Lecture 33: Nov. 30

Last time

- MGF

Today

- Course evaluations (6/38)
- Final exam format
  - Final exam will be take home
  - Open book, open note, not open internet
  - Final exam will be released on Friday (12/09/2022) right after class
  - Final exam due 23:59 pm on Friday 12/16/2022.
  - Scan and submit your exam via email with a single pdf file
  - Send your email to both your instructor and your TA.
  - Submitted exams should be human-readable to receive non-zero scores.
- MGF cont.
- Covariance and Correlation

Moment generating function (Theorem 4.2.12) Let X and Y be independent rvs with mgfs $M_X(\cdot)$ and $M_Y(\cdot)$, respectively. Then the mgf of $Z = X + Y$ is

$$M_Z(t) = M_X(t)M_Y(t)$$

Proof:

$$\begin{aligned} M_Z(t) &= E\exp(Zt) &&= E\{\exp\left[(X+Y)t\right]\} \\ &= E[\exp(Xt)\exp(Yt)] &&= E[\exp(Xt)]\cdot E[\exp(Yt)] \\ &= M_X(t)M_Y(t) \end{aligned}$$

Corollary: If X and Y are independent and $Z = X - Y$,

$$M_Z(t) = M_X(t)M_Y(-t)$$

Example (sum of two independent Poissons) Suppose $X \sim Poisson(\lambda_X)$ and $Y \sim Poisson(\lambda_Y)$ and put $Z = X + Y$. Then, $Z \sim Poisson(\lambda_X + \lambda_Y)$. *Proof:*

$$\begin{aligned} M_Z(t) &= \exp\left[\lambda_X(e^t - 1)\right]\exp\left[\lambda_Y(e^t - 1)\right] \\ &= \exp\left[(\lambda_X + \lambda_Y)(e^t - 1)\right] \end{aligned}$$

Example (sum of two independent normals) Suppose $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$ and X and Y are independent and $Z = X + Y$. Then

$$Z \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$$

Proof:

$$\begin{aligned} M_Z(t) &= \exp\left(\mu_x t + \frac{1}{2}\sigma_x^2 t^2\right) \exp\left(\mu_y t + \frac{1}{2}\sigma_y^2 t^2\right) \\ &= \exp\left[(\mu_x + \mu_y)t + \frac{1}{2}(\sigma_x^2 + \sigma_y^2)t^2\right] \end{aligned}$$

Example (sum of two independent gammas) Suppose $X \sim \Gamma(\alpha_x, \beta)$ and independently $Y \sim \Gamma(\alpha_y, \beta)$. Let $Z = X + Y$. Then $Z \sim \Gamma((\alpha_x + \alpha_y), \beta)$.
Proof:

$$\begin{aligned} M_Z(t) &= \left(\frac{1}{1-\beta t}\right)^{\alpha_x} \left(\frac{1}{1-\beta t}\right)^{\alpha_y} \\ &= \left(\frac{1}{1-\beta t}\right)^{\alpha_x + \alpha_y} \end{aligned}$$

Remember that

- If $\alpha = 1$ we have an exponential with parameter β.
- If $\alpha = n/2$ and $\beta = 2$, we have a $\chi^2(n)$ (with n d.f.). The above result states that $\chi^2(n_1) + \chi^2(n_2) = \chi^2(n_1 + n_2)$.

Covariance and Correlation Let X and Y be two random variables with respective means μ_X, μ_Y and variances $\sigma_X^2 > 0$ and $\sigma_Y^2 > 0$, all assumed to exist.

