

Near-consistent robust estimations of moments for unimodal distributions

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Descriptive statistics for parametric models currently heavily rely on the accuracy of distributional assumptions. Here, based on the invariant structures of unimodal distributions, a series of sophisticated yet efficient estimators, robust to both gross errors and departures from parametric assumptions, are proposed for estimating mean and central moments with insignificant asymptotic biases for common continuous unimodal distributions. This article also illuminates the understanding of the common nature of probability distributions and the measures of them.

orderliness | invariant | unimodal | adaptive estimation | U -statistics

The asymptotic inconsistencies between sample mean (\bar{x}) and nonparametric robust location estimators in asymmetric distributions on the real line have been noticed for more than two centuries (1), yet remain unsolved. Strictly speaking, it is unsolvable as by trimming, some information about the original distribution is removed, making it impossible to estimate the values of the removed parts without distributional assumptions. Newcomb (1886, 1912) provided the first modern approach to this problem by developing a class of estimators that gives "less weight to the more discordant observations" (2, 3). In 1964, Huber (4) used the minimax procedure to obtain M-estimator for the contaminated normal distribution, which has played a pre-eminent role in the later development of robust statistics. However, as previously demonstrated, under growing asymmetric departures from normality, the bias of the Huber M-estimator increases rapidly. This is a common issue in parametric estimations. For example, He and Fung (1999) constructed (5) a robust M-estimator for the two-parameter Weibull distribution, from which all moments can be calculated. Nonetheless, it is inadequate for the gamma, Perato, lognormal, and the generalized Gaussian distributions (SI Dataset S1). Another old and interesting approach is arithmetically computing the parameters using one or more L -statistics as inputs, such as percentile estimators. Examples of percentile estimators for the Weibull distribution, the reader is referred to Menon (1963) (6), Dubey (1967) (7), Hassanein (1971) (8), Marks (2005) (9), and Boudt, Caliskan, and Croux (2011) (10)'s works. At the outset of the study of percentile estimators, it was known that they arithmetically utilizes the invariant structures of probability distributions (6, 11, 12). Maybe such estimators can be named as I -statistics. Formally, an estimator is classified as an I -statistic if it asymptotically satisfies $I(LE_1, \dots, LE_l) = (\theta_1, \dots, \theta_q)$ for the distribution it is consistent, where LEs are calculated with the use of L -statistics, I is defined using arithmetic operations and constants, but it may also incorporate other functions, and θ s are the population parameters it estimates. A subclass of I -statistics, arithmetic I -statistics, is defined as LEs are L -statistics, I is solely defined using arithmetic operations and constants. Since some percentile estimators use the logarithmic func-

tion to transform all random variables before computing the L -statistics, a percentile estimator might not always be an arithmetic I -statistic (7). In this article, two subclasses of I -statistics are introduced, arithmetic I -statistics and quantile I -statistics. Examples of quantile I -statistics will be discussed later. Based on L -statistics, I -statistics are naturally robust. Compared to probability density functions (pdfs) and cumulative distribution functions (cdfs), the quantile functions of many parametric distributions are more elegant. Since the expectation of an L -statistic can be expressed as an integral of the quantile function, I -statistics are often analytically obtainable. However, the performance of the aforementioned examples is often worse than that of the robust M -statistics when the distributional assumption is violated (SI Dataset S1). Even when distributions such as the Weibull and gamma belong to the same larger family, the generalized gamma distribution, a misassumption can still result in substantial biases, rendering the approach ill-suited.

