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A generalization of the Rayleigh–Plesset equation of bubble dynamics

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The classical Rayleigh-Plesset equation of spherical bubble dynamics in an incompressible liquid is generalized to include the non-Newtonian behavior of the liquid and mass exchange processes at the bubble interface.

The well-known Rayleigh–Plesset equation

$$R\ddot{R}+\frac{3}{2}\dot{R}^2=\frac{1}{\rho_l}\left(p_i-p_\infty-\frac{2\sigma}{R}-4\mu\frac{\dot{R}}{R}\right), \tag{1}$$

is the fundamental equation governing the dynamics of a spherical cavity of radius $R(t)$, the center of which is fixed in an unbounded incompressible viscous liquid free of body forces.[1] Here, p_i is the pressure acting on the inner side of the cavity interface, p_∞ is the pressure at large (infinite) distances from the cavity, μ and ρ_l are the viscosity and density of the liquid, and σ is the interfacial tension; the dots denote differentiation with respect to time.

Equation (1) is valid for a Newtonian liquid under conditions of negligible mass exchange at the cavity interface. It is the purpose of this paper to obtain a generalization of Eq. (1) to which these two restrictions do not apply.

I. DERIVATION

From the condition of conservation of mass in the liquid in the spherically symmetric situation of present concern we have

$$u(r,t)=(R^2/r^2)U_l(t), \tag{2}$$

where U_l and u denote the radial liquid velocity at the cavity interface and at a distance r from its center, respectively. The radial component of the momentum equation[2,3] reduces to the form

$$\rho_l\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}\right)=-\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr})-\frac{\tau_{\theta\theta}+\tau_{\varphi\varphi}}{r}, \tag{3}$$

where τ is the stress tensor in the fluid. Since τ is traceless,[2,3] $\tau_{\theta\theta}+\tau_{\varphi\varphi}=-\tau_{rr}$ and integrating Eq. (3) from r to infinity we find, by (2),

$$\rho_l\frac{R}{r}\left(R\dot{U}_l+2\dot{R}U_l-\frac{1}{2}\frac{R^3}{r^3}U_l^2\right)=p(r,t)-p_\infty-\tau_{rr}(r,t)+3\int_r^\infty s^{-1}\tau_{rr}\,ds, \tag{4}$$

which may be viewed as an equation for the liquid pressure provided the other quantities are known. Setting $r=R$ we obtain from (4)

$$R\dot{U}_l+2\dot{R}U_l-\tfrac{1}{2}U_l^2=\frac{1}{\rho_l}\left(p(R,t)-p_\infty-\tau_{rr}(R,t)+3\int_R^\infty r^{-1}\tau_{rr}\,dr\right), \tag{5}$$

where $p(R,t)$ is the pressure acting on the liquid side of the interface.

The conditions of conservation of mass and momentum across the cavity wall are, respectively,[4-6]

$$J\equiv\rho_l(U_l-\dot{R})=\rho_b(U_b-\dot{R}), \tag{6}$$

$$J^2\left(\frac{1}{\rho_b}-\frac{1}{\rho_l}\right)+p_i-p(R,t)+\tau_{rr}(R,t)=\frac{2\sigma}{R}, \tag{7}$$

where J is the mass flux and ρ_b, U_b are the density and velocity on the inner side of the interface. The stress tensor in the bubble has been neglected in (7). Eliminating $p(R,t)$ between (5) and (7) and expressing $\dot{R}$ in terms of U_l and J from (6) we find the main result of this paper, namely,

$$R\dot{U}_l+\frac{3}{2}U_l^2-\frac{J}{\rho_l}\left[2U_l+J\left(\frac{1}{\rho_b}-\frac{1}{\rho_l}\right)\right]=\frac{1}{\rho_l}\left(p_i-p_\infty-\frac{2\sigma}{R}+3\int_R^\infty r^{-1}\tau_{rr}\,dr\right). \tag{8}$$

