Lecture 6: Sept 2

Last time

- Random variables

Today

- Random variables
- Distribution Functions

Induced probability function Suppose we have a sample space $S = \{s_1, s_2, \dots, s_n\}$ with a probability function Pr defined on the original sample space. We define a random variable X with range $\mathcal{X} = \{x_1, \dots, x_m\}$. We can define a probability function $\Pr_X$ on $\mathcal{X}$ in the following way. We will observe $X = x_i$ if an only if the outcome of the random experiment is an $s_j \in S$ such that $X(s_j) = x_i$. Therefore,

$$\Pr_X(X = x_i) = \Pr(\{s_j \in S : X(s_j) = x_i\}),$$

defines an *induced* probability function on $\mathcal{X}$, defined in terms of the original function Pr.

We will write $\Pr(X = x_i)$ rather than $\Pr_X(X = x_i)$ for simplicity. Note on notation: random variables will always be denoted with uppercase leeters and the realized values of the variable (or its range) will be denoted by the corresponding lowercase letters.

Example Consider the experiment of tossing a fair coin three times. Define the random variable X to be the number of heads obtained in the three tosses. A complete enumeration of the value of X for each point in the sample space is

s	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
$X(s)$	4	2	2	2	1	1	1	0

What is the range of X? What is the induced probability function $\Pr_X$?
solution:

So far, we have seen finite S and finite $\mathcal{X}$, and the definition of $\Pr_X$ is straightforward. If $\mathcal{X}$ is uncountable, we define the induced probability function, $\Pr_X$ for anyset $A \subset \mathcal{X}$,

$$\Pr_X(X \in A) = \Pr(\{s \in S : X(s) \in A\}).$$

This defines a legitimate probability function for which the Kolmogorov Axioms can be verified.

Distribution Functions

Distribution Functions are used to describe the behavior of a r.v.

Cumulative distribution function

Definition The *cumulative distribution function* or *cdf* of a random variable X, denoted by $F_X(x)$, is defined by

$$F_X(x) = \Pr_X(X \leqslant x), \text{ for all } x.$$

Definition The *survival function* of a random variable X, is defined by

$$S_X(x) = 1 - F_X(x) = \Pr_X(X > x).$$

Example Consider the experiment of tossing three fair coins, and let X = number of heads observed. The cdf of X is

$$F_X(x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ \frac{1}{8} & \text{if } 0 \leqslant x < 1 \\ \frac{1}{2} & \text{if } 1 \leqslant x < 2 \\ \frac{7}{8} & \text{if } 2 \leqslant x < 3 \\ 1 & \text{if } 3 \leqslant x < \infty \end{cases}$$

Some properties of the cdf:
Let $F(x)$ be a cdf. Then

1. $0 \leqslant F(x) \leqslant 1$
2. $\lim\limits_{x \to -\infty} F(x) = 0$
3. $\lim\limits_{x \to \infty} F(x) = 1$
4. F is nondecreasing: if $a < b$, then $F(a) \leqslant F(b)$
5. F is right-continuous: $\lim\limits_{x \downarrow b} F(x) = F(b)$, or $\lim\limits_{x \to b^+} F(x) = F(b)$
6. $\Pr(a < X \leqslant B) = F(b) - F(a)$

Theorem The function $F(x)$ is a cdf if and only if the following three conditions hold:

1. $\lim\limits_{x \to -\infty} F(x) = 0$ and $\lim\limits_{x \to \infty} F(x) = 1$
2. F is nondecreasing: if $a < b$, then $F(a) \leqslant F(b)$
3. F is right-continuous: $\lim\limits_{x \downarrow b} F(x) = F(b)$, or $\lim\limits_{x \to b^+} F(x) = F(b)$

The cdf does not contain information about the original sample space.

Definition Two random variables X and Y are identically distributed if, for every Borel set $A \subset \mathbb{R}$, $\Pr(X \in A) = \Pr(Y \in A)$.

Example Toss a fair coin n times. The number of heads and the number of tails have the same distribution.

Theorem The following two statements are equivalent:

1. The random variables X and Y are *identically distributed*.
2. $F_X(x) = F_Y(x)$ for every x.