In previous research on semiparametric robust mean estimation, the binomial mean (BM_ϵ) is still inconsistent for any skewed distribution, despite having much smaller asymptotic biases than other weighted averages. All robust location estimators commonly used are symmetric due to the universality of the symmetric distributions. One can construct an asymmetric weighted average that is consistent for a semiparametric class of skewed distributions. This approach has been investigated previously, but its lack of symmetry makes it suitable only for certain applications (13). Shifting from semiparametrics to parametrics, an ideal robust location estimator would have a non-sample-dependent breakdown point (defined in Subsection ??) and be consistent for any symmetric distribution and a skewed distribution with finite second moments. This is called an invariant mean. Based on the mean-symmetric weighted

Significance Statement

Bias, variance, and contamination are the three main errors in statistics. Consistent robust estimation is unattainable without parametric assumptions. Here, based on a paradigm shift inspired by mean-median-mode inequality, Bickel-Lehmann spread, and adaptive estimation, invariant moments are proposed as a means of achieving near-consistent and robust estimations of moments, even in scenarios where moderate violations of distributional assumptions occur, while the variances are sometimes smaller than those of the sample moments.

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average-median inequality, the recombined mean is defined as

$$rm_{d,\epsilon,n} := \lim_{c \rightarrow \infty} \left(\frac{(\text{SWA}_{\epsilon,n} + c)^{d+1}}{(\text{median} + c)^d} - c \right),$$

where d is the key factor for bias correction, $\text{SWA}_{\epsilon,n}$ is $\text{BM}_{\epsilon,n}$ in the first three Subsections, but other symmetric weighted averages can also be used in practice as long as the inequalities hold. The following theorem shows the significance of this arithmetic I -statistic.

Theorem .1. *If the second moments are finite, $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ is a consistent mean estimator for the exponential and any symmetric distributions and the Pareto distribution with quantile function $Q(p) = x_m(1-p)^{-\frac{1}{\alpha}}$, $x_m > 0$, when $\alpha \rightarrow \infty$.*

Proof. Finding d and ϵ that make $rm_{d,\epsilon}$ a consistent mean estimator is equivalent to finding the solution of $E[rm_{d,\epsilon}] = E[X]$. Rearranging the definition, $rm_{d,\epsilon} = \lim_{c \rightarrow \infty} \left(\frac{(\text{BM}_{\epsilon} + c)^{d+1}}{(\text{median} + c)^d} - c \right) = (d+1)\text{BM}_{\epsilon} - d\text{median} = \mu$. So, $d = \frac{\mu - \text{BM}_{\epsilon}}{\text{BM}_{\epsilon} - \text{median}}$. The quantile function of the exponential distribution is $Q(p) = \ln\left(\frac{1}{1-p}\right)\lambda$. $E[x] = \lambda$. $E[\text{median}] = Q\left(\frac{1}{2}\right) = \ln 2\lambda$. For the exponential distribution, the expectation of $\text{BM}_{\frac{1}{8}}$ is $E\left[\text{BM}_{\frac{1}{8}}\right] = \lambda \left(1 + \ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)$. Obviously, the scale parameter λ can be canceled out, $d \approx 0.375$. The proof of the second assertion follows directly from the coincidence property. For any symmetric distribution with a finite second moment, $E[\text{BM}_{\epsilon}] = E[\text{median}] = E[X]$. Then $E[rm_{d,\epsilon}] = \lim_{c \rightarrow \infty} \left(\frac{(E[X] + c)^{d+1}}{(E[X] + c)^d} - c \right) = E[X]$. The proof for the Pareto distribution is more general. The mean of the Pareto distribution is given by $\frac{\alpha x_m}{\alpha - 1}$. The d value with two unknown percentiles p_1 and p_2 for the Pareto distribution is $d_{\text{Pareto}} = \frac{\frac{\alpha x_m}{\alpha - 1} - x_m(1-p_1)^{-\frac{1}{\alpha}}}{x_m(1-p_1)^{-\frac{1}{\alpha}} - x_m(1-p_2)^{-\frac{1}{\alpha}}}$. Since any weighted average can be expressed as an integral of the quantile function, $\lim_{\alpha \rightarrow \infty} \frac{\frac{\alpha}{\alpha-1} - (1-p_1)^{-1/\alpha}}{(1-p_1)^{-1/\alpha} - (1-p_2)^{-1/\alpha}} = -\frac{\ln(1-p_1)+1}{\ln(1-p_1)-\ln(1-p_2)}$, the d value for the Pareto distribution approaches that of the exponential distribution as $\alpha \rightarrow \infty$, regardless of the type of weighted average used. This completes the demonstration. \square

Theorem .1 implies that for the Weibull, gamma, Pareto, lognormal and generalized Gaussian distribution, $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ is consistent for at least one particular case of these two-parameter distributions. The biases of $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ for distributions with skewness between those of the exponential and symmetric distributions are tiny (SI Dataset S1). $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ exhibits excellent performance for all these common unimodal distributions (SI Dataset S1).

