Lecture 15: Sept 26

Last time

- Presentation

Today

- HW3 posted
- Midterm Exam 1 10/10, will have a practice exam
- Probability integral transformation
- Expectations (2.2)

Theorem (Probability integral transformation) Let X have continuous cdf $F_X(x)$ and define the random variable Y as $Y = F_X(X)$. Then Y is uniformly distributed on $(0, 1)$, that is, $\Pr(Y \leqslant y) = y, 0 < y < 1$.

Before we prove this theorem, we will digress for a moment and look at F_X^{-1}, the inverse of the cdf F_X, in some detail. If F_X is strictly increasing, then F_X^{-1} is well defined by

$$F_X^{-1}(y) = x \iff F_X(x) = y.$$

However, if F_X is constant on some interval, then F_X^{-1} is not well defined as Figure 13.1 illustrates. Any $x_1 \leqslant x \leqslant x_2$ satisfies $F_X(x) = y$

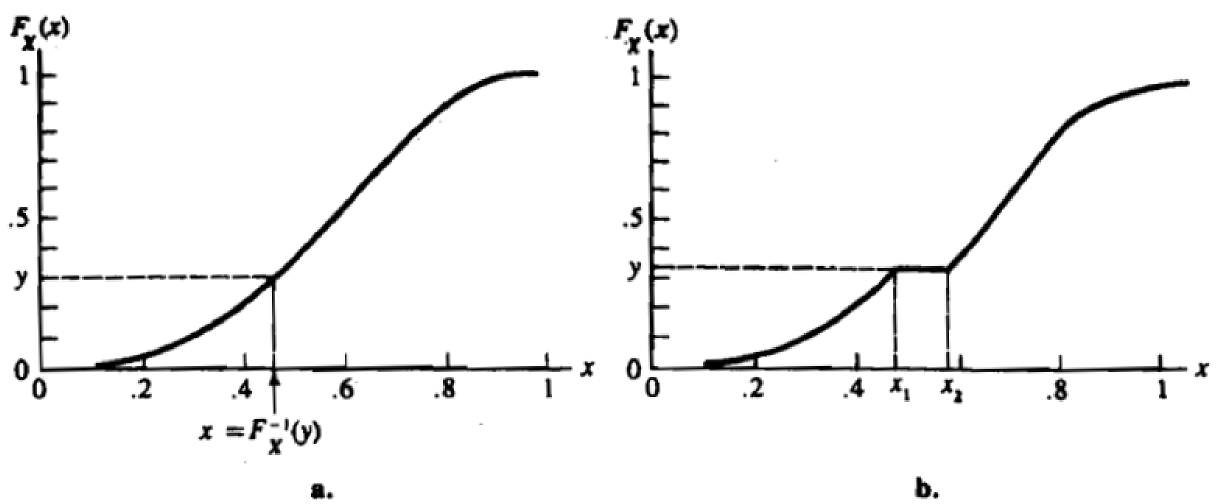

Figure 13.1: Figure 2.1.2. (a) $F_X(x)$ strictly increasing; (b) $F_X(x)$ nondecreasing

This problem is avoided by defining F_X^{-1} for $0 < y < 1$ by

$$F_X^{-1}(y) = \inf\{x : F_X(x) \geqslant y\}.$$

With this definition, for Figure 13.1(b), we have $F_X^{-1}(y) = x_1$.
Proof:
For $Y = F_X(X)$, we have, for $0 < y < 1$,

$$\begin{aligned}
\Pr(Y \leqslant y) &= \Pr(F_X(X) \leqslant y) \\
&= \Pr(F_X^{-1}[F_X(X)] \leqslant F_X^{-1}(y)) \quad (F_X^{-1} \text{ is increasing}) \\
&= \Pr(X \leqslant F_X^{-1}(y)) \\
&= F_X(F_X^{-1}(y)) \qquad (\text{definition of } F_X) \\
&= y.
\end{aligned}$$

One application of the probability integral transformation is in the generation of random samples from a particular distribution. If it is required to generate an observation X from a population with cdf F_X, we need only generate a uniform random number V, between 0 and 1, and solve for x in the equation $F_X(x) = u$.

Expected Values

Definition The *expected value* or *mean* of a random variable $g(X)$, denoted by $Eg(X)$, is

$$Eg(X) = \begin{cases} \int\limits_{-\infty}^{\infty} g(x)f(x)dx & \text{if } X \text{ is continuous} \\ \sum\limits_{x \in \mathcal{X}} g(x)\Pr(X = x) & \text{if } X \text{ is discrete} \end{cases}$$

Provided the integral or summation exists.

