# Cavitation dynamics. I. A mathematical formulation

H. G. Flynn

---

**ARTICLES YOU MAY BE INTERESTED IN**

Cavitation dynamics: II. Free pulsations and models for cavitation bubbles
The Journal of the Acoustical Society of America **58**, 1160 (1975); https://doi.org/10.1121/1.380799

Bubble oscillations of large amplitude
The Journal of the Acoustical Society of America **68**, 628 (1980); https://doi.org/10.1121/1.384720

Nonlinear bubble dynamics
The Journal of the Acoustical Society of America **83**, 502 (1988); https://doi.org/10.1121/1.396145

Sonoluminescence and bubble dynamics for a single, stable, cavitation bubble
The Journal of the Acoustical Society of America **91**, 3166 (1992); https://doi.org/10.1121/1.402855

Numerical investigation of nonlinear oscillations of gas bubbles in liquids
The Journal of the Acoustical Society of America **59**, 283 (1976); https://doi.org/10.1121/1.380884

Acoustic cavitation prediction
The Journal of the Acoustical Society of America **69**, 1624 (1981); https://doi.org/10.1121/1.385939

---

ASA
ACOUSTICAL SOCIETY
OF AMERICA

**Across Acoustics**

The official podcast highlighting authors' research from our publications

# Cavitation dynamics. I. A mathematical formulation*

## H. G. Flynn

*Acoustical Physics Laboratory, Department of Electrical Engineering, The University of Rochester, Rochester, New York 14627*
(Received 16 July 1973)

A set of equations has been derived to describe the dynamical motions of small cavitation bubbles in liquids set into motion by an acoustic pressure field. The mathematical formulation takes into account heat conduction inside a bubble and in the surrounding liquid, and the viscosity, compressibility, and surface tension of the liquid. The effect of vapor pressure may be determined as a function of the interfacial temperature. The formulation consists of nonlinear ordinary differential, integral, and algebraic equations, and has been programmed for solution on a digital computer.

Subject Classification: 25.60; 35.68.

## LIST OF SYMBOLS

| | |
|---|---|
| $a^*$ | speed of sound in liquid |
| $a_n^*$ | equilibrium speed of sound in liquid |
| $a$ | $=a^*/a_n^*=$ dimensionless speed of sound in liquid |
| $A$ | dimensionless speed of sound in liquid at interface |
| $b$ | $=3\gamma$ |
| $c$ | subscript denoting properties of cavity depending only on time |
| $c_n^*$ | $=(a_n^*)^2/\theta_n^*$ |
| $c_v^*$ | specific heat per unit mass at constant volume |
| $c_v$ | $=c_v^*/c_n^*=$ dimensionless specific heat at constant volume |
| $f(t)$ | specified, time-dependent pressure at infinity (Eqs. 10a and 10b) |
| $F$ | auxiliary function (Eqs. 108 and 115) |
| $g_k(t)$ | $=12H_k(t)/(2k+1)\pi c\, c_{cv}$ (Eq. 80) |
| $G(m_l,\ \tau_l\,|\,m'_l,\ \tau'_l)$ | Green's function (Eq. 102) |
| $G(t)$ | auxiliary function (Eq. 82) |
| $h_k(m,\ \tau)$ | auxiliary function (Eq. 70) |
| $h(m,\ \tau)$ | auxiliary function (Eq. 63) |
| $H(\tau)$ | $=h(1,\ \tau)$ |
| $H^*$ | thermodynamic enthalpy per unit mass |
| $H$ | $=H^*/(a_n^*)^2=$ dimensionless thermodynamic enthalpy (Eq. 13) |
| $H_k(\tau)$ | auxiliary function (Eq. 72) |
| $k^*$ | thermal conductivity |
| $k_n^*$ | $=p_{ln}^*R_n^*a_n^*/\theta_n^*$ |
| $k$ | $=k^*/k_n^*=$ dimensionless thermal conductivity |
| $k/c_v$ | $=a_n^*k^*/p_{ln}^*R_n^*c_v^*$ |
| $L_v^*$ | latent heat of evaporation per unit mass |
| $L_v$ | $=L_v^*/a_n^{*2}=$ dimensionless latent heat of evaporation |
| $l$ | subscript denoting properties of liquid |
| $m$ | material mass coordinate (Eqs. 34 and 93) |
| $n$ | subscript denoting equilibrium properties of liquid or cavity at $t=0$ |
| $p^*$ | pressure |
| $p_{ln}^*$ | equilibrium pressure in liquid |
| $p$ | $=p^*/p_{ln}^*=$ dimensionless pressure |
| $p_c(t)$ | dimensionless uniform, time-dependent pressure in cavity |

| | |
|---|---|
| $p_{cn}^*$ | $=p_{gn}^*+p_{vn}^*=$ equilibrium pressure of cavity contents |
| $p_{cn}$ | $=p_{cn}^*/p_{ln}^*=p_{gn}+p_{vn}=$ dimensionless equilibrium pressure of cavity contents |
| $p_g^*$ | pressure of gas in cavity |
| $p_g$ | $=p_g^*/p_{ln}^*=$ dimensionless pressure of gas in cavity |
| $p_{gn}^*$ | equilibrium gas pressure |
| $p_{gn}$ | $=p_{gn}^*/p_{ln}^*=$ dimensionless equilibrium gas pressure |
| $p_v^*$ | vapor pressure of liquid |
| $p_v$ | $=p_v^*/p_{ln}^*=$ dimensionless vapor pressure of liquid $=p_{vn}\theta^{Bv}$ |
| $p_{vn}^*$ | equilibrium vapor pressure at $\theta_n^*$ |
| $p_{vn}$ | dimensionless equilibrium vapor pressure at $\theta=1$ |
| $p_\infty^*(t^*)$ | $=p_{ln}^*-f^*(t^*)=$ time-dependent pressure at infinity |
| $p_\infty(t)$ | $=p_\infty^*(t^*)/p_{ln}^*=1-f(t)=$ dimensionless time-dependent pressure at infinity |
| $P(t)$ | dimensionless pressure in liquid at interface |
| $P_A^*$ | amplitude of acoustic pressure |
| $P_A$ | $=P_A^*/p_{ln}^*=$ dimensionless amplitude of acoustic pressure (Eq. 10a) |
| $Q^*$ | heat transferred across interface |
| $Q_n^*$ | $=p_{ln}^*R_n^{*3}$ |
| $Q$ | $=Q^*/Q_n^*=$ dimensionless heat transferred across interface |
| $r^*$ | radial distance from center of cavity |
| $R_n^*$ | equilibrium radius of cavity |
| $r$ | dimensionless radial distance from center of cavity $=r^*/R_n^*$ |
| $R^*(t^*)$ | radius of cavity |
| $R(t)$ | $=R^*/R_n^*=$ dimensionless radius of cavity |
| $s_c^*$ | entropy of cavity contents |
| $s_n^*$ | $=p_{ln}^*R_n^{*3}/\theta_n^*$ |
| $s_c$ | dimensionless entropy of cavity contents (Eq. 90) |
| $t^*$ | time |
| $t_n^*$ | $=R_n^*/a_n^*$ |
| $t$ | $=t^*/t_n^*=$ dimensionless time |
| $\bar{t}$ | pulse-location parameter (Eq. 10b) |
| $T(t)$ | $=\theta_l(1,\ t)=\theta_l^*(1,\ t^*)/\theta_n^*=$ dimensionless time-dependent temperature in liquid at interface (Eq. 110) |

Copyright © 1975 by the Acoustical Society of America    1379

| | | | | |
|---|---|---|---|---|
| $u^*$ | radial particle velocity | $\psi$ | auxiliary variable (Eq. 46) |
| $u$ | $= u^*/a_n^* =$ dimensionless radial particle velocity | $\psi_0$ | $= T + \psi - 1$ (Eq. 52) |
| $U^*$ | speed of interface | $\rho^*$ | density in mass per unit volume |
| $U$ | $= U^*/a_n^* =$ dimensionless speed of interface | $\rho_n^*$ | $= p_{1n}^*/a_n^{*2}$ |
| $V^*$ | $= 4\pi R^{*3}/3 =$ volume of cavity | $\rho$ | $= \rho^*/\rho_n^* =$ dimensionless density |
| $w^*$ | work done on or by contents of cavity | $\rho_{cn}^*$ | equilibrium density in cavity |
| $w$ | $= w^*/Q_n^* =$ dimensionless work done on contents of cavity (Eq. 118) | $\rho_{cn}$ | $= \rho_{cn}^*/\rho_n^* =$ dimensionless equilibrium density in cavity |
| $W$ | auxiliary variable (Eq. 45) | $\rho_{1n}^*$ | equilibrium density of liquid |
| $\Delta$ | $= \nabla \cdot \mathbf{v}$ | $\rho_{1n}$ | $= \rho_{1n}^*/\rho_n^* =$ dimensionless equilibrium density of liquid |
| $\epsilon^2$ | $= k_c/k_l$ | $\rho_c$ | $= \rho_{cn} R^{-3} =$ dimensionless thermodynamic density in cavity |
| $\gamma$ | $= c_v^*/c_p^*$ | $\sigma^*$ | coefficient of surface tension |
| $\lambda^*$ | pulse-width parameter | $\sigma$ | $= \sigma^*/p_{1n}^* R_n^* =$ dimensionless coefficient of surface tension |
| $\lambda$ | $= \lambda^* t_n^* =$ dimensionless pulse-width parameter (Eq. 10b) | $\tau$ | material time coordinate (Eqs. 39 and 94) |
| $\lambda_A^*$ | acoustic pressure amplitude of pulse | $\theta^*$ | temperature |
| $\lambda_A$ | $= \lambda_A^*/p_{1n}^* =$ dimensionless acoustic pressure amplitude of pulse (Eq. 10b) | $\theta_n^*$ | initial temperature in liquid and cavity |
| $\mu^*$ | coefficient of shear viscosity | $\theta$ | $= \theta^*/\theta_n^* =$ dimensionless temperature |
| $\mu$ | $= a_n^* \mu^*/p_{1n}^* R_n^* =$ dimensionless coefficient of shear viscosity | $\theta_a$ | average dimensionless temperature in cavity (Eq. 81) |
| $\omega^*$ | frequency in radians/sec | $\theta_c$ | thermodynamic temperature in cavity (Eq. 84) |
| $\omega$ | $= \omega^* t_n^* =$ dimensionless frequency | $\theta_1$ | theta function of first kind (Eq. 67) |
| $\phi$ | temperature potential (Eq. 36) | | |

## INTRODUCTION

A central problem in the study of acoustic cavitation is that of understanding the dynamics of small, isolated bubbles set in motion in a liquid by a sound field. The complicated, nonlinear nature of such motions has served in the past to limit investigations to the study of very simple models of such bubbles. However, the advent of large digital computers now makes it possible to construct more realistic models with a reasonable expectation that numerical calculation of their motions will give us a deeper insight into cavitation phenomena.

In this paper we construct a mathematical formulation that enables us to study the motions of such models and the effects of heat conduction, shear viscosity, compressibility, and surface tension on their dynamical behavior. The formulation is a "large amplitude" one in that it is specifically designed to describe the motion of a bubble that expands to some maximum radius and then contracts violently. For such a motion, we accept approximations that considerably simplify our mathematical problems and in the end enable us to place upper or lower bounds on physical quantities of interest, such as the pressure or temperature within the bubble. This formulation consists of a set of nonlinear equations that are solved simultaneously on a digital computer. In obtaining such a formulation, we keep two objectives in mind.

(1) The equations must be such that solutions can be found by a few minutes' computation on a computer.

(2) Predictions made by the solutions must at least give us reliable estimates of the order of magnitude of quantities associated with cavitation bubbles in violent motion.

To obtain the objective of computational enonomy, we introduce a series of approximations and assumptions that enable us to transform the partial differential equations for conservation of mass, momentum, and energy in a fluid into a set of much simpler equations. The validity of these approximations and assumptions is judged *ex post facto* by comparing our solutions with more exact solutions whenever they exist. The second objective will be attained if this comparison with more exact solutions would allow us to predict with confidence whether or not the pressure in a contracting bubble would be 100, 1000, or 10 000 bars, or whether the temperature in it would be 300°, 3000°, or 30 000 °K. Because measurement of these quantities at present eludes experimental techniques, it would be useful to obtain even upper or lower limits on such quantities. In principal it is possible to integrate directly the exact equations of motion for a fluid, but the time required to compute one solution would be in the order of hours rather than minutes.

The formulation is used mainly to study the simultaneous effects of heat conduction, shear viscosity, and compressibility on the dynamics of bubbles. Although all three of these effects are commonly called "dissipative" in the sense that they damp the motion of a freely pulsating bubble, we shall see that heat conduction in general increases the violence of bubble motions, while shear viscosity and compressibility decrease it. In many instances, this competition between the effect of heat conduction and the effects of shear viscosity and compressibility makes it unrealistic to study the effect of heat conduction alone.

It is convenient to use a consistent nomenclature recently suggested for cavitation dynamics.[1] In this no-

menclature, the word "cavity" denotes a model, while the word "bubble" is reserved for the physical system that the model represents. In order to obtain a realistic representation of these small dynamic systems, we need several models. One of these is the "transient cavity" and a second is the "stable cavity." The transient cavity was introduced as a model for bubbles that collapse violently, while the stable cavity was used to represent bubbles that pulsate over relatively long periods of time. In a succeeding paper, we shall use our mathematical formulation to obtain more precise definitions of transient and stable cavities.

