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Casimir energy for dielectrics: Spherical geometry

(action/Green function)

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ABSTRACT **An expression is produced for the Casimir energy of a dielectric medium with a spherically symmetrical dielectric discontinuity. Tests based on dielectric and geometrical limits are successful.**

The remarkable results revealed by precise measurements of coherent sonoluminescence (1) have led me to propose a possible mechanism that is a dynamical counterpart of the static Casimir effect (2). But, before launching a study of the dynamical system, it seemed advisable to clarify the energy structure of the static regime. Ref. 3 does that for parallel plane geometry; this note is concerned with the spherical geometry that is more appropriate to a cavity in water—a spherical hole in a dielectric medium. The two papers have the following in common: source theory normalization concepts (4) are used, and the qualitatively satisfactory replacement of the vector field by two scalar fields is adopted. In this paper, however, the discussion is confined to the electric field—the analogous treatment of the magnetic field is left to Harold† (5).

I start at the point in ref. 3 that still refers to a general space-time varying dielectric constant, $\epsilon(x)$, as contained in the action

$$W_0 = \frac{i}{2}\,\mathrm{Tr}\log H,$$

where

$$H = \partial_0\epsilon(x)\partial_0 - \nabla^2,$$

and there is an unwritten additive constant that is chosen to respect the physical normalization conditions. Then I restrict $\epsilon(x)$ to be a time-independent spherically symmetrical function, one that depends only on the radial coordinate r: $\epsilon(r)$. A frequency dependence of ϵ is left implicit.

The part of the trace that is associated with time, and the complementary angular frequency variable ω, supplies a factor of T, the duration of the measurement, along with the integral from $-\infty$ to ∞ of $d\omega/2\pi$. The accompanying version of H is

$$H = -\omega^2\epsilon(r) - \nabla^2.$$

At this stage, the energy expression, in terms of the spatial part of the trace, Tr_s, is

$$E = -\frac{i}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\mathrm{Tr}_s\log H$$

$$= \frac{1}{2\pi}\int_0^{\infty} d\zeta\,\mathrm{Tr}_s\log H,$$

where the latter form introduces $\zeta = -i\omega$ and assumes that only even powers of ζ occur.

Keeping in mind the unwritten additive constant, one can write

$$\log H = -\int_0^{\infty} dw\,\frac{1}{w+H},$$

so that

$$E = -\frac{1}{2\pi}\int_0^{\infty} d\zeta\int_0^{\infty} dw\,\mathrm{Tr}_s G,$$

in which

$$G = \frac{1}{w+H} = [w + \zeta^2\epsilon(r) - \nabla^2]^{-1}.$$

For given w and ζ, the Green function G has the following spherical harmonic construction:

$$G(\vec{r}, \vec{r}') = \sum_{lm} y_{lm}(\theta\phi)\,\frac{1}{r}\,g_l(r, r')\,\frac{1}{r'}\,Y_{lm}(\theta'\phi')^*,$$

where

$$\left[w + \zeta^2\epsilon(r) + \frac{l(l+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right]g_l(r, r') = \delta(r - r').$$

The normalization of the spherical harmonics then yields

$$E = -\frac{1}{2\pi}\int_0^{\infty} d\zeta\int_0^{\infty} dw\sum_l (2l+1)\int_0^{\infty} dr\, g_l(r, r; \zeta, w).$$

The dielectric geometry of physical interest is characterized by two uniform regions, labeled 1 and 2. Region 1 is $r < r_0$, where $\epsilon(r) = \epsilon_1$; region 2 is $r > r_0$, where $\epsilon(r) = \epsilon_2$. Within the respective regions the structure of the homogeneous part of the differential equation governing the Green function is indicated by

$$\left[w + \zeta^2\epsilon_{1,2} + \frac{l(l+1)}{r^2} - \frac{\partial^2}{\partial r^2}\right]f(r) = 0,$$

or

$$\left[1 + \frac{l(l+1)}{\rho^2} - \frac{\partial^2}{\partial \rho^2}\right]f(\rho) = 0,$$

with

$$\rho_{1,2} = [w + \zeta^2\epsilon_{1,2}]^{1/2}\, r \equiv Z_{1,2}r.$$

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†Harold, hypothetical alert reader of limitless dedication.

