Near-consistent robust estimations of moments for unimodal distributions

Tuban Lee[a,1]

[a]Institute of Biomathematics, Macau SAR 999078, China

Descriptive statistics for parametric models currently heavily rely on the accuracy of distributional assumptions. Here, based on the invariant structures of unimodal distributions, a series of sophisticated yet efficient estimators, robust to both gross errors and departures from parametric assumptions, are proposed for estimating mean and central moments with insignificant asymptotic biases for common unimodal distributions. This article also illuminates the understanding of the common nature of probability distributions and the measures of them.

orderliness | invariant | unimodal | adaptive estimation | U-statistics

The asymptotic inconsistencies between sample mean ($\bar{x}$) and nonparametric robust location estimators in asymmetric distributions on the real line have been noticed for more than two centuries (1), yet remain unsolved. Strictly speaking, it is unsolvable as by trimming, some information about the original distribution is removed, making it impossible to estimate the values of the removed parts without distributional assumptions. Newcomb (1886, 1912) provided the first modern approach to robust parametric estimation by developing a class of estimators that gives "less weight to the more discordant observations" (2, 3). In 1964, Huber (4) used the minimax procedure to obtain M-estimator for the contaminated normal distribution, which has played a pre-eminent role in the later development of robust statistics. However, as previously demonstrated, under growing asymmetric departures from normality, the bias of the Huber M-estimator increases rapidly. This is a common issue in parameter estimations. For example, He and Fung (1999) constructed (5) a robust M-estimator for the two-parameter Weibull distribution, from which all moments can be calculated. Nonetheless, it is inadequate for the gamma, Perato, lognormal, and the generalized Gaussian distributions (SI Dataset S1). Another old and interesting approach is arithmetically computing the parameters using one or more L-statistics as inputs, such as percentile estimators. Examples of percentile estimators for the Weibull distribution, the reader is referred to Menon (1963) (6), Dubey (1967) (7), Hassanein (1971) (8), Marks (2005) (9), and Boudt, Caliskan, and Croux (2011) (10)'s works. At the outset of the study of percentile estimators, it was known that they arithmetically utilizes the invariant structures of probability distributions (6, 11, 12). Maybe such estimators can be named as I-statistics. Formally, an estimator is classified as an I-statistic if it asymptotically satisfies $I(LE_1, \cdots, LE_l) = (\theta_1, \cdots, \theta_q)$ for the distribution it is consistent, where LEs are calculated with the use of L-statistics, I is defined using arithmetic operations and constants, but it may also incorporate other functions, and θs are the population parameters it estimates. A subclass of I-statistics, arithmetic I-statistics, is defined as LEs are L-statistics, I is solely defined using arithmetic operations and constants.

Since some percentile estimators use the logarithmic function to transform all random variables before computing the L-statistics, a percentile estimator might not always be an arithmetic I-statistic (7). In this article, two subclasses of I-statistics are introduced, arithmetic I-statistics and quantile I-statistics. Examples of quantile I-statistics will be discussed later. Based on L-statistics, I-statistics are naturally robust. Compared to probability density functions (pdfs) and cumulative distribution functions (cdfs), the quantile functions of many parametric distributions are more elegant. Since the expectation of an L-statistic can be expressed as an integral of the quantile function, I-statistics are often analytically obtainable. However, the performance of the aforementioned examples is often worse than that of the robust M-statistics when the distributional assumption is violated (SI Dataset S1). Even when distributions such as the Weibull and gamma belong to the same larger family, the generalized gamma distribution, a misassumption can still result in substantial biases, rendering the approach ill-suited.

In previous research on semiparametric robust mean estimation, the binomial mean (BM_ϵ) is still inconsistent for any skewed distribution, despite having much smaller asymptotic biases than other weighted averages. All robust location estimators commonly used are symmetric due to the universality of the symmetric distributions. One can construct an asymmetric weighted average that is consistent for a semiparametric class of skewed distributions. This approach has been investigated previously, but its lack of symmetry makes it suitable only for certain applications (13). Shifting from semiparametrics to parametrics, an ideal robust location estimator would have a non-sample-dependent breakdown point (defined in Subsection **??**) and be consistent for any symmetric distribution and a skewed distribution with finite second moments. This is called an invariant mean. Based on the mean-symmetric weighted

Significance Statement

Bias, variance, and contamination are the three main errors in statistics. Consistent robust estimation is unattainable without parametric assumptions. Here, based on a paradigm shift inspired by mean-median-mode inequality, Bickel-Lehmann spread, and adaptive estimation, invariant moments are proposed as a means of achieving near-consistent and robust estimations of moments, even in scenarios where moderate violations of distributional assumptions occur, while the variances are sometimes smaller than those of the sample moments.

average-median inequality, the recombined mean is defined as

$$rm_{d,\epsilon,n} := \lim_{c \to \infty} \left(\frac{(\mathrm{SWA}_{\epsilon,n} + c)^{d+1}}{(median + c)^d} - c \right),$$

where d is the key factor for bias correction, $\mathrm{SWA}_{\epsilon,n}$ is $\mathrm{BM}_{\epsilon,n}$ in the first three Subsections, but other symmetric weighted averages can also be used in practice as long as the inequalities hold. The following theorem shows the significance of this arithmetic I-statistic.

Theorem .1. *If the second moments are finite, $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ is a consistent mean estimator for the exponential and any symmetric distributions and the Pareto distribution with quantile function $Q(p) = x_m(1-p)^{-\frac{1}{\alpha}}$, $x_m > 0$, when $\alpha \to \infty$.*

Proof. Finding d and ϵ that make $rm_{d,\epsilon}$ a consistent mean estimator is equivalent to finding the solution of $E[rm_{d,\epsilon}] = E[X]$. Rearranging the definition, $rm_{d,\epsilon} = \lim_{c \to \infty} \left(\frac{(\mathrm{BM}_\epsilon + c)^{d+1}}{(median + c)^d} - c \right) = (d+1)\mathrm{BM}_\epsilon - d median = \mu$. So, $d = \frac{\mu - \mathrm{BM}_\epsilon}{\mathrm{BM}_\epsilon - median}$. The quantile function of the exponential distribution is $Q(p) = \ln\left(\frac{1}{1-p}\right)\lambda$. $E[x] = \lambda$. $E[median] = Q\left(\frac{1}{2}\right) = \ln 2\lambda$. For the exponential distribution, the expectation of $\mathrm{BM}_{\frac{1}{8}}$ is $E\left[\mathrm{BM}_{\frac{1}{8}}\right] = \lambda\left(1 + \ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)$. Obviously, the scale parameter λ can be canceled out, $d \approx 0.375$. The proof of the second assertion follows directly from the co-incidence property. For any symmetric distribution with a finite second moment, $E[\mathrm{BM}_\epsilon] = E[median] = E[X]$. Then $E[rm_{d,\epsilon}] = \lim_{c \to \infty} \left(\frac{(E[X] + c)^{d+1}}{(E[X] + c)^d} - c \right) = E[X]$. The proof for the Pareto distribution is more general. The mean of the Pareto distribution is given by $\frac{\alpha x_m}{\alpha - 1}$. The d value with two unknown percentiles p_1 and p_2 for the Pareto distribution is $d_{Perato} = \frac{\frac{\alpha x_m}{\alpha - 1} - x_m(1-p_1)^{-\frac{1}{\alpha}}}{x_m(1-p_1)^{-\frac{1}{\alpha}} - x_m(1-p_2)^{-\frac{1}{\alpha}}}$. Since any weighted average can be expressed as an integral of the quantile function, $\lim_{\alpha \to \infty} \frac{\frac{\alpha}{\alpha - 1} - (1-p_1)^{-1/\alpha}}{(1-p_1)^{-1/\alpha} - (1-p_2)^{-1/\alpha}} = -\frac{\ln(1-p_1) + 1}{\ln(1-p_1) - \ln(1-p_2)}$, the d value for the Pareto distribution approaches that of the exponential distribution as $\alpha \to \infty$, regardless of the type of weighted average used. This completes the demonstration. $\square$

Theorem .1 implies that for the Weibull, gamma, Pareto, lognormal and generalized Gaussian distribution, $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ is consistent for at least one particular case. The biases of $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ for distributions with skewness between those of the exponential and symmetric distributions are tiny (SI Dataset S1). $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ exhibits excellent performance for all these common unimodal distributions (SI Dataset S1).

