Lecture 22: Oct 21

Last time

- Common Discrete Distributions (Chapter 3)

Today

- Presentations
- Common Discrete Distributions (Chapter 3)

Geometric Distribution Consider a series of iid Bernoulli Trials with p = probability of success in each trial. Define a random variable X representing the number of trials until first success. Note X includes the trial at which the success occurs (one parameterization). Then, X has a geometric distribution.

- Sample space: $\{1, 2, \dots\}$
- pmf:

$$f(x) = \Pr(X = x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \dots \\ 0 & otherwise \end{cases}$$

- cdf:

$$F(x) = \Pr(X \leqslant x) = 1 - (1-p)^x$$

- Moments:

$$E(X) = 1/p$$
$$Var(X) = (1-p)/p^2$$

Memoryless property. Suppose $k > i$, then

$$\Pr(X > k | X > i) = \Pr(X > k - i)$$

Proof:

$$\begin{aligned}\Pr(X > k | X > i) &= \frac{\Pr(X > k)}{\Pr(X > i)} = \frac{(1-p)^k}{(1-p)^i} \\ &= (1-p)^{k-i} = \Pr(X > k - i)\end{aligned}$$

Example Suppose X is number of years you live, and X follows a geometric distribution, then

$$\begin{aligned}\Pr(\text{survive two more years}) &= \Pr(X > \text{current age} + 2 | X > \text{current age}) \\ &= \Pr(X > 2)\end{aligned}$$

This model is clearly too simple for human populations (since we do age).

Negative Binomial Distribution Still in the context of iid Bernoulli trials, define a random variable corresponding to the number of trials required to have s successes. We say $X \sim Negbin(s, p)$.

- Sample space: $\{s, (s+1), \dots\}$
- pmf: for $x = s, s+1, s+2, \dots$

$$\begin{aligned} f(x) &= \binom{x-1}{s-1} p^{s-1} q^{x-s} \cdot p \\ &= \binom{x-1}{s-1} p^s q^{x-s} \end{aligned}$$

- cdf: no closed form
- Expectation: $EX = s/p$.
- Variance: $Var(X) = s(1-p)/p^2$

Notes

- Why the name? See Casella & Berger p.95.
- $X \sim Negbin(1, p)$ is the same as $X \sim Geometric(p)$
- $Negbin(n, p)$ is the same as the sum of n $Geometric(p)$ random variables

Other parameterizations The negative binomial distribution is sometimes defined in terms of the random variable Y = number of failures before the rth success. Then

- Sample space: $\{0, 1, 2, \dots\}$
- pmf

$$f(y) = \binom{r+y-1}{y} p^r q^y, \quad y = 0, 1, 2, \dots$$

- cdf: no closed form
- Expectation: $EY = r(1-p)/p$
- Variance: $Var(Y) = r(1-p)/p^2$

Negarive binomial vs. Poisson The negative binomial distribution is often good for modeling count data as an alternative to the Poisson. In the previous parameterization, define

$$\lambda = \frac{r(1-p)}{p} \iff p = \frac{r}{r+\lambda}$$

Then we have

$$\begin{aligned} EX &= \lambda \\ Var(X) &= \frac{\lambda}{p} = \lambda(1 + \frac{\lambda}{r}) = \lambda + \frac{\lambda^2}{r} \end{aligned}$$

For the Poisson we had that the variance equals the mean.
For the negative binomial, the variance is equal to the mean plus a quadratic term. Thus the negative binomial can capture overdispersion in count data.
In the previous parameterization, the pmf becomes

$$\begin{aligned} f(y) &= \binom{r+y-1}{y} p^r q^y = \frac{(r+y-1)!}{y!(r-1)!} \left(\frac{r}{r+\lambda}\right)^s \left(\frac{\lambda}{r+\lambda}\right)^y \\ &= \frac{\lambda^x}{x!} \frac{s(s+1)\dots(s+x-1)}{(s+\lambda)^x} \left(1+\frac{\lambda}{s}\right)^{-s} \end{aligned}$$

Letting $s \to \infty$, we get

$$f(x) \to \frac{\lambda^x}{x!} e^{-\lambda}$$

So for large s, the negative binomial can be approximated by a Poisson with parameter $\lambda = r(1-p)/p$.