

## Lecture 32: Nov. 14

### Last time

- Presentations
- Multiple Random Variables (Chapter 4)

### Today

- Conditional Distributions

#### Example 2

$$F_{X,Y}(x, y) = x - x \log \frac{x}{y} \quad 0 < x \leq y \leq 1$$

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y} =$$

$$f_X(x) =$$

$$f_Y(y) =$$

Note: Once we have  $f_X(x)$  and  $f_Y(y)$ , we can obtain  $F_X(x)$  and  $F_Y(y)$  directly. Double check:  $F_X(x) = F_{X,Y}(x, \infty)$ .

### Conditional Distributions

**Conditional Distributions - Discrete** Recall if  $A$  and  $B$  are two events, the probability of  $A$  conditional on  $B$  is:

$$\Pr(A|B) = \frac{\Pr(A, B)}{\Pr(B)}$$

Defining the events  $A = \{Y = y\}$  and  $B = \{X = x\}$ , it follows that

$$\begin{aligned} \Pr\{Y = y|X = x\} &= \frac{\Pr(X = x, Y = y)}{\Pr(X = x)} \\ &= \frac{f_{X,Y}(x, y)}{f_X(x)} \\ &= f_{Y|X}(y|x) \end{aligned}$$

This is called the *conditional probability mass function* of  $Y$  given  $X$ .

**Example: Discrete** Back to the fair coin example. From the joint pmf of  $X$  and  $Y$ , we can derive all the conditional pmfs:

		Y			
		0	1	2	3
X	0	1/8	1/4	1/8	0
	1	0	1/8	1/4	1/8

**Conditional Distribution - Continuous** If  $F(x, y)$  is absolutely continuous, we define the conditional density of  $Y$  given  $X$  as:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}, \text{ if } f_Y(y) > 0$$

**Example 1**

$$\begin{aligned} F_{XY}(x, y) &= xy & 0 < x < 1, \quad 0 < y < 1 \\ f_{XY}(x, y) &= 1 & 0 < x < 1, \quad 0 < y < 1 \\ f_X(x) &= 1 & 0 < x < 1 \\ f_Y(y) &= 1 & 0 < y < 1 \\ f_{X|Y}(x|y) &= \frac{f_{XY}(x, y)}{f_Y(y)} = 1 & 0 < x < 1 \quad (0 < y < 1) \\ f_{Y|X}(y|x) &= \frac{f_{XY}(x, y)}{f_X(x)} = 1 & 0 < y < 1 \quad (0 < x < 1) \end{aligned}$$

Note: Here we get that the conditional densities are the same as the marginals. This means  $X$  and  $Y$  are independent.

**Example 2**

$$\begin{aligned} F_{XY}(x, y) &= x - x \log \frac{x}{y} & 0 < x \leq y \leq 1 \\ f_{XY}(x, y) &= 1/y & 0 < x \leq y \leq 1 \\ f_X(x) &= -\log x & 0 < x \leq 1 \\ f_Y(y) &= 1 & 0 < y \leq 1 \\ f_{X|Y}(x|y) &= \frac{f_{XY}(x, y)}{f_Y(y)} = 1/y & 0 < x \leq y \quad (0 < y \leq 1) \\ f_{Y|X}(y|x) &= \frac{f_{XY}(x, y)}{f_X(x)} = -\frac{1}{y \log x} & x \leq y \leq 1 \quad (0 < x \leq 1) \end{aligned}$$

- $Y$  is marginally uniform, but not conditionally uniform.
- $X$  is conditionally uniform, but not marginally uniform.

## Independent Random Variables

**Independence** The random variable  $X$  and  $Y$  are said to be *independent* if for any two Borel sets  $A$  and  $B$ ,

$$\Pr(X \in A, Y \in B) = \Pr(X \in A) \Pr(Y \in B)$$

All events defined in terms of  $X$  are independent of all events defined in terms of  $Y$ .

Using the Kolmogorov axioms of probability, it can be shown that  $X$  and  $Y$  are independent if and only if  $\forall(x, y)$  (except possibly for sets of probability 0)

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

or in terms of pmfs (discrete) and pdfs (continuous)

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

### Checking independence

- A necessary condition for independence of  $X$  and  $Y$  is that their joint pdf/pmf has positive probability on a rectangular domain.
- If the domain is rectangular, one can try to write the joint pdf/pmf as a product of functions of  $x$  and  $y$  only.

