Bubble oscillations of large amplitude

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A new equation is derived for large amplitude forced radial oscillations of a bubble in an incident sound field. It includes the effects of acoustic radiation, as in Keller and Kolodner's equation, and the effects of viscosity and surface tension, as in the modified Rayleigh equation due to Plesset, Noltingk and Neppiras, and Poritsky. The free and forced periodic solutions are computed numerically. For large bubbles, such as underwater explosion bubbles, the free oscillations agree with those obtained by Keller and Kolodner. For small bubbles, such as cavitation bubbles, with small or intermediate forcing amplitudes, the results agree with those calculated by Lauterborn from the modified Rayleigh equation of Plesset *et al*. For large forcing amplitudes that equation yielded unsatisfactory results whereas the new equation yields quite satisfactory ones.

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INTRODUCTION

A gas bubble in a liquid performs forced radial oscillations when a sound wave impinges upon it. Large amplitudes result when the acoustic frequency is at or near the bubble's natural frequency, or certain rational multiples of it. Since these large oscillations can be important in cavitation, we have reexamined them by using a new equation which takes account of the acoustic radiation from the bubble. The previous detailed and careful calculations by Lauterborn[1] did not take account of this radiation, and they sometimes yielded unreasonably large amplitudes which did not lie on smooth curves, or they failed to converge to a periodic solution. Our method does not have these defects.

Oscillations of large bubbles were originally analyzed by Rayleigh assuming that the surrounding liquid is incompressible and inviscid, that the bubble remains spherical, and that surface tension is negligible. Plesset,[2] Noltingk and Neppiras,[3] and Poritsky[4] modified this equation to include the effects of viscosity, surface tension, and an incident sound wave, and it was this modified equation which Lauterborn[1] solved. A different modification was made by Keller and Kolodner,[5] who included the effects of acoustic radiation by treating the surrounding liquid as slightly compressible. The same method was used by Epstein and Keller[6] to derive equations for one- and two-dimensional bubbles, for which there are no analogues of the Rayleigh equation. We shall combine these modifications to derive a new equation for the bubble radius. It includes the effects of acoustic radiation, viscosity, surface tension and an incident sound wave.

To solve this new equation we shall first study its free oscillations. As is to be expected, the trajectories in the phase plane are similar to those in the absence of viscosity and surface tension unless the viscosity is very large. The oscillations of a particular underwater explosion bubble are found to be the same as those calculated by Keller and Kolodner,[5] which agree well with experiment.

Then we solve the equation numerically for one of the three bubbles studied by Lauterborn,[1] with various forcing amplitudes. Our results are in very close agreement with his for small and intermediate forcing amplitudes, but not for his largest one. In that case his results suffer from the defects mentioned above, while ours are satisfactory for this and even for larger forcing amplitudes. We have also examined the small amplitude oscillations by the method of averaging, following the procedure used by Prosperetti[7] on the equation of Plesset *et al.*, but we shall not present that analysis here.

I. FORMULATION

We wish to determine $a(t)$, the radius at time t of a spherical bubble of gas and vapor in a fluid of infinite extent. We assume that the pressure $p_b(a)$ within the bubble is uniform and is the sum of the constant vapor pressure p_v and the gas pressure $ka^{-3\gamma}$:

$$p_b(a)=p_v+ka^{-3\gamma}\,. \tag{1.1}$$

Here k is a constant determined by the quantity and type of gas, and γ is the adiabatic exponent of the gas. This pressure $p_b(a)$ exceeds the pressure $p(r,t)$ in the fluid at $r=a$ by the effect of the surface tension σ and the normal component of viscous stress:

$$p_b(a)=p(a,t)+\frac{2\sigma}{a}-\frac{4\mu}{3}\left(\phi_{rr}(a,t)-\frac{\phi_r(a,t)}{a}\right). \tag{1.2}$$

Here $\phi(r,t)$ is the velocity potential of the fluid motion, assumed to be spherically symmetric, and μ is the coefficient of viscosity of the fluid. In addition the velocity a_t of the bubble surface must equal the velocity of the fluid at the surface:

$$a_t=\phi_r(a,t)\,. \tag{1.3}$$

The velocity potential ϕ, the pressure p, and the density $\rho(r,t)$ of the fluid must satisfy conservation of mass, the Navier–Stokes equation, and the equation of state, which are

$$\rho_t+\phi_r\rho_r+\rho\Delta\phi=0\,, \tag{1.4}$$

$$\rho(\phi_{rt}+\phi_r\phi_{rr})+p_r=(4\mu/3)(\Delta\phi)_r\,, \tag{1.5}$$

$$p=p(\rho)\,. \tag{1.6}$$

We seek a, ρ, p, and ϕ satisfying (1.1)–(1.6) given their initial values.

