# Near-consistent robust estimations of moments for unimodal distributions

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Descriptive statistics for parametric models currently rely heavily on the accuracy of distributional assumptions. Here, based on the invariant structures of unimodal distributions, a series of sophisticated yet efficient estimators, robust to both gross errors and departures from parametric assumptions, are proposed for estimating mean and central moments with insignificant asymptotic biases for common continuous unimodal distributions. This article also illuminates the understanding of the common nature of probability distributions and the measures of them.

orderliness | invariant | unimodal | adaptive estimation | $U$-statistics

The asymptotic inconsistencies between sample mean ($\bar{x}$) and nonparametric robust location estimators in asymmetric distributions on the real line have been noticed for more than two centuries (1) but unsolved. Strictly speaking, it is unsolvable because by trimming, some information about the original distribution is removed, making it impossible to estimate the values of the removed parts without distributional assumptions. Newcomb (1886, 1912) provided the first modern approach to this problem by developing a class of estimators that gives "less weight to the more discordant observations" (2, 3). In 1964, Huber (4) used the minimax procedure to obtain M-estimator for contaminated normal distribution, which has played a pre-eminent role in the later development of robust statistics. As previously demonstrated, under growing asymmetric departures from normality, the bias of the Huber M-estimator increases rapidly. This is a common issue in parametric estimations. For example, He and Fung (1999) constructed (5) a robust M-estimator for the two-parameter Weibull distribution. All moments can be calculated from its estimated parameters. As expected, it is inadequate for the gamma, Perato, lognormal, and especially the generalized Gaussian distributions, because the logarithmic function does not produce a result for negative inputs (SI Dataset S1). Instead of minimizing the residuals, another old and interesting approach is arithmetically computing the parameters using one or more $L$-statistics as input values, e.g., the percentile estimators. Examples for the Weibull distribution, the reader is referred to Menon (1963) (6), Dubey (1967) (7), Hassanein (1971) (8), Marks (2005) (9), and Boudt, Caliskan, and Croux (2011) (10)'s works. At the outset of the study of percentile estimators, it was clear that this class of estimators arithmetically utilizes the invariant structures of probability distributions (6, 11, 12). Maybe it can be named as $I$-statistics. Formally, an estimator is classified as an $I$-statistic if asymptotically it satisfies $I(WA_1, \cdots, WA_l) = (\theta_1, \cdots, \theta_q)$ for the distribution it is consistent with, where WAs are weighted averages, $\theta$s are the population parameters it estimates. If the function I is solely defined through addition and/or subtraction, it is also an $L$-statistic. In the previous article, it is shown that quantile average is fundamental for all weighted averages. Based on the quantile function, $I$-statistic is naturally robust. For many parametric distributions, the quantile functions are much more elegant than the pdfs and cdfs. So $I$-statistics are often analytically obtainable. However, the performance of the above examples is often worse than that of the robust $M$-statistics when the distributional assumption is violated (SI Dataset S1). Even when distributions such as the Weibull and gamma belong to the same larger family, the generalized gamma distribution, a misassumption can still result in substantial biases, rendering the approach ill-suited.

In previous work on semiparametric robust mean estimation, although greatly shrinking the asymptotic biases, binomial mean ($BM_\epsilon$) is still inconsistent for any skewed distribution if $\epsilon > 0$ (if $\epsilon \to 0$, since the alternating sum of binomial coefficients is zero, $BM \to \mu$). All robust location estimators commonly used are symmetric due to the universality of the symmetric distributions. One can construct an asymmetric trimmed mean that is consistent for a semiparametric class of skewed distributions. This approach was investigated previously, but it is not symmetric and therefore only suitable for some special applications (13). From semiparametric to parametric, an ideal robust location estimator would have a non-sample-dependent breakdown point (defined in Subsection **??**) and be consistent with any symmetric distribution and a skewed distribution with finite second moments. This is called an invariant mean. Based on the mean-symmetric weighted average-median inequality, the recombined mean is defined as

$$rm_{d,\epsilon,n} := \lim_{c \to \infty} \left( \frac{(SWA_{\epsilon,n} + c)^{d+1}}{(median + c)^d} - c \right),$$

where $d$ is for bias correction, $SWA_{\epsilon,n}$ is $BM_{\epsilon,n}$ in the first three Subsections, while other symmetric weighted averages can also be used in practice as long as the inequalities hold. The next theorem shows the significance of this composite estimator.

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### Significance Statement

Bias, variance, and contamination are the three main errors in statistics. Consistent robust estimation is unattainable without parametric assumptions. Here, based on a paradigm shift inspired by mean-median-mode inequality, Bickel-Lehmann spread, and adaptive estimation, invariant moments are proposed as a means of achieving near-consistent and robust estimations of moments, even in scenarios where moderate violations of distributional assumptions occur, while the variances are sometimes smaller than those of the sample moments.

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**Theorem .1.** *If the second moments are finite, $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ is a consistent mean estimator for the exponential and any symmetric distributions and the Pareto distribution with quantile function $Q(p) = x_m(1-p)^{-\frac{1}{\alpha}}$, $x_m > 0$, when $\alpha \to \infty$.*

