Lecture 12: Sept 19

Last time

- Random variables
- Distribution Functions
- Types of Random Variables

Today

- Counting Techniques
- Transformations of Random Variables

Definition The number of r-tuples we can make $r \leqslant n$, using n different symbols (each only once), is called the *number of permutations of n things r at a time* and is denoted by ${}^nP_r$ which is calculated as

$$ {}^nP_r = n(n-1)\cdots(n-r+1). $$

Example Fifteen cars enter a race. In how many different ways could trophies for first, second, and third place be awarded?
Solutions:

Example How many of the 3-tuples just counted have car number 15 in the first position?
Solutions:

Definition The number of distinct subsets, each of size r, that can be constructed from a set with n elements is called the number of *combinations* of n things r at a time: this number is represented by $\binom{n}{r}$ which reads n choose r.

$$ \binom{n}{r} = \frac{n!}{r!(n-r)!}. $$

Example How many distinct 5-card hands can be dealt from a standard 52-card deck?

$$ \binom{5}{2}5 = \frac{52!}{5!47!} = 2,598,960. $$

Theorem If x and y are any two real numbers and n is a positive integer, then

$$ (x+y)^n = \sum_{i=1}^{n} \binom{n}{i} x^i y^{n-i}, \quad \text{where } \binom{n}{i} = \frac{n!}{(n-i)!i!}. $$

Transformations of Random Variables

Theorem If X is a r.v. with sample space $\mathcal{X} \subset \mathbb{R}$ and cdf $F_X(x)$, then any function of X, say $Y = g(X)$ is also a random variable. The new random variable Y has a new sample space $\mathcal{Y} = g(X) \subset \mathbb{R}$. The objective is to find the cdf $F_Y(y)$ of Y.

Probability mapping: For any set $A \subset \mathcal{Y}$:

$$\begin{aligned}
\Pr(Y \in A) &= \Pr(g(X) \in A) \\
&= \Pr(\{x \in \mathcal{X} : g(x) \in A\}) \\
&= \Pr(X \in g^{-1}(A)),
\end{aligned}$$

where we have defined

$$g^{-1}(A) = \{x \in \mathcal{X} : g(x) \in A\}.$$

Notice that $g^{-1}(A)$ is well defined even if $g(\cdot)$ is not necessarily bijective.

Example (Binomial transformation) A discrete random variable X has a *binomial distribution* if its pmf is of the form

$$f_X(x) = \Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n,$$

where n is a positive integer and $0 \leqslant p \leqslant 1$. Values such as n and p that can be set to different values, producing different probability distributions, are called *parameters*. Consider a random variable $Y = g(X)$, where $g(x) = n - x$; that is, $Y = n - X$. Here $\mathcal{X} = \{0, 1, \ldots, n\}$ and $\mathcal{Y} = \{y : y = g(x), x \in \mathcal{X}\} = \{0, 1, \ldots, n\}$. For any $y \in \mathcal{Y}$, $n - x = g(x) = y$ if and only if $x = n - y$. Therefore, $g^{-1}(y) = n - y$ and

$$\begin{aligned}
f_Y(y) &= \sum_{x \in g^{-1}(y)} f_X(x) \\
&= f_X(n-y) \\
&= \binom{n}{n} - y p^{n-y} (1-p)^{n-(n-y)} \\
&= \binom{n}{y} (1-p)^y p^{n-y}.
\end{aligned}$$

Therefore, Y also has a binomial distribution, but with parameters n and $1 - p$.

Example (exercise 2.3) Suppose X has the geometric pmf $f_X(x) = \frac{1}{3}(\frac{2}{3})^x, x = 0, 1, 2, \ldots$. Determine the probability distribution of $Y = X/(X+1)$. Note that here both X and Y are discrete random variables. To specify the probability distribution of Y, specify its pmf.
Solution:

Theorem Suppose a continuous random variable X has cdf $F_X(x)$, let $Y = g(X)$, and let $\mathcal{X}$ and $\mathcal{Y}$ be defined as

$$\mathcal{X} = \{x : f(x) > 0\} \quad \text{and} \quad \mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

Then,

1. If g is an increasing function on $\mathcal{X}$, $F_Y(y) = F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$.
2. If g is a decreasing function on $\mathcal{X}$, $F_Y(y) = 1 - F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$.

Proof: We start with

$$\begin{aligned} F_Y(y) &= \Pr(Y \leqslant y) \\ &= \Pr(g(X) \leqslant y) \end{aligned}$$