II. SIMPLIFICATION

To solve these equations, we first divide (1.5) by ρ and then integrate it with respect to r from r to infinity to obtain the modified Bernoulli equation

$$-\phi_t-\frac{1}{2}\phi_r^2+\int_p^{p_\infty}\rho^{-1}\,dp-(4\mu/3)\int_r^\infty\rho^{-1}(\Delta\phi)_r\,dr=0\,. \qquad (2.1)$$

Here p_∞ is the constant pressure at $r=\infty$, and we have assumed that ϕ tends to zero at $r=\infty$. Next we differentiate (2.1) with respect to t and write $p_t=p_\rho\rho_t=c^2\rho_t$, where $c^2=p_\rho$ and c is the sound speed:

$$\phi_{tt}+\phi_r\phi_{rt}+c^2\rho_t\rho^{-1}+\frac{4\mu}{3}\int_r^\infty[\rho^{-1}(\Delta\phi)_r]_t\,dr=0\,. \qquad (2.2)$$

Then we use (1.4) in (2.2) to eliminate ρ_t, divide by c^2 and obtain

$$c^{-2}\phi_{tt}-\Delta\phi$$
$$=-c^{-2}\left(\phi_r\phi_{rt}+(4\mu/3)\int_r^\infty[\rho^{-1}(\Delta\phi)_r]_t\,dr\right)+\rho_r\phi_r/\rho\,. \qquad (2.3)$$

To simplify (2.3) we omit the right side, assuming that it is small compared to the terms on the left side, and set c = constant. For a nearly incompressible fluid c^2 is large and ρ_r is small, which tends to justify these assumptions. As a consequence we obtain the wave equation

$$c^{-2}\phi_{tt}-\Delta\phi=0\,. \qquad (2.4)$$

To simplify (2.1) we set ρ = constant in it and obtain

$$p(r,t)=p_\infty-\rho[\phi_t+\tfrac{1}{2}\phi_r^2]+(4\mu/3)\Delta\phi\,. \qquad (2.5)$$

Thus the simplification consists in replacing (1.4) and (1.5) by (2.4) and (2.5).

III. DERIVATION OF THE ORDINARY DIFFERENTIAL EQUATION

We now use (2.5) for p in (1.2) and write that condition in the form

$$\Delta(a)=-\phi_t-\frac{1}{2}\phi_r^2+\frac{4\mu}{\rho a}\phi_r+\frac{2\sigma}{\rho a},\quad \text{at } r=a(t)\,. \qquad (3.1)$$

Here $\Delta(a)$ is the pressure difference times ρ^{-1}:

$$\Delta(a)=\rho^{-1}[p_b(a)-p_\infty]=\rho^{-1}[ka^{-3\gamma}+p_v-p_\infty]\,. \qquad (3.2)$$

Next we write the general solution of (2.4), with f and g arbitrary functions, as

$$\phi(r,t)=r^{-1}f(t-r/c)+r^{-1}g(t+r/c)\,. \qquad (3.3)$$

Then we use (3.3) in (3.1) to obtain an equation for $a(t)$. In doing so we also use (3.3) in (1.3) to eliminate f. In this way we obtain

$$a\Delta(a)-aca_t=c\phi(a,t)-\frac{1}{2}aa_t^2+\frac{4\mu}{\rho}a_t+\frac{2\sigma}{\rho}-2g'\left(t+\frac{a}{c}\right)\,. \qquad (3.3')$$

