Near-consistent robust estimations of moments for unimodal distributions

Tuban Lee[a,1]

[a]Institute of Biomathematics, Macau SAR 999078, China

This manuscript was compiled on March 9, 2023

Descriptive statistics for parametric models currently rely heavily on the accuracy of distributional assumptions. Here, based on the invariant structures of unimodal distributions, a series of sophisticated yet efficient estimators, robust to both gross errors and departures from parametric assumptions, are proposed for estimating mean and central moments with insignificant asymptotic biases for common continuous unimodal distributions. This article also illuminates the understanding of the common nature of probability distributions and the measures of them.

orderliness | invariant | unimodal | adaptive estimation | U-statistics

The asymptotic inconsistencies between sample mean ($\bar{x}$) and nonparametric robust location estimators in asymmetric distributions on the real line have been noticed for more than two centuries (1) but unsolved. Strictly speaking, it is unsolvable because by trimming, some information about the original distribution is removed, making it impossible to estimate the values of the removed parts without distributional assumptions. Newcomb (1886, 1912) provided the first modern approach to this problem by developing a class of estimators that gives "less weight to the more discordant observations" (2, 3). In 1964, Huber (4) used the minimax procedure to obtain M-estimator for contaminated normal distribution, which has played a pre-eminent role in the later development of robust statistics. As previously demonstrated, under growing asymmetric departures from normality, the bias of the Huber M-estimator increases rapidly. This is a common issue in parametric estimations. For example, He and Fung (1999) constructed (5) a robust M-estimator for the two-parameter Weibull distribution. All moments can be calculated from its estimated parameters. As expected, it is inadequate for the gamma, Perato, lognormal, and the generalized Gaussian distributions (SI Dataset S1).

Another old and interesting approach is arithmetically computing the parameters using one or more L-statistics as input values, e.g., the percentile estimators. Examples for the Weibull distribution, the reader is referred to Menon (1963) (6), Dubey (1967) (7), Hassanein (1971) (8), Marks (2005) (9), and Boudt, Caliskan, and Croux (2011) (10)'s works. At the outset of the study of percentile estimators, it was known that percentile estimators arithmetically utilizes the invariant structures of probability distributions (6, 11, 12). Maybe such estimators can be named as I-statistics. Formally, an estimator is classified as an I-statistic if asymptotically it satisfies $\mathrm{I}\left(\mathrm{LE}_1, \cdots, \mathrm{LE}_l\right) = \left(\theta_1, \cdots, \theta_q\right)$ for the distribution it is consistent with, where LEs are calculated with the use of L-statistics, I is defined using arithmetic operations, but it may also incorporate other functions, and θs are the population parameters it estimates. A subclass of I-statistics, arithmetic I-statistics, is defined as LEs are L-statistics, I is solely defined with arithmetic operations. Since some percentile estimators use the logarithmic function to transform all random variables first and then compute the L-statistics, so a percentile estimator might not always be an arithmetic I-statistic (7). In this article, two sub-classes of I-statistics are introduced, arithmetic I-statistics and quantile I-statistics. Examples of quantile I-statistics will be discussed later. In the previous article, it is shown that quantile average is fundamental for all weighted averages. Based on the quantile function, I-statistics are naturally robust. For many parametric distributions, the quantile functions are much more elegant than the pdfs and cdfs. So, I-statistics are often analytically obtainable. However, the performance of the above examples is often worse than that of the robust M-statistics when the distributional assumption is violated (SI Dataset S1). Even when distributions such as the Weibull and gamma belong to the same larger family, the generalized gamma distribution, a misassumption can still result in substantial biases, rendering the approach ill-suited.

In previous work on semiparametric robust mean estimation, although greatly shrinking the asymptotic biases, binomial mean (BM_ϵ) is still inconsistent for any skewed distribution if $\epsilon > 0$ (if $\epsilon \to 0$, since the alternating sum of binomial coefficients is zero, $\mathrm{BM} \to \mu$). All robust location estimators commonly used are symmetric due to the universality of the symmetric distributions. One can construct an asymmetric trimmed mean that is consistent for a semiparametric class of skewed distributions. This approach was investigated previously, but it is not symmetric and therefore only suitable for some special applications (13). From semiparametrics to parametrics, an ideal robust location estimator would have a non-sample-dependent breakdown point (defined in Subsection **??**) and be consistent with any symmetric distribution and a skewed distribution with finite second moments. This is called an invariant mean. Based on the mean-symmetric weighted

Significance Statement

Bias, variance, and contamination are the three main errors in statistics. Consistent robust estimation is unattainable without parametric assumptions. Here, based on a paradigm shift inspired by mean-median-mode inequality, Bickel-Lehmann spread, and adaptive estimation, invariant moments are proposed as a means of achieving near-consistent and robust estimations of moments, even in scenarios where moderate violations of distributional assumptions occur, while the variances are sometimes smaller than those of the sample moments.

T.L. designed research, performed research, analyzed data, and wrote the paper.

The author declares no competing interest.

[1]To whom correspondence should be addressed. E-mail: tl@biomathematics.org

average-median inequality, the recombined mean is defined as

$$rm_{d,\epsilon,n} \coloneqq \lim_{c\to\infty}\left(\frac{(\mathrm{SWA}_{\epsilon,n}+c)^{d+1}}{(median+c)^{d}}-c\right),$$

where d is for bias correction, $\mathrm{SWA}_{\epsilon,n}$ is $\mathrm{BM}_{\epsilon,n}$ in the first three Subsections, while other symmetric weighted averages can also be used in practice as long as the inequalities hold. The next theorem shows the significance of this arithmetic I-statistic.

Theorem .1. *If the second moments are finite, $rm_{d\approx 0.375,\epsilon=\frac{1}{8}}$ is a consistent mean estimator for the exponential and any symmetric distributions and the Pareto distribution with quantile function $Q(p) = x_m(1-p)^{-\frac{1}{\alpha}}$, $x_m > 0$, when $\alpha \to \infty$.*

