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ON THE APPROXIMATION OF COMPLICATED DYNAMICAL BEHAVIOR*

MICHAEL DELLNITZ† AND OLIVER JUNGE†

Abstract. We present efficient techniques for the numerical approximation of complicated dynamical behavior. In particular, we develop numerical methods which allow us to approximate Sinai–Ruelle–Bowen (SRB)-measures as well as (almost) cyclic behavior of a dynamical system. The methods are based on an appropriate discretization of the Frobenius–Perron operator, and two essentially different mathematical concepts are used: our idea is to combine classical convergence results for finite dimensional approximations of compact operators with results from ergodic theory concerning the approximation of SRB-measures by invariant measures of stochastically perturbed systems. The efficiency of the methods is illustrated by several numerical examples.

Key words. computation of invariant measures, approximation of the Frobenius–Perron operator, computation of SRB-measures, almost invariant set, cyclic behavior

AMS subject classifications. 58F11, 65L60, 58F12

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1. Introduction.

The approximation of the behavior of a dynamical system is typically done by direct simulation. This method is particularly useful in the situation where a specific trajectory has to be approximated for a finite period of time. However, if one is interested in the long term behavior and if the underlying system exhibits complicated dynamics, then the information derived from a single trajectory is not always satisfying. Rather in this case it seems more appropriate to determine a statistical description of the dynamical behavior, and this information is encoded in an underlying (natural) invariant measure.

In this paper we describe a numerical method for the approximation of such invariant measures based on a discretization of the *Frobenius–Perron operator*. Using the fact that invariant measures are fixed points of this operator, we first approximate it by a Galerkin projection and then compute eigenvectors of the discretized operator corresponding to the eigenvalue 1. This allows us to identify regions in state space where trajectories are likely to be observed or, on the other hand, hardly observed. In addition to this information we show how to use other parts of the spectrum of the Frobenius–Perron operator to determine the dynamical behavior of the system. First we describe how to decompose an invariant set into components which are *cyclically permuted* by the dynamics. Second we develop techniques for the approximation of *almost invariant sets*, that is, regions in state space which are visited for a "long" period of time before the dynamical process leads to different areas. More generally the same techniques allow us to detect *almost cyclic behavior*, that is, to identify components of invariant sets which are "frequently" cyclically permuted by the dynamical process. Moreover, we can quantify the probability by which the cycle occurs depending on the absolute value of a corresponding eigenvalue of the Frobenius–Perron operator. Roughly speaking, we construct an approximation of the *essential dynamical behavior*,

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†Mathematisches Institut, Universität Bayreuth, D-95440 Bayreuth (dellnitz@uni-bayreuth.de, http://www.uni-bayreuth.de/departments/math/~mdellnitz; junge@uni-bayreuth.de, http://www.uni-bayreuth.de/departments/math/~ojunge).

that is, the dynamics modulo complex (unpredictable) behavior which is due to the presence of chaos.

From a practical point of view the most important invariant measures are the so-called *SRB-measures*. The reason is that for these measures the spatial and temporal averages of observables are identical for a set of initial conditions which has positive Lebesgue measure. The introduction of the underlying concept goes back to Y. Sinai (see [27]), and the existence of these measures has been shown for Axiom A systems by D. Ruelle and R. Bowen [26, 3]. In this article we suggest a numerical method for the approximation of SRB-measures, and in this context their stochastic stability is particularly important: first we use this fact as an analytical tool in our main convergence result in section 4, and second it is of practical importance if we view the numerical approximation as a small random perturbation. Indeed, stochastic stability of SRB-measures is guaranteed for Axiom A systems [17, 18].

More precisely there are two essential mathematical ingredients which allow us to develop a numerical method for the approximation of SRB-measures. We use a result of Yu. Kifer on the convergence of invariant measures in stochastic perturbations of the underlying dynamical system to the SRB-measure (see [17]) and combine this with results on the convergence of eigenspaces of discretized compact operators (see, e.g., [25]). The same technique is used for the approximation of the subsets in state space which are (almost) cyclically permuted by the dynamical process. With respect to the approximation of SRB-measures a similar result has previously been obtained by F. Hunt (see [15]). However, our methods are quite different from the ones used in that work. In particular, the results stated here cover the important situation where the random perturbations have a probability distribution with local support. In fact, this is the relevant case having in mind that the round off error in the numerical approximation can be interpreted as such a local perturbation. Another approach for the computation of SRB-measures—avoiding the approximation of the Frobenius–Perron operator—has recently been suggested by Yu. Kifer [19].

