

# Near-consistent robust estimations of moments for unimodal distributions

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**Descriptive statistics for parametric models currently rely heavily on the accuracy of distributional assumptions. Here, based on the invariant structures of unimodal distributions, a series of sophisticated yet efficient estimators, robust to both gross errors and departures from parametric assumptions, are proposed for estimating mean and central moments with insignificant asymptotic biases for common continuous unimodal distributions. This article also illuminates the understanding of the common nature of probability distributions and the measures of them.**

orderliness | invariant | unimodal | adaptive estimation |  $U$ -statistics

The asymptotic inconsistencies between sample mean ( $\bar{x}$ ) and nonparametric robust location estimators in asymmetric distributions on the real line have been noticed for more than two centuries (1) but unsolved. Strictly speaking, it is unsolvable because by trimming, some information about the original distribution is removed, making it impossible to estimate the values of the removed parts without distributional assumptions. Newcomb (1886, 1912) provided the first modern approach to this problem by developing a class of estimators that gives "less weight to the more discordant observations" (2, 3). In 1964, Huber (4) used the minimax procedure to obtain M-estimator for contaminated normal distribution, which has played a pre-eminent role in the later development of robust statistics. As previously demonstrated, under growing asymmetric departures from normality, the bias of the Huber M-estimator increases rapidly. This is a common issue in parametric estimations. For example, He and Fung (1999) constructed (5) a robust M-estimator for the two-parameter Weibull distribution. All moments can be calculated from its estimated parameters. As expected, it is inadequate for the gamma, Perato, lognormal, and especially the generalized Gaussian distributions, because the logarithmic function does not produce a result for negative inputs (SI Dataset S1). Instead of minimizing the residuals, another old and interesting approach is arithmetically computing the parameters using one or more  $L$ -statistics as input values, e.g., the percentile estimators. Examples for the Weibull distribution, the reader is referred to Menon (1963) (6), Dubey (1967) (7), Hassanein (1971) (8), Marks (2005) (9), and Boudt, Caliskan, and Croux (2011) (10)'s works. At the outset of the study of percentile estimators, it was clear that this class of estimators arithmetically utilizes the invariant structures of probability distributions (6, 11, 12). Maybe it can be named as  $I$ -statistics. Formally, an estimator is classified as an  $I$ -statistic if asymptotically it satisfies  $I(WA_1, \dots, WA_l) = (\theta_1, \dots, \theta_q)$  for the distribution it is consistent with, where WAs are weighted averages,  $\theta$ s are the population parameters it estimates. If the function  $I$  is solely defined through addition and/or subtraction, it is also an  $L$ -statistic. In the previous article, it is shown that quantile average is fundamental for all weighted

averages. Based on the quantile function,  $I$ -statistic is naturally robust. For many parametric distributions, the quantile functions are much more elegant than the pdfs and cdfs. So  $I$ -statistics are often analytically obtainable. However, the performance of the above examples is often worse than that of the robust  $M$ -statistics when the distributional assumption is violated (SI Dataset S1). Even when distributions such as the Weibull and gamma belong to the same larger family, the generalized gamma distribution, a misassumption can still result in substantial biases, rendering the approach ill-suited.

In previous work on semiparametric robust mean estimation, although greatly shrinking the asymptotic biases, binomial mean ( $BM_\epsilon$ ) is still inconsistent for any skewed distribution if  $\epsilon > 0$  (if  $\epsilon \rightarrow 0$ , since the alternating sum of binomial coefficients is zero,  $BM \rightarrow \mu$ ). All robust location estimators commonly used are symmetric due to the universality of the symmetric distributions. One can construct an asymmetric trimmed mean that is consistent for a semiparametric class of skewed distributions. This approach was investigated previously, but it is not symmetric and therefore only suitable for some special applications (13). From semiparametric to parametric, an ideal robust location estimator would have a non-sample-dependent breakdown point (defined in Subsection ??) and be consistent with any symmetric distribution and a skewed distribution with finite second moments. This is called an invariant mean. Based on the mean-symmetric weighted average-median inequality, the recombined mean is defined as

$$rm_{d,\epsilon,n} := \lim_{c \rightarrow \infty} \left( \frac{(SWA_{\epsilon,n} + c)^{d+1}}{(\text{median} + c)^d} - c \right),$$

where  $d$  is for bias correction,  $SWA_{\epsilon,n}$  is  $BM_{\epsilon,n}$  in the first three Subsections, while other symmetric weighted averages can also be used in practice as long as the inequalities hold. The next theorem shows the significance of this composite estimator.