Besides introducing the concept of invariant mean, the purpose of this paper is to demonstrate that, in light of previous works, the estimation of central moments can be transformed into a location estimation problem by using U -statistics, the central moment kernel distributions possess desirable properties, and a series of sophisticated yet efficient robust estimators can be constructed whose biases are typically smaller than the variances (as seen in Table ?? for $n = 5400$) for unimodal distributions.

Background and Main Results

A. Invariant mean. It has long been known that a theoretical model can be adjusted to fit the first two moments of the observed data. A continuous distribution belonging to a location-scale family takes the form $F(x) = F_0\left(\frac{x-\mu}{\lambda}\right)$, where F_0 is a "standard" distribution. Therefore, $F(x) = Q^{-1}(x) \rightarrow x = Q(p) = \lambda Q_0(p) + \mu$. Thus, any weighted average can be expressed as $\lambda \text{WA}_0(\epsilon) + \mu$, where $\text{WA}_0(\epsilon)$ is an integral of $Q_0(p)$ according to the definition of the weighted average. The simultaneous cancellation of μ and λ in $\frac{(\lambda\mu_0 + \mu) - (\lambda\text{BM}_0(\epsilon) + \mu)}{(\lambda\text{BM}_0(\epsilon) + \mu) - (\lambda\text{median}_0 + \mu)}$ assures that d is a constant. Consequently, the roles of BM_{ϵ} and median in $rm_{d,\epsilon}$ can be replaced by any weighted averages, although only symmetric weighted averages are considered in defining the invariant mean.

The performance in heavy-tailed distributions can be further improved by constructing the quantile mean as

$$qm_{d,\epsilon,n} := \hat{Q}_n \left(\left(\hat{F}_n(\text{SWA}_{\epsilon,n}) - \frac{1}{2} \right) d + \hat{F}_n(\text{SWA}_{\epsilon,n}) \right),$$

provided that $\hat{F}_n(\text{SWA}_{\epsilon,n}) \geq \frac{1}{2}$, where $\hat{F}_n(x)$ is the empirical cumulative distribution function of the sample, \hat{Q}_n is the sample quantile function. The most popular method for computing the sample quantile function was proposed by Hyndman and Fan in 1996 (14). To minimize the finite sample bias, here, $\hat{F}_n(x) := \frac{1}{n} \left(\frac{x - X_{sp}}{X_{sp+1} - X_{sp}} + sp \right)$, where $sp = \sum_{i=1}^n \mathbf{1}_{X_i \leq x}$, $\mathbf{1}_A$ is the indicator of event A . The solution of $\hat{F}_n(\text{SWA}_{\epsilon,n}) < \frac{1}{2}$ is reversing the percentile by $1 - \hat{F}_n(\text{SWA}_{\epsilon,n})$, the obtained percentile is also reversed. Without loss of generality, in the following discussion, only the case where $\hat{F}_n(\text{SWA}_{\epsilon,n}) \geq \frac{1}{2}$ is considered. Moreover, in extreme heavy-tailed distributions, the calculated percentile can exceed the breakdown point of SWA_{ϵ} , so the percentile will be modified to $1 - \epsilon$ if this occurs. The quantile mean uses the location-scale invariant in a different way as shown in the following proof.

