Lecture 33: Nov. 30

Last time

- Conditional Distributions

Today

- Course evaluations (4/38)
- Final exam format
  - Final exam will be take home
  - Open book, open note, not open internet
  - Final exam will be released on Friday (12/09/2022) right after class
  - Final exam due 23:59 pm on Friday 12/16/2022.
  - Scan and submit your exam via email with a single pdf file
  - Send your email to both your instructor and your TA.
  - Submitted exams should be human-readable to receive non-zero scores.
- Bivariate Transformation

Bivariate Transformations of Continuous RVs Suppose (X, Y) is continuous and the joint transformation

$$u = g_1(x, y), \quad v = g_2(x, y)$$

is one-to-one and differentiable. Define the inverse mapping

$$x = h_1(u, v), \quad y = h_2(u, v)$$

Then

$$f_{UV}(u, v) = f_{XY}\left(h_1(u, v), h_2(u, v)\right) |J(u, v)|$$

where $J(u, v)$ is the Jacobian of the transformation $(x, y) \to (u, v)$ given by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Example: Rotation of a bivariate normal vector Let $X \sim N(0, 1)$, $Y \sim N(0, 1)$, independent. Define the rotation

$$U = X \cos\theta - Y \sin\theta$$
$$V = X \sin\theta + Y \cos\theta$$

for fixed θ. Then $U \sim N(0, 1)$, $V \sim N(0, 1)$, independent.

Proof:
The range of (X, Y) is $\mathbb{R}^2$. The range of (U, V) is $\mathbb{R}^2$. Need the inverse transformation

$$\begin{aligned} X &= U\cos\theta + V\sin\theta \\ Y &= -U\sin\theta + V\cos\theta \end{aligned}$$

with Jacobian

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1$$

The joint pdf of (X, Y) is

$$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

The joint pdf of (U, V) is

$$\begin{aligned} f_{UV}(u,v) &= \frac{1}{2\pi} e^{-\left[(u\cos\theta+v\sin\theta)^2+(-u\sin\theta+v\cos\theta)^2\right]/2} \cdot |1| \\ &= \frac{1}{2\pi} e^{-(u^2+v^2)/2} = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-v^2/2} \end{aligned}$$

so $U \sim N(0,1)$, $V \sim N(0,1)$, and U and V are independent.

Functions of independent random variables (Theorem 4.3.5) Let X and Y be independent rvs. Let $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be functions. Then the random variables $U = g(X)$ and $V = h(Y)$ are independent.

Sum of two independent rvs Suppose X and Y are independent. What is the distribution of $Z = X + Y$? In general:

$$F_Z(z) = \Pr(X + Y \leqslant z) = \Pr(\{(x,y) \text{ such that } x + y \leqslant z\})$$

Various approaches:

- bivariate transformation method (continuous and discrete)
- Discrete convolution

$$f_Z(z) = \sum_{x+y=z} f_X(x) f_Y(y) = \sum_x f_X(x) f_Y(z-x)$$

- Continuous convolution (Section 5.2)
- MGF method (continuous and discrete)

Example (Sum of two independent Poissons) Define X, Y to be two independent random variables having Poisson distributions with parameters λ_i, $i = 1, 2$. Then:

$$f_{X,Y}(x, y) = \frac{e^{-\lambda_1}\lambda_1^x}{x!}\frac{e^{-\lambda_2}\lambda_2^y}{y!}, x, y = 0, 1, 2, \ldots$$

The distribution of $S = X + Y$ is

$$\begin{aligned} f_S(s) &= \sum_{x=0}^{s} \frac{e^{-\lambda_1}\lambda_1^x}{x!}\frac{e^{-\lambda_2}\lambda_2^{s-x}}{(s-x)!} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{s!}\sum_{x=0}^{s}\binom{s}{x}\lambda_1^x\lambda_2^{s-x} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{s!}(\lambda_1+\lambda_2)^s \end{aligned}$$

Again, S is Poisson with parameter $\lambda = \lambda_1 + \lambda_2$.

Moment generating function (Theorem 4.2.12) Let X and Y be independent rvs with mgfs $M_X(\cdot)$ and $M_Y(\cdot)$, respectively. Then the mgf of $Z = X + Y$ is

$$M_Z(t) = M_X(t)M_Y(t)$$

Proof:

$$\begin{aligned} M_Z(t) &= E\exp(Zt) & &= E\{\exp[(X+Y)t]\} \\ &= E[\exp(Xt)\exp(Yt)] & &= E[\exp(Xt)] \cdot E[\exp(Yt)] \\ &= M_X(t)M_Y(t) \end{aligned}$$

Corollary: If X and Y are independent and $Z = X - Y$,

$$M_Z(t) = M_X(t)M_Y(-t)$$

Example (sum of two independent Poissons) Suppose $X \sim Poisson(\lambda_X)$ and $Y \sim Poisson(\lambda_Y)$ and put $Z = X + Y$. Then, $Z \sim Poisson(\lambda_X + \lambda_Y)$. *Proof:*

$$\begin{aligned} M_Z(t) &= \exp\left[\lambda_X(e^t - 1)\right]\exp\left[\lambda_Y(e^t - 1)\right] \\ &= \exp\left[(\lambda_X + \lambda_Y)(e^t - 1)\right] \end{aligned}$$

Example (sum of two independent normals) Suppose $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ and X and Y are independent and $Z = X + Y$. Then

$$Z \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$$

Proof:

$$\begin{aligned} M_Z(t) &= \exp\left(\mu_x t + \frac{1}{2}\sigma_x^2 t^2\right)\exp\left(\mu_y t + \frac{1}{2}\sigma_y^2 t^2\right) \\ &= \exp\left[(\mu_x + \mu_y)t + \frac{1}{2}(\sigma_x^2 + \sigma_y^2)t^2\right] \end{aligned}$$

Example (sum of two independent gammas) Suppose $X \sim \Gamma(\alpha_x, \beta)$ and independently $Y \sim \Gamma(\alpha_y, \beta)$. Let $Z = X + Y$. Then $Z \sim \Gamma((\alpha_x + \alpha_y), \beta)$.
Proof:

$$\begin{aligned} M_Z(t) &= \left(\frac{1}{1-\beta t}\right)^{\alpha_x} \left(\frac{1}{1-\beta t}\right)^{\alpha_y} \\ &= \left(\frac{1}{1-\beta t}\right)^{\alpha_x+\alpha_y} \end{aligned}$$

Remember that

- If $\alpha = 1$ we have an exponential with parameter β.
- If $\alpha = n/2$ and $\beta = 2$, we have a $\chi^2(n)$ (with n d.f.). The above result states that $\chi^2(n_1) + \chi^2(n_2) = \chi^2(n_1 + n_2)$.