- The *covariance* of X and Y is

$$Cov(X, Y) = E\left[(X - \mu_X)(Y - \mu_Y)\right] = \sigma_{XY}$$

- The *correlation* between X and Y is

$$Cor(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

also written as

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)\left(\frac{Y - \mu_Y}{\sigma_Y}\right)\right]$$

Properties Let c be a constant:

1. $Cov(X,X) = Var(X)$, $\quad Cor(X,X) = 1$
2. $Cov(X,Y) = Cov(Y,X)$, $\quad Cor(X,Y) = Cor(Y,X)$
3. $Cov(X,c) = 0$, $\quad Cor(X,c) = 0$
4. $Cov(X,Y) = E(XY) - E(X)E(Y)$

5. Let $X_c = X - \mu_X$, $Y_c = Y - \mu_Y$. Then

$$Cov(X,Y) = Cov(X_c, Y_c) = E(X_c Y_c)$$
$$Cor(X,Y) = Cor(X_c, Y_c)$$

6. Let $\tilde{X} = (X - \mu_X)/\sigma_X$, $\tilde{Y} = (Y - \mu_Y)/\sigma_Y$. Then,

$$Cor(X,Y) = Cor(\tilde{X}, \tilde{Y}) = Cov(\tilde{X}, \tilde{Y}) = E(\tilde{X}\tilde{Y})$$

Independent vs. Uncorrelated

- X and Y are called *uncorrelated* iff

$$Cov(X,Y) = 0 \quad \text{or equivalently} \quad \rho_{XY} = 0$$

- If X and Y are independent and $Cov(X,Y)$ exists, then $Cov(X,Y) = 0$.
- If X and Y are uncorrelated, this does **not** imply that they are independent.

Example $X \sim U[-1,1]$, $Y = X^2$. Then $Cov(X,Y) = 0$ but X, Y are not independent.

Correlation coefficient For any random variables X and Y,

1. $-1 \leqslant \rho_{XY} \leqslant 1$
2. $|\rho_{XY}| = 1$ if and only if $\exists a \neq 0$ and b such that

$$\Pr(Y = aX + b) = 1.$$

if $\rho_{XY} = 1$ then $a > 0$, and if $\rho_{XY} = -1$, then $a < 0$.

proof:
Let $\tilde{X} = (X - \mu_X)/\sigma_X$, $\tilde{Y} = (Y - \mu_Y)/\sigma_Y$. Then $Cor(X,Y) = E(\tilde{X}\tilde{Y})$,

1.

$$0 \leqslant E(\tilde{X} - \tilde{Y})^2 = 1 + 1 - 2E(\tilde{X}\tilde{Y}) \quad \Rightarrow \quad E(\tilde{X}\tilde{Y}) \leqslant 1$$
$$0 \leqslant E(\tilde{X} + \tilde{Y})^2 = 1 + 1 + 2E(\tilde{X}\tilde{Y}) \quad \Rightarrow \quad -1 \leqslant E(\tilde{X}\tilde{Y})$$

2.

$$\rho_{XY} = 1 \iff \Pr(\tilde{Y} = \tilde{X}) = 1 \quad \Rightarrow \quad a > 0$$
$$\rho_{XY} = -1 \iff \Pr(\tilde{Y} = -\tilde{X}) = 1 \quad \Rightarrow \quad a < 0$$

Random Samples

Definition The random variables $X_1, \ldots, X_n$ are called a *random sample of size n from the population $f(x)$* if $X_1, \ldots, X_n$ are mutually independent and identically distributed (iid) random variables with the same pdf or pmf $f(x)$.

If $X_1, \ldots, X_n$ are iid, then their joint pdf or pmf is

$$f(x_1, \ldots, x_n) = f(x_1)f(x_2)\ldots f(x_n) = \prod_{j=1}^{n} f(x_j)$$

Statistics Let $X_1, \ldots, X_n$ be a random sample and let $T(x_1, \ldots, x_n)$ be a function defined on $\mathbb{R}^n$. Then the random variable $Y = T(X_1, \ldots, X_n)$ is called a *statistic*. The probability distribution of Y is called the *sampling distribution* of Y.

Note: T is only a function of $(x_1, \ldots, x_n)$, no parameters.

