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The Extended Concept-based Multi-objective Path Planning and its A-life implications

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Abstract **– Concept-based multi-objective path planning, involves search and optimization of conceptual path plans by way of their particular solutions. The method, which has been recently introduced by the authors in the context of interactive path planning, is considered here using the ε-Pareto notion. Using this approach the decision maker may explore the decision space for concepts that have performances within a user-defined distance from the Pareto-front. In particular the paper discusses applications such as interactive path planning and re-planning. A simple "safe vs. fast" example is employed to demonstrate the proposed approach. Finally, a short discussion is given to relate the ideas presented here with A-life.**

I. INTRODUCTION

A non-traditional, yet most useful engineering approach to path planning is to view it as Multi-Objective Path Planning (MOPP) problem and to solve it using a Pareto approach (e.g. [1], [2]). Pareto solution approach provides a set of Pareto-optimal path plans that could be used to finally select one such plan. Contradicting objectives for such a planning problem may include criteria such as minimum path length (or associated time), and minimum exposure to a hostile observer (e.g., [2]).

In a recent paper, [2], we have introduced a novel methodology for MOPP, which is based on posing the problem as a concept-based MOPP. In such a concept-based approach the solution space consists of particular path solutions which are related to conceptual path plans. The later are termed conceptual solutions, or in short concepts. Each of the sub-sets (concepts) contains all particular solutions that belong to the same conceptual solution. Figure 1, which is adopted from [2], depicts a typical example of a path-planning situation. The marked paths are particular paths, each associated with one of three concepts. A valid concept, according to the given example, could be a path that begins from the start point, by-pass the steep hill from the left, cross the village, cross the road and go to the target. The conceptual path solutions are clearly abstractive descriptions, whereas the actual path should be a unique path that is associated with one of the concepts.

The concept-based approach has been originated for engineering design applications (e.g., [3] - [5]). Moshaiov and Avigad, [2], has suggested the use of such an approach to allow progressive human intervention in path planning, and demonstrated its applicability. The interactive approach involves the subjective preferences of robot operators' towards conceptual path plans. Such preferences might possibly be incorporated not just directly by concept preferences but also via sub-concepts [2].

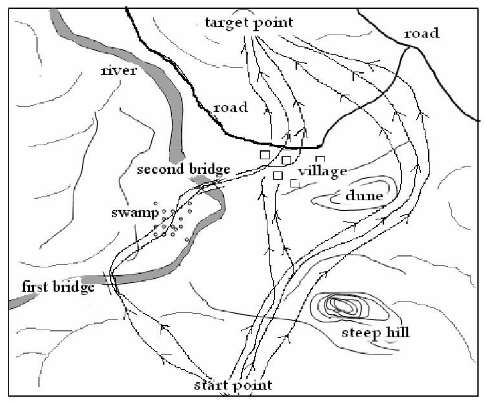

Figure 1: Map of robotic operation area (from [1])

In the current study the interactivity aspects are not explicitly included in the search. Yet, when the problem is posed as a concept-based MOPP, as done here, it opens the way to implicit (post-prior) interactivity. This means that the Decision Maker (DM) may choose a concept, and a related particular solution, from the obtained front based on both objective preferences and concept preferences.