Theorem A.1. *$qm_{d \approx 0.321, \epsilon = \frac{1}{8}}$ is a consistent mean estimator for the exponential, Pareto ($\alpha \rightarrow \infty$) and any symmetric distributions provided that the second moments are finite.*

Proof. Similarly, rearranging the definition, $d = \frac{F(\mu) - F(\text{BM}_{\epsilon})}{F(\text{BM}_{\epsilon}) - \frac{1}{2}}$. The cdf of the exponential distribution is $F(x) = 1 - e^{-\lambda^{-1}x}$, $\lambda \geq 0$, $x \geq 0$, the expectation of BM_{ϵ} can be expressed as $\lambda \text{BM}_0(\epsilon)$, so $F(\text{BM}_{\epsilon})$ is free of λ . When $\epsilon = \frac{1}{8}$, $d = \frac{-e^{-1} + e^{-\left(1 + \ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)}}{\frac{1}{2} - e^{-\left(1 + \ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)}} \approx 0.321$. The proof of the symmetric case is similar. Since for any symmetric distribution with a finite second moment, $F(E[\text{BM}_{\epsilon}]) = F(\mu) = \frac{1}{2}$. Then, the expectation of the quantile mean is $qm_{d,\epsilon} = F^{-1}\left(\left(F(\mu) - \frac{1}{2}\right)d + F(\mu)\right) = F^{-1}\left(0 + F(\mu)\right) = \mu$.

For the assertion related to the Pareto distribution, the cdf of it is $1 - \left(\frac{x_m}{x}\right)^{\alpha}$. So, the d value with two unknown percentile p_1 and p_2 is $d_{\text{Pareto}} = \frac{1 - \left(\frac{x_m}{\frac{x_m}{\alpha-1}}\right)^{\alpha} - \left(1 - \left(\frac{x_m}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^{\alpha}\right)}{\left(1 - \left(\frac{x_m}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^{\alpha}\right) - \left(1 - \left(\frac{x_m}{x_m(1-p_2)^{-\frac{1}{\alpha}}}\right)^{\alpha}\right)} = \frac{1 - \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} - p_1}{p_1 - p_2}$. When $\alpha \rightarrow \infty$, $\left(\frac{\alpha-1}{\alpha}\right)^{\alpha} = \frac{1}{e}$. The d value for the exponential distribution is identical, since $d_{\text{exp}} =$

$$\frac{(1-e^{-1}) - \left(1-e^{-\ln\left(\frac{1}{1-p_1}\right)}\right)}{\left(1-e^{-\ln\left(\frac{1}{1-p_1}\right)}\right) - \left(1-e^{-\ln\left(\frac{1}{1-p_2}\right)}\right)} = \frac{1-\frac{1}{e}-p_1}{p_1-p_2}. \quad \text{All results are now proven.} \quad \square$$

The definitions of location and scale parameters are such that they must satisfy $F(x; \lambda, \mu) = F\left(\frac{x-\mu}{\lambda}; 1, 0\right)$. By recalling $x = \lambda Q_0(p) + \mu$, it follows that the percentile of any weighted average is free of λ and μ , which guarantees the validity of the quantile mean. The quantile mean is a quantile I -statistic. Specifically, an estimator is classified as a quantile I -statistic if LEs are percentiles of a distribution obtained by plugging L -statistics into a cumulative distribution function and I is defined with arithmetic operations, constants and quantile functions. $qm_{d \approx 0.321, \epsilon = \frac{1}{8}}$ works better in the fat-tail scenarios (SI Dataset S1). Theorem .1 and A.1 show that $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ and $qm_{d \approx 0.321, \epsilon = \frac{1}{8}}$ are both consistent mean estimators for any symmetric distribution and a skewed distribution with finite second moments. It's obvious that the breakdown points of $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ and $qm_{d \approx 0.321, \epsilon = \frac{1}{8}}$ are both $\frac{1}{8}$. Therefore they are all invariant means.