At this point it might be well to summarize the plan of this paper. We first find two first-order, nonlinear ordinary differential equations that are equations of motion for the interface of a cavity. In these equations appear two functions, $p_\infty(t)$ and $p_c(t)$. The pressure $p_\infty(t)$ in the liquid at infinity is some specified function of time, while $p_c(t)$ is a spatially uniform, time-dependent pressure of the gas and vapor within the cavity and must be determined in the calculation. These equations of motion explicitly take into account the effects of viscosity, compressibility, and surface tension, and implicitly the effect of heat conduction through $p_c(t)$. We then determine a set of first-order, nonlinear differential equations, an integral solution of another differential equation, and several algebraic equations that enable us to calculate the cavity pressure $p_c(t)$, an average cavity temperature $\theta_c(t)$, an average cavity entropy $s_c(t)$, the temperature $T(t)$ in the liquid at the interface, and $w(t)$, the mechanical work done on the cavity. Calculation of $w(t)$ enables us to predict the mechanical energy supplied to a cavity during collapse and the fraction of that energy dissipated by heat conduction, viscosity, and sound radiation.

This paper is limited mainly to the problem of obtaining this ensemble of equations and of attempting to test its reliability. In succeeding papers, this formulation will be used to study how transient cavities collapse, how stable cavities pulsate, and how the motions of both stable and transient cavities are changed by the onset of Taylor instabilities in the interface between the liquid and the cavity.

It must be emphasized that in all instances we are determining the motion of a pre-existing inhomogeneity in an infinite liquid, and in no case are we attempting to determine how a homogeneous liquid ruptures. Unless otherwise stated, this pre-existing inhomogeneity is taken to be a small spherical cavity, filled with a mixture of gas and vapor and initially at rest in the liquid. It is assumed that this initial amount of gas remains constant throughout the motion, but that at times we may allow the amount of vapor to vary as the cavity expands or contracts. Thus, diffusion of gas is always excluded, but not evaporation and condensation at the interface.

The motion of the liquid, the interface, and the contents of the cavity are everywhere and at all times assumed to be spherically symmetrical about the center of the cavity, which remains at rest at the origin of a spherical coordinate system. The requirement that the center of the cavity remain fixed excludes consideration of any effects of a translation velocity.

The crucial role of heat conduction in the dynamics of a cavity arises from its effect on the pressure within the cavity. We simplify the problem of finding this dependence of the cavity pressure on heat conduction by assuming that the gas-vapor mixture in the cavity is a heat-conducting, but nonviscous, fluid whose pressure is a spatially uniform function of time—$p_c(t)$—and whose particle velocity is a linear function of the radial distance $r$ from the center of the cavity. By these assumptions we avoid calculating any details of the motion within the cavity and can find an average temperature $\theta_c(t)$ in the cavity from the energy equation alone. Then, given $\theta_c(t)$ and the radius $R(r)$ of the cavity, the pressure $p_c(t)$ is easily found from an equation of state for the cavity contents.

Similarly, we simplify the treatment of heat flow in the liquid by decoupling the effect of heat conduction and the effect of compressibility. We achieve this separation by taking the density of the liquid to be constant in any relation where the compressibility of the liquid might affect heat conduction.

In somewhat the same fashion we uncouple the effect of shear viscosity from the effect of compressibility. In a spherically symmetric motion in an incompressible liquid, the stress arising from volume viscosity is zero everywhere, the terms due to both shear and volume viscosity disappear from the momentum equation, and the only effect of viscosity comes from the shear stress at a boundary.[2,3] Therefore, we may greatly simplify our formulation if we represent the effect of viscosity by the relations appropriate to an incompressible liquid.

The interaction of a cavity with an acoustic wave is a complicated scattering problem that we avoid by requiring that a wavelength of the sound in the liquid be much greater than the equilibrium radius of the cavity. When the radius is much less than a wavelength of sound in the liquid, we can reasonably represent the motion of the cavity as being brought about by a spherically symmetric pressure field, in the liquid, which at the interface is equal to the pressure within the cavity and at infinity is equal to a prescribed function of time—$p_\infty(t)$. Thus we replace the sound field by the time-varying pressure $p_\infty(t)$.

The representation of the sound field by a time-dependent pressure at infinity raises a conceptual difficulty in deriving an equation of motion for the cavity interface. We are required to relate this pressure $p_\infty(t)$ to the cavity radius $R(t)$ at the same instant of time. Now, in a compressible liquid, changes in pressure propagate at a finite speed and we should connect events far from and at the cavity by a retarded time. On the other hand, in an incompressible liquid, pressure changes are instantly communicated to every point in the liquid, and this difficulty does not arise. We circumvent this paradox by assuming that in the liquid the speed of sound is so large that changes in $p_\infty(t)$ are effectively communicated to the cavity interface at the same instant of time.

For simplicity we assume in this paper that the gas and vapor in the cavity behave as perfect gases, and that the liquid has a constant speed of sound (that is, the den-

sity of the liquid varies linearly with the pressure). Obviously, such restrictions must be removed if we attempt to predict the maximum temperature, pressures, and speeds attained in the violent collapse of bubbles. Hence, in the subsequent study of cavity collapse, the final phase of collapse is described by still another formulation in which both the liquid and the cavity contents are represented by more realistic equations of state. The values of any extreme temperature or pressure calculated in the present paper may be modified by a subsequent use of different equations of state.

A series of investigators has studied the effects of heat conduction, viscosity, and sound radiation on steady-state, sinusoidal, small-amplitude motions of cavities under the influence of an acoustic field. On the other hand, few have studied the time-dependent, large-amplitude motions of cavities in a sound field, and those few have confined their investigations to nondissipative motions in an incompressible liquid.

The first significant theoretical studies of the damping of free and forced pulsations of cavities were carried out by Pfriem, [4] Saneyosi, [5] and Spitzer. [6] The work of Spitzer was based on a result of H. F. Willis and is referred to as the Willis–Spitzer theory. All of these workers came essentially to the same conclusions about the predicted behavior of cavities, and it has generally been considered that their predictions have been borne out by the experimental work of E. Meyer and a succession of his students. The most recent theoretical work has been the very comprehensive studies by Hsieh and Plesset, [7] Hsieh, [8] and Lauterborn. [9] Most of these theoretical and experimental investigations are summarized in Ref. 1 and in Devan. [10]

Noltingk and Neppiras[11] were the first to derive a differential equation (DE) for the motion of a cavity under the influence of a time-dependent pressure field. They took the liquid to be incompressible and the motions to be nondissipative, and solved their DE on a digital computer. Later Güth[12] found some approximate solutions to the nonlinear motions of a cavity under similar conditions. The most recent calculations of this model have been carried out by Borotnikova and Solovkhin, [13] Woo and Palsey, [14] and Akulichev. [15] A useful review of this work has been published by Kapustina. [16]

The bulk of studies on cavitation dynamics has dealt with the behavior of cavities collapsing under a steady pressure at infinity, the first of these being the pioneer work of Lord Rayleigh. [17] Here we note only the studies of Poritsky[2] and Yang and Yeh[3] on effect of viscosity and surface tension, of Gilmore[18] and Trilling[19] on the effect of compressibility, of Hickling[20] on the effect of heat conduction, of Hickling and Plesset[21] on the effect of compressibility, and of Ivany and Hammitt[22] on the combined effects of viscosity and compressibility. We use the numerical solutions of Hickling, of Hickling and Plesset, and of Ivany and Hammitt as the more "exact" solutions to which we compare our results.

As a convenient reference for the remainder of this paper, let us write down the equations of mass, momentum, and energy for spherically symmetric motions in a

compressible, heat-conducting, nonviscous fluid.

*Mass equation:*

$$\frac{1}{\rho^*}\left[\frac{\partial \rho^*}{\partial t^*} + u^*\frac{\partial \rho^*}{\partial r^*}\right] = -\left[\frac{\partial u^*}{\partial r^*} + \frac{2u^*}{r^*}\right].\tag{1}$$

*Momentum equation:*

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial r^*} = -\frac{1}{\rho^*}\frac{\partial p^*}{\partial r^*}.\tag{2}$$

*Energy equation:*

$$\rho^* c_v^*\left[\frac{\partial \theta^*}{\partial t^*} + u^*\frac{\partial \theta^*}{\partial r^*}\right] = k^*\nabla^2\theta^* - p^*\nabla \cdot \mathbf{u}^*.\tag{3}$$

Here we have assumed that the internal energy of the fluid is given by $\epsilon^* = c_v^*\theta^*$.

In these equations an asterisk attached as a superscript to a symbol denotes a physical quantity expressed in some convenient system of units, such as the MKS system. Thus in such a system of units $\rho^*$ is the density, $u^*$ is the radial velocity, $r^*$ is radial distance, $t^*$ is the time, $p^*$ is the pressure, $\theta^*$ is the temperature, $c_v^*$ is the specific heat at constant volume, and $k^*$ is the thermal conductivity of the fluid.

We rewrite this set of equations by introducing a new set of variables:

$$r = r^*/r_n^*, \quad p = p^*/p_n^*, \quad c_v = c_v^*/c_n^*,$$
$$u = u^*/u_n^*, \quad \theta = \theta^*/\theta_n^*,$$
$$\rho = \rho^*/\rho_n^*, \quad t = t^*/t_n^*, \quad k = k^*/k_n^*,$$

where $r_n^*$, $u_n^*$, $\rho_n^*$, $p_n^*$, $\theta_n^*$, $t_n^*$, $c_n^*$, and $k_n^*$ are constants. It is convenient to make the choices

$$r_n^* = R_n^*, \quad p_n^* = p_{in}^*,$$
$$u_n^* = a_n^*, \quad \theta_n^* = \theta_{cn}^*.$$

Then we have a consistent system of nondimensional equations and variables if we set

$$t_n^* = R_n^*/a_n^*, \quad c_n^* = (a_n^*)^2/\theta_n^*,$$
$$\rho_n^* = p_{in}^*/(a_n^*)^2, \quad k_n^* = \frac{p_{in}^* R_n^* a_n^*}{\theta_n^*}.$$

Here $R_n^*$ is the initial, equilibrium radius of the cavity, $a_n^*$ is the speed of sound in the liquid, $p_{in}^*$ is the equilibrium pressure in the liquid, and $\theta_n^*$ is the equilibrium temperature in both the cavity and the liquid.

In terms of these new variables, Eqs. 1–3 become

$$\frac{1}{\rho}\left[\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r}\right] = -\left[\frac{\partial u}{\partial r} + \frac{2u}{r}\right] = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u),\tag{4}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r},\tag{5}$$

$$\rho\left(\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial r}\right) = \frac{k}{c_v}\nabla^2\theta - \frac{1}{c_v}p(\nabla \cdot \mathbf{v}) = \left(\frac{k}{c_v}\right)\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right)$$
$$-\frac{1}{c_v}\frac{p}{r^2}\frac{\partial}{\partial r}(r^2 u),\tag{6}$$

where

$$\frac{k}{c_v} = \frac{a_n^* k^*}{p_{in}^* R_n^* c_v^*},$$

$$\frac{1}{c_v} = \frac{(a_n^*)^2}{\theta_n^* c_v^*} .$$

We later assume that $k$ and $c_v$ may be functions of pressure and temperature in the cavity.

Quantities that are functions of space and time within the cavity carry no subscripts. Thus, $u(r, t)$ is the particle velocity within the cavity. Quantities associated with the cavity contents that are functions only of time are denoted by a lower case $c$ subscript. Thus, $p_c(t)$ is the uniform pressure within the cavity. We refer to such uniform functions of time as thermodynamic quantities. Quantities that are associated with the liquid are denoted by a lower case $l$. Thus, $p_l(r, t)$ is the pressure in the liquid. Quantities associated with the interface are denoted by capital letters. Thus, $P(t)$ is the pressure in the liquid at the interface and $U(t)$ is the speed of the interface.

Throughout we use the convention that work done by the cavity is positive and work done on the cavity is negative. Heat added to the cavity is positive and heat lost by the cavity negative.

## I. EQUATIONS OF MOTION FOR THE CAVITY INTERFACE

Our first task is to derive a pair of first-order DE's that constitute the equations of motion for the cavity interface. By assumption, the center of the cavity is at rest at the origin of coordinates and the motions of the interface and the liquid are always spherically symmetrical. The liquid is infinite in extent.

We do not attempt to determine the detailed motion of the gas-vapor mixture within the cavity. In addition, we are assuming that the viscosity of the liquid enters the problem only as a boundary condition and that the flow of heat in the liquid itself has negligible effect on the state of the liquid. Consequently, the radius-time curves for the interface are determined by the nondissipative equations of mass and momentum for the liquid. Our assumptions imply that neither viscosity nor heat flow changes the entropy of the liquid. Thus, when we characterize the liquid as having a constant "speed of sound," we mean the usual adiabatic speed of sound.

We start by rewriting Eqs. 4 and 5:

$$\frac{1}{\rho_l}\left[\frac{\partial \rho_l}{\partial t} + u_l \frac{\partial \rho_l}{\partial r}\right] = -\left[\frac{\partial u_l}{\partial r} + \frac{2u_l}{r}\right] = -\Delta_l, \tag{7}$$

$$\frac{\partial u_l}{\partial t} + u_l \frac{\partial u_l}{\partial r} + \frac{1}{\rho_l}\frac{\partial p_l}{\partial r} = 0, \tag{8}$$

where $u_l$, $p_l$, $\rho_l$, and $\Delta_l$ are the particle velocity, pressure, density, and divergence of the liquid, respectively.

