Lecture 10: Sept 14

Last time

- Random variables
- Distribution Functions
- Types of Random Variables

Today

- Continuous Random Variables
- Counting Techniques

Continuous Random Variables

Definition A random variable X is *continuous* if $F_X(x)$ is a continuous function of x.

Definition A random variable X is *absolutely continuous* if $F_X(x)$ is an absolutely continuous function of x.

Definition A function $F(x)$ is *absolutely continuous* if it can be written

$$F(x) = \int_{-\infty}^{x} f(x)dx.$$

Absolute continuity is stronger than continuity but weaker than differentiability. An example of an absolutely continuous function is one that is:

- continuous everywhere
- differentiable everywhere, except possibly for a countable number of points.

Definition The *probability density function* or pdf, $f_X(x)$, of a continuous random variable X is the function that satisfies

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt \quad \text{for all } x.$$

Notation: We write $X \sim F_X(x)$ for the expression "X has a distribution given by $F_X(x)$" where we read the symbol "$\sim$" as "is distributed as". Similarly, we can write $X \sim f_X(x)$ or , if X and Y have the same distribution, $X \sim Y$.

Theorem A function $f_X(x)$ is a pdf (or pmf) of a random variable X if and only if

1. $f_X(x) \geqslant 0$ for all x.
2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$ (pdf) or $\sum_x f_X(x) = 1$ (pmf).

Example Suppose $F(x) = 1 - e^{-\lambda x}$ for $x > 0$ and $F(x) = 0$ otherwise. Is $F(x)$ a cdf? What is the associated pdf?
solution:

Notes

- If X is a continuous random variable, then $f(x)$ is not the probability that $X = x$. In fact, if X is an absolutely continuous random variable with density function $f(x)$, then $\Pr(X = x) = 0$. (Why?)
  proof

- Because $\Pr(X = a) = 0$, all the following are equivalent:

$$\Pr(a \leqslant X \leqslant b), \quad \Pr(a \leqslant X < b) \quad , \quad \Pr(a < X \leqslant b) \quad \text{and} \quad \Pr(a < X < b)$$

- $f(x)$ can exceed one!

Counting Techniques

These sections are from 2.4 of "Introduction to Probability Theory and Statistical Inference" by Harold J. Larson. We employ them to discuss combinatorics.

When the equally likely assumption is made for a finite sample space, the probability of occurrence of any event A is given by the ratio of the number of elements belonging to A to the number of elements belonging to S. For such cases it is useful to be able to count the number of elements belonging to given sets.

A very simple technique that is frequently useful in counting problems is called the *multiplication principle*.

Definition If a first operation can be performed in any of n_1 ways and a second operation can then be performed in any of n_2 ways, both operations can be performed (the second immediately following the first) in $n_1 \cdot n_2$ ways.

Example If we can travel from town A to town B in 3 ways and from town B to town C in 4 ways, then we can travel from A to C via B in a total of $3 \cdot 4 = 12$ ways.

Example If the operation of tossing a die gives rise to 1 of 6 possible outcomes and the operation of tossing a second die gives rise to 1 of 6 possible outcomes, then the operation of tossing a pair of dice gives rise to $6 \cdot 6 = 36$ possible outcomes.

Definition An arrangement of n symbols in a definite order is called a *permutation* of n symbols.

Example Let's consider all possible n-tuples made by n different symbols. In listing all the possible n-tuples, we would perform n natural operations. First we must fill the leftmost position of n-tuples, we have all n symbols to choose from. Then we must fill the second leftmost position, where we have $n-1$ symbols to choose from. Then, the third position with $n-2$ symbols to choose from, and so on. Finally, when we reach the right most position, we have 1 symbol left.

Using the multiplication rule, the total number of ways we can perform all n operations will be

$$n! = n(n-1)(n-2)\cdots 2\cdot 1,$$

where we write $n!$ (read n-factorial) and we define $0! = 1$.

Example Suppose the same 5 people park their cars on the same side of the street in the same block every night. How many different ordering of the 5 cars parked on the street are possible?
Solution:

Example Suppose the same 5 people park their cars on the two sides of the street in the same block every night where one side has 3 slots and the other side has 2. How many different ordering of 3 cars out of 5 can be parked on the 3-slot side?
Solution:

Definition The number of r-tuples we can make $r \leqslant n$, using n different symbols (each only once), is called the *number of permutations of n things r at a time* and is denoted by ${}^nP_r$ which is calculated as

$${}^nP_r = n(n-1)\cdots(n-r+1).$$

Example Fifteen cars enter a race. In how many different ways could trophies for first, second, and third place be awarded?
Solutions:

Example How many of the 3-tuples just counted have car number 15 in the first position?
Solutions:

Definition The number of distinct subsets, each of size r, that can be constructed from a set with n elements is called the number of *combinations* of n things r at a time: this number is represented by $\binom{n}{r}$ which reads n choose r.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Example How many distinct 5-card hands can be dealt from a standard 52-card deck?

$$\binom{5}{2}5 = \frac{52!}{5!47!} = 2,598,960.$$

Theorem If x and y are any two real numbers and n is a positive integer, then

$$(x+y)^n = \sum_{i=1}^{n} \binom{n}{i} x^i y^{n-i}, \qquad \text{where } \binom{n}{i} = \frac{n!}{(n-i)!i!}.$$