Besides introducing the concept of invariant mean, the purpose of this paper is to demonstrate that, in light of previous works, the estimation of central moments can be transformed into a location estimation problem by using U-statistics, the central moment kernel distributions possess desirable properties, and a series of sophisticated yet efficient robust estimators can be constructed whose biases are typically smaller than the variances (as seen in Table **??** for $n = 5400$) for unimodal distributions.

Background and Main Results

A. Invariant mean. It has long been known that a theoretical model can be adjusted to fit the first two moments of the observed data. A continuous distribution belonging to a location–scale family takes the form $F(x) = F_0\left(\frac{x-\mu}{\lambda}\right)$, where F_0 is a "standard" distribution. Therefore, $F(x) = Q^{-1}(x) \to x = Q(p) = \lambda Q_0(p) + \mu$. Thus, any weighted average can be expressed as $\lambda \mathrm{WA}_0(\epsilon) + \mu$, where $\mathrm{WA}_0(\epsilon)$ is an integral of $Q_0(p)$ according to the definition of the weighted average. The simultaneous cancellation of μ and λ in $\frac{(\lambda\mu_0 + \mu) - (\lambda\mathrm{BM}_0(\epsilon) + \mu)}{(\lambda\mathrm{BM}_0(\epsilon)) + \mu - (\lambda median_0 + \mu)}$ assures that d is a constant. Consequently, the roles of BM_ϵ and median in $rm_{d,\epsilon}$ can be replaced by any weighted averages, although only symmetric weighted averages are considered in defining the invariant mean.

The performance in heavy-tailed distributions can be further improved by constructing the quantile mean as

$$qm_{d,\epsilon,n} := \hat{Q}_n\left(\left(\hat{F}_n(\mathrm{SWA}_{\epsilon,n}) - \frac{1}{2}\right)d + \hat{F}_n(\mathrm{SWA}_{\epsilon,n})\right),$$

provided that $\hat{F}_n(\mathrm{SWA}_{\epsilon,n}) \geq \frac{1}{2}$, where $\hat{F}_n(x)$ is the empirical cumulative distribution function of the sample, $\hat{Q}_n$ is the sample quantile function. The most popular method for computing the sample quantile function was proposed by Hyndman and Fan in 1996 (14). To minimize the finite sample bias, here, $\hat{F}_n(x) := \frac{1}{n}\left(\frac{x - X_{sp}}{X_{sp+1} - X_{sp}} + sp\right)$, where $sp = \sum_{i=1}^{n} \mathbf{1}_{X_i \leq x}$, $\mathbf{1}_A$ is the indicator of event A. The solution of $\hat{F}_n(\mathrm{SWA}_{\epsilon,n}) < \frac{1}{2}$ is reversing the percentile by $1 - \hat{F}_n(\mathrm{SWA}_{\epsilon,n})$, the obtained percentile is also reversed. Without loss of generality, in the following discussion, only the case where $\hat{F}_n(\mathrm{SWA}_{\epsilon,n}) \geq \frac{1}{2}$ is considered. Moreover, in extreme heavy-tailed distributions, the calculated percentile can exceed the breakdown point of SWA_ϵ, so the percentile will be modified to $1 - \epsilon$ if this occurs. The quantile mean uses the location-scale invariant in a different way as shown in the following proof.

Theorem A.1. *$qm_{d \approx 0.321, \epsilon = \frac{1}{8}}$ is a consistent mean estimator for the exponential, Pareto ($\alpha \to \infty$) and any symmetric distributions provided that the second moments are finite.*

Proof. Similarly, rearranging the definition, $d = \frac{F(\mu) - F(\mathrm{BM}_\epsilon)}{F(\mathrm{BM}_\epsilon) - \frac{1}{2}}$. The cdf of the exponential distribution is $F(x) = 1 - e^{-\lambda^{-1}x}$, $\lambda \geq 0$, $x \geq 0$, the expectation of BM_ϵ can be expressed as $\lambda\mathrm{BM}_0(\epsilon)$, so $F(\mathrm{BM}_\epsilon)$ is free of λ. When $\epsilon = \frac{1}{8}$, $d = \frac{-e^{-1} + e^{-\left(1 + \ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)}}{\frac{1}{2} - e^{-\left(1 + \ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)}} \approx 0.321$. The proof of the symmetric case is similar. Since for any symmetric distribution with a finite second moment, $F(E[\mathrm{BM}_\epsilon]) = F(\mu) = \frac{1}{2}$. Then, the expectation of the quantile mean is $qm_{d,\epsilon} = F^{-1}\left(\left(F(\mu) - \frac{1}{2}\right)d + F(\mu)\right) = F^{-1}(0 + F(\mu)) = \mu$.

For the assertion related to the Pareto distribution, the cdf of it is $1 - \left(\frac{x_m}{x}\right)^\alpha$. So, the d value with two unknown percentile p_1 and p_2 is

$$d_{Pareto} = \frac{1 - \left(\frac{x_m}{\frac{\alpha x_m}{\alpha - 1}}\right)^\alpha - \left(1 - \left(\frac{x_m}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^\alpha\right)}{\left(1 - \left(\frac{x_m}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^\alpha\right) - \left(1 - \left(\frac{x_m}{x_m(1-p_2)^{-\frac{1}{\alpha}}}\right)^\alpha\right)} =$$

$\frac{1 - \left(\frac{\alpha - 1}{\alpha}\right)^\alpha - p_1}{p_1 - p_2}$. When $\alpha \to \infty$, $\left(\frac{\alpha - 1}{\alpha}\right)^\alpha = \frac{1}{e}$. The d value for the exponential distribution is identical, since $d_{exp} =$

$$\frac{\left(1-e^{-1}\right)-\left(1-e^{-\ln\left(\frac{1}{1-p_1}\right)}\right)}{\left(1-e^{-\ln\left(\frac{1}{1-p_1}\right)}\right)-\left(1-e^{-\ln\left(\frac{1}{1-p_2}\right)}\right)}=\frac{1-\frac{1}{e}-p_1}{p_1-p_2}.$$ All results

are now proven. $\qquad\square$

The definitions of location and scale parameters are such that they must satisfy $F(x;\lambda,\mu)=F(\frac{x-\mu}{\lambda};1,0)$. By recalling $x=\lambda Q_0(p)+\mu$, it follows that the percentile of any weighted average is free of λ and μ, which guarantees the validity of the quantile mean. The quantile mean is a quantile I-statistic. Specifically, an estimator is classified as a quantile I-statistic if LEs are percentiles of a distribution obtained by plugging L-statistics into a cumulative distribution function and I is defined with arithmetic operations, constants and quantile functions. $qm_{d\approx0.321,\epsilon=\frac{1}{8}}$ works better in the fat-tail scenarios (SI Dataset S1). Theorem .1 and A.1 show that $rm_{d\approx0.375,\epsilon=\frac{1}{8}}$ and $qm_{d\approx0.321,\epsilon=\frac{1}{8}}$ are both consistent mean estimators for any symmetric distribution and a skewed distribution with finite second moments. It's obvious that the breakdown points of $rm_{d\approx0.375,\epsilon=\frac{1}{8}}$ and $qm_{d\approx0.321,\epsilon=\frac{1}{8}}$ are both $\frac{1}{8}$. Therefore they are all invariant means.