## Significance Statement

Bias, variance, and contamination are the three main errors in statistics. Consistent robust estimation is unattainable without parametric assumptions. Here, based on a paradigm shift inspired by mean-median-mode inequality, Bickel-Lehmann spread, and adaptive estimation, invariant moments are proposed as a means of achieving near-consistent and robust estimations of moments, even in scenarios where moderate violations of distributional assumptions occur, while the variances are sometimes smaller than those of the sample moments.

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**Theorem .1.** If the second moments are finite,  $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$  is a consistent mean estimator for the exponential and any symmetric distributions and the Pareto distribution with quantile function  $Q(p) = x_m(1-p)^{-\frac{1}{\alpha}}$ ,  $x_m > 0$ , when  $\alpha \rightarrow \infty$ .

*Proof.* Finding  $d, \epsilon$  values that make  $rm_{d, \epsilon}$  a consistent mean estimator is equivalent to finding the solution of  $E[rm_{d, \epsilon}] = E[X]$ . Rearranging the definition,  $rm_{d, \epsilon} = \lim_{c \rightarrow \infty} \left( \frac{(BM_{\epsilon} + c)^{d+1}}{(median + c)^d} - c \right) = (d+1)BM_{\epsilon} - dmedian = \mu$ . So,  $d = \frac{\mu - BM_{\epsilon}}{BM_{\epsilon} - median}$ . The pdf of the exponential distribution is  $f(x) = \lambda^{-1}e^{-\lambda^{-1}x}$ ,  $\lambda \geq 0$ ,  $x \geq 0$ , the cdf is  $F(x) = 1 - e^{-\lambda^{-1}x}$ ,  $x \geq 0$ . The quantile function is  $Q(p) = \ln\left(\frac{1}{1-p}\right)\lambda$ .  $E[x] = \lambda$ .  $E[median] = Q\left(\frac{1}{2}\right) = \ln 2\lambda$ . For the exponential distribution, the expectation of  $BM_{\frac{1}{8}}$  is  $E[BM_{\frac{1}{8}}] = \lambda \left(1 + \ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)$ . Obviously, the scale parameter  $\lambda$  can be canceled out,  $d \approx 0.375$ . The proof of the second assertion follows directly from the coincidence property. For any symmetric distribution with a finite second moment,  $E[BM_{\epsilon}] = E[median] = E[X]$ . Then  $E[rm_{d, \epsilon}] = \lim_{c \rightarrow \infty} \left( \frac{(E[X] + c)^{d+1}}{(E[X] + c)^d} - c \right) = E[X]$ . The proof for the Pareto distribution is more general. The mean of the Pareto distribution is given by  $\frac{\alpha x_m}{\alpha - 1}$ . The  $d$  value with two unknown percentiles  $p_1$  and  $p_2$  for the Pareto distribution is  $d_{Pareto} = \frac{\frac{\alpha x_m}{\alpha - 1} - x_m(1-p_1)^{-\frac{1}{\alpha}}}{x_m(1-p_1)^{-\frac{1}{\alpha}} - x_m(1-p_2)^{-\frac{1}{\alpha}}}$ . Since any weighted average can be expressed as an integral of the quantile function,  $\lim_{\alpha \rightarrow \infty} \frac{\frac{\alpha}{\alpha-1} - (1-p_1)^{-1/\alpha}}{(1-p_1)^{-1/\alpha} - (1-p_2)^{-1/\alpha}} = -\frac{\ln(1-p_1)+1}{\ln(1-p_1)-\ln(1-p_2)}$ , the  $d$  value for the Pareto distribution approaches that of the exponential distribution as  $\alpha \rightarrow \infty$ , regardless of the type of weighted average used. This completes the demonstration.  $\square$

Theorem .1 implies that for the Weibull, gamma, Pareto, lognormal and generalized Gaussian distribution,  $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$  is consistent for at least one particular case of these two-parameter distributions. The biases of  $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$  for distributions with skewness between those of the exponential and symmetric distributions are tiny (SI Dataset S1).  $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$  has excellent performance for all these common unimodal distributions (SI Dataset S1).