Proof. Finding d and ϵ that make $rm_{d,\epsilon}$ a consistent mean estimator is equivalent to finding the solution of $E[rm_{d,\epsilon}] = E[X]$. Rearranging the definition, $rm_{d,\epsilon} = \lim_{c\to\infty}\left(\frac{(\mathrm{BM}_{\epsilon}+c)^{d+1}}{(median+c)^{d}}-c\right) = (d+1)\,\mathrm{BM}_{\epsilon} - d\mathrm{median} = \mu$. So, $d = \frac{\mu-\mathrm{BM}_{\epsilon}}{\mathrm{BM}_{\epsilon}-median}$. The pdf of the exponential distribution is $f(x) = \lambda^{-1}e^{-\lambda^{-1}x}$, $\lambda \geq 0$, $x \geq 0$, the cdf is $F(x) = 1 - e^{-\lambda^{-1}x}$, $x \geq 0$. The quantile function is $Q(p) = \ln\left(\frac{1}{1-p}\right)\lambda$. $E[x] = \lambda$. $E[median] = Q\left(\frac{1}{2}\right) = \ln 2\lambda$. For the exponential distribution, the expectation of $\mathrm{BM}_{\frac{1}{8}}$ is $E\left[\mathrm{BM}_{\frac{1}{8}}\right] = \lambda\left(1+\ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)$. Obviously, the scale parameter λ can be canceled out, $d \approx 0.375$. The proof of the second assertion follows directly from the coincidence property. For any symmetric distribution with a finite second moment, $E[\mathrm{BM}_{\epsilon}] = E[median] = E[X]$. Then $E[rm_{d,\epsilon}] = \lim_{c\to\infty}\left(\frac{(E[X]+c)^{d+1}}{(E[X]+c)^{d}}-c\right) = E[X]$. The proof for the Pareto distribution is more general. The mean of the Pareto distribution is given by $\frac{\alpha x_m}{\alpha-1}$. The d value with two unknown percentiles p_1 and p_2 for the Pareto distribution is $d_{Perato} = \frac{\frac{\alpha x_m}{\alpha-1}-x_m(1-p_1)^{-\frac{1}{\alpha}}}{x_m(1-p_1)^{-\frac{1}{\alpha}}-x_m(1-p_2)^{-\frac{1}{\alpha}}}$. Since any weighted average can be expressed as an integral of the quantile function, $\lim_{\alpha\to\infty}\frac{\frac{\alpha}{\alpha-1}-(1-p_1)^{-1/\alpha}}{(1-p_1)^{-1/\alpha}-(1-p_2)^{-1/\alpha}} = -\frac{\ln(1-p_1)+1}{\ln(1-p_1)-\ln(1-p_2)}$, the d value for the Pareto distribution approaches that of the exponential distribution as $\alpha \to \infty$, regardless of the type of weighted average used. This completes the demonstration. □

Theorem .1 implies that for the Weibull, gamma, Pareto, lognormal and generalized Gaussian distribution, $rm_{d\approx 0.375,\epsilon=\frac{1}{8}}$ is consistent for at least one particular case of these two-parameter distributions. The biases of $rm_{d\approx 0.375,\epsilon=\frac{1}{8}}$ for distributions with skewness between those of the exponential and symmetric distributions are tiny (SI Dataset S1). $rm_{d\approx 0.375,\epsilon=\frac{1}{8}}$ has excellent performance for all these common unimodal distributions (SI Dataset S1).

Besides introducing the concept of invariant mean, the purpose of this paper is to demonstrate that, in light of previous works, the estimation of central moments can be transformed into a location estimation problem by using U-statistics, the central moment kernel distributions have nice properties, and a series of sophisticated yet efficient robust estimators can be constructed whose biases are typically smaller than the variances ($n = 5400$, Table **??**) for unimodal distributions.

Background and Main Results

A. Invariant mean. It has long been known that a theoretical model can be adjusted to fit the first two moments of the observed data. A continuous distribution belonging to a location–scale family has the form $F(x) = F_0\left(\frac{x-\mu}{\lambda}\right)$, where F_0 is a "standard" distribution. Then, $F(x) = Q^{-1}(x) \rightarrow x = Q(p) = \lambda Q_0(p) + \mu$. So, any weighted average can be expressed as $\lambda\mathrm{WA}_0(\epsilon) + \mu$, where $\mathrm{WA}_0(\epsilon)$ is an integral of $Q_0(p)$ according to the definition of the weighted average. The simultaneous cancellation of μ and λ in $\frac{(\lambda\mu_0+\mu)-(\lambda\mathrm{BM}_0(\epsilon)+\mu)}{(\lambda\mathrm{BM}_0(\epsilon))+\mu-(\lambda median_0+\mu)}$ ensures that d is a constant. Consequently, the roles of BM_{ϵ} and median in $rm_{d,\epsilon}$ can be replaced by any weighted averages, although for the definition of invariant mean, only symmetric weighted averages are considered here.

The performance in heavy-tailed distributions can be improved further by constructing the quantile mean as

$$qm_{d,\epsilon,n} \coloneqq \hat{Q}_n\left(\left(\hat{F}_n(\mathrm{SWA}_{\epsilon,n}) - \frac{1}{2}\right)d + \hat{F}_n(\mathrm{SWA}_{\epsilon,n})\right),$$

provided that $\hat{F}_n(\mathrm{SWA}_{\epsilon,n}) \geq \frac{1}{2}$, where $\hat{F}_n(x)$ is the empirical cumulative distribution function of the sample, $\hat{Q}_n$ is the sample quantile function. The most popular method for computing the sample quantile function was proposed by Hyndman and Fan in 1996 (14). To minimize the finite sample bias, here, $\hat{F}_n(x) \coloneqq \frac{1}{n}\left(\frac{x-\hat{Q}_n\left(\frac{sp}{n}\right)}{\hat{Q}_n\left(\frac{1}{n}(sp+1)\right)-\hat{Q}_n\left(\frac{sp}{n}\right)}+sp\right)$, where $sp = \sum_{i=1}^{n}\mathbf{1}_{X_i\leq x}$, $\mathbf{1}_A$ is the indicator of event A. The solution of $\hat{F}_n(\mathrm{SWA}_{\epsilon,n}) < \frac{1}{2}$ is reversing the percentile by $1-\hat{F}_n(\mathrm{SWA}_{\epsilon,n})$, the obtained percentile is also reversed. Without loss of generality, in the following discussion, only the $\hat{F}_n(\mathrm{SWA}_{\epsilon,n}) \geq \frac{1}{2}$ case will be considered. Moreover, in extreme heavy-tailed distributions, the calculated percentile can exceed the breakdown point of SWA_{ϵ}, so the percentile will be modified to $1-\epsilon$ if this happens. The quantile mean uses the location-scale invariant in a different way as shown in the following proof.

Theorem A.1. *$qm_{d\approx 0.321,\epsilon=\frac{1}{8}}$ is a consistent mean estimator for the exponential, Pareto ($\alpha \to \infty$) and any symmetric distributions provided that the second moments are finite.*

Proof. Similarly, rearranging the definition, $d = \frac{F(\mu)-F(\mathrm{BM}_{\epsilon})}{F(\mathrm{BM}_{\epsilon})-\frac{1}{2}}$. Recall the cdf is $F(x) = 1-e^{-\lambda^{-1}x}$, $x \geq 0$, the expectation of BM_{ϵ} can be expressed as $\lambda\mathrm{BM}_0(\epsilon)$, so $F(\mathrm{BM}_{\epsilon})$ is free of λ. When $\epsilon = \frac{1}{8}$, $d = \frac{-e^{-1}+e^{-\left(1+\ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)}}{\frac{1}{2}-e^{-\left(1+\ln\left(\frac{46656}{8575\sqrt{35}}\right)\right)}} \approx 0.321$.

The proof of the symmetric case is similar. Since for any symmetric distribution with a finite second moment, $F(E[\mathrm{BM}_{\epsilon}]) = F(\mu) = \frac{1}{2}$. Then, the expectation of the quantile mean is $qm_{d,\epsilon} = F^{-1}\left(\left(F(\mu)-\frac{1}{2}\right)d + F(\mu)\right) = F^{-1}(0+F(\mu)) = \mu$.

