Lecture 19: Oct 14

Last time

- Presentations
- Moment generating function

Today

- Internal midterm evaluation open
- Moment generating function
- Common Discrete Distributions (Chapter 3)

Definition Let X be a random variable with cdf F_X. The *moment generating function (mgf)* of X (or F_X), denoted by $M_X(t)$, is

$$M_X(t) = Ee^{tX},$$

provided that the expectation exists for t in some neighborhood of 0. That is, there is an $h > 0$ such that, for all t in $-h < t < h$, Ee^{tX} exists. If the expectation does not exist in a neighborhood of 0, we say that the moment generating function does not exist.

More explicitly, we can write the mgf of X as

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx, \quad \text{if } X \text{ is continuous,}$$

or

$$M_X(t) = \sum_x e^{tx} \Pr(X = x), \quad \text{if } X \text{ is discrete.}$$

It is easy to see how the mgf generates moments as in the following theorem.

Theorem If X has mgf $M_X(t)$, then

$$EX^n = M_X^{(n)}(0),$$

where we define

$$M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t)\bigg|_{t=0}.$$

That is, the n^{th} moment is equal to the n^{th} derivative of $M_X(t)$ evaluated at $t = 0$.

Proof:

$$\begin{aligned}
\frac{d}{dt}M_X(t) &= \frac{d}{dt}\int_{-\infty}^{\infty} e^{tx} f_X(x)dx \\
&= \int_{-\infty}^{\infty} \left(\frac{d}{dt}e^{tx}\right) f_X(x)dx \\
&= \int_{-\infty}^{\infty} \left(xe^{tx}\right) f_X(x)dx \\
&= E(Xe^{tX}).
\end{aligned}$$

Therefore,

$$\frac{d}{dt}M_X(t)\bigg|_{t=0} = E(Xe^{tX})\bigg|_{t=0} = EX.$$

Proceeding in an analogous manner, we can establish that

$$\frac{d^n}{dt^n}M_X(t)\bigg|_{t=0} = E(X^n e^{tX})\bigg|_{t=0} = EX^n.$$

Example (Binomial mgf) Let $X \sim Binomial(n,p)$, then its mgf is

$$\begin{aligned}
M_X(t) &= \sum_{x=0}^{n} e^{tx}\binom{n}{x}p^x(1-p)^{n-x} \\
&= \sum_{x=0}^{n}\binom{n}{x}(pe^t)^x(1-p)^{n-x} \\
&= [pe^t + (1-p)]^n.
\end{aligned}$$

Theorem Let $F_X(x)$ and $F_Y(y)$ be two cdfs all of whose moments exist.

1. If X and Y have **bounded support**, then $F_X(u) = F_Y(u)$ for all u if and only if $EX^r = EY^r$ for all integers $r = 0, 1, 2, \ldots$.
2. If the moment generating functions exist and $M_X(t) = M_Y(t)$ for all t in some neighborhood of 0, then $F_X(u) = F_Y(u)$ for all u.

Theorem (Convergence of mgfs) Suppose $\{X_i, i = 1, 2, \ldots\}$ is a sequence of random variables, each with mgf $M_{X_i}(t)$. Furthermore, suppose that

$$\lim_{i\to\infty} M_{X_i}(t) = M_X(t), \qquad \text{for all } t \text{ in a neighborhood of } 0,$$

and $M_X(t)$ is an mgf. Then there is a unique cdf F_X whose moments are determined by $M_X(t)$ and, for all x where $F_X(x)$ is continuous, we have

$$\lim_{i\to\infty} F_{X_i}(x) = F_X(x).$$

That is, *convergence*, for $|t| < h$, of mgfs to an mgf implies *convergence* of cdfs.

Poisson approximation One approximation that is usually taught in elementary statistics courses is that binomial probabilities can be approximated by Poisson probabilities. It is taught that the Poisson approximation is valid "when n is large and np is small", and rules of thumb are sometimes given.

The $Poisson(\lambda)$ pmf is given by

$$\Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots,$$

where λ is a positive constant. The approximation states that if $X \sim Binomial(n, p)$ and $Y \sim Poisson(\lambda)$, with $\lambda = np$, then

$$\Pr(X = x) \approx \Pr(Y = x)$$

for large n and small np. We now show that the mgf converge, lending credence to this approximation. Recall that

$$M_X(t) = [pe^t + (1 - p)]^n.$$

For the $Poisson(\lambda)$ distribution, we can calculate (HW4, exercise 2.33)

$$M_Y(t) = e^{\lambda(e^t - 1)},$$

and if we define $p = \lambda/n$, then $M_X(t) = [1 + (e^t - 1)\lambda/n]^n$ such that $M_X(t) \to M_Y(t)$ as $n \to \infty$.

