Near-consistent robust estimations of moments for unimodal distributions

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Descriptive statistics for parametric models currently heavily rely on the accuracy of distributional assumptions. Here, based on the invariant structures of unimodal distributions, a series of sophisticated yet efficient estimators, robust to both gross errors and departures from parametric assumptions, are proposed for estimating mean and central moments with insignificant asymptotic biases for common unimodal distributions. This article also illuminates the understanding of the common nature of probability distributions and the measures of them.

orderliness | invariant | unimodal | adaptive estimation | U-statistics

The asymptotic inconsistencies between sample mean ($\bar{x}$) and nonparametric robust location estimators in asymmetric distributions on the real line have been noticed for more than two centuries (1), yet remain unsolved. Strictly speaking, it is unsolvable as by trimming, some information about the original distribution is removed, making it impossible to estimate the values of the removed parts without distributional assumptions. Newcomb (1886, 1912) provided the first modern approach to robust parametric estimation by developing a class of estimators that gives "less weight to the more discordant observations" (2, 3). In 1964, Huber (4) used the minimax procedure to obtain M-estimator for the contaminated normal distribution, which has played a pre-eminent role in the later development of robust statistics. However, as previously demonstrated, under growing asymmetric departures from normality, the bias of the Huber M-estimator increases rapidly. This is a common issue in parameter estimations. For example, He and Fung (1999) constructed (5) a robust M-estimator for the two-parameter Weibull distribution, from which all moments can be calculated. Nonetheless, it is inadequate for the gamma, Perato, lognormal, and the generalized Gaussian distributions (SI Dataset S1). Another old and interesting approach is arithmetically computing the parameters using one or more L-statistics as inputs, such as percentile estimators. Examples of percentile estimators for the Weibull distribution, the reader is referred to Menon (1963) (6), Dubey (1967) (7), Hassanein (1971) (8), Marks (2005) (9), and Boudt, Caliskan, and Croux (2011) (10)'s works. At the outset of the study of percentile estimators, it was known that they arithmetically utilizes the invariant structures of probability distributions (6, 11, 12). Maybe such estimators can be named as I-statistics. Formally, an estimator is classified as an I-statistic if it asymptotically satisfies $\mathrm{I}\left(\mathrm{LE}_1, \cdots, \mathrm{LE}_l\right) = \left(\theta_1, \cdots, \theta_q\right)$ for the distribution it is consistent, where LEs are calculated with the use of L-statistics, I is defined using arithmetic operations and constants, but it may also incorporate other functions, and θs are the population parameters it estimates. A subclass of I-statistics, arithmetic I-statistics, is defined as LEs are L-statistics, I is solely defined using arithmetic operations and constants. Since some percentile estimators use the logarithmic function to transform all random variables before computing the L-statistics, a percentile estimator might not always be an arithmetic I-statistic (7). In this article, two subclasses of I-statistics are introduced, arithmetic I-statistics and quantile I-statistics. Examples of quantile I-statistics will be discussed later. Based on L-statistics, I-statistics are naturally robust. Compared to probability density functions (pdfs) and cumulative distribution functions (cdfs), the quantile functions of many parametric distributions are more elegant. Since the expectation of an L-statistic can be expressed as an integral of the quantile function, I-statistics are often analytically obtainable. However, the performance of the aforementioned examples is often worse than that of the robust M-statistics when the distributional assumption is violated (SI Dataset S1). Even when distributions such as the Weibull and gamma belong to the same larger family, the generalized gamma distribution, a misassumption can still result in substantial biases, rendering the approach ill-suited.

In previous research on semiparametric robust mean estimation, the symmetric weighted Hodges-Lehmann mean (SWHLM_ϵ) is still inconsistent for any skewed distribution, despite having much smaller asymptotic biases than symmetric weighted averages. All robust location estimators commonly used are symmetric due to the universality of the symmetric distributions. One can construct an asymmetric weighted average that is consistent for a semiparametric class of skewed distributions. This approach has been investigated previously, but its lack of symmetry makes it suitable only for certain applications (13). Shifting from semiparametrics to parametrics, an ideal robust location estimator would have a non-sample-dependent breakdown point (defined in Subsection **??**) and be consistent for any symmetric distribution and a skewed distribution with finite second moments. This is called an

Significance Statement

Bias, variance, and contamination are the three main errors in statistics. Consistent robust estimation is unattainable without parametric assumptions. Here, based on a paradigm shift inspired by mean-median-mode inequality, Bickel-Lehmann spread, and adaptive estimation, invariant moments are proposed as a means of achieving near-consistent and robust estimations of moments, even in scenarios where moderate violations of distributional assumptions occur, while the variances are sometimes smaller than those of the sample moments.