Besides introducing the concept of invariant mean, the purpose of this paper is to demonstrate that the estimation of central moments can be transformed into a location estimation problem by using  $U$ -statistics, the central moment kernel distributions have nice properties, and, in light of previous works, a series of sophisticated yet efficient robust estimators can be constructed whose biases are typically smaller than the variances ( $n = 5400$ , Table ??) for unimodal distributions.

## Background and Main Results

**A. Invariant mean.** It has long been known that a theoretical model can be adjusted to fit the first two moments of the observed data. A continuous distribution belonging to a location-scale family has the form  $F(x) = F_0\left(\frac{x-\mu}{\lambda}\right)$ , where  $F_0$  is a "standard" distribution. Then,  $F(x) = Q^{-1}(x) \rightarrow x = Q(p) = \lambda Q_0(p) + \mu$ . So, any weighted average can be expressed as  $\lambda WA_0(\epsilon) + \mu$ , where  $WA_0(\epsilon)$  is an integral of  $Q_0(p)$  according to the definition of the weighted average. The simultaneous

cancellation of  $\mu$  and  $\lambda$  in  $\frac{(\lambda\mu_0 + \mu) - (\lambda BM_0(\epsilon) + \mu)}{(\lambda BM_0(\epsilon) + \mu) - (\lambda median_0 + \mu)}$  ensures that  $d$  is a constant. Consequently, the roles of  $BM_{\epsilon}$  and median in  $rm_{d, \epsilon}$  can be replaced by any weighted averages, although for the definition of invariant mean, only symmetric weighted averages are considered here.

The performance in heavy-tailed distributions can be improved further by constructing the quantile mean as

$$qm_{d, \epsilon, n} := \hat{Q}_n \left( \left( \hat{F}_n(SWA_{\epsilon, n}) - \frac{1}{2} \right) d + \hat{F}_n(SWA_{\epsilon, n}) \right),$$

provided that  $\hat{F}_n(SWA_{\epsilon, n}) \geq \frac{1}{2}$ , where  $\hat{F}_n(x)$  is the empirical cumulative distribution function of the sample,  $\hat{Q}_n$  is the sample quantile function. The most popular method for computing the sample quantile function was proposed by Hyndman and Fan in 1996 (14). To minimize the finite sample bias, here,  $\hat{F}_n(x) := \frac{1}{n} \left( \frac{x - Q_n\left(\frac{sp}{n}\right)}{Q_n\left(\frac{1}{n}(sp+1)\right) - Q_n\left(\frac{sp}{n}\right)} + sp \right)$ , where  $sp = \sum_{i=1}^n \mathbf{1}_{X_i \leq x}$ ,  $\mathbf{1}_A$  is the indicator of event  $A$ . The solution of  $\hat{F}_n(SWA_{\epsilon, n}) < \frac{1}{2}$  is reversing the percentile by  $1 - \hat{F}_n(SWA_{\epsilon, n})$ , the obtained percentile is also reversed. Without loss of generality, in the following discussion, only the  $\hat{F}_n(SWA_{\epsilon, n}) \geq \frac{1}{2}$  case will be considered. Moreover, in extreme heavy-tailed distributions, the calculated percentile can exceed the breakdown point of  $SWA_{\epsilon}$ , so the percentile will be modified to  $1 - \epsilon$  if this happens. The quantile mean uses the location-scale invariant in a different way as shown in the following proof.

**Theorem A.1.**  $qm_{d \approx 0.321, \epsilon = \frac{1}{8}}$  is a consistent mean estimator for the exponential, Pareto ( $\alpha \rightarrow \infty$ ) and any symmetric distributions provided that the second moments are finite.