If we let $g(X) = X$, then we get

$$EX = \begin{cases} \int\limits_{-\infty}^{\infty} xf(x)dx & \text{if } X \text{ is continuous} \\ \sum\limits_{x \in \mathcal{X}} x\Pr(X = x) & \text{if } X \text{ is discrete} \end{cases}$$

Example (Exponential mean) Suppose X has an *exponential* (λ) *distribution*, $X \sim Exp(\lambda)$, that is, it has pdf given by

$$f_X(x) = \frac{1}{\lambda}e^{-x/\lambda}, \quad 0 \leqslant x < \infty, \lambda > 0.$$

Find out EX.
Solution:

$$
\begin{aligned}
EX &= \int_0^\infty \frac{1}{\lambda} x e^{-x/\lambda} dx \\
&= -xe^{-x/\lambda}\Big|_0^\infty + \int_0^\infty e^{-x/\lambda} dx \\
&= \int_0^\infty e^{-x/\lambda} dx \\
&= \lambda
\end{aligned}
$$

Example (Binomial mean) if X has a *binomial distribution*, $X \sim Binomial(n, p)$, its pmf is given by

$$
\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n,
$$

where n is a positive integer, $0 \leqslant p \leqslant 1$, and for every fixed pair n and p the pmf sums to 1. Find out EX.
Solution:

$$
EX = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}, = \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x}.
$$

Using the identity $x\binom{n}{x} = n\binom{n-1}{x-1}$, we have

$$
\begin{aligned}
EX &= \sum_{x=1}^{n} n \binom{n-1}{x-1} p^x (1-p)^{n-x} \\
&= \sum_{y=0}^{n-1} n \binom{n-1}{y} p^{y+1} (1-p)^{n-(y+1)} \\
&= np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{n-1-y} \\
&= np,
\end{aligned}
$$

since the last summation must be 1, being the sum over all possible values of a binomial$(n-1, p)$ pmf.

The process of taking expectations is a linear operation, which means that the expectation of a linear function of X can be easily evaluated by noting that for any constants a and b, such that

$$
E(aX + b) = aEX + b
$$

Theorem Let X be a random variable and let a, b, and c be constants. Then for any functions $g_1(x)$ and $g_2(x)$ whose expectations exist,

1. $E(ag_1(X) + bg_2(X) + c) = aEg_1(X) + bEg_2(X) + c$.
2. If $g_1(x) \geqslant 0$ for all x, then $Eg_1(X) \geqslant 0$.
3. If $g_1(x) \geqslant g_2(x)$ for all x, then $Eg_1(X) \geqslant Eg_2(X)$.
4. If $a \leqslant g_1(x) \leqslant b$ for all x, then $a \leqslant Eg_1(X) \leqslant b$.

Proof:
We will give details for only the continuous case, the discrete case being similar. By definition

$$\begin{aligned}
E(ag_1(X) + bg_2(X) + c) &= \int_{-\infty}^{\infty} [ag_1(x) + bg_2(x) + c]\, f_X(x)dx \\
&= \int_{-\infty}^{\infty} ag_1(x) f_X(x)dx + \int_{-\infty}^{\infty} bg_2(x) f_X(x)dx + \int_{-\infty}^{\infty} cf_X(x)dx \\
&= aEg_1(X) + bEg_2(X) + c
\end{aligned}$$

The other three properties are proved in a similar manner (shown in class).

Theorem For a non-negative random variable X (i.e. $f(x) = 0$ for $x < 0$).

$$EX = \begin{cases} \int_0^{\infty}(1 - F(x))dx, & X \text{ continuous} \\ \sum_{x=0}^{\infty}(1 - F(x)), & X \text{ discrete} \end{cases}$$

Proof:
We prove the continuous case first,

$$\begin{aligned}
\int_0^{\infty} [1 - F(x)]\, dx &= \int_0^{\infty} [1 - \Pr(X \leqslant x)]\, dx \\
&= \int_0^{\infty} \Pr(X > x) dx \\
&= \int_0^{\infty} \int_{y=x}^{\infty} f_X(y) dy dx \\
&= \int_0^{\infty} \int_{x=0}^{y} f_X(y) dx dy \\
&= \int_0^{\infty} y f_X(y) dy \\
&= EX.
\end{aligned}$$

Then, for discrete case, we have

$$\begin{aligned}
\sum_{x=0}^{\infty}(1 - F(x)) &= \sum_{x=0}^{\infty} \Pr(X > x) \\
&= \sum_{x=0}^{\infty} \sum_{y=x+1}^{\infty} \Pr(X = y) \\
&= \sum_{y=1}^{\infty} \sum_{x=0}^{y-1} \Pr(X = y) \\
&= \sum_{y=1}^{\infty} y \Pr(X = y) \\
&= EX
\end{aligned}$$