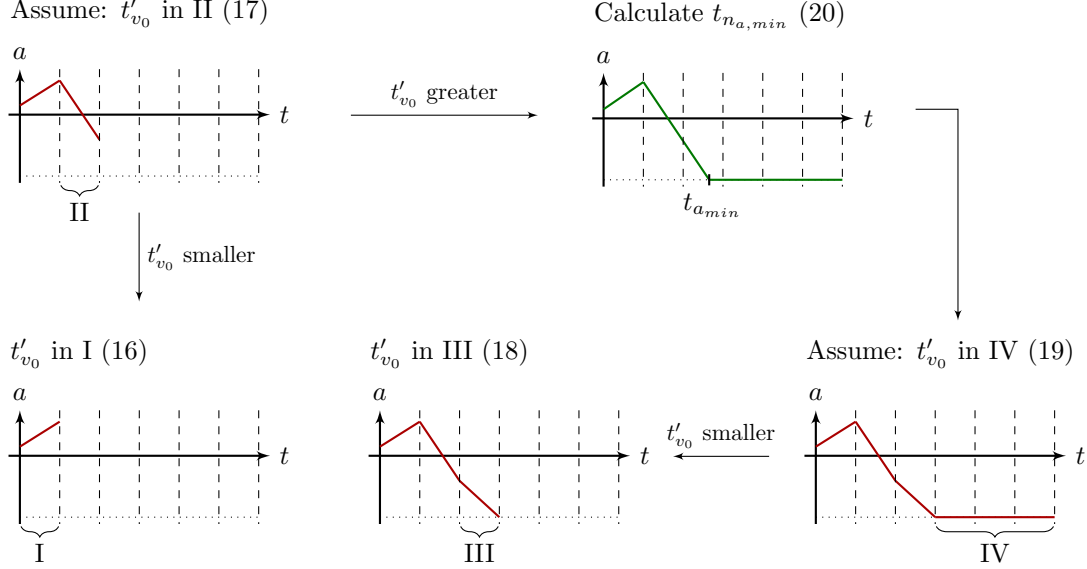


Figure 6: Procedure to determine the value and phase of  $t'_{v_0}$ .

The equation systems (18) and (19) can only be solved if  $t_{n_{a,min}}$  is known.  $t_{n_{a,min}}$  is the time of the discrete decision step at the end of phase II. The duration of phase II can be described as

$$T_{II}(a_1) = \left\lfloor \frac{a_{min} - a_1}{j_{min} \cdot \Delta t} \right\rfloor \cdot \Delta t, \quad \text{with } \Delta t \text{ being the time between decisions steps.} \quad (13)$$

However, equation (13) cannot be solved for  $a_1$  as the floor function is not invertible. To circumvent this problem,  $t_{n_{a,min}}$  is deduced from additional calculations assuming continuous decision steps. Figure 7 shows how an upper bound  $\overline{a'_1}$  and lower bound  $\underline{a'_1}$  for  $a'_1$  can be determined.

The upper bound is computed by assuming continuity of  $t_{a,min}$ . Once  $t_{a,min}$  is known,  $t_{n_{a,min}}$  can be calculated as follows:

$$t_{n_{a,min}} = \Delta t \cdot \left\lceil \frac{t_{a,min}}{\Delta t} \right\rceil$$

The value of  $t_{n_{a,min}}$  remains valid for the kinematic border case with discrete decision steps.

Proof:

If

$$a'_1 \geq \overline{a'_1},$$

the maximum position is exceeded as the velocity is always higher than the velocity of the upper bound.

If

$$a'_1 \leq \underline{a'_1},$$

the maximum position is never reached as the velocity is always lower than the velocity of the upper bound. Consequently, the desired solution  $a'_1$  is required to be between  $\underline{a'_1}$  and  $\overline{a'_1}$ , meaning that the value of  $t_{n_{a,min}}$  does not change.