*Proof.* Similarly, rearranging the definition,  $d = \frac{F(\mu) - F(BM_{\epsilon})}{F(BM_{\epsilon}) - \frac{1}{2}}$ .

Recall the cdf is  $F(x) = 1 - e^{-\lambda^{-1}x}$ ,  $x \geq 0$ , the expectation of  $BM_{\epsilon}$  can be expressed as  $\lambda BM_0(\epsilon)$ , so  $F(BM_{\epsilon})$  is free of  $\lambda$ .

When  $\epsilon = \frac{1}{8}$ ,  $d = \frac{-e^{-1} + e^{-\left(1 + \ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)}}{\frac{1}{2} - e^{-\left(1 + \ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)}} \approx 0.321$ .

The proof of the symmetric case is similar. Since for any symmetric distribution with a finite second moment,  $F(E[BM_{\epsilon}]) = F(\mu) = \frac{1}{2}$ . Then, the expectation of the quantile mean is  $qm_{d, \epsilon} = F^{-1}\left(\left(F(\mu) - \frac{1}{2}\right)d + F(\mu)\right) = F^{-1}\left(0 + F(\mu)\right) = \mu$ .

For the assertion related to the Pareto distribution, the cdf of it is  $1 - \left(\frac{x_m}{x}\right)^{\alpha}$ . So, the  $d$  value with two unknown percentile  $p_1$  and  $p_2$  is

$$d_{Pareto} = \frac{1 - \left(\frac{x_m}{\frac{x_m}{\alpha-1}}\right)^{\alpha} - \left(1 - \left(\frac{x_m}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^{\alpha}\right)}{\left(1 - \left(\frac{x_m}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^{\alpha}\right) - \left(1 - \left(\frac{x_m}{x_m(1-p_2)^{-\frac{1}{\alpha}}}\right)^{\alpha}\right)} = \frac{1 - \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} - p_1}{p_1 - p_2}. \text{ When } \alpha \rightarrow \infty, \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} = \frac{1}{e}. \text{ The } d \text{ value for the exponential distribution is identical, since } d_{exp} = \frac{(1-e^{-1}) - \left(1 - e^{-\ln\left(\frac{1}{1-p_1}\right)}\right)}{\left(1 - e^{-\ln\left(\frac{1}{1-p_1}\right)}\right) - \left(1 - e^{-\ln\left(\frac{1}{1-p_2}\right)}\right)} = \frac{1 - \frac{1}{e} - p_1}{p_1 - p_2}. \text{ All results are now proven. } \square$$

The definitions of location and scale parameters are such that they must satisfy  $F(x; \lambda, \mu) = F\left(\frac{x-\mu}{\lambda}; 1, 0\right)$ . Recall that

152  $x = \lambda Q_0(p) + \mu$ , so the percentile of any weighted average  
 153 is free of  $\lambda$  and  $\mu$ , guaranteeing the validity of the quantile  
 154 mean.  $qm_{d \approx 0.321, \epsilon = \frac{1}{8}}$  works better in the fat-tail scenarios (SI  
 155 Dataset S1). Theorem .1 and A.1 show that  $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$   
 156 and  $qm_{d \approx 0.321, \epsilon = \frac{1}{8}}$  are both consistent mean estimators for  
 157 any symmetric distribution and a skewed distribution with  
 158 finite second moments. It's obvious that the breakdown points  
 159 of  $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$  and  $qm_{d \approx 0.321, \epsilon = \frac{1}{8}}$  are both  $\frac{1}{8}$ . Therefore  
 160 they are all invariant means.

