Supporting Information for

Near-consistent robust estimations of moments for unimodal distributions

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Methods

Analogous to the asymptotic bias, the scaled standard error can be standardized, averaged, and weighted. It should be noted that, in Table 1, for symmetric distributions, the generalized Gaussian, the standard errors were used for location and asymmetry estimators, since when the mean value is close to zero, the scaled standard error will approach infinity and therefore be too sensitive to small changes.

Asymptotic d values for the invariant central moments for the exponential distribution were approximated by a quasi-Monte Carlo study (1, 2) based on generating a large quasi-random sample with sample size 1.8 million from the exponential distribution and quasi-subsampling the sample $1.8k$ million times to approximate the distributions of the kernels of the corresponding U-statistics, then computing the binomial kth central moment (Bkm), median kth central moment (mkm), and corresponding quantiles, finally obtained by the formula $d_{rkm} = \frac{km_{bs} - \mathrm{B}km_{bs}}{\mathrm{B}km_{bs} - mkm_{bs}}$ and $d_{qkm} = \frac{pkm_{bs} - p\mathrm{B}km_{bs}}{p\mathrm{B}km_{bs} - \frac{1}{2}}$, where $p\mathrm{B}km = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{X_i \leq (\mathrm{B}km_{bs})}$, $pkm = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{X_i \leq (km_{bs})}$, bs indicates the bootstrap moments. The accuracy of the estimates was verified by comparing the bootstrap central moments to their asymptotic values (errors), yielding errors of ≈ 0.0003, ≈ 0.001, and ≈ 0.03 for the second, third, and fourth central moments, respectively. The sample standard deviation of the kernel distributions for these moments are 2.234, 9.627, and 60.064, respectively, resulting in standardized errors for the biases that were all smaller than 0.001, thus ensuring the accuracy implied in the number of significant digits of the values in Table 1. The calculations of invariant moments using other symmetric weighted averages, $\mathrm{SQM}_{\frac{1}{8}}$, $\mathrm{BM}_{\nu=2,\epsilon=\frac{1}{8}}$, $\mathrm{WM}_{\frac{1}{8}}$, $\mathrm{BWM}_{\frac{1}{8}}$, and $\mathrm{TM}_{\frac{1}{8}}$, were performed analogously, by substituting of the appropriate values.

For a finite sample size of $n = 5400$, the d values were estimated using 1000 pseudorandom samples with a bootstrap size of 18000. To estimate the errors of d value estimations under finite sample size, first consider the first order Taylor approximation of the d value function, $d = \frac{x_1 - x_2}{x_2 - x_3} \approx d^0 + \frac{\partial d}{\partial x_1}x_1 + \frac{\partial d}{\partial x_2}x_2 + \frac{\partial d}{\partial x_3}x_3$. Then, by applying Bienaymé's identity, the variance of d can be approximated by $\sigma_d^2 \approx \left|\frac{\partial d}{\partial x_1}\right|^2 \sigma_{x_1}^2 + \left|\frac{\partial d}{\partial x_2}\right|^2 \sigma_{x_2}^2 + \left|\frac{\partial d}{\partial x_3}\right|^2 \sigma_{x_3}^2 + 2\left|\frac{\partial d}{\partial x_1}\right|\left|\frac{\partial d}{\partial x_2}\right| Cov\left(X_1, X_2\right) + 2\left|\frac{\partial d}{\partial x_1}\right|\left|\frac{\partial d}{\partial x_3}\right| Cov\left(X_1, X_3\right) + 2\left|\frac{\partial d}{\partial x_2}\right|\left|\frac{\partial d}{\partial x_3}\right| Cov\left(X_2, X_3\right) = \left|\frac{1}{x_2 - x_3}\right|^2 \sigma_{x_1}^2 + \left|-\frac{x_1 - x_2}{(x_2 - x_3)^2} - \frac{1}{x_2 - x_3}\right|^2 \sigma_{x_2}^2 + \left|\frac{x_1 - x_2}{(x_2 - x_3)^2}\right|^2 \sigma_{x_3}^2 + 2\left|\frac{1}{x_2 - x_3}\right|\left|-\frac{x_1 - x_2}{(x_2 - x_3)^2} - \frac{1}{x_2 - x_3}\right| Cov\left(X_1, X_2\right) + 2\left|\frac{1}{x_2 - x_3}\right|\left|\frac{x_1 - x_2}{(x_2 - x_3)^2}\right| Cov\left(X_1, X_3\right) + 2\left|-\frac{x_1 - x_2}{(x_2 - x_3)^2} - \frac{1}{x_2 - x_3}\right|\left|\frac{x_1 - x_2}{(x_2 - x_3)^2}\right| Cov\left(X_2, X_3\right)$. Since for the recombined mean, $\sigma_{x_1}^2 = 0$, so, $\sigma_{d_{rm}}^2 \approx \left|-\frac{x_1 - x_2}{(x_2 - x_3)^2} - \frac{1}{x_2 - x_3}\right|^2 \sigma_{x_2}^2 + \left|\frac{x_1 - x_2}{(x_2 - x_3)^2}\right|^2 \sigma_{x_3}^2 + 2\left(-\frac{x_1 - x_2}{(x_2 - x_3)^2} - \frac{1}{x_2 - x_3}\right)\left(\frac{x_1 - x_2}{(x_2 - x_3)^2}\right) Cov\left(X_2, X_3\right)$, where x_1 is the expected value, x_2 is the weighted average used, x_3 is the median. For quantile mean, since $\sigma_{x_3}^2 = 0$, $\sigma_{d_{qm}}^2 \approx \left|\frac{1}{x_2 - x_3}\right|^2 \sigma_{x_1}^2 + \left|-\frac{x_1 - x_2}{(x_2 - x_3)^2} - \frac{1}{x_2 - x_3}\right|^2 \sigma_{x_2}^2 + 2\left(\frac{1}{x_2 - x_3}\right)\left(-\frac{x_1 - x_2}{(x_2 - x_3)^2} - \frac{1}{x_2 - x_3}\right) Cov\left(X_1, X_2\right)$, where x_1 is the percentile of the expected value, x_2 is the percentile of the weighted average used, x_3 is the percentile of median, $\frac{1}{2}$. Finally, the errors were estimated by the corresponding sample statistics.