As mentioned above, in addition to the approximation of SRB-measures the main development in this paper is a numerical method which allows us to identify (almost) cyclic behavior. To accomplish this we use a Galerkin method to discretize the Frobenius–Perron operator in such a way that the discretization has the same cyclic properties as the operator itself. More precisely, if the underlying dynamical system has a cycle of order r, then the rth roots of unity are eigenvalues of the Frobenius–Perron operator, and we will show that the corresponding eigenmeasures $\nu_0, \ldots, \nu_{r-1}$ yield the desired information on the cyclic components: these components can be identified as supports of probability measures obtained by specific linear combinations of $\nu_0, \ldots, \nu_{r-1}$. Our Galerkin approximation respects the cyclic behavior in the sense that the rth roots of unity are also eigenvalues of the discretized operator and that the corresponding eigenvectors converge to the eigenmeasures $\nu_0, \ldots, \nu_{r-1}$ with increasing dimension of the approximating space. We will illustrate how to use these results to determine the subsets in state space which are almost cyclically permuted.

Finally, let us remark that the results on the approximation of the essential dynamical behavior obtained in this article can also be used to compute other statistical quantities such as the *entropy*, *dimensions* (depending on the particular invariant measure), or *Lyapunov exponents*. In fact, the efficient numerical use of invariant measures for the computation of Lyapunov exponents is currently under investigation.

An outline of the paper is as follows. In section 2 we begin with a brief review of the results on Markov processes which will be needed later on. The Frobenius–Perron

operator is introduced in section 3. In that section we also describe the Galerkin projection that we use in the numerical approximation. In section 4 we use Kifer's result on small random perturbations of diffeomorphisms to prove convergence of the approximations to an SRB-measure in the hyperbolic case (Theorem 4.4). In the main section of this article, section 5, we show how to extract numerically the information on the (almost) cyclic components from the spectrum of the Frobenius–Perron operator. In particular, we present a method to identify regions in phase space where, on average, trajectories stay for a long period of time. Finally, in section 6 we illustrate by several examples the usefulness of our methods as tools in the numerical analysis of dynamical behavior.

2. Stochastic transition functions. For our main theoretical results we are using the concept of *small random perturbations* of dynamical systems. Since we assume that the typical reader is not familiar with this concept we begin by recalling some basic notions and results on Markov processes that will be needed later on. For a detailed introduction the reader is referred to [11].

Invariant measures. Our aim is to approximate the dynamical behavior of discrete dynamical systems of the form

$$x_{i+1} = f(x_i), \quad i = 0, 1, 2, \ldots,$$

where $f : X \to X$ is a diffeomorphism on a compact subset $X \subset \mathbb{R}^n$. We denote by $\mathcal{B}$ the Borel σ-algebra on X and by m the Lebesgue measure on $\mathcal{B}$. Moreover, let $\mathcal{M}$ be the space of probability measures on $\mathcal{B}$. Recall that a measure $\mu \in \mathcal{M}$ is *invariant* if

$$\mu(B) = \mu(f^{-1}(B)) \quad \text{for all } B \in \mathcal{B}.$$

On the other hand, a set $A \in \mathcal{B}$ is *invariant* if

$$f(A) \subset A.$$

We now turn our attention to the more general stochastic framework.

DEFINITION 2.1. *A function $p : X \times \mathcal{B} \to [0, 1]$ is a* stochastic transition function *if*

(i) *$p(x, \cdot)$ is a probability measure for every $x \in X$,*

(ii) *$p(\cdot, A)$ is Lebesgue-measurable for every $A \in \mathcal{B}$.*

Let δ_y denote the Dirac measure supported on the point $y \in X$. Then $p(x, A) = \delta_{h(x)}(A)$ is a stochastic transition function for every m-measurable function h. We will see below that the specific choice $h = f$ represents the deterministic situation in this more general setup.

