Control of Mixing in Fluid Flow: A Maximum Entropy Approach

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***Abstract*—In many technological processes a fundamental stage involves the mixing of two or more fluids. As a result, the design of optimal mixing protocols is a problem of both fundamental and practical importance. In this paper, the authors formulate a prototypical mixing problem in a control framework, where the objective is to determine the sequence of fluid flows that will maximize entropy. By developing the appropriate ergodic–theoretic tools for the determination of entropy of periodic sequences, they derive the form of the protocol which maximizes entropy among all of the possible periodic sequences composed of two shear flows orthogonal to each other. The authors discuss the relevance of their results in the interpretation of previous studies of mixing protocols.**

***Index Terms*—Control of mixing, dynamical systems, entropy, ergodic theory.**

I. Introduction

MIXING is a very important process in many engineering applications. For example, mixing between two or more fluids in a fuel–air mixture in combustion engines enhances the efficiency of the combustion process. Until recently the problem of mixing has been treated without the support of a sufficiently general theory. The approaches used are usually based on *ad hoc* arguments and tailored to the specific situation under consideration. Aref [2] has addressed this question from the point of view of nonlinear dynamical systems theory. The books by Ottino on mixing by chaotic advection [17] and Wiggins [22] are comprehensive general treatments of this subject. The concepts and methods of dynamical systems are used to formulate the theory. Additional examples of this approach can be found in [12] and [14].

Recently there has been a great deal of interest in expanding the traditional domain of control systems theory and designing to encompass new problems in engineering. One of the recent successes is the fusion of control systems ideas with the modern theory of dynamical systems to address problems related to bifurcation control [1]. In this paper we follow a similar path where we show how one of the simplest, and yet very important, problems in fluid mixing can be posed as a control problem. The dynamics of this model is a two-dimensional system that evolves on a torus. The control problem aims at finding the sequence of flows that must be applied in order to maximize the entropy of the system. Since entropy is a measure of randomness, it serves as a good indicator for the quality of mixing in a fluid. In the formulation and solution of this control problem we make use of tools from the ergodic theory of dynamical systems.

The analysis we will carry out in this paper is motivated by the results of [8]. In that paper, mixing in a prototypical problem, the so-called *eggbeater flow*, is considered. This flow provides a simple illustration of the basic stretching-and-folding process involved in mechanical mixing [16]. In particular, in this model the motion of particles in a two-dimensional flow on a torus is assumed to be given by

$$\begin{aligned} \dot{x} &= v_1(y) \\ \dot{y} &= 0 \end{aligned} \tag{1}$$

in a certain interval of time, and by

$$\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= v_2(x) \end{aligned} \tag{2}$$

in another interval of time for given functions v_1 and v_2. The variables x and y vary on the torus $\{(x,y)(\mathrm{mod}\,1)\}$. The reader can think of this as the model of a fluid constrained on a two-dimensional torus equipped with an engine which impresses the velocity profiles given by v_1 and v_2. The problem considered in [8] consists of finding the sequence of actions defined in (1) and (2) which mixes best. The factors to be chosen in the problem are:

1) the *shapes* of the functions v_1, v_2;
2) the *criterion* to be used as a measure of good mixing;
3) the *sequence* of mechanical actions such that the behavior of the system alternates between (1) and (2).

Once 1) and 2) above are specified, the choice of 3) is a control problem for a hybrid system.

The criterion to be used as a measure of mixing must necessarily incorporate two aspects of the mixing problem: firstly, is the whole phase space getting homogeneously mixed by the chosen mixing protocol, and secondly, if it is, then how fast does the process of mixing proceed? Franjione and Ottino [8] have suggested that complicated protocols obtained via symmetry considerations can achieve complete mixing on the phase space, for any chosen v_1, v_2. Ling [12], on the other hand, has argued that tweaking of the system parameters to achieve complete mixing while using simple protocols might be better in practice. In both cases, the following question arises: once the first problem of mixing is resolved, i.e., periodic protocols are found that *are* mixing (in the ergodic

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theory sense [3], [19]), what is the best "superprotocol"? Our approach consists of choosing the protocol which has maximum entropy.

Entropy in the context of dynamical systems theory was introduced as a measure of the disorder created by a transformation. The definition of entropy can be generalized to periodic sequences of transformations (see Section III and the Appendix). We use entropy as a measure of mixing.

We consider the simple linear shear flow case where

$$v_1(u) = v_2(u) = au, \qquad a \neq 0, \qquad a \in \mathrm{R}. \tag{3}$$

We solve the problem of finding a periodic sequence which gives the maximum amount of entropy among the ones defined by the transformations (1) and (2) with (3). We notice that at each step the constant a is a measure of the kinetic energy of the flow, and therefore the problem can be restated as a maximization of entropy with constant energy per step. Another interpretation of the constant a is as a measure of the shear stress acting on the fluid since this is given by $\nu\frac{\partial v_1}{\partial y} = \nu\frac{\partial v_2}{\partial x} = \nu a$, where ν is the viscosity of the fluid. For simplicity of presentation, we will deal with velocity profiles in the form (3) for most of the paper; however, we will provide a generalization of our results to the case of different shear flows, namely

$$\begin{aligned} v_1(y) &= ay, \qquad a \neq 0, \qquad a \in \mathrm{R} \\ v_2(x) &= bx, \qquad b \neq 0, \qquad b \in \mathrm{R} \end{aligned} \tag{4}$$

and a and b have the same sign (see Section VI-C). In addition, we will comment on the case when a and b have different signs.

The paper is organized as follows. In Section II, we review the basic definitions concerning entropy of dynamical systems. In Section III, we define entropy for periodic sequences of measure-preserving automorphisms and state a few general properties that we will use later in the paper. The proofs of these properties are found in the Appendix. In Section IV, we formulate the optimization problem that we want to solve. Some auxiliary results concerning matrices related to the maps that we are considering are given in Section V. The main results are given in Section VI, where we give the expression of the protocol which maximizes entropy among the ones in a given family. Conclusions are presented in Section VII along with the results of numerical experiments.

II. Background

In this section we summarize the basic definitions concerning entropy of automorphisms of dynamical systems that we will need. For a more detailed exposition, the reader is referred to one of the introductory books to ergodic theory such as [10], [3], [19]. For the terminology involving measure theory, the reader is referred to [9].

Recall that a (discrete time) dynamical system consists of a triple (M, μ, Φ), where (M, μ) is a measure space with $\mu(M) = 1$ and Φ is a measure-preserving automorphism (in the following for brevity automorphism), defined on it. Consider a finite measurable partition (in the following for brevity just partition) of the space $\{M\}$, denoted by $\alpha := \{A_i\}_{i\in I}$, where I is a set of indexes of finite cardinality.

Definition 2.1 (Product of Two Partitions): Given a partition $\alpha = \{A_i\}_{i\in I}$ and a partition $\beta = \{B_j\}_{j\in J}$, the product partition $\alpha \vee \beta$ is the partition consisting of sets which are intersections of the sets in α with the sets in β.

Definition 2.2 (Ordering on Partitions): Given two partitions α and β, we write $\alpha \leq \beta$ if each set in β is a subset of a set in α.

Definition 2.3 (Transformation of a Partition Under an Automorphism): Given a partition $\alpha = \{A_i\}_{i\in I}$ and an automorphism Φ, the partition $\Phi\alpha$ is given by

$$\Phi\alpha := \{\Phi A_i\}_{i\in I}. \tag{5}$$

It is immediate to verify that $\Phi(\alpha \vee \beta) = \Phi\alpha \vee \Phi\beta$.