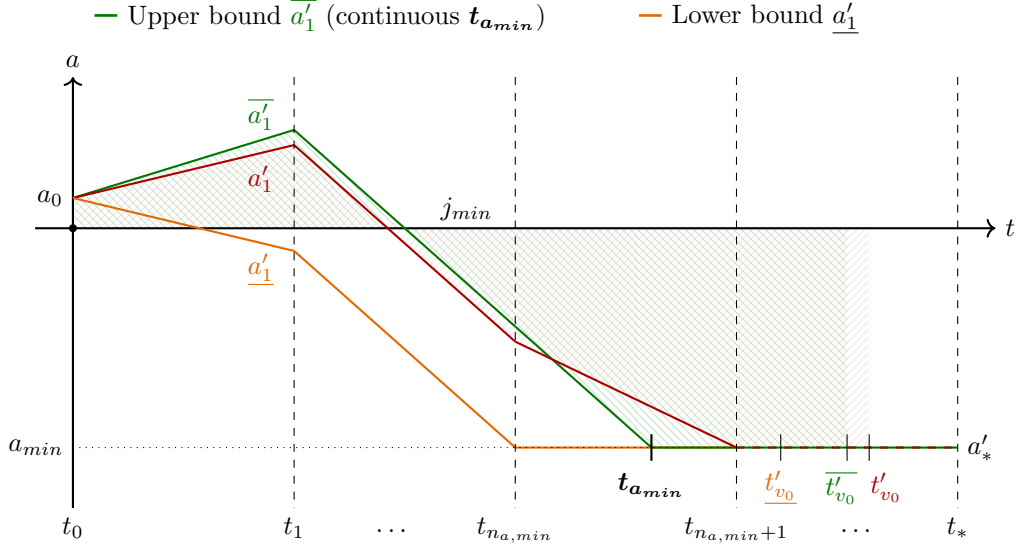


Figure 7: Upper bound  $\overline{a'_1}$  and lower bound  $\underline{a'_1}$  to determine  $t_{n_{a,min}}$  for the kinematic border case. The areas hatched in red and green have to be of equal size in order to ensure a velocity of zero at  $t'_{v_0}$ .

The kinematic border case described above leads to two problems: Firstly, oscillations might occur once the maximum position is reached. Secondly, the velocity  $v'_*$  at decision step  $t_*$  might exceed the minimum velocity  $v_{min}$ . Both problems can be mitigated by calculating the maximum value for  $a''_1$  while ensuring a final velocity  $v''_*$  of zero as indicated by the blue line in Figure 5. The value of  $t_*$  can be derived based on the results from the kinematic border case:

$$t_* = \Delta t \cdot \left\lceil \frac{t'_{v_0}}{\Delta t} \right\rceil \quad (14)$$

As shown in Figure 5, the desired target velocity  $v''_*$  can be reached by increasing the acceleration at time  $t_{n_{j,max}}$ .

Figure 8 illustrates how  $t_{n_{j,max}}$  can be determined by computing the continuous switching time  $t_{j_{max}}$ :

$$t_{n_{j,max}} = \Delta t \cdot \left\lceil \frac{t_{j_{max}}}{\Delta t} \right\rceil \quad (15)$$

