電腦視覺與應用
Computer Vision and Applications

Lecture-04

Estimation for 2D projective transformations

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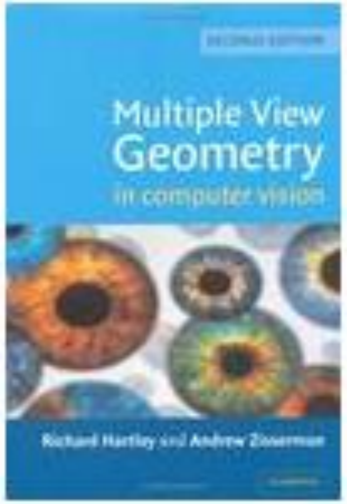
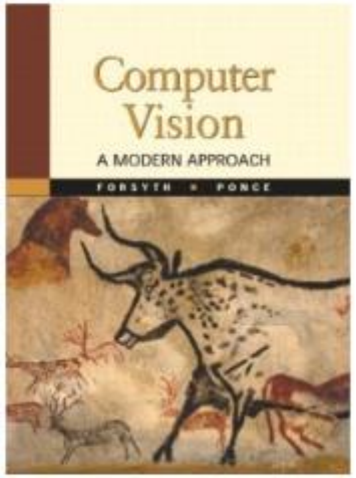
Estimation for 2D projective transformations

- Lecture Reference at:
  - Multiple View Geometry in Computer Vision, Chapter 4. (major)
  - Computer Vision A Modern Approach, (NA).





Parameter estimation

- 2D homography

  Given a set of ($\mathbf{x}_i$, $\mathbf{x}_i'$), compute $\mathbf{H}$ ($\mathbf{x}_i'=\mathbf{H}\mathbf{x}_i$)
- 3D to 2D camera projection

  Given a set of ($\mathbf{X}_i$, $\mathbf{x}_i$), compute $\mathbf{P}$ ($\mathbf{x}_i=\mathbf{P}\mathbf{X}_i$)
- Fundamental matrix

  Given a set of ($\mathbf{x}_i$, $\mathbf{x}_i'$), compute $\mathbf{F}$ ($\mathbf{x}_i'^T\mathbf{F}\mathbf{x}_i=0$)
- Trifocal tensor

  Given a set of ($\mathbf{x}_i$, $\mathbf{x}_i'$, $\mathbf{x}_i''$), compute $\mathbf{T}$

Number of measurements required

- At least as many independent equations as degrees of freedom required

Example:

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

$$\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2 independent equations / point
8 degrees of freedom

$4 \text{x} 2 \geq 8$

Approximate solutions

- Minimal solution

  4 points yield an exact solution for **H**
- More points

  No exact solution, because measurements are inexact ("noise")

  Search for "best" according to some cost function

  Algebraic or geometric/statistical cost

Gold Standard algorithm

- Cost function that is optimal for some assumptions
- Computational algorithm that minimizes it is called “Gold Standard” algorithm
- Other algorithms can then be compared to it

Direct Linear Transformation (DLT)

$$\mathbf{x}'_i = \mathbf{H}\mathbf{x}_i$$

$$\mathbf{x}'_i \times \mathbf{H}\mathbf{x}_i = 0$$ → indeed, $[0\ 0\ 0]^{\mathrm{T}}$

$$\mathbf{x}'_i = \left(x'_i, y'_i, w'_i\right)^{\mathrm{T}}$$

$$\mathbf{H}\mathbf{x}_i = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_i \\ y_i \\ w_i \end{pmatrix} = \begin{pmatrix} h_{11}x_i + h_{12}y_i + h_{13}w_i \\ h_{21}x_i + h_{22}y_i + h_{23}w_i \\ h_{31}x_i + h_{32}y_i + h_{33}w_i \end{pmatrix} = \begin{pmatrix} \mathbf{h}^{1^{\mathrm{T}}}\mathbf{x}_i \\ \mathbf{h}^{2^{\mathrm{T}}}\mathbf{x}_i \\ \mathbf{h}^{3^{\mathrm{T}}}\mathbf{x}_i \end{pmatrix}$$

$$\mathbf{H}\mathbf{x}_i = \begin{pmatrix} \mathbf{h}^{1^{\mathrm{T}}}\mathbf{x}_i \\ \mathbf{h}^{2^{\mathrm{T}}}\mathbf{x}_i \\ \mathbf{h}^{3^{\mathrm{T}}}\mathbf{x}_i \end{pmatrix}$$

here

$$\mathbf{h}^{1^{\mathrm{T}}} = [h_{11}\quad h_{12}\quad h_{13}]$$
$$\mathbf{h}^{2^{\mathrm{T}}} = [h_{21}\quad h_{22}\quad h_{23}]$$
$$\mathbf{h}^{3^{\mathrm{T}}} = [h_{31}\quad h_{32}\quad h_{33}]$$

Direct Linear Transformation (DLT)—cont.

$$\mathbf{x}_i' \times \mathbf{H}\mathbf{x}_i = 0 \rightarrow \begin{pmatrix} x_i' \\ y_i' \\ w_i' \end{pmatrix} \times \begin{pmatrix} \mathbf{h}^{1^\top}\mathbf{x}_i \\ \mathbf{h}^{2^\top}\mathbf{x}_i \\ \mathbf{h}^{3^\top}\mathbf{x}_i \end{pmatrix} = \begin{pmatrix} y_i'\mathbf{h}^{3^\top}\mathbf{x}_i - w_i'\mathbf{h}^{2^\top}\mathbf{x}_i \\ w_i'\mathbf{h}^{1^\top}\mathbf{x}_i - x_i'\mathbf{h}^{3^\top}\mathbf{x}_i \\ x_i'\mathbf{h}^{2^\top}\mathbf{x}_i - y_i'\mathbf{h}^{1^\top}\mathbf{x}_i \end{pmatrix} = 0$$

$$\rightarrow \begin{pmatrix} y_i'\mathbf{x}_i^\top\mathbf{h}^3 - w_i'\mathbf{x}_i^\top\mathbf{h}^2 \\ w_i'\mathbf{x}_i^\top\mathbf{h}^1 - x_i'\mathbf{x}_i^\top\mathbf{h}^3 \\ x_i'\mathbf{x}_i^\top\mathbf{h}^2 - y_i'\mathbf{x}_i^\top\mathbf{h}^1 \end{pmatrix} = \begin{pmatrix} 0^\top\mathbf{h}^1 - w_i'\mathbf{x}_i^\top\mathbf{h}^2 + y_i'\mathbf{x}_i^\top\mathbf{h}^3 \\ w_i'\mathbf{x}_i^\top\mathbf{h}^1 + 0^\top\mathbf{h}^2 - x_i'\mathbf{x}_i^\top\mathbf{h}^3 \\ -y_i'\mathbf{x}_i^\top\mathbf{h}^1 + x_i'\mathbf{x}_i^\top\mathbf{h}^2 + 0^\top\mathbf{h}^3 \end{pmatrix} = 0$$

Direct Linear Transformation (DLT)—cont.

$$\begin{bmatrix} \mathbf{0}^\top & -w_i'\mathbf{x}_i^\top & y_i'\mathbf{x}_i^\top \\ w_i'\mathbf{x}_i^\top & \mathbf{0}^\top & -x_i'\mathbf{x}_i^\top \\ -y_i'\mathbf{x}_i^\top & x_i'\mathbf{x}_i^\top & \mathbf{0}^\top \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} = 0$$

- Indeed, the above equation is an abbreviation of the following format:

$$\begin{bmatrix} [0 \quad 0 \quad 0] & -w_i'[x_i \quad y_i \quad w_i] & y_i'[x_i \quad y_i \quad w_i] \\ w_i'[x_i \quad y_i \quad w_i] & [0 \quad 0 \quad 0] & -x_i'[x_i \quad y_i \quad w_i] \\ -y_i'[x_i \quad y_i \quad w_i] & x_i'[x_i \quad y_i \quad w_i] & [0 \quad 0 \quad 0] \end{bmatrix} \begin{pmatrix} \begin{pmatrix} h_{11} \\ h_{12} \\ h_{13} \end{pmatrix} \\ \begin{pmatrix} h_{21} \\ h_{22} \\ h_{23} \end{pmatrix} \\ \begin{pmatrix} h_{31} \\ h_{32} \\ h_{33} \end{pmatrix} \end{pmatrix} = 0 \Rightarrow \mathbf{A}_i\mathbf{h} = 0$$

Direct Linear Transformation (DLT)—cont.