The definition of entropy involves the following continuous, nonnegative, and concave function $z(t)$, defined in $[0, 1]$:

$$z(t) = \begin{cases} -t\log(t), & 0 < t \leq 1 \\ 0, & t = 0. \end{cases} \tag{6}$$

Definition 2.4 (Entropy of a Partition): The entropy of a partition $\alpha = \{A_i\}_{i\in I}$, denoted by $h(\alpha)$, is defined as

$$h(\alpha) := \sum_{i\in I} z(\mu(A_i)). \tag{7}$$

The properties of the entropy of partitions that will be needed in this paper are summarized in the following theorem. The proof can be found in [3, Th 2.12.5] or [19, Proposition 5.2.5]. In these references a more general setting of conditional entropy is considered.

Theorem 2.5: Let α and β be two arbitrary partitions.

1) $h(\alpha) \geq 0$ with equality iff $\alpha = \{M\}$, where $\{M\}$, denotes the trivial partition consisting only of the set M.
2) $\alpha \leq \beta \rightarrow h(\alpha) \leq h(\beta)$, namely the entropy is a nondecreasing function of its argument.
3) $h(\alpha \vee \beta) \leq h(\alpha) + h(\beta)$, namely entropy is subadditive with respect to the operation of product.
4) Given an automorphism Φ defined on (M, μ), then $h(\Phi\alpha) = h(\alpha)$.

Definition 2.6 (Entropy of a Partition with Respect to an Automorphism): Given a partition α and an automorphism Φ, the entropy of α with respect to Φ, $h(\alpha, \Phi)$, is given by

$$h(\alpha, \Phi) = \lim_{k\rightarrow+\infty} \frac{1}{k} h(\alpha \vee \Phi^{-1}\alpha \vee \Phi^{-2}\alpha \vee \cdots \vee \Phi^{-k+1}\alpha). \tag{8}$$

It is shown in [3, Th. 2.12.21] and [19, Proposition 5.2.10] that this limit exists and therefore this definition is well posed.

Definition 2.7 (Entropy of an Automorphism): The entropy of an automorphism Φ, $h(\Phi)$, is defined as

$$h(\Phi) := \sup_{\alpha} h(\alpha, \Phi). \tag{9}$$

III. Entropy of Sequences

We will be concerned with dynamical systems consisting of triples $(M, \mu, \{\Phi_t\})$, where $\{\Phi_t\}$ is a periodic sequence of automorphisms, with $t = 1, 2, \cdots$. Considering sequences of automorphisms is natural in the context of mixing fluid flows since the case of a single transformation, corresponding to a steady flow, cannot be mixing. This is due to the fact that in incompressible steady flows, the particles never cross the streamlines [17]. In this paper, we consider the problem of finding the sequence that mixes best, out of a set of available sequences. Since entropy is a measure of the disorder created by a transformation, we pose the problem in terms of finding the sequence which maximizes entropy out of a certain set of sequences. Of course, we need to generalize the Definitions 2.6 and 2.7 to sequences. We have the following.

Definition 3.1 (Entropy of a Sequence of Automorphisms with Respect to a Partition): Given a partition α and a sequence of automorphisms $\{\Phi_t\}$, the entropy of $\{\Phi_t\}$ with respect to α, $h(\alpha, \{\Phi_t\})$ is given by the following limit (if it exists):

$$h(\alpha, \{\Phi_t\}) := \lim_{k\to+\infty} \frac{1}{k} h(\alpha \vee (\Phi_1)^{-1}\alpha \vee (\Phi_2 \circ \Phi_1)^{-1}\alpha \vee \cdots \vee (\Phi_{k-1} \circ \cdots \circ \Phi_1)^{-1}\alpha). \tag{10}$$

The following definition is a direct generalization of Definition 2.7.

Definition 3.2 (Entropy of a Sequence of Automorphisms): If the limit in (10) exists for every partition α for a sequence $\{\Phi_t\}$, then the entropy of the sequence $\{\Phi_t\}$, $h(\{\Phi_t\})$, is given by

$$h(\{\Phi_t\}) := \sup_{\alpha} h(\alpha, \{\Phi_t\}). \tag{11}$$

The following proposition states that the limit (10) exists for periodic sequences as considered in this paper. The proof generalizes the standard proof for maps and it is given in the Appendix.

Proposition 3.3: Assume the sequence $\{\Phi_t\}$ is periodic of period n. Then, the limit in (10) exists.

There is a great deal of literature concerning computations of entropy of automorphisms (see, e.g., [18], [19], and the references therein). The following result relates the entropy of periodic sequences to the entropy of an automorphism as in Definitions 2.6. and 2.7. It expresses formally the intuitive idea that entropy is a "per step" quantity. The result is standard when $\Phi_1 = \Phi_2 = \cdots = \Phi_n$ (see, e.g., [19, p. 243]). We relegate the proof to the Appendix.

Lemma 3.4: Let $\{\Phi_t\}$ be a periodic sequence of measure-preserving automorphisms of period n and consider $\Phi := \Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_1$ the composite automorphism. Then we have

$$h(\Phi) = nh(\{\Phi_t\}). \tag{12}$$

IV. Statement of the Problem

Let us write the Poincaré maps for the dynamical system described in (1)–(3). Integration of (1) with (3) gives

$$\begin{aligned} x_{t+1} &= x_t + ay_t \pmod 1 \\ y_{t+1} &= y_t. \end{aligned} \tag{13}$$

Integration of (2) with (3) gives

$$\begin{aligned} x_{t+1} &= x_t \\ y_{t+1} &= y_t + ax_t \pmod 1. \end{aligned} \tag{14}$$

Both of these maps are defined on the unit square $U = \{(x, y) \mid 0 \le x < 1, 0 \le y < 1\}$, and any time a point exits outside U it is projected back to it by the $\mathrm{mod}\ 1$ operation.

Consider now the system $(M, \mu, \{\Phi_t\})$ on the two-dimensional torus M endowed with the Lebesgue measure μ where $\{\Phi_t\}$ is a periodic sequence composed of (13) and (14). By Lemma 3.4, the entropy of $(M, \mu, \{\Phi_t\})$ is related to the entropy of the automorphism Φ on the two-dimensional torus M, where Φ is a finite composition of two measure-preserving automorphisms. These automorphisms can be represented by the action of matrices

$$H := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \qquad V := \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \tag{15}$$

(where a is an element in R), on the unit square U. Whenever the action of either H or V takes a point outside of U, it is projected back to U by subtracting an integer number either from its x coordinate or y coordinate in the plane (we cannot use the usual lift of the map on the torus because a is not necessarily an integer). Notice that both the maps $H \pmod 1$ and $V \pmod 1$ are one-to-one and onto and so is any map which is composed of them. However, they are discontinuous if a is not an integer and the singularity set is given by $\{(x, y) \in M : 0 \le x < 1, y = 0\}$ for $H \pmod 1$ and by $\{(x, y) \in M : 0 \le y < 1, x = 0\}$ for $V \pmod 1$. For a composite map, the singularity set is the finite union of segments in the unit square. This family of maps belongs to the one studied by Chernov [5]. It can be shown to be mixing in the ergodic theoretic sense, unless it is the trivial sequence $(H \pmod 1)^n$ (or $(V \pmod 1)^n$), using the technique developed in [21] for the sawtooth map. More generally, the maps described above belong to the class of maps with singularities considered in [11].

Putting (13) and (14) together, we can write

$$\bar{x}(t+1) = F(t)\bar{x}(t) \pmod 1 \tag{16}$$

where $\bar{x} = [x, y]^T$, and F has the form

$$F(t) = \begin{pmatrix} 1 & u_1(t) \\ u_2(t) & 1 \end{pmatrix} \tag{17}$$

with $u_1, u_2 \in [0, a]$, $u_1 \cdot u_2 = 0$, and $u_1 + u_2 = a$. Notice that the above feedback $F(t)$ is not linear because of the projection. The problem that will be treated here will be of finding a periodic feedback F such that the dynamical system $(M, \mu, \{\Phi_t\})$ has maximum entropy. In other words, the control problem consists of choosing the sequence, among the periodic ones composed by H and V, which has maximum entropy.