Now we take the time derivative of (3.3′) and use (3.1) to eliminate ϕ_t. In this way we obtain

$$a_{tt}\left(\frac{4\mu}{\rho}-a(a_t-c)\right)$$
$$=\frac{1}{2}a_t^3+a_t\Delta(a)-c\left(\frac{3}{2}a_t^2+\frac{4\mu a_t}{\rho a}+\frac{2\sigma}{\rho a}-\Delta(a)\right)$$
$$+aa_t\Delta'(a)+2\left(1+\frac{a_t}{c}\right)g''\left(t+\frac{a}{c}\right)\,. \qquad (3.4)$$

Equation (3.4) is a nonlinear second-order ordinary differential equation for the bubble radius $a(t)$. If we set $\mu=\sigma=g=0$ in it, we obtain the corresponding Eq. (12) of Keller and Kolodner.[5] If instead we divide by c and let c become infinite, we obtain the Rayleigh equation for a bubble in an incompressible liquid, as modified by Plesset[2] and others to include surface tension and viscosity. That is the equation which was solved by Lauterborn[1] with $2c^{-1}g''(t)$ replaced by $\rho^{-1}P\sin\omega t$. In order to solve (3.4) we must specify g and the initial values of a and a_t. Then (1.3) yields f, (3.3) yields ϕ and 2.5 yields p.

In order to specify g we suppose that the bubble is in an incident sound field with velocity potential $\Phi(\mathbf{x},t)$, where $\mathbf{x}=0$ is the center of the bubble. The spherically symmetric part of Φ is of the form $r^{-1}[g(t+r/c)+h(t-r/c)]$. If Φ is regular at $\mathbf{x}=0$ then $h=-g$ and we have

$$\Phi(\mathbf{x},t)=r^{-1}[g(t+r/c)-g(t-r/c)]+\Phi_u(\mathbf{x},t)\,. \qquad (3.5)$$

The unsymmetric part Φ_u must vanish at $\mathbf{x}=0$, so if we let $\mathbf{x}$ and r tend to zero in (3.5) we get

$$\Phi(0,t)=2c^{-1}g'(t)\,. \qquad (3.6)$$

This relation determines $g'(t)$ in terms of the incident potential Φ evaluated at the position of the center of the bubble. Differentiating (3.6) yields

$$2g''(t)=c\Phi_t(0,t)\,. \qquad (3.7)$$

This expression is what is needed in the last term in (3.4).

As an example, let us suppose that the incident field is a plane wave with angular frequency ω and pressure amplitude P. Then $\Phi(\mathbf{x},t)=-(P/\rho\omega)\cos\omega(t-x/c)$ and (3.7) yields

$$2g''(t)=Pc\rho^{-1}\sin\omega t\,. \qquad (3.8)$$

We now substitute (3.8) into (3.4) to obtain

$$a_{tt}\left(\frac{4\mu}{\rho}-a(a_t-c)\right)$$
$$=\frac{1}{2}a_t^3+a_t\Delta(a)-c\left(\frac{3}{2}a_t^2+\frac{4\mu a_t}{\rho a}+\frac{2\sigma}{\rho a}-\Delta(a)\right)$$
$$+aa_t\Delta'(a)+\left(1+\frac{a_t}{c}\right)\frac{Pc}{\rho}\sin\omega\left(t+\frac{a}{c}\right)\,. \qquad (3.9)$$

This is the equation we shall solve. In the Appendix it is rewritten in dimensionless variables.

IV. ANALYSIS OF THE EQUATION

When $P=0$, (3.9) reduces to the autonomous equation for the free oscillations of a bubble. This equation can be analyzed in the phase plane with coordinates a and a_t. It has one unstable critical point, a saddle, at $a=0$,

$a_t = c/(3\gamma - 1)$ and a second critical point at the equilibrium configuration $a = a_e$, $a_t = 0$, where a_e is a root of

$$2\sigma/\rho a_e - \Delta(a_e) = 0\,. \qquad (4.1)$$

Linearization shows that this point is a stable spiral if $\omega_0^2 > 0$ and A stable node if $\omega_0^2 < 0$, where ω_0^2 is given

$$\omega_0^2 = \left[4\left(a_e + \frac{4\mu}{\rho c}\right)\left(\Delta'(a_e) + \frac{2\sigma}{\rho a_e^2}\right) - \left(\frac{4\mu}{\rho a_e} - \frac{\Delta(a_e)}{c} - \frac{a_e\Delta'(a_e)}{c}\right)^2\right]\left[4\left(a_e + \frac{4\mu}{\rho c}\right)^2\right]^{-1}. \qquad (4.2)$$

