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Casimir light: Pieces of the action

(dynamics/Lagrangian)

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ABSTRACT More realistic dynamics for the collapsing dielectric fluid are introduced in stages by adding contributions to the Lagrangian that forms the action. The elements are kinetic energy, Casimir potential energy, air pressure potential energy, and electromagnetic coupling to the moving dielectric. There are successful tests of partial collapse time and of minimum radius.

The simplified treatment of sonoluminescence, based on the dynamical Casimir effect (1–3), has employed the overidealization of instantaneous collapse of an evacuated cavity in water. In this note I begin to remove the more egregious aspects of that treatment.

The physical system consists of (*i*) a continuous, but bounded, dielectric fluid of constant mass density ρ_0 and dielectric constant ϵ and (*ii*) the electromagnetic field. At time $t = 0$ a spherical cavity of radius R exists, within which there is no dielectric material—it is a vacuum. The ensuing collapse for $t > 0$ maintains the spherical symmetry as the dielectric boundary, at $r(t) < R$, decreases.

I adopt the following description of the radially inflowing liquid, which uses the mass density $\rho(\vec{r}t)$ and flux vector $\vec{j}(\vec{r}t)$, where

$$\nabla\cdot\vec{j}(\vec{r}t) + \frac{\partial}{\partial t}\rho(\vec{r}t) = 0, \qquad \vec{j}(\vec{r}t) = \rho(\vec{r}t)\vec{v}(\vec{r}t);$$

namely,

$$v(\vec{r}t) = \frac{\vec{r}}{r}\,v(rt); \qquad \rho(\vec{r}t) = \rho_0\eta[r - r(t)];$$

$$r \geq r(t) : r^2v(rt) = r(t)^2v(t).$$

The symbol η indicates the Heaviside step function, and the simplifying assumption of incompressibility has been adopted.

The implied kinetic energy is

$$T(t) = \frac{1}{2}\int(d\vec{r})\rho(\vec{r}t)[\vec{v}(\vec{r}t)]^2$$

$$= \frac{1}{2}\rho_0\int_{r(t)}^{\infty} dr4\pi r^2\frac{1}{r^4}[r(t)^2v(t)]^2$$

$$= 2\pi\rho_0r(t)^3v(t)^2.$$

A potential energy contribution, V_C, is produced by the Casimir energy excess of the vacuum, relative to the region of uniform dielectric constant ϵ. As such, it involves the spatial integral over the vacuum region, for both field types (2, 3):

$$V_C(t) = \int\frac{(d\vec{r})(d\vec{k})}{(2\pi)^3}k\hbar c\left(1 - \frac{1}{\epsilon^{1/2}}\right),$$

with the integration domain $r < r(t)$. Ignoring dispersion, and with the imposition of a high wavenumber cutoff, K, one gets

$$V_C(t) = \frac{\hbar c}{6\pi}r(t)^3K^4\left(1 - \frac{1}{\sqrt{\epsilon}}\right).$$

Here, then, are two pieces of the action, the time integral of the Lagrangian,

$$L = T - V_C = 2\pi\rho_0r(t)^3[v(t)^2 - \Omega^2],$$

where

$$\Omega^2 = \frac{\hbar c}{12\pi^2}\frac{K^4}{\rho_0}\left(1 - \frac{1}{\epsilon^{1/2}}\right).$$

The stationary property of the action produces the equation of motion in response to the infinitesimal variation $\delta r(t)$, along with

$$\delta v(t) = \frac{d}{dt}\delta r(t).$$

One gets (omitting the explicit reference to t)

$$2r^3\frac{dv}{dt} = -3r^2(v^2 + \Omega^2)$$

or, on writing $dt = dr/v$,

$$\frac{d}{dr}[r^3(v^2 + \Omega^2)] = 0.$$

The immediate solution that obeys the initial condition $v = 0$, for $r = R$, is

$$\frac{v^2}{\Omega^2} = \left(\frac{R}{r}\right)^3 - 1.$$

Experiment (4) tells us that, in the collapse of a bubble from $R = 4 \times 10^{-3}$ cm down to $r = 4 \times 10^{-4}$ cm, the speed of Mach 1 in air is attained: $v^2 \simeq 10^9(\text{cm/s})^2$. That supplies the information ($\rho_0 = 1$ g/cm^3):

$$\Omega^2 = \frac{4}{3}\pi^2\frac{\hbar c}{\Lambda_0^4}\left(1 - \frac{1}{\epsilon^{1/2}}\right) \simeq 10^6(\text{cm/s})^2,$$

which introduces Λ_0, a fundamental cutoff wavelength. With $(1 - \epsilon^{-1/2})$ regarded as of order unity for the dominant short wavelengths, one learns that

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$$\Lambda_0 \simeq \frac{1}{2}\,10^{-5}\mathrm{cm} < \Lambda(\mathrm{H_2O}),$$

consistent with the requirement (3) that any fundamental cutoff lies beyond that produced by the onset of water opacity.