We shall denote the set of periodic sequences composed by H and V by $\mathcal{P}$ and the set of measure-preserving automorphisms obtained by compositions of the maps H and V of length n by $\mathcal{P}_S(n)$. With a minor abuse of notation, we shall sometimes denote with the same symbol a measure-preserving automorphism and the matrix that represents it as in (15). The result of Lemma 3.4 will allow us to compare entropy of periodic sequences by comparing the entropy of the corresponding composite automorphisms. Gathering the above definitions and properties, we can summarize the problem to be solved in the next sections as finding

$$\max_{\{\Phi_t\}\in\mathcal{P}} h(\{\Phi_t\}) \tag{18}$$

or equivalently as finding

$$\max_n \max_{\Phi\in\mathcal{P}_S(n)} \frac{h(\Phi)}{n}. \tag{19}$$

We now state a result on the computation of entropy of maps in $\mathcal{P}_S(n)$. This result is a straightforward consequence of Pesin's formula for computation of entropy [18], as generalized in [11].

Lemma 4.1: Let $\Phi_A \in \mathcal{P}_S(n)$ and A its constant differential be defined almost everywhere and given by a finite product of matrices H and V defined in (15). The entropy of Φ_A is given by

$$h(\Phi_A) = \log|\lambda_{\max}(A)| \tag{20}$$

where $\lambda_{\max}(A)$ is the eigenvalue of A with maximum absolute value.

Proof: It was shown in [11] that the entropy of Φ_A is given by

$$h(\Phi) = \int_M \chi(x,y)d\mu \tag{21}$$

where $\chi(x,y)$ is the positive Lyapunov exponent. This Lyapunov exponent is given by

$$\chi(x,y) = \lim_{n\to+\infty}\frac{1}{n}\log\left\|D_{(x,y)}\Phi_A^n v\right\| \tag{22}$$

for almost every $v\in\mathrm{R}^2$. In our case $D_{(x,y)}\Phi_A^n = A^n$. Note that $\det A = 1$. Assume that A has two eigenvalues with absolute value equal to one (this happens only when $A = H^n$ or $A = V^n$). In that case, the limit in (22), for every v, is zero, and so is the value of the entropy in (21). If A has two different eigenvalues $\lambda_{\min}$ and $\lambda_{\max}$ with $|\lambda_{\min}| = \frac{1}{|\lambda_{\max}|} < 1$, we compute the Lyapunov exponent in (22) using a decomposition of v as $v = \alpha_1 v_{\min} + \alpha_2 v_{\max}$ where $v_{\min}$ and $v_{\max}$ are the eigenvectors corresponding to $\lambda_{\min}$ and $\lambda_{\max}$, respectively. In particular we have

$$\begin{aligned}&\lim_{n\to+\infty}\frac{1}{n}\log\left\|D_{(x,y)}\Phi_A^n v\right\| \\ &\quad= \lim_{n\to+\infty}\frac{1}{n}\log||A^n v|| \\ &\quad= \lim_{n\to+\infty}\frac{1}{n}(\log(|\lambda_{\max}|^n) + \log(||\alpha_2 v_{\max}||)) \\ &\quad= \log|\lambda_{\max}|.\end{aligned}$$

Plugging this into (21), we obtain (20). □

In the following we shall denote by $\mathcal{S}(n)$ the set of matrices obtained from n-products of the matrices H and/or V. There is an obvious one-to-one correspondence between the elements in $\mathcal{S}(n)$ and the ones in $\mathcal{P}_S(n)$. In view of Lemma 4.1, for any fixed n in (19), we look for

$$\max_{S\in\mathcal{S}(n)} |\lambda_{\max}(S)|. \tag{23}$$

Notice that the eigenvalues of matrices with determinant equal to one, such as the ones in $\mathcal{S}(n)$, can be computed by

$$\lambda_{1,2} = \frac{\mathrm{Tr}(S)\pm\sqrt{(\mathrm{Tr}(S))^2-4}}{2}. \tag{24}$$

Notice also that it can be easily shown (see formula (30) below) that for any element S in $\mathcal{S}(n)$, we have $\mathrm{Tr}(S)\geq 2$. Therefore, we have, from formula (24) that the maximum eigenvalue is an increasing real invertible function of the trace. In view of this observation, we can restate problem (23) as

$$\max_{S\in\mathcal{S}(n)} \mathrm{Tr}(S). \tag{25}$$

Once (25) is solved, it will be immediate how to solve (19) and therefore (18).

V. Auxiliary Results

We develop here a few algebraic relations for the n-products of the matrices H and V. We notice that

$$H = V^T \tag{26}$$

and, also

$$H = PVP \tag{27}$$

where P is the permutation matrix $P := \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$. Notice that, using (27) and the fact that the trace does not change with similarity transformations, we have the following property, where S_i, $i = 1,\cdots,n$ are matrices H or V, and $\bar{S}_i$, $i = 1,\cdots,n$ is H if S_i is V and *vice versa*

$$\begin{aligned}\mathrm{Tr}(S_1S_2\cdots S_n) &= \mathrm{Tr}(PS_1S_2\cdots S_nP)\\ &= \mathrm{Tr}(PS_1PPS_2PP\cdots PPS_nP)\\ &= \mathrm{Tr}(\bar{S}_1\bar{S}_2\cdots\bar{S}_n).\end{aligned} \tag{28}$$

In view of the above relation, in maximizing the trace in (25), we can restrict our attention to the elements of $\mathcal{S}(n)$ such that the first matrix from the left is H. Also, from the property of the trace

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \tag{29}$$

when we consider elements of $\mathcal{S}(n)$ other than H^n, we can always restrict our attention to matrix products such that V is the last matrix on the right. Therefore the product which maximizes the trace, for any n, is in the subset of $\mathcal{S}(n)$ given by H^n together with the products of the form $H^{n_1}V^{n_2}\cdots V^{n_s}$, for a certain $s\geq 2$, with $n_1+n_2+\cdots+n_s = n$. The formula for the trace of this matrix can be computed explicitly. We have the following.

Lemma 5.1:

$$\mathrm{Tr}(H^{n_1}V^{n_2}\cdots V^{n_s}) = 2+\left(\sum_{\substack{i_1<i_2\\ i_1\not\sim i_2}} n_{i_1}n_{i_2}\right)a^2+\left(\sum_{\substack{i_1<i_2<i_3<i_4\\ i_1\not\sim i_2\\ i_2\not\sim i_3\\ i_3\not\sim i_4}} n_{i_1}n_{i_2}n_{i_3}n_{i_4}\right)a^4 + \cdots + n_1n_2\cdots n_s a^s \tag{30}$$

where $\not\sim$ denotes the relation of two integers to be neither both odd nor both even.

Proof: First, we notice that

$$H^k = \begin{pmatrix} 1 & ka \\ 0 & 1 \end{pmatrix} \tag{31}$$

and analogously

$$V^k = \begin{pmatrix} 1 & 0 \\ ka & 1 \end{pmatrix} \tag{32}$$

for each integer k. Therefore, we can write

$$\mathrm{Tr}(H^{n_1}V^{n_2}\cdots V^{n_s}) = \mathrm{Tr}((I+N_1)(I+N_2)\cdots(I+N_s)) \tag{33}$$

where the matrix N_i, $i=1,\cdots,s$ is given by

$$N_i = \begin{pmatrix} 0 & n_i a \\ 0 & 0 \end{pmatrix} \tag{34}$$

if i is odd, and by

$$N_i = \begin{pmatrix} 0 & 0 \\ n_i a & 0 \end{pmatrix} \tag{35}$$

if i is even.