**Example** Two points are selected randomly on a line of length  $a$  so as to be on opposite sides of the mid-point of the line. Find the probability that the distance between them is less than  $a/3$ .

*Solution:*

Let  $X$  be the coordinate of a point selected randomly in  $[0, a/2]$  and  $Y$  be the coordinate of a point selected randomly in  $[a/2, a]$ . Assume  $X$  and  $Y$  are independent and uniform over its interval. The joint density is

$$f_{X,Y}(x, y) = 4/a^2, \quad 0 \leq x \leq a/2, a/2 \leq y \leq a$$

Therefore, the solution is

$$\Pr(Y - X < a/3) =$$

**Example: Buffon's Needle** A table is ruled with lines distance 1 unit apart. A needle of length  $L \leq 1$  is thrown randomly on the table. What is the probability that the needle intersects a line?

*Solution:*

Define two random variables:

- $X$ : distance from low end of the needle to the nearest line above
- $\theta$ : angle from the vertical to the needle.

By “random”, we assume  $X$  and  $\theta$  are independent, and

$$X \sim U(0, 1) \quad \text{and} \quad \theta \sim U[-\pi/2, \pi/2].$$

This means that

$$f_{X,\theta}(x, \theta) = 1/\pi, \quad 0 \leq x \leq 1, -\pi/2 \leq \theta \leq \pi/2$$

For the needle to intersect a line, we need  $X < L \cos(\theta)$ .

**Expectations of Independent RVs (Theorem 4.2.10)** Let  $X$  and  $Y$  be independent rvs.

- For any  $A \subset \mathbb{R}$  and  $B \subset \mathbb{R}$ ,

$$\Pr(X \in A, Y \in B) = \Pr(X \in A) \Pr(Y \in B)$$

i.e. the events  $\{X \in A\}$  and  $\{Y \in B\}$  are independent.

- Let  $g(x)$  be a function only of  $x$  and  $h(y)$  be a function only of  $y$ . Then

$$E[g(X)h(Y)] = [Eg(X)][Eh(Y)]$$

*Proof:*

$$\begin{aligned} E[g(X)h(Y)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y)f_{XY}(x,y)dxdy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)dxdy \\ &= \left( \int_{-\infty}^{\infty} g(x)f_X(x)dx \right) \left( \int_{-\infty}^{\infty} h(y)f_Y(y)dy \right) \\ &= [Eg(X)][Eh(Y)] \end{aligned}$$

**Example**  $X, Y$  are independent

$$\begin{aligned} E(X^2Y^3) &= (EX^2)(EY^3) \\ E(Y^2Y^3) &\neq (EY^2)(EY^3) \end{aligned}$$

### Bivariate Transformation

**Functions of random variables** Let  $(X, Y)$  be a bivariate rv with known distributions. Define  $(U, V)$  by

$$U = g_1(X, Y), \quad V = g_2(X, Y)$$

**Probability mapping** For any Borel set  $B \subset \mathbb{R}^2$ ,

$$\Pr[(U, V) \in B] = \Pr[(X, Y) \in A]$$

where  $A$  is the inverse mapping of  $B$ , i.e.

$$A = \{(x, y) \in \mathbb{R}^2 : (g_1(x, y), g_2(x, y)) \in B\}$$

The inverse is well defined even if the mapping is not bijective.

**Example** Let  $g_1(x, y) = x, g_2(x, y) = x^2 + y^2$ .

**Discrete RVs** Suppose that  $(X, Y)$  is a discrete rv, i.e. the pmf is positive on a countable set  $\mathcal{A}$ . Then  $(U, V)$  is also discrete and takes values on a countable set  $\mathcal{B}$ . Define

$$A_{u,v} = \{(x, y) \in \mathcal{A} : g_1(x, y) = u, g_2(x, y) = v\}$$

Then

$$f_{UV}(u, v) = \Pr(U = u, V = v) = \sum_{(x,y) \in A_{u,v}} f_{XY}(x, y)$$

**Sum of two independent Poissons** Let  $X \sim \text{Poisson}(\lambda_1)$ ,  $Y \sim \text{Poisson}(\lambda_2)$ , independent, and define

$$U = X + Y, \quad V = Y$$

- $(X, Y)$  takes values in  $\mathcal{A} = \{0, 1, 2, \dots\}^2$
- $(U, V)$  takes values on  $\mathcal{B} = \{(u, v) : v = 0, 1, 2, \dots, u = v, v + 1, v + 2, \dots\}$ .
- For a particular  $(u, v)$ ,  $A_{uv} = \{(x, y) \in \mathcal{A} : x + y = u, y = v\} = (u - v, u)$ .

