

# Math 3070/6070 Introduction to Probability

Mon/Wed/Fri 11:00am - 11:50am

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## Lecture 1: Aug 22

### Today

- Introduction
- Introduce yourself
- Course logistics

### What is this course about?

This course will provide a calculus-based introduction to probability theory. Material covered will include fundamental axioms of probability, combinatorics, discrete and continuous random variables, multivariate distributions, expectation, and limit theorems, generally following Chapters 1-5 of the textbook. This course is a critical prerequisite for more advanced work in statistical theory and analysis.

### Prerequisite

- Calculus

### Why learn probability

- The subject of probability theory is the foundation upon which all of statistics is built.
- It provides you a tool to model
  - populations
  - experiments
  - almost anything else that could be considered a random phenomenon
  - example topics in [Data Analysis course](#)
- Through these models, statisticians are able to draw inferences about populations based on examination of only a part of the whole.
- A must have for any Data Scientists.

## What this course WILL NOT do for you

It will not help you:

- Beat the casino at blackjack (although it may convince you that it is better not to gamble, or that a casino is a great business).
- Answer your friends' silly questions such as "What are the chances it will rain tomorrow?" (although it might make you think of ways that you might model and compute it).

## Syllabus

Check course website frequently for updates and announcements.

<https://tulane-math-3070-2022.github.io/>

## HW submission

Students are required to submit hand-written homework in recitations to the TA. Homework assignments are expected every two weeks with 4-5 problems at a time.

## Presentations

Do we want to have a 5 bonus point towards the final grade with a presentation?

## Last year comments

Not really, this is my first time teaching this course. There will be an internal mid-term-ish evaluation for this course. Will remember to go over them.

## Lecture 2:Aug 24

### Last time

- Introduction
- Introduce yourself
- Course logistics

### Today

- Set theory (1.1)
- Axiomatic Foundations (1.2)

### Set Theory

One of the main objectives of a statistician is to draw conclusions about a population of objects by conducting an experiment. The first step in this endeavor is to identify the possible outcomes or, in statistical terminology, the *sample space*.

**Definition** The set,  $S$ , of all possible outcomes of a particular experiment is called the *sample space* for the experiment.

**Example** The sample space of

- tossing a coin just once, contains two outcomes, heads and tails

$$S = \{H, T\}$$

- observing reported SAT scores of randomly selected students at a certain university

$$S = \{200, 210, 220, \dots, 780, 790, 800\}$$

- an experiment where the observation is reaction time to a certain stimulus

$$S = (0, \infty)$$

**Definition** An *event* is any collection of possible outcomes of an experiment, that is, any subset of  $S$  (including  $S$  itself).

Let  $A$  be an event,

- $A$  is a subset of  $S$ ,
- event  $A$  occurs if the outcome of the experiment is in the set  $A$ ,
- we generally speak of the probability of an event, rather than a set.

Set operations:

- Containment:

$$A \subset B \iff x \in A \implies x \in B$$

- Equality:

$$A = B \iff A \subset B \text{ and } B \subset A$$

- Union: the union of  $A$  and  $B$ , written as  $A \cup B$ , is the set of elements that belong to either  $A$  or  $B$  or both

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

- Intersection: the intersection of  $A$  and  $B$ , written  $A \cap B$ , is the set of elements that belong to both  $A$  and  $B$ :

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

- Complementation: the complement of  $A$ , written  $A^c$ , is the set of all elements that are not in  $A$ :

$$A^c = \{x : x \notin A\}.$$

**Theorem** For any three events,  $A$ ,  $B$ , and  $C$ , defined on a sample space  $S$ ,

1. Commutativity

$$\begin{aligned} A \cup B &= B \cup A, \\ A \cap B &= B \cap A; \end{aligned}$$

2. Associativity

$$\begin{aligned} A \cup (B \cup C) &= (A \cup B) \cup C, \\ A \cap (B \cap C) &= (A \cap B) \cap C; \end{aligned}$$

3. Distributive Laws

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C), \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C); \end{aligned}$$

4. DeMorgan's Laws

$$\begin{aligned} (A \cup B)^c &= A^c \cap B^c, \\ (A \cap B)^c &= A^c \cup B^c; \end{aligned}$$

We show the proof of  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$  in the distributive laws. Caution: Venn diagrams are helpful in visualization, but they do not constitute a formal proof. To prove that two sets are equal, we need to show that each set contains the other.

*proof:*

- $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ :  
Let  $x \in (A \cap (B \cup C))$ . By definition of intersection,  $x \in (B \cup C)$  that is, either  $x \in B$  or  $x \in C$ . Since  $x$  also must be in  $A$ , we have that either  $x \in (A \cap B)$  or  $x \in (A \cap C)$ ; therefore,  $x \in ((A \cap B) \cup (A \cap C))$ .
- $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$ :  
Let  $x \in ((A \cap B) \cup (A \cap C))$ . This implies that  $x \in (A \cap B)$  or  $x \in (A \cap C)$ . If  $x \in (A \cap B)$ , then  $x$  is in both  $A$  and  $B$ . Since  $x \in B$ , then  $x \in (B \cup C)$  and thus  $x \in (A \cap (B \cup C))$ . It follows the same argument when  $x \in (A \cap C)$ , we still have  $x \in (A \cap (B \cup C))$ .

**Definition** Two events  $A$  and  $B$  are *disjoint* (or *mutually exclusive*) if  $A \cap B = \emptyset$ . The events  $A_1, A_2, \dots$  are *pairwise disjoint* (or *mutually exclusive*) if  $A_i \cap A_j = \emptyset$  for all  $i \neq j$ .

**Definition** If  $A_1, A_2, \dots$  are pairwise disjoint and  $\cup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \dots = S$ , then the collection of  $A_1, A_2, \dots$  forms a *partition* of  $S$ .

**Example** The sets  $A_i = [i, i + 1), i = 0, 1, 2, \dots$  form a partition of  $[0, \infty)$ .

## Basics of Probability Theory

When an experiment is performed, the realization of the experiment is an outcome in the sample space. If the experiment is performed a number of times, then

- different outcomes may occur each time
- some outcomes may repeat
- the “frequency of occurrence” of an outcome can be thought of as a probability

However, we **do not** define probabilities in terms of frequencies but instead take the mathematically simpler axiomatic approach. The axiomatic approach is not concerned with the interpretations of probabilities, but is concerned only that the probabilities are defined by a function satisfying the axioms. Interpretations of the probabilities are quite another matter:

- The “frequency of occurrence” of an event is one example of a particular interpretation of probability.
- Another possible interpretation is a subjective one, where we can think of the probability as a belief in the chance of an event occurring.

## Axiomatic Foundations

For each event  $A$  in the sample space  $S$ , we want to associate with  $A$  a number between zero and one that will be called the probability of  $A$ , denoted by  $\Pr(A)$ . The domain of  $\Pr$  is the set where the arguments of the function  $\Pr(\cdot)$  are defined. It is natural to define the domain of  $\Pr$  as all subsets of  $S$ , that is for each  $A \subset S$ , we define  $\Pr(A)$  as the probability

that  $A$  occurs. However, there are some technical difficulties to overcome which requires us to familiarize with the following.

**Definition** A collection of subsets of  $S$  is called a *sigma algebra* (or *Borel field*), denoted by  $\mathcal{B}$ , if it satisfies the following three properties:

1.  $\emptyset \in \mathcal{B}$  (the empty set is an element of  $\mathcal{B}$ ).
2. If  $A \in \mathcal{B}$ , then  $A^c \in \mathcal{B}$  ( $\mathcal{B}$  is closed under complementation).
3. If  $A_1, A_2, \dots \in \mathcal{B}$ , then  $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$  ( $\mathcal{B}$  is closed under countable unions).

From Property (1) and (2), we see that the empty set and its complement  $S$  (since  $S = \emptyset^c$ ) are always in a sigma algebra. In fact, they construct the *trivial* algebra  $\{\emptyset, S\}$  which is the smallest sigma algebra.

By DeMorgan's Law, (3) can be replaced by:

$$3'. \text{ if } A_1, A_2, \dots \in \mathcal{B}, \text{ then } \cap_{i=1}^{\infty} A_i \in \mathcal{B}.$$

This is because:

$$(\cup_{i=1}^{\infty} A_i^c)^c = \cap_{i=1}^{\infty} A_i.$$

**Example** If  $S$  is finite or countable (where the elements of  $S$  can be put into 1 – 1 correspondence with a subset of the integers), then these technicalities really do not arise, for we define for a given sample space  $S$ ,

$$\mathcal{B} = \{\text{all subsets of } S, \text{ including } S \text{ itself}\}.$$

If  $S$  has  $n$  elements, there are  $2^n$  sets in  $\mathcal{B}$  (why?). [hint: for each element, it is either in or out of a subset, so 2 choices].

**Example** Let  $S = (-\infty, \infty)$ , the real line. Then  $\mathcal{B}$  is chosen to contain all sets of the form

$$[a, b], (a, b], (a, b), \text{ and } [a, b)$$

for all real numbers  $a$  and  $b$ . Also, from the properties of  $\mathcal{B}$ , it follows that  $\mathcal{B}$  contains all sets that can be formed by taking (possibly countably infinite) unions and intersections of sets of the above varieties.

We now define a probability function.

**Definition** Given a sample space  $S$  and an associated sigma algebra  $\mathcal{B}$ , a *probability function* is a function  $\text{Pr}$  with domain  $\mathcal{B}$  that satisfies

1.  $\text{Pr}(A) \geq 0$  for all  $A \in \mathcal{B}$ .
2.  $\text{Pr}(S) = 1$ .

3. If  $A_1, A_2, \dots \in \mathcal{B}$  are pairwise disjoint, then  $\Pr(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pr(A_i)$ .

The above three properties are usually referred to as the Axioms of Probability (or the Kolmogorov Axioms, after A. Kolmogorov, one of the fathers of probability theory). Any function that satisfies the Axioms of Probability is called a probability function.

**Example** Consider the simple experiment of tossing a fair coin (just once), so  $S = \{H, T\}$ . A reasonable probability function is the one that assigns equal probabilities to heads and tails, that is,

$$\Pr(\{H\}) = \Pr(\{T\}).$$

Since  $S = \{H\} \cup \{T\}$ , we have, from Axiom 1,  $\Pr(\{H\} \cup \{T\}) = 1$ . Also,  $\{H\}$  and  $\{T\}$  are disjoint, so  $\Pr(\{H\} \cup \{T\}) = \Pr(\{H\}) + \Pr(\{T\})$ . Collectively, we have

$$\begin{aligned}\Pr(\{H\}) &= \Pr(\{T\}) \\ \Pr(\{H\} \cup \{T\}) &= 1 \\ \Pr(\{H\} \cup \{T\}) &= \Pr(\{H\}) + \Pr(\{T\})\end{aligned}$$

Therefore,  $\Pr(\{H\}) = \Pr(\{T\}) = \frac{1}{2}$ .

## Lecture 3: Aug 26

### Last time

- Set theory (1.1)
- Axiomatic Foundations (1.2)

### Today

- 5 bonus point presentation results
- Axiomatic Foundations (1.2)
- Calculus of Probabilities (1.2)
- Conditional Probability (1.3)

**Example** If  $S$  is finite or countable (where the elements of  $S$  can be put into 1 – 1 correspondence with a subset of the integers), then these technicalities really do not arise, for we define for a given sample space  $S$ ,

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If  $S$  has  $n$  elements, there are  $2^n$  sets in  $\mathcal{B}$  (why?). [hint: for each element, it is either in or out of a subset, so 2 choices].

**Example** Let  $S = (-\infty, \infty)$ , the real line. Then  $\mathcal{B}$  is chosen to contain all sets of the form

$$[a, b], (a, b], (a, b), \text{ and } [a, b)$$

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1.  $\Pr(A) \geq 0$  for all  $A \in \mathcal{B}$ .
2.  $\Pr(S) = 1$ .
3. If  $A_1, A_2, \dots \in \mathcal{B}$  are pairwise disjoint, then  $\Pr(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pr(A_i)$ .