- Equations are linear in h

$$\mathbf{A}_i\mathbf{h} = 0$$

- Only 2 out of 3 are linearly independent (WHY?) (indeed, 2 eq/pt)

$$\begin{bmatrix} 0^\top & -w_i'\mathbf{x}_i^\top & y_i'\mathbf{x}_i^\top \\ w_i'\mathbf{x}_i^\top & 0^\top & -x_i'\mathbf{x}_i^\top \\ -y_i'\mathbf{x}_i^\top & x_i'\mathbf{x}_i^\top & 0^\top \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} = 0$$

- Since equation 3 (3th row) could be a linear combination of equation 1 and equation 2. Example:

Eq1: $-w_i'\mathbf{x}_i^\top\mathbf{h}^2 + y_i'\mathbf{x}_i^\top\mathbf{h}^3 = 0$
Eq2: $w_i'\mathbf{x}_i^\top\mathbf{h}^1 - x_i'\mathbf{x}_i^\top\mathbf{h}^3 = 0$
Eq3: $-y_i'\mathbf{x}_i^\top\mathbf{h}^1 + x_i'\mathbf{x}_i^\top\mathbf{h}^2 = 0$

$\rightarrow$

eliminated

$x_i'(\mathrm{Eq1}) + y_i'(\mathrm{Eq2}) = 0$

$x_i'(-w_i'\mathbf{x}_i^\top\mathbf{h}^2 + \boxed{y_i'\mathbf{x}_i^\top\mathbf{h}^3}) + y_i'(w_i'\mathbf{x}_i^\top\mathbf{h}^1 \boxed{- x_i'\mathbf{x}_i^\top\mathbf{h}^3}) = 0$

$(-w_i')(-y_i'\mathbf{x}_i^\top\mathbf{h}^1 + x_i'\mathbf{x}_i^\top\mathbf{h}^2) = 0$

$(-w_i')(\mathrm{Eq3}) = 0$

Direct Linear Transformation (DLT)—cont.

- Equations are linear in h

$$\mathbf{A}_i\mathbf{h} = 0$$

$$\begin{bmatrix} 0^\top & -w_i'\mathbf{x}_i^\top & y_i'\mathbf{x}_i^\top \\ w_i'\mathbf{x}_i^\top & 0^\top & -x_i'\mathbf{x}_i^\top \end{bmatrix}\begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} = 0$$

- Holds for any homogeneous representation, e.g. $(x_i', y_i', 1)$

Direct Linear Transformation (DLT)—cont.

- Solving for **H**

$$\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \\ \mathbf{A}_4 \end{bmatrix} \mathbf{h} = 0 \qquad \mathbf{Ah} = 0$$

size **A** is 8x9 or 12x9, but rank 8

Trivial solution is $\mathbf{h}=0_9^T$ is not interesting

1-D null-space yields solution of interest

pick for example the one with $\|\mathbf{h}\| = 1$

Direct Linear Transformation (DLT)—cont.

- Over-determined solution

$$\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_n \end{bmatrix} \mathbf{h} = 0 \qquad\qquad \mathbf{Ah} = 0$$

No exact solution because of inexact measurement

i.e. “noise”

Find approximate solution

- Additional constraint needed to avoid 0, e.g. $\|\mathbf{h}\| = 1$
- $Ah = 0$ not possible, so minimize $\|\mathbf{Ah}\|$

DLT algorithm

- Summary:

<u>Objective</u>

Given n≥4 2D to 2D point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}_i'\}$, determine the 2D homography matrix **H** such that $\mathbf{x}_i' = \mathbf{H}\mathbf{x}_i$

<u>Algorithm</u>

1) For each correspondence $\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$ compute $\mathbf{A}_i$. Usually only two first rows needed.
2) Assemble n (2 x 9) matrices $\mathbf{A}_i$ into a single ($2n$ x 9) matrix **A**
3) Obtain SVD of **A**. Solution for **h** is last column of V
4) Determine **H** from **h**

DLT algorithm—in practice:

- For ONE correspondence, you will have TWO equations:

$$\begin{bmatrix} \mathbf{0}^\top & -w'_i\mathbf{x}_i^\top & y'_i\mathbf{x}_i^\top \\ w'_i\mathbf{x}_i^\top & \mathbf{0}^\top & -x'_i\mathbf{x}_i^\top \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 & -w'_i x_i & -w'_i y_i & -w'_i w_i & y' x_i & y' y_i & y' w_i \\ w'_i x_i & w'_i y_i & w'_i w_i & 0 & 0 & 0 & -x'_i x_i & -x'_i y_i & -x'_i w_i \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

DLT algorithm—in practice:

- For n correspondence you have two equations:

$$\mathbf{Ah} = 0$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & -x_1 w_1' & -y_1 w_1' & -w_1 w_1' & x_1 y_1' & y_1 y_1' & w_1 y_1' \\ x_1 w_1' & y_1 w_1' & w_1 w_1' & 0 & 0 & 0 & -x_1 x_1' & -y_1 x_1' & -w_1 x_1' \\ \hline 0 & 0 & 0 & -x_2 w_2' & -y_2 w_2' & -w_2 w_2' & x_2 y_2' & y_2 y_2' & w_2 y_2' \\ x_2 w_2' & y_2 w_2' & w_2 w_2' & 0 & 0 & 0 & -x_2 x_2' & -y_2 x_2' & -w_2 x_2' \\ \hline & & & & \ldots\ldots & & & & \cdots \\ \hline 0 & 0 & 0 & -x_n w_n' & -y_n w_n' & -w_n w_n' & x_n y_n' & y_n y_n' & w_n y_n' \\ x_n w_n' & y_n w_n' & w_n w_n' & 0 & 0 & 0 & -x_n x_n' & -y_n x_n' & -w_n x_n' \end{bmatrix}$$

From 1st correspondence

From 2nd correspondence

From *n*-th correspondence

- Using singular value decomposition (SVD) :

$$\mathbf{A} = \mathbf{USV}^{\mathsf{T}}$$

$$\begin{pmatrix} & \mathrm{A} & \end{pmatrix} = \begin{pmatrix} & \mathrm{U} & \end{pmatrix} \cdot \begin{pmatrix} w_0 & & & \\ & w_1 & & \\ & & \cdots & \\ & & \cdots & \\ & & & w_{N-1} \end{pmatrix} \cdot \begin{pmatrix} & \mathrm{V}^T & \end{pmatrix} \qquad (2.6.2)$$

Reference book: Numerical recipes

DLT algorithm—example:

- For example: a planar card on 3D environment. Six set correspondences are detected in image-AB, image-BC and image-CA

DLT algorithm—example:—cont.