The joint pmf of  $U$  and  $V$  is

$$f_{UV}(u, v) = f_{XY}(u - v, v) = \frac{e^{-\lambda_1} \lambda_1^{u-v}}{(u - v)!} \frac{e^{-\lambda_2} \lambda_2^v}{(v)!}$$

The distribution of  $U = X + Y$  is the marginal

$$\begin{aligned} f + U(u) &= \sum_{v=0}^u \frac{e^{-\lambda_1} \lambda_1^{u-v}}{(u - v)!} \frac{e^{-\lambda_2} \lambda_2^v}{(v)!} \\ &= \frac{e^{-(\lambda_1 + \lambda_2)}}{u!} \sum_{v=0}^u \binom{u}{v} \lambda_1^{u-v} \lambda_2^v \\ &= \frac{e^{-(\lambda_1 + \lambda_2)}}{u!} (\lambda_1 + \lambda_2)^u \end{aligned}$$

We obtain that  $U$  is Poisson with parameter  $\lambda = \lambda_1 + \lambda_2$ .

**Bivariate Transformations of Continuous RVs** Suppose  $(X, Y)$  is continuous and the joint transformation

$$u = g_1(x, y), \quad v = g_2(x, y)$$

is one-to-one and differentiable. Define the inverse mapping

$$x = h_1(u, v), \quad y = h_2(u, v)$$

Then

$$f_{UV}(u, v) = f_{XY}(h_1(u, v), h_2(u, v)) |J(u, v)|$$

where  $J(u, v)$  is the Jacobian of the transformation  $(x, y) \rightarrow (u, v)$  given by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

**Example: Rotation of a bivariate normal vector** Let  $X \sim N(0, 1)$ ,  $Y \sim N(0, 1)$ , independent. Define the rotation

$$\begin{aligned} U &= X \cos \theta - Y \sin \theta \\ V &= X \sin \theta + Y \cos \theta \end{aligned}$$

for fixed  $\theta$ . Then  $U \sim N(0, 1)$ ,  $V \sim N(0, 1)$ , independent.

*Proof:*

The range of  $(X, Y)$  is  $\mathbb{R}^2$ . The range of  $(U, V)$  is  $\mathbb{R}^2$ . Need the inverse transformation

$$\begin{aligned}X &= U \cos \theta + V \sin \theta \\Y &= -U \sin \theta + V \cos \theta\end{aligned}$$

with Jacobian

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{vmatrix} = 1$$

The joint pdf of  $(X, Y)$  is

$$f_{XY}(x, y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2} = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

The joint pdf of  $(U, V)$  is

$$\begin{aligned}f_{UV}(u, v) &= \frac{1}{2\pi} e^{-[(u \cos \theta + v \sin \theta)^2 + (-u \sin \theta + v \cos \theta)^2]/2} \cdot |1| \\&= \frac{1}{2\pi} e^{-(u^2+v^2)/2} = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-v^2/2}\end{aligned}$$

so  $U \sim N(0, 1)$ ,  $V \sim N(0, 1)$ , and  $U$  and  $V$  are independent.

**Functions of independent random variables** (Theorem 4.3.5) Let  $X$  and  $Y$  be independent rvs. Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  be functions. Then the random variables  $U = g(X)$  and  $V = h(Y)$  are independent.