Usually $\omega_0^2 > 0$ and the equilibrium point is a spiral. However if the fluid is extremely viscous it will be a stable node. When ω_0 is real the linearized solution near equilibrium is, with A and θ constants,

$$a(t) - a_e = A\exp\left\{-\left(\frac{4\mu}{\rho a_e} - \frac{\Delta(a_e)}{c} - \frac{a_e\Delta'(a_e)}{c}\right) \times \left[2\left(a_e + \frac{4\mu}{\rho c}\right)\right]^{-1}\right\}\cos(\omega_0 t + \theta)\,. \qquad (4.3)$$

This describes a damped oscillation around equilibrium.

For the nonlinear equation (3.9) with $P = 0$, when $\omega_0^2 > 0$ the trajectories in the phase plane are similar to those for $\sigma = \mu = 0$, which are shown in Ref. 5. We have also solved (3.9) numerically with $P = 0$ for the case of the underwater explosion bubble treated in Fig. 7 of Ref. 5, using the appropriate parameter values. The results are indistinguishable from those given in that figure, which were computed with $\sigma = \mu = 0$. This was to be expected since the dimensionless forms of σ and μ are so small in that case.

The periodic solution of the linearized form of (3.9) with $P \neq 0$ is

$$a(t) - a_e = \frac{Pc}{\rho}\,\mathrm{Im}[A\exp(i\omega(t + c^{-1}a_e))]\,, \qquad (4.4)$$

where

$$A = \left[-\left(a_e + \frac{4\mu}{\rho c}\right)\omega^2 + \left(\frac{4\mu}{\rho a_e} - \frac{\Delta(a_e)}{c} - \frac{a_e\Delta'(a_e)}{c}\right)i\omega - \left(\Delta'(a_e) + \frac{2\sigma}{\rho a_e^2}\right)\right]^{-1}. \qquad (4.5)$$

As $c \to \infty$, A tends to A_∞ given by

$$A_\infty = \left(-a_e\omega^2 - \Delta'(a_e) - \frac{2\sigma}{\rho a_e^2} + \frac{4\mu i\omega}{\rho a_e}\right)^{-1}. \qquad (4.6)$$

We can make A_∞ agree with A by replacing μ in (4.6) by a certain complex effective viscosity. Alternatively we can make $|A_\infty|$ agree with $|A|$ by replacing μ in (4.6) by the real quantity μ_{eff} defined by

