Ch1-Examples

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1 Chapter 1: Polynomial Rings

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```
[2]:  --initial setup
      --%mode=pretty
      --%timeout=10
```

```
[magic succeeded] mode = pretty
[magic succeeded] timeout = 10
```

1.1 Section 1.1: Ideals

In Example 1.4, we look at the following two ideals over $\mathbb{Q}[x]$:

$$I = \langle x^3 + 6x^2 + 12x + 8 \rangle \text{ and } J = \langle x^2 + x - 2 \rangle.$$

Let's try computing their intersection, sum, product, and quotient.

```
[3]:  R = QQ[x]  -- set our base ring
```

```
[7]:  I = ideal((x+2)^3)
      J = ideal((x+2)*(x-1))
      print(concatenate{"I: ", toString I})
      print(concatenate{"J: ", toString J})
```

```
I: ideal(x^3+6*x^2+12*x+8)

J: ideal(x^2+x-2)
```

```
[10]:  InJ = intersect(I,J)
       print(InJ)
       factor InJ_0 -- get the 0th (in this case, only) generator, factored
```

```
            4     3     2
ideal(- x  - 5x  - 6x  + 4x + 8)
```

```
[12]: IplusJ = I+J
      ideal(gens(gb IplusJ))
```

```
[15]: IprodJ = I*J
      print(IprodJ)
      factor IprodJ_0
```

```
         5     4      3     2
ideal(x  + 7x  + 16x  + 8x  - 16x - 16)
```

```
[18]: ImodJ = I:J
      print(ImodJ)
      factor ImodJ_0
```

```
         2
ideal(x  + 4x + 4)
```

In Example 1.7, we consider the ideal generated by the partial derivatives of the cubic $f = 2xyz - x^2 - y^2 - z^2 + 1$. This is the ideal

$$I = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle yz - x, xz - y, xy - z \rangle \subset \mathbb{R}[x, y, z].$$

```
[13]: R = QQ[x,y,z, MonomialOrder => GLex] --doing over QQ since CAS can't represent␣
      ↪RR
```

```
[14]: f = 2*x*y*z-x^2-y^2-z^2+1
```

```
[15]: vars R
```

```
[18]: partials = diff(vars R, f)
      print(partials)
      I = ideal(partials)
```

```
| 2yz-2x 2xz-2y 2xy-2z |
```

```
[19]: p = primaryDecomposition I
```

```
[20]: claim = intersect(p_1, p_2, p_3, p_4)
```

```
[25]: f % claim == 0 -- test membership
```

```
[26]: ideal(gens gb claim)
```

1.2   Section 1.2: Grobner Bases

In the beginning of the section, we make the following claim:

$$\langle 2x + 3y + 5z + 7, 11x + 13y + 17z + 19, 23x + 29y + 31z + 37 \rangle = \langle 7x - 16, 7y + 12, 7z + 9 \rangle.$$

[29]: `I = ideal(2*x+3*y+5*z+7, 11*x+13*y+17*z+19, 23*x+29*y+31*z+37)`

[30]: `gens gb I`

Example 1.18 presents us with multiple applications of Grobner Bases. Here, we provide the code for following such applications. Macaulay2 by default employs a graded reverse lexicographic monomial ordering, and our variables are ordered $x \succ y \succ z$ by order of definition.

[31]: `R = QQ[x,y,z, MonomialOrder=>Lex]`

For $n = 1$, computing the reduced Grobner basis means computing the greatest common divisor of the input polynomials:

[32]: `F = ideal(x^3-6*x^2-5*x-14, 3*x^3 +8*x^2 +11*x+10, 4*x^4 +4*x^3 +7*x^2-x-2)`

[33]: `G = gens gb F`

We can look at the ideal $\mathcal{F} = \{xy - z, xz - y, yz - x\}$ and compute its Grobner basis as well as its standard monomials

[36]:
```
F = ideal(x*y-z,x*z-y,y*z-x)
print(F)
G = gens gb F
```

`ideal (x*y - z, x*z - y, - x + y*z)`

[37]: `staircase = ideal leadTerm F`

[39]:
```
standardMonomials = R/staircase
basis standardMonomials
```

This input is a curve in the (y, z)-plane parametrized by two cubics in one variable x. We write this as $\mathcal{F} = \{y - x^3 + 4x, z - x^3 - x + 1\}$. The Grobner basis has the implicit equation of this curve as its second element.

[41]:
```
use R -- reset the global ring back to R = QQ[x,y,z]
F = ideal(y-x^3+4*x, z-x^3-x+1)
```

[42]: `G = gens gb(F)`

Let z be the sum of $x = \sqrt[3]{7}$ and $y = \sqrt[4]{5}$. We encode this in the set $\mathcal{F} = \{x^3 - 7, y^4 - 5, z - x - y\}$. The real number $z = \sqrt[3]{7} + \sqrt[4]{5}$ is algebraic of degree 12 over $\mathbb{Q}$. Its minimal polynomial is the first element in the Grobner basis.

[43]: `F = ideal(x^3-7, y^4-5, z - x - y)`

```
[44]: G = gens gb F
```

The elementary symmetric polynomials $\mathcal{F} = \{x+y+z, xy+xz+yz, xyz\}$ have the reduced Grobner basis $G = \{x+y+z, y^2+yz+z^2, z^3\}$. There are six standard monomials. The quotient $\mathbb{Q}[x,y,z]/I$ is the regular representation of the symmetric group S_3.

```
[45]: F = ideal(x+y+z,x*y+x*z+y*z, x*y*z)
```

```
[46]: G = gens gb F
```

```
[49]: staircase = ideal leadTerm F
      standardMonomials = R/staircase
      basis standardMonomials
```

```
[50]: S = R/F
```

1.3   Section 1.3: Dimension and Degree

(Example 1.32) Let $n = 2m$ be even and consider the monomial ideal $I = \langle x_1x_2, x_3x_4, x_5x_6, \ldots, x_{2m-3}x_{2m-2}, x_{2m-1}x_{2m}\rangle$. The dimension of I equals m and the degree of I equals $2m$. It is instructive to work out the Hilbert series and the Hilbert polynomial of I for $m = 3, 4$.

```
[51]: R = QQ[x1,x2,x3,x4,x5,x6]
```

```
[52]: I = ideal(x1*x2, x3*x4, x5*x6)
```

```
[53]: dim I
```

```
[54]: degree I
```

```
[55]: poincare I
```

```
[56]: numerator reduceHilbert hilbertSeries I
```