*Proof.* Finding $d, \epsilon$ values that make $rm_{d,\epsilon}$ a consistent mean estimator is equivalent to finding the solution of $E[rm_{d,\epsilon}] = E[X]$. Rearranging the definition, $rm_{d,\epsilon} = \lim_{c \to \infty} \left( \frac{(\mathrm{BM}_\epsilon + c)^{d+1}}{(median + c)^d} - c \right) = (d+1)\mathrm{BM}_\epsilon - d\,\text{median} = \mu$. So, $d = \frac{\mu - \mathrm{BM}_\epsilon}{\mathrm{BM}_\epsilon - median}$. The pdf of the exponential distribution is $f(x) = \lambda^{-1} e^{-\lambda^{-1} x}$, $\lambda \geq 0$, $x \geq 0$, the cdf is $F(x) = 1 - e^{-\lambda^{-1} x}$, $x \geq 0$. The quantile function is $Q(p) = \ln\left(\frac{1}{1-p}\right)\lambda$. $E[x] = \lambda$. $E[median] = Q\left(\frac{1}{2}\right) = \ln 2\lambda$. For the exponential distribution, the expectation of $\mathrm{BM}_{\frac{1}{8}}$ is $E\left[\mathrm{BM}_{\frac{1}{8}}\right] = \lambda\left(1 + \ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)$. Obviously, the scale parameter $\lambda$ can be canceled out, $d \approx 0.375$. The proof of the second assertion follows directly from the coincidence property. For any symmetric distribution with a finite second moment, $E[\mathrm{BM}_\epsilon] = E[median] = E[X]$. Then $E[rm_{d,\epsilon}] = \lim_{c \to \infty} \left( \frac{(E[X]+c)^{d+1}}{(E[X]+c)^d} - c \right) = E[X]$. The proof for the Pareto distribution is more general. The mean of the Pareto distribution is given by $\frac{\alpha x_m}{\alpha - 1}$. The $d$ value with two unknown percentiles $p_1$ and $p_2$ for the Pareto distribution is $d_{Perato} = \frac{\frac{\alpha x_m}{\alpha - 1} - x_m(1-p_1)^{-\frac{1}{\alpha}}}{x_m(1-p_1)^{-\frac{1}{\alpha}} - x_m(1-p_2)^{-\frac{1}{\alpha}}}$. Since any weighted average can be expressed as an integral of the quantile function, $\lim_{\alpha \to \infty} \frac{\frac{\alpha}{\alpha-1} - (1-p_1)^{-1/\alpha}}{(1-p_1)^{-1/\alpha} - (1-p_2)^{-1/\alpha}} = -\frac{\ln(1-p_1)+1}{\ln(1-p_1) - \ln(1-p_2)}$, the $d$ value for the Pareto distribution approaches that of the exponential distribution as $\alpha \to \infty$, regardless of the type of weighted average used. This completes the demonstration. $\square$

Theorem .1 implies that for the Weibull, gamma, Pareto, lognormal and generalized Gaussian distribution, $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ is consistent for at least one particular case of these two-parameter distributions. The biases of $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ for distributions with skewness between those of the exponential and symmetric distributions are tiny (SI Dataset S1). $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ has excellent performance for all these common unimodal distributions (SI Dataset S1).

Besides introducing the concept of invariant mean, the purpose of this paper is to demonstrate that the estimation of central moments can be transformed into a location estimation problem by using $U$-statistics, the central moment kernel distributions have nice properties, and, in light of previous works, a series of sophisticated yet efficient robust estimators can be constructed whose biases are typically smaller than the variances ($n = 5400$, Table **??**) for unimodal distributions.

## Background and Main Results

**A. Invariant mean.** It has long been known that a theoretical model can be adjusted to fit the first two moments of the observed data. A continuous distribution belonging to a location–scale family has the form $F(x) = F_0\left(\frac{x-\mu}{\lambda}\right)$, where $F_0$ is a "standard" distribution. Then, $F(x) = Q^{-1}(x) \to x = Q(p) = \lambda Q_0(p) + \mu$. So, any weighted average can be expressed as $\lambda \mathrm{WA}_0(\epsilon) + \mu$, where $\mathrm{WA}_0(\epsilon)$ is an integral of $Q_0(p)$ according to the definition of the weighted average. The simultaneous

cancellation of $\mu$ and $\lambda$ in $\frac{(\lambda \mu_0 + \mu) - (\lambda \mathrm{BM}_0(\epsilon) + \mu)}{(\lambda \mathrm{BM}_0(\epsilon)) + \mu) - (\lambda median_0 + \mu)}$ ensures that $d$ is a constant. Consequently, the roles of $\mathrm{BM}_\epsilon$ and median in $rm_{d,\epsilon}$ can be replaced by any weighted averages, although for the definition of invariant mean, only symmetric weighted averages are considered here.

The performance in heavy-tailed distributions can be improved further by constructing the quantile mean as

$$qm_{d,\epsilon,n} := \hat{Q}_n \left( \left( \hat{F}_n\left(\mathrm{SWA}_{\epsilon,n}\right) - \frac{1}{2} \right) d + \hat{F}_n\left(\mathrm{SWA}_{\epsilon,n}\right) \right),$$

provided that $\hat{F}_n\left(\mathrm{SWA}_{\epsilon,n}\right) \geq \frac{1}{2}$, where $\hat{F}_n(x)$ is the empirical cumulative distribution function of the sample, $\hat{Q}_n$ is the sample quantile function. The most popular method for computing the sample quantile function was proposed by Hyndman and Fan in 1996 (14). To minimize the finite sample bias, here, $\hat{F}_n(x) := \frac{1}{n}\left( \frac{x - \hat{Q}_n\left(\frac{sp}{n}\right)}{\hat{Q}_n\left(\frac{1}{n}(sp+1)\right) - \hat{Q}_n\left(\frac{sp}{n}\right)} + sp \right)$, where $sp = \sum_{i=1}^{n} \mathbf{1}_{X_i \leq x}$, $\mathbf{1}_A$ is the indicator of event $A$. The solution of $\hat{F}_n\left(\mathrm{SWA}_{\epsilon,n}\right) < \frac{1}{2}$ is reversing the percentile by $1 - \hat{F}_n\left(\mathrm{SWA}_{\epsilon,n}\right)$, the obtained percentile is also reversed. Without loss of generality, in the following discussion, only the $\hat{F}_n\left(\mathrm{SWA}_{\epsilon,n}\right) \geq \frac{1}{2}$ case will be considered. Moreover, in extreme heavy-tailed distributions, the calculated percentile can exceed the breakdown point of $\mathrm{SWA}_\epsilon$, so the percentile will be modified to $1 - \epsilon$ if this happens. The quantile mean uses the location-scale invariant in a different way as shown in the following proof.