161 To study the impact of the choice of SWAs in  $rm$  and  $qm$ ,  
 162 it is constructive to consider a symmetric weighted average  
 163 as a mixture of symmetric quantile averages. Although us-  
 164 ing a less-biased symmetric weighted average can generally  
 165 improve performance (SI Dataset S1), there is a higher risk  
 166 of violation in the semiparametric framework. However, even  
 167 the SQA function is not strictly monotonic, suppose it is  
 168 generally decreasing in  $[0, u]$ , but increasing in  $[u, \frac{1}{2}]$ , since  
 169  $1 - 2\epsilon$  of the symmetric quantile averages will be included  
 170 in the computation of  $SWA_\epsilon$ , as long as  $|u - \frac{1}{2}| \ll 1 - 2\epsilon$ ,  
 171 and other parts of the SQA function satisfy the inequality  
 172 constraints which define the  $\nu$ th orderliness, the mean-SWA-  
 173 median inequality will still be valid (as an example, the SQA  
 174 function is non-monotonic when the shape parameter of the  
 175 Weibull distribution  $\alpha > \frac{1}{1 - \ln(2)} \approx 3.259$  as shown in the  
 176 previous article, yet the mean-BM $_{\frac{1}{8}}$ -median inequality is still  
 177 valid when  $\alpha \leq 3.322$ ). Another key factor determining the  
 178 risk of violation is the skewness of the distribution. In the  
 179 previous article, it is shown that in a family of distributions  
 180 that differ by a skewness-increasing transformation satisfies  
 181  $\frac{d^2 \chi}{dx^2} \geq 0 \wedge \frac{d\chi}{dx} \geq 0$ , the violation of orderliness, if it happens,  
 182 often only occurs when the distribution is near-symmetric  
 183 (15). The over-corrections in  $rm$  and  $qm$  are dependent on  
 184 the  $SWA_\epsilon$ -median difference, which is correlated to the skew-  
 185 ness, so the over-correction is often tiny with a moderate  $d$ .  
 186 This qualitative analysis provides another perspective, in ad-  
 187 dition to the bias bounds (16), that  $rm$  and  $qm$  based on the  
 188 mean-SWA $_\epsilon$ -median inequality are generally safe.

189 **B. Robust estimations of the central moments.** In 1976, Bickel  
 190 and Lehmann, in their third paper of the landmark series *De-*  
 191 *scriptive Statistics for Nonparametric Models* (17), generalized  
 192 a class of estimators called "measures of disperse," which is now  
 193 often named as Bickel-Lehmann dispersion. As an example,  
 194 they proposed a first version of the trimmed standard devi-  
 195 ation,  $\hat{\tau}^2(F; \epsilon) \equiv \tau^2(F; \epsilon)$ , for independent and identically  
 196 distributed random variables  $X_i$  with a distribution  $F$ , where  
 197  $\tau^2(F; \epsilon) = \frac{1}{1-2\epsilon} \int_{Q(\epsilon)}^{Q(1-\epsilon)} y dG(y)$ ,  $Q$  is the quantile function  
 198 of  $G$ ,  $G$  is the distribution of  $Y = X^2$ . Obviously, when  
 199  $\epsilon = 0$ , the result is equivalent to the second raw moment.  
 200 In 1979, in the same series (18), they explored another class  
 201 of estimators called "measures of spread," which "does not  
 202 require the assumption of symmetry." From that, a popular  
 203 efficient scale estimator, the Rousseeuw-Croux scale estimator  
 204 (19), was derived in 1993, but the importance of tackling the  
 205 symmetry assumption has been greatly underestimated. In the  
 206 final section of the paper, they considered another two  
 207 possible versions of the trimmed standard deviations, which

were modified symmetrically here for comparison,

$$\left[ n \left( \frac{1}{2} - \epsilon \right) \right]^{-\frac{1}{2}} \left[ \sum_{k=\frac{n}{2}}^{n(1-\epsilon)} [X_k - X_{n-k+1}]^2 \right]^{\frac{1}{2}}, \quad [1]$$

and

$$\left[ \binom{n}{2} (1 - 2\epsilon) \right]^{-\frac{1}{2}} \left[ \sum_{k=\binom{n}{2}\epsilon}^{\binom{n}{2}(1-\epsilon)} (X - X')_k^2 \right]^{\frac{1}{2}}, \quad [2]$$

where  $(X - X')_1 \leq \dots \leq (X - X')_{\binom{n}{2}}$  are the order statistics  
 of the "pseudo-sample"  $X_i - X_j$ ,  $i < j$ . The paper ended with,  
 "We do not know a fortiori which of the measures [1] or [2] is  
 preferable and leave these interesting questions open."