For an incompressible Newtonian liquid $\tau_{rr}=2\mu(\partial u/\partial r)$ and this equation becomes

$$RU_l+\frac{3}{2}\dot{U}_l^2-\frac{J}{\rho_l}\left[2U_l+J\left(\frac{1}{\rho_b}-\frac{1}{\rho_l}\right)\right]=\frac{1}{\rho_l}\left(p_i-p_\infty-\frac{2\sigma}{R}-4\mu\frac{U_l}{R}\right), \tag{9}$$

which reduces to (1) when the interfacial mass flux J vanishes, i.e., $U_l=\dot{R}$. For a non-Newtonian liquid, again in the case $J=0$, Eq. (8) becomes

$$R\ddot{R}+\frac{3}{2}\dot{R}^2=\frac{1}{\rho_l}\left(p_i-p_\infty-\frac{2\sigma}{R}+3\int_R^\infty r^{-1}\tau_{rr}\,dr\right),$$

a form originally given by Fogler and Goddard.[7,8]

II. DISCUSSION

Equation (8) is in a form particularly useful when the mass flux can be computed independently of conditions within the bubble. For instance, in the process of vapor bubble growth, we have, to a good approximation,

$$LJ=q(R,t),$$

where L is the latent heat and q is the radial heat flux in the liquid at the bubble boundary[9]. A similar relation holds in the case of mass diffusion from a gas bubble.

Of the two terms in brackets in the left-hand side of the complete equation (8), the first one is essentially of kinematic origin and accounts for the fact that the bubble boundary moves not only because of the liquid flow, but also because of the mass flux at the interface. The

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second term is of dynamical origin, and accounts for the additional pressurization of the liquid caused by the difference between its specific volume and that of the bubble contents.

Equation (8) has not been applied to specific cases yet, so that the importance of the terms containing J cannot be assessed quantitatively. However, for the purpose of estimation, we may consider the limiting situations of a vapor bubble growing in an inviscid liquid in the inertia-controlled or in the diffusion-controlled regime.[9] In both cases the bubble remains filled with saturated vapor [respectively at the initial liquid temperature and at the boiling temperature corresponding to p_∞ (Ref. 9)], so that

$$J \simeq \rho_b \dot{R}, \tag{10}$$

and, from (6), $U_l = \dot{R} + J/\rho_l \simeq (1+\rho_b/\rho_l)\dot{R}$. The relative magnitude of the relevant terms is thus of the order of

$$\frac{J^2/\rho_l\rho_b}{U_l^2} \simeq \frac{\rho_b/\rho_l}{(1+\rho_b/\rho_l)^2} \simeq \frac{\rho_b}{\rho_l},$$

and hence small far from the critical point of the liquid. Therefore, it may be concluded that the terms in brackets in Eq. (8) can safely be neglected in many applications. This form of the equation of spherical bubble growth, however, is the correct generalization to be used if interfacial mass transfer and non-Newtonian liquid behavior need be considered in the analysis.

For the case of a Newtonian liquid an equation similar to (8) was derived by Nigmatulin[10-12] who, however, disregarded the second term in brackets on the left-hand side. To judge the correctness of this approximation we consider the ratio of the term neglected to the one retained finding, by (6),

$$J\left(\frac{1}{\rho_b}-\frac{1}{\rho_l}\right)(U_l)^{-1} = \frac{J}{\rho_b\dot{R}}\left(1-\frac{\rho_b}{\rho_l}\right)\left(1+\frac{J}{\rho_b\dot{R}}\right)^{-1}. \tag{11}$$

Unless $|J/\rho_b\dot{R}|$ is small compared with 1 (in which case the effect of both terms can be disregarded), the numerator is seen not to be negligible compared with U_l whenever $\rho_b \ll \rho_l$. For instance, for J given by (10), the ratio (11) has the value $(1-\rho_b/\rho_l)/(1+\rho_b/\rho_l)$.

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