Here the motivation is extended to include not only such an implicit interactive concept-based approach, but also re-planning issues. Re-planning is required when pre-execution plans fails due to the appearance of unexpected obstacle. It is significant for re-planning to take advantage of pre-execution planning as much as possible (e.g., [6]). With respect to this paper, it is important to consider the use of the concept-based path-planning in the context of memorizing useful information for re-planning. This is explained below. Finding a Pareto-optimal set could be viewed as ordering solutions with respect to the objectives of a problem and memorizing the obtained set for future use (selecting a solution). Such a future use, in the case of MOPP, may be influenced not only by future preferences, but also by future changes in the planning domain that may prevent the employment of certain optimal path solutions. For example, suppose that path '1' and path '2', which are depicted in figure 2 are a part of the Pareto-optimal set of a given MOPP. Path '1' has a part that is adjacent to obstacle 'A' and path '2' is not adjacent to obstacle 'A'. Such a situation is common when the problem involves both minimum path length and the objective of maximizing some measure of the distances between the path points and the obstacles. Assuming a small new obstacle 'B' has unexpectedly appeared adjacent to the existing obstacle 'A' would require re-planning. It is further assumed that the existence of obstacle 'B' eliminates the possibility of using path '1' in the re-planning problem. Yet, due to its small size the newly introduced object might not substantially change the planning domain. Therefore, re-planning could benefit from the memorized knowledge, which has been acquired in the original MOPP, namely the memorized Pareto-set. Path '2' might not necessarily be optimal in the new domain of re- planning. Yet, it is still a valid solution that can be immediately used, without a need for further calculations.

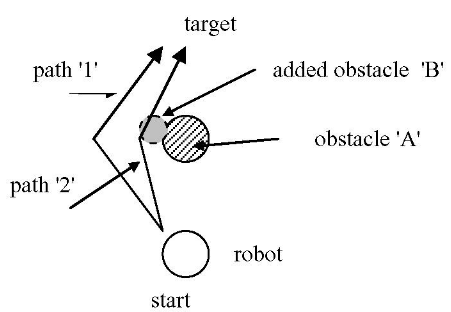

Figure 2: Re-planning example

The above example hints at a need to conceptually distinguish between solutions. Clearly, distinguishing between the optimal solutions, which are remote from obstacle 'A,' from those that are close to it, would support using the memorized Pareto-optimal solutions for a re-planning problem. This calls for the use of the concept-based approach in the original planning problem if re-planning is expected, and in particular for human-robot interaction in the planning. In fact if a conceptual distinction would have been made between the 'far' and the 'near' solutions, the re-planning would become simple. In such a case only the former concept is feasible. It is noted that the re-planning problem is assumed here to be a non-optimal problem by definition, whereas the original problem is to be formulated as a concept-based MOPP. Yet as stated above, with small variations of the planning domain some solutions of the original problem are intuitively expected to be close to the optimal solutions of the re-planning problem. This resembles the use of the original planning to re-plan in the D* algorithm [6].

The interest here is on developing a method to relax the Pareto-optimality condition with respect to the solution of concept-based MOPP problems. In addition to the definition of the relaxed problem, the paper provides a simple example to demonstrate the usefulness of the proposed extended concept-based approach to both interactivity and re-planning issues. Finally, a short discussion is provided to relate the ideas presented here to A-life.

II. METHODOLOGY

This section starts with a clarification of the difference between the regular approach to MOPP and the concept-based one. Next, it provides a formal presentation of the suggested extension to the concept-based approach.

A. Regular vs. Concept-based MOPP

A regular definition of a MOPP problem, which is hereby termed regular MOPP, involves the search for the set of Pareto optimal path solutions from the set of all particular solutions that are feasible particular path solutions. Any particular path is characterized by specific values of the problem decision variables, which represent a point in the problem decision space. The regular MOPP is formulated as a classical Multi-objective Problem (MOP). Such a case involves a comparison between the performances of all particular solutions in the objective space for non-dominancy. The representation, in the objective space, of the set of non-dominated solutions, is known as the Pareto front. The classical MOP is commonly formalized, without losing generality, as follows:

$$\min F(x) \text{ s. t. } \quad x \in X \subseteq S \subseteq R^n \quad (1)$$

where x is the vector of decision variables. In general, x might be subjected to equality and/or inequality constraints, which commonly include some bounds on the decision variables. A solution $x \in X \subseteq S \subseteq R^n$, which satisfies all the constraints, is called a feasible solution. The set X, of all feasible solutions, is called the feasible region in the search space S. In the classical MOP, which is defined in equation 1, y=F(x) is a vector of K objective functions, where,

$$F(x) = [f_1(x), f_2(x), \ldots, f_K(x)]^T \quad (2)$$

and $K \geq 2$. When the objectives are contradicting, there is no single solution to the above problem. The interest, in a classical MOP, is therefore on the trade-offs with respect to the objectives. The well-known concept of Pareto dominance supports the exploration of such trade-offs (see appendix). A path solution that its performances are included in the Pareto front set of a regular MOPP is a Pareto-optimal path solution.