One recognizes the so-called spherical Bessel functions, of imaginary argument. Two standard forms will be adopted: $s_l(\rho)$, which vanishes at $\rho = 0$, and $e_l(\rho)$, which vanishes as $\rho \to \infty$. The normalizations of these functions are such that the Wronskian obeys ($' \equiv \partial/\partial\rho$)

$$s_l(\rho)' e_l(\rho) - s_l(\rho) e_l(\rho)' = 1.$$

The inhomogeneous equation obeyed by $g_l(r, r')$, which contains the dielectric continuity conditions for the discontinuity at r_0, leads to expressions for g at $r = r'$, within each of the two regions. To simplify their presentation, the reference to a particular l is understood, the numbers 1 or 2 indicate an arbitrary point in the corresponding region, and subscript zero, as in 1_0, 2_0, signifies the limit $r \to r_0$, within the designated region. Thus

$$Z_1 g(1, 1) = s(1)\left[e(1) - s(1)\frac{e(1_0)}{s(1_0)}\right] + \left(\frac{s(1)}{s(1_0)}\right)^2 \frac{Z_1}{\lambda},$$

$$Z_2 g(2, 2) = \left[s(2) - e(2)\frac{s(2_0)}{e(2_0)}\right]e(2) + \left(\frac{e(2)}{e(2_0)}\right)^2 \frac{Z_2}{\lambda},$$

where

$$\lambda = Z_1\frac{s(1_0)'}{s(1_0)} - Z_2\frac{e(2_0)'}{e(2_0)} = \left.\frac{\partial s/\partial r}{s}\right|_{1_0} - \left.\frac{\partial e/\partial r}{e}\right|_{2_0}.$$

Any two solutions of the homogeneous ρ-differential equation, f and h, obey the following identity:

$$2fh = \frac{\partial}{\partial\rho}\left\{\left[\frac{1}{2}\rho(fh)'\right]' - 2\rho f'h'\right\},$$

where, of course,

$$\frac{f''}{f} = \frac{h''}{h} = 1 + \frac{l(l+1)}{\rho^2}.$$

Its use, along with the application of a normalization condition, supplies the two ρ-integrals:

$$\int_0^{\rho_{10}} d\rho\, Z_1 g(1, 1) = \frac{1}{2}\left(1 + \frac{Z_1}{\lambda}\frac{\partial}{\partial\rho}\right)\rho\left.\frac{s'}{s}\right|_{1_0}$$

$$= \frac{1}{2}\left(1 + \frac{1}{\lambda}\frac{\partial}{\partial r}\right) r\left.\frac{\partial s/\partial r}{s}\right|_{1_0},$$

and

$$\int_{\rho_{20}}^{\infty} d\rho\, Z_2 g(2, 2) = \frac{1}{2}\left(1 - \frac{Z_2}{\lambda}\frac{\partial}{\partial\rho}\right)\rho\left.\frac{e'}{e}\right|_{2_0}$$

$$= \frac{1}{2}\left(1 - \frac{1}{\lambda}\frac{\partial}{\partial r}\right) r\left.\frac{\partial e/\partial r}{e}\right|_{2_0}.$$

A simple test presents itself. If $\epsilon_1 = \epsilon_2$, $Z_1 = Z_2$, descriptive of an everywhere-uniform medium, does one get an expression for E that is independent of r_0, which has lost its physical meaning? Apart from a numerical factor, the sum of the two ρ-integrals, for $\rho_{10} = \rho_{20} = \rho$, is

$$\rho\left(\frac{s'}{s} + \frac{e'}{e}\right) + \frac{Z}{\lambda}\left[\rho\left(\frac{s'}{s} - \frac{e'}{e}\right)\right]'.$$

The Wronskian, in the form

$$\frac{s'}{s} - \frac{e'}{e} = \frac{1}{se},$$

along with the related expression $Z/\lambda = se$, then yields the value unity, independent of ρ, as it should be, which permits the maintenance, at zero, of the energy for a uniform medium.

If one seeks a more incisive test there is no option but establishing contact with the known result for plane geometry. To that end I install a perfectly conducting medium for all $r < r_1$, with $r_1 < r_0$, and consider the limit in which

$$r_0 - r_1 \equiv \frac{1}{2}a << r_1.$$

That produces nearly plane local geometry over the finite surface area $4\pi r_1^2$.

The energy per unit area, E/A, is dominated by large values of l, which is expressed by

$$\sum_l \frac{2l+1}{4\pi r_1^2} \to \frac{1}{4\pi}\int_0^\infty d\left(\frac{\left(l+\frac{1}{2}\right)^2}{r_1^2}\right) \equiv \frac{1}{4\pi}\int_0^\infty dk^2.$$

The structure of the g-differential equation then effectively involves $w + k^2$, which invites the redefinition

$$w + k^2 \to k^2, \quad \int_0^\infty dw \int_0^\infty dk^2 \to \int_0^\infty dk^2 k^2,$$

so that

$$\frac{E}{A} = -\frac{1}{8\pi^2}\int_0^\infty d\zeta \int_0^\infty dk^2 k^2 \int_{r_1}^\infty dr\, g(r, r; \zeta, k).$$