The boundary conditions that we must satisfy are that (1) at the interface the radial stress must be continuous and (2) at infinity the velocity, the gradient of the velocity, and the acceleration are assumed to vanish, and

$$p_l(r, t) \rightarrow p_\infty(t), \tag{9}$$

where

$$p_\infty(t) = 1 - f(t) \tag{10}$$

and $f(t)$ is a prescribed function of time. In this series of papers, $f(t)$ is either a sinusoidal function of time or a Gaussian pulse. Thus we may have

$$f(t) = P_A \sin \omega t, \tag{10a}$$

where $P_A$ is the pressure amplitude and $\omega$ is the radian frequency, or

$$f(t) = \lambda_A \exp[-\lambda(t - \bar{t})^2], \tag{10b}$$

where $\lambda_A$ is the pressure amplitude, $\lambda$ is a parameter controlling the width of the pulse, and $\bar{t}$ is a parameter that locates the maximum of the pulse in time.

We follow C. Herring[23] in constructing two ordinary DEs relating the motion of a cavity in a linear liquid to a variational pressure at infinity. This derivation is restricted to infinitesimal motions of a cavity, and we infer the proper corrections arising from finite motions from an equation describing unrestricted motions of a cavity in a nonlinear liquid under a constant pressure at infinity. This method of obtaining the correcting factors was adopted here to avoid the difficult problem of treating a cavity driven by a wave coming from infinity. The resulting set of equations is equivalent to a result obtained by Akulichev, Boguslavskii, Ioffe, and Naugol'nykh.[24]

First, let us set $\rho_l$ equal to its equilibrium value $\rho_{l n}$ in the momentum equation, and then integrate that equation with respect to $r$, at a fixed instant, from the interface to infinity. Then we have

$$\int_R^\infty \frac{\partial u_l}{\partial t} dr' - \tfrac{1}{2} U^2 + \frac{1}{\rho_{l n}}[p_\infty(t) - P(t)] = 0, \tag{11}$$

where $U(t)$ is the speed of the interface. Equation 11 related the pressure $p_\infty(t)$ at infinity to the pressure $P(t)$ in the liquid at the same instant of time. The remaining integral may be written as

$$\int_R^\infty \frac{\partial u_l}{\partial t} dr' = \int_R^\infty \frac{1}{r'^2}\frac{\partial}{\partial t}(r'^2 u_l) dr'.$$

Hence, through the definitions

$$\frac{du_l}{dt} = \frac{\partial u_l}{\partial t} + u_l \frac{\partial u_l}{\partial r},$$

$$\Delta_l = \frac{\partial u_l}{\partial r} + \frac{2u_l}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_l),$$

we may obtain, by a partial integration,

$$\int_R^\infty \frac{\partial u_l}{\partial t} dr' = R\frac{dU}{dt} - RU\Delta_R + 2U^2 + \int_R^\infty r'\frac{\partial \Delta_l}{\partial t} dr',$$

where $\Delta_R = \Delta(R, t)$. Hence we have

$$R\frac{dU}{dt} - RU\Delta_R + \tfrac{3}{2}U^2 + \frac{1}{\rho_{l n}}[p_\infty(t) - P(t)] + \int_R^\infty r'\frac{\partial \Delta_l}{\partial t} dr' = 0.$$

Now by Eq. 4, the divergence can also be written as

$$\Delta_l = -\frac{1}{\rho_l}\left[\frac{\partial \rho_l}{\partial t} + u_l \frac{\partial \rho_l}{\partial r}\right] = -\frac{1}{\rho_l}\frac{d\rho_l}{dt} .$$

Since the entropy of the liquid does not change during the motion, the density is a function of the pressure alone, and we have

$$\Delta_l = -\frac{1}{\rho_l}\frac{d\rho_l}{dp_l}\frac{dp_l}{dt} .$$

By definition, the sound speed is

$$\frac{dp_l}{d\rho_l} = a_l^2,$$

where $a_l = a_l/a_n^0$. Hence,

$$\Delta_l = -\frac{1}{\rho_l a_l^2}\frac{dp_l}{dt}.$$

We let $\rho_l = \rho_{ln}$ and $a_l = 1$. Then

$$\Delta_l = -\frac{1}{\rho_{ln}}\frac{dp_l}{dt}$$

and

$$R\frac{dU}{dt} + \frac{3}{2}U^2 + \frac{RU}{\rho_{ln}}\frac{dP}{dt} + \frac{[p_\infty(t) - P(t)]}{\rho_{ln}} + \int_R^\infty r'\frac{\partial\Delta_l}{\partial t}dr' = 0.$$

Herring recognized that in the acoustic approximation the quantity $r\Delta_l$ is a characteristic invariant in the sense that

$$\frac{\partial}{\partial t}(r\Delta_l) + \frac{\partial}{\partial r}(r\Delta_l) = 0.$$

Then the remaining integral may be rewritten as

$$\int_R^\infty r'\frac{\partial\Delta_l}{\partial t}dr' = -[r\Delta_l]_R^\infty = R\Delta_R = -\frac{R}{\rho_{ln}}\frac{dP}{dt}.$$

We may now combine these various equations and obtain

$$R\frac{dU}{dt} + \frac{3}{2}U^2 = \frac{1}{\rho_{ln}}\left[P(t) - p_\infty(t) + R(1-U)\frac{dP}{dt}\right]. \qquad (12)$$

This DE is a first-order ordinary differential equation appropriate for describing the motion of a cavity in a liquid with a constant sound speed when the motions are infinitesimal. The equation contains a term $(1 - U)$ that represents a correction factor arising from the compressibility of the liquid when the motions are infinitesimally small. When such motions are finite, other factors caused by compressibility come into play. Herring,[23] and later Trilling,[19] derived equations containing some of these factors, but their equations are less useful for our purposes than the one to be proposed. Rather than following Herring or Trilling, we determine such factors by drawing an analogy between Eq. 12 and an equation derived by Gilmore[18] as a consequence of the Kirkwood–Bethe assumption.

Gilmore's equation describes the motion of a cavity in a compressible liquid when the pressure $p_\infty$ at infinity is a constant rather than a function of time. It has the form

$$R\left[1 - \frac{U}{A}\right]\frac{dU}{dt} + \frac{3}{2}\left[1 - \frac{1}{3}\frac{U}{A}\right]U^2 = \left[1 + \frac{U}{A}\right]H + \frac{R}{A}\left[1 - \frac{U}{A}\right]\frac{dH}{dt}. \qquad (13)$$

Here the function $A$ is the sound speed and the function $H$ is the enthalpy in the liquid at the interface. It is this equation that we shall use in a later paper to describe a transient cavity in its final stage of collapse. We find that during this final stage of collapse the compression of the cavity contents is essentially adiabatic and that Gilmore's equation is especially useful in describing such adiabatic behavior.

Let us here assume that the sound speed $A$ is a constant and equal to its equilibrium value of unity in our system of variables. We approximate $H(t)$ and $dH/dt$ by

$$H = \frac{[P(t) - p_\infty]}{\rho_{ln}},$$

$$dH/dt = \frac{dP/dt}{\rho_{ln}},$$

where $p_\infty$ is a constant. Gilmore's equation now becomes

$$R(1-U)\frac{dU}{dt} + \frac{3}{2}\left(1 - \frac{1}{3}U\right)U^2$$

$$= \frac{1}{\rho_{ln}}\left\{(1+U)[P(t) - p_\infty] + R(1-U)\frac{dP}{dt}\right\}. \qquad (14)$$

This equation is appropriate for the description of the motion of a cavity under a constant pressure $p_\infty$ at infinity. Except for the new factors on the left-hand side and in the first term on the right-hand side, this equation has the same form as Eq. 12. These three new factors take account of the effect of finite motions in a liquid with a constant sound speed, and we make the assumption that the same additional correction factors appear when the pressure of infinity is a function of time— $p_\infty(t)$, as it is in Eq. 12. The procedure of Herring and Trilling yields factors of $(1 - 2U)$ and $(1 - \frac{4}{3}U)$ in the left-hand side of Eq. 12.

Under such an assumption Eq. 12 becomes

$$R(1-U)\frac{dU}{dt} + \frac{3}{2}\left(1 - \frac{1}{3}U\right)U^2 = \frac{1}{\rho_{ln}}\left\{(1+U)[P(t) - p_\infty(t)]\right.$$

$$\left. + R(1-U)\frac{dP}{dt}\right\}. \qquad (15)$$

This choice of new compressibility factors is motivated by the fact that Gilmore's solution is a surprisingly good approximation to exact solutions obtained by Hickling and Plesset.[21] By use of this proposed equation we are able to extend the calculation of cavity collapse into domains where the compression is essentially adiabatic and Gilmore's equation may be used.

Equation 15 is one of our fundamental equations of motion for a cavity moving under the influence of a time-dependent pressure at infinity. To this equation we add a second:

$$dR/dt = U. \qquad (16)$$

The assumption of a constant sound speed in the liquid leads to solutions that overestimate the effects of compressibility of the motion of a cavity. Compressibility enters the DE for $U$ through the ratio $U/A$, where $A$ is the speed of sound in the liquid at the interface. When $A$ is not a constant and the cavity pressure becomes large enough, $A$ may increase to values greater than unity. The ratio $U/A$ is then reduced in magnitude and the acceleration of the interface is increased. Thus we anticipate that the assumption of a constant sound speed $A = 1$ will give us a lower limit on the magnitude of $U$, and that an "exact" solution would predict speeds for the cavity interface less than those for an incompressible liquid but greater than for a liquid with a constant sound speed.

We have now found two first-order DEs for determining the motion of a cavity. Our problem now reduces to finding an appropriate expression for $P(t)$, the pressure

in the liquid at the interface.

An expression for $P(t)$ can be determined from the requirement of continuity of stress across the interface. This condition implies

$$p_c^*(t) = P^*(t) + 2\sigma^*/R^* + 4\mu^* U^*/R^*,$$

where $\sigma^*$ is the surface-tension coefficient and $\mu^*$ is the shear-viscosity coefficient. In our system of variables, this equation becomes

$$p_c(t) = P(t) + 2\sigma/R + 4\mu U/R .\qquad(17)$$

Here $p_c(t)$ is the spatially uniform pressure in the cavity, $\sigma$ is the dimensionless surface-tension coefficient, and $\mu$ is the dimensionless shear-viscosity coefficient. We note that $\sigma = \sigma^*/p_{ln}^* R_n^*$ and $\mu = a_n^* \mu^*/p_{ln}^* R_n^*$.

The term $2\sigma/R$ is the usual expression for pressure due to surface tension, but the viscosity term merits some explanation. The radial component of the stress tensor can be written as

$$T_{rr} = -p + \lambda \nabla \cdot \mathbf{u} + 2\mu(\partial u/\partial r),$$

where $\lambda =$ dimensionless volume-viscosity coefficient. In conformity with our plan to represent viscous effects by expressions appropriate to a incompressible liquid, we write

$$\nabla \cdot \mathbf{u} = \partial u/\partial r + 2u/r = 0,$$

so that

$$T_{rr} = -p - 4\mu u/r .$$

Thus the term $4\pi\mu/R$ does represent the effect of viscosity in the liquid. Now we write the pressure at infinity as

$$p_\infty(t) = 1 - f(t),$$

the pressure in the liquid at the interface as

$$P(t) = p_c(t) - 4\mu U/R - 2\sigma/R ,\qquad(18)$$

and the derivative of $P(t)$ as

$$\frac{dP}{dt} = \frac{dp_c}{dt} - \frac{4\mu}{R}\frac{dU}{dt} + \frac{4\mu}{R^2}U^2 + \frac{2\sigma}{R^2}U .\qquad(19)$$

Then Eq. 15 becomes

$$R(1-U)\frac{dU}{dt} + \frac{3}{2}(1-\tfrac{1}{3}U)U^2$$

$$= \frac{1}{\rho_{ln}}\left\{(1+U)\left[p_c(t) + f(t) - 1 - \frac{2\sigma}{R} - \frac{4\mu U}{R}\right]\right.$$

$$\left. + R(1-U)\left[\frac{dp_c}{dt} - \frac{4\mu}{R}\frac{dU}{dt} + \frac{4\mu U}{R^2} + \frac{2\sigma}{R^2}U\right]\right\}.$$

We can rearrange this DE into

$$R(1-U)\left[1 + \frac{4\mu}{\rho_{ln}R}\right]\frac{dU}{dt} + \frac{3}{2}(1-\tfrac{1}{3}U)U^2$$

$$= \frac{1}{\rho_{ln}}\left\{(1+U)[p_c(t) + f(t) - 1] - (1+U^2)\left[\frac{2\sigma}{R} + \frac{4\mu U}{R}\right]\right.$$

$$\left. + R(1-U)\frac{dp_c}{dt}\right\}.\qquad(20)$$

Equations 16 and 20 are two DEs that are part of our formulation.

Let us now consider what is meant by the pressure $p_c(t)$ in the cavity, which by one of our assumptions is a spatially uniform function of time. We represent $p_c(t)$ in two different ways, each of which gives us a bound on the effect of vapor pressure of the liquid on the motion of the cavity. One representation is based on the assumption that evaporation and condensation are very slow processes, and the other on the assumption that evaporation and condensation are very rapid processes.

In the first representation we assume that evaporation and condensation are processes so slow that the amount of vapor initially in the cavity remains constant throughout the motion. In this instance, $p_c(t)$ is simply the pressure of a mixture of ideal gases whose equilibrium value is

$$p_{cn} = p_{gn} + p_{vn},$$

where $p_{gn}$ is the initial pressure of the gas and $p_{vn}$ is the vapor pressure of the liquid at the initial temperature of the system.