To study the impact of the choice of SWAs in rm and qm, it is constructive to recall that a symmetric weighted average is a linear combination of symmetric quantile averages. While using a less-biased symmetric weighted average can generally enhance performance (SI Dataset S1), there is a greater risk of violation in the semiparametric framework. However, the mean-SWA-median inequality is robust to slight fluctuations of the SQA function of the underlying distribution. Suppose the SQA function is generally decreasing in $[0,u]$, but increasing in $[u,\frac{1}{2}]$, since $1-2\epsilon$ of the symmetric quantile averages will be included in the computation of SWA_ϵ, as long as $\frac{1}{2}-u\ll1-2\epsilon$, and other portions of the SQA function satisfy the inequality constraints that define the νth orderliness on which the SWA_ϵ is based, the mean-SWA_ϵ-median inequality will still hold. This is due to the violation being bounded (15) and therefore cannot be extreme for unimodal distributions. For instance, the SQA function is non-monotonic when the shape parameter of the Weibull distribution $\alpha>\frac{1}{1-\ln(2)}\approx3.259$ as shown in the previous article, the violation of the third orderliness starts near this parameter as well, yet the mean-$\text{BM}_{\frac{1}{8}}$-median inequality is still valid when $\alpha\leq3.322$. Another key factor in determining the risk of violation is the skewness of the distribution. Previously, it was demonstrated that in a family of distributions differing by a skewness-increasing transformation in van Zwet's sense, the violation of orderliness, if it happens, often only occurs when the distribution is nearly symmetrical (16). The over-corrections in rm and qm are dependent on the SWA_ϵ-median difference, which can be a reasonable measure of skewness (17, 18), implying that the over-correction is often tiny with a moderate d. This qualitative analysis provides another perspective, in addition to the bias bounds (15), that rm and qm based on the mean-SWA_ϵ-median inequality are generally safe.

B. Robust estimations of the central moments. In 1979, Bickel and Lehmann, in their final paper of the landmark series *Descriptive Statistics for Nonparametric Models* (19), generalized a class of estimators called "measures of spread," which "does not require the assumption of symmetry." From that, a popular

efficient scale estimator, the Rousseeuw-Croux scale estimator (20), was derived in 1993, but the importance of tackling the symmetry assumption has been greatly underestimated. While they had already considered one version of the trimmed standard deviation in the third paper of that series (21), in the final section of that paper (19), they explored another two possible versions, which were modified here for comparison,

$$\left[n\left(\frac{1}{2}-\epsilon\right)\right]^{-\frac{1}{2}}\left[\sum_{i=\frac{n}{2}}^{n(1-\epsilon)}\left[X_i-X_{n-i+1}\right]^2\right]^{\frac{1}{2}},\qquad[1]$$

and

$$\left[\binom{n}{2}(1-\epsilon-\gamma\epsilon)\right]^{-\frac{1}{2}}\left[\sum_{i=\binom{n}{2}\gamma\epsilon}^{\binom{n}{2}(1-\epsilon)}\left(X-X'\right)_i^2\right]^{\frac{1}{2}},\quad[2]$$

where $(X-X')_1\leq\ldots\leq(X-X')_{\binom{n}{2}}$ are the order statistics of the "pseudo-sample", X_i-X_j, $i<j$. The paper ended with, "We do not know a fortiori which of the measures [1] or [2] is preferable and leave these interesting questions open."

Observe that the kernel of the unbiased estimation of the second central moment by using U-statistic is $\psi_2(x_1,x_2)=\frac{1}{2}(x_1-x_2)^2$. If adding the $\frac{1}{2}$ term in [2], as $\epsilon\to0$, the result is equivalent to the standard deviation estimated by using U-statistic (also noted by Janssen, Serfling, and Veraverbeke in 1987) (22). In fact, they also showed that, when ϵ is 0, [2] is $\sqrt{2}$ times the standard deviation.

To address their open question, the nomenclature used in this paper is introduced as follows:

Nomenclature. Given a robust estimator $\hat{\theta}$ with an adjustable breakdown point which can be infinitesimal. The name of $\hat{\theta}$ is composed of two parts: the first part denotes the type of estimator, and the second part is the name of the population parameter θ that the estimator is consistent with as $\epsilon\to0$. The abbreviation of the estimator is formed by combining the initial letter(s) of the first part with the common abbreviation of the consistent estimator that measures the population parameter. If the estimator is symmetric, the asymptotic breakdown point, ϵ, is indicated in the subscript of the abbreviation of the estimator, except the median. For asymmetric estimators based on quantile average, the corresponding γ is also indicated after ϵ. Note that ϵ is the right breakdown point (defined in Subsection **??**), while the left breakdown point should be further calculated.

In the previous article on semiparametric robust mean estimation, it was shown that the bias of a robust estimator with an adjustable breakdown point is often monotonic with respect to the breakdown point in a semiparametric distribution. Naturally, the estimator's name should correspond to the population parameter with which it is consistent as $\epsilon\to0$. The trimmed standard deviation following this nomenclature is $\text{T}sd_{\epsilon=1-\sqrt{1-\epsilon_0},\gamma,n}:=\left[\text{TM}_{\epsilon_0,\gamma}\left(\left(\psi_2\left(X_{N_1},X_{N_2}\right)\right)_{N=1}^{\binom{n}{2}}\right)\right]^{-\frac{1}{2}}$, where $\text{TM}_{\epsilon_0,\gamma}(Y)$ denotes the ϵ_0,γ-trimmed mean with the sequence $\left(\psi_2\left(X_{N_1},X_{N_2}\right)\right)_{N=1}^{\binom{n}{2}}$ as an input, the proof of the breakdown point is given in Subsection **??**. Removing the square root yields the trimmed variance ($\text{T}var_{\epsilon,\gamma,n}$). It is now very clear that this definition, essentially the same as

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[2], should be preferable. Not only because it is essentially a trimmed U-statistic for the standard deviation but also because the γ-orderliness of the second central moment kernel distribution is ensured by the next exciting theorem.

Theorem B.1. *The second central moment kernel distribution generated from any unimodal distribution is γ-ordered.*

Proof. The monotonic increasing of the pairwise difference distribution was first implied in its unimodality proof done by Hodges and Lehmann in 1954 (23). Whereas they used induction to get the result in Theorem **??**, Dharmadhikari and Jogdeo in 1982 (24) provided a modern proof of the unimodality using Khintchine's representation (25). Assuming absolute continuity, Purkayastha (26) introduced a much simpler proof in 1998. Transforming the pairwise difference distribution by squaring and multiplying by $\frac{1}{2}$ does not change the monotonicity, making the pdf become monotonically decreasing with mode at zero. In the previous article, it was proven that a right skewed distribution with a monotonic decreasing pdf is always γ-ordered, which gives the desired result. □

Previously, it was shown that any symmetric distribution with a finite second moment is νth ordered, indicating that orderliness does not require unimodality, e.g., a symmetric bimodal distribution is also ordered. An analysis of the Weibull distribution showed that unimodality does not guarantee orderliness. Theorem B.1 reveals another profound relationship between unimodality and orderliness, which is sufficient for trimming inequality.

In 1928, Fisher constructed k-statistics as unbiased estimators of cumulants (27). Halmos (1946) proved that the functional θ admits an unbiased estimator if and only if it is a regular statistical functional of degree k and showed a relation of symmetry, unbiasness and minimum variance (28). In 1948, Hoeffding generalized U-statistics (29) which enable the derivation of a minimum-variance unbiased estimator from each unbiased estimator of an estimable parameter. Heffernan (1997) (30) obtained an unbiased estimator of the kth central moment by using U-statistics and demonstrated that it is the minimum variance unbiased estimator for distributions with finite moments (31, 32). In 1976, Saleh generalized the Hodges-Lehmenn (H-L) estimator (33) to the trimmed H-L mean (which he named "Wilcoxon one-sample statistic") (34). In 1984, Serfling pointed out the speciality of Hodges-Lehmann estimator, which is neither a simple L-statistic nor a U-statistic, and considered the generalized L-statistics and U-statistic structure (35). Also in 1984, Janssen and Serfling and Veraverbeke (36) showed that the Bickel-Lehmann spread also belongs to the same class. It gradually became clear that the Hodges-Lehmenn estimator, trimmed H-L mean and trimmed standard deviation are all trimmed U-statistics (37–39).