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$$\begin{aligned}\Pr(\{H\}) &= \Pr(\{T\}) \\ \Pr(\{H\} \cup \{T\}) &= 1 \\ \Pr(\{H\} \cup \{T\}) &= \Pr(\{H\}) + \Pr(\{T\})\end{aligned}$$

Therefore,  $\Pr(\{H\}) = \Pr(\{T\}) = \frac{1}{2}$ .

## Caculus of Probabilities

We start with some fairly self-evident properties of the probability function when applied to a single event.

**Theorem** If  $\Pr$  is a probability function and  $A$  is any set in  $\mathcal{B}$ , then

1.  $\Pr(\emptyset) = 0$ , where  $\emptyset$  is the empty set;
2.  $\Pr(A) \leq 1$ ;
3.  $\Pr(A^c) = 1 - \Pr(A)$ .

*proof:*

- It's easy to prove (3) first. Since
  - $\Pr(A \cup A^c) = \Pr(S) = 1$ ,
  - $A$  and  $A^c$  are disjoint, by axiom (3),  $\Pr(A \cup A^c) = \Pr(A) + \Pr(A^c)$ .
 so that  $\Pr(A) + \Pr(A^c) = \Pr(S) = 1$
- with (3) proved, (1) is simple. because we know that
  - $S \cup \emptyset = S$ ,
  - $S \cap \emptyset = \emptyset$ , they are disjoint,
 so that  $\Pr(\emptyset) + \Pr(S) = \Pr(\emptyset \cup S) = \Pr(S)$ .
- now for (2),  $\Pr(A) = 1 - \Pr(A^c) \leq 1$ , by axiom (1).

**Theorem** If  $\Pr$  is a probability function and  $A$  and  $B$  are any sets in  $\mathcal{B}$ , then

1.  $\Pr(B \cap A^c) = \Pr(B) - \Pr(A \cap B)$ ;
2.  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ ;

3. If  $A \subset B$ , then  $\Pr(A) \leq \Pr(B)$ .

*proof:*

1. For (1), we have  $B = \{B \cap A\} \cup \{B \cap A^c\}$  and  $\{B \cap A\} \cap \{B \cap A^c\} = \emptyset$ , therefore

$$\Pr(B) = \Pr(\{B \cap A\} \cup \{B \cap A^c\})$$

2. For (2), we plug in (1) first such that we only need to show  $\Pr(A \cup B) = \Pr(A) + \Pr(B \cap A^c)$ . Since  $A \cap \{B \cap A^c\} = \emptyset$  and  $A \cup B = A \cup \{B \cap A^c\}$  (use a Venn diagram, or see Exercise 1.2), we have  $\Pr(A \cup B) = \Pr(A) + \Pr(B \cap A^c)$ .

3. For (3), if  $A \subset B$ , then  $A \cap B = A$ . Then using (1), we have

$$0 \leq \Pr(B \cap A^c) = \Pr(B) - \Pr(A)$$

Formula (2) in the above theorem gives a useful inequality for the probability of an intersection (Bonferroni's Inequality):

$$\Pr(A \cap B) \geq \Pr(A) + \Pr(B) - 1.$$

**Theorem** If  $\Pr$  is a probability function, then

1.  $\Pr(A) = \sum_{i=1}^{\infty} \Pr(A \cap C_i)$  for any partition  $C_1, C_2, \dots$ ;
2.  $\Pr(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \Pr(A_i)$  for any sets  $A_1, A_2, \dots$ .

where (1) is also referred to as "Total probability" and (2) is Boole's inequality.

*proof:*

By definition, since  $C_1, C_2, \dots$  form a partition, we have  $C_i \cap C_j = \emptyset$  for all  $i \neq j$ , and  $S = \cup_{i=1}^{\infty} C_i$ . Therefore,

$$A = A \cap S = A \cap (\cup_{i=1}^{\infty} C_i) = \cup_{i=1}^{\infty} (A \cap C_i),$$

where the last equality follows from the Distributive Law. Since  $\{A \cap C_i\} \cap \{A \cap C_j\} = \emptyset$  (i.e.  $A \cap C_i$  and  $A \cap C_j$  are disjoint), we have

$$\Pr(A) = \Pr(\cup_{i=1}^{\infty} (A \cap C_i)) = \sum_{i=1}^{\infty} \Pr(A \cap C_i).$$

To establish Boole's Inequality, we first construct a disjoint collection  $A_1^*, A_2^*, \dots$ , with the property that  $\cup_{i=1}^{\infty} A_i^* = \cup_{i=1}^{\infty} A_i$ . We define  $A_i^*$  by

$$A_1^* = A_1, \quad A_i^* = A_i \setminus (\cup_{j=1}^{i-1} A_j), \quad i = 2, 3, \dots,$$

where the notation  $A \setminus B$  denotes the part of  $A$  that does not intersect with  $B$ . In other words,  $A \setminus B = A \cap B^c$ . It's easy to see that  $\cup_{i=1}^{\infty} A_i^* = \cup_{i=1}^{\infty} A_i$ , and we have

$$\Pr(\cup_{i=1}^{\infty} A_i) = \Pr(\cup_{i=1}^{\infty} A_i^*) = \sum_{i=1}^{\infty} \Pr(A_i^*)$$

where the last equality holds because  $A_i^*$  are disjoint. To see this, consider any pair of  $A_i^* \cap A_k^*, i > k$ , then

$$\begin{aligned} A_i^* \cap A_k^* &= \{A_i \setminus (\cup_{j=1}^{i-1} A_j)\} \cap \{A_k \setminus (\cup_{j=1}^{k-1} A_j)\} \\ &= \{A_i \cap (\cup_{j=1}^{i-1} A_j)^c\} \cap \{A_k \cap (\cup_{j=1}^{k-1} A_j)^c\} \\ &= \{A_i \cap (\cap_{j=1}^{i-1} A_j^c)\} \cap \{A_k \cap (\cap_{j=1}^{k-1} A_j^c)\} \\ &= \emptyset. \end{aligned}$$

Lastly, we have  $\Pr(A_i^*) \leq \Pr(A_i)$ .

## Conditional Probability

All of the probabilities that we have dealt with thus far have been unconditional probabilities. A sample space was defined and all probabilities were calculated with respect to that sample space. In many instances, however, we are in a position to update the sample space based on new information. In such cases we want to be able to update probability calculations or to calculate *conditional probabilities*.

**Definition** If  $A$  and  $B$  are events in  $S$ , and  $\Pr(B) > 0$ , then the *conditional probability* of  $A$  given  $B$ , written  $\Pr(A|B)$ , is

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Note that  $B$  becomes the sample space now:  $\Pr(B|B) = 1$ .

**Example** Four cards are dealt from the top of a well-shuffled deck. What is the probability that they are the four aces? (there are in total 52 cards)

*solution:*

We define two events first. Let  $A$  be the event {4 aces on top}, and  $B$  be the event {the first card on top is an ace}. For a well-shuffled deck, all groups of 4 cards are equally likely.

In total, there are  $\binom{52}{4} = \frac{52!(52-4)!}{4!} = 270,725$  distinct groups. Therefore, the probability of event  $A$  is  $\Pr(A) = \frac{1}{270,725}$ .

Note,  $\binom{n}{m}$  reads “from  $n$  choose  $m$ ” (for  $m \leq n$ ) and calculates by  $\binom{n}{m} = \frac{n!(n-m)!}{m!}$  that

gives the number of distinct combinations of choosing  $m$  elements from  $n$  total elements.

Now, let's calculate  $\Pr(A|B)$ . First of all,  $A \subset B$ , so that we have  $\Pr(A \cap B) = \Pr(A)$ . For  $\Pr(B)$ , having an ace on top instead of the other 12 kinds,  $\Pr(B) = \frac{1}{13}$ . Then  $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)} = \frac{1}{20,825}$ .

**Theorem** (Bayes' Rule) Let  $A_1, A_2, \dots$  be a partition of the sample space, and let  $B$  be any set. Then, for each  $i = 1, 2, \dots$ ,

$$\Pr(A_i|B) = \frac{\Pr(B|A_i) \Pr(A_i)}{\sum_{j=1}^{\infty} \Pr(B|A_j) \Pr(A_j)}.$$

*proof:*

By "Total probability", we have  $\Pr(B) = \sum_{j=1}^{\infty} \Pr(B \cap A_j)$  which is the denominator. Therefore,  $\Pr(A_i|B) = \frac{\Pr(A_i \cap B)}{\Pr(B)} = \frac{\Pr(B|A_i) \Pr(A_i)}{\sum_{j=1}^{\infty} \Pr(B \cap A_j)}$ .

## Independence

**Definition** Two events,  $A$  and  $B$ , are *statistically independent* if

$$\Pr(A \cap B) = \Pr(A) \Pr(B)$$

Note that independence could have been defined using Bayes' rule by  $\Pr(A|B) = \Pr(A)$  or  $\Pr(B|A) = \Pr(B)$  as long as  $\Pr(A) > 0$  or  $\Pr(B) > 0$ . More notation, often statisticians omit  $\cap$  when writing intersection in a probability function which means  $\Pr(AB) = \Pr(A \cap B)$ . Sometime, statisticians use comma  $(,)$  to replace  $\cap$  inside a probability function too,  $\Pr(A, B) = \Pr(A \cap B)$ .

**Theorem** If  $A$  and  $B$  are independent events, then the following pairs are also independent.

1.  $A$  and  $B^c$ ,
2.  $A^c$  and  $B$ ,
3.  $A^c$  and  $B^c$ .

## Lecture 4: Aug 29

### Last time

- Axiomatic Foundations (1.2)
- Calculus of Probabilities (1.2)

### Today

- HW1 due 09/02, submit in the following recitation
- Conditional Probability (1.3)
- Independence (1.3)

**Theorem** If  $\Pr$  is a probability function and  $A$  and  $B$  are any sets in  $\mathcal{B}$ , then

1.  $\Pr(B \cap A^c) = \Pr(B) - \Pr(A \cap B)$ ;
2.  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ ;
3. If  $A \subset B$ , then  $\Pr(A) \leq \Pr(B)$ .

*proof:*

1. For (1), we have  $B = \{B \cap A\} \cup \{B \cap A^c\}$  and  $\{B \cap A\} \cap \{B \cap A^c\} = \emptyset$ , therefore

$$\Pr(B) = \Pr(\{B \cap A\} \cup \{B \cap A^c\})$$

2. For (2), we plug in (1) first such that we only need to show  $\Pr(A \cup B) = \Pr(A) + \Pr(B \cap A^c)$ . Since  $A \cap \{B \cap A^c\} = \emptyset$  and  $A \cup B = A \cup \{B \cap A^c\}$  (use a Venn diagram, or see Exercise 1.2), we have  $\Pr(A \cup B) = \Pr(A) + \Pr(B \cap A^c)$ .
3. For (3), if  $A \subset B$ , then  $A \cap B = A$ . Then using (1), we have

$$0 \leq \Pr(B \cap A^c) = \Pr(B) - \Pr(A)$$

Formula (2) in the above theorem gives a useful inequality for the probability of an intersection (Bonferroni's Inequality):

$$\Pr(A \cap B) \geq \Pr(A) + \Pr(B) - 1.$$

**Theorem** If  $\Pr$  is a probability function, then

1.  $\Pr(A) = \sum_{i=1}^{\infty} \Pr(A \cap C_i)$  for any partition  $C_1, C_2, \dots$ ;
2.  $\Pr(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \Pr(A_i)$  for any sets  $A_1, A_2, \dots$ .

where (1) is also referred to as “Total probability” and (2) is Boole’s inequality.