In the second representation we assume that evaporation and condensation are so rapid that the vapor pressure is maintained at its equilibrium value determined by the temperature $T(t)$ of the interface. The amount of vapor must then increase during an expansion and decrease during a contraction in order to maintain this equilibrium value. In this instance, the pressure in the cavity is given as the sum of two pressures:

$$p_c(t) = p_g(t) + p_v(T),\qquad(21)$$

where $p_g$ is the pressure due to the gas contained within the cavity and $p_v$ is the vapor pressure of the liquid at the temperature $T(t)$ of the interface.

During a cycle of expansion and compression, vapor molecules will always be able to diffuse in or out of a cavity. It seems likely that near the minimum radius of a cavity evaporation or condensation will virtually cease, and near the maximum radius evaporation or condensation will take place so freely that the vapor pressure will be maintained at its equilibrium value $p_v(T)$. One of our aims is to examine various ways in which this transition might take place and assess the relative importance of the second regime in cavitation dynamics.

In order to simplify the exposition, we assume through most of this paper that the amount of vapor in the cavity does not change during the motion, and then show how we can take into account the effect of a changing vapor pressure maintained at its equilibrium value $p_v(T)$.

In the nomenclature proposed in Ref. 1, a cavity in which there is no mass transport of vapor across its interface is called a "gaseous cavity," and a cavity in which mass transport of vapor maintains $p_v$ at its equilibrium value is called a "vaporous cavity." For each of these two kinds of cavities there are two expressions for $p_c$, depending on whether we use temperature and radius or entropy and radius as variables. The calculated temperature in a cavity will not be uniform, but the pressure will be. Hence, we must relate $p_c$ to some average temperature in the cavity, which we call the thermodynamic temperature and denote by $\theta_c(t)$. A sim-

ilar problem arises when we use entropy as a variable if we use the total entropy of the cavity contents. We denote this total entropy of the cavity contents by the symbol $s_c(t)$.

In a gaseous cavity, the pressure as a function of the thermodynamic temperature $\theta_c$ and the radius $R$ may be written in our dimensionless variables as

$$p_c = p_n \theta_c R^{-3} . \tag{22}$$

As a function of the total entropy of the cavity, and the radius, the pressure in a gaseous cavity is given by

$$p_c(t) = p_{cn} \exp(s_c) R^{-b} , \tag{23}$$

where $s_c$ is the total entropy, $b = 3\gamma$, and $\gamma$ is the ratio of specific heats for the gas–vapor mixture.

It should be noted that, since $s_c$ is the total entropy, the quantity $s_c$ is defined by $s_c^*/s_n^*$, where $s_n^* = \rho_{cn}^*(R_n^*)^3 c_v^*$ and $c_{vn}^*$ is the specific heat at constant volume per unit mass.

A similar pair may be written down for a vaporous cavity. When the temperature and radius are used as variables, we have

$$p_c = p_v + p_{gn} \theta_c R^{-3} , \tag{24}$$

and when the entropy and radius are variables,

$$p_c = p_v + \exp(s_c) R^{-b} . \tag{25}$$

For simplicity we use Eqs. 22 and 23 for $p_c$ in a gaseous cavity. The corresponding equations for a vaporous cavity can be obtained by obvious changes.

We consistently use the word "thermodynamic" to denote several spatially uniform functions of time. Thus, the temperature $\theta_c(t)$ in Eqs. 22 and 24 is here identified as the temperature that would exist in a cavity of radius $R(t)$ if the cavity contents were in equilibrium at a pressure $p_c(t)$. There is another "thermodynamic" quantity that it is convenient to define at this point. This is the thermodynamic density $\rho_c(t)$, which is the density of the cavity contents at a pressure $p_c(t)$ and a temperature $\theta_c(t)$. We can relate this uniform density to the radius by a simple argument. Let $M_0^*$ be the mass of the gas–vapor contents of the cavity. If $M_0^*$ does not change during a motion, then

$$M_0^* = \rho_{cn}^*(4\pi R_n^{*3}/3) = \rho_c(4\pi R^{*3})/3$$

or

$$\rho_c^*/\rho_{cn}^* = [R_n^*/R^*]^3$$

or

$$\rho_c = \rho_{cn}/R^3 . \tag{26}$$

## II. HEAT TRANSFER ACROSS INTERFACE

We now derive a set of equations that enables us to calculate the thermodynamic temperature $\theta_c$ and the entropy $s_c$ of the cavity as functions of time. Once we know either, we can find $p_c(t)$ and $dp_c/dt$, and hence calculate solutions of the equations of motion.

Let $\theta^*(r^*, t^*)$ be the temperature in the cavity and $\theta_l^*(r^*, t^*)$ be the temperature in the liquid. Then $\theta^*$

must satisfy the energy equation for the cavity contents and $\theta_l^*$ must satisfy the energy equation for the liquid. In addition, the two temperatures must satisfy the following boundary and initial conditions at the interface.

(1) The temperature is continuous at the interface:

$$\theta^*(R^*, t^*) = \theta_l(R^*, t^*) = T^*(t^*), \tag{27}$$

where $T^*(t^*)$ is the temperature in the liquid at the interface.

(2) The flux of heat is continuous at the interface:

$$k_c^*\left[\frac{\partial \theta^*}{\partial r^*}\right]_{R^*} = k_l^*\left[\frac{\partial \theta_l^*}{\partial r^*}\right]_{R^*} . \tag{28}$$

(3) Initially, the temperature is the same everywhere in the liquid and in the cavity:

$$\theta^*(r^*, 0) = \theta_l^*(r^*, 0) = \theta_n^* , \tag{29}$$

where $\theta_n^*$ is this uniform, initial temperature.

(4) The temperature in the liquid at infinity remains unchanged:

$$\lim_{r^* \to \infty} \theta_l^*(r^*, t^*) = \theta_n^* . \tag{30}$$

Another term is added to the flux condition whenever we assume that evaporation is so rapid that the vapor pressure within the cavity is maintained at its equilibrium value. How this term arises is seen later.

We now find equations for determining the temperatures in the liquid and in the cavity. It is obvious from the boundary conditions that these equations are coupled and must be solved simultaneously. In Sec. II-A we find an equation for $\theta_c$, and in Sec. II-B an equation for $T$, the temperature in the liquid at the interface.

### A. The temperature and entropy in the cavity

The energy equation for motions within the cavity may be written as

$$\rho\left[\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial r}\right] = \frac{k_c}{c_{cv}}\nabla^2\theta - \frac{1}{c_{cv}}p\nabla \cdot \mathbf{u}, \tag{31}$$

where

$$\frac{k_c}{c_{cv}} = \frac{a_n^*}{p_{ln}^* R_n^*}\frac{k_c^*}{c_v^*} , \tag{32}$$

$$\frac{1}{c_{cv}} = \frac{(a_n^*)^2}{\theta_n^*}\frac{1}{c_{cv}^*} . \tag{33}$$

We assume that both $k_c/c_{cv}$ and $1/c_{cv}$ may be functions of time, but not of space.

In order to derive a set of ordinary differential equations for $\theta_c$ and $s_c$, we start with some transformations suggested by the work of Plesset and Zwick.[25] Our first step is to transform from the spatial coordinates $(r, t)$ to a set of material coordinates $(m, t)$, where

$$m = 3\int_0^r \rho(\xi, t)\xi^2 d\xi . \tag{34}$$

The material coordinate $m$ is such that at the center of the cavity $m = 0$ and at the interface $m = 1$.

In terms of $m$ and $t$, the energy equation for the cavity

contents becomes

$$\frac{\partial \theta}{\partial t} = 9\frac{k_c}{c_{cv}}\frac{\partial}{\partial m}\left[\rho r^4 \frac{\partial}{\partial m}\right] - 3\frac{1}{c_{cv}}p\frac{\partial}{\partial m}(r^2 u). \qquad (35)$$

We introduce a temperature potential $\phi$ such that

$$\partial\phi/\partial m = \theta - 1. \qquad (36)$$

Hence, at $t = 0$, the temperature gradient vanishes everywhere, since then $\phi(m, 0) = 1$. Consistent with this initial gradient, we define the initial value of $\phi$ such that $\phi(m, 0) = 0$. Only the gradients of $\phi$ have physical meaning, and we are free to choose $\phi = 0$ at the origin for all $t$ and $t = 0$ for all $m$. Thus,

$$\phi(m, 0) = 0, \qquad (37a)$$

$$\phi(0, t) = 0. \qquad (37b)$$

In terms of the temperature potential, the energy equation becomes

$$\frac{\partial}{\partial m}\left[\frac{\partial\phi}{\partial t}\right] = 9\frac{k_c}{c_{cv}}\frac{\partial}{\partial m}\left[\rho r^4 \frac{\partial^2\phi}{\partial m^2}\right] - 3\frac{1}{c_{cv}}p\frac{\partial}{\partial m}(r^2 u).$$

We now replace the density $\rho(m, t)$ by the thermodynamic density $\rho_c = \rho_{cn}R^{-3}$ and integrate with respect to $m$. We obtain

$$\frac{\partial\phi}{\partial t} = 9\rho_{cn}\left(\frac{k_c}{c_{cv}}\right)\frac{r^4}{R^3}\frac{\partial^2\phi}{\partial m^2} - \frac{3}{c_{cv}}\int_0^m p\frac{\partial}{\partial m'}(r^2 u)\,dm'.$$

But

$$\int_0^m p\frac{\partial}{\partial m'}(r^2 u)\,dm' = pr^2 u - \int_0^m r^2 u\frac{\partial p}{\partial m'}dm',$$

so that

$$\frac{\partial\phi}{\partial t} = 9\rho_{cn}\left(\frac{k_c}{c_{cv}}\right)\frac{r^4}{R^3}\frac{\partial^2\phi}{\partial m^2} - \frac{3}{c_{cv}}\left[pr^2 u - \int_0^m r^2 u\frac{\partial p}{\partial m'}dm'\right]. \qquad (38)$$

Let us now introduce a new independent variable $\tau$ by use of

$$\tau = 9\rho_{cn}\int_0^t \frac{k_c(\xi')R(\xi')}{c_{cv}(\xi')}d\xi'. \qquad (39)$$

Noting that $u = \partial r/\partial t$, we have

$$\frac{\partial\phi}{\partial\tau} = \left(\frac{r}{R}\right)^4\frac{\partial^2\phi}{\partial m^2} - \frac{3}{c_{cv}}\left[pr^2 u - \int r^2 u\frac{\partial p}{\partial m'}dm'\right]. \qquad (40)$$

We now make the assumption that we can replace the term $(r/R)^4(\partial^2\phi/\partial m^2)$ by the temperature gradient alone. Thus,

$$\frac{\partial\phi}{\partial\tau} = \frac{\partial^2\phi}{\partial m^2} - \frac{3}{c_{cv}}\left[pr^2 u - \int_0^m r^2 u\frac{\partial p}{\partial m'}dm'\right]. \qquad (41)$$

This assumption overestimates the effect of heat conduction within the cavity because we are using the temperature gradient $\partial^2\phi/\partial m^2$ alone, rather than the temperature gradient diminished everywhere by the fraction $(r/R)^4 < 1$.

We now make the further assumption that the pressure gradient within the cavity is zero, so that the integral term vanishes and we may replace $p(m, \tau)$ by $p_c(\tau)$. Thus,

$$\frac{\partial\phi}{\partial\tau} = \frac{\partial^2\phi}{\partial m^2} - \frac{3}{c_{cv}}p_c r^2 u. \qquad (42)$$

For convenience, we may refer to $\partial^2\phi/\partial m^2$ as the gradient term and $-(3/c_{cv})p_c r^2 u$ as the power term. Because $\partial p/\partial m \sim \partial u/\partial t$, we see that the contribution of the integral diminishes the power term when the acceleration is negative and increases it when the acceleration is positive. The integral term is of consequence only in the short interval in which the acceleration is positive. Hence, neglect of the integral term should overestimate the effect of heat conduction.

By making a rough approximation to the integral correction and calculating solutions, it was found that the effect of this term was indeed to decrease the heat transfer. Hence the assumption of a uniform pressure is equivalent to overestimating the effect of heat conduction on a motion. In short, the two assumptions—that we ignore the decrease of the temperature gradient by the factor $(r/R)^4$ and that we ignore the pressure gradient within the cavity—both tend to overestimate the effect on heat flow.

We now introduce the assumption that the velocity within the cavity is a linear function of the radial distance $r$:

$$u = rU/R, \qquad (43)$$

so that

$$r^2 u = \frac{r^3 U}{R} = \frac{mR^2 U}{\rho_{cn}}.$$

The last expression was obtained by the approximation that $m \simeq \rho_c r^3 \simeq \rho_{cn}(r/R)^3$. We then have

$$\frac{\partial\phi}{\partial\tau} = \frac{\partial^2\phi}{\partial m^2} - \frac{3}{c_{cv}\rho_{cn}}p_c R^2 U m. \qquad (44)$$

The form of this DE permits us to define a new dependent variable

$$W(m, \tau) = \phi(m, \tau) + m\psi(\tau), \qquad (45)$$

where

$$\psi(\tau) = \frac{3}{\rho_{cn}}\int_0^\tau \frac{p_c(\xi)R^2(\xi)U(\xi)d\xi}{c_{cv}(\xi)}. \qquad (46)$$

Then the DE reduces to

$$\partial W/\partial\tau = \partial^2 W/\partial m^2. \qquad (47)$$

Thus the use of $m$, $\tau$, $W$, and our various approximations has converted a nonlinear equation into a linear parabolic equation. We now need to restate the boundary and initial conditions in terms of $W$.

The initial condition that $W(m, \tau)$ must satisfy is

$$W(m, 0) = 0, \qquad (48)$$

because of its definition and the initial condition for $\phi$ stated in Eq. 37a.