Extending the trimmed U-statistic to weighted U-statistic, i.e., replacing the trimmed mean with weighted average. The weighted kth central moment $(k \le n)$ is defined as,

$$\mathrm{W}km_{\epsilon=1-(1-\epsilon_0)^{\frac{1}{k}},\gamma,n} := \mathrm{WA}_{\epsilon_0,\gamma,n}\left(\left(\psi_k\left(X_{N_1},\cdots,X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}}\right),$$

where $X_{N_1},\cdots,X_{N_k}$ are the n choose k elements from X, $\psi_k\left(x_1,\cdots,x_k\right) = \sum_{j=0}^{k-2}(-1)^j\left(\frac{1}{k-j}\right)\sum\left(x_{i_1}^{k-j}\ldots x_{i_{(j+1)}}\right) + (-1)^{k-1}(k-1)x_1\ldots x_k$, the second summation is over $i_1,\cdots,i_{j+1} = 1$ to k with $i_1 < \cdots < i_{j+1}$ (30). Despite the

complexity, the structure of the kth central moment kernel distributions can be elucidated by decomposing.

Theorem B.2. *For each pair $(Q(p_i), Q(p_j))$ of the original distribution such that $Q(p_i) < Q(p_j)$, let $x_1 = Q(p_i)$ and $x_k = Q(p_j)$, $\Delta = Q(p_i) - Q(p_j)$, the kth central moment kernel distribution, $k > 2$, can be seen as a mixture distribution and each of the components has the support $\left(-\left(\frac{k}{3+(-1)^k}\right)^{-1}(-\Delta)^k, \frac{1}{k}(-\Delta)^k\right)$.*

Proof. Without loss of generality, generating the distribution of the k-tuple $(Q(p_{i_1}),\ldots,Q(p_{i_k}))$ under continuity, $k > 2$, $i_1 < \ldots < i_k$, $p_{i_1} < \ldots < p_{i_k}$, the corresponding probability density is $f_{X,\ldots,X}(Q(p_{i_1}),\ldots,Q(p_{i_k})) = k!f(Q(p_{i_1}))\ldots f(Q(p_{i_k}))$. Transforming the distribution of the k-tuple by the function $\psi_k(x_1,\ldots,x_k)$, denoting $\bar{\Delta} = \psi_k(Q(p_{i_1}),\ldots,Q(p_{i_k}))$. The probability $f_{\Xi_k}(\bar{\Delta}) = \sum_{\bar{\Delta}=\psi_k(Q(p_{i_1}),\ldots,Q(p_{i_k}))} f_{X,\ldots,X}(Q(p_{i_1}),\ldots,Q(p_{i_k}))$ is the summation of the probabilities of all k-tuples such that $\bar{\Delta}$ is equal to $\psi_k(Q(p_{i_1}),\ldots,Q(p_{i_k}))$. The following Ξ_k is equivalent.

Ξ_k: Every pair with a difference equal to $\Delta = Q(p_{i_1}) - Q(p_{i_k})$ can generate a pseudodistribution (but the integral is not equal to 1, so "pseudo") such that $x_2, \ldots, x_{k-1}$ exhaust all combinations under the inequality constraints, i.e., $Q(p_{i_1}) = x_1 < x_2 < \ldots < x_{k-1} < x_k = Q(p_{i_k})$. The combination of all the pseudodistributions with the same Δ is ξ_Δ. The combination of ξ_Δ, i.e., from $\Delta = 0$ to $Q(0) - Q(1)$, is Ξ_k.

The support of ξ_Δ is the extrema of ψ_k subject to the inequality constraints. Using the Lagrange multiplier, one can easily determine the only critical point at $x_1 = \ldots = x_k = 0$, where $\psi_k = 0$. Other candidates are within the boundaries, i.e., $\psi_k(x_1 = x_1, x_2 = x_k, \cdots, x_k = x_k)$, $\cdots$, $\psi_k(x_1 = x_1, \cdots, x_i = x_1, x_{i+1} = x_k, \cdots, x_k = x_k)$, $\cdots$, $\psi_k(x_1 = x_1, \cdots, x_{k-1} = x_1, x_k = x_k)$. $\psi_k(x_1 = x_1, \cdots, x_i = x_1, x_{i+1} = x_k, \cdots, x_k = x_k)$ can be divided into k groups. If $\frac{k+1-i}{2} \le j \le \frac{k-1}{2}$, from j+1st to $k-j$th group, the gth group has $i\binom{i-1}{g-j-1}\binom{k-i}{j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$, from $k-j$+1th to $i+j$th group, the gth group has $i\binom{i-1}{g-j-1}\binom{k-i}{j} + (k-i)\binom{k-i-1}{j-k+g-1}\binom{i}{k-j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$. If $j < \frac{k+1-i}{2}$, from j+1st to $i+j$th group, the gth group has $i\binom{i-1}{g-j-1}\binom{k-i}{j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$. If $j \ge \frac{k}{2}$, from $k-j+1$st to jth group, the gth group has $(k-i)\binom{k-i-1}{j-k+g-1}\binom{i}{k-j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$, from $j+1$th to $j+i$th group, $i+j < k$, the gth group has $i\binom{i-1}{g-j-1}\binom{k-i}{j} + (k-i)\binom{k-i-1}{j-k+g-1}\binom{i}{k-j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$. The final kth group is the term $(-1)^{k-1}(k-1)x_1^i x_k^{k-i}$. So, if $i + j = k$, $j \ge \frac{k}{2}$, $i \le \frac{k}{2}$, the summed coefficient of $x_1^i x_k^{k-i}$ is $(-1)^{k-1}(k-1) + \sum_{g=i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}(k-i)\binom{k-i-1}{g-i-1} + \sum_{g=k-i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}i\binom{i-1}{g-k+i-1} = (-1)^{k-1}(k-1) + (-1)^{k+1} + (k-i)(-1)^k + (-1)^k(i-1) = (-1)^{k+1}$. The summation identities are $\sum_{g=i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}(k-i)\binom{k-i-1}{g-i-1} = (k-i)\int_0^1\sum_{g=i+1}^{k-1}(-1)^{g+1}\binom{k-i-1}{g-i-1}t^{k-g}dt = (k-i)\int_0^1\left((-1)^i(t-1)^{k-i-1} - (-1)^{k+1}\right)dt =$

370 $(k-i)\left(\frac{(-1)^k}{i-k}+(-1)^k\right)$ $=$ $(-1)^{k+1}$ $+$ $(k-i)(-1)^k$

371 and $\sum_{g=k-i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}i\binom{i-1}{g-k+i-1}$ $=$

372 $\int_0^1\sum_{g=k-i+1}^{k-1}(-1)^{g+1}i\binom{i-1}{g-k+i-1}t^{k-g}dt$ $=$

373 $\int_0^1\left(i(-1)^{k-i}(t-1)^{i-1}-i(-1)^{k+1}\right)dt$ $=$ $(-1)^k(i-1)$.

374 If $j<\frac{k+1-i}{2}$, $i>k-1$, if $i=k$, $\psi_k=0$, if $\frac{k+1-i}{2}\le j\le\frac{k-1}{2}$,

375 $\frac{k+1}{2}\le i\le k-1$, the summed coefficient of $x_1^i x_k^{k-i}$ is

376 $(-1)^{k-1}(k-1)+\sum_{g=k-i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}i\binom{i-1}{g-k+i-1}+$

377 $\sum_{g=i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}(k-i)\binom{k-i-1}{g-i-1}$, the same as above. If

378 $i+j<k$, since $\binom{i}{k-j}=0$, the related terms can be ignored, so,

379 using the binomial theorem and beta function, the summed co-

380 efficient of $x_1^{k-j}x_k^j$ is $\sum_{g=j+1}^{i+j}(-1)^{g+1}\frac{1}{k-g+1}i\binom{i-1}{g-j-1}\binom{k-i}{j}=$

381 $i\binom{k-i}{j}\int_0^1\sum_{g=j+1}^{i+j}(-1)^{g+1}\binom{i-1}{g-j-1}t^{k-g}dt$ $=$

382 $\binom{k-i}{j}i\int_0^1\left((-1)^j t^{k-j-1}\left(\frac{t}{t-1}\right)^{1-i}\right)dt$ $=$

383 $\binom{k-i}{j}i\frac{(-1)^{j+i+1}\Gamma(i)\Gamma(k-j-i+1)}{\Gamma(k-j+1)}$ $=$ $\frac{(-1)^{j+i+1}i!(k-j-i)!(k-i)!}{(k-j)!j!(k-j-i)!}$ $=$

384 $(-1)^{j+i+1}\frac{i!(k-i)!}{k!}\frac{k!}{(k-j)!j!}=\binom{k}{i}^{-1}(-1)^{1+i}\binom{k}{j}(-1)^j$.

