Supporting Information for

Near-consistent robust estimations of moments for unimodal distributions

Tuban Lee.
E-mail: tl@biomathematics.org

This PDF file includes:

Supporting text
Legend for Dataset S1

Other supporting materials for this manuscript include the following:

Dataset S1

Supporting Information Text

Theorem. $\psi_k\left(x_1=\lambda x_1+\mu,\cdots,x_k=\lambda x_k+\mu\right)=\lambda^k\psi_k\left(x_1,\cdots,x_k\right)$.

Proof. ψ_k can be divided into k groups. From 1st to $k-1$th group, the gth group has $\binom{k}{g}\binom{g}{1}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_{i_1}^{k-g+1}\ldots x_{i_g}$. The final kth group is the term $(-1)^{k-1}(k-1)\,x_1\cdots x_k$. Let $x_{i_1}=x_1$, $k\neq g$, the gth group of ψ_k has $\binom{k-l}{g-l}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-g+1}\ x_2\cdots x_l x_{i_1}\cdots x_{i_{g-l}}$, where $x_1,\ x_2,\cdots,x_l$ are fixed, $x_{i_1},\cdots,x_{i_{g-l}}$ are selected such that $i_1,\cdots,\ i_{g-l}\neq 1,2,\cdots,l$. Let $\Psi_k\left(x_1,x_2,\cdots,x_l,x_{i_1},\cdots,x_{i_{g-l}}\right)=\left(\lambda x_1+\mu\right)^{k-g+1}\left(\lambda x_2+\mu\right)\cdots\left(\lambda x_l+\mu\right)\left(\lambda x_{i_1}+\mu\right)\cdots\left(\lambda x_{i_{g-l}}+\mu\right)$, the first group of Ψ_k is $\lambda^k x_1\cdots x_l x_{i_1}\cdots x_{i_{g-l}}$, the hth group of Ψ_k, $h>1$, has $\binom{k-g+1}{k-h-l+2}$ terms having the form $\lambda^{k-h+1}\mu^{h-1}x_1^{k-h-l+2}x_2\cdots x_l$. Transforming ψ_k by Ψ_k, then combing all terms with $\lambda^{k-h+1}\mu^{h-1}x_1^{k-h-l+2}x_2\cdots x_l$, $x_1^{k-h-l+2}\neq x_1$, the summed coefficient is $S1_l=\sum_{g=l}^{h+l-1}(-1)^{g+1}\frac{1}{k-g+1}\binom{k-g+1}{k-h-l+2}\binom{k-l}{g-l}=\sum_{g=l}^{h+l-1}(-1)^{g+1}\frac{(k-l)!}{(h+l-g-1)!(k-h-l+2)!(g-l)!}=0$, since the summation is starting from l, ending at $h+l-1$, the first term includes the factor $g-l=0$, the final term includes the factor $h+l-g-1=0$, the terms in the middle are also zero due to the factorial property. Another possible choice is letting one of $x_{i_2}\ldots x_{i_g}$ equal to x_1, the gth group of ψ_k has $(k-h)\binom{h-1}{g-k+h-1}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1x_2\ldots x_j^{k-g+1}\ldots x_{k-h+1}x_{i_1}\ldots x_{i_{g-k+h-1}}$, provided that $k\neq g$, $2\leq j\leq k-h+1$, where $x_1,\ldots,x_{k-h+1}$ are fixed, x_j^{k-g+1} and $x_{i_1},\cdots,x_{i_{g-k+h-1}}$ are selected. Transforming these terms by $\Psi_k\left(x_1,x_2,\ldots,x_j,\ldots,x_{k-h+1},x_{i_1},\ldots,x_{i_{g-k+h-1}}\right)=\left(\lambda x_1+\mu\right)\left(\lambda x_2+\mu\right)\cdots\left(\lambda x_j+\mu\right)^{k-g+1}\cdots\left(\lambda x_{k-h+1}+\mu\right)\left(\lambda x_{i_1}+\mu\right)\cdots\left(\lambda x_{i_{g-k+h-1}}+\mu\right)$, then, there are $k-g+1$ terms having the form $\lambda^{k-h+1}\mu^{h-1}x_1x_2\ldots x_{k-h+1}$. So, the combined result is $(-1)^{g+1}(k-h)\binom{h-1}{g-k+h-1}\lambda^{k-h+1}\mu^{h-1}x_1x_2\ldots x_{k-h+1}$. Transforming the final kth group of ψ_k by Ψ_k, then, there is one term having the form $(-1)^{k-1}(k-1)\,\lambda^{k-h+1}\mu^{h-1}x_1x_2\ldots x_{k-h+1}$. Another possible combination is that the gth group of ψ_k contains $(g-k+h-1)\binom{h-1}{g-k+h-1}$ terms having the form

$(-1)^{g+1}\frac{1}{k-g+1}x_1x_2\ldots x_{k-h+1}x_{i_1}\ldots x_{i_j}^{k-g+1}\ldots x_{i_{g-k+h-1}}$, there is only one term having the form $\lambda^{k-h+1}\mu^{h-1}x_1x_2\ldots x_{k-h+1}$. The above summation $S1_l$ should also be included, i.e., $x_1^{k-h-l+2}=x_1$, $k=h+l-1$, so, combing all terms with $\lambda^{k-h+1}\mu^{h-1}x_1x_2\ldots x_{k-h+1}$, according to the binomial theorem, the summed coefficient is

$S2_l=\sum_{g=k-h+1}^{k-1}(-1)^{g+1}\binom{h-1}{g-k+h-1}\left(k-h+1+\frac{g-k+h-1}{k-g+1}\right)+(-1)^{k-1}(k-1)=(-1)^k+(-1)^k(k-h)+(h-2)(-1)^k+(-1)^{k-1}(k-1)=0$. The result is the same if replacing x_1 with x_i, where i is from 2 to k, and replacing x_l with other x_i. Thus, all terms including μ can be canceled out. The proof is complete by noticing that the remaining part is $\lambda^k\psi_k\left(x_1,\cdots,x_k\right)$. $\square$

SI Dataset S1 (dataset_one.xlsx)

Raw data of Table 1 in the main text.

References