*proof:*

By definition, since  $C_1, C_2, \dots$  form a partition, we have  $C_i \cap C_j = \emptyset$  for all  $i \neq j$ , and  $S = \cup_{i=1}^{\infty} C_i$ . Therefore,

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where the last equality follows from the Distributive Law. Since  $\{A \cap C_i\} \cap \{A \cap C_j\} = \emptyset$  (i.e.  $A \cap C_i$  and  $A \cap C_j$  are disjoint), we have

$$\Pr(A) = \Pr(\cup_{i=1}^{\infty} (A \cap C_i)) = \sum_{i=1}^{\infty} \Pr(A \cap C_i).$$

To establish Boole’s Inequality, we first construct a disjoint collection  $A_1^*, A_2^*, \dots$ , with the property that  $\cup_{i=1}^{\infty} A_i^* = \cup_{i=1}^{\infty} A_i$ . We define  $A_i^*$  by

$$A_1^* = A_1, \quad A_i^* = A_i \setminus (\cup_{j=1}^{i-1} A_j), \quad i = 2, 3, \dots,$$

where the notation  $A \setminus B$  denotes the part of  $A$  that does not intersect with  $B$ . In other words,  $A \setminus B = A \cap B^c$ . It’s easy to see that  $\cup_{i=1}^{\infty} A_i^* = \cup_{i=1}^{\infty} A_i$ , and we have

$$\Pr(\cup_{i=1}^{\infty} A_i) = \Pr(\cup_{i=1}^{\infty} A_i^*) = \sum_{i=1}^{\infty} \Pr(A_i^*)$$

where the last equality holds because  $A_i^*$  are disjoint. To see this, consider any pair of  $A_i^* \cap A_k^*, i > k$ , then

$$\begin{aligned} A_i^* \cap A_k^* &= \{A_i \setminus (\cup_{j=1}^{i-1} A_j)\} \cap \{A_k \setminus (\cup_{j=1}^{k-1} A_j)\} \\ &= \{A_i \cap (\cup_{j=1}^{i-1} A_j)^c\} \cap \{A_k \cap (\cup_{j=1}^{k-1} A_j)^c\} \\ &= \{A_i \cap (\cap_{j=1}^{i-1} A_j^c)\} \cap \{A_k \cap (\cap_{j=1}^{k-1} A_j^c)\} \\ &= \emptyset. \end{aligned}$$

Lastly, we have  $\Pr(A_i^*) \leq \Pr(A_i)$ .

## Conditional Probability

All of the probabilities that we have dealt with thus far have been unconditional probabilities. A sample space was defined and all probabilities were calculated with respect to that sample space. In many instances, however, we are in a position to update the sample space based on new information. In such cases we want to be able to update probability calculations or to calculate *conditional probabilities*.

**Definition** If  $A$  and  $B$  are events in  $S$ , and  $\Pr(B) > 0$ , then the *conditional probability* of  $A$  given  $B$ , written  $\Pr(A|B)$ , is

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Note that  $B$  becomes the sample space now:  $\Pr(B|B) = 1$ . For disjoint events, if  $A \cap B = \emptyset$ , then  $\Pr(A|B) = 0$  and  $\Pr(B|A) = 0$ .

Conditional probability satisfies the axioms of probability:

1.  $\Pr(S|B) = 1$ ,
2.  $\Pr(A|B) \geq 0$ ,
3. If  $A_1, A_2, \dots$  are mutually exclusive events, then  $\Pr(\cup_{i=1}^{\infty} A_i|B) = \sum_{i=1}^{\infty} \Pr(A_i|B)$

**Example** Four cards are dealt from the top of a well-shuffled deck. What is the probability that they are the four aces? What is the probability of getting four aces at the top if knowing the first card is an ace? (there are in total 52 cards)

*solution:*

We define two events first. Let  $A$  be the event {4 aces on top}, and  $B$  be the event {the first card on top is an ace}. For a well-shuffled deck, all groups of 4 cards are equally likely.

In total, there are  $\binom{52}{4} = \frac{52!(52-4)!}{4!} = 270,725$  distinct groups. Therefore, the probability of event  $A$  is  $\Pr(A) = \frac{1}{270,725}$ .

Note,  $\binom{n}{m}$  reads “from  $n$  choose  $m$ ” (for  $m \leq n$ ) and calculates by  $\binom{n}{m} = \frac{n!}{m!(n-m)!}$  that gives the number of distinct combinations of choosing  $m$  elements from  $n$  total elements. Now, let's calculate  $\Pr(A|B)$ . First of all,  $A \subset B$ , so that we have  $\Pr(A \cap B) = \Pr(A)$ . For  $\Pr(B)$ , having an ace on top instead of the other 12 kinds,  $\Pr(B) = \frac{1}{13}$ . Then  $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A)}{\Pr(B)} = \frac{1}{20,825}$ .

**Theorem** (Bayes' Rule) Let  $A_1, A_2, \dots$  be a partition of the sample space, and let  $B$  be any set. Then, for each  $i = 1, 2, \dots$ ,

$$\Pr(A_i|B) = \frac{\Pr(B|A_i) \Pr(A_i)}{\sum_{j=1}^{\infty} \Pr(B|A_j) \Pr(A_j)}.$$

*proof:*

By “Total probability”, we have  $\Pr(B) = \sum_{j=1}^{\infty} \Pr(B \cap A_j)$  which is the denominator. Therefore,  $\Pr(A_i|B) = \frac{\Pr(A_i \cap B)}{\Pr(B)} = \frac{\Pr(B|A_i) \Pr(A_i)}{\sum_{j=1}^{\infty} \Pr(B \cap A_j)}$ .

## Independence

**Definition** Two events,  $A$  and  $B$ , are *statistically independent* if

$$\Pr(A \cap B) = \Pr(A) \Pr(B)$$

Note that independence could have been defined using Bayes' rule by  $\Pr(A|B) = \Pr(A)$  or  $\Pr(B|A) = \Pr(B)$  as long as  $\Pr(A) > 0$  or  $\Pr(B) > 0$ . More notation, often statisticians

omit  $\cap$  when writing intersection in a probability function which means  $\Pr(AB) = \Pr(A \cap B)$ . Sometime, statisticians use comma  $(,)$  to replace  $\cap$  inside a probability function too,  $\Pr(A, B) = \Pr(A \cap B)$ .



## Lecture 5: Aug 31

### Last time

- Conditional Probability (1.3)
- Independence (1.3)

### Today

- HW1 due 09/02
- Random variables (1.4)
- Distribution Functions (1.5)

**Theorem** If  $A$  and  $B$  are independent events, then the following pairs are also independent.

1.  $A$  and  $B^c$ ,
2.  $A^c$  and  $B$ ,
3.  $A^c$  and  $B^c$ .

*proof:*

For (1),

$$\begin{aligned}\Pr(A, B^c) &= \Pr(A) - \Pr(A, B) \\ &= \Pr(A) - \Pr(A) \Pr(B) \\ &= \Pr(A)(1 - \Pr(B)) \\ &= \Pr(A) \Pr(B^c)\end{aligned}$$

For (2), we just need to switch  $A$  and  $B$ .

For (3), we have  $A^c$  and  $B$  are independent, then we can treat  $A^c$  as  $A'$  and  $B$  as  $B'$ , then  $A'$  and  $B'^c$  are independent which is  $A^c$  and  $B^c$  are independent.

Alternatively, for (2),

$$\begin{aligned}\Pr(A^c, B) &= \Pr(A^c|B) \Pr(B) \\ &= [1 - \Pr(A|B)] \Pr(B) \\ &= [1 - \Pr(A)] \Pr(B) \\ &= \Pr(A^c) \Pr(B).\end{aligned}$$

And for (3),

$$\begin{aligned}\Pr(A^c, B^c) &= \Pr(A^c) - \Pr(A^c, B) \\ &= \Pr(A^c) - \Pr(A^c) \Pr(B) \\ &= \Pr(A^c) \Pr(B^c).\end{aligned}$$

**Example** Let the sample space  $S$  consist of the  $3!$  permutations of the letters  $a$ ,  $b$ , and  $c$  along with the three triples of each letter. Thus,

$$S = \left\{ \begin{array}{ccc} aaa & bbb & ccc \\ abc & bca & cba \\ acb & bac & cab \end{array} \right\}.$$

Furthermore, let each element of  $S$  have probability  $\frac{1}{9}$ . Define

$$A_i = \{i^{th} \text{ place in the triple is occupied by } a\}.$$

What are the values for  $\Pr(A_i), i = 1, 2, 3$ ? Are they pairwise independent?

*solution*

It is easy to count that

$$\Pr(A_i) = \frac{1}{3}, i = 1, 2, 3,$$

and

$$\Pr(A_1, A_2) = \Pr(A_1, A_3) = \Pr(A_2, A_3) = \frac{1}{9}$$

so that  $A_i$ s are pairwise independent.

**Definition\*** A collection of events  $A_1, \dots, A_n$  are *mutually independent* if for any subcollection  $A_{i_1}, \dots, A_{i_k}$ , we have

$$\Pr(\cap_{j=1}^k A_{i_j}) = \prod_{j=1}^k \Pr(A_{i_j}).$$

## Random Variables

In many experiments, it is easier to deal with a summary variable than with the original probability structure.

**Example** consider an opinion poll, we might decide to ask 50 people whether they agree or disagree with a certain issue. If we record a “1” for agree and “0” for disagree, the sample space for this experiment has  $2^{50}$  elements (all length 50 strings consist of 1s and 0s). However, if we are only interested in the number of people who agree, we may define a variable  $X =$  number of 1s recorded out of 50. Then, the sample space for  $X$  is the set of integers  $\{0, 1, 2, \dots, 50\}$ .

**Definition** A *random variable* (r.v.) is a function from a sample space  $S$  into the real numbers.

**Example** In some experiments random variables are implicitly used

Examples of random variables

Experiment	Random variable
Toss two dice	$X$ = sum of numbers
Toss a coin 25 times	$X$ = number of heads in 25 tosses
Apply different amounts of fertilizer to corn plants	$X$ = yield / acre

In defining a random variable, we have also defined a new sample space (the range of the random variable).

## Lecture 6: Sept 2

### Last time

- Random variables

### Today

- HW1 due today
- no class next Monday (Labor day)
- Presentation: Approximate Bayesian Computation
- Presentation: Karl Pearson

## Lecture 7: Sept 7

### Last time

- Presentations

### Today

- HW2 posted (due: Sept 15th)
- No lecture, but reviews on Fridays with two presentations
- Random variables
- Distribution Functions
- Types of Random Variables

**Induced probability function** Suppose we have a sample space  $S = \{s_1, s_2, \dots, s_n\}$  with a probability function  $\Pr$  defined on the original sample space. We define a random variable  $X$  with range  $\mathcal{X} = \{x_1, \dots, x_m\}$ . We can define a probability function  $\Pr_X$  on  $\mathcal{X}$  in the following way. We will observe  $X = x_i$  if and only if the outcome of the random experiment is an  $s_j \in S$  such that  $X(s_j) = x_i$ . Therefore,

$$\Pr_X(X = x_i) = \Pr(\{s_j \in S : X(s_j) = x_i\}),$$

defines an *induced* probability function on  $\mathcal{X}$ , defined in terms of the original function  $\Pr$ .

We will write  $\Pr(X = x_i)$  rather than  $\Pr_X(X = x_i)$  for simplicity. Note on notation: random variables will always be denoted with uppercase letters and the realized values of the variable (or its range) will be denoted by the corresponding lowercase letters.

**Example** Consider the experiment of tossing a fair coin three times. Define the random variable  $X$  to be the number of heads obtained in the three tosses. A complete enumeration of the value of  $X$  for each point in the sample space is

$s$	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
$X(s)$	3	2	2	2	1	1	1	0

What is the range of  $X$ ? What is the induced probability function  $\Pr_X$ ?

*solution:*

The range for the random variable  $X$  is  $\mathcal{X} = \{0, 1, 2, 3\}$ . Assuming all 8 points in  $S$  has probability  $\frac{1}{8}$ . By simply counting, we see that the induced probability function on  $\mathcal{X}$  is

$x$	0	1	2	3
$\Pr_X(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

So far, we have seen finite  $S$  and finite  $\mathcal{X}$ , and the definition of  $\Pr_X$  is straightforward. If  $\mathcal{X}$  is uncountable, we define the induced probability function,  $\Pr_X$  for any set  $A \subset \mathcal{X}$ ,

$$\Pr_X(X \in A) = \Pr(\{s \in S : X(s) \in A\}).$$

This defines a legitimate probability function for which the Kolmogorov Axioms can be verified.