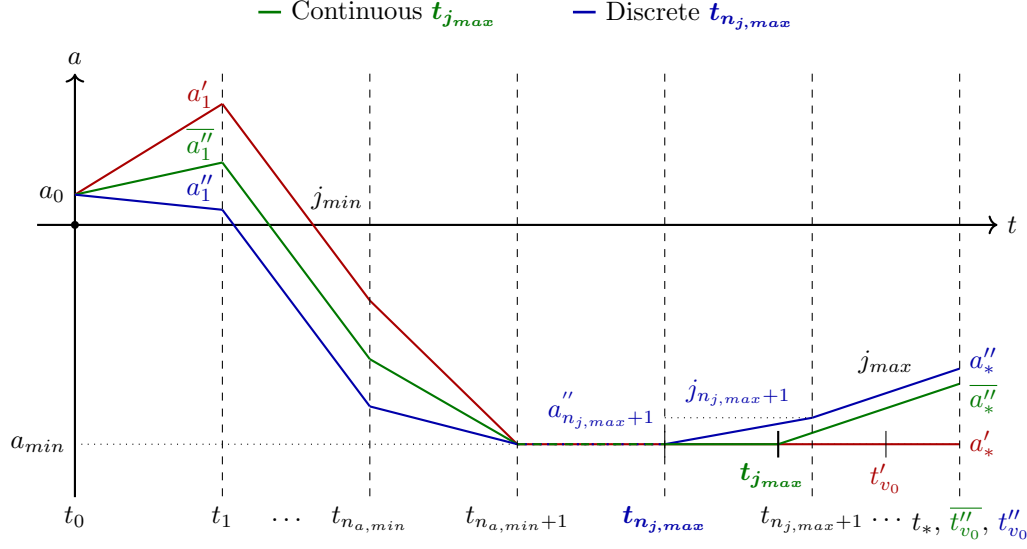


Figure 8: The discrete switching time  $t_{n_{j,max}}$  for oscillation suppression is derived based on the results of the kinematic border case and an additional calculation assuming continuity of the switching time  $t_{j,max}$ .

Based on the phase of  $t_{j,max}$ , two different cases have to be analyzed to calculate  $a''_1$ . Figure 9 illustrates the procedure and refers to the underlying systems of equations.

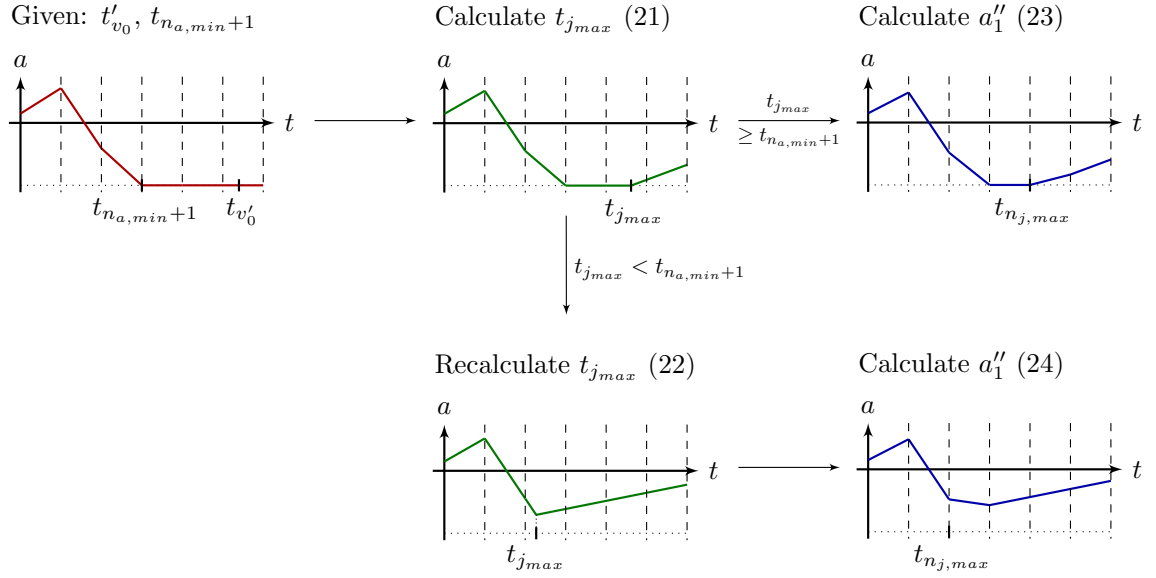


Figure 9: Procedure to suppress oscillations by calculating  $a''_1$ .

## 8 Equations to consider position constraints

### 8.1 Kinematic border case: Phase I

$$v(t'_{v_0}) = v_0 + \int_0^{t'_{v_0}} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \, dt = 0 \quad (16a)$$

$$p(t'_{v_0}) = p_0 + \int_0^{t'_{v_0}} \int_0^t a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \, dt \, dt + t'_{v_0} \cdot v_0 = p_{max} \quad (16b)$$

### 8.2 Kinematic border case: Phase II

$$v(t'_{v_0}) = v_0 + \int_0^{t_1} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \, dt + \int_{t_1}^{t'_{v_0}} a'_1 + j_{min} \cdot (t - t_1) \, dt = 0 \quad (17a)$$

$$p(t'_{v_0}) = p_0 + \int_0^{t_1} \int_0^t a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \, dt \, dt + \int_{t_1}^{t'_{v_0}} \int_{t_1}^t a'_1 + j_{min} \cdot (t - t_1) \, dt \, dt \\ + t'_{v_0} \cdot v_0 + (t'_{v_0} - t_1) \cdot \int_0^{t_1} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \, dt = p_{max} \quad (17b)$$