We set $p^{(1)}(x, A) = p(x, A)$ and define recursively the *i-step stochastic transition function* $p^{(i)} : X \times \mathcal{B} \to \mathbb{R}$ by

$$p^{(i+1)}(x, A) = \int p^{(i)}(y, A)\, p(x, dy), \quad i = 1, 2, \ldots,$$

where $p(x, dy)$ indicates that the integration is done with respect to the measure $p(x, \cdot)$. It is easy to see that $p^{(i)}$ is indeed a stochastic transition function. In particular, for the case where $p(x, A) = \delta_{f(x)}(A)$ we obtain for $i \geq 1$

$$p^{(i)}(x, A) = \delta_{f^i(x)}(A).$$

We now define the notion of an invariant measure in the stochastic setting.

DEFINITION 2.2. *Let p be a stochastic transition function. If $\mu \in \mathcal{M}$ satisfies*

$$\mu(A) = \int p(x, A)\, d\mu(x)$$

for all $A \in \mathcal{B}$, then μ is an invariant measure of p.

Remark 2.3.

(a) In the literature on Markov processes (e.g., [11]) an invariant measure is typically referred to as a *stationary absolute probability measure.* However, having the situation in mind that we consider stochastically perturbed dynamical systems we prefer the notion of an invariant measure (see also [17, 24]).

(b) If μ is an invariant measure of p, then it follows that

$$\mu(A) = \int p^{(i)}(x, A)\, d\mu(x)$$

for all $i = 1, 2, \ldots$.

The following example illustrates the previous remark that we recover the deterministic situation in the case where $p(x, \cdot) = \delta_{f(x)}$.

EXAMPLE 2.4. *Suppose that $p(x, \cdot) = \delta_{f(x)}$ and let μ be an invariant measure of p. Then we compute for $A \in \mathcal{B}$*

$$\mu(A) = \int p(x, A)\, d\mu(x) = \int \delta_{f(x)}(A)\, d\mu(x) = \int \chi_A(f(x))\, d\mu(x) = \mu(f^{-1}(A)),$$

where we denote by χ_A the characteristic function of A. Hence μ is an invariant measure for the diffeomorphism f.

DEFINITION 2.5. *A set $A \in \mathcal{B}$ is called a* consequent set *of x if $p^{(i)}(x, A) = 1$ for all $i \geq 1$. The set A is* invariant *if it is the consequent set of all of its points. Furthermore if $C \in \mathcal{B}$ is a set for which*

$$\lim_{i \to \infty} p^{(i)}(x, C) = 0 \quad \textit{for all } x \in X,$$

then C is called a transient set.

Considering our guiding example let $p(x, \cdot) = \delta_{f(x)}$ and let A be an invariant set for p. Then we have for $y \in A$

$$1 = p(y, A) = \delta_{f(y)}(A).$$

Hence $f(A) \subset A$ and A is an invariant set for the diffeomorphism f.

Absolutely continuous stochastic transition functions. Now we assume that for every $x \in X$ the probability measure $p(x, \cdot)$ is absolutely continuous with respect to the Lebesgue measure m. Hence we may write $p(x, \cdot)$ as

$$p(x, A) = \int_A k(x, y)\, dm(y) \quad \text{for all } A \in \mathcal{B},$$

with an appropriate *transition density function* $k : X \times X \to \mathbb{R}$. Obviously,

$$k(x, \cdot) \in L^1(X, m) \quad \text{and} \quad k(x, y) \geq 0.$$

In this case we also call the stochastic transition function p *absolutely continuous.* Note that

$$\int k(x,y)\,dm(y) = p(x,X) = 1 \quad \text{for all } x \in X.$$

We let $k^{(1)}(x,y) = k(x,y)$ and define the *i-step transition density function* as

$$k^{(i+1)}(x,y) = \int k(x,\xi)k^{(i)}(\xi,y)\,dm(\xi), \quad i = 1,2,\ldots.$$

With this definition we obtain for $A \in \mathcal{B}$

$$p^{(i)}(x,A) = \int_A k^{(i)}(x,y)\,dm(y),$$

that is, the i-step transition density function $k^{(i)}$ is the stochastic transition density function for $p^{(i)}$.

The following theorem provides a characterization of all invariant measures of a certain class of stochastic transition functions.