## Distribution Functions

Distribution Functions are used to describe the behavior of a r.v.

### Cumulative distribution function

**Definition** The *cumulative distribution function* or *cdf* of a random variable  $X$ , denoted by  $F_X(x)$ , is defined by

$$F_X(x) = \Pr_X(X \leq x), \text{ for all } x.$$

**Definition** The *survival function* of a random variable  $X$ , is defined by

$$S_X(x) = 1 - F_X(x) = \Pr_X(X > x).$$

**Example** Consider the experiment of tossing three fair coins, and let  $X$  = number of heads observed. The cdf of  $X$  is

$$F_X(x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ \frac{1}{8} & \text{if } 0 \leq x < 1 \\ \frac{1}{2} & \text{if } 1 \leq x < 2 \\ \frac{7}{8} & \text{if } 2 \leq x < 3 \\ 1 & \text{if } 3 \leq x < \infty \end{cases}$$

Some properties of the cdf:

Let  $F(x)$  be a cdf. Then

1.  $0 \leq F(x) \leq 1$
2.  $\lim_{x \rightarrow -\infty} F(x) = 0$
3.  $\lim_{x \rightarrow \infty} F(x) = 1$
4.  $F$  is nondecreasing: if  $a < b$ , then  $F(a) \leq F(b)$
5.  $F$  is right-continuous:  $\lim_{x \downarrow b} F(x) = F(b)$ , or  $\lim_{x \rightarrow b^+} F(x) = F(b)$
6.  $\Pr(a < X \leq B) = F(b) - F(a)$

**Theorem** The function  $F(x)$  is a cdf if and only if the following three conditions hold:

1.  $\lim_{x \rightarrow -\infty} F(x) = 0$  and  $\lim_{x \rightarrow \infty} F(x) = 1$
2.  $F$  is nondecreasing: if  $a < b$ , then  $F(a) \leq F(b)$
3.  $F$  is right-continuous:  $\lim_{x \downarrow b} F(x) = F(b)$ , or  $\lim_{x \rightarrow b^+} F(x) = F(b)$

The cdf does not contain information about the original sample space.

**Definition** Two random variables  $X$  and  $Y$  are identically distributed if, for every Borel set  $A \subset \mathbb{R}$ ,  $\Pr(X \in A) = \Pr(Y \in A)$ .

**Example** Toss a fair coin  $n$  times. The number of heads and the number of tails have the same distribution.

**Theorem** The following two statements are equivalent:

1. The random variables  $X$  and  $Y$  are *identically distributed*.
2.  $F_X(x) = F_Y(x)$  for every  $x$ .

## Types of Random Variables

**Definition** A random variable  $X$  can be

- *discrete*:
  - $X$  takes on a finite or countably infinite number of values
  - $F_X(x)$  is step-wise constant
- *continuous*:
  - the range of  $X$  consists of subsets of the real line
  - $F_X(x)$  is continuous.
- *mixed*:  $F_X(x)$  is piecewise continuous.

**Example** A random variable has cdf

$$F(x) = \begin{cases} 0 & x < 0 \\ x/2 & 0 \leq x < 1 \\ 2/3 & 1 \leq x < 2 \\ 11/12 & 2 \leq x < 3 \\ 1 & 3 \leq x \end{cases}$$

Is this a valid cdf? Is it a discrete random variable or continuous random variable or mixed?  
*solution:*

$F(x)$  satisfies the three properties of a cdf that

1.  $\lim_{x \rightarrow -\infty} F(x) = 0$  and  $\lim_{x \rightarrow \infty} F(x) = 1$
2.  $F$  is nondecreasing: if  $a < b$ , then  $F(a) \leq F(b)$
3.  $F$  is right-continuous:  $\lim_{x \downarrow b} F(x) = F(b)$ , or  $\lim_{x \rightarrow b^+} F(x) = F(b)$ .

Therefore,  $F(x)$  is a valid cdf. The random variable  $X$  is a mixed type.



## Lecture 8: Sept 9

### Last time

- Random variables
- Distribution Functions
- Types of Random Variables

### Today

- Presentation: Andrey Markov by Ryan Mortonson
- Presentation: Spatial Statistics by Camille Kreisel
- Review part 1

### Review part 1

We briefly review what we have covered so far. We complement this review process with examples/questions taken from the book “Introduction to Probability Theory and Statistical Inference” 3rd ed. by Harold J. Larson.

We started with Set Theory.

**Definition** The set,  $S$ , of all possible outcomes of a particular experiment is called the *sample space* for the experiment.

**Definition** An *event* is any collection of possible outcomes of an experiment, that is, any subset of  $S$  (including  $S$  itself).

An event occurs if any one of its elements is the outcome observed.

**Definition** Two events  $A$  and  $B$  are *disjoint* (or *mutually exclusive*) if  $A \cap B = \emptyset$ . The events  $A_1, A_2, \dots$  are *pairwise disjoint* (or *mutually exclusive*) if  $A_i \cap A_j = \emptyset$  for all  $i \neq j$ .

**Definition** If  $A_1, A_2, \dots$  are pairwise disjoint and  $\cup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \dots = S$ , then the collection of  $A_1, A_2, \dots$  forms a *partition* of  $S$ .

**Theorem** For any three events,  $A$ ,  $B$ , and  $C$ , defined on a sample space  $S$ ,

1. Commutativity

$$\begin{aligned} A \cup B &= B \cup A, \\ A \cap B &= B \cap A; \end{aligned}$$

## 2. Associativity

$$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap C, \\ A \cap (B \cup C) &= (A \cap B) \cup C; \end{aligned}$$

## 3. Distributive Laws

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C), \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C); \end{aligned}$$

## 4. DeMorgan's Laws

$$\begin{aligned} (A \cup B)^c &= A^c \cap B^c, \\ (A \cap B)^c &= A^c \cup B^c; \end{aligned}$$

Then we moved to define a probability function. To establish the domain for the probability function, we start with *sigma algebra*.

**Definition** A collection of subsets of  $S$  is called a *sigma algebra* (or *Borel field*), denoted by  $\mathcal{B}$ , if it satisfies the following three properties:

1.  $\emptyset \in \mathcal{B}$  (the empty set is an element of  $\mathcal{B}$ ).
2. If  $A \in \mathcal{B}$ , then  $A^c \in \mathcal{B}$  ( $\mathcal{B}$  is closed under complementation).
3. If  $A_1, A_2, \dots \in \mathcal{B}$ , then  $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$  ( $\mathcal{B}$  is closed under countable unions).

By DeMorgan's Law, (3) can be replaced by:

$$3'. \text{ if } A_1, A_2, \dots \in \mathcal{B}, \text{ then } \cap_{i=1}^{\infty} A_i \in \mathcal{B}.$$

which means that if we have property (1), (2) and (3) then we have property (1), (2), (3') and vice-versa (if we have property (1), (2) and (3') then we have property (1), (2), (3)).

This is because:

$$(\cup_{i=1}^{\infty} A_i^c)^c = \cap_{i=1}^{\infty} A_i.$$

So that if we have property (3) that  $A_1, A_2, \dots \in \mathcal{B}$  and  $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$ . Then by property (2), we know that  $A_i^c \in \mathcal{B}$  for  $i = 1, 2, \dots$ . And we can apply property (3) again such that if  $A_1^c, A_2^c, \dots \in \mathcal{B}$ , then  $(\cup_{i=1}^{\infty} A_i^c) \in \mathcal{B}$ . Therefore, now we know  $(\cup_{i=1}^{\infty} A_i^c) \in \mathcal{B}$  and we can apply property (2) again to get its complement which is also in the Borel field. Therefore,  $(\cup_{i=1}^{\infty} A_i^c)^c \in \mathcal{B}$  which is  $\cap_{i=1}^{\infty} A_i$ .

For the other direction, we start from property (1), (2) and (3'). With property (3'), we have if  $A_1, A_2, \dots \in \mathcal{B}$ , then  $\cap_{i=1}^{\infty} A_i \in \mathcal{B}$ . We again, first apply property (2) such that if  $A_1, A_2, \dots \in \mathcal{B}$ , then  $A_1^c, A_2^c, \dots \in \mathcal{B}$ . Now, by property (3'), we have  $\cap_{i=1}^{\infty} A_i^c \in \mathcal{B}$ . By applying property (2), we have  $(\cap_{i=1}^{\infty} A_i^c)^c \in \mathcal{B}$ . By substituting  $A_i$  with  $A_i^{*c}$  and taking complement at both side of equation  $(\cup_{i=1}^{\infty} A_i^c)^c = \cap_{i=1}^{\infty} A_i$ , we have  $(\cup_{i=1}^{\infty} A_i^{*c}) = (\cap_{i=1}^{\infty} A_i^{*c})^c$ . Therefore,  $\cup_{i=1}^{\infty} A_i = (\cap_{i=1}^{\infty} A_i^c)^c \in \mathcal{B}$  which is property (3).

## Lecture 9: Sept 12

### Last time

- Presentations

### Today

- HW2 deadline extended (due: Sept 22nd)
- Random variables
- Distribution Functions
- Types of Random Variables

## Distribution Functions

Distribution Functions are used to describe the behavior of a r.v.

### Cumulative distribution function

**Definition** The *cumulative distribution function* or *cdf* of a random variable  $X$ , denoted by  $F_X(x)$ , is defined by

$$F_X(x) = \Pr_X(X \leq x), \text{ for all } x.$$

**Definition** The *survival function* of a random variable  $X$ , is defined by

$$S_X(x) = 1 - F_X(x) = \Pr_X(X > x).$$

**Example** Consider the experiment of tossing three fair coins, and let  $X$  = number of heads observed. The cdf of  $X$  is

$$F_X(x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ \frac{1}{8} & \text{if } 0 \leq x < 1 \\ \frac{1}{2} & \text{if } 1 \leq x < 2 \\ \frac{7}{8} & \text{if } 2 \leq x < 3 \\ 1 & \text{if } 3 \leq x < \infty \end{cases}$$

Some properties of the cdf:

Let  $F(x)$  be a cdf. Then

1.  $0 \leq F(x) \leq 1$
2.  $\lim_{x \rightarrow -\infty} F(x) = 0$
3.  $\lim_{x \rightarrow \infty} F(x) = 1$

4.  $F$  is nondecreasing: if  $a < b$ , then  $F(a) \leq F(b)$
5.  $F$  is right-continuous:  $\lim_{x \downarrow b} F(x) = F(b)$ , or  $\lim_{x \rightarrow b^+} F(x) = F(b)$
6.  $\Pr(a < X \leq B) = F(b) - F(a)$

**Theorem** The function  $F(x)$  is a cdf if and only if the following three conditions hold:

1.  $\lim_{x \rightarrow -\infty} F(x) = 0$  and  $\lim_{x \rightarrow \infty} F(x) = 1$
2.  $F$  is nondecreasing: if  $a < b$ , then  $F(a) \leq F(b)$
3.  $F$  is right-continuous:  $\lim_{x \downarrow b} F(x) = F(b)$ , or  $\lim_{x \rightarrow b^+} F(x) = F(b)$

The cdf does not contain information about the original sample space.

**Definition** Two random variables  $X$  and  $Y$  are identically distributed if, for every Borel set  $A \subset \mathbb{R}$ ,  $\Pr(X \in A) = \Pr(Y \in A)$ .

**Example** Toss a fair coin  $n$  times. The number of heads and the number of tails have the same distribution.

**Theorem** The following two statements are equivalent:

1. The random variables  $X$  and  $Y$  are *identically distributed*.
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## Types of Random Variables

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- *mixed*:  $F_X(x)$  is piecewise continuous.

**Example** A random variable has cdf

$$F(x) = \begin{cases} 0 & x < 0 \\ x/2 & 0 \leq x < 1 \\ 2/3 & 1 \leq x < 2 \\ 11/12 & 2 \leq x < 3 \\ 1 & 3 \leq x \end{cases}$$

Is this a valid cdf? Is it a discrete random variable or continuous random variable or mixed?  
*solution:*

$F(x)$  satisfies the three properties of a cdf that

1.  $\lim_{x \rightarrow -\infty} F(x) = 0$  and  $\lim_{x \rightarrow \infty} F(x) = 1$
2.  $F$  is nondecreasing: if  $a < b$ , then  $F(a) \leq F(b)$
3.  $F$  is right-continuous:  $\lim_{x \downarrow b} F(x) = F(b)$ , or  $\lim_{x \rightarrow b^+} F(x) = F(b)$ .