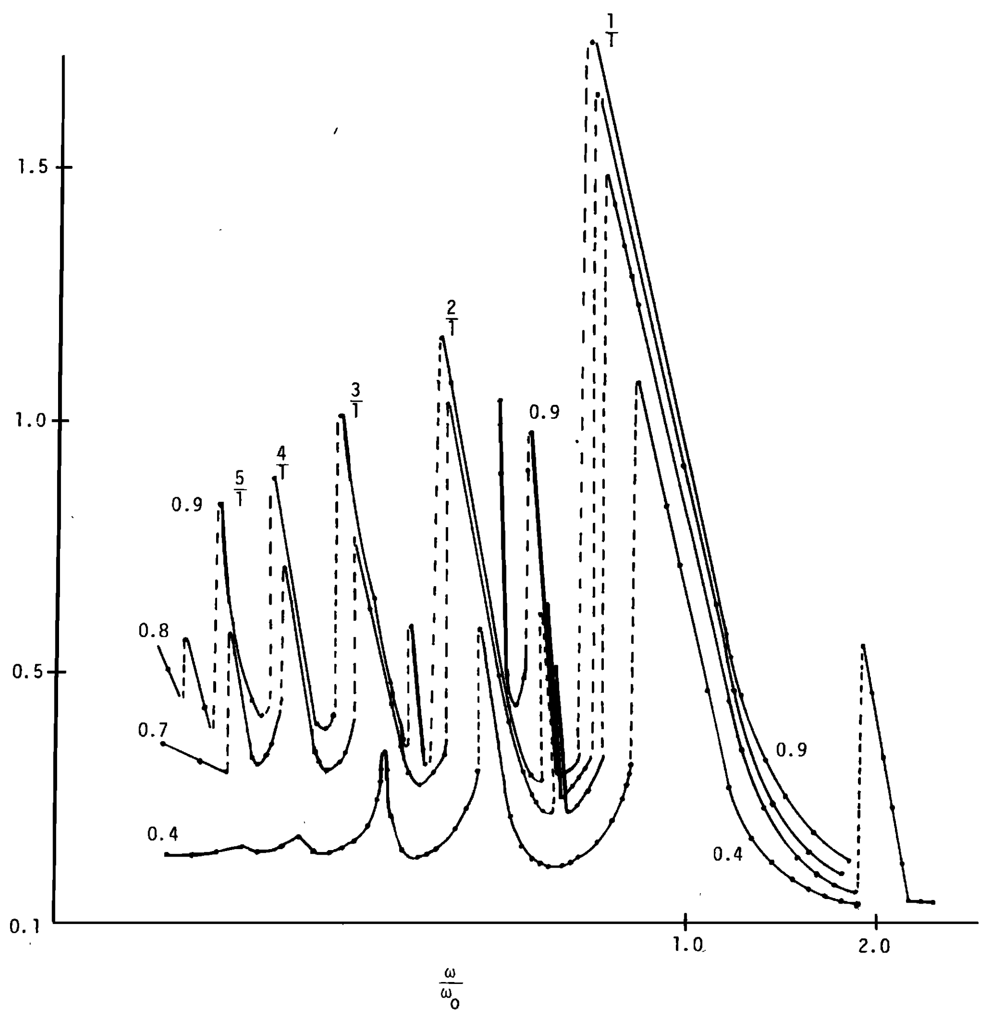

FIG. 1. Frequency response curves for a bubble in water with equilibrium radius $a_e = 10\,\mu$m in incident sound waves of amplitudes $P = 0.4$, 0.7, 0.8, and 0.9 bar. The ordinate is $(a_{max} - a_e)/a_e$ and the abscissa is ω/ω_0. The curves are computed from (3.49) and ω_0 is given by Eq. (4.2).