At the origin $(m = 0)$, $W(m, \tau)$ vanishes because of its definition and the condition on $\phi$ given by Eq. 37b. Thus,

$$W(0, \tau) = 0. \qquad (49)$$

At the interface, we require continuity of temperature. Hence, when $m = 1$, we have

$$\theta(1, \tau) = \theta_l(1, \tau) = T(\tau).$$

In terms of $\phi$, this condition is

$$\partial\phi/\partial m\big|_{m=1} = \theta_1(1,\ \tau) - 1 = T(\tau) - 1\ .$$

But, by the definition of $W$,

$$\partial\phi/\partial m\big|_{m=1} = [\partial W/\partial m]_{m=1} - \psi(\tau)\ ,$$

so that at $m=1$ the condition on $W$ is

$$[\partial W/\partial m]_{m=1} = T(\tau) + \psi(\tau) - 1\ . \tag{50}$$

It is convenient to write this boundary condition as

$$[\partial W/\partial m]_{m=1} = \psi_0(\tau)\ , \tag{51}$$

where

$$\psi_0(\tau) = T(\tau) + \psi(\tau) - 1\ . \tag{52}$$

We shall employ a Laplace transform method to solve this boundary-value problem. We define the Laplace transform of $W$ as

$$L[W(m,\ \tau)] = \overline{W}(m,\ s) = \int_0^\infty e^{-s\tau'} W(m,\ \tau')d\tau'\ . \tag{53}$$

By virtue of the initial condition on $W$, the partial differential equation is reduced to

$$d^2W/dm^2 - s\overline{W} = 0\ . \tag{54}$$

The boundary conditions on $\overline{W}$ are

$$\overline{W}(0,\ s) = 0\ , \tag{55}$$

$$dW(1,\ s)/dm = \overline{\psi}_0\ , \tag{56}$$

where

$$\overline{\psi}_0 = L[\psi_0(\tau)]\ .$$

In order to satisfy the boundary conditions, we choose the solution of this DE of the form

$$\overline{W} = A\sinh(\sqrt{s}m)\ , \tag{57}$$

where $A$ is a function of $s$ but not of $m$.

The condition on $\overline{W}$ at $m=0$ is then satisfied, and we determine $A(s)$ by the condition on $\overline{W}$ at $m=1$. Thus,

$$A = \frac{\overline{\psi}_0}{\sqrt{s}\cosh\sqrt{s}}\ .$$

Hence we have

$$\overline{W}(m,\ s) = \frac{\overline{\psi}_0\sinh(\sqrt{s}m)}{\sqrt{s}\cosh\sqrt{s}}\ . \tag{58}$$

But

$$L^{-1}[\overline{\psi}_0] = \psi_0(\tau)$$

and

$$L^{-1}\left[\frac{\sinh(m\sqrt{s})}{\sqrt{s}\cosh\sqrt{s}}\right] = \theta_1\left[\frac{m}{2}\bigg| i\pi\tau\right],$$

where $\theta_1$ is a theta function of the first kind.[26]

Hence, by the use of the convolution theorem,

$$W(m,\ \tau) = \int_0^\tau \theta_1\left[\frac{m}{2}\bigg| i\pi\xi\right]\psi_0(\tau - \xi)d\xi\ . \tag{59}$$

Equation 59 is a solution of our problem, but it is hardly in a convenient form for inclusion in a scheme of numerical calculation.

From Eq. 59 we want to determine the gradient of the

temperature in the cavity. Now,

$$\partial\theta/\partial m = \partial^2\phi/\partial m^2\ ,$$

but by Eq. 44 we have

$$\frac{\partial^2\phi}{\partial m^2} = \frac{\partial\phi}{\partial\tau} + \frac{3}{c_{cv}\rho_{cn}}p_c mR^2\frac{dR}{d\tau}\ .$$

Hence,

$$\frac{\partial\theta}{\partial m} = \frac{\partial\phi}{\partial\tau} + \frac{3}{c_{cv}\rho_{cn}}p_c mR^2\frac{dR}{d\tau}\ .$$

Because of our definition of $W(m,\ \tau)$

$$\frac{\partial\phi}{\partial\tau} = \frac{\partial W}{\partial\tau} - \frac{3}{c_{cv}\rho_{cn}}p_c mR^2\frac{dR}{d\tau}\ , \tag{42}$$

we see that

$$\partial\theta/\partial m = \partial W/\partial\tau\ . \tag{60}$$

We obtain the derivative of $W$ from our solution in the form

$$\frac{\partial W}{\partial\tau} = \theta_1\left[\frac{m}{2}\bigg| i\pi\xi\right]\psi_0(\tau - \xi)\bigg|_{\xi=0}^{\xi=\tau} + \int_0^\tau \theta_1\left[\frac{m}{2}\bigg| i\pi\xi\right]\frac{d\psi_0(\tau - \xi)}{d\tau}d\xi\ . \tag{61}$$

The first term in the derivative vanishes at both limits, at the upper because of the definition of $\psi_0(\tau)$ and at the lower because of the definition of the theta function of the first kind. Under the integral we write

$$\frac{d\psi_0(\tau - \xi)}{d\tau} = \frac{d\psi_0(\tau - \xi)}{d(\tau - \xi)}$$

and obtain

$$\frac{\partial W}{\partial\tau} = \int_0^\tau \theta_1\left[\frac{m}{2}\bigg| i\pi\xi\right]\frac{d\psi_0(\tau - \xi)}{d(\tau - \xi)}d\xi\ . \tag{62}$$

It is convenient to define an auxiliary function $h(m,\ \tau)$ by

$$h(m,\ \tau) = \frac{c_{cv}}{6}\int_0^\tau \theta_1\left[\frac{m}{2}\bigg| i\pi\xi\right]\frac{d\psi_0(\tau - \xi)}{d(\tau - \xi)}d\xi, \tag{63}$$

whose value at $m=1$ we denote by

$$H(\tau) = h(1,\ \tau)\ . \tag{64}$$

Then

$$\frac{\partial W}{\partial\tau} = \frac{6h(m,\ \tau)}{c_{cv}}, \tag{65}$$

and finally we have, as an expression for the temperature gradient in the cavity,

$$\frac{\partial\theta}{\partial m} = \frac{6h(m,\ \tau)}{c_{cv}}\ . \tag{66}$$

We now introduce an approximation for the theta function by use of the series[27]

$$\theta_1\left[\frac{m}{2}\bigg| i\pi\tau\right] = i\sum_{n=-\infty}^{\infty}(-1)^n\exp[-\pi^2(n-\tfrac{1}{2})^2\tau + i\pi(n-\tfrac{1}{2})m]\ . \tag{67}$$

The first $k$ terms of this series can be written as

$$\theta_1\left(\frac{m}{2}\middle| i\pi\tau\right) = 2e^{-\pi^2\tau/4}\sin(\pi m/2) - 2e^{-9\pi^2\tau/4}\sin(3\pi m/2)$$

$$+ 2e^{-25\pi^2\tau/4}\sin(5\pi m/2)$$

$$+ \cdots + 2(-1)^k e^{-(2k+1)^2\pi^2\tau/4}\sin\{[2k+1]\pi m/2\} + \cdots,$$

$$k = 0,\ 1,\ 2,\ldots.\tag{68}$$

For each term of the series that we include in an approximation for $\theta_1$, there will appear an additional first-order DE in our mathematical formulation. As a result of such an approximation for $\theta_1$, the function $h(m,\ \tau)$ may be written as

$$h(m,\ \tau) = h_0(m,\ \tau) - h_1(m,\ \tau) + \cdots + (-1)^k h_k(m,\ \tau),\tag{69}$$

where the $k$ term corresponds to

$$h_k(m,\ \tau) = \frac{c_{cv}\sin\{[2k+1]\pi m/2\}}{3}\int_0^\tau e^{-(2k+1)^2\pi^2 t/4}\frac{d\psi_0(\tau-\xi)}{d(\tau-\xi)}d\xi.\tag{70}$$

We transform the integral by a change of variable,

$$\chi = \tau - \xi,$$

and obtain

$$h_k(m,\ \tau) = \sin\{[2k+1]\pi m/2\}H_k(\tau),\tag{71}$$

where

$$H_k(\tau) = \frac{c_{cv}}{3}e^{-(2k+1)^2\pi^2\tau/4}\int_0^\tau e^{(2k+1)^2\pi^2\chi/4}\frac{d\psi_0}{dx}dx.\tag{72}$$

We write our definition of $H_k$ as

$$e^{(2k+1)^2\pi^2\tau/4}H_k(\tau) = \frac{c_v}{3}\int_0^\tau e^{(2k+1)^2\pi^2\chi/4}\frac{d\psi_0}{dx},$$

and, on differentiating this equation, obtain the DE

$$\frac{dH_k}{d\tau} + \frac{(2k+1)^2\pi^2}{4}H_k = \frac{c_{cv}}{3}\frac{d\psi_0}{d\tau}.\tag{73}$$

We use this to obtain $k$ DEs for the $H$s, and obtain $h(m,\tau)$ from

$$h(m,\ \tau) = H_0(\tau)\sin(\pi m/2) - H_1(\tau)\sin(3\pi m/2)$$

$$+ H_2(\tau)\sin(5\pi m/2) - \cdots + \cdots.\tag{74}$$

At the interface $(m = 1)$

$$h(0,\ \tau) = H(\tau) = [H_0 + H_1 + H_2 + H_3 + \cdots].\tag{75}$$

Now

$$\psi_0(\tau) = T(\tau) + \psi(\tau) - 1$$

and

$$\frac{d\psi_0}{d\tau} = 3\frac{p_c(\tau)R^2(\tau)U(\tau)}{\rho_{cn}c_{cv}(\tau)} + \frac{dT}{d\tau}.$$

Thus we obtain

$$\frac{dH_k}{d\tau} + \frac{(2k+1)^2\pi^3}{4}H_k = \frac{1}{\rho_{cn}}p_c(\tau)R^2(\tau)U(\tau) + \frac{c_{cv}}{3}\frac{dT}{d\tau},$$

and hence, by our definition of $\tau$,

$$\frac{dH_k}{dt} + \frac{9(2k+1)^2\pi^2\rho_{cn}k_c}{4c_{cv}}RH_k = \frac{1}{\rho_{cn}}p_cR^2U + \frac{c_{cv}}{3}\frac{dT}{dt}.\tag{76}$$

The first two DEs of this sequence are

$$\frac{dH_0}{dt} + \frac{9\pi^2\rho_{cn}k_c}{4c_{cv}}RH_0 = \frac{1}{\rho_{cn}}p_cR^2U + \frac{c_{cv}}{3}\frac{dT}{dt},\tag{77a}$$

$$\frac{dH_1}{dt} + \frac{81\pi^2\rho_{cn}k_c}{4c_{cv}}RH_1 = \frac{1}{\rho_{cn}}p_cR^2U + \frac{c_{cv}}{3}\frac{dT}{dt}.\tag{77b}$$

Numerical calculation of solutions shows that the functions $H_{k_1}$ are such that

$$H_0 > H_1 \gg H_2 \gg H_3 \cdots,$$

and in actual computations of solutions we usually retain only the first two functions, $H_0$ and $H_1$, of this sequence.

We may now obtain the gradient of $\theta$ from

$$\frac{\partial\theta}{\partial m} = \frac{6}{c_{cv}}h(m,\ \tau)$$

$$= \frac{6}{c_{cv}}\left\{H_0(\tau)\sin\frac{\pi m}{2} - H_1(\tau)\sin\frac{3\pi m}{2}\right.$$

$$\left. + H_2(\tau)\sin\frac{5\pi m}{2} - \cdots\right\}.\tag{78}$$

Hence, on integrating and replacing $\tau$ by $t$ we have

$$\theta(m,\ t) = c_0(t) - \frac{12}{\pi c_{cv}}H_0(t)\cos(\pi m/2)$$

$$+ \frac{12}{3\pi c_{cv}}H_1(t)\cos(3\pi m/2)$$

$$- \frac{12}{5\pi c_{cv}}H_2(t)\cos(5\pi m/2) + \cdots.$$

At $m = 1$, $\theta(1,\ t) = c_0(t)$. Hence we must have $c_0(t) = T(t)$ so that

$$\theta(m,\ t) = T(t) - g_0\cos(\pi m/2) + g_1(t)\cos(3\pi m/2)$$

$$- g_2(t)\cos(5\pi m/2) + \cdots,\tag{79}$$

where

$$g_0 = \frac{12}{\pi}\frac{H_0}{c_{cv}},\quad g_2 = \frac{12H_2}{5\pi c_{cv}},$$

$$g_1 = \frac{12}{3\pi}\frac{H_1}{c_{cv}},\quad g_k(t) = \frac{12H_k(t)}{(2k+1)\pi c_{cv}}.\tag{80}$$

We now calculate an average temperature $\theta_a(t)$ for the cavity by

$$\theta_a(t) = \frac{3}{R^3}\int_0^R \theta(m,\ t)r'^2\,dr'.\tag{81}$$

When we substitute our expression for $\theta(m,\ t)$, we obtain

$$\theta_a(t) = T(t) - \frac{3}{R^3}\int_0^R r'^2\,dr'\left[g_0(t)\cos\left(\frac{\pi m}{2}\right) - g_1(t)\cos\left(\frac{3\pi m}{2}\right)\right.$$

$$\left. + g_2(t)\cos\left(\frac{5\pi m}{2}\right) + \cdots\right].$$

Now

$$3r^2\,dr = \frac{dm}{\rho} = \left(\frac{p_{cn}}{\rho_{cn}}\right)\frac{\theta(m,\ t)\,dm}{p_c(t)}.$$