**Sum of two independent rvs** Suppose  $X$  and  $Y$  are independent. What is the distribution of  $Z = X + Y$ ? In general:

$$F_Z(z) = \Pr(X + Y \leq z) = \Pr(\{(x, y) \text{ such that } x + y \leq z\})$$

Various approaches:

- bivariate transformation method (continuous and discrete)
- Discrete convolution

$$f_Z(z) = \sum_{x+y=z} f_X(x) f_Y(y) = \sum_x f_X(x) f_Y(z-x)$$

- Continuous convolution (Section 5.2)
- MGF method (continuous and discrete)

**Example** (Sum of two independent Poissons) Define  $X, Y$  to be two independent random variables having Poisson distributions with parameters  $\lambda_i, i = 1, 2$ . Then:

$$f_{X,Y}(x, y) = \frac{e^{-\lambda_1} \lambda_1^x}{x!} \frac{e^{-\lambda_2} \lambda_2^y}{y!}, x, y = 0, 1, 2, \dots$$

The distribution of  $S = X + Y$  is

$$\begin{aligned} f_S(s) &= \sum_{x=0}^s \frac{e^{-\lambda_1} \lambda_1^x}{x!} \frac{e^{-\lambda_2} \lambda_2^{s-x}}{(s-x)!} \\ &= \frac{e^{-(\lambda_1 + \lambda_2)}}{s!} \sum_{x=0}^s \binom{s}{x} \lambda_1^x \lambda_2^{s-x} \\ &= \frac{e^{-(\lambda_1 + \lambda_2)}}{s!} (\lambda_1 + \lambda_2)^s \end{aligned}$$

Again,  $S$  is Poisson with parameter  $\lambda = \lambda_1 + \lambda_2$ .

**Moment generating function** (Theorem 4.2.12) Let  $X$  and  $Y$  be independent rvs with mgfs  $M_X(\cdot)$  and  $M_Y(\cdot)$ , respectively. Then the mgf of  $Z = X + Y$  is

$$M_Z(t) = M_X(t)M_Y(t)$$

*Proof:*

$$\begin{aligned} M_Z(t) &= E \exp(Zt) &&= E \{ \exp[(X + Y)t] \} \\ &= E[\exp(Xt) \exp(Yt)] &&= E[\exp(Xt)] \cdot E[\exp(Yt)] \\ &= M_X(t)M_Y(t) \end{aligned}$$

**Corollary:** If  $X$  and  $Y$  are independent and  $Z = X - Y$ ,

$$M_Z(t) = M_X(t)M_Y(-t)$$

**Example** (sum of two independent Poissons) Suppose  $X \sim \text{Poisson}(\lambda_X)$  and  $Y \sim \text{Poisson}(\lambda_Y)$  and put  $Z = X + Y$ . Then,  $Z \sim \text{Poisson}(\lambda_X + \lambda_Y)$ . *Proof:*

$$\begin{aligned} M_Z(t) &= \exp[\lambda_X(e^t - 1)] \exp[\lambda_Y(e^t - 1)] \\ &= \exp[(\lambda_X + \lambda_Y)(e^t - 1)] \end{aligned}$$

**Example** (sum of two independent normals) Suppose  $X \sim N(\mu_x, \sigma_x^2)$  and  $Y \sim N(\mu_y, \sigma_y^2)$  and  $X$  and  $Y$  are independent and  $Z = X + Y$ . Then

$$Z \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$$

*Proof:*

$$\begin{aligned} M_Z(t) &= \exp\left(\mu_x t + \frac{1}{2}\sigma_x^2 t^2\right) \exp\left(\mu_y t + \frac{1}{2}\sigma_y^2 t^2\right) \\ &= \exp\left[(\mu_x + \mu_y)t + \frac{1}{2}(\sigma_x^2 + \sigma_y^2)t^2\right] \end{aligned}$$

**Example** (sum of two independent gammas) Suppose  $X \sim \Gamma(\alpha_x, \beta)$  and independently  $Y \sim \Gamma(\alpha_y, \beta)$ . Let  $Z = X + Y$ . Then  $Z \sim \Gamma((\alpha_x + \alpha_y), \beta)$ .

*Proof:*

$$\begin{aligned} M_Z(t) &= \left( \frac{1}{1 - \beta t} \right)^{\alpha_x} \left( \frac{1}{1 - \beta t} \right)^{\alpha_y} \\ &= \left( \frac{1}{1 - \beta t} \right)^{\alpha_x + \alpha_y} \end{aligned}$$

Remember that

- If  $\alpha = 1$  we have an exponential with parameter  $\beta$ .
- If  $\alpha = n/2$  and  $\beta = 2$ , we have a  $\chi^2(n)$  (with  $n$  d.f.). The above result states that  $\chi^2(n_1) + \chi^2(n_2) = \chi^2(n_1 + n_2)$ .