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invariant mean. Based on the mean-symmetric weighted L-statistic-median inequality, the recombined mean is defined as

$$rm_{d,\epsilon,n} \coloneqq \lim_{c\to\infty}\left(\frac{(\mathrm{SWL}_{\epsilon,n}+c)^{d+1}}{(m_n+c)^d}-c\right),$$

where d is the key factor for bias correction, m_n is the sample median, $\mathrm{SWL}_{\epsilon,n}$ is either a symmetric weighted average or a symmetric weighted H-L mean. $\mathrm{BM}_{\epsilon,n}$ is used in the first Subsection, but other symmetric weighted L-statistics can also be used in practice as long as the inequalities hold. The following theorem shows the significance of this arithmetic I-statistic.

Theorem .1. *If the second moments are finite, $rm_{d\approx 0.163,\epsilon=\frac{1}{16}}$ is a consistent mean estimator for the exponential and any symmetric distributions and the Pareto distribution with quantile function $Q(p) = x_m(1-p)^{-\frac{1}{\alpha}}$, $x_m > 0$, when $\alpha \to \infty$.*

Proof. Finding d and ϵ that make $rm_{d,\epsilon}$ a consistent mean estimator is equivalent to finding the solution of $E\left[rm_{d,\epsilon,n}\right] = E\left[X\right]$. Rearranging the definition, $rm_{d,\epsilon} = \lim_{c\to\infty}\left(\frac{(\mathrm{BM}_\epsilon+c)^{d+1}}{(m+c)^d}-c\right) = (d+1)\,\mathrm{BM}_\epsilon - dm = \mu$. So, $d = \frac{\mu-\mathrm{BM}_\epsilon}{\mathrm{BM}_\epsilon-m}$. The quantile function of the exponential distribution is $Q(p) = \ln\left(\frac{1}{1-p}\right)\lambda$. $E\left[X\right] = \lambda$. $E\left[m_n\right] = Q\left(\frac{1}{2}\right) = \ln 2\lambda$. For the exponential distribution, $E\left[\mathrm{BM}_{\frac{1}{16},n}\right] = \lambda\left(1+\ln\left(\frac{16866160640\sqrt[4]{\frac{5}{39}}}{1744156557\ 11^{3/4}}\right)\right)$. Obviously, the scale parameter λ can be canceled out, $d \approx 0.163$. The proof of the second assertion follows directly from the coincidence property. For any symmetric distribution with a finite second moment, $E\left[\mathrm{BM}_{\epsilon,n}\right] = E\left[m_n\right] = E\left[X\right]$. Then $E\left[rm_{d,\epsilon,n}\right] = \lim_{c\to\infty}\left(\frac{(E[X]+c)^{d+1}}{(E[X]+c)^d}-c\right) = E\left[X\right]$. The proof for the Pareto distribution is more general. The mean of the Pareto distribution is given by $\frac{\alpha x_m}{\alpha-1}$. The d value with two unknown percentiles p_1 and p_2 for the Pareto distribution is $d_{Perato} = \frac{\frac{\alpha x_m}{\alpha-1}-x_m(1-p_1)^{-\frac{1}{\alpha}}}{x_m(1-p_1)^{-\frac{1}{\alpha}}-x_m(1-p_2)^{-\frac{1}{\alpha}}}$. Since any weighted L-statistic can be expressed as an integral of the quantile function, $\lim_{\alpha\to\infty}\frac{\frac{\alpha}{\alpha-1}-(1-p_1)^{-1/\alpha}}{(1-p_1)^{-1/\alpha}-(1-p_2)^{-1/\alpha}} = -\frac{\ln(1-p_1)+1}{\ln(1-p_1)-\ln(1-p_2)}$, the d value for the Pareto distribution approaches that of the exponential distribution as $\alpha\to\infty$, regardless of the type of weighted L-statistic used. This completes the demonstration. $\square$

Theorem .1 implies that for the Weibull, gamma, Pareto, lognormal and generalized Gaussian distribution, $rm_{d\approx 0.163,\epsilon=\frac{1}{16}}$ is consistent for at least one particular case. The biases of $rm_{d\approx 0.163,\epsilon=\frac{1}{16}}$ for distributions with skewness between those of the exponential and symmetric distributions are tiny (SI Dataset S1). $rm_{d\approx 0.163,\epsilon=\frac{1}{16}}$ exhibits excellent performance for all these common unimodal distributions (SI Dataset S1).

Besides introducing the concept of invariant mean, the purpose of this paper is to demonstrate that, in light of previous works, the estimation of central moments can be transformed into a location estimation problem by using U-statistics, the central moment kernel distributions possess desirable properties, and a series of sophisticated yet efficient robust estimators can be constructed whose biases are typically smaller than the variances (as seen in Table **??** for $n = 5400$) for unimodal distributions.

Data Availability. Data for Table **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

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