Near-consistent robust estimations of moments for unimodal distributions

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Descriptive statistics for parametric models currently rely heavily on the accuracy of distributional assumptions. Here, based on the invariant structures of unimodal distributions, a series of sophisticated yet efficient estimators, robust to both gross errors and departures from parametric assumptions, are proposed for estimating mean and central moments with insignificant asymptotic biases for common continuous unimodal distributions. This article also illuminates the understanding of the common nature of probability distributions and the measures of them.

orderliness | invariant | unimodal | adaptive estimation | U-statistics

The asymptotic inconsistencies between sample mean ($\bar{x}$) and nonparametric robust location estimators in asymmetric distributions on the real line have been noticed for more than two centuries (1) but unsolved. Strictly speaking, it is unsolvable because by trimming, some information about the original distribution is removed, making it impossible to estimate the values of the removed parts without distributional assumptions. Newcomb (1886, 1912) provided the first modern approach to this problem by developing a class of estimators that gives "less weight to the more discordant observations" (2, 3). In 1964, Huber (4) used the minimax procedure to obtain M-estimator for contaminated normal distribution, which has played a pre-eminent role in the later development of robust statistics. As previously demonstrated, under growing asymmetric departures from normality, the bias of the Huber M-estimator increases rapidly. This is a common issue in parametric estimations. For example, He and Fung (1999) constructed (5) a robust M-estimator for the two-parameter Weibull distribution. All moments can be calculated from its estimated parameters. As expected, it is inadequate for the gamma, Perato, lognormal, and the generalized Gaussian distributions (SI Dataset S1). Instead of minimizing the residuals, another old and interesting approach is arithmetically computing the parameters using one or more L-statistics as input values, e.g., the percentile estimators. Examples for the Weibull distribution, the reader is referred to Menon (1963) (6), Dubey (1967) (7), Hassanein (1971) (8), Marks (2005) (9), and Boudt, Caliskan, and Croux (2011) (10)'s works. At the outset of the study of percentile estimators, it was known that this class of estimators arithmetically utilizes the invariant structures of probability distributions (6, 11, 12). Maybe it can be named as I-statistics. Formally, an estimator is classified as an I-statistic if asymptotically it satisfies $\mathrm{I}\left(\mathrm{WA}_1, \cdots, \mathrm{WA}_l\right) = \left(\theta_1, \cdots, \theta_q\right)$ for the distribution it is consistent with, where WAs are weighted averages, θs are the population parameters it estimates. I-statistics have two subclasses, arithmetic I-statistics and quantile I-statistics. The percentile estimators belong to the arithmetic I-statistics. Examples of quantile I-statistics will be discussed later. In the previous article, it is shown that quantile average is fundamental for all weighted averages. Based on the quantile function, I-statistic is naturally robust. For many parametric distributions, the quantile functions are much more elegant than the pdfs and cdfs. So, I-statistics are often analytically obtainable. However, the performance of the above examples is often worse than that of the robust M-statistics when the distributional assumption is violated (SI Dataset S1). Even when distributions such as the Weibull and gamma belong to the same larger family, the generalized gamma distribution, a misassumption can still result in substantial biases, rendering the approach ill-suited.

In previous work on semiparametric robust mean estimation, although greatly shrinking the asymptotic biases, binomial mean (BM_ϵ) is still inconsistent for any skewed distribution if $\epsilon > 0$ (if $\epsilon \to 0$, since the alternating sum of binomial coefficients is zero, $\mathrm{BM} \to \mu$). All robust location estimators commonly used are symmetric due to the universality of the symmetric distributions. One can construct an asymmetric trimmed mean that is consistent for a semiparametric class of skewed distributions. This approach was investigated previously, but it is not symmetric and therefore only suitable for some special applications (13). From semiparametrics to parametrics, an ideal robust location estimator would have a non-sample-dependent breakdown point (defined in Subsection **??**) and be consistent with any symmetric distribution and a skewed distribution with finite second moments. This is called an invariant mean. Based on the mean-symmetric weighted average-median inequality, the recombined mean is defined as

$$rm_{d,\epsilon,n} \coloneqq \lim_{c\to\infty}\left(\frac{\left(\mathrm{SWA}_{\epsilon,n}+c\right)^{d+1}}{\left(median+c\right)^{d}}-c\right),$$

where d is for bias correction, $\mathrm{SWA}_{\epsilon,n}$ is $\mathrm{BM}_{\epsilon,n}$ in the first three Subsections, while other symmetric weighted averages can also be used in practice as long as the inequalities hold.