As explained in the introduction section and detailed in [2], in a concept-based MOPP sub-sets of the entire set of path solutions are associated with concepts. Such an association could be, in principle, disregarded and the search for solutions could be done as in a regular MOPP. Once found the Pareto-optimal solutions could be divided in accordance with their concept associations. Yet, it is beneficial from the search point-of-view (see [7]) to explicitly pose the problem as a concept-based MOPP.

The concept-based MOP and its solution has been first formalized in [3], in conjunction with engineering design problems, where the term s-Pareto has been used. These are re-stated here, with respect to the concept-based MOPP, for the sake of completeness and clarity. We define n_c sets of path decision variables, one set for each conceptual path plan, where the variables describe the paths of the concepts. The m-th set of all the feasible path solutions, of the m-th path concept, is denoted X_m, where $X_m \subseteq S_m \subseteq R^{n_m}$, S_m is the search space of the m-th path concept, and n_m is the space dimension that is associated with the m-th path concept. X_m contains the decision variable vectors, x^m, of the m-th path concept, $x^m \in X_m$ for m = 1,….,n_c , where the dimension n_m of the vectors, x^m, could in general be concept dependent. The set X is the union of these n_c sets such that

$$X = \bigcup_{m=1}^{n_c} X_m \qquad (3)$$

The vector of objective functions $F: X \rightarrow Y$ is given as follows

$$F(x) = \left\{ F^m\left(x \right) ,\text{for } x = x^m,\ m = 1,..,n_c \right\} \qquad (4)$$

where $F^m(x^m) = [f_1^m(x^m),..,f_K^m(x^m)]^T$: $X_m \rightarrow Y$, for m = 1,..., n_c , is the mapping, into the objective space, of the particular solutions that are associated with the m-th concept. The mapping of the m-th concept is done by using a set of concept related objective functions with $f_k^m(x^m)$ (or in short f_k^m) as the k-th objective function.

The above exposition supports the definition of a concept-based MOPP similarly to a regular MOPP. The concept-based MOPP is defined as the problem stated in (1), with the minimization of F(x) as defined in (4), subject to (3). Concept-based MOPs involve finding the Pareto-optimal concepts. A Pareto-optimal concept has been defined as a concept with at least one member of its sub-set to be a non-dominated solution with respect to the entire feasible set of solutions [7]. We term the solution to the concept-based MOPP as the concept-based Pareto set, and designate it as P_C^*. Similarly, the associated front is termed the concept-based Pareto front (s-Pareto in [3]), and is designated as PF_C^*. The end result, which is a Pareto set and its associated front, should help to understand the distribution of concepts' representatives on the front, rather than just specific path solutions as done in the regular MOPP.

B. The extended concept-based MOPP

The extension of concept-based MOPs has been originally suggested in [8] in the context of multi-objective design in nature and the artificial. Here the application of such an extension is studied in the context of MOPP. The extension is defined based on a combination of two ideas, namely the definition of the concept-based MOP, and the notion of ε-dominancy, (e.g., [8]). The general motivation, as presented in [9], for the suggested extension in the context of design in nature and the artificial, is that when dealing with concepts a resilient approach seems appropriate. Citing from [9] "Concepts are associated with clusters of points in the objective space, which are not restricted to the Pareto-front. Solving for the concept-based Pareto set is therefore too restrictive as it does not contain a full representation of the concept's associated performances. This means that any comparison among concepts based on the resulting front of a concept-based MOP is inherently limited." This is illustrated in figure 3, which is adopted from [9]. The concept represented by stars could be viewed as optimal with respect to a certain region of the Pareto front, yet the one designated by circles covers a similar region but spans to a larger part of the front. With respect to changes of objectives from the upper part of the front to the lower part of the front the "circle concept" appears to be more robust. Yet, the "star concept" is expected to be more robust to uncertainties on the goals of the design as related to the upper part of the front when the solutions associated with the first rank and second rank are to be disregarded due to some design uncertainties. It should be noted that human preferences, towards concepts and sub-concepts, may push the solution away from the concept-based Pareto-set, as explained in [5] where a subjective-objective approach have been applied. All of this means that one should not be satisfied just by obtaining the front when dealing with concepts.

