1. Probability

a)
b)

ⓐ if X & Y are independent

$\Rightarrow E(XY) = E(X)E(Y)$

$\Rightarrow Cov(X,Y) = E(XY) - E(X)E(Y)$

$= E(X)E(Y) - E(X)E(Y)$

$= 0$ //

ⓑ (i) $P(+) = \frac{72}{75}$

(ii) $P(-) = \frac{3}{75}$

(iii) $P(\overset{\text{has}}{+}) = \frac{P(+|HIV) \cdot P(HIV)}{P(+)}$

$P(+) \rightarrow P(+|HIV)P(HIV) + P(+|no)P(no)$

$= 0.0297$

$\Rightarrow \frac{P(HIV)P(+|HIV)}{P(+)} = 0.475 \Rightarrow P(HIV|+) = 47.5\%$ //

$P(-) = P(-|HIV)P(HIV) + P(-|No) \cdot P(No)$

$= 0.7576$

$P(HIV|-) = 0.0063$

2. Bayes Theorem

ⓐ $P(C)+\frac{1}{2}P(C)+\frac{1}{2}P(C)+\frac{1}{2}P(C)=1$

$(1+\frac{3}{2})P(C)=1$, $P(C)=\frac{2}{5}$, $P(A)=\frac{1}{5}$, $P(B)=\frac{1}{5}$, $P(D)=\frac{1}{5}$

$P(A|B)=0.3$

$P(D|B)=0.3$

$P(C|B)=0.4$

$P(A)+P(B)+P(D)=\frac{3}{5}$

open door B $\Rightarrow P(B)$ and $P(A\cdot)P(D)$

$\Rightarrow P(A)$ and $P(D) \Rightarrow \frac{3}{5}$

ⓑ

	surprised	un	total
cheat	1.8	0.12	2
no	14.6	58.4	73
Total	16.4	58.6	75

$\Rightarrow P(\text{cheating}|\text{surprised})=\frac{1.8}{16.4}$

$=0.1097=10.97\%$

3. Linear Algebra

ⓐ $(A^T)^{-1}=(A^{-1})^T$

$AA^{-1}=I$

$(A^{-1})^T A^T = I^T = I$

$(A^{-1})^T A = I$

$(A^{-1})^T = A^{-1}$

$I = I^T$, $(AA^{-1})=I$

$\Rightarrow AA^{-1}=(AA^{-1})^T$, $(AB)^T=B^TA^T$

$\Rightarrow AA^{-1}=(A^{-1})^T A^T$, $AA^{-1}=A^{-1}A=I$

$\Rightarrow A^{-1}A=(A^{-1})^T A^T$, $A=A^T$

$\Rightarrow A^{-1}A=(A^{-1})^T A$

$A^{-1}A(A^{-1})=(A^{-1})^T A(A^{-1})$

$A^{-1}I=(A^{-1})^T I$

$\Rightarrow A^{-1}=(A^{-1})^T$

(b) $\begin{vmatrix} 3-\lambda & 4 & -1 \\ -1 & -2-\lambda & 1 \\ 3 & 9 & 0-\lambda \end{vmatrix} = -(\lambda-2)(\lambda-2)(\lambda+3)$

$\Rightarrow \lambda = 2, 2, -3$ $\Rightarrow$ not positive definite

$\lambda_1 = 2$ $\quad\quad \lambda = -3$

$\Rightarrow v_1 = \begin{vmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{vmatrix} \quad \Rightarrow v_2 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$

4 Probability Distribution

2.2

$$p(x|\mu) \; \cancel{\ldots} = \frac{1+\mu}{2} + \frac{1-\mu}{2} = 1$$

mean $p(x) = -1\left(\frac{1-\mu}{2}\right) + \frac{(1+\mu)}{2} = \mu$

variance $= (-1)^2\left(\frac{1-\mu}{2}\right) + 1^2\left(\frac{1+\mu}{2}\right) + \mu^2 = 1-\mu^2$

entropy $= -\left(\left(\left(\frac{1-\mu}{2}\right)\ln\left(\left(\frac{1-\mu}{2}\right)\right)\right) + \left(\left(\frac{1+\mu}{2}\right)\ln\left(\left(\frac{1+\mu}{2}\right)\right)\right)\right)$

$= -\left(\left(\frac{1-\mu}{2}\right)\ln\left(\left(\frac{1-\mu}{2}\right)\right)\right) - \left(\left(\frac{1+\mu}{2}\right)\ln\left(\frac{1+\mu}{2}\right)\right)$

2.10

$$\int \prod_{k=2}^{M} \mu_k^{\alpha_k - 1} \, d\mu = \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_M)}{\Gamma(\alpha_0)}$$

$$E[\mu_j] = \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_M)} \int \mu_j \prod_{k=1}^{M} \mu_k^{\alpha_k - 1} \, d\mu$$

$$= \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_M)} \cdot \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_j+1)\cdots\Gamma(\alpha_M)}{\Gamma(\alpha_0+1)}$$

$E[\mu_j] = \frac{\alpha_j}{\alpha_0}$

$\mathrm{var}[\mu_j] = E[\mu_j^2] - E[\mu_j]^2 = \frac{\alpha_j(\alpha_j+1)}{\alpha_0(\alpha_0+2)} - \frac{\alpha_j^2}{\alpha_0^2} = \frac{\alpha_j(\alpha_0-\alpha_j)}{\alpha_0^2(\alpha_0+1)}$

$$\mathrm{cov}[\mu_j \mu_l] = E(\mu_j \mu_l) - E(\mu_j) E[\mu_l]$$

$$\rightarrow \mathrm{cov}[\mu_j \mu_l] = \frac{\alpha_j \alpha_l}{\alpha_0(\alpha_0+1)} - \frac{\alpha_j \alpha_l}{\alpha_0 \alpha_0} = -\frac{\alpha_j \alpha_l}{\alpha_0^2(\alpha_0+1)}$$

2.12

$$\int_a^b \frac{1}{b-a}\, dx = \frac{b-a}{b-a} = 1$$

$$E(x) = \int_a^b \frac{1}{b-a} x\, dx = \left(\frac{x^2}{2(b-a)}\right)_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$$

$$E(x^2) = \int_a^b \frac{1}{b-a} x^2\, dx = \left(\frac{x^3}{3(b-a)}\right)_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3}$$

$$\mathrm{Var}(x) = E(x^2) - E(x)^2 = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{(b-a)^2}{12}$$

2.15

$$H(x) = -\int N(x|\mu,\Sigma) \ln N(x|\mu,\Sigma)\, dx$$

$$= \frac{1}{2}\left(D\ln(2\pi) + \ln|\Sigma| + \mathrm{Tr}[\Sigma^{-1}\Sigma]\right)$$

$$H(x) = \frac{1}{2}\left(D\ln(2\pi) + \ln|\Sigma| + D\right)$$