### 8.3 Kinematic border case: Phase III

$$v(t'_{v_0}) = v_0 + \int_0^{t_1} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \, dt + \int_{t_1}^{t_{n_{a,min}}} a'_1 + j_{min} \cdot (t - t_1) \, dt \\ + \int_{t_{n_{a,min}}}^{t'_{v_0}} a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot (t - t_{n_{a,min}}) \, dt = 0 \quad (18a)$$

$$p(t'_{v_0}) = p_0 + \int_0^{t_1} \int_0^t a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \, dt \, dt + \int_{t_1}^{t_{n_{a,min}}} \int_{t_1}^t a'_1 + j_{min} \cdot (t - t_1) \, dt \, dt \\ + \int_{t_{n_{a,min}}}^{t'_{v_0}} \int_{t_{n_{a,min}}}^t a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot (t - t_{n_{a,min}}) \, dt \, dt \\ + t'_{v_0} \cdot v_0 + (t'_{v_0} - t_1) \cdot \int_0^{t_1} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \, dt \\ + (t'_{v_0} - t_{n_{a,min}}) \cdot \int_{t_1}^{t_{n_{a,min}}} a'_1 + j_{min} \cdot (t - t_1) \, dt = p_{max} \quad (18b)$$



#### 8.4 Kinematic border case: Phase IV

$$\begin{aligned}
v(t'_{v_0}) &= v_0 + \int_0^{t_1} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \, dt + \int_{t_1}^{t_{n_{a,min}}} a'_1 + j_{min} \cdot (t - t_1) \, dt \\
&+ \int_{t_{n_{a,min}}}^{t_{n_{a,min}}+1} a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot (t - t_{n_{a,min}}) \, dt \quad (19a) \\
&+ \int_{t_{n_{a,min}}+1}^{t'_{v_0}} a_{min} \, dt = 0
\end{aligned}$$

$$\begin{aligned}
p(t'_{v_0}) &= p_0 + \int_0^{t_1} \int_0^t a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \, dt \, dt + \int_{t_1}^{t_{n_{a,min}}} \int_{t_1}^t a'_1 + j_{min} \cdot (t - t_1) \, dt \, dt \\
&+ \int_{t_{n_{a,min}}}^{t_{n_{a,min}}+1} \int_{t_{n_{a,min}}}^t a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot (t - t_{n_{a,min}}) \, dt \, dt \\
&+ \int_{t_{n_{a,min}}+1}^{t'_{v_0}} \int_{t_{n_{a,min}}+1}^t a_{min} \, dt \, dt + t'_{v_0} \cdot v_0 + (t'_{v_0} - t_1) \cdot \int_0^{t_1} a_0 + \frac{a'_1 - a_0}{\Delta t} \cdot t \, dt \\
&+ (t'_{v_0} - t_{n_{a,min}}) \cdot \int_{t_1}^{t_{n_{a,min}}} a'_1 + j_{min} \cdot (t - t_1) \, dt + (t'_{v_0} - t_{n_{a,min}}+1) \\
&\cdot \int_{t_{n_{a,min}}}^{t_{n_{a,min}}+1} a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a'_1 + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot (t - t_{n_{a,min}}) \, dt \\
&= p_{max} \quad (19b)
\end{aligned}$$

For the sake of clarity,  $t_{n_{a,min}}+1$  is defined as a shortened form of  $t_{n_{a,min}} + \Delta t$ .