I note here the time interval, T_0, for total collapse down to $r = 0$, if this description remains valid over the whole range:

$$T_0 = \frac{1}{\Omega}\int_0^R dr\left[\left(\frac{R}{r}\right)^3 - 1\right]^{-1/2}.$$

With $r/R = x^{1/3}$, one has

$$T_0 = \frac{R}{\Omega}\frac{1}{3}B\left(\frac{5}{6},\frac{1}{2}\right) \simeq \frac{3}{4}\frac{R}{\Omega},$$

through the intermediary of the Beta function. For the more relevant situation of collapse down to $\mathbf{r}$, $0 < \mathbf{r} \ll R$, one gets

$$T_{\mathbf{r}} \simeq T_0 - \frac{1}{\Omega}\int_0^{\mathbf{r}} dr\left(\frac{r}{R}\right)^{3/2}$$

$$= \frac{3}{4}\frac{R}{\Omega}\left[1 - \frac{8}{15}\left(\frac{\mathbf{r}}{R}\right)^{5/2}\right].$$

For the example (4) $R = 4 \times 10^{-3}$ cm, $\mathbf{r}/R = 1/10$, with $\Omega = 10^3$ cm/s,

$$T_{\mathbf{r}} \simeq 3 \times 10^{-6}\ \mathrm{s},$$

which is quite compatible with the observation of several microseconds.

It is time to confront the fact that all this is really not happening in a vacuum—the bubble is sustained by air pressure.

I assume the adiabatic relation between air pressure (p) and volume ($4\pi r^3/3$, as normalized by unit atmospheric pressure (p_0) at the equilibrium radius (r_0):

$$p(t)(r(t)^3)^\gamma = p_0(r_0^3)^\gamma,$$

where $\gamma = 1.4$ is the ratio of specific heats. (The short-range van der Waals repulsive forces are not included here.)

The air pressure addition to the potential energy, V_p, expressed as the spatial integral over the "vacuum" (now the gaseous region free of dielectric) is

$$V_p(t) = \int (d\vec{r})p(t)$$

$$= \frac{4}{3}\pi r_0^3 p_0\left(\frac{r_0}{r(t)}\right)^{3(\gamma-1)}.$$

It is useful to compare the repulsive force derived from V_p with that implied by the attractive Casimir energy, V_C. In doing so one encounters the dimensionless ratio (using cgs units)

$$\frac{p_0}{\rho_0\Omega^2} \simeq \frac{10^6}{10^6} \simeq 1.$$

Then one gets (omitting explicit reference to t)

$$-\frac{\partial V_p/\partial r}{\partial V_C/\partial r} = (\gamma - 1)\frac{V_p}{V_C}$$

$$= \frac{2}{3}(\gamma-1)\left(\frac{r_0}{r}\right)^{3\gamma}.$$

So, for $r = r_0$, the repulsive pressure force is still only about a fourth (0.27) of the attractive Casimir force. At half that distance, $r = 1/2r_0$, the ratio is more than inverted (4.9).

The equation of motion—as modified by the introduction of V_p in the action—is conveyed by the constancy of (it is the energy, divided by $2\pi\rho_0$)

$$r^3(v^2 + \Omega^2) + \Omega^2\frac{2}{3}r_0^3\left(\frac{r_0}{r}\right)^{3(\gamma-1)}.$$

With the initial condition of $v = 0$ for $r = R$, one gets

$$\frac{v^2}{\Omega^2} = (1+\beta)\left(\frac{R}{r}\right)^3 - 1 - \frac{2}{3}\left(\frac{r_0}{r}\right)^{3\gamma},$$

where

$$\beta = \frac{2}{3}\left(\frac{r_0}{R}\right)^{3\gamma}.$$

Note that, in the range $R \geq r \geq r_0$, $R/r_0 \gg 1$, the previous result effectively reemerges. The turning point produced by air pressure—where $v = 0$ for the radius $\mathbf{r} < r_0$, again with the simplifying ratio $R/r_0 \gg 1$—is contained in

$$\left(\frac{\mathbf{r}}{r_0}\right)^{(\gamma-1)} = \left(\frac{2}{3}\right)^{1/3}\frac{r_0}{R}.$$

Figure 3*b* of ref. 4 presents data for a "non-light-emitting bubble," which would seem to be the situation just discussed. Unfortunately, the limited data have been extrapolated by means of a dynamics that differs from the one I am using. Also, the dissipation that is evident in the decreasing amplitudes of successive "bounces" may have a contribution from electromagnetic effects to which neither the photomultiplier tube nor the naked eye is sensitive. Also, some electromagnetic emissions could be absorbed in the water.