385 The coefficient of $x_1^i x_k^{k-i}$ in $\binom{k}{i}^{-1}(-1)^{1+i}(x_1-x_k)^k$

386 is $\binom{k}{i}^{-1}(-1)^{1+i}\binom{k}{i}(-1)^{k-i}$ $=$ $(-1)^{k+1}$, same as the

387 summed coefficient if $i+j=k$. If $i+j<$

388 k, the coefficient of $x_1^{k-j}x_k^j$ is $\binom{k}{i}^{-1}(-1)^{1+i}\binom{k}{j}(-1)^j$,

389 same as the corresponding summed coefficient. There-

390 fore, $\psi_k(x_1=x_1,\ldots,x_i=x_1,x_{i+1}=x_k,\ldots,x_k=x_k)$ $=$

391 $\binom{k}{i}^{-1}(-1)^{1+i}(x_1-x_k)^k$, the maximum and minimum of ψ_k

392 follow directly from the properties of the binomial coeffi-

393 cient. $\qquad\square$

394 ξ_Δ is closely related to $f_\Xi(\Delta)$, which is the pairwise differ-

395 ence distribution, since the probability density of ξ_Δ can be ex-

396 pressed as $f_{\Xi_k}(\bar\Delta|\Delta)$ and $\sum_{\bar\Delta=-\left(\frac{3+(-1)^k}{2}\right)^{-1}(-\Delta)^k}^{\frac{1}{k}(-\Delta)^k}f_{\Xi_k}(\bar\Delta|\Delta)=$

397 $f_\Xi(\Delta)=\int_0^\infty 2f(t)f(t-\Delta)dt$. The support of the original

398 distribution is assumed to be $[0,\infty)$ for simplicity. Recall that

399 $f_\Xi(\Delta)$ is monotonic increasing with a mode at the origin if the

400 original distribution is unimodal. Thus, in general, ignoring

401 the shape of ξ_Δ, Ξ_k is monotonic left and right around zero.

402 In fact, the median of Ξ_k also exhibits a strong tendency to be

403 close to zero, as it can be cast as a weighted mean of the medi-

404 ans of ξ_Δ. When Δ is small, all values of ξ_Δ are close to zero,

405 resulting in the median of ξ_Δ being close to zero as well. When

406 Δ is large, the median of ξ_Δ depends on its skewness, but the

407 corresponding weight is much smaller, so even if ξ_Δ is highly

408 skewed, the median of Ξ_k will only be slightly shifted from

409 zero. Denote the median of Ξ_k as m_{Ξ_k}, for the five parametric

410 distributions here, $|m_{\Xi_k}|$s are all $\le 0.1\sigma$ for Ξ_3 and Ξ_4 (SI

411 Dataset S1). Assuming $m_{\Xi_k}=0$, for the even ordinal central

412 moment kernel distribution, the average probability density on

413 the left side of zero is greater than that on the right side, since

414 $\frac{\frac{1}{2}}{\binom{k}{2}^{-1}(Q(0)-Q(1))^k}>\frac{\frac{1}{2}}{\frac{1}{k}(Q(0)-Q(1))^k}$. This means that, on aver-

415 age, the inequality $f(Q(\epsilon))\ge f(Q(1-\epsilon))$ holds. For the odd

416 ordinal distribution, the discussion is more challenging since

417 it is generally symmetric. Just consider Ξ_3, let $x_1=Q(p_i)$

418 and $x_3=Q(p_j)$, changing the value of x_2 from $Q(p_i)$ to

419 $Q(p_j)$ will monotonically change the value of $\psi_3(x_1,x_2,x_3)$,

420 since $\frac{\partial\psi_3(x_1,x_2,x_3)}{\partial x_2}=-\frac{x_1^2}{2}-x_1x_2+2x_1x_3+x_2^2-x_2x_3-\frac{x_3^2}{2}$,

421 $-\frac{3}{4}(x_1-x_3)^2\le\frac{\partial\psi_3(x_1,x_2,x_3)}{\partial x_2}\le-\frac{1}{2}(x_1-x_3)^2\le 0$. If the

422 original distribution is right-skewed, ξ_Δ will be left-skewed,

423 so, for Ξ_3, the average probability density of the right side of

424 zero will be greater than that of the left side, which means,

425 on average, the inequality $f(Q(\epsilon))\le f(Q(1-\epsilon))$ holds (the

426 same result can be inferred from the definition of central mo-

427 ments, where the positivity of the odd order central moment

428 is directly related to the left-skewness of the corresponding

429 kernel distribution). In all, the monotonicity of the pairwise

430 difference distribution guides the general shape of the kth

431 central moment kernel distribution, $k>2$, forcing it to be

432 unimodal-like with the mode and median close to zero, then,

433 the inequality $f(Q(\epsilon))\le f(Q(1-\epsilon))$ or $f(Q(\epsilon))\ge f(Q(1-\epsilon))$

434 holds in general. If a distribution is ordered and all of its cen-

435 tral moment kernel distributions are also ordered, it is called

436 completely ordered. Although strict complete orderliness is

437 difficult to prove, even if the inequality may be violated in

438 a small range, as discussed in Subsection A, the mean-SWA-

439 median inequality remains valid, in most cases, for the central

440 moment kernel distribution.

441 Another crucial property of the central moment kernel dis-

442 tribution, location invariant, is introduced in the next theorem.

443 The proof is provided in the SI Text.

444 **Theorem B.3.** $\psi_k(x_1=\lambda x_1+\mu,\cdots,x_k=\lambda x_k+\mu)$ $=$

445 $\lambda^k\psi_k(x_1,\cdots,x_k)$.

A direct result of Theorem B.3 is that, Wkm after stan-
dardization is invariant to location and scale. So, the weighted
standardized kth moment is defined to be

$$\mathrm{W}skm_{\epsilon,\gamma,n}\coloneqq\frac{\mathrm{W}km_{\epsilon,\gamma,n}}{\mathrm{W}var_{\epsilon_2,\gamma,n}^{\frac{k}{2}}}.$$

Consider two continuous distributions belonging to the
same location–scale family, their corresponding kth central
moment kernel distributions only differ in scaling. So d is
invariant, as shown in Subsection A. The recombined kth
central moment, based on rm, is defined by,

$$rkm_{d,\epsilon=1-(1-\epsilon_0)^{\frac{1}{k}},n}\coloneqq(d+1)\,\mathrm{SW}km_{\epsilon,n}-dmkm_n,$$

where $\mathrm{SW}km_{\epsilon,n}$ is using the binomial kth central moment
($\mathrm{B}km_{\epsilon_0,n}$) here, mkm_n is the median kth central moment.
Since $\mathrm{SW}km_{\epsilon,n}$ is an L-statistic, the resulting $rkm_{d,\epsilon,n}$ is an
arithmetic I-statistic. Similarly, the quantile will not change
after scaling. The quantile kth central moment is thus defined
as

$$qkm_{d,\epsilon,n}\coloneqq\hat Q_n\left(\left(p\mathrm{SW}km_{\epsilon,n}-\frac{1}{2}\right)d+p\mathrm{SW}km_{\epsilon,n}\right),$$

446 where $p\mathrm{SW}km_{\epsilon,n}=\hat F_{\psi,n}(\mathrm{SW}km_{\epsilon,n})$, $\hat F_{\psi,n}$ is the empirical

447 cumulative distribution function of the corresponding central

448 moment kernel distribution. $qkm_{d,\epsilon,n}$ is a quantile I-statistic.