Hence,

$$\theta_a(t) = T(t) - \frac{p_{cn}}{\rho_{cn}R^3p_c}\int_0^1 \theta(m',\ t)\,dm'\left[g_0(t)\cos\left(\frac{\pi m'}{2}\right)\right.$$

$$\left. - g_1(t)\cos\left(\frac{3\pi m'}{2}\right) + \cdots\right].$$

Again substituting our expression for $\theta(m, t)$, we integrate and obtain

$$\theta_a(t) = T(t) - G(t)/\theta_c(t),$$

where we have used

$$\theta_c = \frac{p_c(t)R^3}{p_{cn}}$$

and

$$G(t) = \frac{2}{\pi \rho_{cn}} \left\{ g_0(t) + \frac{g_1(t)}{3} + \frac{g_2(t)}{5} + \frac{g_3(t)}{7} + \cdots \right\} T(t)$$

$$- \frac{1}{2\rho_{cn}} \left\{ g_0^2(t) + g_1^2(t) + g_2^2(t) + g_3^2(t) + \cdots \right\}. \quad (82)$$

Let us now identify the mean temperature $\theta_a(t)$ with the thermodynamic temperature $\theta_c(t)$. We then have

$$\theta_c^2 - T(t)\theta_c + G(t) = 0 \quad (83)$$

and

$$\theta_c(t) = \tfrac{1}{2}\left[ T(t) + \sqrt{T^2 - 4G} \right]. \quad (84)$$

In terms of the $H_k$s, the function $G(t)$ becomes

$$G(t) = \frac{24}{2c_{cv}\rho_{cn}} \left\{ H_0(t) + \tfrac{1}{9}H_1(t) + \tfrac{1}{25}H_2(t) + \tfrac{1}{49}H_3(t) \right\} T$$

$$- \frac{1}{2\rho_{cn}} \left( \frac{12}{\pi c_{cv}} \right)^2 \left\{ H_0^2 + \tfrac{1}{9}H_1^2 + \tfrac{1}{25}H_2^2 + \tfrac{1}{49}H_3^2 \right\}. \quad (85)$$

At the interface the gradient is

$$\left( \frac{\partial \theta}{\partial m} \right)_{m=1} = \frac{6}{c_{cv}} H.$$

We transform this equation into

$$\left( \frac{\partial \theta}{\partial r} \right)_{R=1} = \frac{6H}{c_{cv}} \frac{dm}{dr}.$$

But $dm/dr = 3\rho r^2$, so

$$\left( \frac{\partial \theta}{\partial r} \right)_{R=1} = \frac{18\rho R^2 H}{c_{cv}}.$$

We approximate $\rho$ by $\rho_{cn} R^{-3}$ and obtain

$$\left( \frac{\partial \theta}{\partial r} \right)_{R=1} = \frac{18\rho_{cn} H}{c_{cv} R}. \quad (86)$$

Knowing the gradient in the cavity at the interface, we can now calculate the rate at which heat is transported across the interface from

$$\frac{dQ^*}{dt^*} = 4\pi k_c^* R^{*2} \left[ \frac{\partial \theta^*}{\partial r^*} \right]_{R^*},$$

where $Q^*$ is the heat transported across the interface, or from

$$\frac{dQ}{dt} = 4\pi k_c R^2 \left[ \frac{\partial \theta}{\partial r} \right]_R,$$

where $Q = Q^*/Q_n^*$ and $Q_n^* = p_{1n}^* R_n^{*3}$. By use of our expression for $(\partial \theta/\partial r)_R$, we then have

$$\frac{dQ}{dt} = \frac{72\pi \rho_{cn} k_c R H}{c_{cv}}. \quad (87)$$

It proves useful to have a measure of the entropy of the cavity contents. We could obtain an estimate of the rate of change of the entropy by dividing the equation for the

heat flux, Eq. 87, by the average temperature of the cavity. However, the flux of heat through any spherical surface in the cavity is not constant, but is a function of position, as is the temperature in the cavity. Hence the entropy change in the cavity actually is less than that calculated in the above manner because the temperature is in general greater and the heat flux less the closer a spherical surface is to the center of the cavity. We obtain a more realistic estimate of the rate of change of entropy in the cavity even though in the end we are forced to use the average temperature $\theta_c(t)$ rather than the local temperture $\theta(m, t)$ in evaluating an expression for $ds_c/dt$. Here $s_c(t)$ is the total entropy of the cavity contents. Consider a volume element formed by spherical surfaces of radii $r_1$ and $r_2 (r_2 > r_1)$. The flux of heat into the element is

$$4\pi k^* \left[ (r_2^*)^2 \left( \frac{\partial \theta^*}{\partial r^*} \right)_2 - (r_1^*)^2 \left( \frac{\partial \theta^*}{\partial r^*} \right)_1 \right]$$

and the rate of change of entropy in the element is

$$\frac{ds^*}{dt^*} = 4\pi k_c^* \left[ \frac{r_2^{*2}}{\theta_2^*} \left( \frac{\partial \theta^*}{\partial r^*} \right)_2 - \frac{r_1^{*2}}{\theta_1^*} \left( \frac{\partial \theta^*}{\partial r^*} \right)_1 \right].$$

Here $(\partial \theta^*/\partial r^*)_2$ is the temperature gradient at $r_2^*$ and $\theta_2^*$ is the temperature at that surface. In our dimensionless variables, we can write this equation as

$$\frac{ds}{dt} = 4\pi k_c \left[ \frac{r_2^2}{\theta_2} \left( \frac{\partial \theta}{\partial r} \right)_2 - \frac{r_1^2}{\theta_1} \left( \frac{\partial \theta}{\partial r} \right)_1 \right], \quad (88)$$

where we have defined $s = s^*/s_n^*$ and $s_n = R_n^{*3} p_{1n}^*/\theta_n^*$. In an $(m, t)$ set of variables,

$$\frac{ds}{dt} = 4\pi k_c \left[ \frac{3\rho_2 r_2^4}{\theta_2} \left( \frac{\partial \theta}{\partial m} \right)_2 - \frac{3\rho_1 r_1^4}{\theta_1} \left( \frac{\partial \theta}{\partial m} \right)_1 \right].$$

We now multiply and divide this derivative by $dm$, form the limit as $dm \to 0$, and

$$r_1 \to r_2 \to r,$$

$$\rho_1 \to \rho_2 \to \rho,$$

$$\theta_1 \to \theta_2 \to \theta,$$

and obtain

$$\frac{ds}{dt} = 12\pi k_c \frac{\rho r^4}{\theta} \frac{\partial^2 \theta}{\partial m^2} dm,$$

where we have set

$$\frac{\partial^2 \theta}{\partial m^2} = \lim_{dm \to 0} \left[ \frac{(\partial \theta/\partial m)_2 - (\partial \theta/\partial m)_1}{dm} \right].$$

We now obtain the rate of change of the entropy in the entire cavity by integration

$$\frac{ds_c}{dt} = 12\pi k_c \int_0^1 \frac{\rho r^4}{\theta} \frac{\partial^2 \theta}{\partial m'^2} dm'. \quad (89)$$

We approximate this integral by replacing $\theta$ by $\theta_c$, the thermodynamic temperature, and $\rho$ by $m/r^3$, and the remaining $r$ by $R$.

$$\frac{ds_c}{dt} = \frac{12\pi k_c R}{\theta_c} \int_0^1 m \frac{\partial^2 \theta}{\partial m'^2} dm'.$$

But, on obtaining the second derivative from

$$\frac{\partial \theta}{\partial m} = \frac{6h(m, t)}{c_{cv}},$$

we have

$$\frac{ds_c}{dt} = \left(\frac{72\pi k_c}{c_{cv}}\right)\frac{RH}{\theta_c}\left(1 - \frac{2}{\pi}\right). \tag{90}$$

The factor $(1 - 2/\pi)$ represents the effect of the change of heat flux with position within the cavity, and, as we had expected, decreases the rate of change of entropy within the cavity. Equation 90 is another member of the equations in our formulation.

## B. The temperature in the liquid

We anticipate that the change in the temperature $T$ in the liquid at the interface will usually be small compared to the cavity temperature, but it would be reassuring to find by calculation that the change, $T - 1$, does not significantly change most motions. We initially assume that any change in $T$ is caused by a flow of heat and not by condensation or evaporation.

The problem of determining the temperature in the liquid at the interface is solved by a Green's function method. The solution that we obtain is a convolution integral of the heat flux through the interface over all previous instants of time.

From Eq. 6 we have, as an energy equation for an incompressible liquid in spherically symmetrical motion,

$$\frac{\partial \theta_l}{\partial t} + u\frac{\partial \theta_l}{\partial r} = \frac{k_l}{\rho_{ln}c_{lv}}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial \theta_l}{\partial r}\right], \tag{91}$$

where $\theta_l(r, t)$ is the temperature in the liquid at any point and

$$\frac{k_l}{\rho_{ln}c_{lv}} = \frac{1}{a_n^* R_n^*}\frac{k_l^*}{\rho_{ln}^* c_{lv}^*}. \tag{92}$$

We introduce the material coordinates

$$m_l = 1 + 3\rho_{ln}\int_R^r \xi^2 d\xi = 1 + \rho_{ln}(r^3 - R^3), \tag{93}$$

$$\tau_l = 9\rho_{ln}^2\frac{k_l}{c_{lv}}\int_0^t R^4(\xi)d\xi, \tag{94}$$

where $\rho_{ln} = \rho_{ln}^*/\rho_n^*$. Then the energy equation becomes

$$\frac{\partial \theta_l}{\partial \tau_l} = \frac{\partial^2 \theta_l}{\partial m_l^2}, \tag{95}$$

where we have put $(r/R)^4 = 1$.

The temperature $\theta_l$ and its gradient must satisfy the following conditions:

$$\theta_l(1, \tau_l) = \theta(1, \tau), \quad \text{for all } \tau, \tag{96}$$

$$\left[\frac{\partial \theta_l}{\partial r}\right]_{r=R} = \epsilon^2\left[\frac{\partial \theta}{\partial r}\right]_{r=R}, \quad \text{for all } \tau, \tag{97}$$

(where $\epsilon^2 = k_c/k_l$)

$$\theta_l(m_l, 0) = 1, \tag{98}$$

$$\lim_{m_l \to \infty} \theta_l(m_l, \tau_l) = 1. \tag{99}$$

The first condition is that there be continuity of temperature at the interface. The second condition is that there be continuity of heat flux at the interface. The third condition is the initial condition that the temperature everywhere be equal to the equilibrium value $\theta_n^*$. The fourth condition is that the temperature at infinity remain unchanged.

The second condition is Eq. 28 put into dimensionless variables. When it is transformed into an equation in $m_l$ and $m$, the condition becomes

$$\left[\frac{\partial \theta_l}{\partial m_l}\right]_{m_l=1} = \epsilon^2\frac{\rho}{\rho_{ln}}\left[\frac{\partial \theta}{\partial m}\right]_{m=1}. \tag{100}$$

We use a Green's function technique to solve this boundary-value problem. From Morse and Feshbach[28] we take, as the appropriate formalism for this problem, with obvious changes in notation,

$$\theta_l(m_l, \tau_l) = -\frac{1}{4\pi}\int_0^{\tau_l}\left[\frac{\partial \theta_l}{\partial m_l'}\right]_{m_l'=1}G(m_l, \tau_l| 1, \tau_l')d\tau_l'$$

$$+\frac{1}{4\pi}\int_1^\infty \theta_l(m_l', 0)G(m_l, \tau_l| m_l', 0)dm_l', \tag{101}$$

where $G(m_l, \tau_l|m_l', \tau_l')$ is some appropriate Green's function, $(m_l, \tau_l)$ are the field coordinates, and $(m_l', \tau_l')$ are the source coordinates with respect to which the indicated integrations are to be carried out. The initial condition enables us to simplify the second integral into

$$\frac{1}{4\pi}\int_1^\infty G(m_l, \tau_l| m_l', 0)dm_l'.$$

We choose as the proper Green's function for our problem the function

$$G(m_l, \tau_l| m_l', \tau_l') = 2\left(\frac{\pi}{\tau_l - \tau_l'}\right)^{1/2}\left\{\exp\left[\frac{-(m_l + m_l' - 2)^2}{4(\tau_l - \tau_l')}\right]\right.$$

$$\left. + \exp\left[\frac{-(m_l - m_l')^2}{4(\tau_l - \tau_l')}\right]\right\}. \tag{102}$$

This Green's function can be derived by the method of images from the Green's function for the heat equation in a semi-infinite domain[29] terminated at $m_l = 1$.