Examples

$$\begin{aligned} \text{sample mean} \quad & \bar{X} = \tfrac{1}{n} \textstyle\sum_{j=1}^{n} X_j \\ \text{sample variance} \quad & S^2 = \tfrac{1}{n-1} \textstyle\sum_{j=1}^{n} (X_j - \bar{X})^2 \\ \text{sample standard deviation} \quad & S = \sqrt{S^2} \\ \text{minimum} \quad & X_{(1)} = \min_{1 \leqslant i \leqslant n} X_i \end{aligned}$$

Properties Let $x_1, \ldots, x_n$ be n numbers and define

$$\bar{x} = \frac{1}{n} \sum_{j=1}^{n} x_j, \quad s^2 = \frac{1}{n-1} \sum_{j=1}^{n} (x_j - \bar{x})^2$$

Then

$$\begin{aligned} \min_a \sum_{j=1}^{n} (x_j - a)^2 &= \sum_{j=1}^{n} (x_j - \bar{x})^2 \\ (n-1)s^2 &= \sum_{j=1}^{n} (x_j - \bar{x})^2 = \sum_{j=1}^{n} x_j^2 - n\bar{x}^2 \end{aligned}$$

Residuals Lemma: Let $X_1, \ldots, X_n$ be a random sample from a population with mean μ and variance σ^2. Define the residuals $R_i = X_i - \bar{X}$. Then

$$\begin{aligned} E(R_i) = 0, \quad & Var(R_i) = \tfrac{n-1}{n}\sigma^2 \\ Cov(R_i, \bar{X}) = 0, \quad & Cov(R_i, R_j) = -\sigma^2/n \text{ if } i \neq j \end{aligned}$$

Theorem Let $X_1, \ldots, X_n$ be a random sample from a population with mgf $M_X(t)$. Then the mgf of the sample mean is

$$M_{\bar{X}}(t) = [M_X(t/n)]^n$$

Convergence

Convergence in Probability A sequence of random variables $X_1, \ldots, X_n$ *converges in probability* to a random variable X, denoted

$$X_n \xrightarrow{p} X$$

if for every $\epsilon > 0$,

$$\lim_{n \to \infty} \Pr(|X_n - X| < \epsilon) = 1$$

or equivalently

$$\lim_{n \to \infty} \Pr(|X_n - X| > \epsilon) = 0$$

In other words, X_n is more and more likely to be close to X, or less and less likely to be far from X.

Example Let $X_n = X + \epsilon_n$, where $\epsilon_n \sim N(0, 1/n)$ and X is an arbitrary random variable. Then, as $n \to \infty$,

$$X_n \xrightarrow{p} X$$

Weak law of large numbers (WLLN) Let $Y_1, \ldots, Y_n$ be iid with common mean μ and variance σ^2. Then, as $n \to \infty$,

$$\bar{Y}_n = \frac{1}{n} \sum_{j=1} Y_j \xrightarrow{p} \mu$$

Proof:
The proof is quite simple, being a straightforward application of Chebychev's Inequality. We have, for every $\epsilon > 0$,

$$\Pr(|\bar{Y}_n - \mu| \geqslant \epsilon) = \Pr(|\bar{Y}_n - \mu|^2 \geqslant \epsilon^2) \leqslant \frac{E(\bar{X} - \mu)^2}{\epsilon^2} = \frac{Var(\bar{X})}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \to 0 \text{ as } n \to \infty$$

Convergence in Distribution A sequence of random variables $X_1, \ldots, X_n$ *converges in distribution* to a random variable X, denoted

$$X_N \xrightarrow{d} X$$

if

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$

This is also called *convergence in law* or *weak convergence*. In other words, the distribution of X_n is closer and closer to the distribution of X.