**Theorem A.1.** *$qm_{d \approx 0.321, \epsilon = \frac{1}{8}}$ is a consistent mean estimator for the exponential, Pareto ($\alpha \to \infty$) and any symmetric distributions provided that the second moments are finite.*

*Proof.* Similarly, rearranging the definition, $d = \frac{F(\mu) - F(\mathrm{BM}_\epsilon)}{F(\mathrm{BM}_\epsilon) - \frac{1}{2}}$. Recall the cdf is $F(x) = 1 - e^{-\lambda^{-1} x}$, $x \geq 0$, the expectation of $\mathrm{BM}_\epsilon$ can be expressed as $\lambda \mathrm{BM}_0(\epsilon)$, so $F(\mathrm{BM}_\epsilon)$ is free of $\lambda$. When $\epsilon = \frac{1}{8}$, $d = \frac{-e^{-1} + e^{-\left(1 + \ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)}}{\frac{1}{2} - e^{-\left(1 + \ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)}} \approx 0.321$.

The proof of the symmetric case is similar. Since for any symmetric distribution with a finite second moment, $F(E[\mathrm{BM}_\epsilon]) = F(\mu) = \frac{1}{2}$. Then, the expectation of the quantile mean is $qm_{d,\epsilon} = F^{-1}\left( \left( F(\mu) - \frac{1}{2} \right) d + F(\mu) \right) = F^{-1}(0 + F(\mu)) = \mu$.

For the assertion related to the Pareto distribution, the cdf of it is $1 - \left(\frac{x_m}{x}\right)^\alpha$. So, the $d$ value with two unknown percentile $p_1$ and $p_2$ is

$$d_{Pareto} = \frac{1 - \left(\frac{x_m}{\frac{\alpha x_m}{\alpha - 1}}\right)^\alpha - \left(1 - \left(\frac{x_m}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^\alpha\right)}{\left(1 - \left(\frac{x_m}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^\alpha\right) - \left(1 - \left(\frac{x_m}{x_m(1-p_2)^{-\frac{1}{\alpha}}}\right)^\alpha\right)} =$$

$\frac{1 - \left(\frac{\alpha-1}{\alpha}\right)^\alpha - p_1}{p_1 - p_2}$. When $\alpha \to \infty$, $\left(\frac{\alpha-1}{\alpha}\right)^\alpha = \frac{1}{e}$. The $d$ value for the exponential distribution is identical, since $d_{exp} =$

$$\frac{(1 - e^{-1}) - \left(1 - e^{-\ln\left(\frac{1}{1-p_1}\right)}\right)}{\left(1 - e^{-\ln\left(\frac{1}{1-p_1}\right)}\right) - \left(1 - e^{-\ln\left(\frac{1}{1-p_2}\right)}\right)} = \frac{1 - \frac{1}{e} - p_1}{p_1 - p_2}.$$ All results

are now proven. $\square$

The definitions of location and scale parameters are such that they must satisfy $F(x; \lambda, \mu) = F\left(\frac{x-\mu}{\lambda}; 1, 0\right)$. Recall that

<sup>152</sup> $x = \lambda Q_0(p) + \mu$, so the percentile of any weighted average
<sup>153</sup> is free of $\lambda$ and $\mu$, guaranteeing the validity of the quantile
<sup>154</sup> mean. $qm_{d \approx 0.321, \epsilon = \frac{1}{8}}$ works better in the fat-tail scenarios (SI
<sup>155</sup> Dataset S1). Theorem .1 and A.1 show that $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$
<sup>156</sup> and $qm_{d \approx 0.321, \epsilon = \frac{1}{8}}$ are both consistent mean estimators for
<sup>157</sup> any symmetric distribution and a skewed distribution with
<sup>158</sup> finite second moments. It's obvious that the breakdown points
<sup>159</sup> of $rm_{d \approx 0.375, \epsilon = \frac{1}{8}}$ and $qm_{d \approx 0.321, \epsilon = \frac{1}{8}}$ are both $\frac{1}{8}$. Therefore
<sup>160</sup> they are all invariant means.

<sup>161</sup> To study the impact of the choice of SWAs in $rm$ and $qm$,
<sup>162</sup> it is constructive to consider a symmetric weighted average
<sup>163</sup> as a mixture of symmetric quantile averages. Although us-
<sup>164</sup> ing a less-biased symmetric weighted average can generally
<sup>165</sup> improve performance (SI Dataset S1), there is a higher risk
<sup>166</sup> of violation in the semiparametric framework. However, even
<sup>167</sup> the SQA function is not strictly monotonic, suppose it is
<sup>168</sup> generally decreasing in $[0, u]$, but increasing in $[u, \frac{1}{2}]$, since
<sup>169</sup> $1 - 2\epsilon$ of the symmetric quantile averages will be included
<sup>170</sup> in the computation of $\text{SWA}_\epsilon$, as long as $|u - \frac{1}{2}| \ll 1 - 2\epsilon$,
<sup>171</sup> and other parts of the SQA function satisfy the inequality
<sup>172</sup> constraints which define the $\nu$th orderliness, the mean-$\text{SWA}_\epsilon$-
<sup>173</sup> median inequality will still be valid (as an example, the SQA
<sup>174</sup> function is non-monotonic when the shape parameter of the
<sup>175</sup> Weibull distribution $\alpha > \frac{1}{1 - \ln(2)} \approx 3.259$ as shown in the
<sup>176</sup> previous article, yet the mean-$\text{BM}_{\frac{1}{8}}$-median inequality is still
<sup>177</sup> valid when $\alpha \leq 3.322$). Another key factor determining the
<sup>178</sup> risk of violation is the skewness of the distribution. In the
<sup>179</sup> previous article, it is shown that in a family of distributions
<sup>180</sup> that differ by a skewness-increasing transformation in van
<sup>181</sup> Zwet's sense, the violation of orderliness, if it happens, of-
<sup>182</sup> ten only occurs when the distribution is near-symmetric (15).
<sup>183</sup> The over-corrections in $rm$ and $qm$ are dependent on the
<sup>184</sup> $\text{SWA}_\epsilon$-median difference, which is correlated to the skewness
<sup>185</sup> (16, 17), so the over-correction is often tiny with a moderate
<sup>186</sup> $d$. This qualitative analysis provides another perspective, in
<sup>187</sup> addition to the bias bounds (18), that $rm$ and $qm$ based on
<sup>188</sup> the mean-$\text{SWA}_\epsilon$-median inequality are generally safe.