#### 8.5 Kinematic border case: Calculation of $t_{n_{a,min}}$

$$\begin{aligned}
v(\overline{t'_{v_0}}) &= v_0 + \int_0^{t_1} a_0 + \frac{\overline{a'_1} - a_0}{\Delta t} \cdot t \, dt + \int_{t_1}^{t_{a_{min}}} \overline{a'_1} + j_{min} \cdot (t - t_1) \, dt \\
&+ \int_{t_{a_{min}}}^{\overline{t'_{v_0}}} a_{min} \, dt = 0 \quad (20a)
\end{aligned}$$

$$\begin{aligned}
p(\overline{t'_{v_0}}) &= p_0 + \int_0^{t_1} \int_0^t a_0 + \frac{\overline{a'_1} - a_0}{\Delta t} \cdot t \, dt \, dt + \int_{t_1}^{t_{a_{min}}} \int_{t_1}^t \overline{a'_1} + j_{min} \cdot (t - t_1) \, dt \, dt \\
&+ \int_{t_{a_{min}}}^{\overline{t'_{v_0}}} \int_{t_{a_{min}}}^t a_{min} \, dt \, dt + \overline{t'_{v_0}} \cdot v_0 + (\overline{t'_{v_0}} - t_1) \cdot \int_0^{t_1} a_0 + \frac{\overline{a'_1} - a_0}{\Delta t} \cdot t \, dt \\
&+ (\overline{t'_{v_0}} - t_{a_{min}}) \cdot \int_{t_1}^{t_{a_{min}}} \overline{a'_1} + j_{min} \cdot (t - t_1) \, dt = p_{max} \quad (20b)
\end{aligned}$$

$$t_{a_{min}} = \Delta t \cdot \left( 1 + \frac{a_{min} - \overline{a_1'}}{j_{min} \cdot \Delta t} \right) \quad (20c)$$

## 8.6 Oscillation suppression: Calculation of $t_{n_{j,max}}$

### 8.6.1 Case analysis: $t_{j_{max}} \geq t_{n_{a,min}+1}$

$$\begin{aligned} v(t_*) &= v_0 + \int_0^{t_1} a_0 + \frac{\overline{a_1''} - a_0}{\Delta t} \cdot t \, dt + \int_{t_1}^{t_{n_{a,min}}} \overline{a_1''} + j_{min} \cdot (t - t_1) \, dt \\ &+ \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}} \overline{a_1''} + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (\overline{a_1''} + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot (t - t_{n_{a,min}}) \, dt \quad (21a) \\ &+ \int_{t_{n_{a,min}+1}}^{t_{j_{max}}} a_{min} \, dt + \int_{t_{j_{max}}}^{t_*} a_{min} + j_{max} \cdot (t - t_{j_{max}}) \, dt = 0 \end{aligned}$$

$$\begin{aligned} p(t_*) &= p_0 + \int_0^{t_1} \int_0^t a_0 + \frac{\overline{a_1''} - a_0}{\Delta t} \cdot t \, dt \, dt + \int_{t_1}^{t_{n_{a,min}}} \int_{t_1}^t \overline{a_1''} + j_{min} \cdot (t - t_1) \, dt \, dt \\ &+ \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}} \int_{t_{n_{a,min}}}^t \overline{a_1''} + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (\overline{a_1''} + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot (t - t_{n_{a,min}}) \, dt \, dt \\ &+ \int_{t_{n_{a,min}+1}}^{t_{j_{max}}} \int_{t_{n_{a,min}+1}}^t a_{min} \, dt \, dt + \int_{t_{j_{max}}}^{t_*} \int_{t_{j_{max}}}^t a_{min} + j_{max} \cdot (t - t_{j_{max}}) \, dt \, dt \\ &+ t_* \cdot v_0 + (t_* - t_1) \cdot \int_0^{t_1} a_0 + \frac{\overline{a_1''} - a_0}{\Delta t} \cdot t \, dt \\ &+ (t_* - t_{n_{a,min}}) \cdot \int_{t_1}^{t_{n_{a,min}}} \overline{a_1''} + j_{min} \cdot (t - t_1) \, dt \\ &+ (t_* - t_{n_{a,min}+1}) \\ &\cdot \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}} \overline{a_1''} + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (\overline{a_1''} + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot (t - t_{n_{a,min}}) \, dt \\ &+ (t_* - t_{j_{max}}) \cdot \int_{t_{n_{a,min}+1}}^{t_{j_{max}}} a_{min} \, dt \\ &= p_{max} \quad (21b) \end{aligned}$$

### 8.6.2 Case analysis: $t_{j_{max}} < t_{n_{a,min}+1}$

Note: These equations also apply if  $t'_{v_0}$  is within phase II.