Formula (33) is easily seen (by induction on s) to give

$$\mathrm{Tr}(H^{n_1}V^{n_2}\cdots V^{n_s}) = \mathrm{Tr}\left(I + \sum_{k=1}^{s}\bar{\sum} N_{i_1}\cdots N_{i_k}\right) \tag{36}$$

where the sum $\bar{\sum}$ is taken over all the $\binom{s}{k}$ combinations of the indexes $i_1,\cdots,i_k$ in $1,\cdots,s$ with $i_1<\cdots<i_s$. Now notice that from the definitions (34) and (35), $N_iN_j=0$ if i and j are both odd or both even. Therefore, we can consider in the sum $\bar{\sum}$ in (36) only products where neighboring matrices correspond to indexes that are not both even and not both odd. If k is odd in the term $N_{i_1}\cdots N_{i_k}$, because of the previous observation, either i_1 and i_k are both odd or they are both even. In one case the product matrix takes the form $\left(\begin{smallmatrix} 0 & \star \\ 0 & 0 \end{smallmatrix}\right)$, in the other, it takes the form $\left(\begin{smallmatrix} 0 & 0 \\ \star & 0 \end{smallmatrix}\right)$. In both cases the trace of the resulting matrix is zero. Finally, for k even, it is easily seen that the trace of $N_{i_1}\cdots N_{i_k}$ is given by $n_{i_1}\cdots n_{i_k}a^k$. These observations, along with the linearity of the trace, show that we can write (33) as (30). □

A consequence of (30) is that the only elements of $\mathcal{P}$ that have zero entropy are $HHH\cdots H\cdots$ and $VVV\cdots V\cdots$. Also

$$\frac{\partial\, \mathrm{Tr}\, A}{\partial a} > 0, \qquad \text{for } a>0$$

implies that the entropy of a particular periodic sequence is increasing with the modulus of a. We will not perform a direct maximization of the coefficients of a in (30), to solve our optimization problem (25), but we shall proceed along a different route.

Given a matrix A, A^σ denotes the antitranspose of the matrix A, namely the matrix obtained by reflecting the elements of A along the secondary diagonal. It is easily seen that $(A^n)^\sigma = (A^\sigma)^n$. The following two matrices will be important in the sequel:

$$HV = \begin{pmatrix} (1+a^2) & a \\ a & 1 \end{pmatrix} \tag{37}$$

$$VH = \begin{pmatrix} 1 & a \\ a & (1+a^2) \end{pmatrix}. \tag{38}$$

It is clear by inspection that

$$HV = (VH)^\sigma \tag{39}$$

and we shall denote, for a given integer $k\ge 0$

$$(HV)^k = ((VH)^k)^\sigma = \begin{pmatrix} u & v \\ v & z \end{pmatrix}. \tag{40}$$

The following result will be very useful.

Lemma 5.2: Consider $(HV)^k$ with $k\ge 0$, as in (40). Then we have

$$u = z + va. \tag{41}$$

Proof: The proof is by induction on k. Equation (41) is true when $k=0$. Assume that for $k-1$ we have

$$(HV)^{k-1} = \begin{pmatrix} u' & v' \\ v' & z' \end{pmatrix} \tag{42}$$

with

$$u' = z' + av'. \tag{43}$$

Using (37), we obtain

$$\begin{aligned}(HV)^k &= (HV)^{k-1}(HV) \\ &= \begin{pmatrix} ((1+a^2)u' + v'a) & (u'a+v') \\ (v' + z'a + v'a^2) & (z' + av') \end{pmatrix}\end{aligned} \tag{44}$$

and using (43), we have

$$(HV)^k := \begin{pmatrix} u & v \\ v & z \end{pmatrix} = \begin{pmatrix} ((1+a^2)u' + v'a) & (u'a+v') \\ (u'a+v') & u' \end{pmatrix} \tag{45}$$

from which it is immediate to verify that (41) holds. □

VI. Sequence with Maximum Entropy

In this section, we will solve the problem stated in Section IV. In particular, we will show that the sequence which maximizes entropy, in the set of periodic sequences constructed with H and V in (15), is the alternate sequence $H\circ V\circ H\circ V\cdots$. Moreover, we will see that this is actually an absolute maximum in the set of all the periodic sequences constructed with H and V in (15). We will first work with $n = n_1+n_2+\cdots+n_s$ even, as the case of n odd easily follows from the solution of the problem with n even.

A. Case $|a| \geq 1$

Lemma 6.1: Assume in (15) $|a| \geq 1$. Consider a matrix

$$S := H^{n_1} V^{n_2} \cdots V^{n_s} \tag{46}$$

in $\mathcal{S}(n)$, and assume that there are some exponents n_i, $i \in \{1, \cdots, s\}$ such that $n_i \geq 3$. Then there exists a matrix

$$\bar{S} := H^{\bar{n}_1} V^{\bar{n}_2} \cdots \bar{V}^{\bar{n}_{\bar{s}}} \tag{47}$$

with $\bar{n}_i \leq 2$, for all $i = 1, \cdots, \bar{s}$, in $\mathcal{S}(n)$ such that

$$\operatorname{Tr}(S) \leq \operatorname{Tr}(\bar{S}). \tag{48}$$

Proof: It follows, from formula (30), that the trace of every element in $\mathcal{S}(n)$ does not change if we replace the parameter a with $-a$. Therefore we assume without loss of generality $a \geq 1$.

Assume that, for a certain j, $n_j \geq 3$ in (46). Assume also, without loss of generality, that n_j is an exponent of H in S; otherwise, we can use the property in (28) and (29) to generate another matrix (with H and V as first and last element, respectively), with the same trace, such that this holds, and the following computations will go through in the same way. We have

$$\begin{aligned}\operatorname{Tr}(S) &= \operatorname{Tr}(H^{n_1} V^{n_2} \cdots V^{n_{j-1}} H^{n_j} V^{n_{j+1}} \cdots V^{n_s}) \\ &= \operatorname{Tr}(H^{n_j} V^{n_{j+1}} \cdots V^{n_s} H^{n_1} \cdots V^{n_{j-1}})\end{aligned} \tag{49}$$

where we have used the property of the trace in (29). Now define

$$L := \begin{pmatrix} \lambda & \mu \\ \nu & \gamma \end{pmatrix} = (H^{n_j - 3} V^{n_{j+1}} \cdots V^{n_s} H^{n_1} \cdots V^{n_{j-1}}) \tag{50}$$

and notice that, since $a \geq 1$, L is composed of nonnegative elements. We write

$$\operatorname{Tr}(S) = \operatorname{Tr}(H^3 L) = \operatorname{Tr}(HLH^2) \tag{51}$$

and notice that the product in the matrix L ends with a matrix V. Now we show by direct computation that $\operatorname{Tr}(HLH^2) \leq \operatorname{Tr}(HLHV)$. We use the fact that L is a matrix of nonnegative elements and the relations in (31) and (37). We have

$$\begin{aligned}\operatorname{Tr}(S) &= \operatorname{Tr}(HLH^2) \\ &= \operatorname{Tr}\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & \mu \\ \nu & \gamma \end{pmatrix} \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}\right) \\ &= \lambda + 3a\nu + \gamma\end{aligned} \tag{52}$$

$$\begin{aligned}\operatorname{Tr}(HLHV) &= \operatorname{Tr}\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & \mu \\ \nu & \gamma \end{pmatrix} \begin{pmatrix} (1+a^2) & a \\ a & 1 \end{pmatrix}\right) \\ &= (1+a^2)\lambda + (1+a^2)\gamma + (2a+a^3)\nu + a\mu.\end{aligned} \tag{53}$$

Comparing (53) and (52), and recalling that $a \geq 1$, we have that $\operatorname{Tr}(HLHV) \geq \operatorname{Tr}(HLH^2)$. Therefore, we have constructed a matrix in $\mathcal{S}(n)$, which has trace at least as large as the original one and such that the exponent $n_j \geq 3$ has been changed to $n_j - 2$. Proceeding iteratively in this way we end up with a matrix $\bar{S}$ as in (47), whose exponents $\bar{n}_i$, $i = 1, \cdots, \bar{s}$ are all one or two, which has trace greater or equal than the trace of the original matrix. □

Lemma 6.2: Assume $|a| \geq 1$. For every matrix $\bar{S}$ with exponents ≤ 2 as in (47), the following holds:

$$\operatorname{Tr}(\bar{S}) \leq \operatorname{Tr}\left((HV)^{\frac{n}{2}}\right) \tag{54}$$

with equality if and only if $\bar{S} = (HV)^{\frac{n}{2}}$.