<sup>189</sup> **B. Robust estimations of the central moments.** In 1976, Bickel
<sup>190</sup> and Lehmann, in their third paper of the landmark series *De-*
<sup>191</sup> *scriptive Statistics for Nonparametric Models* (19), generalized
<sup>192</sup> a class of estimators called "measures of disperse," which is now
<sup>193</sup> often named as Bickel-Lehmann dispersion. As an example,
<sup>194</sup> they proposed a first version of the trimmed standard devi-
<sup>195</sup> ation, $\hat{\tau}^2(F; \epsilon) \equiv \tau^2(F; \epsilon)$, for independent and identically
<sup>196</sup> distributed random variables $X_i$ with a distribution $F$, where
<sup>197</sup> $\tau^2(F; \epsilon) = \frac{1}{1 - 2\epsilon} \int_{Q(\epsilon)}^{Q(1-\epsilon)} y dG(y)$, $Q$ is the quantile function
<sup>198</sup> of $G$, $G$ is the distribution of $Y = X^2$. Obviously, when
<sup>199</sup> $\epsilon = 0$, the result is equivalent to the second raw moment.
<sup>200</sup> In 1979, in the same series (20), they explored another class
<sup>201</sup> of estimators called "measures of spread," which "does not
<sup>202</sup> require the assumption of symmetry." From that, a popular
<sup>203</sup> efficient scale estimator, the Rousseeuw-Croux scale estimator
<sup>204</sup> (21), was derived in 1993, but the importance of tackling the
<sup>205</sup> symmetry assumption has been greatly underestimated. In
<sup>206</sup> the final section of the paper, they considered another two
<sup>207</sup> possible versions of the trimmed standard deviations, which
<sup>208</sup> were modified here for comparison,

$$\left[ n \left( \frac{1}{2} - \epsilon \right) \right]^{-\frac{1}{2}} \left[ \sum_{k=\frac{n}{2}}^{n(1-\epsilon)} [X_k - X_{n-k+1}]^2 \right]^{\frac{1}{2}}, \qquad [1]$$

<sup>209</sup>

and <sup>210</sup>

$$\left[ \binom{n}{2} (1 - \epsilon - \gamma\epsilon) \right]^{-\frac{1}{2}} \left[ \sum_{k=\binom{n}{2}\epsilon}^{\binom{n}{2}(1-\gamma\epsilon)} \left( X - X' \right)_k^2 \right]^{\frac{1}{2}}, \quad [2]$$

<sup>211</sup>

<sup>212</sup> where $(X - X')_1 \leq \ldots \leq (X - X')_{\binom{n}{2}}$ are the order statistics
<sup>213</sup> of the "pseudo-sample" $X_i - X_j$, $i < j$. The paper ended with,
<sup>214</sup> "We do not know a fortiori which of the measures [1] or [2] is
<sup>215</sup> preferable and leave these interesting questions open."

<sup>216</sup> Observe that the kernel of the unbiased estimation of the
<sup>217</sup> second central moment by using $U$-statistic is $\psi_2(x_1, x_2) =$
<sup>218</sup> $\frac{1}{2}(x_1 - x_2)^2$. If adding the $\frac{1}{2}$ term in [2], as $\epsilon \to 0$, the result
<sup>219</sup> is equivalent to the standard deviation estimated by using
<sup>220</sup> $U$-statistic (also noted by Janssen, Serfling, and Veraverbeke
<sup>221</sup> in 1987) (22). In fact, they also implied that, when $\epsilon$ is 0, [2]
<sup>222</sup> is $\sqrt{2}$ times the standard deviation.

<sup>223</sup> To address their open questions, the nomenclature used in
<sup>224</sup> this paper is introduced as follows:

<sup>225</sup> *Nomenclature.* Given a robust estimator $\hat{\theta}$. The first part
<sup>226</sup> of the name of the robust statistic defined in this paper is a
<sup>227</sup> prefix that indicates the type of estimator, and the second
<sup>228</sup> part is the name of the population parameter $\theta$ that the
<sup>229</sup> estimator is consistent with as $\epsilon \to 0$. The abbreviation
<sup>230</sup> of the estimator is the initial letter(s) of the first part plus
<sup>231</sup> the common abbreviation of the consistent estimator that
<sup>232</sup> measures the population parameter. If the estimator is not
<sup>233</sup> a $U$-statistic, the breakdown point, $\epsilon$, is indicated in the
<sup>234</sup> subscript of the abbreviation of the estimator. If the estimator
<sup>235</sup> is a weighted $U$-statistic, the breakdown point of the location
<sup>236</sup> estimator is indicated (except the median).

