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## 2 **Supporting Information for**

### 3 **Near-consistent robust estimations of moments for unimodal distributions**

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#### 6 **This PDF file includes:**

- 7     Supporting text
- 8     Legend for Dataset S1
- 9     SI References

#### 10 **Other supporting materials for this manuscript include the following:**

- 11     Dataset S1

## Supporting Information Text

**Theorem 0.1.**  $\psi_k(x_1 = \lambda x_1 + \mu, \dots, x_k = \lambda x_k + \mu) = \lambda^k \psi_k(x_1, \dots, x_k)$ .

*Proof.*  $\psi_k$  can be divided into  $k$  groups. From 1st to  $k-1$ th group, the  $g$ th group has  $\binom{k}{g}\binom{g}{1}$  terms having the form  $(-1)^{g+1} \frac{1}{k-g+1} x_{i_1}^{k-g+1} \dots x_{i_g}$ . The final  $k$ th group is the term  $(-1)^{k-1} (k-1) x_1 \dots x_k$ . Let  $x_{i_1} = x_1$ ,  $k \neq g$ , the  $g$ th group of  $\psi_k$  has  $\binom{k-l}{g-l}$  terms having the form  $(-1)^{g+1} \frac{1}{k-g+1} x_1^{k-g+1} x_2 \dots x_l x_{i_1} \dots x_{i_{g-l}}$ , where  $x_1, x_2, \dots, x_l$  are fixed,  $x_{i_1}, \dots, x_{i_{g-l}}$  are selected such that  $i_1, \dots, i_{g-l} \neq 1, 2, \dots, l$ . Let  $\Psi_k(x_1, x_2, \dots, x_l, x_{i_1}, \dots, x_{i_{g-l}}) = (\lambda x_1 + \mu)^{k-g+1} (\lambda x_2 + \mu) \dots (\lambda x_l + \mu) (\lambda x_{i_1} + \mu) \dots (\lambda x_{i_{g-l}} + \mu)$ , the first group of  $\Psi_k$  is  $\lambda^k x_1 \dots x_l x_{i_1} \dots x_{i_{g-l}}$ , the  $h$ th group of  $\Psi_k$ ,  $h > 1$ , has  $\binom{k-g+1}{k-h-l+2}$  terms having the form  $\lambda^{k-h+1} \mu^{h-1} x_1^{k-h-l+2} x_2 \dots x_l, x_1^{k-h-l+2} \neq x_1$ , the summed coefficient is  $S1_l = \sum_{g=l}^{h+l-1} (-1)^{g+1} \frac{1}{k-g+1} \binom{k-g+1}{k-h-l+2} \binom{k-l}{g-l} = \sum_{g=l}^{h+l-1} (-1)^{g+1} \frac{(k-l)!}{(h+l-g-1)!(k-h-l+2)!(g-l)!} = 0$ , since the summation is starting from  $l$ , ending at  $h+l-1$ , the first term includes the factor  $g-l=0$ , the final term includes the factor  $h+l-g-1=0$ , the terms in the middle are also zero due to the factorial property.

