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Casimir light: Photon pairs

(vacuum/probability)

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ABSTRACT **Expressions are developed for weak single pair emission probability and strong emission average number of pairs. The water transparency cutoff is closely realized, showing that the fundamental time scale is even shorter.**

An earlier note (1) identified the release of Casimir energy as the source of the electromagnetic radiation that is observed in the phenomenon of coherent sonoluminescence (2, 3). It is the purpose of this note to replace that qualitative relationship with a quantitative statement.

I begin with the precise version of the pair emission probability in ref. 4, specifically that of a collapsing spherical cavity in the circumstance $|\epsilon - 1| \ll 1$.

The infinitesimal variation of $\epsilon(x)$ in the electric field action W_e (4) produces

$$\delta w_e = \int (dx)\, \frac{1}{2}\, \partial_0 A \delta\epsilon(x) \partial_0 A.$$

The introduction of the fields for two distinct particles, along with their detection sources,

$$A = A_{\vec{k}} + A_{\vec{k}'},$$

where, for example, $(\omega \simeq |\vec{k}| \equiv k)$,

$$A_{\vec{k}} = iJ^*_{\vec{k}} \left[\frac{(d\vec{k})}{(2\pi)^3} \frac{1}{2\omega}\right]^{1/2} e^{-i\vec{k}\cdot\vec{r}} e^{i\omega t},$$

yields the probability amplitude

$$\langle 1_{\vec{k}} 1_{\vec{k}'} | 0 \rangle = -i \left[\frac{(d\vec{k})}{(2\pi)^3} \frac{(d\vec{k}')}{(2\pi)^3} \frac{\omega\omega'}{4}\right]^{1/2} \times \int (d\vec{r}) dt \delta\epsilon(x) e^{-i\vec{r}\cdot(\vec{k}+\vec{k}')} e^{it(\omega+\omega')}.$$

The physical situation is the following: $\epsilon(x)$ is the constant ϵ for all t when $r > R$, but, for $r < R$, it is unity for $t < 0$, and is ϵ for $t > 0$. Thus, relative to the totally uniform medium of dielectric constant ϵ, and, with $|\epsilon - 1| \ll 1$,

$$\delta\epsilon(x) = -(\epsilon - 1)\eta(R - r)\eta(-t).$$

The required time integral is

$$\int_{-\infty}^{0} dt e^{it(\omega+\omega')} = [i(\omega + \omega')]^{-1},$$

and the spatial integral, divided by $(2\pi)^3$, is a quasi-delta function, for $kR \gg 1$. Its square is well approximated by

$$\frac{1}{(2\pi)^3} V\delta(\vec{k} + \vec{k}'), \qquad V = \frac{4\pi}{3} R^3,$$

which describes a pair of photons with nearly equal and opposite momenta. Then the probability of emitting such a pair within a differential momentum range is

$$V \frac{(d\vec{k})}{(2\pi)^3} \left(\frac{\epsilon - 1}{4}\right)^2,$$

and the total probability of an electric field pair is

$$p_e = \frac{1}{2} V \int \frac{(d\vec{k})}{(2\pi)^3} \left(\frac{\epsilon - 1}{4}\right)^2.$$

In the analogous magnetic field discussion, one begins with (5)

$$\delta W_m = \int (dx)\, \frac{1}{2} \vec{\nabla} A \frac{\delta\epsilon(x)}{(\epsilon(x))^2} \cdot \vec{\nabla} A.$$

The effective substitution in the known probability amplitude is

$$\omega\omega' \simeq k^2 \to \vec{k}\cdot\vec{k}' \simeq -k^2.$$

Thus, the probabilities p_m and p_e are equal, giving the sum

$$p = p_e + p_m = V \int \frac{(d\vec{k})}{(2\pi)^3} \left(\frac{\epsilon - 1}{4}\right)^2.$$

Seeking to remove the restriction $|\epsilon - 1| \ll 1$, I turn to the vacuum persistence probability amplitude, $\exp(iW_0)$, where the real part of W_0 refers to energy and the imaginary to change of state, as produced by photon emission. Ref. 5 also provides the differential information

$$\delta W_0 = \frac{i}{2} \mathrm{Tr}(\delta H G), \qquad G = H^{-1},$$

where

$$H_e = \partial_0 \epsilon \partial_0 - \nabla^2, \qquad H_m = \partial_0^2 - \vec{\nabla} \frac{1}{\epsilon} \cdot \vec{\nabla}.$$

I invoke the dominance of the spatial volume effect by writing

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$$\mathrm{Tr}(\delta HG) = V\int \frac{(d\vec{k})}{(2\pi)^3}\int_{-\infty}^{\infty} dt\delta Hg_{\vec{k}}(t, t),$$

in which

$$\delta H_e = \partial_0\delta\epsilon\partial_0, \qquad \delta H_m = -k^2\frac{\delta\epsilon}{\epsilon^2},$$

and

$$[\partial_0\epsilon(t)\partial_0 + k^2]g_e(t, t') = \delta(t - t'),$$

$$\left[\partial_0^2 + \frac{k^2}{\epsilon(t)}\right]g_m(t, t') = \delta(t - t').$$