<sup>237</sup> In the previous semiparametric robust mean article, it is
<sup>238</sup> shown that the bias of a reasonable robust estimator should
<sup>239</sup> be monotonic with respect to the breakdown point in a semi-
<sup>240</sup> parametric distribution and naturally, its name should align
<sup>241</sup> with the consistent estimator. Naturally, the trimmed stan-
<sup>242</sup> dard deviation following this nomenclature is $\text{T}sd_{\epsilon,\gamma,n} :=$
$\left[ \text{TM}_{\epsilon,\gamma} \left( (\psi_2(X_{N_1}, X_{N_2}))_{N=1}^{\binom{n}{2}} \right) \right]^{-\frac{1}{2}}$, where $\text{TM}_{\epsilon,\gamma}(Y)$ denotes <sup>243</sup>
<sup>244</sup> the $\epsilon, \gamma$-trimmed mean with the sequence $(\psi_2(X_{N_1}, X_{N_2}))_{N=1}^{\binom{n}{2}}$
<sup>245</sup> as an input. If the square root is removed, it is named as the
<sup>246</sup> trimmed variance ($\text{T}var_{\epsilon,\gamma,n}$). It is now very clear that this
<sup>247</sup> definition, essentially the same as [2], should be preferable.
<sup>248</sup> Not only because it is essentially a trimmed $U$-statistic for
<sup>249</sup> the standard deviation but also because the $\gamma$-orderliness of
<sup>250</sup> the pseudo-sample distribution is ensured by the next exciting
<sup>251</sup> theorem.

**Theorem B.1.** *The second central moment kernel distribution* <sup>252</sup>
*generated from any continuous unimodal distribution is $\gamma$-* <sup>253</sup>
*ordered, if $\gamma \geq 1$.* <sup>254</sup>

*Proof.* Let $Q(p)$, $0 \leq p \leq 1$, denote the quantile of the contin- <sup>255</sup>
uous unimodal distribution $f_X(x)$. The corresponding proba- <sup>256</sup>
bility density is $f(Q(p))$. Generating the distribution of the <sup>257</sup>

pair $(Q(p_i), Q(p_j))$, $i < j$, $p_i < p_j$, the corresponding probability density is $f_{X,X}(Q(p_i), Q(p_j)) = 2f(Q(p_i))f(Q(p_j))$. Transforming the pair $(Q(p_i), Q(p_j))$, $i < j$, by the function $\Phi(x_1, x_2) = x_1 - x_2$, the pairwise difference distribution has a mode that is arbitrary close to $M - M = 0$. The monotonic increasing of the pairwise difference distribution was first implied in its unimodality proof done by Hodges and Lehmann in 1954 (23). Whereas they used induction to get the result, Dharmadhikari and Jogdeo in 1982 (24) gave a modern proof of the unimodality using Khintchine's representation (25). Assuming absolute continuity, Purkayastha (26) introduced a much simpler proof in 1998. Transforming the pairwise difference distribution by squaring and multiplying $\frac{1}{2}$ does not change the monotonicity, making the pdf become monotonically decreasing with mode at zero. In the previous semiparametric robust mean estimation article, it is proven that a right skewed distribution with a monotonic decreasing pdf is always $\gamma$-ordered, which gives the desired result. □

*Remark.* The assumption of continuity of distributions is important for monotonicity because, unlike in the continuous case, it is possible to get pairs with the same value for a discrete distribution. For example, let the probabilities of the singletons $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$ of a probability mass function of a discrete probability distribution be $\frac{1}{11}$, $\frac{4}{11}$, $\frac{3}{11}$, $\frac{2}{11}$, and $\frac{1}{11}$, respectively. This is a unimodal distribution, but the corresponding $\psi_2$ distribution is non-monotonic, whose singletons $\{0\}$, $\{0.5\}$, $\{2\}$, $\{4.5\}$ and $\{8\}$ have probabilities $\frac{21}{66}$, $\frac{24}{66}$, $\frac{2}{14}$, $\frac{6}{66}$, and $\frac{1}{66}$, respectively.

Previously, it was shown that any symmetric distribution with a finite second moment is $\nu$th ordered. That means the orderliness does not require unimodality, e.g., for a symmetric bimodal distribution, it is also ordered. Examples from the Weibull distribution show that unimodality does not guarantee orderliness. Theorem B.1 reveals another profound relationship between unimodality and orderliness, which is sufficient for trimming inequality.

In 1928, Fisher constructed $k$-statistics as unbiased estimators of cumulants (27). Halmos (1946) proved that the functional $\theta$ admits an unbiased estimator if and only if it is a regular statistical functional of degree $k$ and showed a relation of symmetry, unbiasness and minimum variance (28). In 1948, Hoeffding generalized $U$-statistics (29) which enable the derivation of a minimum-variance unbiased estimator from each unbiased estimator of an estimable parameter. Heffernan (1997) (30) obtained an unbiased estimator of the $k$th central moment by using $U$-statistics and demonstrated that it is the minimum variance unbiased estimator for distributions with finite moments (31, 32). In 1976, Saleh generalized the Hodges-Lehmenn estimator (33) to the trimmed H-L mean (he named "Wilcoxon one-sample statistic") (34). In 1984, Serfling pointed out the speciality of Hodges-Lehmann estimator, which is neither a simple $L$-statistics nor $U$-statistic, and considered the generalized $L$-statistics and $U$-statistic structure (35). Also in 1984, Janssen and Serfling and Veraverbeke (36) showed that the Bickel-Lehmann spread also belongs to the same class. It was gradually clear that the Hodges-Lehmenn estimator, trimmed H-L mean and trimmed standard deviation are all trimmed $U$-statistics (37–39). Due to the combinatorial explosion, the bootstrap (40), introduced by Efron in 1979, is indispensable in large sample studies. In 1981, Bickel and Freedman (41) showed that the bootstrap

is asymptotically valid to approximate the original distribution in a wide range of situations, including $U$-statistics. The limit laws of bootstrapped trimmed $U$-statistics was proven by Helmers, Janssen, and Veraverbeke (1990) (42).

