# Lecture 18: Oct 3

## Last time

- Practice examples

## Today

- Midterm 1 practice exam posted on canvas
- Moments and moment generating function

### Moments

Example (Minimizing distance) The expected value of a random variable has another property, one that we can think of as relating to the interpretation of $EX$ as a good guess at a value of $X$.

Suppose we measure the distance between a random variable $X$ and a constant $b$ by $(X-b)^2$. The closer $b$ is to $X$, the smaller this quantity is. We can now determine the value of $b$ that minimizes $E\left[(X-b)^2\right]$ and, hence, will provide us with a good predictor of $X$. (Note that it does no good to look for a value of $b$ that minimizes $(X-b)^2$, since the answer would depend on $X$, making it a useless predictor of $X$.)

We could proceed with the minimization of $E(X-b)^2$ by using calculus, but there is a simpler method:

$$\begin{aligned} E(X-b)^2 &= E(X-EX+EX-b)^2 \\ &= E\left[(X-EX)+(EX-b)\right]^2 \\ &= E(X-EX)^2+(EX-b)^2+2E\left[(X-EX)(EX-b)\right], \end{aligned}$$

where we have expanded the square. Note that $E\left[(X-EX)(EX-b)\right]=(EX-b)E(X-EX)=0$, since $EX-b$ is constant and comes out of the expectation, $E(X-EX)=EX-EX=0$. This means

$$E(X-b)^2=E(X-EX)^2+(EX-b)^2.$$

Such that $E(X-b)^2$ is minimized at $b=EX$. And $E(X-EX)^2$ is actually the variance of $X$ ($VarX=E(X-EX)^2$).

The various moments of a distribution are an important class of expectations.

Definition For each integer $n$, the $n$th *moment* of $X$ (or $F_X(x)$), $\mu_n'$, is

$$\mu_n'=EX^n.$$

The $n$th *central moment* of $X$, $\mu_n$, is

$$\mu_n=E(X-\mu)^n,$$

where $\mu = \mu'_1 = EX$.

Notes:

- $\mu'_0 = EX^0 = 1$
- $\mu'_1$ is the *mean*, usually denoted by $\mu$.
- $\mu_0 = E(X - \mu)^0 = 1$
- $\mu_1 = 0$
- $\mu_2 = E(X - EX)^2$ is the *variance*
- $\mu_3 = E(X - EX)^3$ is related to the *skewness*.
- $\mu_4 = E(X - EX)^4$ is related to the *kurtosis*.

**Definition** The *variance* of a random variable $X$ is its second central moment, $\operatorname{Var}(X) = E\left[(X - EX)^2\right]$. The positive square root of $\operatorname{Var}(X)$ is the *standard deviation* of $X$.

The variance gives a measure of the degree of spread of a distribution around its mean. Figure 18.1 shows a plot of two samples, one sample draws 100 numbers from a normal distribution with mean 0 and variance 1, $N(0,1)$. The other sample draws 100 numbers from a normal distribution with mean 0 and variance 100, $N(0,100)$.

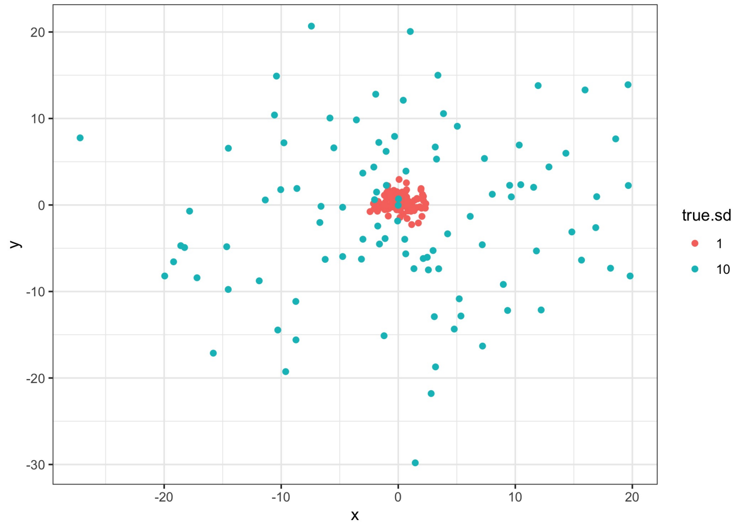


Figure 18.1: Figure 2.1.2. Two samples of 100 numbers drawn from $N(0,1)$ and $N(0,100)$.

**Example** (Exponential variance) Let $X$ have the exponential($\lambda$) distribution. We can calculate the variance of $X$ now.
*Solution:*

$$\begin{aligned}
\mathrm{Var}\,(X) &= E(X-\lambda)^2 \\
&= \int_0^\infty (x-\lambda)^2 \frac{1}{\lambda} e^{-x/\lambda} dx
\end{aligned}$$

**Theorem** If $X$ is a random variable with finite variance, then for any constants $a$ and $b$,

$$\mathrm{Var}\,(aX+b) = a^2 \mathrm{Var}\,(X)\,.$$

*Proof:*
From the definition, we have

$$\begin{aligned}
\mathrm{Var}\,(aX+b) &= E\left[(aX+b) - E(aX+b)\right]^2 \\
&= E(aX - aEX)^2 \\
&= a^2 E(X-EX)^2 \\
&= a^2 \mathrm{Var}\,(X)\,.
\end{aligned}$$

It is sometimes to use an alternative formula for the variance, given by

$$\mathrm{Var}\,(X) = E(X^2) - (EX)^2,$$

which is easily established by

$$\begin{aligned}
\mathrm{Var}\,(X) &= E(X-EX)^2 = E\left[X^2 - 2XEX + (EX)^2\right] \\
&= EX^2 - 2(EX)^2 + (EX)^2 \\
&= EX^2 - (EX)^2.
\end{aligned}$$

**Example** (Binomial variance) Let $X \sim Binomial(n,p)$, that is ,

$$\Pr(X=x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

What is the variance of $X$?
*Solutions:*

**Definition** Let $X$ be a random variable with cdf $F_X$. The *moment generating function (mgf)* of $X$ (or $F_X$), denoted by $M_X(t)$, is

$$M_X(t) = Ee^{tX},$$

provided that the expectation exists for $t$ in some neighborhood of 0. That is, there is an $h > 0$ such that, for all $t$ in $-h < t < h$, $Ee^{tX}$ exists. If the expectation does not exist in a neighborhood of 0, we say that the moment generating function does not exist.

More explicitly, we can write the mgf of $X$ as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx, \quad \text{if } X \text{ is continuous,}$$

or

$$M_X(t) = \sum_x e^{tx} \Pr(X = x), \quad \text{if } X \text{ is discrete.}$$

It is easy to see how the mgf generates moments as in the following theorem.

Theorem If $X$ has mgf $M_X(t)$, then

$$EX^n = M_X^{(n)}(0),$$

where we define

$$M_X^{(0)} = \frac{d^n}{dt^n} M_X(t) \bigg|_{t=0}.$$

That is, the $n^{th}$ moment is equal to the $n^{th}$ derivative of $M_X(t)$ evaluated at $t = 0$.

*Proof:*