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Casimir energy for dielectrics

(fields/sources)

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ABSTRACT **A treatment in terms of two scalar fields supplies the electromagnetic energy of attraction between two similar dielectric slabs with parallel plane interfaces.**

The static Casimir effect (1–3) as introduced in ref. 1 is "an observable non-classical electromagnetic force of attraction between two parallel [perfectly] conducting plates" (quoted from ref. 2). Inasmuch as this force varies as the inverse fourth power of the distance between the plates, the interaction energy is immediately inferred. The Casimir force has been extended to "the more general circumstance of parallel interfaces between dielectric media" (ref. 4 and references therein). Again—if not so transparently—the interaction energy can be inferred. But no general procedure has been available that yields the energy directly.

The impetus to fill that gap comes from recent discoveries in coherent sonoluminescence (5) that I interpret as a dynamical Casimir effect wherein dielectric media are accelerated and emit light. In the cited experiments a bubble in water—a hole in a dielectric medium—undergoes contraction and expansion in response to the positive and negative pressures of a strong acoustical field.

What the static and dynamic Casimir effects share is the reference to the quantum probability amplitude for the preservation of the photon vacuum state:

$$\langle 0t_1|0t_2\rangle = \exp[iW_0].$$

That the vacuum persistence probability is less than one, in a dynamical situation where photons can be emitted, is expressed by a nonzero imaginary part of W_0:

$$|\langle 0t_1|0t_2\rangle|^2 = \exp[-2\ \mathrm{Im}\ W_0].$$

In a static situation, where W_0 is real, the shift in phase associated with a time lapse, $t_1 - t_2 = T$, identifies E, the energy of the system, by

$$W_0 = -ET,\ \hbar = 1.$$

Thus, if one has the ambition to find the imaginary part of W_0 for a dynamical realm, it might be advisable to begin with the simpler real part appropriate to a static regime.

As a glance at the details of ref. 4 will attest, the vectorial nature of the electromagnetic field produces complications of structure that may not be of significance, at least for a qualitative understanding. Indeed, the use of a scalar field in ref. 2 produced a result that required only a corrective factor of 2, appropriate to the photon polarization multiplicity. In the general dielectric situation of ref. 4, this simple doubling is replaced by the additive contributions of an electric scalar field and a magnetic scalar field.

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The two types of scalar fields, $A(X)$, and sources, $J(X)$, are distinguished by the labels e (electric) and m (magnetic). The space-time coordinates, X, are given individually by $X^0 = (c = 1)t$, and $X_{1,2,3} = x, y, z$. The initial action, W, is the sum of

$$W_e = \int (dX)[{}^1\!/_2\epsilon(X)(\partial_0 A)^2 - {}^1\!/_2(\vec{\nabla}A)^2 + AJ]_e$$

and

$$W_m = \int (dX)\left[{}^1\!/_2(\partial_0 A)^2 - {}^1\!/_2\frac{1}{\epsilon(X)}(\vec{\nabla}A)^2 + AJ\right]_m,$$

where $\epsilon(X)$ is a general space-time varying dielectric constant. The solutions of the implied field equations, in terms of the respective Green's functions, are compactly presented by

$$A = GJ,\ HG = 1,$$

where

$$\begin{aligned} &\mathrm{e}: H = \partial_0\epsilon\partial_0 - \nabla^2 \\ &\mathrm{m}: H = \partial_0^2 - \vec{\nabla}\cdot\frac{1}{\epsilon}\vec{\nabla}. \end{aligned} \qquad \epsilon = \epsilon(X)$$

The explicit expression of the vacuum amplitude in terms of sources is indicated by (coordinates are omitted)

$$\exp[iW(J)] = \exp\left[\frac{i}{2}\int JGJ\right],$$

which can be thought of as referring to either individual type or as a sum of both. For a causal arrangement, with two disjoint source components, $J = J_1 + J_2$, the factor that represents multiphoton exchange is

$$\exp\left[i\int J_1GJ_2\right] = \exp\left[i\int JA\right],$$

where, in the latter form, the field and the source refer to different source components. In particular, the exchange of two photons is represented by

$$\frac{1}{2}\left[i\int JA\right]^2.$$

Now I return to the general form and consider the effect on $\exp[iW]$ of an infinitesimal change in G. It is given by the factor

$$\frac{i}{2}\int J\delta GJ = -\frac{i}{2}\int A\delta HA,$$

which can be interpreted as an additional two-photon emission from the infinitesimal source:

$$\delta(JJ) = i\delta H.$$

This new process will also act within $W(J)$ to yield the new contribution

$$\delta W_o = \frac{i}{2}\operatorname{Tr}(H^{-1}\delta H),$$

where Tr indicates the trace appropriate to the space-time coordinates. The implied finite change in W_o is

$$W_o = \frac{i}{2}\operatorname{Tr}\log H + \text{const.},$$

where the additive constant is chosen to respect the physical normalization conditions.