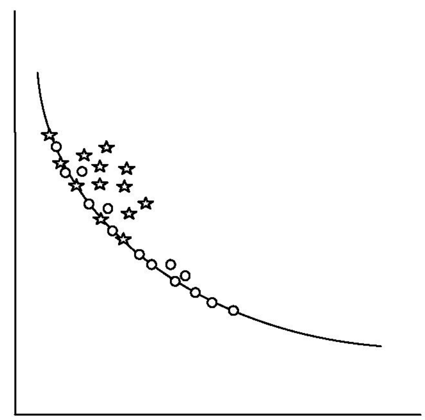

Figure 3: Concept performances (from [9])

The present exposition of the extended concept-based MOPP is motivated not only by the above two main reasons that appear to be applicable to both engineering design and path planning problems. Here an additional motivation is due to the understanding that re-planning is likely to occur in robot applications, and therefore, memorizing solutions of the concept-based MOPP could be too restrictive (see additional discussion on re-planning in the introduction).

In the extended concept-based MOPP the interest is not on the performances of particular solutions along the front, as done in the regular MOPP, nor on the association of such solutions with concepts as done in the concept-based MOPP. The focus of the extension done here is rather on obtaining the set of particular paths that are associated with concepts that have not only performances along the Pareto-front but also at its vicinity. For this purpose a relaxed version of the Pareto front, namely the ε-Pareto front is used (see [8]). Similarly to the notion of dominancy as detailed in the appendix, a vector $u=(u_1,\ldots,u_k)$ is said to ε-dominate v, denoted by $u \prec_\varepsilon v$, iff $\forall i \in \{1,\ldots,k\}$, $u_i - \varepsilon_i \le v_i$, and $\varepsilon_i > 0$. Without losing generality the extended problem is defined with respect to the min-min optimization problem as:

$$\min_\varepsilon F(x) \quad , \text{ s. t. } \quad x \in X \qquad (5)$$

The minimization operator uses the notion of the ε-dominancy to find the extended concept-based Pareto set, P^*_{eC}, and its associated extended "front" PF^*_{eC}, which are defined via the Pareto set of the concept-based MOPP, P^+_C and its front, PF^+_C, as follows.

$$P^*_{eC} := \{x^*_m \in X \mid \forall\ x^*_m\ \exists\ x^+_i \in P^+_C : F^m(x^*_m) \prec_\varepsilon F^i(x^+_i),\ m \in \{1,..,n_c\}\ \ and\ \ i=1,..,n_c\} \qquad (6)$$

and

,

$$PF^*_{eC} := \{y^* = F^m(x^*_m) \ : x^*_m \in P^*_{eC}\} \qquad (7)$$

The above definition of the extended concept-based MOPP allows humans to define their vicinity of interest with respect to the behavior of path concepts near the Pareto front. This is done by way of the ε vector, which is used in the definition of the ε-dominancy. As discussed above this allows flexibility with respect to both interactivity and path re-planning. The following section provides a demonstration of the applicability of the extended concept-based MOPP.