THEOREM 2.6. *Let p be an absolutely continuous stochastic transition function with density function k. Suppose that $k(x,y) \le M$ for some positive constant $M > 0$ and all $x, y \in X$.*

Then X can be decomposed into finitely many disjoint invariant sets $E_1, E_2, \ldots, E_e$ and a transient set $F = X - \cup_{j=1}^{e} E_j$ such that for each E_j there is a unique probability measure $\pi_j \in \mathcal{M}$ with $\pi_j(E_j) = 1$ and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} p^{(i)}(x,A) = \pi_j(A) \quad \textit{for all } A \in \mathcal{B} \textit{ and all } x \in E_j. \tag{2.1}$$

Furthermore the left-hand side in (2.1) *exists uniformly in x and defines for every fixed $x \in X$ an invariant measure. Finally, every invariant measure of p is a convex combination of the π_j's.*

A proof of this theorem can be found in [11].

Remark 2.7.

(a) The E_j's are called the *ergodic sets* of p.

(b) One can show that the invariant measures π_j are absolutely continuous with density functions $\kappa_j \in L^1$, that is, we have

$$\pi_j(A) = \int_A \kappa_j(y)\,dm(y), \quad j = 1,\ldots,e.$$

It follows that for every fixed $x \in X$ the limit in (2.1) is also absolutely continuous with a density function $\ell(x,\cdot) \in L^1$.

3. Approximation of the Frobenius–Perron operator. The main purpose of this section is to describe an appropriate Galerkin method for the approximation of the Frobenius–Perron operator. But first we introduce this operator and derive certain spectral properties.

The Frobenius–Perron operator.

DEFINITION 3.1. *Let p be a stochastic transition function. Then the* Frobenius–Perron operator $P : \mathcal{M}_{\mathbb{C}} \to \mathcal{M}_{\mathbb{C}}$ *is defined by*

$$P\mu(A) = \int p(x,A)\,d\mu(x),$$

where $\mathcal{M}_{\mathbb{C}}$ is the space of bounded complex valued measures on $\mathcal{B}$. If p is absolutely continuous with density function k, then we may define the Frobenius–Perron operator P on L^1 by

$$Pg(y) = \int k(x,y)g(x)\,dm(x) \quad \textit{for all } g \in L^1.$$

Remark 3.2.

(a) By definition a measure $\mu \in \mathcal{M}$ is invariant if and only if it is a fixed point of P. In other words, invariant measures correspond to eigenmeasures of P for the eigenvalue 1.
Moreover, let $\lambda \in \mathbb{C}$ be an eigenvalue of P with corresponding eigenmeasure ν, that is, $P\nu = \lambda\nu$. Then in particular

$$\lambda\nu(X) = P\nu(X) = \int p(x,X)\,d\nu(x) = \nu(X)$$

since $p(x,X) = 1$ for all $x \in X$. It follows that $\nu(X) = 0$ if $\lambda \neq 1$.

(b) Observe that in the deterministic situation where $p(x,\cdot) = \delta_{f(x)}$ we obtain

$$P\mu(A) = \int p(x,A)\,d\mu(x) = \mu(f^{-1}(A))$$

(cf. Example 2.4). This is indeed the standard definition of the Frobenius–Perron operator in the deterministic setting (see, e.g., [21]).

(c) Note that in the case where p is absolutely continuous we have $P : L^1 \to L^1$ since for each $g \in L^1$

$$\begin{aligned}\int Pg(y)\,dm(y) &= \iint k(x,y)g(x)\,dm(x)\,dm(y)\\ &= \int g(x)\int k(x,y)\,dm(y)\,dm(x)\\ &= \int g(x)\,dm(x) < +\infty.\end{aligned}$$

Correspondingly, a nonnegative fixed point $g \in L^1$ of P with $\|g\|_1 = 1$ is the density of an invariant probability measure and, conversely, the density of every absolutely continuous invariant probability measure is a fixed point of P.

We are particularly interested in approximating cyclic dynamical behavior of the underlying dynamical system. In the stochastic setting this corresponds to the situation where there are disjoint compact subsets $X_j \subset X$, $j = 0,\dots,r-1$, such that

$$X = \bigcup_{j=0}^{r-1} X_j,$$

and for which the stochastic transition function p satisfies

$$p(x, X_{j+1 \bmod r}) = \begin{cases} 1 & \text{if } x \in X_j,\\ 0 & \text{otherwise.}\end{cases} \tag{3.1}$$

We now relate the cyclic dynamical behavior described by (3.1) to spectral properties of the corresponding Frobenius–Perron operator P.

