Math 3070/6070 Introduction to Probability

Mon/Wed/Fri 11:00am - 11:50am

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1 Lecture 1:Aug 22

Today

- Introduction
- Introduce yourself
- Course logistics

What is this course about?

This course will provide a calculus-based introduction to probability theory. Material covered will include fundamental axioms of probability, combinatorics, discrete and continuous random variables, multivariate distributions, expectation, and limit theorems, generally following Chapters 1-5 of the textbook. This course is a critical prerequisite for more advanced work in statistical theory and analysis.

Prerequisite

- Calculus

Why learn probability

- The subject of probability theory is the foundation upon which all of statistics is built.
- It provides you a tool to model
    - populations
    - experiments
    - almost anything else that could be considered a random phenomenon
    - example topics in Data Analysis course
- Through these models, statisticians are able to draw inferences about populations based on examination of only a part of the whole.
- A must have for any Data Scientists.

What this course WILL NOT do for you

It will not help you:

- Beat the casino at blackjack (although it may convince you that it is better not to gamble, or that a casino is a great business).

- Answer your friends' silly questions such as "What are the chances it will rain tomorrow?" (although it might make you think of ways that you might model and compute it).

Syllabus

Check course website frequently for updates and announcements.

https://tulane-math-3070-2022.github.io/

HW submission

Students are required to submit hand-written homework in recitations to the TA. Homework assignments are expected every two weeks with 4-5 problems at a time.

Presentations

Do we want to have a 5 bonus point towards the final grade with a presentation?

Last year comments

Not really, this is my first time teaching this course. There will be an internal mid-term-ish evaluation for this course. Will remember to go over them.

2 Lecture 2:Aug 24

Last time

- Introduction
- Introduce yourself
- Course logistics

Today

- Set theory (1.1)
- Axiomatic Foundations (1.2)

Set Theory

One of the main objectives of a statistician is to draw conclusions about a population of objects by conducting an experiment. The first step in this endeavor is to identify the possible outcomes or, in statistical terminology, the *sample space.*

Definition   The set, S, of all possible outcomes of a particular experiment is called the *sample space* for the experiment.

Example   The sample space of

- tossing a coin just once, contains two outcomes, heads and tails

$$S = \{H, T\}$$

- observing reported SAT scores of randomly selected students at a certain university

$$S = \{200, 210, 220, \ldots, 780, 790, 800\}$$

- an experiment where the observation is reaction time to a certain stimulus

$$S = (0, \infty)$$

Definition   An *event* is any collection of possible outcomes of an experiment, that is , any subset of S (including S itself).

Let A be an event,

- A is a subset of S,
- event A occurs if the outcome of the experiment is in the set A,
- we generally speak of the probability of an event, rather than a set.

Set operations:

- Containment:
$$A \subset B \iff x \in A \implies x \in B$$

- Equality:
$$A = B \iff A \subset B \text{ and } B \subset A$$

- Union: the union of A and B, written as $A \cup B$, is the set of elements that belong to either A or B or both
$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

- Intersection: the intersection of A and B, written $A \cap B$, is the set of elements that belong to both A and B:
$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

- Complementation: the complement of A, written A^c, is the set of all elements that are not in A:
$$A^C = \{x : x \notin A\}.$$

Theorem For any three evnets, A, B, and C, defined on a sample space S,

1. Commutativity
$$A \cup B = B \cup A,$$
$$A \cap B = B \cap A;$$

2. Associativity
$$A \cup (B \cup C) = (A \cup B) \cup C,$$
$$A \cap (B \cap C) = (A \cap B) \cap C;$$

3. Distributive Laws
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C);$$

4. DeMorgan's Laws
$$(A \cup B)^c = A^c \cap B^c,$$
$$(A \cap B)^c = A^c \cup B^c;$$

We show the proof of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ in the distributive laws. Caution: Venn diagrams are helpful in visualization, but they do not constitute a formal proof. To prove that two sets are equal, we need to show that each set contains the other.
proof:

- $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$:
  Let $x \in (A \cap (B \cup C))$. By definition of intersection, $x \in (B \cup C)$ that is, either $x \in B$ or $x \in C$. Since x also must be in A, we have that either $x \in (A \cap B)$ or $x \in (A \cap C)$; therefore, $x \in ((A \cap B) \cup (A \cap C))$.

- $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$:
  Let $x \in ((A \cap B) \cup (A \cap C))$. This implies that $x \in (A \cap B)$ or $x \in (A \cap C)$. If $x \in (A \cap B)$, then x is in both A and B. Since $x \in B$, then $x \in (B \cup C)$ and thus $x \in (A \cap (B \cup C))$. It follows the same argument when $x \in (A \cap C)$, we still have $x \in (A \cap (B \cup C))$.

Definition Two events A and B are *disjoint* (or *mutually exclusive*) if $A \cap B = \varnothing$. The events $A_1, A_2, \ldots$ are *pairwise disjoint* (or *mutually exclusive*) if $A_i \cap A_j = \varnothing$ for all $i \neq j$.