449 For standardized moments, quantile skewness and quan-

450 tile kurtosis are defined to be $qskew_{d,\epsilon,n}\coloneqq\frac{qtm_{d,\epsilon,n}}{qsd_{d,\epsilon_2,n}^3}$ and

451 $qkurt_{d,\epsilon,n}\coloneqq\frac{qfm_{d,\epsilon,n}}{qsd_{d,\epsilon_2,n}^4}$. Quantile standard deviation ($qsd_{d,\epsilon,n}$),

452 recombined standard deviation ($rsd_{d,\epsilon,n}$), quantile third cen-

453 tral moment ($qtm_{d,\epsilon,n}$), quantile fourth central moment

454 ($qfm_{d,\epsilon,n}$), recombined third central moment ($rtm_{d,\epsilon,n}$), re-

455 combined fourth central moment ($rfm_{d,\epsilon,n}$), recombined skew-

456 ness ($rskew_{d,\epsilon,n}$), and recombined kurtosis ($rkurt_{d,\epsilon,n}$) are all

457 defined similarly as above and not repeated here. The trans-

458 formation to a location problem can also empower related

statistical tests. From the better performance of the quantile mean in heavy-tailed distributions, quantile central moments are generally better than recombined central moments regarding asymptotic bias.

To avoid confusion, it should be noted that the robust location estimations of the kernel distributions discussed in this paper differ from the approach taken by Joly and Lugosi (2016) (40), which is computing the median of all U-statistics from different disjoint blocks. Compared to bootstrap median U-statistics, this approach can produce two additional kinds of finite sample bias, one arises from the limited numbers of blocks, another is due to the size of the U-statistics (consider the mean of all U-statistics from different disjoint blocks, it is definitely not identical to the original U-statistic, except when the kernel is the Hodges-Lehmann kernel). Laforgue, Clemencon, and Bertail (2019)'s median of randomized U-statistics (41) is more sophisticated and can overcome the limitation of the number of blocks, but the second kind of bias remains unsolved.

C. Congruent distribution.

In the realm of nonparametric statistics, the precise values of robust estimators are of secondary importance. What is of primary importance is their relative differences or orders. Based on this principle, in the absence of contamination, as the parameters of the distribution vary, all reasonable nonparametric location estimates should asymptotically change in the same direction. Otherwise if the results obtained based on the trimmed mean are completely different from those based on the median, a contradiction arises. However, such contradictions are possible, as in the case of the Weibull distribution, $E[m] = \lambda \sqrt[\alpha]{\ln(2)}$, $E[\mu] = \lambda \Gamma \left(1 + \frac{1}{\alpha}\right)$, then, when $\alpha = 1$, $E[m] = \lambda \ln(2) \approx 0.693\lambda$, $E[\mu] = \lambda$, but when $\alpha = \frac{1}{2}$, $E[m] = \lambda \ln^2(2) \approx 0.480\lambda$, $E[\mu] = 2\lambda$, the mean increases, but the median decreases. To study the conditions that avoid such scenarios by classifying distributions through the signs of derivatives, let the quantile average function of a parametric distribution be denoted as $\mathrm{QA}\left(\epsilon, \gamma, \alpha_1, \cdots, \alpha_i, \cdots, \alpha_k\right)$, where α_i represent the parameters of the distribution, then, a distribution is γ-congruent if and only if the sign of $\frac{\partial \mathrm{QA}}{\partial \alpha_i}$ remains the same for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$. If this partial derivative is equal to zero or undefined, it can be considered both positive and negative, and thus does not impact the analysis. Asymptotically, any weighted average can be expressed as an integral of the quantile average function. Since the sign does not change after integration, the sign of $\frac{\partial \mathrm{QA}}{\partial \alpha_i}$ remains the same for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$ implies that all γ-weighted averages change in the same direction as the parameters change, as long as they are not undefined. A distribution is completely γ-congruent if and only if it is γ-congruent and all its central moment kernel distributions are also γ-congruent. Setting $\gamma = 1$ constitutes the definitions of congruence and complete congruence. Chebyshev's inequality implies that, for any probability distribution with finite moments, even if some weighted averages change in a direction different from that of the sample mean, the deviations are bounded. Furthermore, distributions with infinite moments can be γ-congruent, since the definition is based on the quantile average, not the sample mean.

The following theorems show the conditions that a distribution is congruent or γ-congruent.

Theorem C.1. *A symmetric distribution with a finite second moment is always congruent.*

Proof. For any symmetric distribution with a finite second moment, all symmetric quantile averages coincide. The conclusion follows immediately. □

Theorem C.2. *A positive define location-scale distribution with a finite second moment is always γ-congruent.*

Proof. As shown in discussions in Subsection A, for a location-scale distribution, any weighted average can be expressed as $\lambda \mathrm{WA}_0(\epsilon) + \mu$, where $\mathrm{WA}_0(\epsilon)$ is an integral of $Q_0(p)$ according to the definition of the weighted average. Therefore, the derivatives with respect to the parameters λ or μ are always positive. By application of the definition, the desired outcome is obtained. □

Theorem C.3. *The second central moment kernal distribution derived from a continuous location-scale unimodal distribution with a finite second moment is always γ-congruent.*

Proof. Theorem B.3 shows that the corresponding central moment kernel distribution is also a location-scale family distribution. Theorem B.1 shows that it is positively defined. Implementing Theorem C.2 yields the desired result. □

For the Pareto distribution, $\frac{\partial Q(p,\alpha)}{\partial \alpha} = \frac{x_m(1-p)^{-1/\alpha} \ln(1-p)}{\alpha^2}$. Since $\ln(1 - p) < 0$ for all $0 < p < 1$, $(1 - p)^{-1/\alpha} > 0$ for all $0 < p < 1$ and $\alpha > 0$, so $\frac{\partial Q(p,\alpha)}{\partial \alpha} < 0$, and therefore $\frac{\partial \mathrm{QA}(\epsilon,\gamma,\alpha)}{\partial \alpha} < 0$, the Pareto distribution is γ-congruent. The derivative for the lognormal distribution is $\frac{\partial \mathrm{SQA}(\epsilon,\sigma)}{\partial \sigma} = \frac{-\mathrm{erfc}^{-1}(2\epsilon)e^{\mu - \sqrt{2}\sigma \mathrm{erfc}^{-1}(2\epsilon)} - \mathrm{erfc}^{-1}(2-2\epsilon)e^{\mu - \sqrt{2}\sigma \mathrm{erfc}^{-1}(2-2\epsilon)}}{\sqrt{2}}$. Since the inverse complementary error function is positive when the input is smaller than 1, and negative when the input is larger than 1, $\mathrm{erfc}^{-1}(2\epsilon) = -\mathrm{erfc}^{-1}(2 - 2\epsilon)$, $e^{\mu - \sqrt{2}\sigma \mathrm{erfc}^{-1}(2-2\epsilon)} > e^{\mu - \sqrt{2}\sigma \mathrm{erfc}^{-1}(2\epsilon)}$, $\frac{\partial \mathrm{SQA}(\epsilon,\sigma)}{\partial \sigma} > 0$, the lognormal distribution is congruent. Theorem C.1 implies that the generalized Gaussian distribution is congruent. For the Weibull distribution, when α changes from 1 to $\frac{1}{2}$, the average probability density on the left side of the median increases, since $\frac{\frac{1}{2}}{\lambda \ln(2)} < \frac{\frac{1}{2}}{\lambda \ln^2(2)}$, but the mean increases, indicating that the distribution is more heavy-tailed, the probability density of large values will also increase. The main reason for non-congruence of a right-skewed smooth partial bounded probability distribution lies in the simultaneous increase of probability densities on two opposite sides: one approaching the bound and the other approaching infinity. Note that the gamma distribution does not have this issue, it looks to be congruent.