PROPOSITION 3.3. *If the stochastic transition function p satisfies* (3.1), *then we have the following for the corresponding Frobenius–Perron operator P:*

(a) *The rth power P^r has an eigenvalue 1 of multiplicity at least r. Moreover, there are r corresponding invariant measures $\mu_k \in \mathcal{M}$, $k = 0, 1, \ldots, r-1$, with support on X_k, that is,* $\operatorname{supp}(\mu_k) \subset X_k$. *These measures can be chosen to satisfy*

$$\mu_k = P^k \mu_0, \quad k = 0, 1, \ldots, r-1.$$

(b) *The rth roots of unity ω_r^k, $k = 0, 1, \ldots, r-1$, where $\omega_r = e^{2\pi i/r}$, are eigenvalues of P.*

Proof.

(a) Observing that for each $\mu \in \mathcal{M}_{\mathbb{C}}$

$$P^j \mu(A) = \int p^{(j)}(x, A)\, d\mu(x),$$

where $p^{(j)}$ is the j-step stochastic transition function, the existence of the measures μ_k, $k = 0, 1, \ldots, r-1$, follows from standard results on Markov processes (see, e.g., [11, Chap. V]). Moreover, these measures can be chosen so that

$$\mu_{k+1 \bmod r} = P\mu_k, \quad k = 0, 1, \ldots, r-1.$$

Simply note that if μ_k is invariant for P^r, then

$$P^r(P\mu_k) = P(P^r \mu_k) = P\mu_k$$

and hence $P\mu_k$ is an invariant measure with support on X_{k+1}.

(b) Let μ be one of the probability measures which exist by part (a). We show that for $k \in \{0, 1, \ldots, r-1\}$

$$\nu_k = \sum_{j=0}^{r-1} \omega_r^{-kj} P^j \mu \in \mathcal{M}_{\mathbb{C}} \tag{3.2}$$

is an eigenmeasure of P for the eigenvalue ω_r^k. Indeed, using the fact that $P^r \mu = \mu$ we compute

$$\begin{aligned} P\nu_k &= P\mu + \omega_r^{-k} P^2 \mu + \cdots + \omega_r^{-k(r-2)} P^{r-1} \mu + \omega_r^{-k(r-1)} \mu \\ &= \omega_r^k \left(\mu + \omega_r^{-k} P\mu + \cdots + \omega_r^{-k(r-1)} P^{r-1} \mu \right) \\ &= \omega_r^k \nu_k. \end{aligned}$$

Finally, $\nu_k \neq 0$ since $\nu_k(X_j) \neq 0$ for $j = 0, 1, \ldots, r-1$. □

Approximation by a Galerkin method. We begin with the following observation which immediately follows from standard results on integral operators (see, e.g., [28, p. 277]).

LEMMA 3.4. *Suppose that the transition density function k satisfies*

$$\iint |k(x, y)|^2 \, dm(x) dm(y) < \infty. \tag{3.3}$$

Then the Frobenius–Perron operator $P : L^2 \to L^2$ is compact.

From now on we consider the case where P is given by a dynamical process with a transition density function k satisfying the condition (3.3). The aim is to use a Galerkin method for the approximation of such a Frobenius–Perron operator together with its spectrum. More precisely, let V_d, $d \geq 1$, be a sequence of d-dimensional subspaces of L^2 and let $Q_d : L^2 \to V_d$ be a projection such that Q_d converges pointwise to the identity on L^2. If we define the approximating operators by $P_d = Q_d P$, then we have

$$\|P_d - P\|_2 \to 0 \quad \text{as } d \to \infty.$$

Denote by $\sigma(P)$ and $\rho(P)$ the spectrum and resolvent set of P, respectively, and by $R_z = (zI - P)^{-1}$, $z \in \rho(P)$, the resolvent operator. Let $\lambda \neq 0 \in \sigma(P)$ be a nonzero eigenvalue of P and let $\Gamma \subset \mathbb{C}$ be a circle in $\rho(P)$ with center λ such that no other point of $\sigma(P)$ is inside Γ. Then the operator defined by

$$E = E(\lambda) = \frac{1}{2\pi i} \int_\Gamma R_z(P)\, dz$$

is a projection onto the space of generalized eigenvectors associated with λ and P. The following theorem—which is a specific application of the main result of [25] on compact operators—allows us to approximate eigenvectors of P by eigenvectors of P_d.