Proof: Assume from (30), without loss of generality, $a \geq 1$ as in Lemma 6.1. The result is easily verified by comparing the traces of (31), (32), and (37), if $n = 2$. Therefore, we have to prove the result only for $n \geq 4$. Assume by (29) and (28) that H^2 is at the first position from the left. Since n is even this is not the only square term and there exists at least another square factor in the product. There are two cases: 1) the closest square element to H^2 on the right is V^2 and 2) the closest square element H^2 on the right is H^2. Let us consider these two cases separately.

Case 1): The matrix $\bar{S}$ has the form

$$\bar{S} = H^2 (VH)^k V^2 L \tag{55}$$

for a suitable $k \geq 0$. L is a general matrix with nonnegative elements obtained by some multiplications of H and V, for which we will use the same notation with Greek letters as in (50). L starts with H and ends with V. We claim that

$$\operatorname{Tr}(\bar{S}) = \operatorname{Tr}(H^2 (VH)^k V^2 L) < \operatorname{Tr}(L(HV)^{k+2}). \tag{56}$$

Recalling the definitions of $(HV)^k$ of (40) and the fact that $(VH)^k$ is the antitranspose of $(HV)^k$ and (31) and (32), we have

$$\begin{aligned}&\operatorname{Tr}(H^2 (VH)^k V^2 L) \\ &\quad = \operatorname{Tr}\left(\begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & v \\ v & u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2a & 1 \end{pmatrix} \begin{pmatrix} \lambda & \mu \\ \nu & \gamma \end{pmatrix}\right) \\ &\quad = (z + 4av + 4a^2 u)\lambda + (v + 2au)\nu + (v + 2au)\mu + u\gamma\end{aligned} \tag{57}$$

and

$$\begin{aligned}\operatorname{Tr}(L(HV)^{k+2}) &= \operatorname{Tr}\left(\begin{pmatrix} \lambda & \mu \\ \nu & \gamma \end{pmatrix} \begin{pmatrix} u & v \\ v & z \end{pmatrix} \begin{pmatrix} (1+a^2) & a \\ a & 1 \end{pmatrix}^2\right) \\ &= ((1+3a^2+a^4)u + (2a+a^3)v)\lambda \\ &\quad + ((1+3a^2+a^4)v + (2a+a^3)z)\mu \\ &\quad + ((2a+a^3)u + (1+a^2)v)\nu \\ &\quad + ((2a+a^3)v + (1+a^2)z)\gamma.\end{aligned} \tag{58}$$

Using the expression of z obtained from (41) in (57) and (58), we obtain

$$\begin{aligned}\operatorname{Tr}(H^2 (VH)^k V^2 L) &= ((1+4a^2)u + 3av)\lambda + (v+2au)\nu \\ &\quad + (v+2au)\mu + u\gamma\end{aligned} \tag{59}$$

and

$$\begin{aligned}\operatorname{Tr}(L(HV)^{k+2}) &= ((1+3a^2+a^4)u + (2a+a^3)v)\lambda \\ &\quad + ((1+a^2)v + (2a+a^3)u)\mu \\ &\quad + ((2a+a^3)u + (1+a^2)v)\nu \\ &\quad + (av + (1+a^2)u)\gamma.\end{aligned} \tag{60}$$

Comparing the coefficients of λ, μ, ν, and γ in (59) and (60), recalling that $a \geq 1$, it is easily seen that every coefficient in

(60) is greater than or equal to the corresponding coefficient in (59). Moreover, notice that the element γ in L and the element u in HV are always greater than or equal to zero. They are actually greater than or equal to one since they come from products of matrices H and V and $a \geq 1$. Moreover, v is also greater than or equal to zero. With these facts in mind, it is easy to verify that the element containing γ in (60) is *strictly* greater than the element containing γ in (59), which proves the claim in (56).

Case 2): The matrix $\bar{S}$ has the form

$$\bar{S} = H^2(VH)^k VH^2 L \tag{61}$$

for a suitable $k \geq 0$. Here also L is a general matrix with nonnegative elements, obtained by some multiplications of H and V. Notice also that in this case, L starts with V and ends with V. We claim that

$$\mathrm{Tr}(\bar{S}) = \mathrm{Tr}(H^2(VH)^k VH^2 L) < \mathrm{Tr}(HL(HV)^{k+2}). \tag{62}$$

As in Case 1), we compute explicitly the traces of these two matrices and compare the coefficients of λ, μ, ν, and γ, with the help of (41). We obtain

$$\begin{aligned}\mathrm{Tr}&(H^2(VH)^k VH^2 L)\\ &= (z + 3av + 2a^2u)\lambda\\ &\quad + (2az + (1+6a^2)v + 2a(1+2a^2)u)\nu\\ &\quad + (v + au)\mu + (2av + (1+2a^2)u)\gamma\end{aligned} \tag{63}$$

and

$$\begin{aligned}\mathrm{Tr}&(HL(HV)^{k+2})\\ &= ((1+3a^2+a^4)u + (2a+a^3)v)\lambda\\ &\quad + ((1+3a^2+a^4)v + (2a+a^3)z)\mu\\ &\quad + ((3a+4a^3+a^5)u + (1+3a^2+a^4)v)\nu\\ &\quad + ((3a+4a^3+a^5)v + (1+3a^2+a^4)z)\gamma.\end{aligned} \tag{64}$$

Replacing z as given in (41) in (63) and (64), we obtain

$$\begin{aligned}\mathrm{Tr}&(H^2(VH)^k VH^2 L)\\ &= ((1+2a^2)u + 2av)\lambda + (v+au)\mu + ((4a+4a^3)u\\ &\quad + (1+4a^2)v)\nu + (2av + (1+2a^2)u)\gamma\end{aligned} \tag{65}$$

and

$$\begin{aligned}\mathrm{Tr}&(HL(HV)^{k+2})\\ &= ((1+3a^2+a^4)u + (2a+a^3)v)\lambda\\ &\quad + ((1+a^2)v + (2a+a^3)u)\mu\\ &\quad + ((3a+4a^3+a^5)u + (1+3a^2+a^4)v)\nu\\ &\quad + ((2a+a^3)v + (1+3a^2+a^4)u)\gamma.\end{aligned} \tag{66}$$

Comparing the coefficients of (65) and (66), and recalling that $a \geq 1$ along with a closer look at the term containing γ as in Case 1), gives (62).

The result of the lemma follows by iterating the above procedure, in particular eliminating possible square powers in the matrix L in $L(HV)^{k+2}$ and $HL(HV)^{k+2}$.

B. Case $|a| < 1$

Lemmas 6.1 and 6.2 solve the subproblem (25) when $|a| \geq 1$ as in the case where is n odd can be solved from the analysis developed above: the sequence of period 3, $\cdots HVV \cdots$, for example, has the same entropy as the sequence of period 6, $\cdots HVVHVV \cdots$, and all the sequences of period 6 have entropy strictly less than $\cdots HVHVHV \cdots$. We summarize the analysis in the following theorem. In the proof, we only need to consider the case $|a| < 1$.