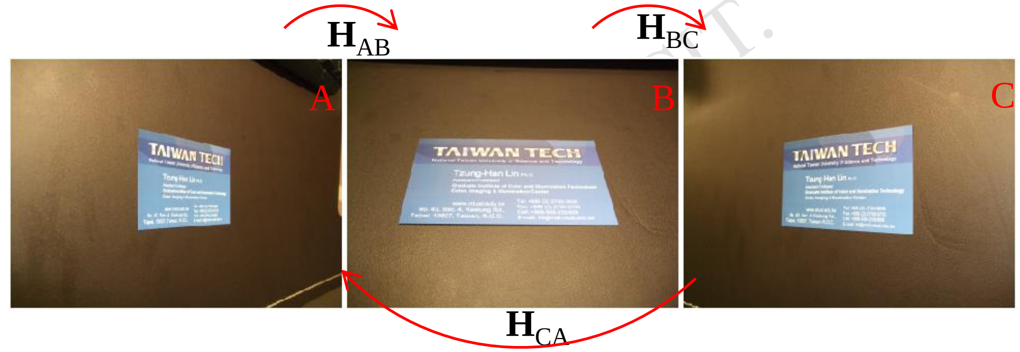
- Find all correspondences:

$pA1=[651,386,1]^T$	$pB1=[459,392,1]^T$	$pC1=[522,406,1]^T$
$pA2=[576,696,1]^T$	$pB2=[282,667,1]^T$	$pC2=[446,688,1]^T$
$pA3=[730,651,1]^T$	$pB3=[592,629,1]^T$	$pC3=[605,657,1]^T$
$pA4=[859,686,1]^T$	$pB4=[913,677,1]^T$	$pC4=[801,708,1]^T$
$pA5=[784,509,1]^T$	$pB5=[711,484,1]^T$	$pC5=[682,499,1]^T$
$pA6=[916,460,1]^T$	$pB6=[1009,424,1]^T$	$pC6=[918,402,1]^T$

DLT algorithm—example:—cont.

- Equation:

$$\mathbf{Ah} = 0$$




```
Aab=[0 0 0 -pB1(3)*pA1' pB1(2)*pA1'; pB1(3)*pA1' 0 0 0 -pB1(1)*pA1';
    0 0 0 -pB2(3)*pA2' pB2(2)*pA2'; pB2(3)*pA2' 0 0 0 -pB2(1)*pA2';
    0 0 0 -pB3(3)*pA3' pB3(2)*pA3'; pB3(3)*pA3' 0 0 0 -pB3(1)*pA3';
    0 0 0 -pB4(3)*pA4' pB4(2)*pA4'; pB4(3)*pA4' 0 0 0 -pB4(1)*pA4';
    0 0 0 -pB5(3)*pA5' pB5(2)*pA5'; pB5(3)*pA5' 0 0 0 -pB5(1)*pA5';
    0 0 0 -pB6(3)*pA6' pB6(2)*pA6'; pB6(3)*pA6' 0 0 0 -pB6(1)*pA6';]
Abc=[0 0 0 -pC1(3)*pB1' pC1(2)*pB1'; pC1(3)*pB1' 0 0 0 -pC1(1)*pB1';
    0 0 0 -pC2(3)*pB2' pC2(2)*pB2'; pC2(3)*pB2' 0 0 0 -pC2(1)*pB2';
    0 0 0 -pC3(3)*pB3' pC3(2)*pB3'; pC3(3)*pB3' 0 0 0 -pC3(1)*pB3';
    0 0 0 -pC4(3)*pB4' pC4(2)*pB4'; pC4(3)*pB4' 0 0 0 -pC4(1)*pB4';
    0 0 0 -pC5(3)*pB5' pC5(2)*pB5'; pC5(3)*pB5' 0 0 0 -pC5(1)*pB5';
    0 0 0 -pC6(3)*pB6' pC6(2)*pB6'; pC6(3)*pB6' 0 0 0 -pC6(1)*pB6';]
Aca=[0 0 0 -pA1(3)*pC1' pA1(2)*pC1'; pA1(3)*pC1' 0 0 0 -pA1(1)*pC1';
    0 0 0 -pA2(3)*pC2' pA2(2)*pC2'; pA2(3)*pC2' 0 0 0 -pA2(1)*pC2';
    0 0 0 -pA3(3)*pC3' pA3(2)*pC3'; pA3(3)*pC3' 0 0 0 -pA3(1)*pC3';
    0 0 0 -pA4(3)*pC4' pA4(2)*pC4'; pA4(3)*pC4' 0 0 0 -pA4(1)*pC4';
    0 0 0 -pA5(3)*pC5' pA5(2)*pC5'; pA5(3)*pC5' 0 0 0 -pA5(1)*pC5';
    0 0 0 -pA6(3)*pC6' pA6(2)*pC6'; pA6(3)*pC6' 0 0 0 -pA6(1)*pC6';]
```

```
Aab =
     0       0     0    -651    -386    -1   255192   151312    392
   651     386     1       0       0     0  -298809  -177174   -459
     0       0     0    -576    -696    -1   384192   464232    667
   576     696     1       0       0     0  -162432  -196272   -282
     0       0     0    -730    -651    -1   459170   409479    629
   730     651     1       0       0     0  -432160  -385392   -592
     0       0     0    -859    -686    -1   581543   464422    677
   859     686     1       0       0     0  -784267  -626318   -913
     0       0     0    -784    -509    -1   379456   246356    484
   784     509     1       0       0     0  -557424  -361899   -711
     0       0     0    -916    -460    -1   388384   195040    424
   916     460     1       0       0     0  -924244  -464140  -1009
Abc =
     0       0     0    -459    -392    -1   185436   158368    404
   459     392     1       0       0     0  -221238  -188944   -482
     0       0     0    -282    -667    -1   194016   458896    688
   282     667     1       0       0     0  -125772  -297482   -446
     0       0     0    -592    -629    -1   388944   413253    657
   592     629     1       0       0     0  -358160  -380545   -605
     0       0     0    -913    -677    -1   646404   479316    708
   913     677     1       0       0     0  -731313  -542277   -801
     0       0     0    -711    -484    -1   354789   241516    499
   711     484     1       0       0     0  -484902  -330088   -682
     0       0     0   -1009    -424    -1   405618   170448    402
  1009     424     1       0       0     0  -926262  -389232   -918
Aca =
     0       0     0    -482    -404    -1   186052   155944    386
   482     404     1       0       0     0  -313782  -263004   -651
     0       0     0    -446    -688    -1   310416   478848    696
   446     688     1       0       0     0  -256896  -396288   -576
     0       0     0    -605    -657    -1   393855   427707    651
   605     657     1       0       0     0  -441650  -479610   -730
     0       0     0    -801    -708    -1   549486   485688    686
   801     708     1       0       0     0  -688059  -608172   -859
     0       0     0    -682    -499    -1   347138   253991    509
   682     499     1       0       0     0  -534688  -391216   -784
     0       0     0    -918    -402    -1   422280   184920    460
   918     402     1       0       0     0  -840888  -368232   -916
```

DLT algorithm—example:—cont.

- Solve by SVD

$$\mathbf{Ah} = 0$$

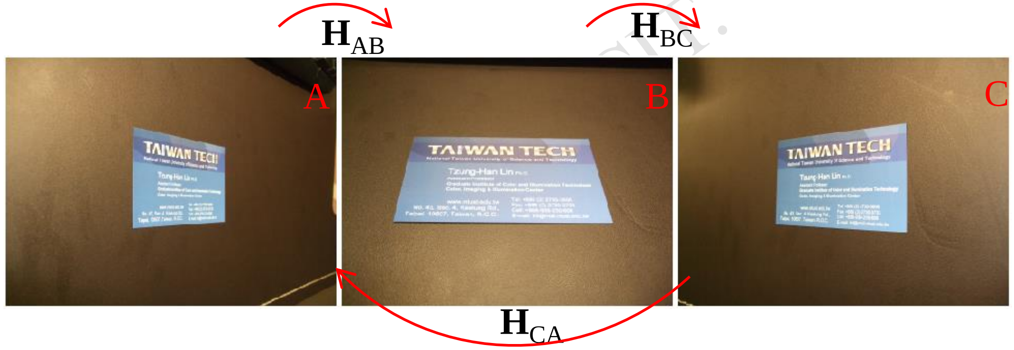
Solve by SVD method.