$$\begin{aligned} v(t_*) &= v_0 + \int_0^{t_1} a_0 + \frac{\overline{a_1''} - a_0}{\Delta t} \cdot t \, dt + \int_{t_1}^{t_{j_{max}}} \overline{a_1''} + j_{min} \cdot (t - t_1) \, dt \\ &+ \int_{t_{j_{max}}}^{t_*} \overline{a_1''} + j_{min} \cdot (t_{j_{max}} - t_1) + j_{max} \cdot (t - t_{j_{max}}) \, dt = 0 \quad (22a) \end{aligned}$$

$$\begin{aligned}
p(t_*) &= p_0 + \int_0^{t_1} \int_0^t a_0 + \frac{\overline{a_1''} - a_0}{\Delta t} \cdot t \, dt \, dt + \int_{t_1}^{t_{j_{max}}} \int_{t_1}^t \overline{a_1''} + j_{min} \cdot (t - t_1) \, dt \, dt \\
&+ \int_{t_{j_{max}}}^{t_*} \int_{t_{j_{max}}}^t \overline{a_1''} + j_{min} \cdot (t_{j_{max}} - t_1) + j_{max} \cdot (t - t_{j_{max}}) \, dt \, dt \\
&+ t_* \cdot v_0 + (t_* - t_1) \cdot \int_0^{t_1} a_0 + \frac{\overline{a_1''} - a_0}{\Delta t} \cdot t \, dt \\
&+ (t_* - t_{j_{max}}) \cdot \int_{t_1}^{t_{j_{max}}} \overline{a_1''} + j_{min} \cdot (t - t_1) \, dt \\
&= p_{max}
\end{aligned} \tag{22b}$$

## 8.7 Oscillation suppression: Calculation of $a_1''$

Given  $t_{n_{j,max}}$  from (8.6), the following systems of equations can be solved for  $a_1''$  and  $j_{n_{j,max}+1}$ . For the sake of clarity,  $t_{n_{j,max}+1}$  is defined as a shortened form of  $t_{n_{j,max}} + \Delta t$ .

### 8.7.1 Case analysis: $t_{j_{max}} \geq t_{n_{a,min}+1}$

$$\begin{aligned}
v(t_*) &= v_0 + \int_0^{t_1} a_0 + \frac{a_1'' - a_0}{\Delta t} \cdot t \, dt + \int_{t_1}^{t_{n_{a,min}}} a_1'' + j_{min} \cdot (t - t_1) \, dt \\
&+ \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}} a_1'' + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a_1'' + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot (t - t_{n_{a,min}}) \, dt \\
&+ \int_{t_{n_{a,min}+1}}^{t_{n_{j,max}}} a_{min} \, dt + \int_{t_{n_{j,max}}}^{t_{n_{j,max}+1}} a_{min} + j_{n_{j,max}+1} \cdot (t - t_{n_{j,max}}) \, dt \\
&+ \int_{t_{n_{j,max}+1}}^{t_*} a_{min} + j_{n_{j,max}+1} \cdot \Delta t + j_{max} \cdot (t - t_{n_{j,max}+1}) \, dt = v_{*,min}''
\end{aligned} \tag{23a}$$