For the assertion related to the Pareto distribution, the cdf of it is $1-\left(\frac{x_m}{x}\right)^{\alpha}$. So, the d value with two unknown percentile p_1 and p_2 is $d_{Pareto} = \frac{1-\left(\frac{x_m}{\frac{\alpha x_m}{\alpha-1}}\right)^{\alpha}-\left(1-\left(\frac{x_m}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^{\alpha}\right)}{\left(1-\left(\frac{x_m}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^{\alpha}\right)-\left(1-\left(\frac{x_m}{x_m(1-p_2)^{-\frac{1}{\alpha}}}\right)^{\alpha}\right)} = \frac{1-\left(\frac{\alpha-1}{\alpha}\right)^{\alpha}-p_1}{p_1-p_2}$. When $\alpha \to \infty$, $\left(\frac{\alpha-1}{\alpha}\right)^{\alpha} = \frac{1}{e}$. The d value for the exponential distribution is identical, since $d_{exp} = \frac{\left(1-e^{-1}\right)-\left(1-e^{-\ln\left(\frac{1}{1-p_1}\right)}\right)}{\left(1-e^{-\ln\left(\frac{1}{1-p_1}\right)}\right)-\left(1-e^{-\ln\left(\frac{1}{1-p_2}\right)}\right)} = \frac{1-\frac{1}{e}-p_1}{p_1-p_2}$. All results are now proven. $\square$

The definitions of location and scale parameters are such that they must satisfy $F(x;\lambda,\mu) = F(\frac{x-\mu}{\lambda};1,0)$. Recall that $x = \lambda Q_0(p) + \mu$, so the percentile of any weighted average is free of λ and μ, guaranteeing the validity of the quantile mean. quantile mean is a quantile I-statistic. Formally, an estimator is classified as a quantile I-statistic, if LEs are percentiles of a distribution obtained by plugging L-statistics into a cumulative distribution function and I is defined with arithmetic operations and quantile functions. $qm_{d\approx 0.321,\epsilon=\frac{1}{8}}$ works better in the fat-tail scenarios (SI Dataset S1). Theorem .1 and A.1 show that $rm_{d\approx 0.375,\epsilon=\frac{1}{8}}$ and $qm_{d\approx 0.321,\epsilon=\frac{1}{8}}$ are both consistent mean estimators for any symmetric distribution and a skewed distribution with finite second moments. It's obvious that the breakdown points of $rm_{d\approx 0.375,\epsilon=\frac{1}{8}}$ and $qm_{d\approx 0.321,\epsilon=\frac{1}{8}}$ are both $\frac{1}{8}$. Therefore they are all invariant means.

To study the impact of the choice of SWAs in rm and qm, it is constructive to consider a symmetric weighted average as a mixture of symmetric quantile averages. Although using a less-biased symmetric weighted average can generally improve performance (SI Dataset S1), there is a higher risk of violation in the semiparametric framework. However, the mean-SWA-median inequality is robust to small fluctuations of the SQA function of the underlying distribution. Suppose the SQA function is generally decreasing in $[0,u]$, but increasing in $[u,\frac{1}{2}]$, since $1-2\epsilon$ of the symmetric quantile averages will be included in the computation of SWA_ϵ, as long as $|u-\frac{1}{2}| \ll 1-2\epsilon$, and other parts of the SQA function satisfy the inequality constraints which define the νth orderliness, the mean-SWA_ϵ-median inequality will still be valid (as an example, the SQA function is non-monotonic when the shape parameter of the Weibull distribution $\alpha > \frac{1}{1-\ln(2)} \approx 3.259$ as shown in the previous article, yet the mean-$\text{BM}_{\frac{1}{8}}$-median inequality is still valid when $\alpha \leq 3.322$). Another key factor determining the risk of violation is the skewness of the distribution. In the previous article, it is shown that in a family of distributions that differ by a skewness-increasing transformation in van Zwet's sense, the violation of orderliness, if it happens, often only occurs when the distribution is near-symmetric (15). The over-corrections in rm and qm are dependent on the SWA_ϵ-median difference, which is correlated to the skewness (16, 17), so the over-correction is often tiny with a moderate d. This qualitative analysis provides another perspective, in addition to the bias bounds (18), that rm and qm based on the mean-SWA_ϵ-median inequality are generally safe.

B. Robust estimations of the central moments. In 1976, Bickel and Lehmann, in their third paper of the landmark series *Descriptive Statistics for Nonparametric Models* (19), generalized a class of estimators called "measures of disperse," which is now often named as Bickel-Lehmann dispersion. As an example, they proposed a first version of the trimmed standard deviation, $\hat{\tau}^2(F;\epsilon) \equiv \tau^2(F;\epsilon)$, for independent and identically distributed random variables X with a distribution F, where $\tau^2(F;\epsilon) = \frac{1}{1-2\epsilon}\int_{Q(\epsilon)}^{Q(1-\epsilon)} ydG(y)$, Q is the quantile function of G, G is the distribution of $Y = X^2$. Obviously, when $\epsilon = 0$, the result is equivalent to the second raw moment. In 1979, in the same series (20), they explored another class of estimators called "measures of spread," which "does not require the assumption of symmetry." From that, a popular efficient scale estimator, the Rousseeuw-Croux scale estimator (21), was derived in 1993, but the importance of tackling the symmetry assumption has been greatly underestimated. In the final section of the paper, they considered another two possible versions of the trimmed standard deviations, which were modified here for comparison,

$$\left[n\left(\frac{1}{2}-\epsilon\right)\right]^{-\frac{1}{2}}\left[\sum_{i=\frac{n}{2}}^{n(1-\epsilon)}\left[X_i - X_{n-i+1}\right]^2\right]^{\frac{1}{2}}, \qquad [1]$$

and

$$\left[\binom{n}{2}(1-\epsilon-\gamma\epsilon)\right]^{-\frac{1}{2}}\left[\sum_{i=\binom{n}{2}\epsilon}^{\binom{n}{2}(1-\gamma\epsilon)}\left(X-X'\right)_i^{\,2}\right]^{\frac{1}{2}}, \qquad [2]$$

where $(X-X')_1 \leq \ldots \leq (X-X')_{\binom{n}{2}}$ are the order statistics of the "pseudo-sample". The paper ended with, "We do not know a fortiori which of the measures [1] or [2] is preferable and leave these interesting questions open."

Observe that the kernel of the unbiased estimation of the second central moment by using U-statistic is $\psi_2(x_1,x_2) = \frac{1}{2}(x_1-x_2)^2$. If adding the $\frac{1}{2}$ term in [2], as $\epsilon \to 0$, the result is equivalent to the standard deviation estimated by using U-statistic (also noted by Janssen, Serfling, and Veraverbeke in 1987) (22). In fact, they also showed that, when ϵ is 0, [2] is $\sqrt{2}$ times the standard deviation.

To address their open questions, the nomenclature used in this paper is introduced as follows:

Nomenclature. Given a robust estimator $\hat{\theta}$. The first part of the name of the robust statistic defined in this paper is a prefix that indicates the type of estimator, and the second part is the name of the population parameter θ that the estimator is consistent with as $\epsilon \to 0$. The abbreviation of the estimator is the initial letter(s) of the first part plus the common abbreviation of the consistent estimator that measures the population parameter. If the estimator is symmetric and not a U-statistic, ϵ is indicated in the subscript of the abbreviation of the estimator. If the estimator is asymmetric, the corresponding γ is also indicated after ϵ. If the estimator is a weighted U-statistic, the breakdown point of the location estimator is indicated (except the median).