Another possible choice is letting one of  $x_{i_2} \dots x_{i_g}$  equal to  $x_1$ , the  $g$ th group of  $\psi_k$  has  $(k-h) \binom{h-1}{g-k+h-1}$  terms having the form  $(-1)^{g+1} \frac{1}{k-g+1} x_1 x_2 \dots x_j^{k-g+1} \dots x_{k-h+1} x_{i_1} \dots x_{i_{g-k+h-1}}$ , provided that  $k \neq g$ ,  $2 \leq j \leq k-h+1$ , where  $x_1, \dots, x_{k-h+1}$  are fixed,  $x_j^{k-g+1}$  and  $x_{i_1}, \dots, x_{i_{g-k+h-1}}$  are selected. Transforming these terms by  $\Psi_k(x_1, x_2, \dots, x_j, \dots, x_{k-h+1}, x_{i_1}, \dots, x_{i_{g-k+h-1}}) = (\lambda x_1 + \mu) (\lambda x_2 + \mu) \dots (\lambda x_j + \mu)^{k-g+1} \dots (\lambda x_{k-h+1} + \mu) (\lambda x_{i_1} + \mu) \dots (\lambda x_{i_{g-k+h-1}} + \mu)$ , then, there are  $k-g+1$  terms having the form  $\lambda^{k-h+1} \mu^{h-1} x_1 x_2 \dots x_{k-h+1}$ . So, the combined result is  $(-1)^{g+1} (k-h) \binom{h-1}{g-k+h-1} \lambda^{k-h+1} \mu^{h-1} x_1 x_2 \dots x_{k-h+1}$ . Transforming the final  $k$ th group of  $\psi_k$  by  $\Psi_k$ , then, there is one term having the form  $(-1)^{k-1} (k-1) \lambda^{k-h+1} \mu^{h-1} x_1 x_2 \dots x_{k-h+1}$ . Another possible combination is that the  $g$ th group of  $\psi_k$  contains  $(g-k+h-1) \binom{h-1}{g-k+h-1}$  terms having the form  $(-1)^{g+1} \frac{1}{k-g+1} x_1 x_2 \dots x_{k-h+1} x_{i_1} \dots x_{i_j}^{k-g+1} \dots x_{i_{g-k+h-1}}$ , there is only one term having the form  $\lambda^{k-h+1} \mu^{h-1} x_1 x_2 \dots x_{k-h+1}$ . The above summation  $S1_l$  should also be included, i.e.,  $x_1^{k-h-l+2} = x_1$ ,  $k = h+l-1$ , so, combining all terms with  $\lambda^{k-h+1} \mu^{h-1} x_1 x_2 \dots x_{k-h+1}$ , according to the binomial theorem, the summed coefficient is  $S2_l = \sum_{g=k-h+1}^{k-1} (-1)^{g+1} \binom{h-1}{g-k+h-1} (k-h+1 + \frac{g-k+h-1}{k-g+1}) + (-1)^{k-1} (k-1) = (k-h+1) \sum_{g=k-h+1}^{k-1} (-1)^{g+1} \binom{h-1}{g-k+h-1} + \sum_{g=k-h+1}^{k-1} (-1)^{g+1} \binom{h-1}{g-k+h-1} \frac{(g-k+h-1)}{k-g+1} + (-1)^{k-1} (k-1) = (-1)^k (k-h+1) + (h-2)(-1)^k + (-1)^{k-1} (k-1) = 0$ . The summation identities are  $\sum_{g=k-h+1}^{k-1} (-1)^{g+1} \binom{h-1}{g-k+h-1} = (-1)^k$  and  $\sum_{g=k-h+1}^{k-1} (-1)^{g+1} \binom{h-1}{g-k+h-1} \frac{(g-k+h-1)}{k-g+1} = (h-2)(-1)^k$ . These two summation identities are proven in Lemma 0.2 and 0.3. The result is the same if replacing  $x_1$  with  $x_i$ , where  $i$  is from 2 to  $k$ , and replacing  $x_l$  with other  $x_i$ . Thus, all terms including  $\mu$  can be canceled out. The proof is complete by noticing that the remaining part is  $\lambda^k \psi_k(x_1, \dots, x_k)$ .  $\square$

**Lemma 0.2.**  $\sum_{g=k-h+1}^{k-1} (-1)^{g+1} \binom{h-1}{g-k+h-1} = (-1)^k$ .

*Proof.* Let  $u = k-h+1$ , then the above identity becomes  $\sum_{g=u}^{k-1} (-1)^{g+1} \binom{k-u}{g-u} = (-1)^k$ . Then, by deducing,

$$\begin{aligned} \sum_{g=u}^{k-1} (-1)^{g+1} \binom{k-u}{g-u} &= \sum_{i=0}^{k-u-1} (-1)^{i+u+1} \binom{k-u}{i} && \text{(Substitute } i = g-u) \\ &= (-1)^{k+2} + \sum_{i=0}^{k-u} (-1)^{i+u+1} \binom{k-u}{i} \\ &= (-1)^{k+2} + (-1)^{u+1} \sum_{i=0}^{k-u} (-1)^i \binom{k-u}{i} \\ &= (-1)^k && \text{(Apply the alternating sum identity),} \end{aligned}$$