At $m_l' = 1$, the derivative of this function vanishes, as it should, and the function itself becomes

$$G(m_l, \tau_l| 1, \tau_l') = 4\left(\frac{\pi}{\tau_l - \tau_l'}\right)^{1/2}\exp\left[\frac{-(m_l - 1)^2}{4(\tau_l - \tau_l')}\right]. \tag{103}$$

Now we want to determine $T$, the value of $\theta_l$ at $m_l = 1$, rather than the entire temperature field in the liquid. Hence we set $m_l = 1$, and the Green's function for our problem finally reduces to the simple expression

$$G(1, \tau_l| 1, \tau_l') = 4\left(\frac{\pi}{\tau_l - \tau_l'}\right)^{1/2}. \tag{104}$$

When we substitute this function and Eqs. 98 and 103 into Eq. 101, we have

$$T(\tau_l) = \theta_l(1, \tau_l) = -\frac{1}{\sqrt{\pi}}\int_0^{\tau_l}\left[\frac{\partial \theta_l}{\partial m_l'}\right]_{m_l'=1}\frac{d\tau_l'}{(\tau_l - \tau_l')^{1/2}}$$

$$+\frac{1}{\sqrt{\pi}}\int_1^\infty \exp\left[\frac{-(m_l' - 1)^2}{4\tau_l}\right]\frac{dm_l'}{\sqrt{\tau_l}}. \tag{105}$$

The second integral in this equation can be evaluated

explicitly:

$$\int_1^\infty \exp \frac{-(m_1' - 1)^2}{4\tau_1} \frac{dm_1'}{\sqrt{\tau_1}} = \sqrt{\pi} . \qquad (106)$$

Hence we have

$$T(\tau_1) = 1 + \int_0^{\tau_1} \frac{F(\tau_1') d\tau_1'}{(\tau_1 - \tau_1')^{1/2}} , \qquad (107)$$

where we have introduced the boundary condition on $[\partial\theta_1 / \partial m_1]_{m_1=1}$ from Eq. 100 and written

$$F(\tau_1) = -\frac{\epsilon^2}{\sqrt{\pi}} \frac{\rho}{\rho_{1n}} \left[\frac{\partial\theta}{\partial m}\right]_{m=1} . \qquad (108)$$

By our definition of $\tau_1(t)$, we transform the integral for $T$ into

$$T(t) = 1 + \int_0^{\tau_1(t)} \frac{F_0(\epsilon) d\tau(\epsilon)}{[\tau_1(t) - \tau_1(\epsilon)]^{1/2}} , \qquad (109)$$

where

$$F_0(t) = -\frac{6}{\sqrt{\pi}} \left(\frac{k_c}{k_1}\right) \left(\frac{\rho_{cn}}{\rho_{1n}}\right) \frac{1}{c_{cv}} \qquad (110)$$

and

$$\tau_1(t) = 9\rho_{1n}^2 \left(\frac{k_1}{c_{1v}}\right) \int_0^t R^4(\epsilon) d\epsilon . \qquad (94)$$

Equations 109 and 110 are part of the mathematical formalism for cavitation dynamics that we are constructing.

The convolution integral in Eq. 109 is an exact solution to our linearized boundary-value problem in the liquid. It states that a value of $T$ at a given instant $t$ is the integral of all previous values of $F_0(\xi)$ at instants $\xi$, each weighted or "damped" by the function $[\tau_1(t) - \tau_1(\xi)]^{-1/2}$, from $\xi = 0$ to $\xi = t$.

It is obvious that the temperature $T(t)$ in the liquid at the interface should decrease when heat flows into the cavity during expansion. Similarly, $T(t)$ should increase when heat flows from the cavity during compression. In either case, the change in $T$ serves to decrease the temperature gradient in the cavity, and hence reduces the amount of heat transferred across the cavity interface. Thus, if we were to make the assumption that $T$ remains constant at its initial value of 1 throughout the motion, the heat transfer would be greater than if $T$ were allowed to vary with the time. A constant $T$ gives us an upper bound on the effect of $T$ on heat transfer.

There is an additional, compelling reason for seeking a method by which the temperature $T$ in the liquid at the interface may be calculated. Once we know the interfacial temperature $T$, we would then be able to explore the effects of evaporation and condensation on the motion of a cavity, at least during that part of any motion in which it could be characterized as a vaporous cavity. We shall see that simply by adding another term to the expression for $F(\tau_1)$, as given by Eq. 108, it is possible to predict the motion of a vaporous cavity, that is, a model in which evaporation and condensation maintains the vapor pressure at its equilibrium value $p_v(T)$, determined solely by the interfacial temperature $T$.

The dynamical effects of evaporation and condensation

for vapor-filled cavities in superheated liquids have been investigated by several authors, but reference is made here only to the work of Plesset and Zwick. [25] In these investigations, the vapor pressure within the cavity is the driving force for the cavity motion; but for a cavity moving under the influence of an acoustic pressure field, the effect of vapor pressure is usually not so crucial.

During evaporation and condensation, there is a flux of mass and energy across the interface, and, as a consequence, both the interfacial temperature $T$ and the number of molecules of vapor in the cavity change. This transfer of molecules into the cavity is in general a diffusive process, but we make no attempt to follow this process in detail. Instead we confine our treatment to the simple model of a vaporous cavity.

So far we have assumed that evaporation and condensation are such slow processes that the amount of vapor remains constant at its initial value. We now assume that, at least during part of a cycle of expansion and contraction, evaporation and condensation are rapid enough to maintain the vapor pressure in the cavity at its equilibrium value $p_v(T)$, which is a function solely of the interfacial temperature $T$. We anticipate that during evaporation molecules leave the interface and transport energy into the cavity, causing an inward flux of heat and consequently a drop in the interfacial temperature $T$. During condensation the converse will tend to take place and there will be an outward flux of heat and an increase in $T$. Our problem is that of introducing the effect of this additional heat flux into our boundary conditions.

During an expansion, the flux of heat into the cavity is given by

$$I_v^* = \frac{4\pi}{3} L_v^* \frac{d}{dt} (\rho_v^* R^{*3}) , \qquad (111)$$

where $L_v^*$ is the latent heat of evaporation per unit mass and $\rho_v^*$ is the density of the vapor.

If we assume that $\rho_v^*$ does not change very much during a motion, we have

$$I_v^* = 4\pi L_v^* \rho_{vn}^* R^{*2} U^* ,$$

where $\rho_{vn}^*$ is the initial density of the vapor. Hence the heat flux per unit area is

$$L_v^* \rho_{vn}^* U^* .$$

We use this result to add a term to the boundary condition on heat flux. Thus,

$$k_1^* \left(\frac{\partial\theta^*}{\partial r^*}\right)_{R^*} = k_c^* \left(\frac{\partial\theta^*}{\partial r^*}\right)_{R^*} + L_v^* \rho_{vn}^* U^* .$$

We put this equation into dimensionless form:

$$k_1 \left(\frac{\partial\theta}{\partial r}\right)_R = k_c \left(\frac{\partial\theta}{\partial r}\right)_R + \left(\frac{L_v^* \rho_n^* a_n^* R_n^*}{k_n^* \theta_n^*}\right) U\rho_{vn} .$$

Now

$$k_n^* = \frac{p_{1n}^* R_n^* a_n^*}{\theta_n^*} , \quad \rho_n^* = p_{1n}^* / a_n^{*2} ,$$

so that we have

$$k_I \left(\frac{\partial \theta_I}{\partial r}\right)_R = k_c \left(\frac{\partial \theta}{\partial r}\right)_R + L_v U \rho_{vn}, \qquad (112)$$

where

$$L_v = L_v^* / a_n^{*2}. \qquad (113)$$

We transform this equation by use of the material coordinates $m$ and $m_I$:

$$3\rho_{In} R^2 k_I \left(\frac{\partial \theta_I}{\partial m_I}\right)_1 = 3\rho R^2 k \left(\frac{\partial \theta}{\partial m}\right)_1 + L_v U \rho_{vn}$$

or

$$\left(\frac{\partial \theta_I}{\partial m_I}\right)_1 = \left(\frac{k_c}{k_I}\right)\left(\frac{\rho}{\rho_{In}}\right)\left(\frac{\partial \theta}{\partial m}\right)_1 + \frac{L_v \rho_{vn}}{3\rho_{In} k_I}\left(\frac{U}{R^2}\right)$$

$$= \epsilon^2 \left(\frac{\rho}{\rho_{In}}\right)\left(\frac{\partial \theta}{\partial m}\right)_1 + \frac{L_v \rho_{vn}}{3\rho_{In} k_I}\left(\frac{U}{R^2}\right). \qquad (114)$$

This boundary condition is the same as that in Eq. 100, with an additional term arising from heat flux due to vapor transfer. Hence,

$$F(\tau_I) = -\frac{\epsilon^2}{\sqrt{\pi}}\left(\frac{\rho}{\rho_{In}}\right)\left(\frac{\partial \theta}{\partial m}\right)_{m=1} - \frac{1}{\sqrt{\pi}}\left(\frac{L_v \rho_{vn}}{3\rho_{In} k_I}\right)\left(\frac{U}{R^2}\right). \qquad (115)$$

This new expression for $F$ is now used in the integral for the interfacial temperature $T$ whenever we wish to take account of evaporation and condensation within a vaporous cavity.

The vapor pressure in the cavity is calculated from an empirical equation

$$p_v(T) = p_{vn} T^{B_v}. \qquad (116)$$

Finally, the pressure in the cavity is

$$p_c = \frac{p_{gn} \theta_c}{R^3} + p_v(T). \qquad (117)$$

We distinguish the model by the term "vaporous cavity."

## III. A FORMULATION FOR CAVITATION DYNAMICS

We need one more equation in order to complete our mathematical formulation for cavitation dynamics. This equation will enable us to calculate the work done when the interface of a cavity moves. We use the sign convention that such work is positive when work is done on the liquid by an expanding cavity.

When the radius of a cavity increases by $dR^*$, the increment of work done by the cavity is

$$dw^* = 4\pi R^{*2} p_c^* dR^*.$$

But $dR^* = U^* dt^*$; hence we can write

$$dw^* = 4\pi R^{*2} p_c^* U^* dt.$$

We make $w^*$ dimensionless by use of the same quantity $Q_n^*$ that we used in making the heat $Q^*$ dimensionless. Thus we have

$$dw = dw^*/Q_n^* = 4\pi \left[\frac{R_n^{*3} p_{In}^*}{Q_n^*}\right] R^2 U p_c \, dt.$$

But $Q_n^* = R_n^{*3} p_{In}^*$, so that

$$dw = 4\pi R^2 U p_c \, dt$$

and

$$dw/dt = 4\pi R^2 U p_c. \qquad (118)$$

Let us now assemble the set of equations that constitute our mathematical formulation for studying cavitation dynamics. This formulation is shown in Table I.

The first objective of this study was to obtain a set of equations whose solutions could be found by a few minutes' calculation on a digital computer. The formulation displayed in Table I has four different variants depending on whether $f(t)$ is a sinusoidal function of time (Eq. 10a) or a Gaussian pulse (Eq. 10b) and on whether

TABLE I. Mathematical formulation for studying cavitation dynamics.

(a) $R(1-U)\left[1 + \frac{4\mu}{\rho_{In}R}\right]\frac{dU}{dt} + \frac{3}{2}(1 - \frac{1}{3}U)U^2$

$$= \frac{1}{\rho_{In}}\left\{(1+U)[p_c(t) + f(t) - 1] - (1 + U^2)\right.$$

$$\left. \times \left\{\frac{2\sigma}{R} + \frac{4\mu U}{R}\right\} + R(1-U)\frac{dp_c}{dt}\right\}. \qquad (20)$$

(b) $dR/dt = U$. $\qquad (16)$

(c) $\frac{dH_k}{dt} + \left\{\frac{9(2k+1)^2\pi^2\rho_{cn}k_c}{4c_{cv}}\right\}RH_k = \frac{1}{\rho_{cn}}\rho_c R^2 U + \frac{c_{cv}}{3}\frac{dT}{dt}. \qquad (76)$

(d) $H = \sum_{k=0} H_k$. $\qquad (75)$

(e) $\theta_c = \frac{1}{2}[T + (T^2 - 4G)^{1/2}]$, $\qquad (84)$

where

$$G = \frac{2}{\pi\rho_{cn}}\left\{g_0 + \frac{g_1}{3} + \frac{g_2}{5} + \frac{g_3}{7}\right\}T - \frac{1}{2\rho_{cn}}\{g_0^2 + g_1^2 + g_2^2 + g_3^2\}, \qquad (82)$$

$$g_0 = 12H_0/\pi c_{cv}, \quad g_2 = 12H_2/5\pi c_{cv},$$
$$\qquad (80)$$
$$g_1 = 12H_1/3\pi c_{cv}, \quad g_3 = 12H_3/7\pi c_{cv}.$$

(f) $\frac{ds_c}{dt} = \left(\frac{72\pi k_c}{c_{cv}}\right)\frac{RH}{\theta_c}\left(1 - \frac{2}{\pi}\right). \qquad (90)$

(g) $dw/dt = 4\pi R^2 U p_c. \qquad (118)$

(h) $T(t) = 1 + \int_0^{\tau_I(t)}\frac{F_0(\epsilon)d\tau(\epsilon)}{[\tau_I(t) - \tau_I(\epsilon)]^{1/2}}. \qquad (109)$

(i) $F_0(t) = -\frac{6}{\sqrt{\pi}}\left(\frac{k_c}{k_I}\right)\left(\frac{\rho_{cn}}{\rho_{In}}\right)\frac{1}{c_{cv}}\frac{H(t)}{R^3}. \qquad (110)$

(j) $\tau_I(t) = 9\rho_{In}^2\left(\frac{k_I}{c_{Iv}}\right)\int_0^t R^4(\epsilon)d\epsilon. \qquad (94)$

(k) $f(t) = P_A \sin\omega t \qquad (10a)$

or

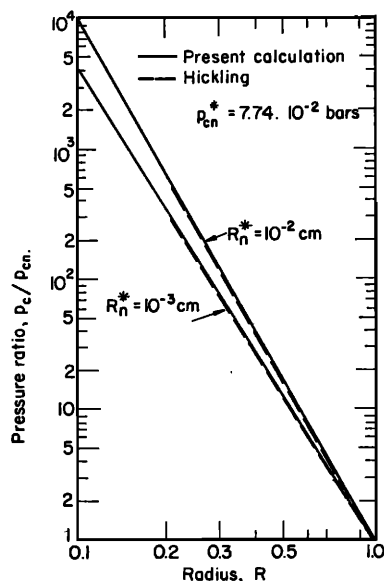$$f(t) = \lambda_A \exp[-\lambda(t - \bar{t})^2]. \qquad (10b)$$

(l) $p_c = p_n\theta_c/R^3 \qquad (22)$

or

$$p_c = p_n \exp[s_c]/R^b. \qquad (23)$$

*Note:* As written here, $p_c$ describes a gaseous cavity. For a vaporous cavity, Eqs. 116 and 117 would be used so as to include the vapor pressure $p_v$ explicitly. An extra term would be introduced in the equation for $T$ through use of Eq. 115.