Therefore,  $F(x)$  is a valid cdf. The random variable  $X$  is a mixed type.

### Discrete Random Variables

Suppose a random variable  $X$  takes only a finite or countable number of values. Let the sample space of  $X$  be  $S = \{x_1, x_2, \dots\}$ . Then the cdf can be expressed as:

$$F(x) = \sum_{x_i \leq x} \Pr(X = x_i).$$

**Definition** The *probability mass function* (pmf) of a discrete random variable  $X$  is given by

$$f_X(x) = \Pr(X = x) \text{ for all } x.$$

If the sample space of  $X$  is  $X = \{x_1, x_2, \dots\}$ , then

$$f(x_i) = \Pr(X = x_i) = \Pr(x_{i-1} < X \leq x_i) = F(x_i) - F(x_{i-1}).$$

**Example** (Geometric probabilities) Suppose we do an experiment that consists of tossing a coin until a head appears. Let  $p$  = probability of a head on any given toss, and define a random variable  $X$  = number of tosses required to get a head. Then for any  $x = 1, 2, \dots$ ,

$$\Pr(X = x) = (1 - p)^{x-1}p,$$

since we must get  $x - 1$  tails followed by a head for the event to occur and all trials are independent. What is the pmf of the above Geometric distribution? What is the cdf?

*solution:*

We have the pmf

$$f(x) = \Pr(X = x) = \begin{cases} (1-p)^{x-1}p & \text{for } x = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

For cdf, we have

$$\begin{aligned} F(x) &= \Pr(X \leq x) = \sum_{i=1}^{\lfloor x \rfloor} f(i) \\ &= \begin{cases} f(1) + f(2) + \dots + f(\lfloor x \rfloor) & \text{for } x \geq 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 - (1-p)^{\lfloor x \rfloor} & \text{for } x \geq 1 \\ 0 & \text{for } x < 1 \end{cases} \end{aligned}$$

where  $\lfloor x \rfloor$  denote the floor function that returns the largest integer smaller or equal to  $x$  and we used the summation of a geometric sequence.

**Definition** The *domain* of a random variable  $X$  is the set of all values of  $x$  for which  $f(x) > 0$ . This is also called *range* or *sample space*.

Properties of the pmf:

1.  $f(x) > 0$  for at most a countable number of values  $x$ . For all other values  $x$ ,  $f(x) = 0$ .
2. Let  $\{x_1, x_2, \dots\}$  denote the domain of  $X$ . Then

$$\sum_{i=1}^{\infty} f(x_i) = 1.$$

An obvious consequence is that  $f(x) \leq 1$  over the domain.

**Example** What is the pmf of a deterministic random variable (a constant)?

*solution:*

$$f(x) = \Pr(X = x) = \begin{cases} 1 & \text{for } x = c \\ 0 & \text{otherwise.} \end{cases}$$

This is equivalent as a constant of value  $c$ .

**Example** In many applications, a formula can be used to represent the pmf of a random variable. Suppose  $X$  can take values  $1, 2, \dots$  with pmf

$$f(x) = \begin{cases} \frac{1}{x(x+1)} & \text{for } x = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

How would we determine if this is an allowable pmf?

*solution:*

We show that  $f(x)$  satisfies the properties of pmf.

1.  $f(x) > 0$  for a countable number of values  $x$ . For all other values  $x$ ,  $f(x) = 0$ .
2. Let  $\{x_1, x_2, \dots\}$  denote the domain of  $X$ . Then

$$\sum_{i=1}^{\infty} f(x_i) = \sum_{i=1}^{\infty} f(i) = \sum_{i=1}^{\infty} \left( \frac{1}{x} - \frac{1}{x+1} \right) = 1.$$

## Lecture 10: Sept 14

### Last time

- Random variables
- Distribution Functions
- Types of Random Variables

### Today

- Continuous Random Variables
- Counting Techniques

### Continuous Random Variables

**Definition** A random variable  $X$  is *continuous* if  $F_X(x)$  is a continuous function of  $x$ .

**Definition** A random variable  $X$  is *absolutely continuous* if  $F_X(x)$  is an absolutely continuous function of  $x$ .

**Definition** A function  $F(x)$  is *absolutely continuous* if it can be written

$$F(x) = \int_{-\infty}^x f(x)dx.$$

Absolute continuity is stronger than continuity but weaker than differentiability. An example of an absolutely continuous function is one that is:

- continuous everywhere
- differentiable everywhere, except possibly for a countable number of points.

**Definition** The *probability density function* or pdf,  $f_X(x)$ , of a continuous random variable  $X$  is the function that satisfies

$$F_X(x) = \int_{-\infty}^x f_X(t)dt \quad \text{for all } x.$$

**Notation:** We write  $X \sim F_X(x)$  for the expression “ $X$  has a distribution given by  $F_X(x)$ ” where we read the symbol “ $\sim$ ” as “is distributed as”. Similarly, we can write  $X \sim f_X(x)$  or, if  $X$  and  $Y$  have the same distribution,  $X \sim Y$ .

**Theorem** A function  $f_X(x)$  is a pdf (or pmf) of a random variable  $X$  if and only if

1.  $f_X(x) \geq 0$  for all  $x$ .
2.  $\int_{-\infty}^{\infty} f_X(x)dx = 1$  (pdf)    or     $\sum_x f_X(x) = 1$  (pmf).



**Example** Suppose  $F(x) = 1 - e^{-\lambda x}$  for  $x > 0$  and  $F(x) = 0$  otherwise. Is  $F(x)$  a cdf? What is the associated pdf?

*solution:*

$F(x)$  satisfies the three properties of cdf

1.  $\lim_{x \rightarrow -\infty} F(x) = 0$  and  $\lim_{x \rightarrow \infty} F(x) = 1$
2.  $F$  is nondecreasing: if  $a < b$ , then  $F(a) \leq F(b)$
3.  $F$  is right-continuous:  $\lim_{x \downarrow b} F(x) = F(b)$ , or  $\lim_{x \rightarrow b^+} F(x) = F(b)$ .

$F(x)$  is a cdf. Actually,  $F(x)$  is the cdf of exponential distribution.

To get the pdf, we only need to differentiate the cdf.

$$f(x) = \frac{dF(x)}{dx} = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Notes

- If  $X$  is a continuous random variable, then  $f(x)$  is not the probability that  $X = x$ . In fact, if  $X$  is an absolutely continuous random variable with density function  $f(x)$ , then  $\Pr(X = x) = 0$ . (Why?)

*proof*

$$\begin{aligned} \Pr(X = x) &= \lim_{h \rightarrow 0} \int_{x-h}^{x+h} f(u) du \\ &= \lim_{h \rightarrow 0} F(x+h) - F(x-h) \\ &= F(x+) - F(x-) \\ &= 0 \end{aligned}$$

- Because  $\Pr(X = a) = 0$ , all the following are equivalent:

$$\Pr(a \leq X \leq b), \quad \Pr(a \leq X < b) \quad , \quad \Pr(a < X \leq b) \quad \text{and} \quad \Pr(a < X < b)$$

- $f(x)$  can exceed one!

## Counting Techniques

These sections are from 2.4 of “Introduction to Probability Theory and Statistical Inference” by Harold J. Larson. We employ them to discuss combinatorics.

When the equally likely assumption is made for a finite sample space, the probability of occurrence of any event  $A$  is given by the ratio of the number of elements belonging to  $A$  to the number of elements belonging to  $S$ . For such cases it is useful to be able to count the number of elements belonging to given sets.

A very simple technique that is frequently useful in counting problems is called the *multiplication principle*.

**Definition** If a first operation can be performed in any of  $n_1$  ways and a second operation can then be performed in any of  $n_2$  ways, both operations can be performed (the second immediately following the first) in  $n_1 \cdot n_2$  ways.

**Example** If we can travel from town  $A$  to town  $B$  in 3 ways and from town  $B$  to town  $C$  in 4 ways, then we can travel from  $A$  to  $C$  via  $B$  in a total of  $3 \cdot 4 = 12$  ways.

**Example** If the operation of tossing a die gives rise to 1 of 6 possible outcomes and the operation of tossing a second die gives rise to 1 of 6 possible outcomes, then the operation of tossing a pair of dice gives rise to  $6 \cdot 6 = 36$  possible outcomes.

**Definition** An arrangement of  $n$  symbols in a definite order is called a *permutation* of  $n$  symbols.

**Example** Let's consider all possible  $n$ -tuples made by  $n$  different symbols. In listing all the possible  $n$ -tuples, we would perform  $n$  natural operations. First we must fill the leftmost position of  $n$ -tuples, we have all  $n$  symbols to choose from. Then we must fill the second leftmost position, where we have  $n - 1$  symbols to choose from. Then, the third position with  $n - 2$  symbols to choose from, and so on. Finally, when we reach the right most position, we have 1 symbol left.

Using the multiplication rule, the total number of ways we can perform all  $n$  operations will be

$$n! = n(n - 1)(n - 2) \cdots 2 \cdot 1,$$

where we write  $n!$  (read  $n$ -factorial) and we define  $0! = 1$ .

**Example** Suppose the same 5 people park their cars on the same side of the street in the same block every night. How many different ordering of the 5 cars parked on the street are possible?

*Solution:*

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

**Example** Suppose the same 5 people park their cars on the two sides of the street in the same block every night where one side has 3 slots and the other side has 2. How many different ordering of 3 cars out of 5 can be parked on the 3-slot side?

*Solution:*

For the first slot, we have 5 possible cars to choose from. For the second slot, we have 4 cars to choose from (one is taken for the first slot). For the third slot, we have 3 cars to choose from (two cars are taken for the other two slots). In total, there are

$$5 \cdot 4 \cdot 3 = 60$$

ways.

**Definition** The number of  $r$ -tuples we can make  $r \leq n$ , using  $n$  different symbols (each only once), is called the *number of permutations of  $n$  things  $r$  at a time* and is denoted by  ${}^nP_r$ , which is calculated as

$${}^nP_r = n(n-1) \cdots (n-r+1).$$

**Example** Fifteen cars enter a race. In how many different ways could trophies for first, second, and third place be awarded?

*Solutions:*

$${}^{15}P_3 = 15 \cdot 14 \cdot 13 = 2730.$$

**Example** How many of the 3-tuples just counted have car number 15 in the first position?

*Solutions:*

$${}^{14}P_2 = 14 \cdot 13 = 182.$$

**Definition** The number of distinct subsets, each of size  $r$ , that can be constructed from a set with  $n$  elements is called the number of *combinations of  $n$  things  $r$  at a time*: this number is represented by  $\binom{n}{r}$  which reads  $n$  choose  $r$ .

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

**Example** How many distinct 5-card hands can be dealt from a standard 52-card deck?

$$\binom{52}{5} = \frac{52!}{5!47!} = 2,598,960.$$

**Theorem** If  $x$  and  $y$  are any two real numbers and  $n$  is a positive integer, then

$$(x+y)^n = \sum_{i=1}^n \binom{n}{i} x^i y^{n-i}, \quad \text{where } \binom{n}{i} = \frac{n!}{(n-i)!i!}.$$

## Lecture 11: Sept 16

### Last time

- Continuous Random Variables

### Today

- Presentations
- Review part 2

### Review part 2

**Definition** Given a sample space  $S$  and an associated sigma algebra  $\mathcal{B}$ , a *probability function* is a function  $\Pr$  with domain  $\mathcal{B}$  that satisfies

1.  $\Pr(A) \geq 0$  for all  $A \in \mathcal{B}$ .
2.  $\Pr(S) = 1$ .
3. If  $A_1, A_2, \dots \in \mathcal{B}$  are pairwise disjoint, then  $\Pr(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pr(A_i)$ .