Although some common parametric distributions are not congruent, Theorem C.2 establishes that γ-congruence always holds for a positive define location-scale family distribution and thus for the second central moment kernel distribution generated from a continuous location-scale unimodal distribution as shown in Theorem C.3. Theorem B.2 demonstrates that all their central moment kernel distributions are unimodal-like with mode and median close to zero, as long as they are unimodal distributions. Assuming finite moments and constant $Q(0) - Q(1)$, increasing the mean of the kernel distribution will result in a more heavy-tailed distribution, i.e., the probability density of the values close to $\frac{1}{k}(-\Delta)^k$ increases. While the total probability density on either side of zero remains

unchanged as the median is generally close to zero and much less impacted by increasing the mean, the probability density of the values close to zero decreases. This transformation will increase nearly all symmetric weighted averages, in the general sense. Therefore, except for the median, which is assumed to be zero, nearly all symmetric weighted averages for all central moment kernel distributions derived from unimodal distributions should change in the same direction when the parameters change. Therefore, they are valid measures for nonparametric descriptive statistics.

D. A shape-scale distribution as the consistent distribution.
Up to this point, in this article, the consistent robust estimation has been limited to a location-scale distribution, with the location parameter often being omitted for simplicity. To construct probability distributions can be made to fit the observed skewness and kurtosis arbitrarily well, in 1894, Pearson (42) introduced a family of continuous probability distributions that are now often characterized by the square of the skewness and the kurtosis. If the skewness and the kurtosis are interrelated by a shape parameter, a distribution specified by a shape parameter (denoted as α) and a scale parameter (denoted as λ) is often referred to as a shape-scale distribution. Weibull, gamma, Pareto, lognormal, and generalized Gaussian distributions (when μ is a constant) are all shape-scale unimodal distributions. Moreover, if α or skewness or kurtosis is a constant, the shape-scale distribution is reduced to a location-scale distribution. The above discussion shows that, due to the invariant property, if a location-scale distribution is chosen as the consistent distribution, the type of invarient moments and their related weighted moments are given, there should exist a unique k-tuple $(d_{im}, \ldots, d_{ikm})$ calibrated by the distribution and the corresponding kernel distributions generated from this distribution. For a right skewed shape-scale distribution, let $D(|skewness|, kurtosis, k, etype, dtype, n) = d_{ikm}$ denote these relations, where the first input is the absolute value of the skewness, the second input is the kurtosis, the third is the order of the central moment (if $k = 1$, the mean), the fourth is the type of estimator, the fifth is the type of consistent distribution, and the sixth input is the sample size. For simplicity, the last three inputs will be omitted in the following discussion. Hold in awareness that due to the invariant property of scale, specifying d values for a shape-scale distribution only requires either skewness or kurtosis, while the other may be also omitted. Since many common shape-scale distributions are always right skewed (if not, only the right skewed or left skewed part is used for calibration, while the other part is omitted), the absolute value of the skewness should be identical to the skewness for them and it can also handle the left skew scenario well.

For recombined moments, the object of using a shape-scale distribution as the consistent distribution is to find solutions for the system of equations
$$\begin{cases} rm\left(\mathrm{SWA}, median, D(|rskew|, rkurt, 1)\right) = \mu \\ rvar\left(\mathrm{SW}var, mvar, D(|rskew|, rkurt, 2)\right) = \mu_2 \\ rtm\left(\mathrm{SW}tm, mtm, D(|rskew|, rkurt, 3)\right) = \mu_3 \\ rfm\left(\mathrm{SW}fm, mfm, D(|rskew|, rkurt, 4)\right) = \mu_4 \\ rskew = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} \\ rkurt = \frac{\mu_4}{\mu_2^2} \end{cases},$$
where μ_2, μ_3 and μ_4 are the population second,

third and fourth central moments. $|rskew|$ and $rkurt$ should be the invariant points of the functions $\varsigma(|rskew|) = \left| \frac{rtm(\mathrm{SW}tm, mtm, D(|rskew|, 3))}{rvar(\mathrm{SW}var, mvar, D(|rskew|, 2))^{\frac{3}{2}}} \right|$ and $\varkappa(rkurt) = \frac{rfm(\mathrm{SW}fm, mfm, D(rkurt, 4))}{rvar(\mathrm{SW}var, mvar, D(rkurt, 2))^2}$. Clearly, this is an overdetermined nonlinear system of equations, given that the skewness and kurtosis are interrelated for a shape-scale distribution. Since an overdetermined system constructed with random coefficients is almost always inconsistent, it is natural to optimize them separately using the fixed-point iteration (see Algorithm 1, only $rkurt$ is provided, others are the same).

Algorithm 1 $rkurt$ for a shape-scale distribution

Input: D; SWvar; SWfm; $mvar$; mfm; $maxit$; δ
Output: $rkurt_{i-1}$
   $i = 0$
2: $rkurt_i \leftarrow \varkappa(kurtosis_{max})$ ▷ Using the maximum kurtosis available in D as an initial guess.
   repeat
4:    $i = i + 1$
   $rkurt_{i-1} \leftarrow rkurt_i$
6:    $rkurt_i \leftarrow \varkappa(rkurt_{i-1})$
   until $i > maxit$ or $|rkurt_i - rkurt_{i-1}| < \delta$ ▷ $maxit$ is the maximum number of iterations, δ is a small positive number.

The following theorem shows the validaty of Algorithm 1.

Theorem D.1. *Assuming mkms are all equal to zero, $|rskew|$ and $rkurt$, defined as the largest attracting fix points of the functions $\varsigma(|rskew|)$ and $\varkappa(rkurt)$, are consistent estimators of $\tilde{\mu}_3$ and $\tilde{\mu}_4$ for a shape-scale distribution whose central moment kernel distributions are all congruent, as long as they are within the domain of D, where $\tilde{\mu}_3$ and $\tilde{\mu}_4$ are the population skewness and kurtosis.*

Proof. Without loss of generality, only $rkurt$ is considered here, while the logic for $|rskew|$ is the same. Also, according to the property of invariance, the second central moments of the sample and consistent distribution are all assumed to be 1. From the definition of D, $\frac{\varkappa(rkurt_D)}{rkurt_D} = \frac{\frac{fm_D - \mathrm{SW}fm_D}{\mathrm{SW}fm_D - mfm_D}(\mathrm{SW}fm - mfm) + \mathrm{SW}fm}{rkurt_D \left(\frac{var_D - \mathrm{SW}var_D}{\mathrm{SW}var_D - mvar_D}(\mathrm{SW}var - mvar) + \mathrm{SW}var \right)^2}$, where the subscript D indicates that the estimates are from the central moment kernel distributions generated from the consistent distribution used to calibrate the d values, while other estimates are from the sample.

Then, assuming the mkms are all equal to zero, $\frac{\varkappa(rkurt_D)}{rkurt_D} = \frac{\frac{fm_D - \mathrm{SW}fm_D}{\mathrm{SW}fm_D}(\mathrm{SW}fm) + \mathrm{SW}fm}{rkurt_D \left(\frac{\mathrm{SW}var}{\mathrm{SW}var_D} \right)^2} = \frac{\left(\frac{fm_D - \mathrm{SW}fm_D}{\mathrm{SW}fm_D} + 1 \right)(\mathrm{SW}fm)}{fm_D \left(\frac{\mathrm{SW}var}{\mathrm{SW}var_D} \right)^2} = \frac{\mathrm{SW}fm\mathrm{SW}var_D^2}{\mathrm{SW}fm_D\mathrm{SW}var^2} = \frac{\frac{\mathrm{SW}fm}{\mathrm{SW}var^2}}{\frac{\mathrm{SW}fm_D}{\mathrm{SW}var_D^2}} = \frac{\mathrm{SW}kurt}{\mathrm{SW}kurt_D}$. Since SW$fm_D$ are from the same kernel distribution as $fm_D = rkurt_D var_D^2$, according to the congruence, an increase in fm_D will also result in an increase in SWfm_D. Combining with Theorem B.3, SW$kurt$ is a measure of kurtosis that is invariant to location and scale, so $\lim_{rkurt_D \to \infty} \frac{\varkappa(rkurt_D)}{rkurt_D} < 1$. As a result, if there is at least one fix point, let the largest one be

666 fix_{max}, then it is attracting since $|\frac{\partial(\varkappa(rkurt_D))}{\partial(rkurt_D)}| < 1$ for all
667 $rkurt_D \in [fix_{max}, kurtosis_{max}]$.