THEOREM 3.5 (see [25]). *Let λ_d be an eigenvalue of P_d such that $\lambda_d \to \lambda$ for $d \to \infty$, and let g_d be a corresponding eigenvector of unit length. Then there is a vector $h_d \in R(E)$ and a constant $C > 0$ such that $(\lambda I - P)h_d = 0$ and*

$$\|h_d - g_d\|_2 \leq C\|(P - P_d)|_{R(E)}\|_2,$$

where $R(E)$ denotes the range of E.

Next we use Theorem 3.5 to approximate the eigenvalues of P which are lying on the unit circle. For this we construct a Galerkin projection which possesses the same cyclic behavior in the approximation. Suppose that (3.1) holds and let $\{\varphi_i^j\}$, $j = 0, 1, \ldots, r-1$, $i = 1, 2, \ldots, d_j$, be a basis of V_d with the following properties:

$$\begin{aligned} &\text{(i)} \quad \operatorname{supp}(\varphi_i^j) \subset X_j \quad (j = 0, 1, \ldots, r-1, \quad i = 1, 2, \ldots, d_j), \\ &\text{(ii)} \quad \sum_{i=1}^{d_j} \varphi_i^j(x) = 1 \quad \text{for all } x \in X_j,\ j = 0, 1, \ldots, r-1. \end{aligned} \tag{3.4}$$

Remark 3.6.

(a) In section 5 we will see how to generate a basis satisfying (3.4). In that case, V_d consists of functions which are locally constant.

(b) Observe that by construction

$$\sum_{i,j} \varphi_i^j(x) = 1 \quad \text{for all } x \in X.$$

The Galerkin projection $Q_d g$ of $g \in L^2$ is defined by

$$(Q_d g, \varphi_i^j) = (g, \varphi_i^j) \quad \text{for all } i, j,$$

where $(\cdot, \cdot)$ is the usual inner product in L^2. The following result is a generalization of Lemma 8 in [10], where just the fixed point of P is considered. Recall that $\omega_r = e^{2\pi i/r}$.

PROPOSITION 3.7. *Suppose that the Galerkin projection satisfies* (3.4). *Then the approximating operators $P_d = Q_d P$ possess the eigenvalues ω_r^k, $k = 0, 1, \ldots, r-1$.*

Proof. Suppose that λ is an eigenvalue of P_d with corresponding eigenvector $\psi(x) = \sum_{i,j} \beta_i^j \varphi_i^j(x)$. Then $P_d\psi = \lambda\psi$ is equivalent to

$$\sum_{i_1,k_1} \beta_{i_1}^{k_1}(P\varphi_{i_1}^{k_1}, \varphi_{i_2}^{k_2}) = \lambda \sum_{i_1,k_1} \beta_{i_1}^{k_1}(\varphi_{i_1}^{k_1}, \varphi_{i_2}^{k_2}) \quad \text{for all } i_2, k_2.$$

Introducing the coefficient vector $\beta = (\beta_i^j)$ we may write this equation in matrix form as

$$M_1\beta - \lambda M_2\beta = 0, \tag{3.5}$$

where both M_1 and M_2 have nonnegative entries. Moreover, noting that

$$\int Q_d g \, dm = \int P_d g \, dm = \int g \, dm \quad \text{for every } g \in L^2,$$

and using the fact that $\sum_{i,j} \varphi_i^j(x) = 1$ we can proceed in the same way as in [10, Lemma 8], to see that $(1, 1, \ldots, 1)$ is a left eigenvector with eigenvalue 1 for the generalized eigenvalue problem (3.5). The fact that M_2 is invertible—since $\{\varphi_i^j\}$ is a basis of V_d—now implies that there is an eigenvector α with

$$M_2^{-1}M_1\alpha = \alpha.$$

We claim that $M_2^{-1}M_1$ has a cyclic structure so that $(M_2^{-1}M_1)^r$ is of block diagonal form where the blocks have the dimensions d_j, $j = 0, 1, \ldots, r-1$. Decomposing α with respect to this block structure we may proceed as in the proof of part (b) of Proposition 3.3 to show that ω_r^k, $k = 0, 1, \ldots, r-1$, are eigenvalues of (3.5) and hence of P_d as desired.