The above three properties are usually referred to as the Axioms of Probability (or the Kolmogorov Axioms, after A. Kolmogorov, one of the fathers of probability theory). Any function that satisfies the Axioms of Probability is called a probability function.

**Theorem** If  $\Pr$  is a probability function and  $A$  is any set in  $\mathcal{B}$ , then

1.  $\Pr(\emptyset) = 0$ , where  $\emptyset$  is the empty set;
2.  $\Pr(A) \leq 1$ ;
3.  $\Pr(A^c) = 1 - \Pr(A)$ .

**Theorem** If  $\Pr$  is a probability function and  $A$  and  $B$  are any sets in  $\mathcal{B}$ , then

1.  $\Pr(B \cap A^c) = \Pr(B) - \Pr(A \cap B)$ ;
2.  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ ;
3. If  $A \subset B$ , then  $\Pr(A) \leq \Pr(B)$ .

**Theorem** If  $\Pr$  is a probability function, then

1.  $\Pr(A) = \sum_{i=1}^{\infty} \Pr(A \cap C_i)$  for any partition  $C_1, C_2, \dots$ ;
2.  $\Pr(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \Pr(A_i)$  for any sets  $A_1, A_2, \dots$

where (1) is also referred to as “Total probability” and (2) is Boole’s inequality.

**Definition** If  $A$  and  $B$  are events in  $S$ , and  $\Pr(B) > 0$ , then the *conditional probability* of  $A$  given  $B$ , written  $\Pr(A|B)$ , is

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

Note that  $B$  becomes the sample space now:  $\Pr(B|B) = 1$ .

**Theorem** (Bayes' Rule) Let  $A_1, A_2, \dots$  be a partition of the sample space, and let  $B$  be any set. Then, for each  $i = 1, 2, \dots$ ,

$$\Pr(A_i|B) = \frac{\Pr(B|A_i) \Pr(A_i)}{\sum_{j=1}^{\infty} \Pr(B|A_j) \Pr(A_j)}.$$

## Lecture 12: Sept 19

### Last time

- Random variables
- Distribution Functions
- Types of Random Variables

### Today

- Counting Techniques
- Transformations of Random Variables

**Definition** The number of  $r$ -tuples we can make  $r \leq n$ , using  $n$  different symbols (each only once), is called the *number of permutations of  $n$  things  $r$  at a time* and is denoted by  ${}^nP_r$ , which is calculated as

$${}^nP_r = n(n-1) \cdots (n-r+1).$$

**Example** Fifteen cars enter a race. In how many different ways could trophies for first, second, and third place be awarded?

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**Example** How many distinct 5-card hands can be dealt from a standard 52-card deck?

$$\binom{52}{5} = \frac{52!}{5!47!} = 2,598,960.$$

**Theorem** If  $x$  and  $y$  are any two real numbers and  $n$  is a positive integer, then

$$(x+y)^n = \sum_{i=0}^n \binom{n}{i} x^i y^{n-i}, \quad \text{where } \binom{n}{i} = \frac{n!}{(n-i)!i!}.$$

## Transformations of Random Variables

**Theorem** If  $X$  is a r.v. with sample space  $\mathcal{X} \subset \mathbb{R}$  and cdf  $F_X(x)$ , then any function of  $X$ , say  $Y = g(X)$  is also a random variable. The new random variable  $Y$  has a new sample space  $\mathcal{Y} = g(\mathcal{X}) \subset \mathbb{R}$ . The objective is to find the cdf  $F_Y(y)$  of  $Y$ .

**Probability mapping:** For any set  $A \subset \mathcal{Y}$ :

$$\begin{aligned}\Pr(Y \in A) &= \Pr(g(X) \in A) \\ &= \Pr(\{x \in \mathcal{X} : g(x) \in A\}) \\ &= \Pr(X \in g^{-1}(A)),\end{aligned}$$

where we have defined

$$g^{-1}(A) = \{x \in \mathcal{X} : g(x) \in A\}.$$

Notice that  $g^{-1}(A)$  is well defined even if  $g(\cdot)$  is not necessarily bijective.

**Example** (Binomial transformation) A discrete random variable  $X$  has a *binomial distribution* if its pmf is of the form

$$f_X(x) = \Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n,$$

where  $n$  is a positive integer and  $0 \leq p \leq 1$ . Values such as  $n$  and  $p$  that can be set to different values, producing different probability distributions, are called *parameters*. Consider a random variable  $Y = g(X)$ , where  $g(x) = n - x$ ; that is,  $Y = n - X$ . Here  $\mathcal{X} = \{0, 1, \dots, n\}$  and  $\mathcal{Y} = \{y : y = g(x), x \in \mathcal{X}\} = \{0, 1, \dots, n\}$ . For any  $y \in \mathcal{Y}$ ,  $n - x = g(x) = y$  if and only if  $x = n - y$ . Therefore,  $g^{-1}(y) = n - y$  and

$$\begin{aligned}f_Y(y) &= \sum_{x \in g^{-1}(y)} f_X(x) \\ &= f_X(n - y) \\ &= \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)} \\ &= \binom{n}{y} (1-p)^y p^{n-y}.\end{aligned}$$

Therefore,  $Y$  also has a binomial distribution, but with parameters  $n$  and  $1 - p$ .

**Example** (exercise 2.3) Suppose  $X$  has the geometric pmf  $f_X(x) = \frac{1}{3}(\frac{2}{3})^x, x = 0, 1, 2, \dots$ . Determine the probability distribution of  $Y = X/(X + 1)$ . Note that here both  $X$  and  $Y$  are discrete random variables. To specify the probability distribution of  $Y$ , specify its pmf.  
*Solution:*

$$\Pr(Y = y) = \Pr\left(\frac{X}{X+1} = y\right) = \Pr\left(X = \frac{y}{1-y}\right) = \frac{1}{3}\left(\frac{2}{3}\right)^{y/(1-y)}, y = 0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{x}{x+1}, \dots$$

## Lecture 13: Sept 21

### Last time

- Counting Techniques
- Transformations of Random Variables

### Today

- Transformations of continuous random variables

**Theorem** Suppose a continuous random variable  $X$  has cdf  $F_X(x)$ , let  $Y = g(X)$ , and let  $\mathcal{X}$  and  $\mathcal{Y}$  be defined as

$$\mathcal{X} = \{x : f(x) > 0\} \quad \text{and} \quad \mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

Then,

1. If  $g$  is an increasing function on  $\mathcal{X}$ ,  $F_Y(y) = F_X(g^{-1}(y))$  for  $y \in \mathcal{Y}$ .
2. If  $g$  is a decreasing function on  $\mathcal{X}$ ,  $F_Y(y) = 1 - F_X(g^{-1}(y))$  for  $y \in \mathcal{Y}$ .

*Proof:* We start with

$$\begin{aligned} F_Y(y) &= \Pr(Y \leq y) \\ &= \Pr(g(X) \leq y) \end{aligned}$$

1. If  $g$  is an increasing function, then  $g(X) \leq y$  if and only if  $X \leq g^{-1}(y)$ . Therefore,  $F_Y(y) = \Pr(g(X) \leq y) = \Pr(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$ .
2. Similarly, if  $g$  is a decreasing function, then  $g(X) \leq y$  if and only if  $X \geq g^{-1}(y)$ . And  $F_Y(y) = \Pr(g(X) \leq y) = \Pr(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y))$ .

**Theorem** Let  $X$  have pdf  $f_X(x)$  and let  $Y = g(X)$ , where  $g$  is a monotone function. Let  $\mathcal{X}$  and  $\mathcal{Y}$  be defined as

$$\mathcal{X} = \{x : f(x) > 0\} \quad \text{and} \quad \mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}.$$

Suppose that  $f_X(x)$  is continuous on  $\mathcal{X}$  and that  $g^{-1}(y)$  has a continuous derivative on  $\mathcal{Y}$ . Then the pdf of  $Y$  is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & y \in \mathcal{Y} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof:*

From last theorem, we have the cdf forms  $F_Y(y)$ . Then  $f_Y(y) = \frac{d}{dy} F_Y(y)$ . (finish the proof)  
From last theorem, we have

$$F_Y(y) = \begin{cases} F_X(g^{-1}(y)) & \text{if } g \text{ is increasing} \\ 1 - F_X(g^{-1}(y)) & \text{if } g \text{ is decreasing.} \end{cases}$$



We have, by the chain rule,

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y) & \text{if } g \text{ is increasing} \\ -f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y) & \text{if } g \text{ is decreasing,} \end{cases}$$

where  $\frac{d}{dy}g^{-1}(y) < 0$  when  $g$  is decreasing such that  $-\frac{d}{dy}g^{-1}(y) = |\frac{d}{dy}g^{-1}(y)|$ .

**Example** (Square transformation) Suppose  $X$  is a continuous random variable. For  $y > 0$ , the cdf of  $Y = X^2$  is

$$F_Y(y) = \Pr(Y \leq y) = \Pr(X^2 \leq y) = \Pr(-\sqrt{y} \leq X \leq \sqrt{y}).$$

Because  $x$  is continuous, we can drop the equality from the left endpoint and obtain

$$\begin{aligned} F_Y(y) &= \Pr(-\sqrt{y} < X \leq \sqrt{y}) \\ &= \Pr(X \leq \sqrt{y}) - \Pr(X \leq -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}). \end{aligned}$$

The pdf of  $Y$  can now be obtained from the cdf by differentiation:

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) \\ &= \frac{d}{dy} [F_X(\sqrt{y}) - F_X(-\sqrt{y})] \\ &= \frac{1}{2\sqrt{y}}f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}}f_X(-\sqrt{y}), \end{aligned}$$

where we use the chain rule to differentiate  $F_X(\sqrt{y})$  and  $F_X(-\sqrt{y})$ .

**Example** (Linear transformation) Suppose  $X$  is a continuous random variable with pdf  $f_X(x)$ . Let

$$Y = a + bX, \quad \frac{dy}{dx} = b.$$

Then

$$f_Y(y) = f_X[g^{-1}(y)] \left| \frac{dx}{dy} \right| = f_X\left(\frac{y-a}{b}\right) \frac{1}{|b|}.$$

This transformation is often used when  $X$  has mean 0 and standard deviation 1. The linear transformation above creates a random variable  $Y$  with a distribution that has the same shape as that of  $X$  but has mean  $a$  and variance  $b^2$ .

Conversely, if  $Y$  has mean  $a$  and standard deviation  $b$ , then  $X = (Y - a)/b$  has mean 0 and standard deviation 1. This is called sometimes the “Studentized” transformation.

**Example** (Normal distribution) Let  $X \sim N(0, 1)$ :

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty.$$

The transformation

$$Y = \mu + \sigma X, \quad X = \frac{Y - \mu}{\sigma}$$

yields

$$f_Y(y) = f_X\left(\frac{y - \mu}{\sigma}\right) \frac{1}{\sigma} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y - \mu)^2}{2\sigma^2}}.$$

More generally, a distribution is a member of the class of *location-scale* distributions if the distribution of a linear transformation of a random variable with that distribution has the same distribution, but with different parameters.

**Example** (Square root of an exponential RV) Suppose  $X \sim \exp(\lambda)$ , so that

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and consider the distribution of  $Y = \sqrt{X}$ . The transformation

$$y = g(x) = \sqrt{x}, \quad x \geq 0$$

is one-to-one and has an inverse  $x = y^2$  with  $dx/dy = 2y$ . Thus

$$f_Y(y) = f_X(y^2) 2y = 2\lambda y e^{-\lambda y^2}, \quad y \geq 0.$$

This distribution is a particular form of the Rayleigh distribution and is a special case of the Weibull distribution.

## Lecture 14: Sept 23

Last time

- Transformations of continuous random variables

Today

- Practice examples

**Example** A random variable  $X$  has a discrete uniform  $(1, N)$  distribution,  $X \sim Unif\{1, N\}$ , if

$$\Pr(X = x|N) = \frac{1}{N}, \quad x = 1, 2, \dots, N,$$

where  $N$  is a specified integer. This distribution puts equal mass on each of the outcomes  $1, 2, \dots, N$ . Question: what is the cdf of this r.v.?