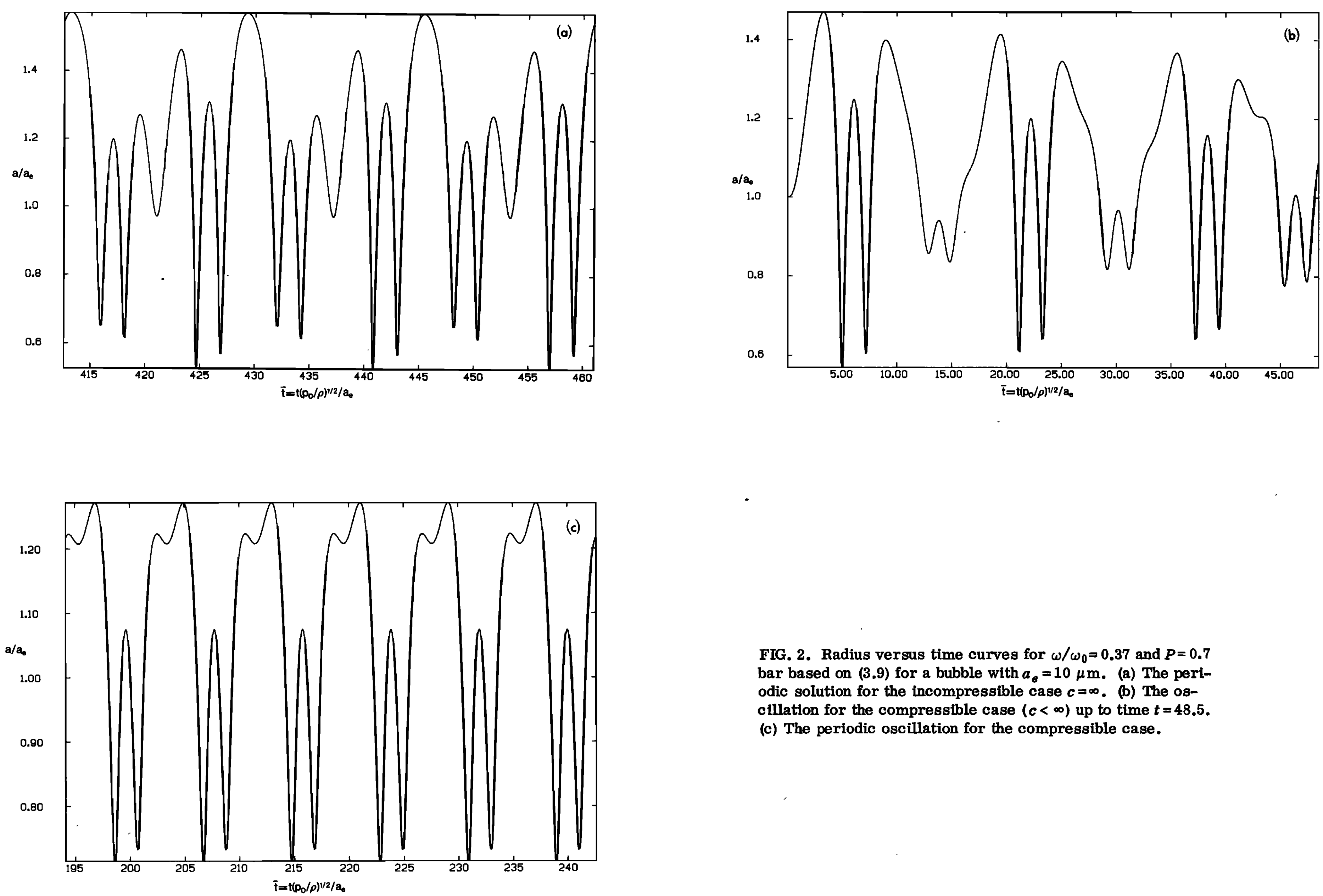

FIG. 2. Radius versus time curves for $\omega/\omega_0 = 0.37$ and $P = 0.7$ bar based on (3.9) for a bubble with $a_e = 10\ \mu$m. (a) The periodic solution for the incompressible case $c = \infty$. (b) The oscillation for the compressible case ($c < \infty$) up to time $t = 48.5$. (c) The periodic oscillation for the compressible case.