Example in Matlab:

```
 [U,S,V]=svd(Aab)
Hab=[V(1:3,9)';V(4:6,9)';V(7:9,9)']        →last column
```

```
   Hab =
     -0.0027    0.0007    0.6249
      0.0007   -0.0010   -0.7807
      0.0000    0.0000   -0.0031
   Hbc =
      0.0023    0.0020    0.9096
     -0.0011    0.0077   -0.4155
     -0.0000    0.0000    0.0047
   Hca =
     -0.0032   -0.0001    0.5620
     -0.0014   -0.0020    0.8271
     -0.0000   -0.0000   -0.0006
```

$$\mathbf{H}_{\mathrm{CA}} \approx (\mathbf{H}_{\mathrm{BC}}\mathbf{H}_{\mathrm{AB}})^{-1}$$ Up to scale



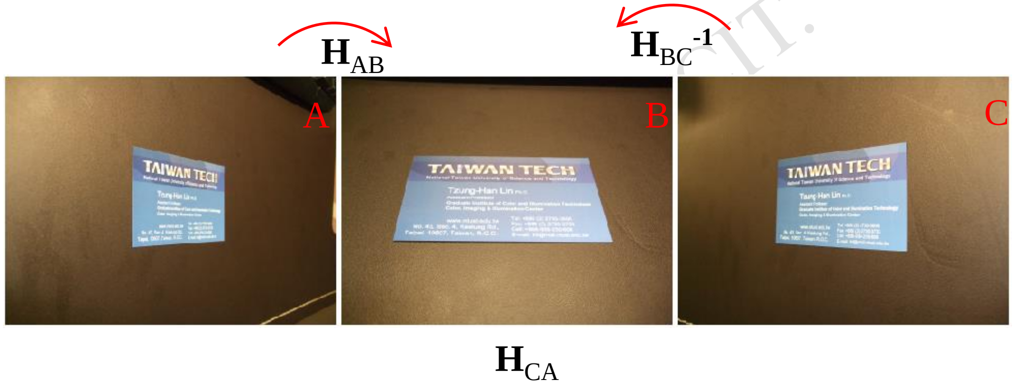
estimated points   measured points

$$\boxed{\bar{p}_B} = \mathbf{H}_{\mathrm{AB}}\boxed{(p_A)}$$
$$\boxed{\bar{p}_C} = \mathbf{H}_{\mathrm{BC}}\boxed{(p_B)}$$
$$\boxed{\bar{p}_A} = \mathbf{H}_{\mathrm{CA}}\boxed{(p_C)}$$

$$\boxed{\bar{\bar{p}}_C} = \mathbf{H}_{\mathrm{BC}}\mathbf{H}_{\mathrm{AB}}\boxed{(p_A)}$$

DLT algorithm—example:—cont.

- Verify points



Measurement

$pB1=[459,392,1]^T$
$pB2=[282,667,1]^T$
$pB3=[592,629,1]^T$
$pB4=[913,677,1]^T$
$pB5=[711,484,1]^T$
$pB6=[1009,424,1]^T$

Estimation from image A to B

$H_{AB}*pA1=$	459.3547	391.963
$H_{AB}*pA2=$	281.7844	667.7953
$H_{AB}*pA3=$	593.3631	628.0585
$H_{AB}*pA4=$	912.6841	677.0122
$H_{AB}*pA5=$	708.9633	483.3161
$H_{AB}*pA6=$	1009.858	424.8571

Note: 3rd element is 1

Estimation from image C to B

$H_{BC}^{-1*}pC1=$	458.6791	392.246
$H_{BC}^{-1*}pC2=$	282.0898	667.0208
$H_{BC}^{-1*}pC3=$	592.38	629.2953
$H_{BC}^{-1*}pC4=$	912.6669	677.023
$H_{BC}^{-1*}pC5=$	710.8009	483.0512
$H_{BC}^{-1*}pC6=$	1009.367	424.3639