To study the impact of the choice of SWAs in rm and qm , it is constructive to recall that a symmetric weighted average is a linear combination of symmetric quantile averages. While using a less-biased symmetric weighted average can generally enhance performance (SI Dataset S1), there is a greater risk of violation in the semiparametric framework. However, the mean-SWA-median inequality is robust to slight fluctuations of the SQA function of the underlying distribution. Suppose the SQA function is generally decreasing in $[0, u]$, but increasing in $[u, \frac{1}{2}]$, since $1-2\epsilon$ of the symmetric quantile averages will be included in the computation of SWA_ϵ , as long as $\frac{1}{2} - u \ll 1-2\epsilon$, and other portions of the SQA function satisfy the inequality constraints that define the ν th orderliness on which the SWA_ϵ is based, the mean-SWA $_\epsilon$ -median inequality will still hold. This is due to the violation being bounded (15) and therefore cannot be extreme for unimodal distributions. For instance, the SQA function is non-monotonic when the shape parameter of the Weibull distribution $\alpha > \frac{1}{1-\ln(2)} \approx 3.259$ as shown in the previous article, the violation of the third orderliness starts near this parameter as well, yet the mean-BM $_{\frac{1}{8}}$ -median inequality is still valid when $\alpha \leq 3.322$. Another key factor in determining the risk of violation is the skewness of the distribution. Previously, it was demonstrated that in a family of distributions differing by a skewness-increasing transformation in van Zwet's sense, the violation of orderliness, if it happens, often only occurs when the distribution is nearly symmetrical (16). The over-corrections in rm and qm are dependent on the SWA_ϵ -median difference, which can be a reasonable measure of skewness (17, 18), implying that the over-correction is often tiny with a moderate d . This qualitative analysis provides another perspective, in addition to the bias bounds (15), that rm and qm based on the mean-SWA $_\epsilon$ -median inequality are generally safe.

B. Robust estimations of the central moments. In 1976, Bickel and Lehmann, in their third paper of the landmark series *Descriptive Statistics for Nonparametric Models* (19), generalized a class of estimators called "measures of disperse," which is now often named as Bickel-Lehmann dispersion. As an example,

they proposed a first version of the trimmed standard deviation, $\hat{\tau}^2(F; \epsilon) \equiv \tau^2(F; \epsilon)$, for independent and identically distributed random variables X with a distribution F , where $\tau^2(F; \epsilon) = \frac{1}{1-2\epsilon} \int_{Q(\epsilon)}^{Q(1-\epsilon)} y dG(y)$, Q is the quantile function of G , G is the distribution of $Y = X^2$. Obviously, when $\epsilon = 0$, the result is equivalent to the second raw moment. In 1979, in the same series (20), they explored another class of estimators called "measures of spread," which "does not require the assumption of symmetry." From that, a popular efficient scale estimator, the Rousseeuw-Croux scale estimator (21), was derived in 1993, but the importance of tackling the symmetry assumption has been greatly underestimated. In the final section of that paper (20), they considered another two possible versions of the trimmed standard deviations, which were modified here for comparison,

$$\left[n \left(\frac{1}{2} - \epsilon\right)\right]^{-\frac{1}{2}} \left[\sum_{i=\frac{n}{2}}^{n(1-\epsilon)} [X_i - X_{n-i+1}]^2 \right]^{\frac{1}{2}}, \quad [1]$$

and

$$\left[\binom{n}{2} (1 - \epsilon - \gamma\epsilon)\right]^{-\frac{1}{2}} \left[\sum_{i=\binom{n}{2}\epsilon}^{\binom{n}{2}(1-\gamma\epsilon)} (X - X')_i^2 \right]^{\frac{1}{2}}, \quad [2]$$

where $(X - X')_1 \leq \dots \leq (X - X')_{\binom{n}{2}}$ are the order statistics of the "pseudo-sample". The paper ended with, "We do not know a fortiori which of the measures [1] or [2] is preferable and leave these interesting questions open."