III. CASE STUDY

This study involves the objectives of minimizing path length and exposure to a hostile observer. The robot, which is taken as a point in this example, is allowed to move along the grid lines of a 10x10 working space without crossing through any of the four line-obstacles (marked by bold lines) as shown in figure 4. Each path is defined by a list of four points. All paths begin at the same starting point (5,0), and finish at the same goal (5,10). The paths are distinguished by the coordinates of the second and third points, which are given in table 1 per all paths of the example. The robot is restricted to move from each point of the path to the next one by moving first in the x direction and then in the y direction. This restriction is used here to simplify the example and make it tractable for hand-calculations. The last restriction also means that a Manhattan distance is used to calculate the path length. The first index of the path number, as given in the path index columns of the table, indicates the conceptual path number. The second number of that index refers to the particular path that belongs to the concept. For example both paths 1.2 and 1.4 belong to the 1st concept.

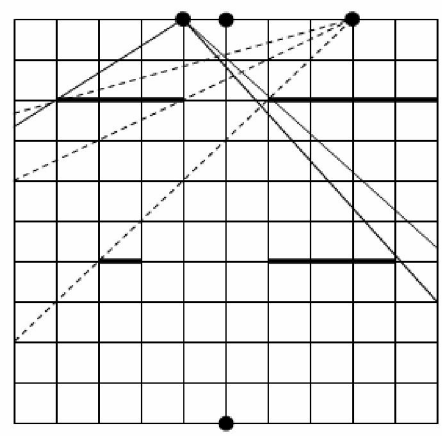

Figure 4: Problem representation

Table 1: Path Description

Path index	2nd Point	3rd Point	**Path index**	2nd Point	3rd Point
1.1	1,4	0,8	**5.1**	5,4	0,8
1.2	1,4	1,8	**5.2**	5,4	1,8
1.3	2,4	0,8	**5.3**	6,4	0,8
1.4	2,4	1,8	**5.4**	6,4	1,8
2.1	1,4	4,8	**6.1**	5,4	4,8
2.2	1,4	5,8	**6.2**	5,4	5,8
2.3	1,4	6,8	**6.3**	5,4	6,8
2.4	2,4	4,8	**6.4**	6,4	4,8
2.5	2,4	5,8	**6.5**	6,4	5,8
2.6	2,4	6,8	**6.6**	6,4	6,8
3.1	3,4	0,8	**7.1**	9,4	0,8
3.2	3,4	1,8	**7.2**	9,4	1,8
3.3	4,4	0,8	**7.3**	10,4	0,8
3.4	4,4	1,8	**7.4**	10,4	1,8
4.1	3,4	4,8	**8.1**	9,4	4,8
4.2	3,4	5,8	**8.2**	9,4	5,8
4.3	3,4	6,8	**8.3**	9,4	6,8
4.4	4,4	4,8	**8.4**	10,4	4,8
4.5	4,4	5,8	**8.5**	10,4	5,8
4.6	4,4	6,8	**8.6**	10,4	6,8

Table 2, and 3 show the Manhattan distance of the path and the exposed traveling distances (approximated) for two different detectors (Case 'A' and 'B' respectively).

Table 2: Path Performances (case 'A')

Path index	Dist.	Exp. (app.)	Path index	Dist.	Exp. (app.)
1.1	20	8.7	**5.1**	20	12.7
1.2	18	7	**5.2**	18	11
1.3	20	8.7	**5.3**	22	14.7
1.4	18	7	**5.4**	20	13
2.1	18	8	**6.1**	12	12
2.2	18	8	**6.2**	10	10
2.3	20	10	**6.3**	12	12
2.4	16	8	**6.4**	14	14
2.5	16	9	**6.5**	12	12
2.6	18	10	**6.6**	12	12
3.1	20	8.7	**7.1**	28	15
3.2	18	7	**7.2**	26	13.3
3.3	20	12.7	**7.3**	30	17
3.4	18	11	**7.4**	28	15.3
4.1	16	10	**8.1**	20	14.3
4.2	14	8	**8.2**	18	12.3
4.3	16	10	**8.3**	18	12.3
4.4	12	12	**8.4**	22	16.3
4.5	12	12	**8.5**	20	14.3
4.6	14	14	**8.6**	20	14.3

Table 3: Path Performances (case 'B')