Note: 3rd element is 1

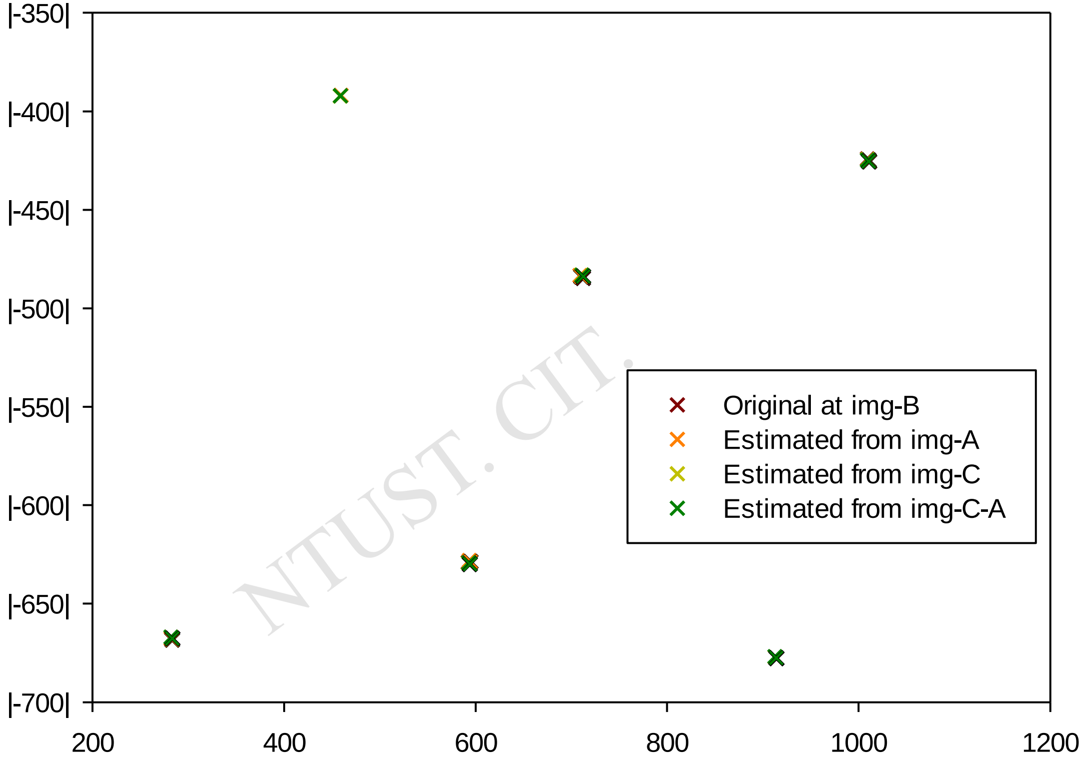
Estimation from images C to A to B

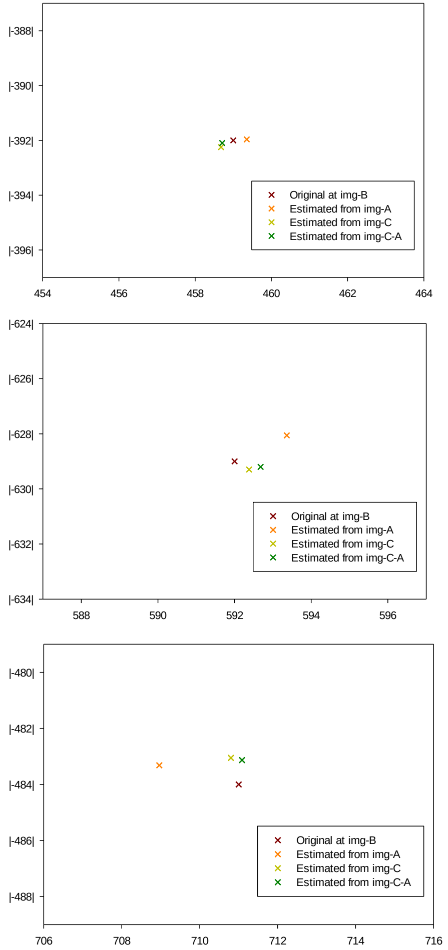
$H_{AB}{}^{*}H_{CA}{}^{*}pC1=$	458.7103	392.0953
$H_{AB}{}^{*}H_{CA}{}^{*}pC2=$	281.851	666.9389
$H_{AB}{}^{*}H_{CA}{}^{*}pC3=$	592.6809	629.201
$H_{AB}{}^{*}H_{CA}{}^{*}pC4=$	912.5832	676.8083
$H_{AB}{}^{*}H_{CA}{}^{*}pC5=$	711.0876	483.1301
$H_{AB}{}^{*}H_{CA}{}^{*}pC6=$	1009.163	424.7749

Note: 3rd element is 1

DLT algorithm—example:—cont.

- Reprojection error:





DLT algorithm—example:—cont.

- With comparison to openCV result.

Matlab (using SVD)

Hab =

0.8816	-0.2139	-204.4555
-0.2171	0.3386	255.4139
-0.0004	-0.0003	1.0000

Hbc =

0.4706	0.4011	199.6538
-0.2304	1.5985	-75.7620
-0.0004	0.0007	1.0000

Hca =

1.0e+003 *

0.0058	0.0002	-1.0419
0.0025	0.0036	-1.5110
0.0000	0.0000	0.0010

OpenCV (stored as float)

Homography Matrix (A to B):
0.879630 -0.214684 -203.041306
-0.217263 0.337555 255.723053
-0.000377 -0.000339 1.000000

Homography Matrix (B to C):
0.471623 0.402092 199.173584
-0.230184 1.600397 -76.327538
-0.000360 0.000672 1.000000

Homography Matrix (C to A):
5.713303 0.233439 -1023.777222
2.430560 3.548679 -1491.053589
0.003936 0.000247 1.000000

OpenCV (stored as double)

Homography Matrix (A to B):
0.879630 -0.214684 -203.041299
-0.217263 0.337555 255.723051
-0.000377 -0.000339 1.000000

Homography Matrix (B to C):
0.471623 0.402092 199.173589
-0.230184 1.600397 -76.327540
-0.000360 0.000672 1.000000

Homography Matrix (C to A):
5.713303 0.233439 -1023.777224
2.430560 3.548679 -1491.053634
0.003936 0.000247 1.000000

Inhomogeneous solution

- Since **h** can only be computed up to scale, pick h_{33}=1, and solve for 8-vector

$$\begin{bmatrix} 0 & 0 & 0 & -x_i w_i' & -y_i w_i' & -w_i w_i' & x_i y_i' & y_i y_i' \\ x_i w_i' & y_i w_i' & w_i w_i' & 0 & 0 & 0 & x_i x_i' & y_i x_i' \end{bmatrix} \tilde{\mathbf{h}} = \begin{pmatrix} -w_i y_i' \\ w_i x_i' \end{pmatrix}$$

- Solve using Gaussian elimination (4 points) or using linear least-squares (more than 4 points)
- However, if h_{33}=0 this approach fails also poor results if h_9 close to zero. Therefore, not recommended.
- Note h_{33}=0 if origin is mapped to infinity $\mathbf{l}_\infty^{\mathsf{T}} \mathbf{H} \mathbf{x}_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{H} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$

Inhomogeneous solution—cont.

- Least square method: (n>4)

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -x_1w_1' & -y_1w_1' & -w_1w_1' & x_1y_1' & y_1y_1' \\ x_1w_1' & y_1w_1' & w_1w_1' & 0 & 0 & 0 & x_1x_1' & y_1x_1' \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 & -x_2w_2' & -y_2w_2' & -w_2w_2' & x_2y_2' & y_2y_2' \\ x_2w_2' & y_2w_2' & w_2w_2' & 0 & 0 & 0 & x_2x_2' & y_2x_2' \end{bmatrix} \\ \cdots \\ \begin{bmatrix} 0 & 0 & 0 & -x_nw_n' & -y_nw_n' & -w_nw_n' & x_ny_n' & y_ny_n' \\ x_nw_n' & y_nw_n' & w_nw_n' & 0 & 0 & 0 & x_nx_n' & y_nx_n' \end{bmatrix} \end{bmatrix} \tilde{\mathbf{h}} = \begin{pmatrix} -w_1y_1' \\ -w_1x_1' \\ -w_2y_2' \\ -w_2x_2' \\ \cdots \\ -w_ny_n' \\ -w_nx_n' \end{pmatrix} \begin{array}{l} \text{From 1st correspondence} \\ \\ \text{From 2nd correspondence} \\ \\ \cdots \\ \\ \text{From } n\text{-th correspondence} \end{array}$$

- Rewrite as a matrix form:

$$\mathbf{N}_{2n\times 8}\tilde{\mathbf{h}}_{8\times 1} = \mathbf{d}_{2n\times 1}$$

Inhomogeneous solution—cont.

- Solution for the matrix

known (given, or calculated by feature detection algorithms)

$$\mathbf{N}_{2n\times 8}\tilde{\mathbf{h}}_{8\times 1} = \mathbf{d}_{2n\times 1}$$

unknown (to be determined)

- Apply a transpose of N to the equation : $[\mathbf{N}^{\mathrm{T}}]_{8\times 2n}[\mathbf{N}]_{2n\times 8}\tilde{\mathbf{h}}_{8\times 1} = [\mathbf{N}^{\mathrm{T}}]_{8\times 2n}\mathbf{d}_{2n\times 1}$

- Let:

$$\mathbf{M}_{8\times 8} = [\mathbf{N}^{\mathrm{T}}]_{8\times 2n}[\mathbf{N}]_{2n\times 8}$$

$$\tilde{\mathbf{d}}_{8\times 1} = [\mathbf{N}^{\mathrm{T}}]_{8\times 2n}\mathbf{d}_{2n\times 1}$$

- Then,

$$\mathbf{M}_{8\times 8}\tilde{\mathbf{h}}_{8\times 1} = \tilde{\mathbf{d}}_{8\times 1}$$

$$\tilde{\mathbf{h}}_{8\times 1} = [\mathbf{M}^{-1}]_{8\times 8}\tilde{\mathbf{d}}_{8\times 1} = \left([\mathbf{N}^{\mathrm{T}}]_{8\times 2n}[\mathbf{N}]_{2n\times 8}\right)^{-1}[\mathbf{N}^{\mathrm{T}}]_{8\times 2n}\mathbf{d}_{2n\times 1}$$

Inhomogeneous solution—example

- For example (the same with previous example):

$$\mathbf{N}_{2n\times 8}\tilde{\mathbf{h}}_{8\times 1} = \mathbf{d}_{2n\times 1}$$