Extending the trimmed $U$-statistic to weighted $U$-statistic, i.e., replacing the trimmed mean with weighted average. The weighted $k$th central moment ($k \leq n$) is defined as,

$$\mathrm{W}km_{\epsilon,\gamma,n} \coloneqq \mathrm{WA}_{\epsilon,\gamma,n}\left( \left(\psi_k\left(X_{N_1}, \cdots, X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}} \right),$$

where $X_{N_1}, \cdots, X_{N_k}$ are the $n$ choose $k$ elements from $X$, $\psi_k(x_1, \cdots, x_k) = \sum_{j=0}^{k-2}(-1)^j\left(\frac{1}{k-j}\right)\sum\left(x_{i_1}^{k-j}\ldots x_{i_{(j+1)}}\right) + (-1)^{k-1}(k-1)x_1\ldots x_k$, the second summation is over $i_1, \cdots, i_{j+1} = 1$ to $k$ with $i_1 < \cdots < i_{j+1}$ (30). Despite the complexity, the structure of the $k$th central moment kernel distributions can be elucidated by decomposing.

**Theorem B.2.** *For each pair $(Q(p_i), Q(p_j))$ of the original distribution, let $x_1 = Q(p_i)$ and $x_k = Q(p_j)$, $\Delta = Q(p_i) - Q(p_j)$. The $k$th central moment kernel distribution, $k > 2$, can be seen as a mixture distribution and each of the components has the support $\left(-\left(\frac{k}{3+(-1)^k}\right)^{-1}(-\Delta)^k, \frac{1}{k}(-\Delta)^k\right)$.*

*Proof.* Generating the distribution of the $k$-tuple $(Q(p_{i_1}), \ldots, Q(p_{i_k}))$, $k > 2$, $i_1 < \ldots < i_k$, $p_{i_1} < \ldots < p_{i_k}$, the corresponding probability density is $f_{X,\ldots,X}(Q(p_{i_1}), \ldots, Q(p_{i_k})) = k!f(Q(p_{i_1}))\ldots f(Q(p_{i_k}))$. Transforming the distribution of the $k$-tuple by the function $\psi_k(x_1, \ldots, x_k)$, denoting $\bar{\Delta} = \psi_k(Q(p_{i_1}), \ldots, Q(p_{i_k}))$. The probability $f_{\Xi_k}(\bar{\Delta}) = \sum_{\bar{\Delta}=\psi_k(Q(p_{i_1}),\ldots,Q(p_{i_k}))} f_{X,\ldots,X}(Q(p_{i_1}), \ldots, Q(p_{i_k}))$ is the summation of the probabilities of all $k$-tuples such that $\bar{\Delta}$ is equal to $\psi_k(Q(p_{i_1}), \ldots, Q(p_{i_k}))$. The following $\Xi_k$ is equivalent.

$\Xi_k$: Every pair with a difference equal to $\Delta = Q(p_{i_1}) - Q(p_{i_k})$ can generate a pseudodistribution (but the integral is not equal to 1, so "pseudo") such that $x_2, \ldots, x_{k-1}$ exhaust all combinations under the inequality constraints, i.e., $Q(p_{i_1}) = x_1 < x_2 < \ldots < x_{k-1} < x_k = Q(p_{i_k})$. The combination of all the pseudodistributions with the same $\Delta$ is $\xi_\Delta$. The combination of $\xi_\Delta$, i.e., from $\Delta = 0$ to $Q(0) - Q(1)$, is $\Xi_k$.

The support of $\xi_\Delta$ is the extrema of $\psi_k$ subject to the inequality constraints. Using the Lagrange multiplier, one can easily determine the only critical point at $x_1 = \ldots = x_k = 0$, where $\psi_k = 0$. Other candidates are within the boundaries, i.e., $\psi_k(x_1 = x_1, x_2 = x_k, \cdots, x_k = x_k)$, $\ldots$, $\psi_k(x_1 = x_1, \cdots, x_i = x_1, x_{i+1} = x_k, \cdots, x_k = x_k)$, $\ldots$, $\psi_k(x_1 = x_1, \cdots, x_{k-1} = x_1, x_k = x_k)$. $\psi_k(x_1 = x_1, \cdots, x_i = x_1, x_{i+1} = x_k, \cdots, x_k = x_k)$ can be divided into $k$ groups. If $\frac{k+1-i}{2} \leq j \leq \frac{k-1}{2}$, from $j+1$st to $k-j$th group, the $g$th group has $i\binom{i-1}{g-j-1}\binom{k-i}{j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$, from $k-j+1$th to $i+j$th group, the $g$th group has $i\binom{i-1}{g-j-1}\binom{k-i}{j} + (k-i)\binom{k-i-1}{j-k+g-1}\binom{i}{k-j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$. If $j < \frac{k+1-i}{2}$, from $j+1$st to $i+j$th group, the $g$th group has $i\binom{i-1}{g-j-1}\binom{k-i}{j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$. If $j \geq \frac{k}{2}$, from $k-j+1$st to $j$th group, the $g$th group has $(k-i)\binom{k-i-1}{j-k+g-1}\binom{i}{k-j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$, from $j+1$th to $j+i$th group, $i+j < k$, the $g$th group