Relation between "in distribution" and "in probability" Theorem:

1. Convergence in probability implies convergence in distribution:

$$X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{d} X$$

2. Suppose $X_n \xrightarrow{d} X$ where X has a degenerate distribution, i.e. $\Pr\{X = a\} = 1$ for some $a \in \mathbb{R}$. Then,

$$X_n \xrightarrow{d} a \Rightarrow X_n \xrightarrow{p} a$$

Convergence in Distribution via Convergence of Mgfs Theorem: Suppose the mgf $M_n(t)$ of Y_n exists for $|t| < h$, and the mgf $M(t)$ of Y exists for $|t| < h_1 < h$. Then,

$$Y_n \xrightarrow{d} Y \iff \lim_{n\to\infty} M_n(t) = M(t), \quad |t| < h_1$$

Example Let $X_\lambda \sim Poisson(\lambda)$. Then, as $\lambda \to \infty$,

$$\frac{X_\lambda - \lambda}{\lambda} \xrightarrow{p} 0$$
$$\frac{X_\lambda - \lambda}{\sqrt{\lambda}} \xrightarrow{d} N(0,1)$$

Central Limit Theorem Let $X_1, X_2, \ldots, X_n$ be a sequence of iid random variables whose mgfs exist in a neighborhood of 0 (that is, $M_{X_i}(t)$ exists for $|t| < h$, for some positive $h > 0$). Let $EX_i = \mu$ and $Var(X_i) = \sigma^2 > 0$. (Both μ and σ^2 are finite since the mgf exists) Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Let $G_n(x)$ denote the cdf of $\sqrt{n}(\bar{X}_n - \mu)/\sigma$. Then, for any x, $-\infty < x < \infty$,

$$\lim_{n\to\infty} G_n(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy;$$

that is, $\sqrt{n}(\bar{X}_n - \mu)/\sigma$ has a limiting standard normal distribution, in other words, $\sqrt{n}(\bar{X}_n - \mu)/\sigma \xrightarrow{d} N(0,1)$

Proof:
Define $Y_i = (X_i - \mu)/\sigma$, and let $M_Y(t)$ denote the common mgf of Y_is, which exists for $|t| < \sigma h$ and $M_Y(t) = M_{\frac{1}{\sigma}X_i - \mu/\sigma}(t) = e^{-\frac{\mu}{\sigma}t} M_X(\frac{t}{\sigma})$. Since

$$\frac{\sqrt{n}(\bar{X}_n)}{\sigma} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Y_i,$$

we have,

$$\begin{aligned} M_{\sqrt{n}(\bar{X}_n - \mu)/\sigma}(t) &= M_{\sum_{i=1}^{n} Y_i/\sqrt{n}}(t) \\ &= M_{\sum_{i=1}^{n} Y_i}(t/\sqrt{n}) \\ &= \left[M_Y(t/\sqrt{n})\right]^n. \end{aligned}$$

We now expand $M_Y(t/\sqrt{n})$ in a Taylor series (power series) around 0.

$$M_Y(\frac{t}{\sqrt{n}}) = \sum_{k=0}^{\infty} M_Y^{(k)}(0)\frac{(t/\sqrt{n})^k}{k!},$$

where $M_Y^{(k)}(0) = (d^k/dt^k)M_Y(t)|_{t=0}$. Since the mgfs exist for $|t| < h$, the power series expansion is valid if $t < \sqrt{n}\sigma h$.

Using the facts that $M_Y^{(0)} = 1$, $M_Y^{(1)} = 0$, and $M_Y^{(2)} = 1$ (by construction, the mean and variance of Y are 0 and 1), we have

$$M_Y(\frac{t}{\sqrt{n}}) = 1 + \frac{(t/\sqrt{n})^2}{2!} + R_Y(\frac{t}{\sqrt{n}}),$$

where R_Y is the remainder term in the Taylor expansion such that

$$\lim_{n\to\infty} \frac{R_Y(t/\sqrt{n})}{(t/\sqrt{n})^2} = 0.$$

Therefore, for any fixed t, we can write

$$\begin{aligned}
\lim_{n\to\infty} \left[M_Y(\frac{t}{\sqrt{n}})\right]^n &= \lim_{n\to\infty} \left[1 + \frac{(t/\sqrt{n})^2}{2!} + R_Y(\frac{t}{\sqrt{n}})\right]^n \\
&= \lim_{n\to\infty} \left[1 + \frac{1}{n}\left(\frac{t^2}{2} + nR_Y(\frac{t}{\sqrt{n}})\right)\right]^n \\
&= e^{t^2/2}
\end{aligned}$$