The time has come for specializing to the static situation with x–y translational symmetry. The part of the trace that refers to the x, y, t variables, with their continuous complementary variables k_x, k_y, ω, is given by phase-space integrals:

$$\operatorname{Tr}_{xyt}\log H = \frac{A}{(2\pi)^2}\int_{-\infty}^{\infty} dk_x dk_y \frac{T}{2\pi}\int_{-\infty}^{\infty} d\omega \log H,$$

in which the x–y integrals have been limited to a finite area A, and the time integral equals T, the duration of the measurement. The appropriate forms of H in this integral are ($k^2 = k_x^2 + k_y^2$)

$$H_e = k^2 - \omega^2\epsilon(z) - \partial_z^2,$$

$$H_m = \frac{k^2}{\epsilon(z)} - \omega^2 - \partial_z\frac{1}{\epsilon(z)}\partial_z.$$

At this point the energy per unit area appears as

$$\frac{E}{A} = \frac{1}{(4\pi)^2}\int_0^{\infty} dk^2 \int_{-\infty}^{\infty} d\zeta \operatorname{Tr}_z \log H,$$

in which the variable $\zeta = -i\omega$ has been introduced, and an additive constant is understood.

Now it behooves one to perform a partial integration on the variable k^2:

$$\int_0^{\infty} dk^2 \operatorname{Tr}_z \log H = -\int_0^{\infty} dk^2 k^2 \operatorname{Tr}_z\left(\frac{\partial H}{\partial k^2}\frac{1}{H}\right),$$

in which the indicated derivative of H is unity for H_e and $1/\epsilon(z)$ for H_m. The Green's functions symbolized here by $1/H$ are made explicit in

$$[k^2 + \zeta^2\epsilon(z) - \partial_z^2]g_e(z, z'; k, \zeta) = \delta(z - z'),$$

$$\left[\frac{k^2}{\epsilon(z)} + \zeta^2 - \partial_z\frac{1}{\epsilon(z)}\partial_z\right]g_m(z, z'; k, \zeta) = \delta(z - z').$$

At this stage the energy expressions are

$$\frac{E_e}{A} = -\frac{1}{(4\pi)^2}\int_0^{\infty} dk^2 k^2 \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} dz\, g_e(z, z)$$

$$\frac{E_m}{A} = -\frac{1}{(4\pi)^2}\int_0^{\infty} dk^2 k^2 \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} dz \frac{1}{\epsilon(z)} g_m(z, z).$$

Note that if $\epsilon(z)$ were everywhere constant, g_e and g_m/ϵ would be identical.

The actual dielectric geometry is characterized by two values of $\epsilon(z)$: 1, for $|z| < \frac{1}{2}a$, and ϵ, for $|z| > \frac{1}{2}a$. The solutions of the Green's function equations utilize these symbols:

$$\kappa = (k^2 + \zeta^2)^{1/2},\ \kappa' = (k^2 + \epsilon\zeta^2)^{1/2}$$

$$D_e = \frac{\kappa' + \kappa}{\kappa' - \kappa},\ D_m = \frac{\epsilon\kappa + \kappa'}{\epsilon\kappa - \kappa'}.$$

After constructing the $g(z, z')$ and performing the integrations involving $g(z, z)$, one finds contributions to E that, for example, are proportional to aA, the volume enclosed between the slabs. The implied constant energy density—independent of the separation of the slabs—violates the normalization of the vacuum energy density to zero. Accordingly, the additive constant has a piece that maintains the vacuum energy normalization. There is also a contribution to E that is proportional to A, energy associated with the individual slabs. The normalization to zero of the energy for an isolated slab is maintained by another part of the additive constant. The outcome for either type is presented by

$$\int_{-\infty}^{\infty} dz \frac{\partial H}{\partial k^2} g(z, z) = \frac{B}{D^2 e^{2\kappa a} - 1},$$

where

$$B = \frac{a}{\kappa} + (D - D^{-1})\frac{\kappa'^2 - \kappa^2}{2\kappa'^2\kappa^2},$$

and then

$$\frac{E}{A} = -\frac{1}{(4\pi)^2}\int_0^{\infty} dk^2 k^2 \int_{-\infty}^{\infty} d\zeta \frac{B}{D^2 e^{2\kappa a} - 1}.$$

(A somewhat different derivation for the electric field is in ref. 6.)

The individual versions of B are

$$B_e = \frac{a}{\kappa} + \frac{2}{\kappa'\kappa},$$

$$B_m = \frac{a}{\kappa} - \frac{2\epsilon}{\kappa'\kappa}\frac{\kappa'^2 - \kappa^2}{\epsilon^2\kappa^2 - \kappa'^2}.$$

Contact with a known result appears on simulating the perfect conductor idealization by letting $\epsilon \to \infty$. Then $D^2 \to 1$, $B \to a/\kappa$, and the energy per unit area, for either type, is

$$\frac{E}{A} = -\frac{1}{8\pi^2}\int_0^{\infty} dk^2 k^2 d\zeta \frac{a}{\kappa}\frac{1}{e^{2\kappa a} - 1}$$

$$= -\frac{1}{a^3}\frac{\zeta(4)}{(4\pi)^2} = -\frac{1}{a^3}\frac{\pi^2}{16 \times 90},$$

as expected. The a^{-3} behavior holds for any constant ϵ, that is, in the absences of dispersion. The energy formula is more general, however. As an additive combination of different frequencies, it accommodates a frequency dependence of the dielectric constant.

Finally, I note the results for a non-dispersive ϵ, with $|\epsilon - 1| << 1$

$$\frac{E}{A}(e, m) = -\frac{1}{a^3}\left(\frac{\epsilon - 1}{16\pi}\right)^2 \left(\frac{1}{5}, \frac{43}{15}\right).$$

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