We now prove the claim. Since the matrix M_2 in (3.5) already has the desired block diagonal form (by (i) in (3.4)), it remains to show that the basis functions are cyclically permuted by P_d respecting the block structure of M_2. More precisely, we will show that $(P\varphi_{i_1}^{k_1}, \varphi_{i_2}^{k_2}) = 0$ if $k_2 \neq (k_1 + 1) \bmod r$.

By (3.1) we have

$$\int_{X_{j+1 \bmod r}} k(x, y) \, dm(y) = \begin{cases} 1 & \text{if } x \in X_j \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\int_{X_{k_2}} \int_{X_{k_1}} k(x, y) \, dm(x) dm(y) = \int_{X_{k_1}} \int_{X_{k_2}} k(x, y) \, dm(y) dm(x) = 0$$

if $k_2 \neq (k_1 + 1) \bmod r$, and therefore

$$(P\varphi_{i_1}^{k_1}, \varphi_{i_2}^{k_2}) = \int_{X_{k_2}} \int_{X_{k_1}} k(x, y)\varphi_{i_1}^{k_1}(x)\varphi_{i_2}^{k_2}(y) \, dm(x) dm(y) = 0$$

if $k_2 \neq (k_1 + 1) \bmod r$ as desired. □

Now we may combine Theorem 3.5 and Proposition 3.7 to obtain a convergence result for eigenvectors corresponding to eigenvalues of P of modulus one.

COROLLARY 3.8. *Suppose that P and its approximation P_d satisfy the hypotheses stated above. Then each simple eigenvalue $e^{2\pi ik/r}$ of P on the unit circle is an eigenvalue of P_d and there are corresponding eigenvectors g_d of P_d converging to an eigenfunction h of P. More precisely, there is a constant $C > 0$ such that for all $d \geq 1$*

$$\|h - g_d\|_2 \leq C\|P_d - P\|_2.$$

4. The computation of SRB-measures.

SRB-measures. Let us briefly recall the notion of an SRB-measure. In the existing literature several different definitions can be found which are all equivalent in the case where the underlying dynamical behavior is Axiom A. This is precisely the situation we will consider, and hence we can, without loss of generality, work with just one of them.

DEFINITION 4.1. *An ergodic measure μ is an* SRB-measure *if there exists a subset $U \subset X$ with $m(U) > 0$ and such that for each continuous function ψ*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \psi(f^j(x)) = \int \psi \, d\mu \tag{4.1}$$

for all $x \in U$.

Remark 4.2.

(a) Recall that (4.1) always holds for μ-a.e. $x \in X$ by the Birkhoff ergodic theorem. The crucial difference for an SRB-measure is that the temporal average equals the spatial average for a set of initial points $x \in X$ which has positive Lebesgue-measure. This is the reason why this measure is also referred to as the *natural* or the *physically relevant* invariant measure.

(b) The concept of SRB-measures in the context of Anosov systems had been introduced by Y. Sinai in the 1960s (e.g., [27]). Later the existence of SRB-measures was shown for Axiom A systems by R. Bowen and D. Ruelle (see [26, 3]). More recently, M. Benedicks and L.-S. Young have shown that the Hénon-map has an SRB-measure for a "large" set of parameter values [1]. However, it is still one of the major problems in ergodic theory to establish the existence of SRB-measures for a more general class of dynamical systems.

Small random perturbations. We specify concretely the stochastic transition function p underlying the numerical realization. Recall that the purpose is to approximate the Frobenius–Perron operator of a *deterministic* dynamical system represented by a diffeomorphism f. Hence the *stochastic* system that we consider should be a small perturbation of this original deterministic system.