the proof is complete.  $\square$

**Lemma 0.3.**  $\sum_{g=k-h+1}^{k-1} (-1)^{g+1} \binom{h-1}{g-k+h-1} \frac{(g-k+h-1)}{k-g+1} = (h-2)(-1)^k$ .

*Proof.* Let  $u = k-h+1$ , then the above identity becomes  $\sum_{g=u}^{k-1} (-1)^{g+1} \binom{k-u}{g-u-1} = (-1)^k (k-u-1)$ . Then by deducing,

$$\begin{aligned}
\sum_{g=u}^{k-1} (-1)^{g+1} \binom{k-u}{g-u-1} &= \sum_{i=-1}^{k-u-2} (-1)^{u+i+2} \binom{k-u}{i} && \text{(Substitute } i = g - u - 1\text{)} \\
&= \sum_{i=0}^{k-u} (-1)^{u+i+2} \binom{k-u}{i} - \sum_{i=k-u-1}^{k-u} (-1)^{u+i+2} \binom{k-u}{i} && \text{(Apply the alternating sum identity)} \\
&= - \sum_{i=k-u-1}^{k-u} (-1)^{u+i+2} \binom{k-u}{i} \\
&= (-1)^{k+2} \binom{k-u}{k-u-1} + (-1)^{k+3} \binom{k-u}{k-u} \\
&= (-1)^{k+2} (k-u) + (-1)^{k+3} \\
&= (-1)^{k+2} (k-u-1),
\end{aligned}$$

the proof is complete.  $\square$

## Methods

**A. d value calibration.** Asymptotic  $d$  values for the invariant central moments for the exponential distribution were approximated by a quasi-Monte Carlo study (1, 2) based on generating a large quasi-random sample with sample size 1.8 million from the exponential distribution and quasi-subsampling the sample 1.8k million times to approximate the distributions of the kernels of the corresponding  $U$ -statistics, then computing the binomial  $k$ th central moment ( $Bkm$ ), median  $k$ th central moment ( $mkm$ ), and corresponding quantiles, finally obtained by the formulae  $d_{rkm} = \frac{km_{bs} - Bkm_{bs}}{Bkm_{bs} - mkm_{bs}}$  and  $d_{qkm} = \frac{pkm_{bs} - pBkm_{bs}}{pBkm_{bs} - \frac{1}{2}}$ , where  $pBkm = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{X_i \leq (Bkm_{bs})}$ ,  $pkm = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{X_i \leq (km_{bs})}$ ,  $bs$  indicates the quasi-bootstrap moments. The accuracy of the estimates was verified by comparing the quasi-bootstrap central moments to their asymptotic values, yielding errors of  $\approx 0.0003$ ,  $\approx 0.001$ , and  $\approx 0.03$  for the second, third, and fourth central moments, respectively. The sample standard deviation of the kernel distributions for these moments are 2.234, 9.627, and 60.064, respectively, resulting in standardized errors for the biases that were all smaller than 0.001, thus ensuring the accuracy implied in the number of significant digits of the values in Table 1. The calculations of invariant moments using other symmetric weighted averages,  $SQM_{\frac{1}{8}}$ ,  $BM_{\nu=2, \epsilon=\frac{1}{8}}$ ,  $WM_{\frac{1}{8}}$ ,  $BWM_{\frac{1}{8}}$ , and  $TM_{\frac{1}{8}}$ , were performed analogously by substituting of the appropriate values. For invariant mean, the procedure is the same, just without quasi-bootstrap.