Path index	Dist.	Exp. (app.)	Path index	Dist.	Exp. (app.)
1.1	20	11.3	**5.1**	20	11.3
1.2	18	9.5	**5.2**	18	9.5
1.3	20	10.3	**5.3**	22	10.3
1.4	18	9.5	**5.4**	20	9.5
2.1	18	7	**6.1**	12	5
2.2	18	5	**6.2**	10	3
2.3	20	5	**6.3**	12	3
2.4	16	5	**6.4**	14	5
2.5	16	3	**6.5**	12	3
2.6	18	3	**6.6**	12	3
3.1	20	11	**7.1**	28	11.3
3.2	18	8.5	**7.2**	26	9
3.3	20	11.3	**7.3**	30	11.3
3.4	18	9.5	**7.4**	28	9.5
4.1	16	5	**8.1**	20	5
4.2	14	3	**8.2**	18	3
4.3	16	3	**8.3**	18	3
4.4	12	5	**8.4**	22	5
4.5	12	3	**8.5**	20	3
4.6	14	3	**8.6**	20	5

In case 'A' a detector is located at (4,8) and in case 'B' at (8,8), as depicted in figure 4. It is assumed that the detection is obtained within the lines of sight, which are blocked by the line-obstacles. Also assumed is that the robot can move attached to the obstacles, and that borders of the vision are visible. Figures 5 and 6 show the performances in the objective space for case 'A' and 'B' respectively. In these figures concepts 1 – 5 are marked by the symbols as given in table 4. Although the information in tables 2 and 3 appears somewhat redundant to that of figures 5 and 6, it provides a clear association of the performances with the path index. The front in case 'A' , as depicted in figure 5, contains paths from concepts 1, 3, 4 and 6. In particular the optimal paths 1.2, 1.4, and 3.2 all have the same performances of length 18 and exposure of 7, path 4.2 has a length of 14 and an exposure of 8 and path 6.2 has both performances with the value of 10. Case 'B', as depicted in figure 6, demonstrates how sensitive the results could be with respect to a change of the detector position. In this case only concept 6 is optimal with a single path (#6.2) having the optimal performances of length=10 and exposure=3.

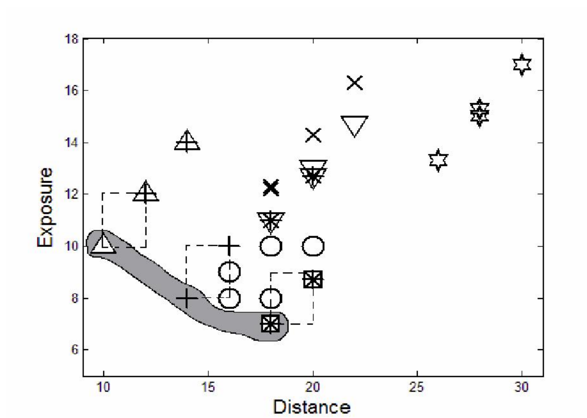

Figure 5: Performances in case 'A'

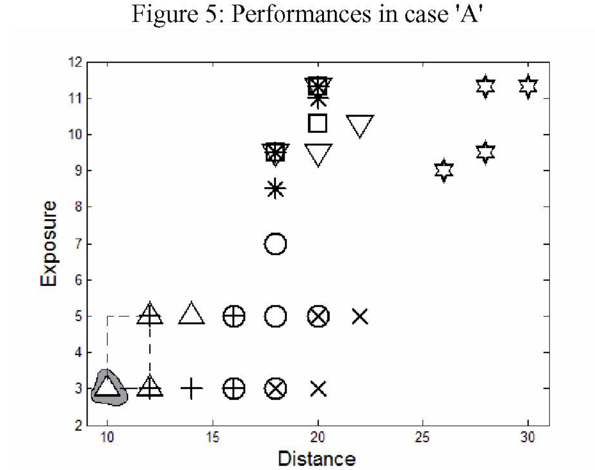

Figure 6: Performances in case 'B'

Table 4: Legend for figures 5-8

Concept #	Legend
1	square
2	circle
3	star
4	plus
5	Triangle (pointing down)
6	Triangle (pointing up)
7	hexagon
8	cross