The simplified physical situation has $\epsilon(t < 0) = 1$, and $\epsilon(t > 0) = \epsilon$. One can realize this arrangement by beginning with a totally uniform dielectric medium, and then, in infinitesimal steps, introduce the inhomogeneity. Begin, for example, with the uniform dielectric value ϵ, and then reduce that value to 1, for $t < 0$. An elementary form of this was used in the single two-photon emission study. An alternative, which I shall now adopt, begins with the uniform dielectric value 1, and then raises it to ϵ for $t > 0$. This will be done using the magnetic version.

To simplify the notation, I use ϵ to signify both the transitory value during the integration and the final value. One needs $g_m(t, t')$, for $t, t' > 0$, to complete the trace evaluation. It is

$$g_m(t, t') = \frac{i\,\epsilon^{1/2}}{2k}\left[\exp(-ik\epsilon^{-1/2}|t - t'|) - \frac{\epsilon^{1/2} - 1}{\epsilon^{1/2} + 1}\exp(-ik\epsilon^{-1/2}(t + t'))\right],$$

and then one gets

$$\int_0^T dtg_m(t, t) = T\frac{i\,\epsilon^{1/2}}{2k} - \frac{\epsilon}{4k^2}\frac{\epsilon^{1/2} - 1}{\epsilon^{1/2} + 1},$$

where T is a relatively long observation time. At this point one has

$$\delta W_0 = V\int \frac{(d\vec{k})}{(2\pi)^3}\left[T\frac{k}{2}\frac{\delta\epsilon}{2\epsilon^{3/2}} + \frac{i}{8}\frac{\delta\epsilon}{\epsilon}\frac{\epsilon^{1/2} - 1}{\epsilon^{1/2} + 1}\right].$$

The outcome of integration is presented as

$$W_0 = -ET + \frac{i}{2}N,$$

which gives the vacuum persistence probability amplitude in the form

$$e^{iW_0} = e^{-iET}e^{-N/2},$$

where

$$E = -V\int \frac{(d\vec{k})}{(2\pi)^3}\frac{1}{2}k(1 - \epsilon^{-1/2})$$

and

$$N = V\int \frac{(d\vec{k})}{(2\pi)^3}\log\frac{\epsilon^{1/4} + \epsilon^{-1/4}}{2}.^{\dagger}$$

E, the energy of the filled cavity relative to the evacuated cavity, reproduces the result in ref. 1. As for N, which gives the vacuum persistence probability

$$|e^{iW_0}|^2 = e^{-N},$$

one anticipates a consistency check with a known result. For ϵ sufficiently close to 1 that $N \ll 1$, the vacuum persistence probability in that circumstance, $1 - N$, identifies N as the probability for emitting a single photon pair. Then,

$$|\epsilon - 1| \ll 1:\ \log\frac{\epsilon^{1/4} + \epsilon^{-1/4}}{2} \simeq \frac{(\epsilon - 1)^2}{32}$$

confirms that

$$N_m\ (= N_e) = p_m\ (= p_e).$$

Outside these restrictive circumstances, N, for each field type, retains its interpretation as the average number of photon pairs, whereas the probability of emitting n photon pairs is (7)

$$p(n) = e^{-N}\frac{N^n}{n!}.$$

The expected total number of photon pairs, of both types, is

$$N = N_e + N_m = 2V\int \frac{(d\vec{k})}{(2\pi)^3}\log\frac{\epsilon^{1/4} + \epsilon^{-1/4}}{2} = 2\left(\frac{4\pi}{3}\right)^2\left(\frac{R}{\Lambda}\right)^3\log\frac{\epsilon^{1/4} + \epsilon^{-1/4}}{2},$$

where the second version ignores dispersion and introduces the cutoff wavelength Λ. Apart from the mathematical cutoff that gives a simplified description of the short-time behavior of the system, there is a physical cutoff at a wavelength near 1.9×10^{-5} cm; for shorter wavelengths water becomes opaque.

With the choice $\epsilon^{1/2} = 4/3$, the logarithm that enters N has the value 0.010. Together with $N = 3 \times 10^6$, as observed (3) for the temperature 3°C, and (ref. 1) $R = 4 \times 10^{-3}$ cm, one gets

$$\Lambda = 2 \times 10^{-5}\ \mathrm{cm},$$

strikingly close to the edge of transparency.

It would seem that the fundamental mechanism operates at an even smaller time scale, as expressed by the cutoff frequency inequality

$$\overline{\omega} > c\frac{2\pi}{\Lambda} \simeq 10^{16}\ \mathrm{s}^{-1}.$$

†Harold (see ref. 6), being well aware of the limit that is implicit in the trace, will have no difficulty in showing that the same results appear for the electric field.

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