One is thereby invited to include in the action, through an additive contribution to the Lagrangian, the description of the electromagnetic field in the presence of the moving dielectric material.

For the example of the e(lectric) field, at a given time t,

$$L_e(t) = \int (d\vec{r})\,\frac{1}{2}\left\{\frac{1}{c^2}\,\epsilon(\vec{r}t)\left[\frac{\partial}{\partial t}A(\vec{r}t)\right]^2 - [\vec{\nabla}A(\vec{r}t)]^2\right\}.$$

This is decomposed into two additive parts: (*i*) L_{ea}, in which the dielectric constant is the everywhere uniform ϵ, and (*ii*) the reduction of the dielectric constant to unity in the "vacuum," as produced by the subtraction of $(\epsilon - 1)$ from ϵ in the region $r < \mathrm{r}(t)$:

$$L_{eb}(t) = -\int (d\vec{r})\eta[r(t) - r]\frac{1}{2}\frac{\epsilon - 1}{c^2}\left[\frac{\partial}{\partial t}A(\vec{r}t)\right]^2.$$

It is not my intention now to delve into the intricacies of quantum field theory. Rather, I proceed to use available experimental data in a probe of the electromagnetic coupling. It begins with the observation that the pulse of radiation is emitted within a time interval that is shorter than 5×10^{-11} s. Indeed, to judge by the spectrum, it may be orders of magnitude smaller.

It is plausible that the energy drained away in that short time interval is extracted from the kinetic energy of the collapsing dielectric fluid. For simplicity in the discussion of the major example of ref. 4, I take the radius at which Mach 1 speed is reached, the radius at which the energy release occurs, and the equilibrium radius r_0 all to be the same: 4×10^{-4} cm. I want to know the value of v_0, the speed of the dielectric wall immediately after the blow-off of electromagnetic energy.

To that end, I look at Figure 4*a* of ref. 4 and read that r_1, the maximum bounce radius, is 7×10^{-4} cm. Now consider the part of the first bounce that takes one outward from r_0, at the speed v_0, to the stopping point at r_1. What is constant over this interval is expressed well enough $\left[\frac{2}{3}\left(\frac{r_0}{r_1}\right)^{3\gamma} = 0.06\right]$ by

$$r_0^3\left(\frac{v_0^2}{\Omega^2}+\frac{5}{3}\right)=r_1^3,$$

or

$$\frac{v_0^2}{\Omega^2}=\left(\frac{r_1}{r_0}\right)^3-\frac{5}{3}=3.7,$$

which is to say,

$$v_0 = 1.9 \times 10^3 \text{ cm/s}.$$

Finally, consider the part of the initial collapse that begins at r_0, immediately after the radiation burst, and ends at the minimum radius $\mathbf{r}$. What is constant over that stretch is conveyed by

$$\left(\frac{r_1}{r_0}\right)^3=\frac{2}{3}\left(\frac{r_0}{\mathbf{r}}\right)^{3(\gamma-1)},$$

which is the analogue of an earlier result where R appears in the place of r_1. Its use in the present context is arguably more justified. Indeed, the value of $\mathbf{r}$ that emerges, $\mathbf{r} = 0.7 \times 10^{-4}$ cm, agrees with observation.

A question: If, as it would seem, a mechanism exists that transfers kinetic energy of a macroscopic body into energy of microscopic entities, could there not be—in a different circumstance—a mechanism that transfers energy of microscopic entities into kinetic energy of a macroscopic body?

1. Schwinger, J. (1993) *Proc. Natl. Acad. Sci. USA* **90,** 958–959.
2. Schwinger, J. (1993) *Proc. Natl. Acad. Sci. USA* **90,** 2105–2106.
3. Schwinger, J. (1993) *Proc. Natl. Acad. Sci. USA* **90,** 4505–4507.
4. Barber, B. & Putterman, S. (1992) *Phys. Rev. Lett.* **69,** 3839–3842.