In the previous semiparametric robust mean article, it is shown that the bias of a reasonable robust estimator should be monotonic with respect to the breakdown point in a semiparametric distribution and, naturally, its name should align with the consistent estimator. The trimmed standard deviation following this nomenclature is $\text{T}sd_{\epsilon,\gamma,n} :=$ $\left[\text{TM}_{\epsilon,\gamma}\left(\left(\psi_2\left(X_{N_1},X_{N_2}\right)\right)_{N=1}^{\binom{n}{2}}\right)\right]^{-\frac{1}{2}}$, where $\text{TM}_{\epsilon,\gamma}(Y)$ denotes the ϵ,γ-trimmed mean with the sequence $\left(\psi_2\left(X_{N_1},X_{N_2}\right)\right)_{N=1}^{\binom{n}{2}}$ as an input. If the square root is removed, it is named as the trimmed variance ($\mathrm{T}var_{\epsilon,\gamma,n}$). It is now very clear that this definition, essentially the same as [2], should be preferable. Not only because it is essentially a trimmed U-statistic for the standard deviation but also because the γ-orderliness of the second central moment kernel distribution is ensured by the next exciting theorem.

Theorem B.1. *The second central moment kernel distribution generated from any continuous unimodal distribution is γ-ordered, if $\gamma \geq 1$.*

Proof. Let $Q(p)$, $0 \leq p \leq 1$, denote the quantile of the continuous unimodal distribution $f_X(x)$. The corresponding probability density is $f(Q(p))$. Generating the distribution of the pair $(Q(p_i), Q(p_j))$, $i < j$, $p_i < p_j$, the corresponding probability density is $f_{X,X}(Q(p_i), Q(p_j)) = 2f(Q(p_i))f(Q(p_j))$. Transforming the pair $(Q(p_i), Q(p_j))$, $i < j$, by the function $\Phi(x_1, x_2) = x_1 - x_2$, the pairwise difference distribution has a mode that is arbitrary close to $M - M = 0$. The monotonic increasing of the pairwise difference distribution was first implied in its unimodality proof done by Hodges and Lehmann in 1954 (23). Whereas they used induction to get the result, Dharmadhikari and Jogdeo in 1982 (24) gave a modern proof of the unimodality using Khintchine's representation (25). Assuming absolute continuity, Purkayastha (26) introduced a much simpler proof in 1998. Transforming the pairwise difference distribution by squaring and multiplying $\frac{1}{2}$ does not change the monotonicity, making the pdf become monotonically decreasing with mode at zero. In the previous semiparametric robust mean estimation article, it is proven that a right skewed distribution with a monotonic decreasing pdf is always γ-ordered, if $\gamma \geq 1$, which gives the desired result. □

Remark. The assumption of continuity of distributions is important for monotonicity because, unlike in the continuous case, it is possible to get pairs with the same value for a discrete distribution. For example, let the probabilities of the singletons $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ and $\{5\}$ of a probability mass function of a discrete probability distribution be $\frac{1}{11}$, $\frac{4}{11}$, $\frac{3}{11}$, $\frac{2}{11}$, and $\frac{1}{11}$, respectively. This is a unimodal distribution, but the corresponding ψ_2 distribution is non-monotonic, whose singletons $\{0\}$, $\{0.5\}$, $\{2\}$, $\{4.5\}$ and $\{8\}$ have probabilities $\frac{21}{66}$, $\frac{24}{66}$, $\frac{2}{14}$, $\frac{6}{66}$, and $\frac{1}{66}$, respectively.

Previously, it was shown that any symmetric distribution with a finite second moment is νth ordered. That means the orderliness does not require unimodality, e.g., for a symmetric bimodal distribution, it is also ordered. Examples from the Weibull distribution show that unimodality does not guarantee orderliness. Theorem B.1 reveals another profound relationship between unimodality and orderliness, which is sufficient for trimming inequality.

In 1928, Fisher constructed k-statistics as unbiased estimators of cumulants (27). Halmos (1946) proved that the functional θ admits an unbiased estimator if and only if it is a regular statistical functional of degree k and showed a relation of symmetry, unbiasness and minimum variance (28). In 1948, Hoeffding generalized U-statistics (29) which enable the derivation of a minimum-variance unbiased estimator from each unbiased estimator of an estimable parameter. Heffernan (1997) (30) obtained an unbiased estimator of the kth central moment by using U-statistics and demonstrated that it is the minimum variance unbiased estimator for distributions with finite moments (31, 32). In 1976, Saleh generalized the Hodges-Lehmenn estimator (33) to the trimmed H-L mean (he named "Wilcoxon one-sample statistic") (34). In 1984, Serfling pointed out the speciality of Hodges-Lehmann estimator, which is neither a simple L-statistics nor U-statistic, and considered the generalized L-statistics and U-statistic structure (35). Also in 1984, Janssen and Serfling and Veraverbeke (36) showed that the Bickel-Lehmann spread also belongs to the same class. It was gradually clear that the Hodges-Lehmenn estimator, trimmed H-L mean and trimmed standard deviation are all trimmed U-statistics (37–39). Due to the combinatorial explosion, the bootstrap (40), introduced by Efron in 1979, is indispensable in large sample studies. In 1981, Bickel and Freedman (41) showed that the bootstrap is asymptotically valid to approximate the original distribution in a wide range of situations, including U-statistics. The limit laws of bootstrapped trimmed U-statistics was proven by Helmers, Janssen, and Veraverbeke (1990) (42).

Extending the trimmed U-statistic to weighted U-statistic, i.e., replacing the trimmed mean with weighted average. The weighted kth central moment ($k \leq n$) is defined as,

$$\mathrm{W}km_{\epsilon,\gamma,n} := \mathrm{WA}_{\epsilon,\gamma,n}\left(\left(\psi_k\left(X_{N_1},\cdots,X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}}\right),$$

where $X_{N_1},\cdots,X_{N_k}$ are the n choose k elements from X, $\psi_k\left(x_1,\cdots,x_k\right) = \sum_{j=0}^{k-2}(-1)^j\left(\frac{1}{k-j}\right)\sum\left(x_{i_1}^{k-j}\ldots x_{i_{(j+1)}}\right) + (-1)^{k-1}(k-1)\,x_1\ldots x_k$, the second summation is over $i_1,\cdots,i_{j+1} = 1$ to k with $i_1 < \cdots < i_{j+1}$ (30). Despite the complexity, the structure of the kth central moment kernel distributions can be elucidated by decomposing.

Theorem B.2. *For each pair $\left(Q\left(p_i\right), Q\left(p_j\right)\right)$ of the original distribution, let $x_1 = Q(p_i)$ and $x_k = Q(p_j)$, $\Delta = Q(p_i) - Q(p_j)$. The kth central moment kernel distribution, $k > 2$, can be seen as a mixture distribution and each of the components has the support $\left(-\left(\frac{k}{\frac{3+(-1)^k}{2}}\right)^{-1}(-\Delta)^k, \frac{1}{k}(-\Delta)^k\right)$.*

Proof. Generating the distribution of the k-tuple $(Q(p_{i_1}),\ldots,Q(p_{i_k}))$, $k > 2$, $i_1 < \ldots < i_k$, $p_{i_1} < \ldots < p_{i_k}$, the corresponding probability density is $f_{X,\ldots,X}(Q(p_{i_1}),\ldots,Q(p_{i_k})) = k!f(Q(p_{i_1}))\ldots f(Q(p_{i_k}))$. Transforming the distribution of the k-tuple by the function $\psi_k\left(x_1,\ldots,x_k\right)$, denoting $\bar{\Delta} = \psi_k\left(Q(p_{i_1}),\ldots,Q(p_{i_k})\right)$. The probability $f_{\Xi_k}(\bar{\Delta}) = \sum_{\bar{\Delta}=\psi_k\left(Q(p_{i_1}),\ldots,Q(p_{i_k})\right)} f_{X,\ldots,X}(Q(p_{i_1}),\ldots,Q(p_{i_k}))$ is the summation of the probabilities of all k-tuples such that $\bar{\Delta}$ is equal to $\psi_k\left(Q(p_{i_1}),\ldots,Q(p_{i_k})\right)$. The following Ξ_k is equivalent.