has $i\binom{i-1}{g-j-1}\binom{k-i}{j} + (k-i)\binom{k-i-1}{j-k+g-1}\binom{i}{k-j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$. The final $k$th group is the term $(-1)^{k-1}(k-1)x_1^ix_k^{k-i}$. So, if $i+j=k$, $j \geq \frac{k}{2}$, $i \leq \frac{k}{2}$, the summed coefficient of $x_1^ix_k^{k-i}$ is $(-1)^{k-1}(k-1) + \sum_{g=i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}(k-i)\binom{k-i-1}{g-i-1} + \sum_{g=k-i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}i\binom{i-1}{g-k+i-1} = (-1)^{k-1}(k-1) + (-1)^{k+1} + (k-i)(-1)^k + (-1)^k(i-1) = (-1)^{k+1}$. The summation identities are $\sum_{g=i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}(k-i)\binom{k-i-1}{g-i-1} = (k-i)\int_0^1\sum_{g=i+1}^{k-1}(-1)^{g+1}\binom{k-i-1}{g-i-1}t^{k-g}dt = (k-i)\int_0^1\left((-1)^i(t-1)^{k-i-1}-(-1)^{k+1}\right)dt = (k-i)\left(\frac{(-1)^k}{i-k}+(-1)^k\right) = (-1)^{k+1} + (k-i)(-1)^k$. $\sum_{g=k-i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}i\binom{i-1}{g-k+i-1} = \int_0^1\sum_{g=k-i+1}^{k-1}(-1)^{g+1}i\binom{i-1}{g-k+i-1}t^{k-g}dt = \int_0^1\left(i(-1)^{k-i}(t-1)^{i-1}-i(-1)^{k+1}\right)dt = (-1)^k(i-1)$. If $j < \frac{k+1-i}{2}$, $i > k-1$, if $i = k$, $\psi_k = 0$, if $\frac{k+1-i}{2} \leq j \leq \frac{k}{2}$, $\frac{k+1}{2} \leq i \leq k-1$, the summed coefficient of $x_1^ix_k^{k-i}$ is $(-1)^{k-1}(k-1) + \sum_{g=k-i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}i\binom{i-1}{g-k+i-1} + \sum_{g=i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}(k-i)\binom{k-i-1}{g-i-1}$, the same as above. If $i+j < k$, since $\binom{i}{k-j} = 0$, the related terms can be ignored, so, using the binomial theorem and beta function, the summed coefficient of $x_1^{k-j}x_k^j$ is $\sum_{g=j+1}^{i+j}(-1)^{g+1}\frac{1}{k-g+1}i\binom{i-1}{g-j-1}\binom{k-i}{j} = i\binom{k-i}{j}\int_0^1\sum_{g=j+1}^{i+j}(-1)^{g+1}\binom{i-1}{g-j-1}t^{k-g}dt = \binom{k-i}{j}i\int_0^1\left((-1)^jt^{k-j-1}\left(\frac{t}{t-1}\right)^{1-i}\right)dt = \binom{k-i}{j}i\frac{(-1)^{j+i+1}\Gamma(i)\Gamma(k-j-i+1)}{\Gamma(k-j+1)} = \frac{(-1)^{j+i+1}i!(k-j-i)!(k-i)!}{(k-j)!j!(k-j-i)!} = (-1)^{j+i+1}\frac{i!(k-i)!}{k!}\frac{k!}{(k-j)!j!} = \binom{k}{i}^{-1}(-1)^{1+i}\binom{k}{j}(-1)^j$.

The coefficient of $x_1^ix_k^{k-i}$ in $\binom{k}{i}^{-1}(-1)^{1+i}(x_1-x_k)^k$ is $\binom{k}{i}^{-1}(-1)^{1+i}\binom{k}{i}(-1)^{k-i} = (-1)^{k+1}$, same as the summed coefficient if $i+j=k$. If $i+j < k$, the coefficient of $x_1^{k-j}x_k^j$ is $\binom{k}{i}^{-1}(-1)^{1+i}\binom{k}{j}(-1)^j$, same as the corresponding summed coefficient. Therefore, $\psi_k(x_1 = x_1,\ldots,x_i = x_1,x_{i+1} = x_k,\ldots,x_k = x_k) = \binom{k}{i}^{-1}(-1)^{1+i}(x_1-x_k)^k$, the maximum and minimum of $\psi_k$ follow directly from the properties of the binomial coefficient. □

$\xi_\Delta$ is closely related to $f_\Xi(\Delta)$, which is the pairwise difference distribution, since the probability density of $\xi_\Delta$ is $f_{\Xi_k}(\bar{\Delta}|\Delta)$, $\sum_{\bar{\Delta}=-\left(\frac{3+(-1)^k}{2}\right)^{-1}(-\Delta)^k}^{\frac{1}{k}(-\Delta)^k}f_{\Xi_k}(\bar{\Delta}|\Delta) = f_\Xi(\Delta)$. Recall that $f_\Xi(\Delta)$ is monotonic increasing with a mode at the origin if the original distribution is unimodal. Thus, in general, ignoring the shape of $\xi_\Delta$, $\Xi_k$ is monotonic left and right around zero. In fact, the median of $\Xi_k$ is also close to zero, as it can be cast as a weighted mean of the medians of $\xi_\Delta$. When $\Delta$ is small, all values of $\xi_\Delta$ are close to zero, resulting in the median of $\xi_\Delta$ close to zero, When $\Delta$ is large, the median of $\xi_\Delta$ depends on its skewness, but the corresponding weight is much smaller, so even if $\xi_\Delta$ is highly skewed, the median of $\Xi_k$ will only be slightly shifted from zero (denote the median of $\Xi_k$ as $m_{\Xi_k}$, for five parametric distributions here, $|m_{\Xi_k}|$s are all $\leq 0.1\sigma$ for $\Xi_3$ and $\Xi_4$, SI Dataset S1). Assuming $m_{\Xi_k} = 0$, for the even ordinal central moment kernel distribution, the average probability density on the left side of zero is greater than that on the right