*Solutions:*

$$F(x) = \Pr(X \leq x|N) = \begin{cases} 0, & x < 1 \\ \frac{1}{N}, & 1 \leq x < 2 \\ \frac{2}{N}, & 2 \leq x < 3 \\ \frac{3}{N}, & 3 \leq x < 4 \\ \vdots & \\ \frac{N-1}{N}, & N-1 \leq x < N \\ 1, & N \leq x \end{cases}$$

**Example** The continuous uniform distribution is defined by spreading mass uniformly over an interval  $[a, b]$ . A random variable  $X$  has a continuous uniform  $[a, b]$  distribution,  $X \sim Unif(a, b)$ , if its pdf is given by

$$f(x|a, b) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

Question: what is the cdf?

*Solutions:*

$$F(x) = \Pr(X \leq x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & b \leq x. \end{cases}$$

## Lecture 15: Sept 26

### Last time

- Presentation

### Today

- HW3 posted
- Midterm Exam 1 10/10, will have a practice exam
- Probability integral transformation
- Expectations (2.2)

**Theorem** (Probability integral transformation) Let  $X$  have continuous cdf  $F_X(x)$  and define the random variable  $Y$  as  $Y = F_X(X)$ . Then  $Y$  is uniformly distributed on  $(0, 1)$ , that is,  $\Pr(Y \leq y) = y, 0 < y < 1$ .

Before we prove this theorem, we will digress for a moment and look at  $F_X^{-1}$ , the inverse of the cdf  $F_X$ , in some detail. If  $F_X$  is strictly increasing, then  $F_X^{-1}$  is well defined by

$$F_X^{-1}(y) = x \iff F_X(x) = y.$$

However, if  $F_X$  is constant on some interval, then  $F_X^{-1}$  is not well defined as Figure 13.1 illustrates. Any  $x_1 \leq x \leq x_2$  satisfies  $F_X(x) = y$



Figure 13.1: Figure 2.1.2. (a)  $F_X(x)$  strictly increasing; (b)  $F_X(x)$  nondecreasing

This problem is avoided by defining  $F_X^{-1}$  for  $0 < y < 1$  by

$$F_X^{-1}(y) = \inf\{x : F_X(x) \geq y\}.$$

With this definition, for Figure 13.1(b), we have  $F_X^{-1}(y) = x_1$ .

*Proof:*

For  $Y = F_X(X)$ , we have, for  $0 < y < 1$ ,

$$\begin{aligned}\Pr(Y \leq y) &= \Pr(F_X(X) \leq y) \\ &= \Pr(F_X^{-1}[F_X(X)] \leq F_X^{-1}(y)) \quad (F_X^{-1} \text{ is increasing}) \\ &= \Pr(X \leq F_X^{-1}(y)) \\ &= F_X(F_X^{-1}(y)) \quad (\text{definition of } F_X) \\ &= y.\end{aligned}$$

One application of the probability integral transformation is in the generation of random samples from a particular distribution. If it is required to generate an observation  $X$  from a population with cdf  $F_X$ , we need only generate a uniform random number  $V$ , between 0 and 1, and solve for  $x$  in the equation  $F_X(x) = u$ .

## Expected Values

**Definition** The *expected value* or *mean* of a random variable  $g(X)$ , denoted by  $Eg(X)$ , is

$$Eg(X) = \begin{cases} \int_{-\infty}^{\infty} g(x)f(x)dx & \text{if } X \text{ is continuous} \\ \sum_{x \in \mathcal{X}} g(x) \Pr(X = x) & \text{if } X \text{ is discrete} \end{cases}$$

Provided the integral or summation exists.

If we let  $g(X) = X$ , then we get

$$EX = \begin{cases} \int_{-\infty}^{\infty} xf(x)dx & \text{if } X \text{ is continuous} \\ \sum_{x \in \mathcal{X}} x \Pr(X = x) & \text{if } X \text{ is discrete} \end{cases}$$

**Example** (Exponential mean) Suppose  $X$  has an *exponential* ( $\lambda$ ) *distribution*,  $X \sim \text{Exp}(\lambda)$ , that is, it has pdf given by

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}, \quad 0 \leq x < \infty, \lambda > 0.$$

Find out  $EX$ .

*Solution:*

$$\begin{aligned}
EX &= \int_0^{\infty} \frac{1}{\lambda} x e^{-x/\lambda} dx \\
&= -x e^{-x/\lambda} \Big|_0^{\infty} + \int_0^{\infty} e^{-x/\lambda} dx \\
&= \int_0^{\infty} e^{-x/\lambda} dx \\
&= \lambda
\end{aligned}$$

**Example** (Binomial mean) if  $X$  has a *binomial distribution*,  $X \sim \text{Binomial}(n, p)$ , its pmf is given by

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n,$$

where  $n$  is a positive integer,  $0 \leq p \leq 1$ , and for every fixed pair  $n$  and  $p$  the pmf sums to 1. Find out  $EX$ .

*Solution:*

$$EX = \sum_{x=0}^n x \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^n x \binom{n}{x} p^x (1-p)^{n-x}.$$

Using the identity  $x \binom{n}{x} = n \binom{n-1}{x-1}$ , we have

$$\begin{aligned}
EX &= \sum_{x=1}^n n \binom{n-1}{x-1} p^x (1-p)^{n-x} \\
&= \sum_{y=0}^{n-1} n \binom{n-1}{y} p^{y+1} (1-p)^{n-(y+1)} \\
&= np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{n-1-y} \\
&= np,
\end{aligned}$$

since the last summation must be 1, being the sum over all possible values of a binomial( $n-1, p$ ) pmf.

## Lecture 16: Sept 28

### Last time

- HW3 posted
- Midterm Exam 1 10/10, will have a practice exam
- Probability integral transformation
- Expectations (2.2)

### Today

- Exam 1 covers up to next Monday's lecture
- Expectations (2.2)
- Moments and moment generating function

### Expectation

The process of taking expectations is a linear operation, which means that the expectation of a linear function of  $X$  can be easily evaluated by noting that for any constants  $a$  and  $b$ , such that

$$E(aX + b) = aEX + b$$

**Theorem** Let  $X$  be a random variable and let  $a$ ,  $b$ , and  $c$  be constants. Then for any functions  $g_1(x)$  and  $g_2(x)$  whose expectations exist,

1.  $E(ag_1(X) + bg_2(X) + c) = aEg_1(X) + bEg_2(X) + c$ .
2. If  $g_1(x) \geq 0$  for all  $x$ , then  $Eg_1(X) \geq 0$ .
3. If  $g_1(x) \geq g_2(x)$  for all  $x$ , then  $Eg_1(X) \geq Eg_2(X)$ .
4. If  $a \leq g_1(x) \leq b$  for all  $x$ , then  $a \leq Eg_1(X) \leq b$ .

*Proof:*

We will give details for only the continuous case, the discrete case being similar. By definition

$$\begin{aligned} E(ag_1(X) + bg_2(X) + c) &= \int_{-\infty}^{\infty} [ag_1(x) + bg_2(x) + c] f_X(x) dx \\ &= \int_{-\infty}^{\infty} ag_1(x) f_X(x) dx + \int_{-\infty}^{\infty} bg_2(x) f_X(x) dx + \int_{-\infty}^{\infty} cf_X(x) dx \\ &= aEg_1(X) + bEg_2(X) + c \end{aligned}$$

The other three properties are proved in a similar manner (shown in class).

**Example** (Method of indicators) An example of how the above properties are useful. Let  $X \sim \text{Binomial}(n, p)$  for  $n$  positive integer and  $0 \leq p \leq 1$  ( $n$  is the number of independent identical binary trials and  $p$  is the probability of success). We can write

$$X = \sum_{i=1}^n I_i$$

where  $I_i$  is the indicator that  $i^{\text{th}}$  trial is a success (i.e.  $I_i \stackrel{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$ ). We have

$$EI_i = 1 \cdot p + 0 \cdot (1 - p) = p.$$

Therefore,

$$EX = \sum_{i=1}^n EI_i = \sum_{i=1}^n p = np.$$

**Theorem** For a non-negative random variable  $X$  (i.e.  $f(x) = 0$  for  $x < 0$ ).

$$EX = \begin{cases} \int_0^\infty (1 - F(x))dx, & X \text{ continuous} \\ \sum_{x=0}^\infty (1 - F(x)), & X \text{ discrete} \end{cases}$$

*Proof:*

We prove the continuous case first,

$$\begin{aligned} \int_0^\infty [1 - F(x)] dx &= \int_0^\infty [1 - \Pr(X \leq x)] dx \\ &= \int_0^\infty \Pr(X > x) dx \\ &= \int_0^\infty \int_{y=x}^\infty f_X(y) dy dx \\ &= \int_0^\infty \int_{x=0}^y f_X(y) dx dy \\ &= \int_0^\infty y f_X(y) dy \\ &= EX. \end{aligned}$$



Then, for discrete case, we have

$$\begin{aligned}
\sum_{x=0}^{\infty} (1 - F(x)) &= \sum_{x=0}^{\infty} \Pr(X > x) \\
&= \sum_{x=0}^{\infty} \sum_{y=x+1}^{\infty} \Pr(X = y) \\
&= \sum_{y=1}^{\infty} \sum_{x=0}^{y-1} \Pr(X = y) \\
&= \sum_{y=1}^{\infty} y \Pr(X = y) \\
&= EX
\end{aligned}$$

## Moments

**Example** (Minimizing distance) The expected value of a random variable has another property, one that we can think of as relating to the interpretation of  $EX$  as a good guess at a value of  $X$ .

Suppose we measure the distance between a random variable  $X$  and a constant  $b$  by  $(X - b)^2$ . The closer  $b$  is to  $X$ , the smaller this quantity is. We can now determine the value of  $b$  that minimizes  $E[(X - b)^2]$  and, hence, will provide us with a good predictor of  $X$ . (Note that it does no good to look for a value of  $b$  that minimizes  $(X - b)^2$ , since the answer would depend on  $X$ , making it a useless predictor of  $X$ .)

We could proceed with the minimization of  $E(X - b)^2$  by using calculus, but there is a simpler method:

$$\begin{aligned}
E(X - b)^2 &= E(X - EX + EX - b)^2 \\
&= E[(X - EX) + (EX - b)]^2 \\
&= E(X - EX)^2 + (EX - b)^2 + 2E[(X - EX)(EX - b)],
\end{aligned}$$

where we have expanded the square. Note that  $E[(X - EX)(EX - b)] = (EX - b)E(X - EX) = 0$ , since  $EX - b$  is constant and comes out of the expectation,  $E(X - EX) = EX - EX = 0$ . This means

$$E(X - b)^2 = E(X - EX)^2 + (EX - b)^2.$$

Such that  $E(X - b)^2$  is minimized at  $b = EX$ . And  $E(X - EX)^2$  is actually the variance of  $X$  ( $Var X = E(X - EX)^2$ ).

The various moments of a distribution are an important class of expectations.

**Definition** For each integer  $n$ , the  $n$ th *moment* of  $X$  (or  $F_X(x)$ ),  $\mu'_n$ , is

$$\mu'_n = EX^n.$$

The  $n$ th *central moment* of  $X$ ,  $\mu_n$ , is

$$\mu_n = E(X - \mu)^n,$$

where  $\mu = \mu'_1 = EX$ .

Notes:

- $\mu'_0 = EX^0 = 1$
- $\mu'_1$  is the *mean*, usually denoted by  $\mu$ .
- $\mu_0 = E(X - \mu)^0 = 1$
- $\mu_1 = 0$
- $\mu_2 = E(X - EX)^2$  is the *variance*
- $\mu_3 = E(X - EX)^3$  is related to the *skewness*.
- $\mu_4 = E(X - EX)^4$  is related to the *kurtosis*.

**Definition** The *variance* of a random variable  $X$  is its second central moment,  $\text{Var}(X) = E[(X - EX)^2]$ . The positive square root of  $\text{Var}(X)$  is the *standard deviation* of  $X$ .