FIG. 1.  Pressure within contracting $N_2$ cavity.

the cavity pressure $p_c$ is a function of temperature $\theta_c$ (Eq. 22) or entropy $s_c$ (Eq. 23). These four variants have been programmed for solution on an IBM 360. Calculations made with these programs show that this first objective has been reached, the actual machine time running from a few seconds to a few minutes for the calculation of many cycles of the motion of a cavity.

The second objective of this study was that predictions made by such numerical solutions should give reliable estimates of the order of magnitude of quantities associated with cavitation bubbles in violent motion. We must now see whether this second objective has been reached by comparing such quantities as radius–time curves, the velocity-radius curves, cavity pressures, and cavity temperatures, predicted by this formulation, with those of more exact solutions.

The work of Hickling[20] provides us with a standard against which to compare predictions of our formulation when heat conduction within the cavity is taken into account, but the viscosity and compressibility of the liquid are not. Hickling carried out an extensive machine calculation in which he used the exact equations for mass, momentum, and energy within the cavity, and his work is here regarded as "more exact" than the solutions obtained with the present formulation.

The pressure $p_c$ within a cavity predicted by the present formulation is compared with the pressure found by Hickling in Fig. 1. The deviation of our values from those of Hickling is usually less than 1% and never more than 3%. The cavity pressure is the quantity that enters directly into the equation of motion of the interface and determines the effect of heat transfer on cavity motion.

The mathematical formulation developed here enables one to calculate an average temperature $\theta_c$ within a cavity through the use of Eq. 84. Hickling found a distribution of temperature with radial distance with the cavities for which he carried out numerical calculations. However, one can calculate an average temperature

from Hickling's temperature distribution using the same rule for averaging as was used in Eq. 81. When such averages, denoted by $\langle \theta_H \rangle$, are found, they may be compared with results from the present formulation. The result is shown in Fig. 2 for a nitrogen-filled cavity of radius $10^{-2}$ cm. The average temperature $\theta_c$ found through the present formulation always lies below the Hickling average $\langle \theta_H \rangle$, as one would expect, because by and large the present formulation overestimates the effect of heat conduction. The results of the two calculations differ by 3% to 6%. Radius-time curves obtained by the two methods of calculation agree within 5%.

The numerical calculations of Hickling and Plesset[21] on a cavity collapse in a compressible liquid afford us a standard by which we can judge the usefulness of our solutions in predicting the effect of compressibility on cavity motions. Figure 3 compares a solution obtained from the present formulation with a curve calculated from the published diagrams of Hickling and Plesset for a cavity contracting adiabatically from an initial size in which the gas pressure was $10^{-3}$ bars.

The two calculations shown in Fig. 3 start to diverge when the pressure within the cavity exceeds 3000 bars, and as a consequence the dimensionless speed of sound in the liquid at the interface rises to a value of about 1.4. In Fig. 4 a similar comparison is made for a cavity contracting isothermally. Here again the solution obtained from the present formulation diverges from the "exact" solution of Hickling and Plesset when the pressure within the cavity exceeds about 3000 bars. For both cavities, the effect of an increasing sound
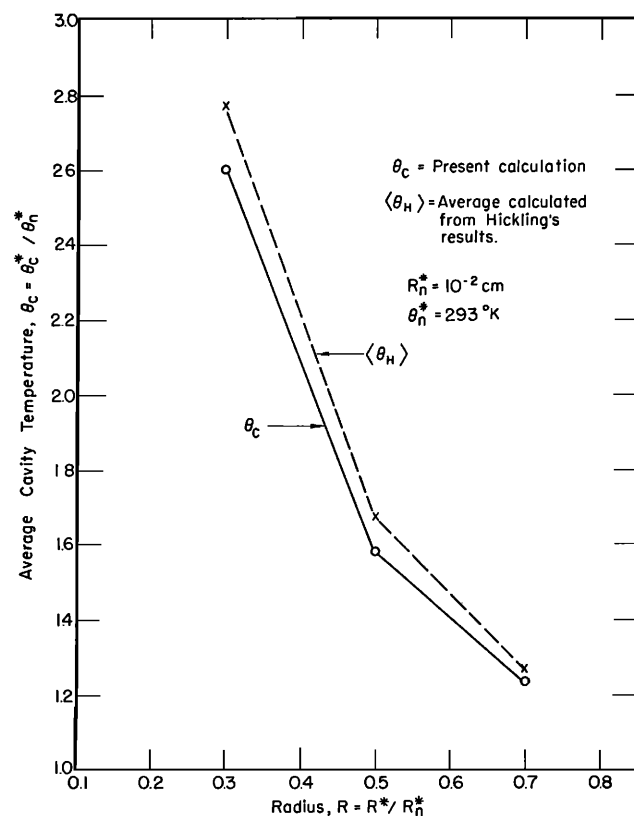


FIG. 2.  Collapse of $N_2$ cavity in water.

speed in the liquid at the interface causes the exact solution to predict higher cavity speeds of collapse. Figure 4 shows how our assumption of a constant sound speed in the liquid causes an abrupt reduction of the speed of collapse when $U$ becomes greater than two.

The conclusion that one reaches then is that the present formulation takes adequate account of compressibility whenever the cavity pressure is less than about 3000 bars. When this limit is exceeded, we shall continue our calculations by the use of Gilmore's[18] Eq. 13, rather than by our Eq. 20. The Hickling–Plesset solutions indicate that the Gilmore equation yields surprisingly good agreement with the exact solutions in problems of interest.

The numerical calculations of Ivany and Hammitt[22] give us a check on the accuracy with which our formulation takes the effect of viscosity of the liquid into account as a factor in cavitation dynamics. Ivany and Hammitt obtained numerical solutions for the motions of a cavity in a compressible, viscous liquid when heat conduction is ignored. Here we shall compare our solutions only with those of Ivany and Hammitt, in which they assumed the liquid was incompressible and viscous. For such motions, the formulation obtained here is the same as that used by Ivany and Hammitt. For a cavity with an equilibrium radius of 0.127 cm and an initial gas pressure of $10^{-3}$ atm, the two calculations gave the same maximum pressure and maximum velocity obtained during collapse.

It should be noted that the parameters $k_c/c_{cv}$ and $c_{cv}$ shown in Eq. 6 are not necessarily constants, but may vary with time, as we pointed out immediately below Eqs. 32 and 33. In other words, we are free to make the thermal conductivity and the specific heat of the cavity contents to be functions of time. In the programs for obtaining numerical solutions of our mathematical formulation, $k_c/c_{cv}$ and $c_{cv}$ are assumed to be powers
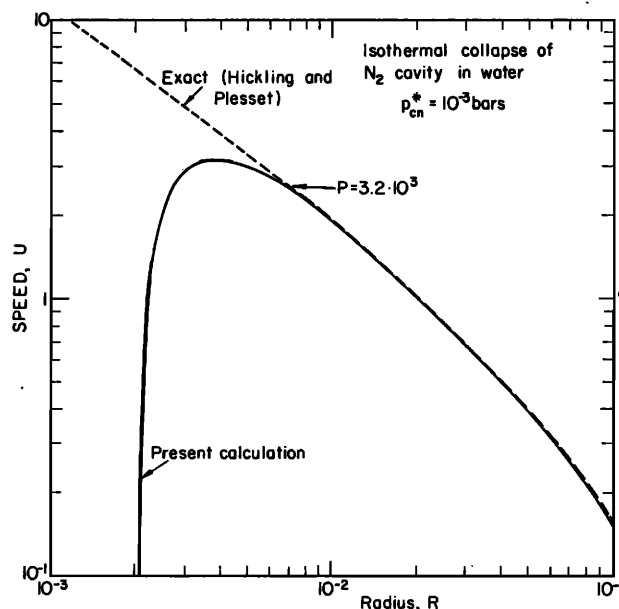


FIG. 4. Speed of cavity in an isothermal contraction.

of the temperature $\theta_c$ and the pressure $p_c$ in the form $\theta_c^m p_c^d$, where $m$ and $d$ are constants. In the calculations described in this paper, $k_c/c_{cv}$ and $c_{cv}$ were taken to be constants in order to compare our results with other published calculations. Later, in studying the collapse of cavities in which $\theta_c$ rises appreciably, we shall assume that $k_c/c_{cv}$ and $c_{cv}$ vary as some power of $\theta_c$ or $p_c$ or both.

## ACKNOWLEDGMENTS

FIG. 3. Speed of cavity in an adiabatic contraction.

[1] H. G. Flynn, Physical Acoustics, W. P. Mason, Ed. (Academic, New York, 1964), Vol. IB, p. 62.
[2] H. Poritsky, Proc. First U. S. National Congress on Applied Mechanics, E. Sternberg, Ed. (Am. Soc. Mech. Eng., New
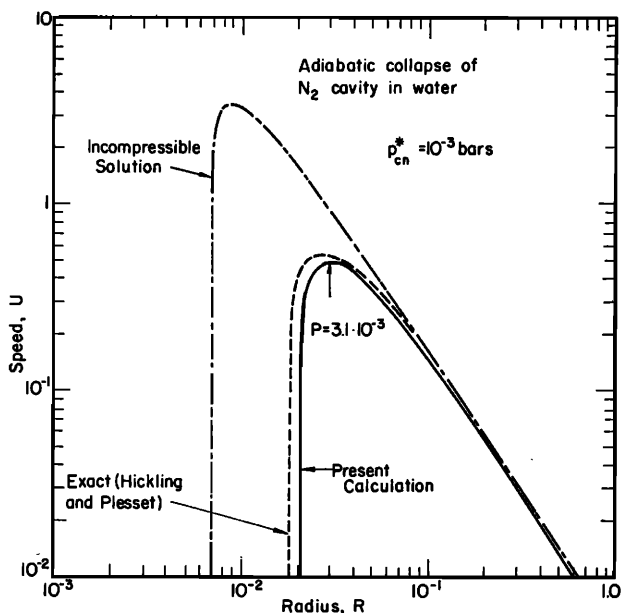
York, 1952), p. 813.

[3]W.-J. Yang and H.-C. Yeh, AIChE J. **12**, 927 (1966).

[4]H. Pfriem, Akust. Z. **5**, 202 (1940).

[5]Z. Saneyosi, Electrotech. J. (Jpn.) **5**, 49 (1941).

[6]Lyman Spitzer Jr., OSRD Rep. No. 1705 (NDRC Sec. 601-sr-20-918) (1943).

[7]M. S. Plesset and D.-Y. Hsieh, Phys. Fluids **3**, 882 (1960), and D.-Y. Hsieh and M. S. Plesset, Rep. No. 85-19, Eng. Div., Calif. Inst. Tech. (1961).

[8]D.-Y. Hsieh, J. Basic Eng. (Trans. ASME) **87**, 991 (1965).

[9]W. Lauterborn, Acustica **20**, 14 (1968).

[10]C. Devan Jr., J. Acoust. Soc. Am. **31**, 1654 (1959).

[11]B. E. Noltingk and E. A. Neppiras, Proc. Phys. Soc. Lond. B**63**, 674 (1950); B**64**, 1032 (1951).

[12]W. Güth, Acustica **6**, 532 (1956).

[13]M. I. Borotnikova and R. I. Solovkhin, Akust. Zh. **10**, 34 (1964) [Sov. Phys.—Acoust. **10**, 28 (1964)].

[14]S. H. Woo and P. R. Palsey, J. Acoust. Soc. Am. **42**, 114 (1967).

[15]V. A. Akulichev, Akust. Zh. **13**, 170 (1967) [Sov. Phys.—Acoust. **13**, 149 (1967)], and Akust. Zh. **13**, 533 (1967) [Sov. Phys.—Acoust. **13**, 455 (1968)].

[16]O. A. Kapustina, Akust. Zh. **15**, 489–504 (1969) [Sov. Phys.—

Acoust. **15**, 427–438 (1970)].

[17]Lord Rayleigh, Phil. Mag. (Series 6) **34**, 94 (1917).

[18]F. R. Gilmore, Rep. No. 26-4, Hydrodyn. Lab., Calif. Inst. Tech. (1952).

[19]L. Trilling, J. Appl. Phys. **23**, 14 (1952).

[20]Robert Hickling, J. Acoust. Soc. Am. **35**, 967 (1963).

[21]Robert Hickling and M. S. Plesset, Phys. Fluids **7**, 7 (1964).

[22]R. D. Ivany and F. G. Hammitt, J. Basic Eng. (Trans. ASME) **87**, 977 (1965).

[23]Conyers Herring, OSRD Rep. No. 236 (NDRC C4-sr-20-010) (1941).

[24]V. A. Akulichev, Yu. Ya. Boguslavskii, A. I. Ioffe, and K. A. Naugol'nykh, Akust. Zh. **13**, 321 (1967) [Sov. Phys.—Acoust. **13**, 281 (1968)].

[25]M. S. Plesset and S. A. Zwick, J. Appl. Phys. **23**, 95 (1952).

[26]A. Erdélyi, Ed., *Tables of Integral Transforms* (McGraw–Hill, New York, 1954), Vol. 1.

[27]A. Erdélyi, Ed., *Higher Transcendental Functions* (McGraw–Hill, New York, 1953), Vol. 2.

[28]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw–Hill, New York, 1953), Vol. 1.

[29]H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (Oxford U.P., London, 1959), 2nd ed.