Ξ_k: Every pair with a difference equal to $\Delta = Q(p_{i_1}) - Q(p_{i_k})$ can generate a pseudodistribution (but the integral is not equal to 1, so "pseudo") such that $x_2, \ldots, x_{k-1}$ exhaust all combinations under the inequality constraints, i.e., $Q(p_{i_1}) = x_1 < x_2 < \ldots < x_{k-1} < x_k = Q(p_{i_k})$. The combination of all the pseudodistributions with the same Δ is ξ_Δ. The combination of ξ_Δ, i.e., from $\Delta = 0$ to $Q(0) - Q(1)$, is Ξ_k.

The support of ξ_Δ is the extrema of ψ_k subject to the inequality constraints. Using the Lagrange multiplier, one can easily determine the only critical point at $x_1 = \ldots = x_k = 0$, where $\psi_k = 0$. Other candidates are within the boundaries, i.e., $\psi_k\left(x_1 = x_1, x_2 = x_k, \cdots, x_k = x_k\right)$,

$\cdots$, $\psi_k(x_1 = x_1, \cdots, x_i = x_1, x_{i+1} = x_k, \cdots, x_k = x_k)$, $\cdots$, $\psi_k(x_1 = x_1, \cdots, x_{k-1} = x_1, x_k = x_k)$. $\psi_k(x_1 = x_1, \cdots, x_i = x_1, x_{i+1} = x_k, \cdots, x_k = x_k)$ can be divided into k groups. If $\frac{k+1-i}{2} \leq j \leq \frac{k-1}{2}$, from $j+1$st to $k-j$th group, the gth group has $i\binom{i-1}{g-j-1}\binom{k-i}{j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$, from $k-j+1$th to $i+j$th group, the gth group has $i\binom{i-1}{g-j-1}\binom{k-i}{j} + (k-i)\binom{k-i-1}{j-k+g-1}\binom{i}{k-j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$. If $j < \frac{k+1-i}{2}$, from $j+1$st to $i+j$th group, the gth group has $i\binom{i-1}{g-j-1}\binom{k-i}{j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$. If $j \geq \frac{k}{2}$, from $k-j+1$st to jth group, the gth group has $(k-i)\binom{k-i-1}{j-k+g-1}\binom{i}{k-j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$, from $j+1$th to $j+i$th group, $i+j<k$, the gth group has $i\binom{i-1}{g-j-1}\binom{k-i}{j} + (k-i)\binom{k-i-1}{j-k+g-1}\binom{i}{k-j}$ terms having the form $(-1)^{g+1}\frac{1}{k-g+1}x_1^{k-j}x_k^j$. The final kth group is the term $(-1)^{k-1}(k-1)x_1^i x_k^{k-i}$. So, if $i+j=k$, $j \geq \frac{k}{2}$, $i \leq \frac{k}{2}$, the summed coefficient of $x_1^i x_k^{k-i}$ is $(-1)^{k-1}(k-1) + \sum_{g=i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}(k-i)\binom{k-i-1}{g-i-1} + \sum_{g=k-i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}i\binom{i-1}{g-k+i-1} = (-1)^{k-1}(k-1) + (-1)^{k+1} + (k-i)(-1)^k + (-1)^k(i-1) = (-1)^{k+1}$. The summation identities are $\sum_{g=i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}(k-i)\binom{k-i-1}{g-i-1} = (k-i)\int_0^1\sum_{g=i+1}^{k-1}(-1)^{g+1}\binom{k-i-1}{g-i-1}t^{k-g}dt = (k-i)\int_0^1\left((-1)^i(t-1)^{k-i-1} - (-1)^{k+1}\right)dt = (k-i)\left(\frac{(-1)^k}{i-k} + (-1)^k\right) = (-1)^{k+1} + (k-i)(-1)^k$ and $\sum_{g=k-i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}i\binom{i-1}{g-k+i-1} = \int_0^1\sum_{g=k-i+1}^{k-1}(-1)^{g+1}i\binom{i-1}{g-k+i-1}t^{k-g}dt = \int_0^1\left(i(-1)^{k-i}(t-1)^{i-1} - i(-1)^{k+1}\right)dt = (-1)^k(i-1)$. If $j < \frac{k+1-i}{2}$, $i > k-1$, if $i=k$, $\psi_k = 0$, if $\frac{k+1-i}{2} \leq j \leq \frac{k-1}{2}$, $\frac{k+1}{2} \leq i \leq k-1$, the summed coefficient of $x_1^i x_k^{k-i}$ is $(-1)^{k-1}(k-1) + \sum_{g=k-i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}i\binom{i-1}{g-k+i-1} + \sum_{g=i+1}^{k-1}(-1)^{g+1}\frac{1}{k-g+1}(k-i)\binom{k-i-1}{g-i-1}$, the same as above. If $i+j<k$, since $\binom{i}{k-j} = 0$, the related terms can be ignored, so, using the binomial theorem and beta function, the summed coefficient of $x_1^{k-j}x_k^j$ is $\sum_{g=j+1}^{i+j}(-1)^{g+1}\frac{1}{k-g+1}i\binom{i-1}{g-j-1}\binom{k-i}{j} = i\binom{k-i}{j}\int_0^1\sum_{g=j+1}^{i+j}(-1)^{g+1}\binom{i-1}{g-j-1}t^{k-g}dt = \binom{k-i}{j}i\int_0^1\left((-1)^j t^{k-j-1}\left(\frac{t}{t-1}\right)^{1-i}\right)dt = \binom{k-i}{j}i\frac{(-1)^{j+i+1}\Gamma(i)\Gamma(k-j-i+1)}{\Gamma(k-j+1)} = \frac{(-1)^{j+i+1}i!(k-j-i)!(k-i)!}{(k-j)!j!(k-j-i)!} = (-1)^{j+i+1}\frac{i!(k-i)!}{k!}\frac{k!}{(k-j)!j!} = \binom{k}{i}^{-1}(-1)^{1+i}\binom{k}{j}(-1)^j$.