Theorem 6.3: Consider $a \neq 0$ in (15). The sequence of period 2, $H \circ V$ is the one which maximizes entropy, among the periodic sequences composed of H and V. Moreover, every other periodic sequence gives strictly less entropy.

Proof: We only have to prove that for $|a| < 1$ and given an even n, $\mathrm{Tr}(HV)^{\frac{n}{2}} > \mathrm{Tr}(S)$, where S is any matrix different from $(HV)^{\frac{n}{2}}$ of the form $S = H^{n_1}V^{n_2} \cdots V^{n_s}$, with $n_1 + n_2 + \cdots + n_s = n$, for a certain $s \leq n$. The traces of S and $(HV)^{\frac{n}{2}}$ can be expressed as polynomials in a by (30). From (30), we notice that $\mathrm{Tr}((HV)^{\frac{n}{2}})$ is a monic polynomial of degree n with integer coefficients. $\mathrm{Tr}(S)$ is a polynomial with integer coefficients and of degree s strictly less than n. The difference

$$\mathrm{Tr}\big((HV)^{\frac{n}{2}}\big) - \mathrm{Tr}(S) := p(a) \tag{67}$$

is a monic polynomial of degree n with integer coefficients. Let $\bar{p}(a)$ be defined by the factorization $p(a) := a^k \bar{p}(a)$, where $k \geq 0$ is the number of roots at zero if $p(a) - \bar{p}(a)$ is a monic polynomial.

From Lemma 6.1 and Lemma 6.2, $p(a) > 0$ if $|a| \geq 1$. For $|a| < 1$ we argue by contradiction. Assume that $p(a) \leq 0$ for some a in $(-1, 1)$. Then, by continuity, there must exist at least one root of $\bar{p}(a)$ different from zero which must necessarily be in $(-1, 1)$. Let us denote these roots as $a_1, \cdots, a_r$ with $r > 0$. We have, since $\bar{p}(a)$ is monic, $\prod_{i=1}^{r} |a_i| = |c|$, where c is the constant term in $\bar{p}(a)$. But this cannot be, since c is an integer and all of the a_i are in $(-1, 1)$. □

C. Case of Different Linear Shear Flows

In this subsection, based on the above we prove a result analogous to Theorem 6.3 for the compositions of shear flows with different shear strengths. We have

$$\begin{aligned} v_1(y) &= ay, \qquad a \neq 0, \qquad a \in \mathrm{R}\\ v_2(x) &= bx, \qquad b \neq 0, \qquad b \in \mathrm{R}\end{aligned} \tag{68}$$

and a and b have the same sign. In this case, the matrices H and V are defined as

$$H := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \qquad V := \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}. \tag{69}$$

The maps composed of H and V with a and b different, but of the same sign, are mixing, in the ergodic theoretic sense [21], [5]. We have

$$H = P^{-1}VP \tag{70}$$

where

$$P := \begin{pmatrix} 0 & 1 \\ b/a & 0 \end{pmatrix}$$

and thus property (28) is preserved. This, together with (29), allows us to consider again only products of the form

$H^{n_1}V^{n_2}\cdots V^{n_s}$. The formula for the trace of this matrix can again be computed explicitly. We have the following.

Lemma 6.4:

$$\mathrm{Tr}(H^{n_1}V^{n_2}\cdots V^{n_s}) = 2 + \left(\sum_{\substack{i_1<i_2 \\ i_1 \not\sim i_2}} n_{i_1}n_{i_2}\right)ab + \left(\sum_{\substack{i_1<i_2<i_3<i_4 \\ i_1\not\sim i_2 \\ i_2\not\sim i_3 \\ i_3\not\sim i_4}} n_{i_1}n_{i_2}n_{i_3}n_{i_4}\right)a^2b^2 + \cdots + n_1n_2\cdots n_s a^{s/2}b^{s/2}. \tag{71}$$

Since a and b are of the same sign, we are allowed to let $c^2 = ab$. For given a, b the sequence with maximal entropy is the one for which $n_1, \cdots, n_s$ maximize

$$2 + \left(\sum_{\substack{i_1<i_2 \\ i_1\not\sim i_2}} n_{i_1}n_{i_2}\right)c^2 + \left(\sum_{\substack{i_1<i_2<i_3<i_4 \\ i_1\not\sim i_2 \\ i_2\not\sim i_3 \\ i_3\not\sim i_4}} n_{i_1}n_{i_2}n_{i_3}n_{i_4}\right)c^4 + \cdots + n_1n_2\cdots n_s c^s. \tag{72}$$

We have already shown that $n_1 = 1$, $n_2 = 1$ maximizes the above expression for a fixed c. Thus, we have the following.

Theorem 6.4: The sequence of period 2, $H \circ V$ is the one which maximizes entropy, among the periodic ones composed of H and V given in (69), with a and b of equal sign. Moreover, every other periodic sequence gives strictly less entropy.

Clearly, Theorem 6.4 includes Theorem 6.3 as a special case, when $a = b$. The case when a and b are of different sign can lead to maps on the torus that mix poorly. In particular, consider $b = -a = -2$. In this case, the map HV is represented by

$$A := \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$$

so $\mathrm{Tr}(A) = -2$. There are two coinciding eigenvalues at one. Therefore, both of the Lyapunov exponents are zero.

Note that it makes a huge difference in mixing properties whether a shear flow with upwards or downwards orientation is chosen after the horizontal shear flow. In particular, the (a, a) case is mixing (from an ergodic theoretic point of view) while $(a, -a)$ may not be as in the above-mentioned case of $a = 2$, where the composite transformation is not even ergodic (see, e.g., [19, pp. 60–61]).

VII. Concluding Remarks

The problem considered in this paper has been motivated by recent results in the control of mixing [8], [12], [14]. We have shown that the maximum entropy achievable with a periodic sequence of maps $H \circ V$ in (69), is obtained with the alternate sequence $H \circ V \circ H \circ V \cdots$. Its expression can be given explicitly using Pesin's formula (20) and Lemma 3.4 and it is

$$h(\{H, V\}) = \frac{1}{2}\log\frac{(2+ab)+\sqrt{ab(4+ab)}}{2}. \tag{73}$$

In previous work, three approaches to the design of mixing protocols have been used.

1) Franjione and Ottino [8] have advocated the use of aperiodic mixing protocols that destroy the symmetries in the flow to achieve complete mixing over the phase space.
2) Ling and collaborators [12], [13] searched for the *mixing windows* of simple protocols, namely conditions for the nonexistence of low period elliptic fixed points of associated maps. These points are in general responsible for the formation of large islands and consequent poor mixing
3) Liu *et al.* [14] proposed the use of aperiodic sequences to achieve mixing throughout the phase space.

We have restricted our attention to periodic protocols and selected the ones that are mixing in the ergodic theoretic sense. Among them we chose the one which has maximum entropy. Figs. 1 and 2 show the mixing behavior of the protocols H^8V^2 and $(HV)^5$. Initial points are placed on a grid in the rectangle $0 \le x \le 1$, $0 \le y \le 0.5$. We used the values $a = b = 0.28$. Although both protocols mix well, both being mixing in the ergodic theoretic sense, the one with larger entropy creates disorder in a faster manner. In general, the speed of mixing improves dramatically by increasing the parameters a and b, but for different initial configurations the difference between the two protocols is not always evident. In general, the mixing behavior will be dependent on the initial configuration. From Pesin's formula (21) we know that entropy is equal to the stretching rate (given by the Lyapunov exponents) averaged over the whole phase space. In choosing the maximizing entropy we did not refer to any particular configuration of the fluids to be mixed. For some particular configurations, we would need more stretching to break bonds in particular directions or in particular points of the flow. In this sense, the protocol proposed here is the best if we do not consider particular initial configurations or if we would like to design a mixer that works well for various initial conditions.