$$\begin{aligned}
p(t_*) &= p_0 + \int_0^{t_1} \int_0^t a_0 + \frac{a_1'' - a_0}{\Delta t} \cdot t \, dt \, dt + \int_{t_1}^{t_{n_{a,min}}} \int_{t_1}^t a_1'' + j_{min} \cdot (t - t_1) \, dt \, dt \\
&+ \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}} \int_{t_{n_{a,min}}}^t a_1'' + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a_1'' + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot (t - t_{n_{a,min}}) \, dt \, dt \\
&+ \int_{t_{n_{a,min}+1}}^{t_{n_{j,max}}} \int_{t_{n_{a,min}+1}}^t a_{min} \, dt \, dt + \int_{t_{n_{j,max}}}^{t_{n_{j,max}+1}} \int_{t_{n_{j,max}}}^t a_{min} + j_{n_{j,max}+1} \cdot (t - t_{n_{j,max}}) \, dt \, dt \\
&+ \int_{t_{n_{j,max}+1}}^{t_*} \int_{t_{n_{j,max}+1}}^t a_{min} + j_{n_{j,max}+1} \cdot \Delta t + j_{max} \cdot (t - t_{n_{j,max}+1}) \, dt \, dt \\
&+ t_* \cdot v_0 + (t_* - t_1) \cdot \int_0^{t_1} a_0 + \frac{a_1'' - a_0}{\Delta t} \cdot t \, dt \\
&+ (t_* - t_{n_{a,min}}) \cdot \int_{t_1}^{t_{n_{a,min}}} a_1'' + j_{min} \cdot (t - t_1) \, dt \\
&+ (t_* - t_{n_{a,min}+1}) \\
&\cdot \int_{t_{n_{a,min}}}^{t_{n_{a,min}+1}} a_1'' + j_{min} \cdot (t_{n_{a,min}} - t_1) + \frac{a_{min} - (a_1'' + j_{min} \cdot (t_{n_{a,min}} - t_1))}{\Delta t} \cdot (t - t_{n_{a,min}}) \, dt \\
&+ (t_* - t_{n_{j,max}}) \cdot \int_{t_{n_{a,min}+1}}^{t_{n_{j,max}}} a_{min} \, dt + (t_* - t_{n_{j,max}+1}) \cdot \int_{t_{n_{j,max}}}^{t_{n_{j,max}+1}} a_{min} + j_{n_{j,max}+1} \cdot (t - t_{n_{j,max}}) \, dt \\
&= p_{max}
\end{aligned} \tag{23b}$$

### 8.7.2 Case analysis: $t_{j,max} < t_{n_{a,min}+1}$

Note: These equations also apply if  $t'_{v_0}$  is within phase II.

$$\begin{aligned}
v(t_*) &= v_0 + \int_0^{t_1} a_0 + \frac{a_1'' - a_0}{\Delta t} \cdot t \, dt + \int_{t_1}^{t_{n_{j,max}}} a_1'' + j_{min} \cdot (t - t_1) \, dt \\
&+ \int_{t_{n_{j,max}}}^{t_{n_{j,max}+1}} a_1'' + j_{min} \cdot (t_{n_{j,max}} - t_1) + j_{n_{j,max}+1} \cdot (t - t_{n_{j,max}}) \, dt \\
&+ \int_{t_{n_{j,max}+1}}^{t_*} a_1'' + j_{min} \cdot (t_{n_{j,max}} - t_1) + j_{n_{j,max}+1} \cdot \Delta t + j_{max} \cdot (t - t_{n_{j,max}+1}) \, dt = v_{*,min}''
\end{aligned} \tag{24a}$$

$$\begin{aligned}
p(t_*) &= p_0 + \int_0^{t_1} \int_0^t a_0 + \frac{a_1'' - a_0}{\Delta t} \cdot t \, dt \, dt + \int_{t_1}^{t_{n_j, max}} \int_{t_1}^t a_1'' + j_{min} \cdot (t - t_1) \, dt \, dt \\
&+ \int_{t_{n_j, max}}^{t_{n_j, max}+1} \int_{t_{n_j, max}}^t a_1'' + j_{min} \cdot (t_{n_j, max} - t_1) + j_{n_j, max+1} \cdot (t - t_{n_j, max}) \, dt \, dt \\
&+ \int_{t_{n_j, max}+1}^{t_*} \int_{t_{n_j, max}+1}^t a_1'' + j_{min} \cdot (t_{n_j, max} - t_1) + j_{n_j, max+1} \cdot \Delta t + j_{max} \cdot (t - t_{n_j, max}+1) \, dt \, dt \\
&+ t_* \cdot v_0 + (t_* - t_1) \cdot \int_0^{t_1} a_0 + \frac{a_1'' - a_0}{\Delta t} \cdot t \, dt \\
&+ (t_* - t_{n_j, max}) \cdot \int_{t_1}^{t_{n_j, max}} a_1'' + j_{min} \cdot (t - t_1) \, dt \\
&+ (t_* - t_{n_j, max}+1) \cdot \int_{t_{n_j, max}}^{t_{n_j, max}+1} a_1'' + j_{min} \cdot (t_{n_j, max} - t_1) + j_{n_j, max+1} \cdot (t - t_{n_j, max}) \, dt \\
&= p_{max}
\end{aligned} \tag{24b}$$