The computations of ABs and AABs for invariant central moments were just the same as mentioned previously. The SSE was computed by approximating the sampling distribution with 1000 pseudorandom samples for $n = 5400$ and 30 pseudorandom samples for $n = 1.8 \times 10^6$. Common random numbers were used for better comparison. The errors of AB and SSE were estimated by $se\left(\bar{x}\right) = \frac{\sigma}{\sqrt{n}} \approx \frac{usb}{\sqrt{n}}$, $se\left(sd\right) \approx \frac{1}{2\sigma} se\left(var\right) = \sqrt{\frac{\mu_4}{4n\sigma^2} - \frac{n-3}{4n(n-1)}\sigma^2} \approx \sqrt{\frac{fm}{4nvar} - \frac{n-3}{4n(n-1)}var}$, where usb is unbiased standard deviation of the sampling distribution with normality assumption (3). The computational methods used for two-parameter distributions were identical.

For simplicity, a brute force approach is used to estimate the maximum biases of SWAs and SWkms for five unimodal distributions. A wide range is set to roughly estimate the parameter ranges in which the maximum bias might occur (the corresponding maximum kurtoses are all larger than 500). Then, the parameter range is broken to 100 parts, combining with the above results for AAB estimations, the maximum of both is determined to be very close to the true maximum. Pseudo-maximum bias is the same as described in the main text.

The brute force approach is generally valid, i.e., the maximum is the global maximum, not local maximum, even when the the corresponding maximum kurtosis is finite. Because all five distributions here have the property that, as the kurtosis of the distribution increases to infinity, the standardized biases of SWAs approach zero.

For example, for the Perato distribution,

$$B_{\mathrm{SQA}}(\epsilon, \alpha) = \frac{\frac{1}{2}\left(x_m\left(1-\epsilon\right)^{-\frac{1}{\alpha}} + x_m\epsilon^{-\frac{1}{\alpha}}\right) - \frac{\alpha x_m}{\alpha - 1}}{\sqrt{\frac{\alpha {x_m}^2}{(1-\alpha)^2(\alpha-2)}}}.$$

$\lim_{\alpha\to 2} B_{\mathrm{SQA}}(\epsilon, \alpha) = \lim_{\alpha\to 2} \frac{\frac{1}{2}\left(x_m(1-\epsilon)^{-\frac{1}{\alpha}} + x_m\epsilon^{-\frac{1}{\alpha}}\right)}{\sqrt{\frac{\alpha {x_m}^2}{(1-\alpha)^2(\alpha-2)}}} - \frac{\frac{\alpha x_m}{\alpha-1}}{\sqrt{\frac{\alpha {x_m}^2}{(1-\alpha)^2(\alpha-2)}}} = \lim_{\alpha\to 2} \frac{\frac{1}{2}\left(\frac{1}{\sqrt{\epsilon}} + \frac{1}{\sqrt{1-\epsilon}}\right)}{\sqrt{\frac{\alpha}{(1-\alpha)^2(\alpha-2)}}} + \lim_{\alpha\to 2} \frac{-\alpha}{\sqrt{\frac{\alpha}{(\alpha-2)}}} = 0.$

In the previous article, it is proven that when the kurtoses of the distributions approach infinity, all distributions will be ordered, that means the SWAs based on the orderliness will follow the mean-SWA-median inequality, thus, proving the limits of the ratios between μ and σ, as well as m and σ is enough.