Nab =

0	0	0	-651	-386	-1	255192	151312
651	386	1	0	0	0	-298809	-177174
0	0	0	-576	-696	-1	384192	464232
576	696	1	0	0	0	-162432	-196272
0	0	0	-730	-651	-1	459170	409479
730	651	1	0	0	0	-432160	-385392
0	0	0	-859	-686	-1	581543	464422
859	686	1	0	0	0	-784267	-626318
0	0	0	-784	-509	-1	379456	246356
784	509	1	0	0	0	-557424	-361899
0	0	0	-916	-460	-1	388384	195040
916	460	1	0	0	0	-924244	-464140

dab =

-392
459
-667
282
-629
592
-677
913
-484
711
-424
1009

$$\tilde{\mathbf{h}}_{8\times 1} = \left([\mathbf{N}^{\mathrm{T}}]_{8\times 2n}[\mathbf{N}]_{2n\times 8}\right)^{-1}[\mathbf{N}^{\mathrm{T}}]_{8\times 2n}\mathbf{d}_{2n\times 1}$$

Implement in Matlab:

```
hab=inv(Nab'*Nab)*Nab'*dab
```

hab =

0.8803
-0.2141
-203.6746
-0.2172
0.3381
255.5480
-0.0004
-0.0003

Hab =

0.8803	-0.2141	-203.6746
-0.2172	0.3381	255.5480
-0.0004	-0.0003	1.0000

Inhomogeneous solution—example

- Compare result with SVD method:

Matlab (using inhomogenous sol.)

Hab =

0.8803	-0.2141	-203.6746
-0.2172	0.3381	255.5480
-0.0004	-0.0003	1.0000

Hbc =

0.4710	0.4017	199.4039
-0.2304	1.5995	-76.0091
-0.0004	0.0007	1.0000

Hca =

1.0e+003 *

0.0056	0.0002	-1.0019
0.0024	0.0035	-1.4654
0.0000	0.0000	0.0010

Matlab (using SVD)

Hab =

0.8816	-0.2139	-204.4555
-0.2171	0.3386	255.4139
-0.0004	-0.0003	1.0000

Hbc =

0.4706	0.4011	199.6538
-0.2304	1.5985	-75.7620
-0.0004	0.0007	1.0000

Hca =

1.0e+003 *

0.0058	0.0002	-1.0419
0.0025	0.0036	-1.5110
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OpenCV (stored as float)

Homography Matrix (A to B):
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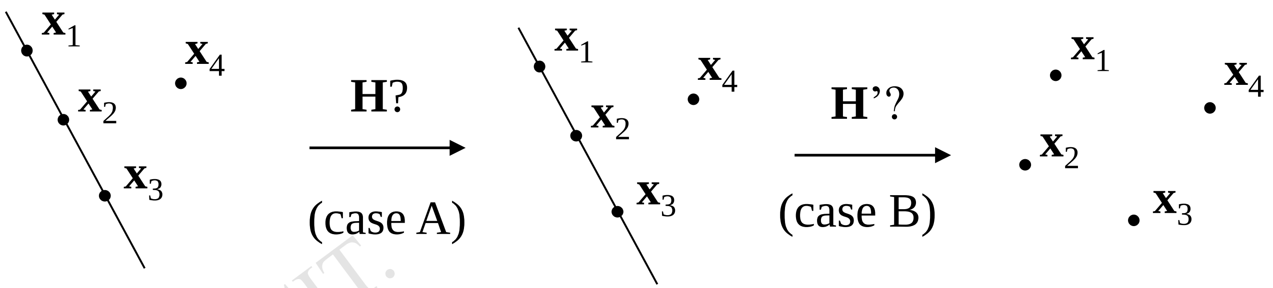
-0.000360 0.000672 1.000000

Homography Matrix (C to A):
5.713303 0.233439 -1023.777222
2.430560 3.548679 -1491.053589
0.003936 0.000247 1.000000

Degenerate configurations

- Sometimes, degeneration happens…



Cost functions (for evaluation)

- Algebraic distance
- Geometric distance
- Re-projection error

Cost functions (for evaluation)

- Algebraic distance

DLT minimizes $\|\mathbf{Ah}\|$

$\boldsymbol{\varepsilon} = \mathbf{Ah}$ residual vector

algebraic error vector

$\boldsymbol{\varepsilon}_i$ partial vector for each ($\mathbf{x}_i \leftrightarrow \mathbf{x}_i'$)

algebraic distance

$$\sum_i d_{\text{alg}}(\mathbf{x}_i', \mathbf{H}\mathbf{x}_i)^2 = \sum_i \|\boldsymbol{\varepsilon}_i\|^2 = \|\mathbf{Ah}\|^2 = \|\boldsymbol{\varepsilon}\|^2$$

Not geometrically/statistically meaningful, but given good normalization it works fine and is very fast (use for initialization)

Cost functions (for evaluation)

- Algebraic distance—cont.

DLT –original format

(one correspondence)

$$\begin{bmatrix} 0 & 0 & 0 & -w_i'x_i & -w_i'y_i & -w_i'w_i & y'x_i & y'y_i & y'w_i \\ w_i'x_i & w_i'y_i & w_i'w_i & 0 & 0 & 0 & -x_i'x_i & -x_i'y_i & -x_i'w_i \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

governing equation,
or so called constraint

unknown (to be determined)

known (given, or by feature detection)

known (has been determined)

evaluated cost values

Algebraic distance

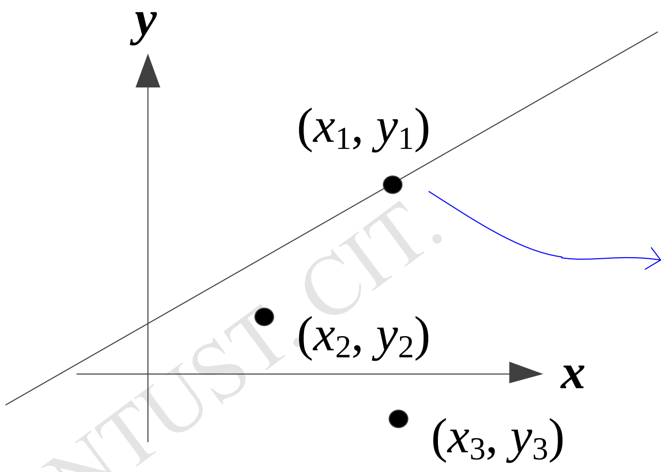
(one correspondence)

$$\begin{bmatrix} 0 & 0 & 0 & -w_i'x_i & -w_i'y_i & -w_i'w_i & y'x_i & y'y_i & y'w_i \\ w_i'x_i & w_i'y_i & w_i'w_i & 0 & 0 & 0 & -x_i'x_i & -x_i'y_i & -x_i'w_i \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \\ h_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xi} \\ \varepsilon_{yi} \end{bmatrix} = \boldsymbol{\varepsilon}_i$$

Cost function

Cost functions (for evaluation)

- Algebraic distance—cont.
  - A very simple example by the line equation



line equation:

$$ax + by + c = 0$$

$$ax_1 + by_1 + c = 0$$

$$ax_2 + by_2 + c = \varepsilon_2 \neq 0$$

$$ax_3 + by_3 + c = \varepsilon_3 \neq 0$$

Cost functions (for evaluation)

- Geometric distance
  - Error in one image

$$\sum_i d\left(\mathbf{x}'_i, \mathbf{H}\tilde{\mathbf{x}}_i\right)^2$$

  - Symmetric transfer error

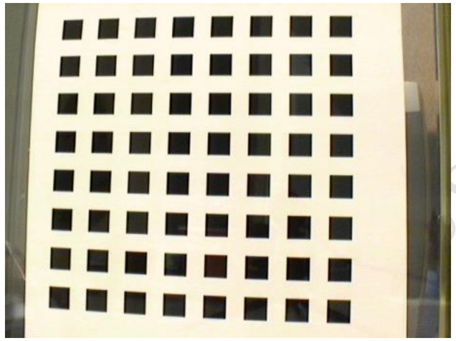
$$\sum_i [d(\mathbf{x}_i, \mathbf{H}^{-1}\mathbf{x}'_i)^2 + d(\mathbf{x}'_i, \mathbf{H}\mathbf{x}_i)^2]$$