For finite sample, the  $d$  values were estimated using 1000 pseudorandom samples with sample size  $n = 5400$  with a quasi-bootstrap size of 18000. To estimate the errors of  $d$  value estimations under finite sample size, first consider the first order Taylor approximation of the  $d$  value function,  $d = \frac{x_1 - x_2}{x_2 - x_3} \approx d^0 + \frac{\partial d}{\partial x_1} x_1 + \frac{\partial d}{\partial x_2} x_2 + \frac{\partial d}{\partial x_3} x_3$ . Then, by applying Bienaymé's identity, the variance of  $d$  can be approximated by  $\sigma_d^2 \approx \left| \frac{\partial d}{\partial x_1} \right|^2 \sigma_{x_1}^2 + \left| \frac{\partial d}{\partial x_2} \right|^2 \sigma_{x_2}^2 + \left| \frac{\partial d}{\partial x_3} \right|^2 \sigma_{x_3}^2 + 2 \left| \frac{\partial d}{\partial x_1} \right| \left| \frac{\partial d}{\partial x_2} \right| Cov(X_1, X_2) + 2 \left| \frac{\partial d}{\partial x_1} \right| \left| \frac{\partial d}{\partial x_3} \right| Cov(X_1, X_3) + 2 \left| \frac{\partial d}{\partial x_2} \right| \left| \frac{\partial d}{\partial x_3} \right| Cov(X_2, X_3) = \left| \frac{1}{x_2 - x_3} \right|^2 \sigma_{x_1}^2 + \left| -\frac{x_1 - x_2}{(x_2 - x_3)^2} - \frac{1}{x_2 - x_3} \right|^2 \sigma_{x_2}^2 + \left| \frac{x_1 - x_2}{(x_2 - x_3)^2} \right|^2 \sigma_{x_3}^2 + 2 \left| \frac{1}{x_2 - x_3} \right| \left| -\frac{x_1 - x_2}{(x_2 - x_3)^2} - \frac{1}{x_2 - x_3} \right| Cov(X_1, X_2) + 2 \left| \frac{1}{x_2 - x_3} \right| \left| \frac{x_1 - x_2}{(x_2 - x_3)^2} \right| Cov(X_1, X_3) + 2 \left| -\frac{x_1 - x_2}{(x_2 - x_3)^2} - \frac{1}{x_2 - x_3} \right| \left| \frac{x_1 - x_2}{(x_2 - x_3)^2} \right| Cov(X_2, X_3).$

Since for the recombined mean,  $\sigma_{x_1}^2 = 0$ , so,  $\sigma_{d_{rm}}^2 \approx \left| -\frac{x_1 - x_2}{(x_2 - x_3)^2} - \frac{1}{x_2 - x_3} \right|^2 \sigma_{x_2}^2 + \left| \frac{x_1 - x_2}{(x_2 - x_3)^2} \right|^2 \sigma_{x_3}^2 + 2 \left( -\frac{x_1 - x_2}{(x_2 - x_3)^2} - \frac{1}{x_2 - x_3} \right) \left( \frac{x_1 - x_2}{(x_2 - x_3)^2} \right) Cov(X_2, X_3)$ , where  $x_1$  is the expected value,  $x_2$  is the weighted average used,  $x_3$  is the median. For quantile mean, since  $\sigma_{x_3}^2 = 0$ ,  $\sigma_{d_{qm}}^2 \approx \left| \frac{1}{x_2 - x_3} \right|^2 \sigma_{x_1}^2 + \left| -\frac{x_1 - x_2}{(x_2 - x_3)^2} - \frac{1}{x_2 - x_3} \right|^2 \sigma_{x_2}^2 + 2 \left( \frac{1}{x_2 - x_3} \right) \left( -\frac{x_1 - x_2}{(x_2 - x_3)^2} - \frac{1}{x_2 - x_3} \right) Cov(X_1, X_2)$ , where  $x_1$  is the percentile of the expected value,  $x_2$  is the percentile of the weighted average used,  $x_3$  is the percentile of median,  $\frac{1}{2}$ . Finally, the errors were estimated by the corresponding sample statistics. The results of error estimation were included in the SI Dataset S1.