Another way the initial positions of the fluids to be mixed can be taken into account is as follows: we can look at an initial configuration of fluids as a finite measurable partition α as introduced in Section II. The more the transformation Φ moves the sets of α in a disordered way around the phase space, the smaller the elements of the partition $\alpha^n := \vee_{k=0}^{n}\Phi^k\alpha$. One consequence of Shannon–McMillan–Breiman entropy equipartition theorem (see, e.g., [19, Corollary 6.2.5]) is that the sets in α^n decrease in size exponentially in n as a function of the entropy $h(\alpha, \Phi)$ (see Definition 2.6). Further discussion on the physical interpretation of entropy in the mixing of fluids as well as an application a different physical problem is presented in [6]. Whether or not the use of *aperiodic* protocols can improve mixing in the ergodic theoretic sense is an open problem. In this context, one needs

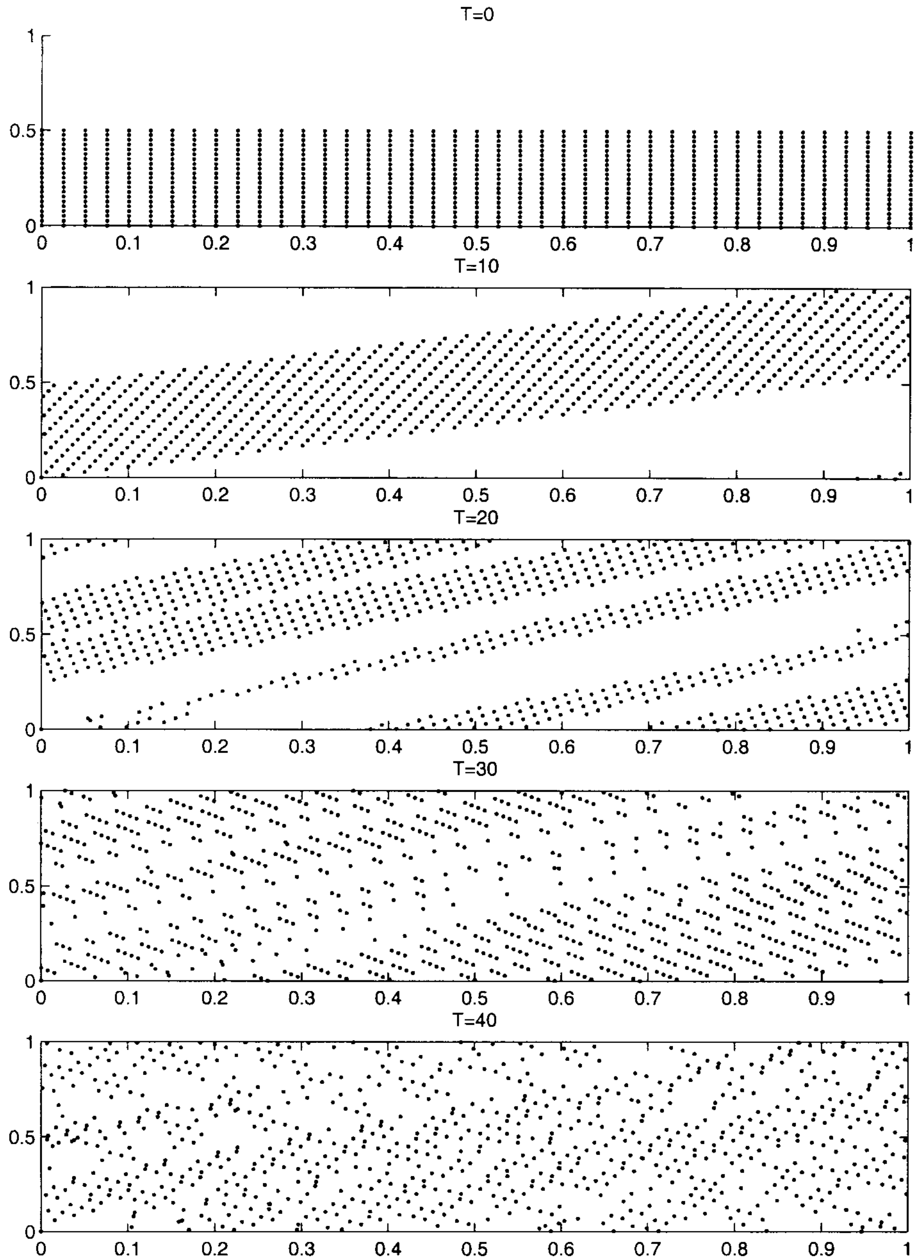

Fig. 1. Mixing behavior of the sequence of period 10 H^8V^2; $a = b = 0.28$.

to develop some ergodic theory for general sequences of transformations. Some work in this direction is done in [4] and [7].

APPENDIX

Here we give the proofs of Proposition 3.3 and Lemma 3.4.

Proposition 3.3: Assume the sequence $\{\Phi_t\}$ is periodic of period n. Then, the limit in (10) exists for every partition α.

Proof: Let α be a partition and consider the sequence of positive numbers

$$h_k = h\big(\alpha \vee \Phi_1^{-1}\alpha \vee \cdots \vee (\Phi_{k-1} \circ \cdots \circ \Phi_1)^{-1}\alpha\big). \quad (74)$$

We have to prove that the limit

$$\lim_{k\to\infty} \frac{h_k}{k} \quad (75)$$

exists. We notice that $h_1, \cdots, h_n$ are finite, since we are working with finite partitions. It is easily seen from properties 3) and 4) of Theorem 2.5, that for any $j \geq 0$, $0 \leq l \leq n-1$, we have

$$h_{jn+l} \leq h_{jn} + h_l \quad (76)$$

where we set $h_0 := 0$. Now consider the following subsequences of the sequence in (74):

$$h_{jn+l}, \qquad l = 0, 1, \cdots, n-1.$$

We prove that each sequence $\frac{h_{jn+l}}{jn+l}$ has a limit, when j goes to infinity, which is the same for each l. First, we notice that $\frac{h_{jn}}{jn}$ has a limit $\frac{L}{n}$. The proof relies on the fact that h_{jn} is a subadditive and nondecreasing sequence and it is exactly the same as the one given in [19, Proposition 5.2.10] for the standard case.