  - Reprojection error

$$\sum_i [d(\mathbf{x}_i, \hat{\mathbf{x}}_i)^2 + d(\mathbf{x}'_i, \hat{\mathbf{x}}'_i)^2]$$

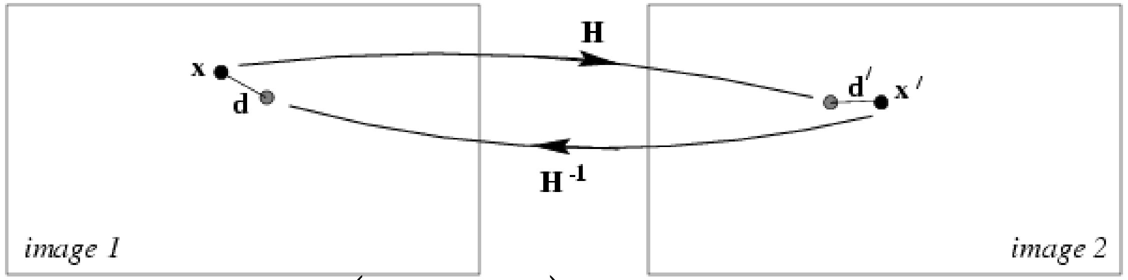
$\mathbf{x}$ measured coordinates
$\hat{\mathbf{x}}$ estimated coordinates
$\tilde{\mathbf{x}}$ true coordinates
$d(.,.)$ Euclidean distance (in image)

e.g. calibration pattern

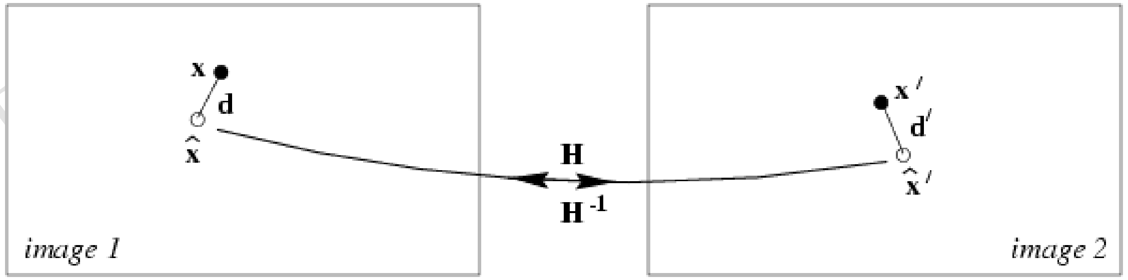


Cost functions (for evaluation)

- Geometric distance
  - Reprojection error



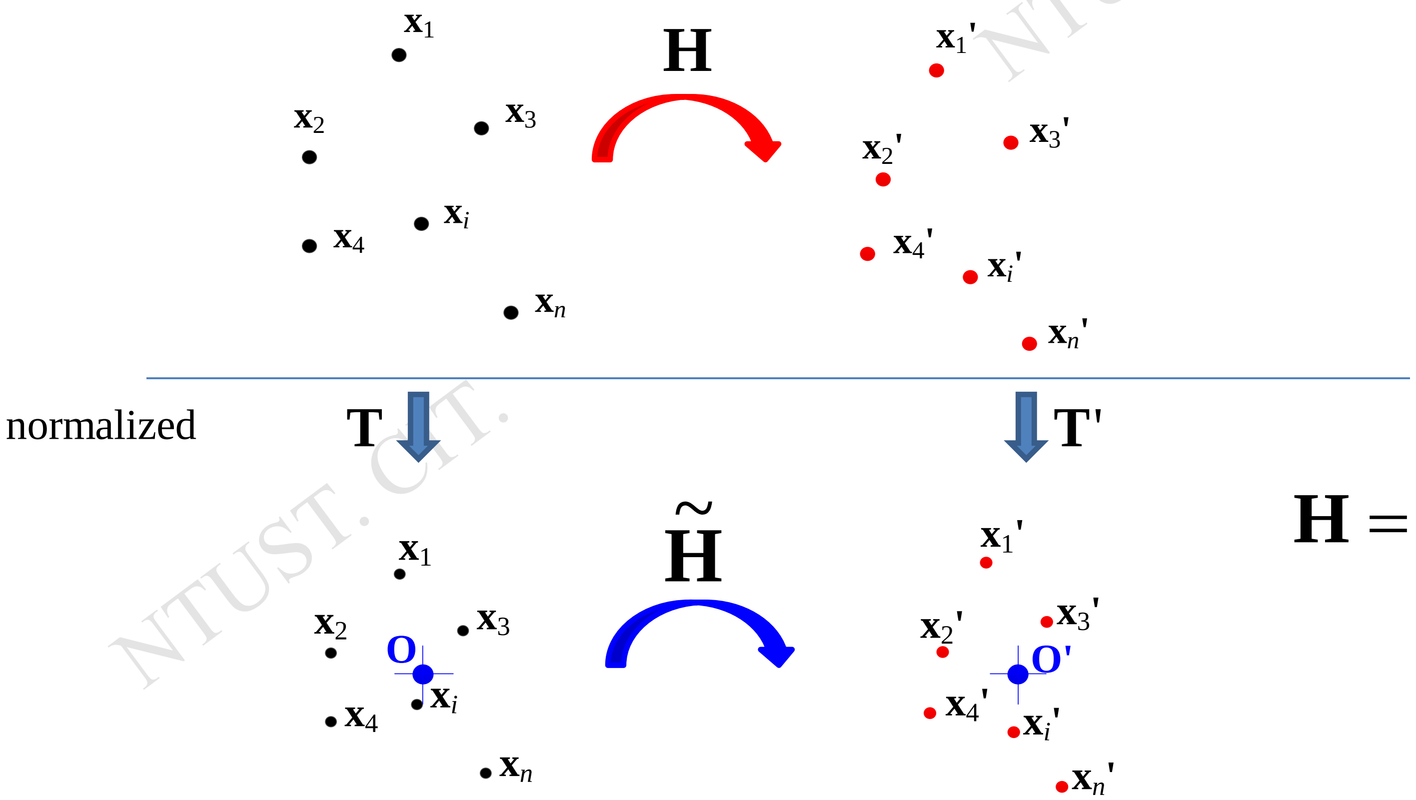
$$d\left(\mathbf{x}, \mathbf{H}^{-1}\mathbf{x}'\right)^2 + d\left(\mathbf{x}', \mathbf{H}\mathbf{x}\right)^2$$



$$d\left(\mathbf{x}, \hat{\mathbf{x}}\right)^2 + d\left(\mathbf{x}', \hat{\mathbf{x}}'\right)^2$$

Normalizing transformations

- To have a better solution for DLT algorithm (much stable)



$$\mathbf{H} = \mathbf{T}'^{-1}\tilde{\mathbf{H}}\mathbf{T}$$

Normalizing transformations—cont.