Observe that the kernel of the unbiased estimation of the  
 second central moment by using  $U$ -statistic is  $\psi_2(x_1, x_2) =$   
 $\frac{1}{2}(x_1 - x_2)^2$ . If adding the  $\frac{1}{2}$  term in [2], as  $\epsilon \rightarrow 0$ , the result  
 is equivalent to the standard deviation estimated by using  
 $U$ -statistic (also noted by Janssen, Serfling, and Veraverbeke  
 in 1987) (20). In fact, they also implied that, when  $\epsilon$  is 0, [2]  
 is  $\sqrt{2}$  times the standard deviation.

To address their open questions, the nomenclature used in  
 this paper is introduced as follows:

**Nomenclature.** Given a robust estimator  $\hat{\theta}$ . The first part  
 of the name of the robust statistic defined in this paper is a  
 prefix that indicates the type of estimator, and the second  
 part is the name of the population parameter  $\theta$  that the  
 estimator is consistent with as  $\epsilon \rightarrow 0$ . The abbreviation  
 of the estimator is the initial letter(s) of the first part plus  
 the common abbreviation of the consistent estimator that  
 measures the population parameter. If the estimator is not  
 a  $U$ -statistic, the breakdown point,  $\epsilon$ , is indicated in the  
 subscript of the abbreviation of the estimator. If the estimator  
 is a robustified  $U$ -statistic, the breakdown point of the location  
 estimator is indicated (except the median).

Naturally, the trimmed standard deviation following this  
 nomenclature is  $Tsd_{\epsilon, n} := \left[ TM_\epsilon \left( (\psi_2(X_{N_1}, X_{N_2}))_{N=1}^{\binom{n}{2}} \right) \right]^{-\frac{1}{2}}$ ,  
 where  $TM_\epsilon(Y)$  denotes the  $\epsilon$ -symmetric trimmed mean with  
 the sequence  $(\psi_2(X_{N_1}, X_{N_2}))_{N=1}^{\binom{n}{2}}$  as an input. If the square  
 root is removed, it is named as the trimmed variance ( $Tvar_{\epsilon, n}$ ).  
 It is now very clear that this definition, essentially the same  
 as [2], should be preferable. Not only because it is essentially  
 a trimmed  $U$ -statistic for the standard deviation but also  
 because the orderliness of the pseudo-sample distribution is  
 ensured by the next exciting theorem.

**Theorem B.1.** *The second central moment kernel distribu-*  
*tion generated from any continuous unimodal distribution is*  
*ordered.*

*Proof.* Let  $Q(p)$ ,  $0 \leq p \leq 1$ , denote the quantile of the contin-  
 uous unimodal distribution  $f_X(x)$ . The corresponding proba-  
 bility density is  $f(Q(p))$ . Generating the distribution of the  
 pair  $(Q(p_i), Q(p_j))$ ,  $i < j$ ,  $p_i < p_j$ , the corresponding proba-  
 bility density is  $f_{X,X}(Q(p_i), Q(p_j)) = 2f(Q(p_i))f(Q(p_j))$ .  
 Transforming the pair  $(Q(p_i), Q(p_j))$ ,  $i < j$ , by the function  
 $\Phi(x_1, x_2) = x_1 - x_2$ , the pairwise difference distribution has  
 a mode that is arbitrary close to  $M - M = 0$ . The mono-  
 tonic increasing of the pairwise difference distribution was



first implied in its unimodality proof done by Hodges and Lehmann in 1954 (21). Whereas they used induction to get the result, Dharmadhikari and Jogdeo in 1982 (22) gave a modern proof of the unimodality using Khintchine's representation (23). Assuming absolute continuity, Purkayastha (24) introduced a much simpler proof in 1998. Transforming the pairwise difference distribution by squaring and multiplying  $\frac{1}{2}$  does not change the monotonicity, making the pdf become monotonically decreasing with mode at zero. In the previous semiparametric robust mean estimation article, it is proven that a right skewed distribution with a monotonic decreasing pdf is always ordered, which gives the desired result.  $\square$