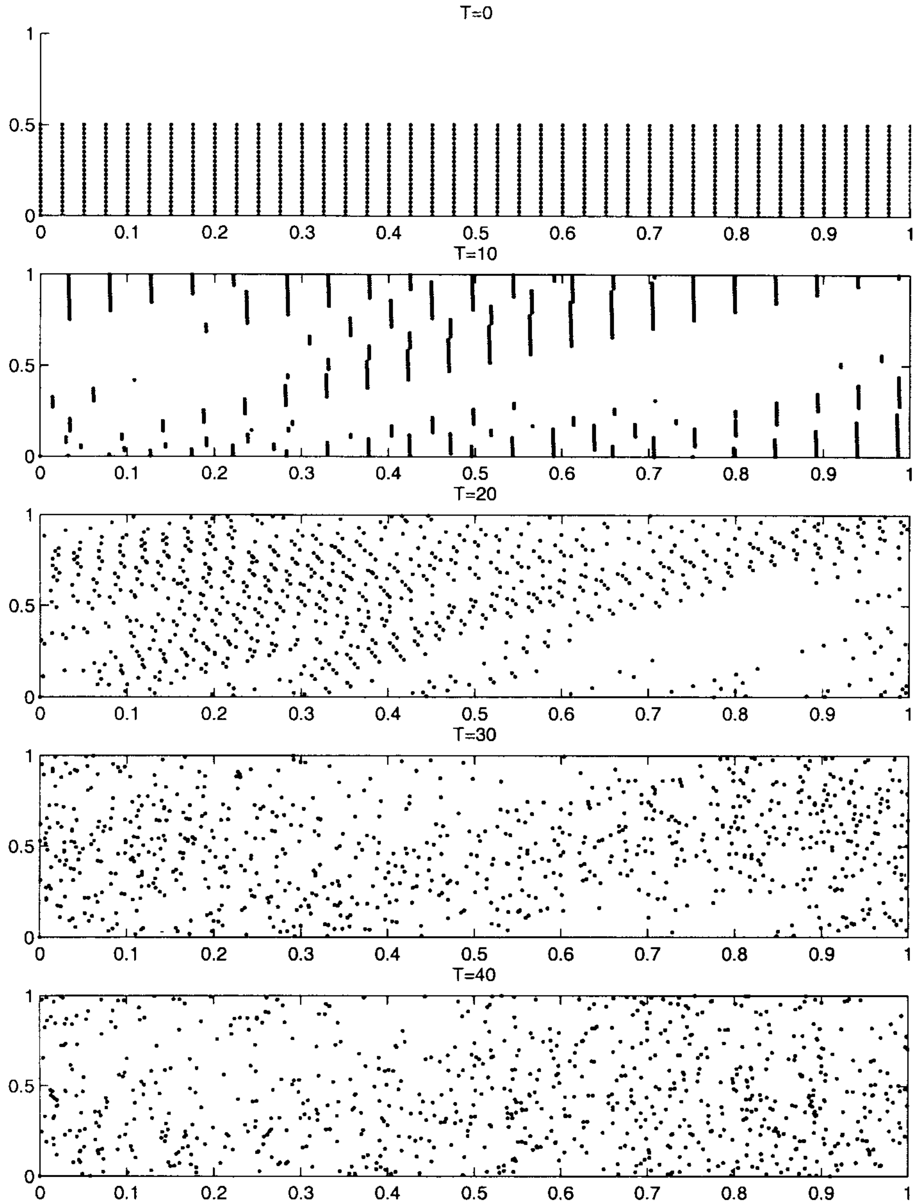

Fig. 2. Mixing behavior of the sequence of period 10 $(HV)^5$; $a = b = 0.28$.

To prove the convergence of $\frac{h_{jn+l}}{jn+l}$, for each $l = 1, \cdots, n-1$, notice that, from (76) and 2) and 3) of Theorem 2.5, we have

$$\frac{h_{jn}}{jn+l} \leq \frac{h_{jn+l}}{jn+l} \leq \frac{h_{jn}}{jn+l} + \frac{h_l}{jn+l} \tag{77}$$

and letting j go to infinity, we have that all subsequences $\frac{h_{jn+l}}{jn+l}$ tend to the same limit $\frac{L}{n}$. Therefore, the sequence $\frac{h_k}{k}$ also tends to the same limit $\frac{L}{n}$ when k goes to infinity. □

Lemma 3.4: Let $\{\Phi_t\}$ be a periodic sequence of measure-preserving automorphisms of period n, and consider $\Phi := \Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_1$ the composite automorphism. Then we have

$$h(\Phi) = nh(\{\Phi_t\}). \tag{78}$$

Proof: First we prove that

$$h(\Phi) \leq nh(\{\Phi_t\}). \tag{79}$$

Consider, for a given partition α, the sequence (80), as shown at the top of the next page, and notice that this is a subsequence of the sequence (81), also shown at the top of the next page, considered in (10). Therefore, the sequences (81) and (80) have the same limit, when k goes to infinity, which is by definition $h(\alpha, \{\Phi_t\})$. Moreover, it is

$$h(\alpha, \{\Phi_t\}) = \lim_{j \to +\infty} \frac{\bar{h}_j}{n(j-1)+1} = \lim_{j \to +\infty} \frac{\bar{h}_j}{nj}. \tag{82}$$

Consider the sequence (83), also shown at the top of the next page, which when j goes to infinity, tends to $h(\alpha, \Phi)$, by the definition of Φ. From 2) of Theorem 2.5 we get

$$\frac{h'_j(\alpha)}{j} \leq \frac{\bar{h}_j(\alpha)}{j} \tag{84}$$

$$\frac{\bar{h}_j(\alpha)}{n(j-1)+1} := \frac{h\big(\alpha \vee (\Phi_1)^{-1}\alpha \vee \big(\Phi_2 \circ \Phi_1\big)^{-1}\alpha \vee \cdots \vee (\Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_1)^{-j+1}\alpha\big)}{n(j-1)+1} \tag{80}$$

$$\frac{h_k(\alpha)}{k} := \frac{h\big(\alpha \vee \Phi_1^{-1}\alpha \vee (\Phi_2 \circ \Phi_1)^{-1}\alpha \vee \cdots \vee (\Phi_{k-1} \circ \cdots \circ \Phi_1)^{-1}\alpha\big)}{k} \tag{81}$$

$$\frac{h_j'(\alpha)}{j} := \frac{h(\alpha \vee (\Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_1)^{-1}\alpha \vee \cdots \vee (\Phi_n \circ \Phi_{n-1} \circ \cdots \circ \Phi_1)^{-j+1}\alpha)}{j} \tag{83}$$

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and using this and (82), we obtain

$$\frac{1}{n}h(\alpha, \Phi) = \frac{1}{n}\lim_{j\to+\infty}\frac{h_j'}{j} \le \lim_{j\to+\infty}\frac{\bar{h}_j}{nj} = h(\alpha, \{\Phi_t\}). \tag{85}$$

Taking the suprema over all the partitions α of the terms of (85), we obtain (79).

We now prove that

$$h(\Phi) \ge nh(\{\Phi_t\}). \tag{86}$$

Choose a partition α. We show that there exists a partition $\tilde{\alpha}$, such that

$$h(\tilde{\alpha}, \Phi) = nh(\alpha, \{\Phi_t\}). \tag{87}$$

Pick

$$\tilde{\alpha} := \alpha \vee (\Phi_1)^{-1}\alpha \vee \cdots \vee (\Phi_{n-1} \circ \cdots \circ \Phi_1)^{-1}\alpha. \tag{88}$$

We have

$$\begin{aligned} &h(\tilde{\alpha}, \Phi) \\ &\quad = n\lim_{k\to+\infty}\frac{1}{nk}h\big(\alpha \vee \Phi_1^{-1}\alpha \vee \cdots \vee (\Phi_{n-1} \circ \cdots \circ \Phi_1)^{-1}\alpha \\ &\qquad \vee (\Phi_n \circ \cdots \circ \Phi_1)^{-1}\alpha \vee (\Phi_n \circ \cdots \circ \Phi_1)^{-1} \circ \Phi_1^{-1}\alpha \\ &\qquad \vee \cdots \vee (\Phi_n \circ \cdots \circ \Phi_1)^{-k+1} \circ (\Phi_{n-1} \circ \cdots \circ \Phi_1)^{-1}\alpha\big). \end{aligned} \tag{89}$$

The limit on the right-hand side of (89) is $h(\alpha, \{\Phi_t\})$; therefore, we have

$$h(\tilde{\alpha}, \Phi) = nh(\alpha, \{\Phi_t\}). \tag{90}$$

Therefore, for each partition α, there exists a partition $\tilde{\alpha}$ such that (90) holds. Therefore, we have

$$h(\Phi) = \sup_{\alpha} h(\alpha, \Phi) \ge n\sup_{\alpha} h(\alpha, \{\Phi_t\}) = nh(\{\Phi_t\}) \tag{91}$$

which is the inequality in (86). $\square$

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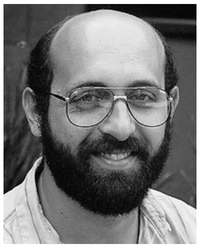

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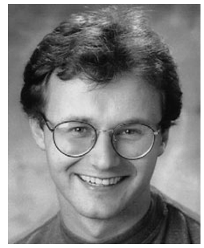

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