- What the **T** means?:
  - A composition of "Translation" and "Scale".
  - Indeed,

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}_{Normalized} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\bar{x} \\ 0 & 1 & -\bar{y} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}_{Original}$$

$$\mathbf{T} = \begin{bmatrix} s_x & 0 & -s_x\bar{x} \\ 0 & s_y & -s_y\bar{y} \\ 0 & 0 & 1 \end{bmatrix}$$

(i) The points are translated so that their centroid is at the origin.
(ii) The points are then scaled so that the average distance from the origin is equal to $\sqrt{2}$.
(iii) This transformation is applied to each of the two images independently.

$(\bar{x}, \bar{y}) \longrightarrow$ centroid of all points

$(s_x, s_y) \longrightarrow$ scale (could be s_x=s_y), and be suggested to be $\sqrt{2}/l$

l is the average distance to centroid.

Normalized DLT algorithm

<u>Objective</u>

Given n≥4 2D to 2D point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}_i'\}$, determine the 2D homography matrix $\mathbf{H}$ such that $\mathbf{x}_i' = \mathbf{H}\mathbf{x}_i$

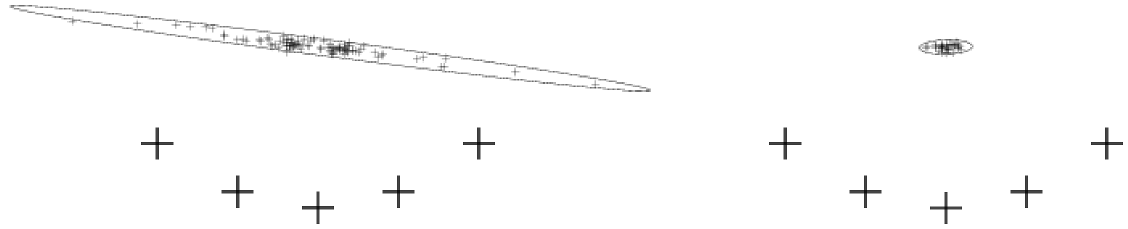
<u>Algorithm</u>

(i) Normalize points, here we have and

(ii) Apply DLT algorithm to determine

(iii) Denormalize solution, then get

Algorithm 4.2 [Hartley04]



RANSAC (RANdom SAmple Consensus)

- A robust estimation

<u>Objective</u>

Robust fit of model to data set S which contains outliers

<u>Algorithm</u>

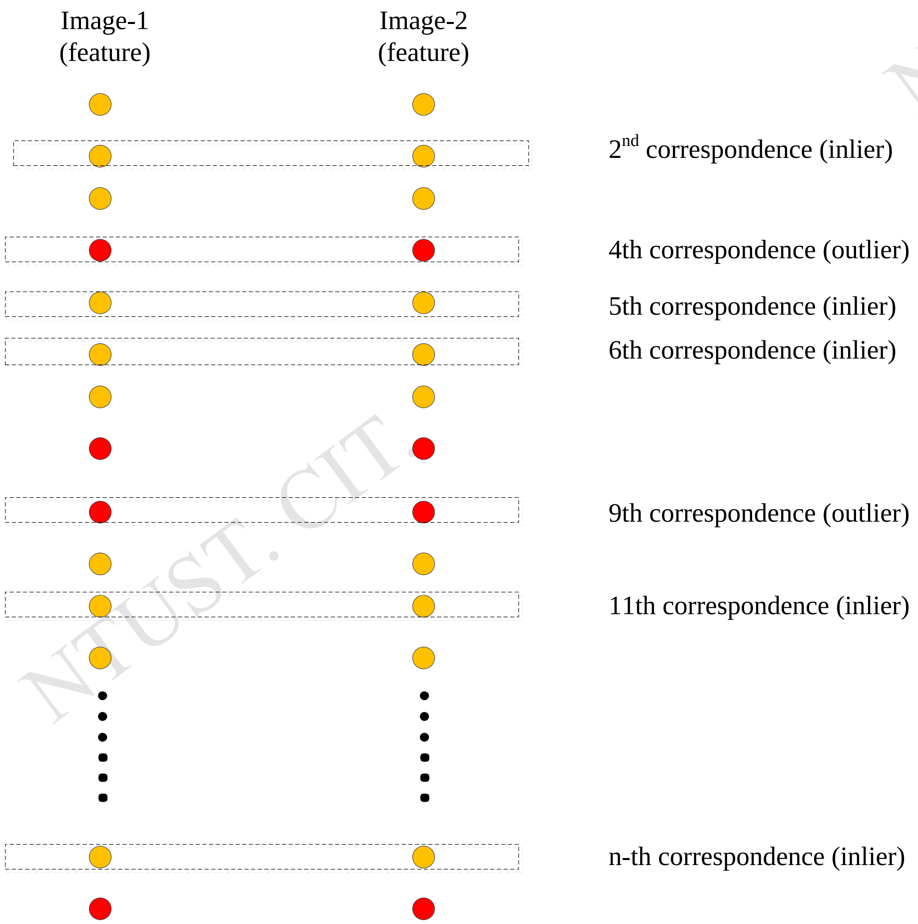
1) Randomly select a sample of *s* data points from S and instantiate the model from this subset.
2) Determine the set of data points S_i which are within a distance threshold *t* of the model. The set S_i is the consensus set of samples and defines the inliers of *S*.
3) If the subset of S_i (the number of inliers) is greater than some threshold *T*, re-estimate the model using all the points in S_i and terminate
4) If the size of S_i is less than *T*, select a new subset and repeat the above.
5) After *N* trials the largest consensus set $\mathrm{S_i}$ is selected, and the model is re-estimated using all the points in the subset S_i

Algorithm 4.4 [Hartley04]

Fischler, M.A. and Bolles, R.C. 1981. Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. *Communications of the ACM*. 24, 6 (1981).

RANSAC—cont. (RANdom SAmple Consensus)

For example, randomly select 7 correspondences, then calculate homography (H).



RANSAC—cont. (RANdom SAmple Consensus)

- Let p be the probability that the RANSAC algorithm in some iteration selects only inliers from the input data set when it chooses the n points from which the model parameters are estimated. Let w be the probability of choosing an inlier each time a single point is selected. And let k be the maximum number of iterations allowed in the algorithm

$$1 - p = (1 - w^n)^k$$

RANSAC—cont.
(RANdom SAmple Consensus)

- Summary:
  - Two thresholds are assigned
    - Distance of error (誤差距離)
    - Number of inliers (正確的對應點數量)
  - Randomly select correspondences
  - Unique solution ?

Automatic estimation of a homography (RANSAC)

Objective

Compute the 2D homography between two images.

Algorithm

(i) **Interest points:** Compute interest points in each image.
(ii) **Putative correspondences:** Compute a set of interest point matches based on proximity and similarity of their intensity neighbourhood.
(iii) **RANSAC robust estimation:** Repeat for N samples, where N is determined adaptively as in algorithm 4.5:
  (a) Select a random sample of 4 correspondences and compute the homography $\mathtt{H}$.
  (b) Calculate the distance $d_\perp$ for each putative correspondence.
  (c) Compute the number of inliers consistent with $\mathtt{H}$ by the number of correspondences for which $d_\perp < t = \sqrt{5.99}\,\sigma$ pixels.

  Choose the $\mathtt{H}$ with the largest number of inliers. In the case of ties choose the solution that has the lowest standard deviation of inliers.
(iv) **Optimal estimation:** re-estimate $\mathtt{H}$ from all correspondences classified as inliers, by minimizing the ML cost function (4.8–*p*95) using the Levenberg–Marquardt algorithm of section A6.2(*p*600).
(v) **Guided matching:** Further interest point correspondences are now determined using the estimated $\mathtt{H}$ to define a search region about the transferred point position.

The last two steps can be iterated until the number of correspondences is stable.

Algorithm 4.6. *Automatic estimation of a homography between two images using RANSAC.*

Algorithm 4.6 [Hartley04]

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