Lecture 19: Oct 14

Last time

- Midterm exam 1 review

Today

- Presentations
- Moment generating function

Definition Let X be a random variable with cdf F_X. The *moment generating function (mgf)* of X (or F_X), denoted by $M_X(t)$, is

$$M_X(t) = Ee^{tX},$$

provided that the expectation exists for t in some neighborhood of 0. That is, there is an $h > 0$ such that, for all t in $-h < t < h$, Ee^{tX} exists. If the expectation does not exist in a neighborhood of 0, we say that the moment generating function does not exist.

More explicitly, we can write the mgf of X as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx, \quad \text{if } X \text{ is continuous,}$$

or

$$M_X(t) = \sum_x e^{tx} \Pr(X = x), \quad \text{if } X \text{ is discrete.}$$

It is easy to see how the mgf generates moments as in the following theorem.

Theorem If X has mgf $M_X(t)$, then

$$EX^n = M_X^{(n)}(0),$$

where we define

$$M_X^{(0)} = \left.\frac{d^n}{dt^n} M_X(t)\right|_{t=0}.$$

That is, the n^{th} moment is equal to the n^{th} derivative of $M_X(t)$ evaluated at $t = 0$.

Proof:

$$\begin{aligned}
\frac{d}{dt} M_X(t) &= \frac{d}{dt} \int_{-\infty}^{\infty} e^{tx} f_X(x)dx \\
&= \int_{-\infty}^{\infty} \left(\frac{d}{dt} e^{tx}\right) f_X(x)dx \\
&= \int_{-\infty}^{\infty} \left(xe^{tx}\right) f_X(x)dx \\
&= E(Xe^{tX}).
\end{aligned}$$

Therefore,

$$\frac{d}{dt}M_X(t)\bigg|_{t=0} = E(Xe^{tX})\bigg|_{t=0} = EX.$$

Proceeding in an analogous manner, we can establish that

$$\frac{d^n}{dt^n}M_X(t)\bigg|_{t=0} = E(X^n e^{tX})\bigg|_{t=0} = EX^n.$$

Example (Binomial mgf) Let $X \sim Binomial(n, p)$, then its mgf is

$$\begin{aligned} M_X(t) &= \sum_{x=0}^{n} e^{tx}\binom{n}{x}p^x(1-p)^{n-x} \\ &= \sum_{x=0}^{n}\binom{n}{x}(pe^t)^x(1-p)^{n-x} \\ &= [pe^t + (1-p)]^n. \end{aligned}$$

Theorem Let $F_X(x)$ and $F_Y(y)$ be two cdfs all of whose moments exist.

1. If X and Y have **bounded support**, then $F_X(u) = F_Y(u)$ for all u if and only if $EX^r = EY^r$ for all integers $r = 0, 1, 2, \ldots$.
2. If the moment generating functions exist and $M_X(t) = M_Y(t)$ for all t in some neighborhood of 0, then $F_X(u) = F_Y(u)$ for all u.

Theorem (Convergence of mgfs) Suppose $\{X_i, i = 1, 2, \ldots\}$ is a sequence of random variables, each with mgf $M_{X_i}(t)$. Furthermore, suppose that

$$\lim_{i\to\infty} M_{X_i}(t) = M_X(t), \qquad \text{for all } t \text{ in a neighborhood of } 0,$$

and $M_X(t)$ is an mgf. Then there is a unique cdf F_X whose moments are determined by $M_X(t)$ and, for all x where $F_X(x)$ is continuous, we have

$$\lim_{i\to\infty} F_{X_i}(x) = F_X(x).$$

That is, *convergence*, for $|t| < h$, of mgfs to an mgf implies *convergence* of cdfs.

Poisson approximation One approximation that is usually taught in elementary statistics courses is that binomial probabilities can be approximated by Poisson probabilities. It is taught that the Poisson approximation is valid "when n is large and np is small", and rules of thumb are sometimes given.

The $Poisson(\lambda)$ pmf is given by

$$\Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots,$$

where λ is a positive constant. The approximation states that if $X \sim Binomial(n, p)$ and $Y \sim Poisson(\lambda)$, with $\lambda = np$, then

$$\Pr(X = x) \approx \Pr(Y = x)$$

for large n and small np. We now show that the mgf converge, lending credence to this approximation. Recall that

$$M_X(t) = [pe^t + (1-p)]^n.$$

For the $Poisson(\lambda)$ distribution, we can calculate (HW4, exercise 2.33)

$$M_Y(t) = e^{\lambda(e^t - 1)},$$

and if we define $p = \lambda/n$, then $M_X(t) = [1 + (e^t - 1)\lambda/n]^n$ such that $M_X(t) \to M_Y(t)$ as $n \to \infty$.

Theorem For any constant a and b, the mgf of the random variable $aX + b$ is given by

$$M_{aX+b} = e^{bt} M_X(at).$$

Proof:
By definition,

$$\begin{aligned} M_{aX+b} &= E\left(e^{(aX+b)t}\right) \\ &= E\left(e^{(aX)t} e^{bt}\right) \\ &= e^{bt} E\left(e^{(aX)t}\right) \\ &= e^{bt} M_X(at). \end{aligned}$$