**Remark.** The assumption of continuity of distributions is important for monotonicity because, unlike in the continuous case, it is possible to get pairs with the same value for a discrete distribution. For example, let the probabilities of the singletons  $\{1\}$ ,  $\{2\}$ ,  $\{3\}$ ,  $\{4\}$  and  $\{5\}$  of a probability mass function of a discrete probability distribution be  $\frac{1}{11}$ ,  $\frac{4}{11}$ ,  $\frac{3}{11}$ ,  $\frac{2}{11}$ , and  $\frac{1}{11}$ , respectively. This is a unimodal distribution, but the corresponding  $\psi_2$  distribution is non-monotonic, whose singletons  $\{0\}$ ,  $\{0.5\}$ ,  $\{2\}$ ,  $\{4.5\}$  and  $\{8\}$  have probabilities  $\frac{21}{66}$ ,  $\frac{24}{66}$ ,  $\frac{2}{14}$ ,  $\frac{6}{66}$ , and  $\frac{1}{66}$ , respectively.

Previously, it was shown that any symmetric distribution with a finite second moment is  $\nu$ th ordered. That means the orderliness does not require unimodality, e.g., for a symmetric bimodal distribution, it is also ordered. Examples from the Weibull distribution show that unimodality does not guarantee orderliness. Theorem B.1 reveals another profound relationship between unimodality and orderliness, which is sufficient for trimming inequality.

In 1928, Fisher constructed  $k$ -statistics as unbiased estimators of cumulants (25). Halmos (1946) proved that the functional  $\theta$  admits an unbiased estimator if and only if it is a regular statistical functional of degree  $k$  and showed a relation of symmetry, unbiasedness and minimum variance (26). In 1948, Hoeffding generalized  $U$ -statistics (27) which enable the derivation of a minimum-variance unbiased estimator from each unbiased estimator of an estimable parameter. Heffernan (1997) (28) obtained an unbiased estimator of the  $k$ th central moment by using  $U$ -statistics and demonstrated that it is the minimum variance unbiased estimator for distributions with finite moments (29, 30). In 1984, to study generalized  $L$ -statistics, Serfling considered the  $U$ -statistic structure (31). Gijbels, Janssen and Veraverbeke generalized the trimmed  $U$ -statistics in 1988 (32). Due to the combinatorial explosion, the bootstrap (33), introduced by Efron in 1979, is indispensable in large sample studies. In 1981, Bickel and Freedman (34) showed that the bootstrap is asymptotically valid to approximate the original distribution in a wide range of situations, including  $U$ -statistics. After that, the limit laws of bootstrapped  $U$ -statistics have been intensively studied and proven by Athreya, Ghosh, Low, and Sen (1984) and Helmers, Janssen, and Veraverbeke (1990) (35, 36). Users can check the accuracy by comparing the unbiased central moments (37) to the bootstrap central moments. The weighted  $k$ th central moment ( $k \leq n$ ) is defined as,

$$Wkm_{\epsilon, \gamma, n} := WA_{\epsilon, \gamma, n} \left( (\psi_k(X_{N_1}, \dots, X_{N_k}))_{N=1}^{\binom{n}{k}} \right),$$

where  $X_{N_1}, \dots, X_{N_k}$  are the  $n$  choose  $k$  elements from  $X$ ,  $\psi_k(x_1, \dots, x_k) = \sum_{j=0}^{k-2} (-1)^j \binom{1}{k-j} \sum (x_{i_1}^{k-j} \dots x_{i_{j+1}}) +$

$(-1)^{k-1} (k-1) x_1 \dots x_k$ , the second summation is over  $i_1, \dots, i_{j+1} = 1$  to  $k$  with  $i_1 < \dots < i_{j+1}$  (18, 25, 26, 28). Despite the complexity, the structure of the  $k$ th central moment kernel distributions can be elucidated by decomposing.

**Data Availability.** Data for Table ?? are given in SI Dataset S1. All codes have been deposited in [GitHub](#).

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