Definition If $A_1, A_2, \ldots$ are pairwise disjoint and $\cup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots = S$, then the collection of $A_1, A_2, \ldots$ forms a *partition* of S.

Example The sets $A_i = [i, i+1), i = 0, 1, 2, \ldots$ form a partition of $[0, \infty)$.

Basics of Probability Theory

When an experiment is performed, the realization of the experiment is an outcome in the sample space. If the experiment is performed a number of times, then

- different outcomes may occur each time

- some outcomes may repeat

- the "frequency of occurrence" of an outcome can be thought of as a probability

However, we **do not** define probabilities in terms of frequencies but instead take the mathematically simpler axiomatic approach. The axiomatic approach is not concerned with the interpretations of probabilities, but is concerned only that the probabilities are defined by a function satisfying the axioms. Interpretations of the probabilities are quite another matter:

- The "frequency of occurrence" of an event is one example of a particular interpretation of probability.

- Another possible interpretation is a subjective one, where we can think of the probability as a belief in the chance of an event occurring.

Axiomatic Foundations

For each event A in the sample space S, we want to associate with A a number between zero and one that will be called the probability of A, denoted by $\Pr(A)$. The domain of $\Pr$ is the set where the arguments of the function $\Pr()$ are defined. It is natural to define the domain of $\Pr$ as all subsets of S, that is for each $A \subset S$, we define $\Pr(A)$ as the probability

that A occurs. However, there are some technical difficulties to overcome which requires us to familiarize with the following.

Definition   A collection of subsets of S is called a *sigma algebra* (or *Borel field*), denoted by $\mathcal{B}$, if it satisfies the following three properties:

1. $\varnothing \in \mathcal{B}$ (the empty set is an element of $\mathcal{B}$).

2. If $A \in \mathcal{B}$, then $A^c \in \mathcal{B}$ ($\mathcal{B}$ is closed under complementation).

3. If $A_1, A_2, \cdots \in \mathcal{B}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$ ($\mathcal{B}$ is closed under countable unions).

From Property (1) and (2), we see that the empty set and its complement S (since $S = \varnothing^c$) are always in a sigma algebra. In fact, they construct the *trivial* algebra $\{\varnothing, S\}$ which is the smallest sigma algebra.

By DeMorgan's Law, (3) can be replaced by:

$$3'. \text{ if } A_1, A_2, \cdots \in \mathcal{B}, \text{ then } \cap_{i=1}^{\infty} A_i \in \mathcal{B}.$$

This is because:

$$(\cup_{i=1}^{\infty} A_i^c)^c = \cap_{i=1}^{\infty} A_i.$$

Example   If S is finite or countable (where the elements of S can be put into $1-1$ correspondence with a subset of the integers), then these technicalities really do not arise, for we we define for a given sample space S,

$$\mathcal{B} = \{\text{all subsets of } S, \text{ including } S \text{ itself}\}.$$

If S has n elements, there are 2^n sets in $\mathcal{B}$ (why?).[hint: for each element, it is either in or out of a subset, so 2 choices].

Example   Let $S = (-\infty, \infty)$, the real line. Then $\mathcal{B}$ is chosen to contain all sets of the form

$$[a, b], (a, b], (a, b), \text{ and } [a, b)$$

for all real numbers a and b. Also, from the properties of $\mathcal{B}$, it follows that $\mathcal{B}$ contains all sets that can be formed by taking (possibly countably infinite) unions and intersections of sets of the above varieties.

We now define a probability function.

Definition   Given a sample space S and an associated sigma algebra $\mathcal{B}$, a *probability function* is a function Pr with domain $\mathcal{B}$ that satisfies

1. $\Pr(A) \geqslant 0$ for all $A \in \mathcal{B}$.

2. $\Pr(S) = 1$.

3. If $A_1, A_2, \cdots \in \mathcal{B}$ are pairwise disjoint, then $\Pr(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pr(A_i)$.

The above three properties are usually referred to as the Axioms of Probability (or the Kolmogorov Axioms, after A. Kolmogorov, one of the fathers of probability theory). Any function that satisfies the Axioms of Probability is called a probability function.

Example   Consider the simple experiment of tossing a fair coin (just once), so $S = \{H, T\}$. A reasonable probability function is the one that assigns equal probabilities to heads and tails, that is,
$$\Pr(\{H\}) = \Pr(\{T\}).$$
Since $S = \{H\} \cup \{T\}$, we have , from Axiom 1, $\Pr(\{H\} \cup \{T\}) = 1$. Also, $\{H\}$ and $\{T\}$ are disjoint, so $\Pr(\{H\} \cup \{T\}) = \Pr(\{H\}) + \Pr(\{T\})$. Collectively, we have

$$\Pr(\{H\}) = \Pr(\{T\})$$
$$\Pr(\{H\} \cup \{T\}) = 1$$
$$\Pr(\{H\} \cup \{T\}) = \Pr(\{H\}) + \Pr(\{T\})$$

Therefore, $\Pr(\{H\}) = \Pr(\{T\}) = \frac{1}{2}$.