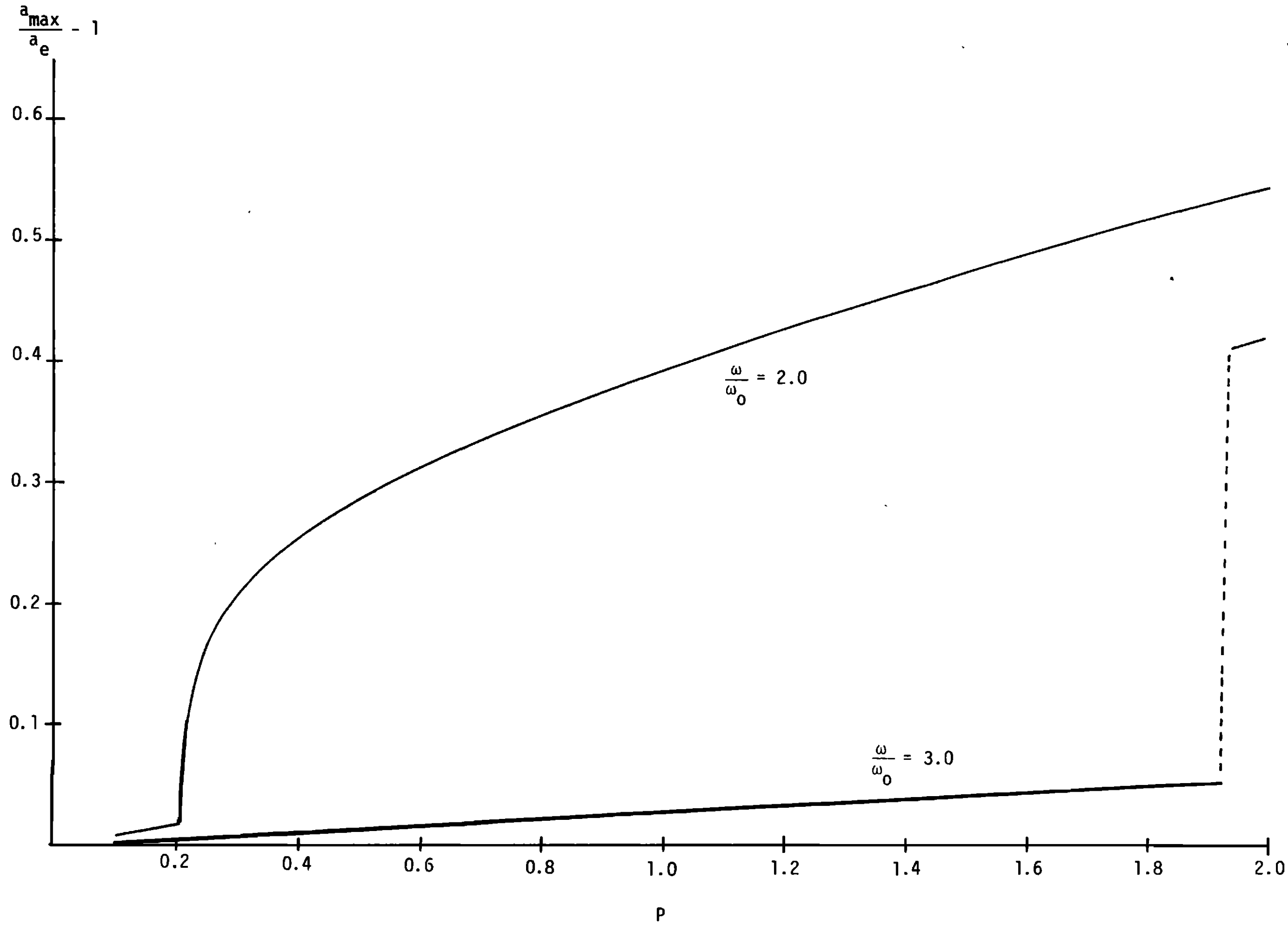

FIG. 3. Amplitude of the subharmonic response as a function of the amplitude of the incident sound wave for $\omega/\omega_0=2$ and $\omega/\omega_0=3$.

$$\mu_{eff} = \mu + \frac{1}{4c}\left[\rho a_e^2\left(a_e\omega^2 + \Delta'(a_e) + \frac{2\sigma}{\rho a_e^2}\right)\right.$$

$$\left. -\rho a_e\left(\Delta(a_e) + a_e\Delta'(a_e)\right)\right] + O(c^{-2}) \tag{4.7}$$

An approximate form of (4.7) appropriate for high frequencies, $\mu_{eff} = \mu + \rho a_e^3\omega^2/4c$, was derived by Devin,[8] Chapman and Plesset[9] and Prosperetti.[10]

V. NUMERICAL SOLUTIONS

In solving (3.9) numerically we shall follow the procedure used by Lauterborn[1] on the corresponding equation for an incompressible fluid, in order to show that the differences in results are due to differences in the equations, and not due to differences in methods of solution. Thus we start with a bubble at rest at its equilibrium radius, i.e. $a_t(0)=0, a(0)=a_e$. Then for given P and ω, we follow the solution until it becomes periodic, and plot $(a_{max}-a_e)/a_e$ vs ω/ω_0. The constants are chosen to represent a bubble in water at 20° C with a static pressure $p_\infty = 1$ bar and a polytropic exponent $\gamma = 1.33$. Thus $\rho = 0.998$ g/cm^3, $\sigma = 72.5$ dyn/cm, $\mu = 0.01$ g/cm/s, $p_v = 0.0233$ bar and $c = 1.484 \times 10^5$ cm/s. We shall study Lauterborn's largest bubble, for which $a_e = 10^{-3}$ cm, since acoustic radiation is most important for it.

Figure 1 shows our frequency response diagram for four values of the incident pressure amplitude: $P=0.4$, 0.7, 0.8, and 0.9 bar. It corresponds to Lauterborn's Fig. 3. For $P=0.4$ bar our curve and his appear to be identical, while for $P=0.7$ bar they are also identical except for the ultraharmonic resonance of order 5/2 which occurs in his curve at about $\omega/\omega_0 = 0.37$, but is absent in ours. To see how this difference arises, we shall consider the radius time curve $a(t)$ for $P=0.7$ bar and $\omega/\omega_0 = 0.37$ in both the incompressible and compressible cases. Figure 2(a) shows the steady state or periodic oscillation in the incompressible case and Fig. 2(b) shows some early oscillations in the compressible case. We note that the two curves are somewhat similar. However as time goes on, in the compressible case the shape of the oscillation changes to the periodic form shown in Fig. 2(c), which does not contain the 5/2 ultraharmonic.

