Lecture 27: Nov. 2

Last time

- Common Continuous Distributions

Today

- Families of Distributions

Exponential Families A family of pdfs or pmfs with vector parameter $\boldsymbol{\theta}$ is called an *exponential family* if it can be expressed as

$$f(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta})exp\left(\sum_{j=1}^{k} w_j(\boldsymbol{\theta})t_j(x)\right), \quad x \in S \subset \mathbb{R}$$

where S is not defined in terms of $\boldsymbol{\theta}$, $h(x)$, $c(\boldsymbol{\theta}) \geqslant 0$ and the functions are just functions of the parameters specified; i.e. h is free of $\boldsymbol{\theta}$, $c(\boldsymbol{\theta})$ is free of x, etc...

Examples:

- One-dimensional: Exponential, Poisson
- Two-dimensional: Gaussian

Exponential family parameterizations are unique except for multiplying constant factors.

Example: Gaussian Let $X \sim N(\mu, \sigma^2)$.

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(-\frac{x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2}\right) \end{aligned}$$

Thus

$$\begin{aligned} h(x) &= \tfrac{1}{\sqrt{2\pi}} & c(\mu, \sigma) &= \tfrac{1}{\sigma} \exp\left(-\tfrac{\mu^2}{2\sigma^2}\right) \\ w_1(\mu, \sigma) &= -\tfrac{1}{2\sigma^2} & w_2(\mu, \sigma) &= \tfrac{\mu}{\sigma^2} \\ t_1(x) &= x^2 & t_2(x) &= x \end{aligned}$$

The parameter space is $(\mu, \sigma^2) \in \mathbb{R} \times (0, \infty)$.

Example: Binomial Let $X \sim Binomial(n, p)$, $0 < p < 1$.

$$\begin{aligned} f(x|p) &= \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} (1-p)^n \left[\frac{p}{1-p}\right]^x \\ &= \binom{n}{x} (1-p)^n \exp\left[\log\left(\frac{p}{1-p}\right) x\right] \end{aligned}$$

Thus,

$$\begin{array}{ll} h(x) = \binom{n}{x}, \quad x = 0, \ldots, n & w_1(p) = \log\left(\frac{p}{1-p}\right) \\ c(p) = (1-p)^n, 0 < p < 1 & t_1(x) = x \end{array}$$

Note that this works when p is considered the parameter, while n is fixed. Also, p cannot be 0 or 1. Otherwise, the range changes.

More examples The following distributions below to Exponential families:

- Continuous: exponential, Gaussian, gamma, beta, χ^2
- Discrete: Poisson, geometric, binomial (fixed # trials), negative binomial (fixed # successes)

The following distributions