Clearly a decision that is based on optimality can be taken in both case 'A' and 'B.' Yet, practical considerations that might not be included in the model, such as the sudden appearance of an unforeseen obstacle, calls for a less restrictive optimality analysis. Allowing ε=2 for both objectives and calculating the extended concept-based Pareto set and front gives, for both case 'A' and 'B,' a much wider spectrum of the extended concept-based optimal path options. Such additional options could be useful and might be memorized as reasonable solutions for later decision making. The effect of the extension is depicted in the areas marked by dashed rectangles in figures 5 and 6 for case 'A' and 'B' respectively. Interesting to note is that the new fronts contains concept 2 (in case 'A') and concept 4 (in case 'B') that do not appear in the original fronts. This is a significant result which means that slightly reducing the optimality restriction might bring up new and worthy concepts.

IV. A-LIFE IMPLICATIONS

In nature both caution from dangers as well as swift action to catch food might influence survival chances, and could occur in a simultaneous and contradicting manners. In fact a close review on MOPP related references (e.g., [10]) reveals that survival criteria other than those used here are common to the problem (e.g., weather threat and fuel consumptions). In some cases the relevance to military applications is apparent [11]. References, such as the above, emphasize the survival nature of the MOPP problem, which makes it relevant to A-life studies at least from a comparative point of view. Most MOPP studies aim at optimal solutions, namely the Pareto-front. However it is not clear to what a degree such a situation exists in Nature. In fact the term fittest is misleading and survival in Nature may involve non-optimal survivors. Solving MOPP with the relaxation of the Pareto notion as done here has therefore a potential for a closer resemblance to Nature than that of studies with the regular Pareto notion. Safe vs. fast problems, such as in the example of [2], and in the current paper, could possibly be related to such life-like survival situations. In spite of this apparent resemblance the approach presented here needs some modifications to make its relation to A-life clearer. The development of the concept-based approach has been motivated by engineering design. In such a case it is assumed that concepts are predefined by the designers. In A-life studies this should be avoided. From a strong A-life position, changing the concept-based approach to accommodate for self organization of concepts should be attempted. This, with the employment of a MOEA approach to the search, as in [7], would support a claim that a life form is created in the computation media. Relating the concept-based approach to weak A-life studies would require a better understanding of the role of non-dominancy and Pareto-optimality in Nature, as discussed in [9].

V. SUMMARY AND CONCLUSIONS

The concept-based MOPP approach is discussed and extended. It is demonstrated by a simple example that such a relaxation could provide a better insight to concepts that are otherwise would not be included in the resulting front. Future work may include an extension of our methodology to multi-robots planning and piloting, and the incorporation of evolutionary computations for searching the decision space of the extended concept-based MOPP. Both implicit and explicit interactive techniques should be studied for making the proposed approach a human-machine decision-making tool, and for its extension to other robotic tasks. Finally, there is a need for further work to relate the concept-based approach to both strong and weak A-life standpoints.

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APPENDIX

The development of an optimality-based Pareto front is based on a comparison between solutions using the idea of vector domination. Under minimization a vector $u = (u_1,...,u_n)$ is said to dominate $v = (v_1,...,v_n)$, denoted by $u \preceq v$, iff u is partially less than v, i.e., $\forall i \in \{1,...,n\}, u_i \le v_i \wedge \exists i \in \{1,...,n\} : u_i < v_i$. If u dominates v in the objective space then the corresponding solution of u is considered a better solution than the one corresponding to v (with respect to the minimization problem). The *Pareto optimal set,* P^*, is the set of optimal solutions such that:

$$P^* := \{x^* \in X | \neg\exists x' \in X : F(x') \preceq F(x^*)\} \quad \text{(A1)}$$

The performances of the optimal solutions constitute the Pareto front set PF^* which is defined as:

$$PF^* := \{y^* = F(x^*) \quad : x^* \in P^*\}. \quad \text{(A2)}$$