The coefficient of $x_1^i x_k^{k-i}$ in $\binom{k}{i}^{-1}(-1)^{1+i}(x_1-x_k)^k$ is $\binom{k}{i}^{-1}(-1)^{1+i}\binom{k}{i}(-1)^{k-i} = (-1)^{k+1}$, same as the summed coefficient if $i+j=k$. If $i+j<k$, the coefficient of $x_1^{k-j}x_k^j$ is $\binom{k}{i}^{-1}(-1)^{1+i}\binom{k}{j}(-1)^j$, same as the corresponding summed coefficient. Therefore, $\psi_k(x_1 = x_1, \ldots, x_i = x_1, x_{i+1} = x_k, \ldots, x_k = x_k) = \binom{k}{i}^{-1}(-1)^{1+i}(x_1-x_k)^k$, the maximum and minimum of ψ_k follow directly from the properties of the binomial coefficient. □

ξ_Δ is closely related to $f_\Xi(\Delta)$, which is the pairwise difference distribution, since the probability density of ξ_Δ is $f_{\Xi_k}(\bar{\Delta}|\Delta)$, $\sum_{\bar{\Delta}=-\binom{k}{\frac{3+(-1)^k}{2}}^{-1}(-\Delta)^k}^{\frac{1}{k}(-\Delta)^k} f_{\Xi_k}(\bar{\Delta}|\Delta) = f_\Xi(\Delta)$. Recall that $f_\Xi(\Delta)$ is monotonic increasing with a mode at the origin if the original distribution is unimodal. Thus, in general, ignoring the shape of ξ_Δ, Ξ_k is monotonic left and right around zero. In fact, the median of Ξ_k is also close to zero, as it can be cast as a weighted mean of the medians of ξ_Δ. When Δ is small, all values of ξ_Δ are close to zero, resulting in the median of ξ_Δ close to zero, When Δ is large, the median of ξ_Δ depends on its skewness, but the corresponding weight is much smaller, so even if ξ_Δ is highly skewed, the median of Ξ_k will only be slightly shifted from zero (denote the median of Ξ_k as m_{Ξ_k}, for five parametric distributions here, $|m_{\Xi_k}|$s are all $\leq 0.1\sigma$ for Ξ_3 and Ξ_4, SI Dataset S1). Assuming $m_{\Xi_k} = 0$, for the even ordinal central moment kernel distribution, the average probability density on the left side of zero is greater than that on the right side, since $\frac{\frac{1}{2}}{\binom{k}{2}^{-1}(Q(0)-Q(1))^k} > \frac{\frac{1}{2}}{\frac{1}{k}(Q(0)-Q(1))^k}$. This means that, on average, the inequality $f(Q(\epsilon)) \geq f(Q(1-\epsilon))$ holds. For the odd ordinal distribution, the discussion is harder since it is generally symmetric. Just consider Ξ_3, let $x_1 = Q(p_i)$ and $x_3 = Q(p_j)$, changing the value of x_2 from $Q(p_i)$ to $Q(p_j)$ will monotonically change the value of $\psi_3(x_1, x_2, x_3)$, since $\frac{\partial\psi_3(x_1,x_2,x_3)}{\partial x_2} = -\frac{x_1^2}{2} - x_1x_2 + 2x_1x_3 + x_2^2 - x_2x_3 - \frac{x_3^2}{2}$, $-\frac{3}{4}(x_1-x_3)^2 \leq \frac{\partial\psi_3(x_1,x_2,x_3)}{\partial x_2} \leq -\frac{1}{2}(x_1-x_3)^2 \leq 0$. If the original distribution is right-skewed, ξ_Δ will be left-skewed, so, for Ξ_3, the average probability density of the right side of zero will be greater than that of the left side, which means, on average, the inequality $f(Q(\epsilon)) \leq f(Q(1-\epsilon))$ holds (the same result can be inferred from the definition of central moments, the positive of odd order central moment is directly related to the left-skewness of the corresponding kernel distribution). In all, the monotonicity of the pairwise difference distribution guides the general shape of the kth central moment kernel distribution, $k>2$, forcing it to be unimodal-like with mode and median close to zero, then, the inequality $f(Q(\epsilon)) \leq f(Q(1-\epsilon))$ or $f(Q(\epsilon)) \geq f(Q(1-\epsilon))$ holds in general. If a distribution is ordered and its all central moment kernel distributions are also ordered, it is called completely ordered. Although strict complete orderliness is hard to prove, the inequality may be violated in a small range, as discussed in Subsection A, the mean-SWA$_\epsilon$-median inequality remains valid, in most cases, for the central moment kernel distribution.

Another key property of the central moment kernel distribution, location invariant, is introduced in the next theorem. The proof is given in the SI Text.

Theorem B.3. $\psi_k(x_1 = \lambda x_1 + \mu, \cdots, x_k = \lambda x_k + \mu) = \lambda^k\psi_k(x_1, \cdots, x_k)$.

Consider two continuous distributions belonging to the same location–scale family, their corresponding kth central moment kernel distributions only differ in scaling. So d is invariant, as shown in Subsection A. The recombined kth central moment, based on rm, is defined by,

$$rkm_{d,\epsilon,n} := (d+1)\,\mathrm{SW}km_{\epsilon,n} - dmkm_{\epsilon,n},$$

where $\mathrm{SW}km_{\epsilon,n}$ is using the binomial kth central moment ($\mathrm{B}km_{\epsilon,n}$) here, $mkm_{\epsilon,n}$ is the median kth central moment. Since $\mathrm{SW}km_{\epsilon,n}$ is an L-statistic, $rkm_{d,\epsilon,n}$ is an arithmetic I-statistic. Similarly, the quantile will not change after scaling.

The quantile kth central moment is thus defined as

$$qkm_{d,\epsilon,n} \coloneqq \hat{Q}_n\left(\left(p\mathrm{SW}km - \frac{1}{2}\right)d + p\mathrm{SW}km\right),$$

where $p\mathrm{SW}km = \hat{F}_{\psi,n}\left(\mathrm{SW}km_{\epsilon,n}\right)$, $\hat{F}_{\psi,n}$ is the empirical cumulative distribution function of the corresponding central moment kernel distribution. $qkm_{d,\epsilon,n}$ is a quantile I-statistic.

Finally, for standardized moments, quantile skewness and quantile kurtosis are defined to be $qskew_{d,\epsilon,n} \coloneqq \frac{qtm_{d,\epsilon,n}}{qsd^3_{d,\epsilon,n}}$ and $qkurt_{d,\epsilon,n} \coloneqq \frac{qfm_{d,\epsilon,n}}{qsd^4_{d,\epsilon,n}}$. Quantile standard deviation ($qsd_{d,\epsilon,n}$), recombined standard deviation ($rsd_{d,\epsilon,n}$), quantile third central moment ($qtm_{d,\epsilon,n}$), quantile fourth central moment ($qfm_{d,\epsilon,n}$), recombined third central moment ($rtm_{d,\epsilon,n}$), recombined fourth central moment ($rfm_{d,\epsilon,n}$), recombined skewness ($rskew_{d,\epsilon,n}$), and recombined kurtosis ($rkurt_{d,\epsilon,n}$) are all defined similarly as above and not repeated here. The transformation to a location problem can also empower related statistical tests. From the better performance of the quantile mean in heavy-tailed distributions, quantile central moments are generally better than recombined central moments regarding asymptotic bias.

To avoid confusion, the robust location estimations of the kernel distributions here are different from Joly and Lugosi (2016) (43)'s approach, which is computing the median of all U-statistics from different disjoint blocks based on the median of means technique, although asymptotically, as discussed in the previous article, it can be equivalent to the median U-statistic if the size of each block is equal to the degree of the kernel. Laforgue, Clemencon, and Bertail (2019) proposed the median of randomized U-statistics (43, 44), which is more sophisticated and closer to the median U-statistic if setting an additional constraint on the block size.