The variance gives a measure of the degree of spread of a distribution around its mean. Figure 18.3 shows a plot of two samples, one sample draws 100 numbers from a normal distribution with mean 0 and variance 1,  $N(0, 1)$ . The other sample draws 100 numbers from a normal distribution with mean 0 and variance 100,  $N(0, 100)$ .



Figure 16.2: Figure 2.1.2. Two samples of 100 numbers drawn from  $N(0, 1)$  and  $N(0, 100)$ .

**Example** (Exponential variance) Let  $X$  have the exponential( $\lambda$ ) distribution. We can calculate the variance of  $X$  now.

*Solution:*

$$\begin{aligned}\text{Var}(X) &= E(X - \lambda)^2 \\ &= \int_0^\infty (x - \lambda)^2 \frac{1}{\lambda} e^{-x/\lambda} dx\end{aligned}$$

$$\begin{aligned}\text{Var}(X) &= \int_0^\infty (x^2 - 2x\lambda + \lambda^2) \frac{1}{\lambda} e^{-x/\lambda} dx \\ &= \int_0^\infty x^2 \frac{1}{\lambda} e^{-x/\lambda} dx - 2 \int_0^\infty x\lambda \frac{1}{\lambda} e^{-x/\lambda} dx + \lambda^2 \\ &= EX^2 - \lambda^2 \\ &= \lambda^2\end{aligned}$$

**Theorem** If  $X$  is a random variable with finite variance, then for any constants  $a$  and  $b$ ,

$$\text{Var}(aX + b) = a^2 \text{Var}(X).$$

*Proof:*

From the definition, we have

$$\begin{aligned}\text{Var}(aX + b) &= E[(aX + b) - E(aX + b)]^2 \\ &= E(aX - aEX)^2 \\ &= a^2 E(X - EX)^2 \\ &= a^2 \text{Var}(X).\end{aligned}$$

It is sometimes to use an alternative formula for the variance, given by

$$\text{Var}(X) = E(X^2) - (EX)^2,$$

which is easily established by

$$\begin{aligned}\text{Var}(X) &= E(X - EX)^2 = E[X^2 - 2XEX + (EX)^2] \\ &= EX^2 - 2(EX)^2 + (EX)^2 \\ &= EX^2 - (EX)^2.\end{aligned}$$

**Example** (Binomial variance) Let  $X \sim \text{Binomial}(n, p)$ , that is ,

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}.$$

What is the variance of  $X$ ?

*Solutions:*

Method #1:

We want to find  $EX^2$  first. We use the

$$EX^2 = \sum_{x=0}^n x^2 \binom{n}{x} p^x (1-p)^{n-x}.$$

we use the same property  $x^2 \binom{n}{x} = xn \binom{n-1}{x-1}$ . We then have

$$\begin{aligned} EX^2 &= n \sum_{x=1}^n x \binom{n-1}{x-1} p^x (1-p)^{n-x} \\ &= n \sum_{y=0}^{n-1} (y+1) \binom{n-1}{y} p^{y+1} (1-p)^{n-1-y} \\ &= np \sum_{y=0}^{n-1} y \binom{n-1}{y} p^y (1-p)^{n-1-y} + np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{n-1-y} \\ &= np \cdot (n-1)p + np \\ &= n(n-1)p^2 + np. \end{aligned}$$

And now

$$\begin{aligned} \text{Var}(X) &= EX^2 - (EX)^2 \\ &= n(n-1)p^2 + np - (np)^2 \\ &= np - np^2 \\ &= np(1-p). \end{aligned}$$

Method #2:

Recall that we could write  $X = \sum_{i=1}^n I_i$ , where  $I_i \stackrel{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$ . Then

$$\begin{aligned} \text{Var}(X) &= \text{Var}\left(\sum_{i=1}^n I_i\right) \\ &= \sum_{i=1}^n \text{Var}(I_i) \quad (I_i\text{'s are independent}) \\ &= n \text{Var}(I_1) \quad (I_i\text{'s are identically distributed}) \\ &= n [E(I_1^2) - (EI_1)^2] \\ &= n [p - p^2] \\ &= np(1-p). \end{aligned}$$

## Lecture 17: Sept 30

### Last time

- Exam 1 covers up to next Monday's lecture
- Expectations (2.2)
- Moments and moment generation function

### Today

- Practice examples

**Example** A random variable  $X$  has a discrete uniform  $(1, N)$  distribution,  $X \sim Unif\{1, N\}$ , if

$$\Pr(X = x|N) = \frac{1}{N}, \quad x = 1, 2, \dots, N,$$

where  $N$  is a specified integer. This distribution puts equal mass on each of the outcomes  $1, 2, \dots, N$ . Question: what is the cdf of this r.v.?

*Solutions:*

$$F(x) = \Pr(X \leq x|N) = \begin{cases} 0, & x < 1 \\ \frac{1}{N}, & 1 \leq x < 2 \\ \frac{2}{N}, & 2 \leq x < 3 \\ \frac{3}{N}, & 3 \leq x < 4 \\ \vdots \\ \frac{N-1}{N}, & N-1 \leq x < N \\ 1, & N \leq x \end{cases}$$

**Example** The continuous uniform distribution is defined by spreading mass uniformly over an interval  $[a, b]$ . A random variable  $X$  has a continuous uniform  $[a, b]$  distribution,  $X \sim Unif(a, b)$ , if its pdf is given by

$$f(x|a, b) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

Question: what is the cdf?

*Solutions:*

$$F(x) = \Pr(X \leq x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & b \leq x. \end{cases}$$

## Lecture 18: Oct 3

Last time

- Practice examples

Today

- Moments and moment generating function

Moments

**Example** (Minimizing distance) The expected value of a random variable has another property, one that we can think of as relating to the interpretation of  $EX$  as a good guess at a value of  $X$ .

Suppose we measure the distance between a random variable  $X$  and a constant  $b$  by  $(X - b)^2$ . The closer  $b$  is to  $X$ , the smaller this quantity is. We can now determine the value of  $b$  that minimizes  $E[(X - b)^2]$  and, hence, will provide us with a good predictor of  $X$ . (Note that it does no good to look for a value of  $b$  that minimizes  $(X - b)^2$ , since the answer would depend on  $X$ , making it a useless predictor of  $X$ .)

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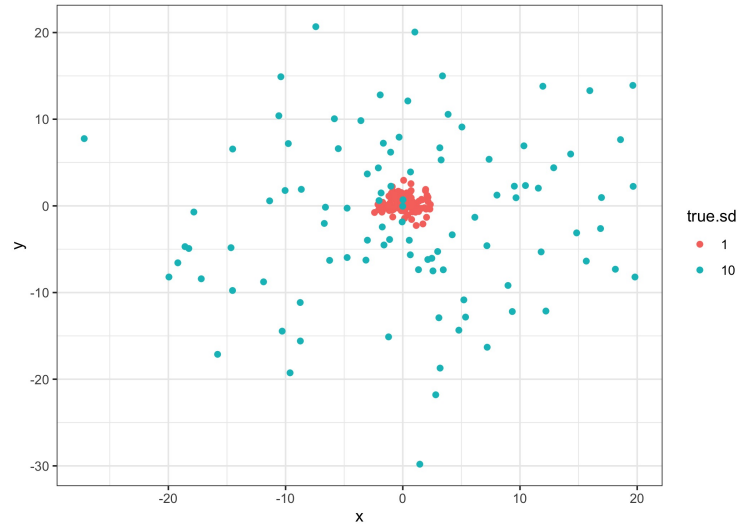


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$$\begin{aligned}
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&= \int_0^\infty x^2 \frac{1}{\lambda} e^{-x/\lambda} dx - 2 \int_0^\infty x\lambda \frac{1}{\lambda} e^{-x/\lambda} dx + \lambda^2 \\
&= EX^2 - \lambda^2 \\
&= \lambda^2
\end{aligned}$$

**Theorem** If  $X$  is a random variable with finite variance, then for any constants  $a$  and  $b$ ,

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*Proof:*

From the definition, we have

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&= E(aX - aEX)^2 \\
&= a^2 E(X - EX)^2 \\
&= a^2 \text{Var}(X).
\end{aligned}$$

It is sometimes to use an alternative formula for the variance, given by

$$\text{Var}(X) = E(X^2) - (EX)^2,$$

which is easily established by

$$\begin{aligned}
\text{Var}(X) &= E(X - EX)^2 = E[X^2 - 2XEX + (EX)^2] \\
&= EX^2 - 2(EX)^2 + (EX)^2 \\
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$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}.$$

What is the variance of  $X$ ?

*Solutions:*

Method #1:

We want to find  $EX^2$  first. We use the

$$EX^2 = \sum_{x=0}^n x^2 \binom{n}{x} p^x (1 - p)^{n-x}.$$



we use the same property  $x^2 \binom{n}{x} = xn \binom{n-1}{x-1}$ . We then have

$$\begin{aligned}
EX^2 &= n \sum_{x=1}^n x \binom{n-1}{x-1} p^x (1-p)^{n-x} \\
&= n \sum_{y=0}^{n-1} (y+1) \binom{n-1}{y} p^{y+1} (1-p)^{n-1-y} \\
&= np \sum_{y=0}^{n-1} y \binom{n-1}{y} p^y (1-p)^{n-1-y} + np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y (1-p)^{n-1-y} \\
&= np \cdot (n-1)p + np \\
&= n(n-1)p^2 + np.
\end{aligned}$$

And now

$$\begin{aligned}
\text{Var}(X) &= EX^2 - (EX)^2 \\
&= n(n-1)p^2 + np - (np)^2 \\
&= np - np^2 \\
&= np(1-p).
\end{aligned}$$

Method #2:

Recall that we could write  $X = \sum_{i=1}^n I_i$ , where  $I_i \stackrel{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$ . Then

$$\begin{aligned}
\text{Var}(X) &= \text{Var}\left(\sum_{i=1}^n I_i\right) \\
&= \sum_{i=1}^n \text{Var}(I_i) \quad (I_i\text{'s are independent}) \\
&= n \text{Var}(I_1) \quad (I_i\text{'s are identically distributed}) \\
&= n [E(I_1^2) - (EI_1)^2] \\
&= n [p - p^2] \\
&= np(1-p).
\end{aligned}$$

**Definition** Let  $X$  be a random variable with cdf  $F_X$ . The *moment generating function (mgf)* of  $X$  (or  $F_X$ ), denoted by  $M_X(t)$ , is

$$M_X(t) = Ee^{tX},$$

provided that the expectation exists for  $t$  in some neighborhood of 0. That is, there is an  $h > 0$  such that, for all  $t$  in  $-h < t < h$ ,  $Ee^{tX}$  exists. If the expectation does not exist in a neighborhood of 0, we say that the moment generating function does not exist.

More explicitly, we can write the mgf of  $X$  as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx, \quad \text{if } X \text{ is continuous,}$$

or

$$M_X(t) = \sum_x e^{tx} \Pr(X = x), \quad \text{if } X \text{ is discrete.}$$

It is easy to see how the mgf generates moments as in the following theorem.

**Theorem** If  $X$  has mgf  $M_X(t)$ , then

$$EX^n = M_X^{(n)}(0),$$

where we define

$$M_X^{(0)} = \frac{d^n}{dt^n} M_X(t) \Big|_{t=0}.$$

That is, the  $n^{th}$  moment is equal to the  $n^{th}$  derivative of  $M_X(t)$  evaluated at  $t = 0$ .

*Proof:*

$$\begin{aligned} \frac{d}{dt} M_X(t) &= \frac{d}{dt} \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \\ &= \int_{-\infty}^{\infty} \left( \frac{d}{dt} e^{tx} \right) f_X(x) dx \\ &= \int_{-\infty}^{\infty} (x e^{tx}) f_X(x) dx \\ &= E(X e^{tX}). \end{aligned}$$

Therefore,

$$\frac{d}{dt} M_X(t) \Big|_{t=0} = E(X e^{tX}) \Big|_{t=0} = EX.$$

Proceeding in an analogous manner, we can establish that

$$\frac{d^n}{dt^n} M_X(t) \Big|_{t=0} = E(X^n e^{tX}) \Big|_{t=0} = EX^n.$$