**B. AAB, AB, and SSE.** The computations of ABs and AABs for invariant central moments were described in the Main Text. The SSE was computed by approximating the sampling distribution with 1000 pseudorandom samples for  $n = 5400$  and 30 pseudorandom samples for  $n = 1.8 \times 10^6$ . Common random numbers were used for better comparison. Analogous to the asymptotic bias, the scaled standard error can be standardized, averaged, and weighted. It should be noted that, in Table 1, for symmetric distributions, the generalized Gaussian, the standard errors were used for location and asymmetry estimators, since when the mean value is close to zero, the scaled standard error will approach infinity and therefore be too sensitive to small changes. The errors of AB and SSE were estimated by  $se(\bar{x}) = \frac{\sigma}{\sqrt{n}} \approx \frac{usb}{\sqrt{n}}$ ,  $se(sd) \approx \frac{1}{2\sigma} se(var) = \sqrt{\frac{\mu_4}{4n\sigma^2} - \frac{n-3}{4n(n-1)}\sigma^2} \approx$

78  $\sqrt{\frac{fm}{4nvar} - \frac{n-3}{4n(n-1)}var}$ , where  $usb$  is unbiased standard deviation of the sampling distribution with normality assumption (3).  
 79 The computational methods used for two-parameter distributions were identical. The computations of invariant moments were  
 80 described in the Main Text. The results of error estimation were included in the SI Dataset S1.

81 **C. The impact of bootstrap size on variance.** The study of the impact of the bootstrap size, from  $n = 1.8 \times 10^2$  to  $n = 1.8 \times 10^4$ ,  
 82 on the variance for the exponential distribution was done the same as above.

83 **D. Comparisons to M-estimator and percentile estimator.** Within the same kurtosis range and four two-parametric distributions  
 84 (except the generalized Gaussian distribution, since the logarithmic function does not produce results for negative values)  
 85 as the above, the percentile estimators were computed using the method proposed by Marks (2005) (4) (consistent with the  
 86 Weibull distribution) and the parameter setting proposed by Boudt, Caliskan, and Croux (2011) (5). The results were then  
 87 transformed to the first four moments to compute AABs. The robust M-estimators were also computed in the same way using  
 88 the method proposed by Huber (6) (consistent with the Gaussian distribution) and He and Fung (1999) (7) (consistent with  
 89 the Weibull distribution). Bisection is used to find the solution of the key equation in (7), while the results from the percentile  
 90 estimator were used as initial values (-0.3 and +0.3).

91 **E. Maximum asymptotic biases.** For simplicity, a brute force approach was used to estimate the maximum biases of SWAs and  
 92 SWkms for five unimodal distributions. A wide range was set to roughly estimate the parameter ranges in which the maximum  
 93 bias might occur (the corresponding maximum kurtoses are all larger than 500). Then, the parameter range was broken to 100  
 94 parts, combining with the above results for AAB estimations, the maximum of both was determined to be very close to the  
 95 true maximum. Pseudo-maximum bias was described in the Main Text.

96 The brute force approach is generally valid, i.e., the maximum is the global maximum, not local maximum, even when the  
 97 the corresponding maximum kurtosis is finite. Because all five distributions here have the property that, as the kurtosis of the  
 98 distribution increases to infinity, the standardized biases of SWAs approach zero.

For example, for the Perato distribution,

$$B_{\text{SQA}}(\epsilon, \alpha) = \frac{\frac{1}{2} \left( x_m (1 - \epsilon)^{-\frac{1}{\alpha}} + x_m \epsilon^{-\frac{1}{\alpha}} \right) - \frac{\alpha x_m}{\alpha - 1}}{\sqrt{\frac{\alpha x_m^2}{(1 - \alpha)^2 (\alpha - 2)}}}.$$