For example, for the Weibull distribution, the ratio of μ and σ is $\lim_{\alpha\to 0}\frac{\Gamma\left(1+\frac{1}{\alpha}\right)}{\sqrt{\Gamma\left(\frac{\alpha+2}{\alpha}\right)}}=\lim_{\alpha\to 0}\frac{\left(1+\frac{1}{\alpha}-1\right)!}{\sqrt{\left(\frac{\alpha+2}{\alpha}-1\right)!}}=\lim_{\alpha\to 0}\frac{\left(\frac{1}{\alpha}\right)!}{\sqrt{\left(2\times\frac{1}{\alpha}\right)!}}=0$, the ratio of m and σ is $\lim_{\alpha\to 0}\frac{\sqrt[\alpha]{\ln(2)}}{\sqrt{\Gamma\left(\frac{\alpha+2}{\alpha}\right)}}=\lim_{\alpha\to 0}\frac{0}{\sqrt{\Gamma\left(\frac{\alpha+2}{\alpha}\right)}}=0$.

Similarly, for the gamma distribution, the ratio of μ and σ is $\lim_{\alpha\to 0}\frac{\alpha}{\sqrt{\alpha}}=\lim_{\alpha\to 0}\frac{1}{\sqrt{\alpha}}=0$, the ratio of m and σ is $\lim_{\alpha\to 0}\frac{P^{-1}\left(\alpha,\frac{1}{2}\right)}{\sqrt{\alpha}}=0$ (4).

The lognormal distribution is the same, the ratio of μ and σ is $\lim_{\sigma\to\infty}\frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{\left(e^{\sigma^2}-1\right)e^{2\mu+\sigma^2}}}=\lim_{\sigma\to\infty}\frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{e^{2\mu+2\sigma^2}}}=\lim_{\sigma\to\infty}\frac{e^{\frac{\sigma^2}{2}}}{e^{\sigma^2}}=0$, the ratio of m and σ is $\lim_{\sigma\to\infty}\frac{e^{\mu}}{\sqrt{\left(e^{\sigma^2}-1\right)e^{2\mu+\sigma^2}}}=0$.

As demonstrated, the growth rate of the standard deviation greatly exceeds that of the mean and that of the median. This phenomenon is closely tied to the Taylor's law and is more widespread than these examples suggest.

Theorem 0.1. *$qm_{d\approx 0.321,\epsilon=\frac{1}{8}}$ is a consistent mean estimator for the exponential, Pareto ($\alpha\to\infty$) and any symmetric distributions provided that the second moments are finite.*

Proof. Similarly, rearranging the definition, $d=\frac{F(\mu)-F(\mathrm{BM}_\epsilon)}{F(\mathrm{BM}_\epsilon)-\frac{1}{2}}$. Recall the cdf is $F(x)=1-e^{-\lambda^{-1}x},\ x\geq 0$, the expectation of BM_ϵ can be expressed as $\lambda\mathrm{W}_{\mathrm{BM}_\epsilon}(\epsilon)$, so $F(\mathrm{BM}_\epsilon)$ is free of λ. When $\epsilon=\frac{1}{8}$, $d=\frac{-e^{-1}+e^{-\left(1+\ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)}}{\frac{1}{2}-e^{-\left(1+\ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)}}\approx 0.321$.

The proof of the symmetric case is similar. Since for any symmetric distribution with a finite second moment, $F\left(E\left[\mathrm{BM}_\epsilon\right]\right)=F(\mu)=\frac{1}{2}$. Then, the expectation of the quantile mean is $qm_{d,\epsilon}=F^{-1}\left(\left(F(\mu)-\frac{1}{2}\right)d+F(\mu)\right)=F^{-1}\left(0+F(\mu)\right)=\mu$.

For the assertion related to the Pareto distribution, the cdf of it is $1-\left(\frac{x_m}{x}\right)^\alpha$. So, the d value with two unknown percentile p_1 and p_2 is $d_{Pareto}=\frac{1-\left(\frac{x_m}{\frac{\alpha x_m}{\alpha-1}}\right)^\alpha-\left(1-\left(\frac{x_m}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^\alpha\right)}{\left(1-\left(\frac{x_m}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^\alpha\right)-\left(1-\left(\frac{x_m}{x_m(1-p_2)^{-\frac{1}{\alpha}}}\right)^\alpha\right)}=\frac{1-\left(\frac{\alpha-1}{\alpha}\right)^\alpha-p_1}{p_1-p_2}$. When $\alpha\to\infty$, $\left(\frac{\alpha-1}{\alpha}\right)^\alpha=\frac{1}{e}$. The d value for the exponential distribution is identical, since $d_{exp}=\frac{\left(1-e^{-1}\right)-\left(1-e^{-\ln\left(\frac{1}{1-p_1}\right)}\right)}{\left(1-e^{-\ln\left(\frac{1}{1-p_1}\right)}\right)-\left(1-e^{-\ln\left(\frac{1}{1-p_2}\right)}\right)}=\frac{1-\frac{1}{e}-p_1}{p_1-p_2}$. All results are now proven. □

SI Dataset S1 (dataset_one.xlsx)

Raw data of Table 1 in the main text.

References

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