Observe that the kernel of the unbiased estimation of the second central moment by using U -statistic is $\psi_2(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^2$. If adding the $\frac{1}{2}$ term in [2], as $\epsilon \rightarrow 0$, the result is equivalent to the standard deviation estimated by using U -statistic (also noted by Janssen, Serfling, and Veraverbeke in 1987) (22). In fact, they also showed that, when ϵ is 0, [2] is $\sqrt{2}$ times the standard deviation.

To address their open question, the nomenclature used in this paper is introduced as follows:

Nomenclature. Given a robust estimator $\hat{\theta}$. The first part of the name of the robust statistic defined in this paper is a name that indicates the type of estimator, and the second part is the name of the population parameter θ that the estimator is consistent with as $\epsilon \rightarrow 0$. The abbreviation of the estimator is formed by combining the initial letter(s) of the first part with the common abbreviation of the consistent estimator that measures the population parameter. If the estimator is symmetric and not a U -statistic, ϵ is indicated in the subscript of the abbreviation of the estimator. For asymmetric estimators, the corresponding γ is also indicated after ϵ . For weighted U -statistics, the breakdown point of the location estimator is indicated, except the median.

In the previous article on semiparametric robust mean estimation, it was shown that if the breakdown point of a robust estimator is adjustable, then its bias is often monotonic with respect to the breakdown point in a semiparametric distribution and consistent with the population parameter when the breakdown point is zero. Naturally, its name should align with the consistent estimator. The trimmed standard deviation following this nomenclature is $Tsd_{\epsilon, \gamma, n} :=$

262 $\left[\text{TM}_{\epsilon, \gamma} \left((\psi_2(X_{N_1}, X_{N_2}))_{N=1}^{(n)} \right) \right]^{-\frac{1}{2}}$, where $\text{TM}_{\epsilon, \gamma}(Y)$ denotes
 263 the ϵ, γ -trimmed mean with the sequence $(\psi_2(X_{N_1}, X_{N_2}))_{N=1}^{(n)}$
 264 as an input. Removing the square root yields the trimmed
 265 variance $(\text{Tvar}_{\epsilon, \gamma, n})$. It is now very clear that this definition,
 266 essentially the same as [2], should be preferable. Not only
 267 because it is essentially a trimmed U -statistic for the standard
 268 deviation but also because the γ -orderliness of the second central
 269 moment kernel distribution is ensured by the next exciting
 270 theorem.

271 **Theorem B.1.** *The second central moment kernel distribution*
 272 *generated from any continuous unimodal distribution is γ -*
 273 *ordered, if $\gamma \geq 1$.*

274 *Proof.* Let $Q(p)$, $0 \leq p \leq 1$, denote the quantile of the continuous
 275 unimodal distribution $f_X(x)$. The corresponding probability
 276 density is $f(Q(p))$. Generating the distribution of the pair
 277 $(Q(p_i), Q(p_j))$, $i < j$, $p_i < p_j$, the corresponding probability
 278 density is $f_{X,X}(Q(p_i), Q(p_j)) = 2f(Q(p_i))f(Q(p_j))$.
 279 Transforming the pair $(Q(p_i), Q(p_j))$, $i < j$, by the function
 280 $\Phi(x_1, x_2) = x_1 - x_2$, the pairwise difference distribution has
 281 a mode that is arbitrary close to $M - M = 0$. The monotonic
 282 increasing of the pairwise difference distribution was
 283 first implied in its unimodality proof done by Hodges
 284 and Lehmann in 1954 (23). Whereas they used induction to get
 285 the result, Dharmadhikari and Jogdeo in 1982 (24) provided a
 286 modern proof of the unimodality using Khintchine's representation
 287 (25). Assuming absolute continuity, Purkayastha (26)
 288 introduced a much simpler proof in 1998. Transforming the
 289 pairwise difference distribution by squaring and multiplying by
 290 $\frac{1}{2}$ does not change the monotonicity, making the pdf become
 291 monotonically decreasing with mode at zero. In the previous
 292 article, it was proven that a right skewed distribution with a
 293 monotonic decreasing pdf is always γ -ordered, if $\gamma \geq 1$, which
 294 gives the desired result. \square