C. Congruent distribution. In the realm of nonparametric statistics, the precise values of robust estimators are of secondary importance. What is of primary importance is their relative differences, or orders. In the absence of contamination, as the parameters of the distribution vary, a reasonable nonparametric location estimator will asymptotically change in the same direction as the other location estimators. Otherwise if the results based on trimmed mean are completely different from those based on median, a contradiction arises. A distribution satisfying this property for any symmetric weighted average is called a congruent distribution. If extending to any ϵ,γ-weighted average, it is γ-congruent. A distribution is completely congruent if and only if it is congruent and its all central moment kernel distributions are also congruent. Complete γ-congruence is analogous. Chebyshev's inequality implies that, for any probability distribution with finite moments, even if some weighted averages change in a direction different from that of the sample mean, the deviations are bounded. Also, distributions with infinite moments can be congruent, since it is defined here that from infinity to infinity, the direction change can be interpreted as both increasing and decreasing. The following theorems show the conditions that a distribution is congruent or γ-congruent.

Theorem C.1. *Let the symmetric quantile average function of a parametric distribution be denoted as $SQA\left(\epsilon, \alpha_1, \cdots, \alpha_i, \cdots, \alpha_k\right)$, where α_i represent the parameters of the distribution. This distribution is congruent if and only if the sign of $\frac{\partial SQA(\epsilon,\alpha_i)}{\partial \alpha_i}$ remains the same (if equal to zero, it can be seen as both positive and negative and thus also impact the analysis) for all $0 < \epsilon \leq \frac{1}{2}$. Replacing SQA with $QA_{\epsilon,\gamma}$ constitutes a necessary and sufficient condition for the distribution to be considered γ-congruent.*

Proof. Asymptotically, any symmetric weighted average can be expressed as an integral of the symmetric quantile average function. Since the sign won't change after integration, from definition, the sign of $\frac{\partial \mathrm{SQA}(\epsilon,\alpha_i)}{\partial \alpha_i}$ remains the same for all $0 < \epsilon \leq \frac{1}{2}$ is equivalent to all symmetric weighted averages change in the same direction as the parameters change. The same logic applies to the γ-congruence case, as the constancy of the sign of $\frac{\partial \mathrm{QA}(\epsilon,\alpha_i)}{\partial \alpha_i}$ for all $0 < \epsilon \leq \frac{1}{1+\gamma}$ is equivalent to the statement that all γ-weighted averages also change in the same direction. The proof is finished. □

Theorem C.2. *If a distribution is γ-congruent, it is congruent.*

Proof. Any symmetric weighted average is also a weighted average. This concludes the proof. □

Theorem C.3. *A symmetric distribution with a finite second moment is always congruent.*

Proof. For any symmetric distribution with a finite second moment, all symmetric quantile averages coincide. The conclusion follows immediately. □

Theorem C.4. *A positive define location-scale distribution with a finite second moment is always γ-congruent.*

Proof. As shown in discussions in Subsection A, for a location-scale distribution, any weighted average can be expressed as $\lambda\mathrm{WA}_0(\epsilon) + \mu$, where $\mathrm{WA}_0(\epsilon)$ is an integral of $\mathrm{Q}_0(p)$ according to the definition of the weighted average. Therefore, the derivatives with respect to the parameters λ or μ are always positive. By application of Theorem C.1,the desired outcome is obtained. □

Theorem C.5. *The second central moment kernal distribution derived from a continuous location-scale unimodal distribution with a finite second moment is always γ-congruent.*

Proof. Theorem B.3 shows that the corresponding central moment kernel distribution is also a location-scale family distribution. Theorem B.1 shows that it is positively defined. Implementing Theorem C.4 yields the desired result. □

For the Pareto distribution, $\frac{\partial Q(p,\alpha)}{\partial \alpha} = \frac{x_m(1-p)^{-1/\alpha}\ln(1-p)}{\alpha^2}$. Since $\ln(1-p) < 0$ for all $0 < p < 1$, $(1-p)^{-1/\alpha} > 0$ for all $0 < p < 1$ and $\alpha > 0$, so $\frac{\partial Q(p,\alpha)}{\partial \alpha} < 0$, and therefore $\frac{\partial QA(\epsilon,\gamma,\alpha)}{\partial \alpha} < 0$, the Pareto distribution is γ-congruent. The derivative for the lognormal distribution is $\frac{\partial \mathrm{SQA}(\epsilon,\sigma)}{\partial \sigma} = \frac{-\mathrm{erfc}^{-1}(2\epsilon)e^{\mu-\sqrt{2}\sigma\mathrm{erfc}^{-1}(2\epsilon)} - \mathrm{erfc}^{-1}(2-2\epsilon)e^{\mu-\sqrt{2}\sigma\mathrm{erfc}^{-1}(2-2\epsilon)}}{\sqrt{2}}$. Since the inverse complementary error function is positive when the input is smaller than 1, and negative when the input is larger than 1, $\mathrm{erfc}^{-1}(2\epsilon) = -\mathrm{erfc}^{-1}(2-2\epsilon)$, $e^{\mu-\sqrt{2}\sigma\mathrm{erfc}^{-1}(2-2\epsilon)} > e^{\mu-\sqrt{2}\sigma\mathrm{erfc}^{-1}(2\epsilon)}$, $\frac{\partial \mathrm{SQA}(\epsilon,\sigma)}{\partial \sigma} > 0$, the lognormal distribution is congruent. Theorem C.3 implies that the generalized Gaussian distribution is congruent. For the Weibull distribution, just consider the median and mean, $E[m] = \lambda \sqrt[\alpha]{\ln(2)}$, $E[\mu] = \lambda\Gamma\left(1+\frac{1}{\alpha}\right)$, then, when $\alpha = 1$, $E[m] = \lambda\ln(2) \approx 0.693\lambda$, $E[\mu] = \lambda$, but when $\alpha = \frac{1}{2}$, $E[m] = \lambda\ln^2(2) \approx 0.480\lambda$, $E[\mu] = 2\lambda$, the mean increases, but the median decreases. Therefore, it is not congruent. When α changes from 1 to $\frac{1}{2}$, the average probability density on the left side of median increases, since $\frac{\frac{1}{2}}{\lambda\ln(2)} < \frac{\frac{1}{2}}{\lambda\ln^2(2)}$, but the mean increases, meaning that the distribution is more heavy-tailed, the probability density of large values will also increase. The reason for non-congruence lies in the simultaneous increase of probability densities on two opposite sides: one approaching zero and the other approaching infinity. Note that the gamma distribution does not have this issue, it looks to be congruent.

Although many common parametric distributions are not congruent, Theorem C.4 establishes that γ-congruence always holds for a positive define location-scale family distribution and thus for the second central moment kernel distribution generated from a continuous location-scale unimodal distribution as shown in Theorem C.5. Theorem B.2 demonstrates that all their central moment kernel distributions are unimodal-like with mode and median close to zero, as long as they are unimodal distributions. This implies, align with Theorem B.3, that different kernel distributions mainly differ in scale and they are, in some senses, reduced to a location-scale family distribution. Assuming finite moments, if $Q(0) - Q(1)$ remains constant, increasing the mean of the kernel distribution will result in a more heavy-tailed distribution, i.e., the probability density closer to $\frac{1}{k}(-\Delta)^k$ will increase. While the total probability density on either side of zero will remain unchanged as the median is generally close to zero and much less impacted during the mean increasing, the probability density close to zero will decrease. This transformation will increases nearly all symmetric weighted averages, in the general sense, due to the heavy tail. As a result, nearly all symmetric weighted averages, except the median since it is assumed to be zero, for all central moment kernel distributions derived from unimodal distributions should change in the same direction when the parameters change.