Theorem For any constant a and b, the mgf of the random variable $aX + b$ is given by

$$M_{aX+b} = e^{bt} M_X(at).$$

Proof:
By definition,

$$\begin{aligned} M_{aX+b} &= E\left(e^{(aX+b)t}\right) \\ &= E\left(e^{(aX)t} e^{bt}\right) \\ &= e^{bt} E\left(e^{(aX)t}\right) \\ &= e^{bt} M_X(at). \end{aligned}$$

Common Discrete Distribution

Why parametric models?

- *Parametric models* or *distribution families* have a specific form but can change according to a fixed number of parameters.
- The objective is to model a population. Parametric models are often appropriate in common situations with similar mechanisms.
- Parametric models have many known and useful properties and are easy to work with. When fitting a population, only a few parameters need to be estimated: *parametric inference*.

- Sometimes one does not want to make parametric assumptions and would rather work with non-parametric models. But non-parametric models can be infinite dimensional.
- In this course, we emphasize parametric models.

Discrete uniform X has the discrete unifrom$(1, N)$ distribution if X is equally likely to be one of $\{1, 2, \ldots, N\}$.

- Sample space: $\{1, 2, \ldots, N\}$
- pmf:
$$f_X(x) = \frac{1}{N}, \quad x = 1, 2, \ldots, N$$
- cdf:
$$F_X(x) = \Pr(X \leqslant x) = \frac{x}{N}, \quad x = 1, 2, \ldots, N$$
- moments:
$$EX = \frac{N+1}{2}$$

Bernoulli Distribution Consider an experiment where outcomes are binary (say, Success or Failure) and the probability of success is p. Define the following random variable

$$Y = \begin{cases} 1 & \text{outcome is success} \\ 0 & \text{outcome is failure} \end{cases}$$

Then, Y has a Bernoulli Distribution.

- Sample space: $\{0, 1\}$.
- pmf: $\Pr(Y = 1) = p$ and $\Pr(Y = 0) = 1 - p$. We can write this as:
$$f(y) = \Pr(Y = y) = \begin{cases} p^y(1-p)^{1-y} & y = 0, 1 \\ 0 & othersie \end{cases}$$
- what are the cdf, mean and variance?

Binomial Distribution A $Binomial(n, p)$ random variable X is defined as the number of successes in n i.i.d. (independent, identically distributed) Bernoulli trials, each with probability p of success:

$$X = \sum_{i=1}^{n} Y_i, \quad Y_1, \ldots, Y_n \overset{\text{i.i.d.}}{\sim} Bernoulli(p)$$

- Sample space: $\{0, 1, \ldots, n\}$
- pmf:
$$f_X(s) = \begin{cases} \binom{n}{s} p^x (1-p)^{n-s} & s = 0, 1, \ldots, n \\ 0 & otherwise \end{cases}$$

- cdf:

$$F_X(x) = \sum_{s=0}^{x} \binom{n}{s} p^s (1-p)^{n-s} \quad \text{(no closed form)}$$

Poisson Distribution The Poisson distribution was derived by the French mathematician Poisson in 1837 as a limiting version of the binomial distribution. The Poisson distribution is often used to model the number of occurrences in a given time interval. One of the basic assumptions on which the Poisson distribution is built is that, for small time intervals, the probability of an arrival is proportional to the length of waiting time. This makes it a reasonable model for situations such as waiting for a bus, waiting for customers to arrive in a bank.

The Poisson distribution has a single parameter λ, sometimes called the intensity parameter. A Poisson random variable X, takes values in the nonnegative integers with pmf

$$\Pr(X = x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \ldots$$

To see that $\sum_{x=0}^{\infty} P(X = x|\lambda) = 1$, recall the Taylor series expansion of $e^{\lambda} = \sum\limits_{i=0}^{\infty} \frac{\lambda^i}{i!}$. Thus

$$\sum_{x=0}^{\infty} \Pr(X = x|\lambda) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1$$

What is the mean and variance of X?

$$\begin{aligned}
EX &= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!} \\
&= \lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} \\
&= \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} \\
&= \lambda
\end{aligned}$$

Similarly

$$
\begin{aligned}
EX^2 &= \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{(x-1)!} \\
&= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!} + \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-2)!} \\
&= \lambda + \lambda^2
\end{aligned}
$$

So that

$$Var(X) = EX^2 - (EX)^2 = \lambda$$

- Sample space: $\{0, 1, \ldots\}$
- pmf: $\Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$
- cdf: $F_X(x) = \sum_{s=0}^{x} \frac{e^{-\lambda}\lambda^s}{s!}$