Significance Statement

Bias, variance, and contamination are the three main errors in statistics. Consistent robust estimation is unattainable without parametric assumptions. Here, based on a paradigm shift inspired by mean-median-mode inequality, Bickel-Lehmann spread, and adaptive estimation, invariant moments are proposed as a means of achieving near-consistent and robust estimations of moments, even in scenarios where moderate violations of distributional assumptions occur, while the variances are sometimes smaller than those of the sample moments.

T.L. designed research, performed research, analyzed data, and wrote the paper.

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The next theorem shows the significance of this composite estimator.

Theorem .1. *If the second moments are finite, $rm_{d\approx 0.375,\epsilon=\frac{1}{8}}$ is a consistent mean estimator for the exponential and any symmetric distributions and the Pareto distribution with quantile function $Q(p)=x_m(1-p)^{-\frac{1}{\alpha}}$, $x_m>0$, when $\alpha\to\infty$.*

Proof. Finding d and ϵ that make $rm_{d,\epsilon}$ a consistent mean estimator is equivalent to finding the solution of $E[rm_{d,\epsilon}]=E[X]$. Rearranging the definition, $rm_{d,\epsilon}=\lim_{c\to\infty}\left(\frac{(\mathrm{BM}_\epsilon+c)^{d+1}}{(median+c)^d}-c\right)=(d+1)\,\mathrm{BM}_\epsilon-d\mathrm{median}=\mu$. So, $d=\frac{\mu-\mathrm{BM}_\epsilon}{\mathrm{BM}_\epsilon-median}$. The pdf of the exponential distribution is $f(x)=\lambda^{-1}e^{-\lambda^{-1}x}$, $\lambda\geq 0$, $x\geq 0$, the cdf is $F(x)=1-e^{-\lambda^{-1}x}$, $x\geq 0$. The quantile function is $Q(p)=\ln\left(\frac{1}{1-p}\right)\lambda$. $E[x]=\lambda$. $E[median]=Q\left(\frac{1}{2}\right)=\ln 2\lambda$. For the exponential distribution, the expectation of $\mathrm{BM}_{\frac{1}{8}}$ is $E\left[\mathrm{BM}_{\frac{1}{8}}\right]=\lambda\left(1+\ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)$. Obviously, the scale parameter λ can be canceled out, $d\approx 0.375$. The proof of the second assertion follows directly from the coincidence property. For any symmetric distribution with a finite second moment, $E[\mathrm{BM}_\epsilon]=E[median]=E[X]$. Then $E[rm_{d,\epsilon}]=\lim_{c\to\infty}\left(\frac{(E[X]+c)^{d+1}}{(E[X]+c)^d}-c\right)=E[X]$. The proof for the Pareto distribution is more general. The mean of the Pareto distribution is given by $\frac{\alpha x_m}{\alpha-1}$. The d value with two unknown percentiles p_1 and p_2 for the Pareto distribution is $d_{Perato}=\frac{\frac{\alpha x_m}{\alpha-1}-x_m(1-p_1)^{-\frac{1}{\alpha}}}{x_m(1-p_1)^{-\frac{1}{\alpha}}-x_m(1-p_2)^{-\frac{1}{\alpha}}}$. Since any weighted average can be expressed as an integral of the quantile function, $\lim_{\alpha\to\infty}\frac{\frac{\alpha}{\alpha-1}-(1-p_1)^{-1/\alpha}}{(1-p_1)^{-1/\alpha}-(1-p_2)^{-1/\alpha}}=-\frac{\ln(1-p_1)+1}{\ln(1-p_1)-\ln(1-p_2)}$, the d value for the Pareto distribution approaches that of the exponential distribution as $\alpha\to\infty$, regardless of the type of weighted average used. This completes the demonstration. □

Theorem .1 implies that for the Weibull, gamma, Pareto, lognormal and generalized Gaussian distribution, $rm_{d\approx 0.375,\epsilon=\frac{1}{8}}$ is consistent for at least one particular case of these two-parameter distributions. The biases of $rm_{d\approx 0.375,\epsilon=\frac{1}{8}}$ for distributions with skewness between those of the exponential and symmetric distributions are tiny (SI Dataset S1). $rm_{d\approx 0.375,\epsilon=\frac{1}{8}}$ has excellent performance for all these common unimodal distributions (SI Dataset S1).

Besides introducing the concept of invariant mean, the purpose of this paper is to demonstrate that, in light of previous works, the estimation of central moments can be transformed into a location estimation problem by using U-statistics, the central moment kernel distributions have nice properties, and a series of sophisticated yet efficient robust estimators can be constructed whose biases are typically smaller than the variances ($n=5400$, Table **??**) for unimodal distributions.

Data Availability. Data for Table **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

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