Data Availability. Data for Table **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

ACKNOWLEDGMENTS. I gratefully acknowledge the constructive comments made by the editor which substantially improved the clarity and quality of this paper.

1. CF Gauss, *Theoria combinationis observationum erroribus minimis obnoxiae.* (Henricus Dieterich), (1823).
2. S Newcomb, A generalized theory of the combination of observations so as to obtain the best result. *Am. journal Math.* **8**, 343–366 (1886).
3. S Newcomb, Researches on the motion of the moon. part ii, the mean motion of the moon and other astronomical elements derived from observations of eclipses and occultations extending from the period of the babylonians until ad 1908. *United States. Naut. Alm. Off. Astron. paper; v. 9* **9**, 1 (1912).
4. PJ Huber, Robust estimation of a location parameter. *Ann. Math. Stat.* **35**, 73–101 (1964).
5. X He, WK Fung, Method of medians for lifetime data with weibull models. *Stat. medicine* **18**, 1993–2009 (1999).
6. M Menon, Estimation of the shape and scale parameters of the weibull distribution. *Technometrics* **5**, 175–182 (1963).
7. SD Dubey, Some percentile estimators for weibull parameters. *Technometrics* **9**, 119–129 (1967).
8. KM Hassanein, Percentile estimators for the parameters of the weibull distribution. *Biometrika* **58**, 673–676 (1971).
9. NB Marks, Estimation of weibull parameters from common percentiles. *J. applied Stat.* **32**, 17–24 (2005).
10. K Boudt, D Caliskan, C Croux, Robust explicit estimators of weibull parameters. *Metrika* **73**, 187–209 (2011).
11. SD Dubey, *Contributions to statistical theory of life testing and reliability.* (Michigan State University of Agriculture and Applied Science. Department of statistics), (1960).
12. LJ Bain, CE Antle, Estimation of parameters in the weibdl distribution. *Technometrics* **9**, 621–627 (1967).
13. RV Hogg, Adaptive robust procedures: A partial review and some suggestions for future applications and theory. *J. Am. Stat. Assoc.* **69**, 909–923 (1974).
14. RJ Hyndman, Y Fan, Sample quantiles in statistical packages. *The Am. Stat.* **50**, 361–365 (1996).
15. WR van Zwet, *Convex Transformations of Random Variables: Nebst Stellingen.* (1964).
16. AL Bowley, *Elements of statistics.* (King) No. 8, (1926).
17. RA Groeneveld, G Meeden, Measuring skewness and kurtosis. *J. Royal Stat. Soc. Ser. D (The Stat.* **33**, 391–399 (1984).
18. C Bernard, R Kazzi, S Vanduffel, Range value-at-risk bounds for unimodal distributions under partial information. *Insur. Math. Econ.* **94**, 9–24 (2020).
19. PJ Bickel, EL Lehmann, Descriptive statistics for nonparametric models. iii. dispersion in *Selected works of EL Lehmann.* (Springer), pp. 499–518 (2012).
20. PJ Bickel, EL Lehmann, Descriptive statistics for nonparametric models iv. spread in *Selected Works of EL Lehmann.* (Springer), pp. 519–526 (2012).
21. PJ Rousseeuw, C Croux, Alternatives to the median absolute deviation. *J. Am. Stat. association* **88**, 1273–1283 (1993).
22. P Janssen, R Serfling, N Veraverbeke, Asymptotic normality of u-statistics based on trimmed samples. *J. statistical planning inference* **16**, 63–74 (1987).
23. J Hodges, E Lehmann, Matching in paired comparisons. *The Annals Math. Stat.* **25**, 787–791 (1954).
24. S Dharmadhikari, K Jogdeo, Unimodal laws and related in *A Festschrift For Erich L. Lehmann.* (CRC Press), p. 131 (1982).
25. AY Khintchine, On unimodal distributions. *Izv. Nauchno-Isled. Inst. Mat. Mech.* **2**, 1–7 (1938).
26. S Purkayastha, Simple proofs of two results on convolutions of unimodal distributions. *Stat. & probability letters* **39**, 97–100 (1998).
27. RA Fisher, Moments and product moments of sampling distributions. *Proc. Lond. Math. Soc.* **2**, 199–238 (1930).
28. PR Halmos, The theory of unbiased estimation. *The Annals Math. Stat.* **17**, 34–43 (1946).
29. W Hoeffding, A class of statistics with asymptotically normal distribution. *The Annals Math. Stat.* **19**, 293–325 (1948).
30. PM Heffernan, Unbiased estimation of central moments by using u-statistics. *J. Royal Stat. Soc. Ser. B (Statistical Methodol.* **59**, 861–863 (1997).
31. D Fraser, Completeness of order statistics. *Can. J. Math.* **6**, 42–45 (1954).
32. AJ Lee, *U-statistics: Theory and Practice.* (Routledge), (2019).
33. J Hodges Jr, E Lehmann, Estimates of location based on rank tests. *The Annals Math. Stat.* **34**, 598–611 (1963).
34. A Ehsanes Saleh, Hodges-lehmann estimate of the location parameter in censored samples. *Annals Inst. Stat. Math.* **28**, 235–247 (1976).
35. RJ Serfling, Generalized l-, m-, and r-statistics. *The Annals Stat.* **12**, 76–86 (1984).
36. P Janssen, R Serfling, N Veraverbeke, Asymptotic normality for a general class of statistical functions and applications to measures of spread. *The Annals Stat.* **12**, 1369–1379 (1984).
37. MG Akritas, Empirical processes associated with v-statistics and a class of estimators under random censoring. *The Annals Stat.* **14**, 619–637 (1986).
38. I Gijbels, P Janssen, N Veraverbeke, Weak and strong representations for trimmed u-statistics. *Probab. theory related fields* **77**, 179–194 (1988).
39. J Choudhury, R Serfling, Generalized order statistics, bahadur representations, and sequential nonparametric fixed-width confidence intervals. *J. Stat. Plan. Inference* **19**, 269–282 (1988).
40. B Efron, Bootstrap methods: Another look at the jackknife. *The Annals Stat.* **7**, 1–26 (1979).
41. PJ Bickel, DA Freedman, Some asymptotic theory for the bootstrap. *The annals statistics* **9**, 1196–1217 (1981).
42. R Helmers, P Janssen, N Veraverbeke, *Bootstrapping U-quantiles.* (CWI. Department of Operations Research, Statistics, and System Theory [BS]), (1990).
43. E Joly, G Lugosi, Robust estimation of u-statistics. *Stoch. Process. their Appl.* **126**, 3760–3773 (2016).
44. P Laforgue, S Clémençon, P Bertail, On medians of (randomized) pairwise means in *International Conference on Machine Learning.* (PMLR), pp. 1272–1281 (2019).