On introducing

$$\kappa_{1,2}^2 = k^2 + \zeta^2\epsilon_{1,2},$$

the g-equation reads

$$\left(\kappa_{1,2}^2 - \frac{\partial^2}{\partial r^2}\right) g(r, r') = \delta(r - r').$$

Thus, the analogue of ρ is $\kappa(r - r_1)$, and that of Z is κ. The basic functions are given explicitly by

$$s(\rho) = \sinh(\rho), \quad e(\rho) = e^{-\rho},$$

which satisfy the Wronskian condition. The structure of λ is now explicit as

$$\lambda = \kappa_1 \coth\left(\frac{1}{2}\kappa_1 a\right) + \kappa_2,$$

or as

$$\frac{1}{\lambda} = \frac{1}{\kappa_2 - \kappa_1}\frac{e^{\kappa_1 a} - 1}{De^{\kappa_1 a} - 1}$$

$$= \frac{1}{\kappa_2 + \kappa_1} - \frac{2\kappa_1}{\kappa_2^2 - \kappa_1^2}\frac{1}{De^{\kappa_1 a} - 1},$$

wherein

$$D = \frac{\kappa_2 + \kappa_1}{\kappa_2 - \kappa_1}.$$

Another simple test now offers itself. Let ϵ_2, and therefore κ_2, $\to \infty$, which is the well-known situation of two parallel, perfectly conducting plates, here a distance ½a apart. Only region 1 contributes, and, with $\lambda \to \infty$ (and $\kappa_1 \to \kappa$),

$$\int_{r_1}^{r_0} dr\, g(r, r) = \frac{1}{\kappa^2} \int_0^{\rho_{10}} d\rho\, \kappa g(1, 1)$$

$$= \frac{1}{\kappa}\left(\frac{a}{4} + \frac{\frac{1}{2}a}{e^{\kappa a} - 1}\right).$$

The normalization condition on a then yields

$$\frac{E}{A} = -\frac{1}{8\pi^2} \int_0^\infty d\zeta \int_0^\infty dk^2\, k^2 \frac{\frac{1}{2}a}{\kappa} \frac{1}{e^{\kappa a} - 1},$$

which is just the result presented, for example, in the next-to-last equation of ref. 3, but with a replaced by ½a, the appropriate distance between the plates. Of course, this limiting test result is produced more easily by constructing g directly for the perfect conductor circumstance.

The outcome for finite ϵ_2 is

$$\left.\frac{E}{A}\right|_{\mathbf{a}} = -\frac{1}{8\pi^2} \int_0^\infty d\zeta \int_0^\infty dk^2\, k^2 \left(\frac{\frac{1}{2}a}{\kappa_1} + \frac{1}{\kappa_2 \kappa_1}\right) \frac{1}{De^{\kappa_1 a} - 1},$$

where the label $\mathbf{a}$ = antisymmetrical anticipates that $E_{\mathbf{a}}$ is not directly comparable with the known parallel plate energy (ref. 3, with $\kappa \to \kappa_1$, $\kappa' \to \kappa_2$):

$$\frac{E}{A} = -\frac{1}{8\pi^2} \int_0^\infty d\zeta \int_0^\infty dk^2\, k^2 \left(\frac{a}{\kappa_1} + \frac{2}{\kappa_2 \kappa_1}\right) \frac{1}{D^2 e^{2\kappa_1 a} - 1}.$$

Indeed, the latter refers, not to a semi-infinite region, but to an infinite one, comprising the two semi-infinite slabs of dielectric constant ϵ_2 and the separating slab, of thickness a, with dielectric constant ϵ_1. That arrangement is symmetrical about the bisecting plane that produces two ϵ_1 slabs, each of thickness ½a.

Accordingly, Green's function can be decomposed into two additive parts, one that is antisymmetrical about that plane, the other symmetrical. The respective parts vanish, or have a vanishing normal derivative, on that plane. As one can easily verify in the circumstance $\epsilon_2 \to \infty$, and Harold will confirm for finite ϵ_2, the symmetrical counterpart of the antisymmetrical energy $E_{\mathbf{a}}$ is produced by the substitution

$$\frac{1}{De^{\kappa_1 a} - 1} \to -\frac{1}{De^{\kappa_1 a} + 1},$$

and the sum of the two,

$$\frac{1}{De^{\kappa_1 a} - 1} - \frac{1}{De^{\kappa_1 a} + 1} = \frac{2}{D^2 e^{2\kappa_1 a} - 1},$$

yields the energy E.

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