668 Asymptotically, consider any $SWkurt_D > SWkurt$,
669 $\frac{\varkappa(rkurt_D)}{rkurt_D} < 1$, the same logic applies, a consistent estimator
670 must be the last attracting fix point, fix_{max} is the consistent
671 estimator. □

672 As a result of Theorem D.1, assuming continuity, mkms
673 are all equal to zero, and congruence of the central moment
674 kernel distributions, Algorithm 1 converges surely provided
675 that a fix point exists within the domain of D. At this stage,
676 D can only be approximated through a Monte Carlo study.
677 Continuity can be ensured by using linear interpolation. One
678 common encountered problem is that the domain of D de-
679 pends on both the consistent distribution and the Monte Carlo
680 study, so the iteration may halt at the boundary if the fix
681 point is not within the domain. However, by setting a proper
682 maximum number of iterations, the algorithm can return the
683 optimal boundary value. For quantile moments, the logic is
684 similar, if the percentiles do not exceed the breakdown point.
685 If this is the case, consistent estimation is impossible, and
686 the algorithm will stop due to the maximum number of iter-
687 ations. The fix point iteration is, in principle, similar to the
688 iterative reweighing in M-estimator, but an advantage of this
689 algorithm is that the optimization is solely related to the d
690 value function and is independent of the sample size (except
691 for the quantile moments, which require re-computation of the
692 quantile function, but this operation has a time complexity of
693 $O(1)$ for a sorted sample). Since $|rskew|$ can specify d_{rm} after
694 optimization, this algorithm enables the robust estimations of
695 all four moments to reach a near-consistent level for common
696 unimodal distributions (Table **??**, SI Dataset S1), just using
697 the Weibull distribution as the consistent distribution.

698 **E. Variance.** As the fundamental theorem in statistics, the
699 central limit theorem declares that the standard deviation of
700 the limiting form of the sampling distribution of the sample
701 mean is $\frac{\sigma}{\sqrt{n}}$. The principle, asymptotic normality, was later
702 applied to the sampling distributions of robust location esti-
703 mators (2, 36, 43–50) and it was found that the efficiencies
704 of the robust location estimators can differ significantly from
705 that of the arithmetic mean. Daniell (1920) stated (43) that
706 comparing the efficiencies of various kinds of estimators is
707 useless unless they all tend to coincide asymptotically. Bickel
708 and Lehmann, also in the landmark series (49, 50), argued
709 that meaningful comparisons can be made by studying the
710 standardized variances, asymptotic variances, and sharp lower
711 bounds of these estimators. Here, the scaled standard error
712 (SSE) is proposed to estimate the variances of all estimators, in-
713 cluding recombined/quantile moments, on a scale comparable
714 to that of the sample mean.

715 *Definition* E.1 (Scaled standard error). Let $\mathcal{M}_{s_i s_j} \in \mathbb{R}^{i \times j}$
716 denote the sample-by-statistics matrix, i.e., the first column
717 is the main statistics of interest, $\widehat{\theta_m}$, the second to the jth
718 column are $j - 1$ statistics required to scale, $\widehat{\theta_{r_1}}$, $\widehat{\theta_{r_2}}$, ...,
719 $\widehat{\theta_{r_{j-1}}}$. Then, the scaling factor $\mathcal{S} = \left[1, \frac{\theta_{r_1}^-}{\theta_m^-}, \frac{\theta_{r_2}^-}{\theta_m^-}, \ldots, \frac{\theta_{r_{j-1}}^-}{\theta_m^-}\right]^T$
720 is a $j \times 1$ matrix, which $\bar{\theta}$ is the mean of the column. The
721 normalized matrix is $\mathcal{M}_{s_i s_j}^N = \mathcal{M}_{s_i s_j}\mathcal{S}$. The SSEs are the
722 unbiased standard deviations of the corresponding columns.

723 Setting the bootstrap moments as the main statistics of
724 interest, the SSEs of all robust estimators proposed here are
725 often between those of the median moments and those of the
726 unbiased sample moments (SI Dataset S1). This is because
727 similar monotonic relations between robustness and variance
728 are also very common, e.g., Bickel and Lehmann (50) proved
729 that a lower bound for the efficiency of TM_ϵ to sample mean
730 is $(1 - 2\epsilon)^2$ and this monotonic bound holds true for any dis-
731 tribution. Lai, Robbins, and Yu (1983) proposed an estimator
732 that adaptively chooses the mean or median in a symmetric
733 distribution and showed that the results were typically as
734 good as the better of the sample mean and median regarding
735 variance (51). It can be interpreted as an attempt to use the
736 variance version of the mean-SWA-median inequality. While
737 they used bootstrap standard error as the criterion, another
738 approach can be dated back to Laplace (1812) (52) is using
739 a linear combination of median and mean and the weight is
740 deduced to achieve minimum variance; examples for symmetric
741 distributions see Samuel-Cahn (1994), Chan and He (1994),
742 and Damilano and Puig (2004)'s papers (53–55).

743 Scaled standard error allows for direct comparison of vari-
744 ances for different location estimators in asymmetric distri-
745 butions. In this study, twelve possible combinations were
746 created using two invariant means and related symmetric
747 weighted averages ($BM_{\frac{1}{8}}$, $SQM_{\frac{1}{8}}$, $BM_{\nu=2,\epsilon=\frac{1}{8}}$, $WM_{\frac{1}{8}}$, $BWM_{\frac{1}{8}}$,
748 and $TM_{\frac{1}{8}}$ used here). Each combination has a SSE for a single-
749 parameter distribution, which can be inferred through a Monte
750 Carlo study. Then, the combination with the smallest SSE
751 is chosen (if the percentiles of quantile moments exceed the
752 breakdown point, this combination will be excluded). Theo-
753 retically, bootstrap is the optimal way to infer the variance-
754 optimal choice without distributional assumptions, however,
755 its computational cost is high. Similar to Subsection D, let
756 $I(|skewness|, kurtosis, k, dtype, n) = ikm_{WA}$ denote these re-
757 lations. Then, since $\lim_{rkurt \to \infty} \frac{I(rkurt, 4)}{I(rkurt, 2)^2 rkurt} < 1$, the same
758 fix point iteration algorithm can be used to choose the variance-
759 optimum combination. The only difference is that unlike D,
760 I is defined to be discontinuous but linear interpolation can
761 also ensure continuity. This approach yields results that are
762 often nearly optimal (SI Dateset S1).

763 Due to combinatorial explosion, the bootstrap (56), in-
764 troduced by Efron in 1979, is indispensable for computing
765 invariant moments in practice. In 1981, Bickel and Freed-
766 man (57) showed that the bootstrap is asymptotically valid to
767 approximate the original distribution in a wide range of situa-
768 tions, including U-statistics. The limit laws of bootstrapped
769 trimmed U-statistics were proven by Helmers, Janssen, and
770 Veraverbeke (1990) (58). In the previous article, the advan-
771 tages of quasi-bootstrap were discussed (59–61). By using
772 quasi-sampling, the impact of the number of repetitions of
773 the bootstrap, or bootstrap size, on variance is negligible
774 (SI Dataset S1). An estimator based on the quasi-bootstrap
775 approach can be seen as a complex deterministic estimator
776 that is not only computationally efficient but also statistical
777 efficient. The only drawback of quasi-bootstrap compared
778 to non-bootstrap is that a small bootstrap size can produce
779 additional finite sample bias but this can be corrected by recal-
780 ibrating the d values (SI Text). The default bootstrap size is
781 set as 18 thousand here, as it balances computational cost and
782 finite sample bias, except for the asymptotic value calculation.
783 In general, the variances of invariant central moments are

much smaller than those of corresponding unbiased sample central moments (deduced by Cramér ([62])), except that of the corresponding second central moment (Table **??**).

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Lee

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