Lecture 8: Sept 9

Last time

- Random variables
- Distribution Functions
- Types of Random Variables

Today

- Presentation: Andrey Markov by Ryan Mortonson
- Presentation: Spatial Statistics by Camille Kreisel
- Review part 1

Review and more practice

We briefly review what we have covered so far. We complement this review process with examples/questions taken from the book "Introduction to Probability Theory and Statistical Inference" 3rd ed. by Harold J. Larson.

We started with Set Theory.

Definition The set, S, of all possible outcomes of a particular experiment is called the *sample space* for the experiment.

Definition An *event* is any collection of possible outcomes of an experiment, that is , any subset of S (including S itself).

An event occurs if any one of its elements is the outcome observed.

Definition Two events A and B are *disjoint* (or *mutually exclusive*) if $A \cap B = \varnothing$. The events $A_1, A_2, \ldots$ are *pairwise disjoint* (or *mutually exclusive*) if $A_i \cap A_j = \varnothing$ for all $i \neq j$.

Definition If $A_1, A_2, \ldots$ are pairwise disjoint and $\cup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots = S$, then the collection of $A_1, A_2, \ldots$ forms a *partition* of S.

Theorem For any three events, A, B, and C, defined on a sample space S,

1. Commutativity
$$A \cup B = B \cup A,$$
$$A \cap B = B \cap A;$$

2. Associativity
$$A \cup (B \cup C) = (A \cup B) \cup C,$$
$$A \cap (B \cap C) = (A \cap B) \cap C;$$

3. Distributive Laws
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C);$$

4. DeMorgan's Laws
$$(A \cup B)^c = A^c \cap B^c,$$
$$(A \cap B)^c = A^c \cup B^c;$$

Then we moved to define a probability function. To establish the domain for the probability function, we start with *sigma algebra*.

Definition A collection of subsets of S is called a *sigma algebra* (or *Borel field*), denoted by $\mathcal{B}$, if it satisfies the following three properties:

1. $\varnothing \in \mathcal{B}$ (the empty set is an element of $\mathcal{B}$).
2. If $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$ ($\mathcal{B}$ is closed under complementation).
3. If $A_1, A_2, \cdots \in \mathcal{B}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$ ($\mathcal{B}$ is closed under countable unions).

By DeMorgan's Law, (3) can be replaced by:

$$3'. \text{ if } A_1, A_2, \cdots \in \mathcal{B}, \text{ then } \cap_{i=1}^{\infty} A_i \in \mathcal{B}.$$

which means that if we have property (1), (2) and (3) then we have property (1), (2), (3') and vise-versa (if we have property (1), (2) and (3') then we have property (1), (2), (3)).

This is because:
So that if we have property (3) that $A_1, A_2, \cdots \in \mathcal{B}$ and $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$. Then by property (2), we know that $A_i^c \in \mathcal{B}$ for $i = 1, 2, \ldots$. And we can apply property (3) again such that if $A_1^c, A_2^c, \cdots \in \mathcal{B}$, then $(\cup_{i=1}^{\infty} A_i^c) \in \mathcal{B}$. Therefore, now we know $(\cup_{i=1}^{\infty} A_i^c) \in \mathcal{B}$ and we can apply property (2) again to get its complement which is also in the Borel field. Therefore, $(\cup_{i=1}^{\infty} A_i^c)^c \in \mathcal{B}$ which is $\cap_{i=1}^{\infty} A_i$.

For the other direction, we start from property (1), (2) and (3'). With property (3'), we have if $A_1, A_2, \cdots \in \mathcal{B}$, then $\cap_{i=1}^{\infty} A_i \in \mathcal{B}$. We again, first apply property (2) such that if $A_1, A_2, \cdots \in \mathcal{B}$, then $A_1^c, A_2^c, \cdots \in \mathcal{B}$. Now, by property (3'), we have $\cap_{i=1}^{\infty} A_i^c \in \mathcal{B}$. By applying property (2), we have $(\cap_{i=1}^{\infty} A_i^c)^c \in \mathcal{B}$. By substituting A_i with A_i^{*c} and taking complement at both side of equation $(\cup_{i=1}^{\infty} A_i^c)^c = \cap_{i=1}^{\infty} A_i$, we have $(\cup_{i=1}^{\infty} A_i^*) = (\cap_{i=1}^{\infty} A_i^{*c})^c$. Therefore, $\cup_{i=1}^{\infty} A_i = (\cap_{i=1}^{\infty} A_i^c)^c \in \mathcal{B}$ which is property (3).