side, since $\frac{\frac{1}{2}}{\binom{k}{2}^{-1}(Q(0)-Q(1))^k} > \frac{\frac{1}{2}}{\frac{1}{k}(Q(0)-Q(1))^k}$. This means that, on average, the inequality $f(Q(\epsilon)) \geq f(Q(1-\epsilon))$ holds. For the odd ordinal distribution, the discussion is harder since it is generally symmetric. Just consider $\Xi_3$, let $x_1 = Q(p_i)$ and $x_3 = Q(p_j)$, changing the value of $x_2$ from $Q(p_i)$ to $Q(p_j)$ will monotonically change the value of $\psi_3(x_1,x_2,x_3)$, since $\frac{\partial\psi_3(x_1,x_2,x_3)}{\partial x_2} = -\frac{x_1^2}{2} - x_1x_2 + 2x_1x_3 + x_2^2 - x_2x_3 - \frac{x_3^2}{2}$, $-\frac{3}{4}(x_1-x_3)^2 \leq \frac{\partial\psi_3(x_1,x_2,x_3)}{\partial x_2} \leq -\frac{1}{2}(x_1-x_3)^2 \leq 0$. If the original distribution is right-skewed, $\xi_\Delta$ will be left-skewed, so, for $\Xi_3$, the average probability density of the right side of zero will be greater than that of the left side, which means, on average, the inequality $f(Q(\epsilon)) \leq f(Q(1-\epsilon))$ holds (the same result can be inferred from the definition of central moments, the positive of odd order central moment is directly related to the left-skewness of the corresponding kernel distribution). In all, the monotonicity of the pairwise difference distribution guides the general shape of the $k$th central moment kernel distribution, $k > 2$, forcing it to be unimodal-like with mode and median close to zero, then, the inequality $f(Q(\epsilon)) \leq f(Q(1-\epsilon))$ or $f(Q(\epsilon)) \geq f(Q(1-\epsilon))$ holds in general. If a distribution is ordered and its all central moment kernel distributions are also ordered, it is called completely ordered. Although strict complete orderliness is hard to prove, the inequality may be violated in a small range, as discussed in Subsection A, the mean-SWA$_\epsilon$-median inequality remains valid, in most cases, for the central moment kernel distribution.

Another key property of the central moment kernel distribution, location invariant, is introduced in the next theorem. The proof is given in the SI Text.

**Theorem B.3.** $\psi_k(x_1 = \lambda x_1 + \mu, \cdots, x_k = \lambda x_k + \mu) = \lambda^k\psi_k(x_1, \cdots, x_k)$.

Consider two continuous distributions belonging to the same location–scale family, their corresponding $k$th central moment kernel distributions only differ in scaling. So $d$ is invariant, as shown in Subsection A. The recombined $k$th central moment, based on $rm$, is defined by,

$$rkm_{d,\epsilon,n} := (d+1)\,\mathrm{SW}km_{\epsilon,n} - dmkm_{\epsilon,n},$$

where $\mathrm{SW}km_{\epsilon,n}$ is using the binomial $k$th central moment ($\mathrm{B}km_{\epsilon,n}$) here, $mkm_{\epsilon,n}$ is the median $k$th central moment. Similarly, the quantile will not change after scaling. The quantile $k$th central moment is thus defined as

$$qkm_{d,\epsilon,n} := \hat{Q}_n\left(\left(p\mathrm{SW}km - \frac{1}{2}\right)d + p\mathrm{SW}km\right),$$

where $p\mathrm{SW}km = \hat{F}_{\psi,n}(\mathrm{SW}km_{\epsilon,n})$, $\hat{F}_{\psi,n}$ is the empirical cumulative distribution function of the corresponding central moment kernel distribution.

Finally, for standardized moments, quantile skewness and quantile kurtosis are defined to be $qskew_{d,\epsilon,n} := \frac{qtm_{d,\epsilon,n}}{qsd_{d,\epsilon,n}^3}$ and $qkurt_{d,\epsilon,n} := \frac{qfm_{d,\epsilon,n}}{qsd_{d,\epsilon,n}^4}$. Quantile standard deviation ($qsd_{d,\epsilon,n}$), recombined standard deviation ($rsd_{d,\epsilon,n}$), quantile third central moment ($qtm_{d,\epsilon,n}$), quantile fourth central moment ($qfm_{d,\epsilon,n}$), recombined third central moment ($rtm_{d,\epsilon,n}$), recombined fourth central moment ($rfm_{d,\epsilon,n}$), recombined skewness ($rskew_{d,\epsilon,n}$), and recombined kurtosis ($rkurt_{d,\epsilon,n}$) are all defined similarly as above and not repeated here. The transformation to a location problem can also empower related

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statistical tests. From the better performance of the quantile mean in heavy-tailed distributions, quantile central moments are generally better than recombined central moments regarding asymptotic bias.

To avoid confusion, the robust location estimations of the kernel distributions here are different from Joly and Lugosi (2016) ([43])'s approach, which is computing the median of all $U$-statistics from different disjoint blocks based on the median of means technique, although asymptotically, as discussed in the previous article, it can be equivalent to the median $U$-statistc if the size of each block is equal to the degree of the kernel. The drawbacks of this approach were realized by Laforgue, Clemencon, and Bertail (2019) and they proposed the medians of randomized $U$-statistics ([43], [44]) which is more sophisticated, and more close to the median $U$-statistc if setting an additional constraint on the block size.

**Data Availability.** Data for Table **??** are given in SI Dataset S1. All codes have been deposited in [GitHub](GitHub).

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