295 *Remark.* The assumption of continuity of distributions is important
 296 for monotonicity because, unlike in the continuous
 297 case, it is possible to obtain pairs with the same value for a
 298 discrete distribution. For example, let the probabilities of the
 299 singletons $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$ of a probability mass
 300 function of a discrete probability distribution be $\frac{1}{11}$, $\frac{4}{11}$, $\frac{3}{11}$,
 301 $\frac{2}{11}$, and $\frac{1}{11}$, respectively. This is a unimodal distribution, but
 302 the corresponding ψ_2 distribution is non-monotonic, whose
 303 singletons $\{0\}$, $\{0.5\}$, $\{2\}$, $\{4.5\}$ and $\{8\}$ have probabilities
 304 $\frac{21}{66}$, $\frac{24}{66}$, $\frac{2}{14}$, $\frac{6}{66}$, and $\frac{1}{66}$, respectively.

305 Previously, it was shown that any symmetric distribution
 306 with a finite second moment is ν th ordered, indicating that
 307 orderliness does not require unimodality, e.g., a symmetric
 308 bimodal distribution is also ordered. On the other hand, an
 309 analysis of the Weibull distribution shows that unimodality
 310 does not guarantee orderliness. Theorem B.1 reveals another
 311 profound relationship between unimodality and orderliness,
 312 which is sufficient for trimming inequality.

313 In 1928, Fisher constructed k -statistics as unbiased estimators
 314 of cumulants (27). Halmos (1946) proved that the functional θ
 315 admits an unbiased estimator if and only if it is a regular statistical
 316 functional of degree k and showed a relation of symmetry, unbiasedness
 317 and minimum variance (28). In 1948, Hoeffding generalized U -statistics (29)
 318 which enable the derivation of a minimum-variance unbiased estimator from

each unbiased estimator of an estimable parameter. Heffernan (1997) (30)
 obtained an unbiased estimator of the k th central moment by using
 320 U -statistics and demonstrated that it is the minimum variance
 321 unbiased estimator for distributions with finite moments (31, 32).
 322 In 1976, Saleh generalized the Hodges-Lehmann (H-L) estimator (33)
 323 to the trimmed H-L mean (which he named "Wilcoxon one-sample statistic")
 324 (34). In 1984, Serfling pointed out the speciality of Hodges-Lehmann
 325 estimator, which is neither a simple L -statistic nor a U -statistic,
 326 and considered the generalized L -statistics and U -statistic structure (35).
 327 Also in 1984, Janssen and Serfling and Veraverbeke (36) showed that the
 328 Bickel-Lehmann spread also belongs to the same class. It gradually became
 329 clear that the Hodges-Lehmann estimator, trimmed H-L mean and trimmed
 330 standard deviation are all trimmed U -statistics (37–39). 334

Extending the trimmed U -statistic to weighted U -statistic,
 i.e., replacing the trimmed mean with weighted average. The
 weighted k th central moment ($k \leq n$) is defined as,

$$Wkm_{\epsilon, \gamma, n} := \text{WA}_{\epsilon, \gamma, n} \left((\psi_k(X_{N_1}, \dots, X_{N_k}))_{N=1}^{(n)} \right),$$

where X_{N_1}, \dots, X_{N_k} are the n choose k elements from X ,
 335 $\psi_k(x_1, \dots, x_k) = \sum_{j=0}^{k-2} (-1)^j \binom{k-1}{j} \sum (x_{i_1}^{k-j} \dots x_{i_{j+1}}) +$
 336 $(-1)^{k-1} (k-1) x_1 \dots x_k$, the second summation is over
 337 $i_1, \dots, i_{j+1} = 1$ to k with $i_1 < \dots < i_{j+1}$ (30). Despite the
 338 complexity, the structure of the k th central moment kernel
 339 distributions can be elucidated by decomposing. 340

Data Availability. Data for Table ?? are given in SI Dataset S1.
 All codes have been deposited in [GitHub](#). 342

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