Our response curve for $P=0.8$ bar is very similar to that for $P=0.7$ bar, but it is quite different from Lauterborn's curve. His incompressible result and our compressible one are similar for $\omega/\omega_0>0.6$, but not for $\omega/\omega_0<0.6$. The incompressible result shows many peaks corresponding to harmonic and ultraharmonic resonances, at some frequencies the amplitudes were too large to fit in the figure, and at some frequencies no periodic solution was obtained. The compressible result always yielded periodic solutions with amplitudes which fit into the figure, and some harmonic and ultraharmonic resonances.

For $P = 0.9$ bar our result is again similar to that for $P = 0.8$ bar, but with more resonances. We computed it only for $\omega/\omega_0 > 0.5$ to save computer time, since the oscillation went through many cycles before becoming periodic.

In Fig. 1 we also see a subharmonic resonance at $\omega/\omega_0 = 2$. This resonance did not occur for small values of P, but instead there is a threshold value which P must exceed in order for it to appear. This threshold phenomenon is shown in Fig. 3. For $\omega/\omega_0 = 2$ we find that the threshold value of P lies between 0.20 and 0.22 bar while Lauterborn found it to be between 0.1 and 0.15 bar. Similarly for $\omega/\omega_0 = 3$ the compressible threshold lies between 1.9 and 1.94 bar while in the incompressible case Lauterborn found it to be between 1.3 and 1.5 bar. Thus in both cases the inclusion of compressibility increases the threshold, as one might expect.

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APPENDIX

In terms of the equilibrium radius a_e which satisfies (4.1), we define dimensionless variables as follows:

$$\begin{aligned} &x = a/a_e, \quad \bar{r} = r/a_e, \\ &p_0 = p_\infty - p_v, \quad \bar{t} = t(p_0/\rho)^{1/2}/a_e, \\ &\bar{\phi} = \phi a_e(p_0/\rho)^{1/2}, \quad \bar{k} = k/p_0 a_e^{3\gamma}, \\ &\bar{\sigma} = 2\sigma/a_e p_0, \quad \bar{c} = c(\rho/p_0)^{1/2} \\ &\nu = 4\mu/a_e(\rho p_0)^{1/2}, \quad \bar{g}'' = (\rho/p_0)^{3/2} g'', \\ &\bar{\omega} = \omega a_e(\rho/p_0)^{1/2}. \end{aligned} \tag{A1}$$

We now introduce these variables into (3.4) and then omit the bars to obtain

$$\begin{aligned} x_{tt}[\nu - x(x_t - c)] = {} & \frac{x_t^3}{2} + x_t[(1-3\gamma)kx^{-3\gamma} - 1] \\ & - c\left(\frac{3x_t^2}{2} + \frac{\nu x_t}{x} + \frac{\sigma}{x} - kx^{-3\gamma} + 1\right) \\ & + 2\left(1 + \frac{x_t}{c}\right) g''\left[\left(t + \frac{x}{c}\right) a_e\left(\frac{\rho}{p_0}\right)^{1/2}\right]. \end{aligned} \tag{A2}$$

In the same way we obtain from (4.1)

$$k = 1 + \sigma. \tag{A3}$$

Similarly we get from (3.9)

$$\begin{aligned} x_{tt}[\nu - x(x_t - c)] = {} & \frac{x_t^3}{2} + x_t[(1-3\gamma)kx^{-3\gamma} - 1] \\ & - c\left(\frac{3x_t^2}{2} + \frac{\nu x_t}{x} + \frac{\sigma}{x} - kx^{-3\gamma} + 1\right) \\ & + \left(1 + \frac{x_t}{c}\right)\frac{Pc}{p_0}\sin\omega\left(t + \frac{x}{c}\right). \end{aligned} \tag{A4}$$

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