$$99 \lim_{\alpha \rightarrow 2} B_{\text{SQA}}(\epsilon, \alpha) = \lim_{\alpha \rightarrow 2} \frac{\frac{1}{2} \left( x_m (1 - \epsilon)^{-\frac{1}{\alpha}} + x_m \epsilon^{-\frac{1}{\alpha}} \right) - \frac{\alpha x_m}{\alpha - 1}}{\sqrt{\frac{\alpha x_m^2}{(1 - \alpha)^2 (\alpha - 2)}}} - \frac{\frac{\alpha x_m}{\alpha - 1}}{\sqrt{\frac{\alpha x_m^2}{(1 - \alpha)^2 (\alpha - 2)}}} = \lim_{\alpha \rightarrow 2} \frac{\frac{1}{2} \left( \frac{1}{\sqrt{\epsilon}} + \frac{1}{\sqrt{1 - \epsilon}} \right)}{\sqrt{\frac{\alpha}{(1 - \alpha)^2 (\alpha - 2)}}} + \lim_{\alpha \rightarrow 2} \frac{-\alpha}{\sqrt{\frac{\alpha}{(\alpha - 2)}}} = 0.$$

100 In the previous article, it is proven that when the kurtoses of the distributions approach infinity, all the five parametric  
 101 distributions will be ordered, that means the SWAs based on the orderliness will follow the mean-SWA-median inequality, thus,  
 102 proving the limits of the ratios between  $\mu$  and  $\sigma$ , as well as  $m$  and  $\sigma$  is enough.

103 For example, for the Weibull distribution, the ratio of  $\mu$  and  $\sigma$  is  $\lim_{\alpha \rightarrow 0} \frac{\Gamma(1 + \frac{1}{\alpha})}{\sqrt{\Gamma(\frac{\alpha + 2}{\alpha})}} = \lim_{\alpha \rightarrow 0} \frac{(1 + \frac{1}{\alpha} - 1)!}{\sqrt{(\frac{\alpha + 2}{\alpha} - 1)!}} = \lim_{\alpha \rightarrow 0} \frac{(\frac{1}{\alpha})!}{\sqrt{(2 \times \frac{1}{\alpha})!}} =$   
 104  $0$ , the ratio of  $m$  and  $\sigma$  is  $\lim_{\alpha \rightarrow 0} \frac{\sqrt[3]{\ln(2)}}{\sqrt{\Gamma(\frac{\alpha + 2}{\alpha})}} = \lim_{\alpha \rightarrow 0} \frac{0}{\sqrt{\Gamma(\frac{\alpha + 2}{\alpha})}} = 0$ .

105 Similarly, for the gamma distribution, the ratio of  $\mu$  and  $\sigma$  is  $\lim_{\alpha \rightarrow 0} \frac{\alpha}{\sqrt{\alpha}} = \lim_{\alpha \rightarrow 0} \frac{1}{\sqrt{\alpha}} = 0$ , the ratio of  $m$  and  $\sigma$  is  
 106  $\lim_{\alpha \rightarrow 0} \frac{P^{-1}(\alpha, \frac{1}{2})}{\sqrt{\alpha}} = 0$  (8).

107 The lognormal distribution is the same, the ratio of  $\mu$  and  $\sigma$  is  $\lim_{\sigma \rightarrow \infty} \frac{e^{\mu + \frac{\sigma^2}{2}}}{\sqrt{(e^{\sigma^2} - 1)e^{2\mu + \sigma^2}}} = \lim_{\sigma \rightarrow \infty} \frac{e^{\mu + \frac{\sigma^2}{2}}}{\sqrt{e^{2\mu + 2\sigma^2}}} = \lim_{\sigma \rightarrow \infty} \frac{e^{\frac{\sigma^2}{2}}}{e^{\sigma^2}} =$   
 108  $0$ , the ratio of  $m$  and  $\sigma$  is  $\lim_{\sigma \rightarrow \infty} \frac{e^{\mu}}{\sqrt{(e^{\sigma^2} - 1)e^{2\mu + \sigma^2}}} = 0$ .

109 As demonstrated, the growth rate of the standard deviation greatly exceeds that of the mean and that of the median. This  
 110 phenomenon is closely tied to the Taylor's law and is more widespread than these examples suggest.

#### 111 SI Dataset S1 (dataset\_one.xlsx)

112 Raw data of Table 1 in the Main Text.

## 113 References

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