For $\varepsilon > 0$ we set

$$k_\varepsilon(x,y) = \frac{1}{\varepsilon^n m(B)} \chi_B\left(\frac{1}{\varepsilon}\Big(y - x\Big)\right), \quad x, y \in X. \tag{4.2}$$

Here $B = B_0(1)$ denotes the open ball in $\mathbb{R}^n$ of radius one and χ_B is the characteristic function of B. Obviously $k_\varepsilon(f(x), y)$ is a transition density function and we may define a stochastic transition function p_ε by

$$p_\varepsilon(x, A) = \int_A k_\varepsilon(f(x), y) \, dm(y). \tag{4.3}$$

Remark 4.3. Note that $p_\varepsilon(x, \cdot) \to \delta_{f(x)}$ for $\varepsilon \to 0$ uniformly in x in a weak*-sense. Hence the Markov process defined by any initial probability measure μ and the transition function p_ε is a *small random perturbation* of the deterministic system f in the sense of Yu. Kifer [17].

Observe that we can apply the results from section 3 since

$$\iint |k_\varepsilon(f(x), y)|^2 \, dm(x) dm(y) \le \left(\frac{m(X)}{\varepsilon^n m(B)}\right)^2 < \infty,$$

and therefore the Frobenius–Perron operator $P_\varepsilon : L^2 \to L^2$ is compact (see Lemma 3.4).

Approximation of SRB-measures. We now combine Corollary 3.8 with a result of Yu. Kifer [17] to show that the approximations of the invariant measures converge to an SRB-measure with decreasing magnitude of the random perturbations.

Let us be more precise. Suppose that the diffeomorphism f possesses a hyperbolic attractor Λ with an SRB-measure μ_{SRB}, and let p_ε be a small random perturbation of f. Then, under certain hypotheses on p_ε, it is shown in [17] that the invariant measures of p_ε converge in a weak*-sense to μ_{SRB} as $\varepsilon \to 0$. On the other hand we can approximate the relevant eigenmeasures of P_ε by Corollary 3.8 and this leads to the desired result.

THEOREM 4.4. *Suppose that the diffeomorphism f has a hyperbolic attractor Λ, and that there exists an open set $U_\Lambda \supset \Lambda$ such that*

$$k_\varepsilon(x,y) = 0 \quad \text{if } x \in \overline{f(U_\Lambda)} \text{ and } y \notin U_\Lambda.$$

Then the transition function p_ε in (4.3) has a unique invariant measure π_ε with support on Λ, and the approximating measures

$$\mu_d^\varepsilon(A) = \int_A g_d^\varepsilon \, dm$$

converge in a weak-sense to the SRB-measure μ_{SRB} of f as $\varepsilon \to 0$ and $d \to \infty$,*

$$\lim_{\varepsilon \to 0} \lim_{d \to \infty} \mu_d^\varepsilon = \mu_{SRB}. \tag{4.4}$$

Proof. It is straightforward to check that the conditions of Theorem 1 in [17] are satisfied for the densities

$$q_x^\varepsilon(y) = k_\varepsilon(x,y),$$

provided $\varepsilon < 1$. Hence—denoting the unique invariant measure of the transition function p_ε with support in $\overline{U_\Lambda}$ by π_ε—this theorem implies that

$$\pi_\varepsilon \overset{\text{weak*}}{\longrightarrow} \mu_{SRB} \quad \text{for } \varepsilon \to 0. \tag{4.5}$$

By Remark 2.7 we know that π_ε is absolutely continuous, and we denote its density function by κ_ε. Then Corollary 3.8 guarantees that the fixed points g_d^ε of P_d^ε converge to κ_ε as $d \to \infty$. Therefore

$$\left| \int h \, d\mu_d^\varepsilon - \int h \, d\pi_\varepsilon \right| = \left| \int h(g_d^\varepsilon - \kappa_\varepsilon) \, dm \right| \leq \|h\|_2 \|g_d^\varepsilon - \kappa_\varepsilon\|_2 \to 0$$

as $d \to \infty$ for every $h \in L^2$ and, in particular,

$$\mu_d^\varepsilon \overset{\text{weak*}}{\longrightarrow} \pi_\varepsilon \quad \text{as } d \to \infty.$$

Combining this with (4.5) leads to (4.4), as desired. □

5. Extracting dynamical behavior. In the previous section we have seen that we can approximate the physically relevant invariant measure—the SRB-measure—of our original deterministic system by the computation of the invariant measure of a randomly perturbed system. However, dynamically the nonstationary behavior is also interesting, and we will now describe how to detect numerically components in state space which are (almost) cyclically permuted.

Extraction of cyclic behavior. Suppose that the stochastic transition function of the randomly perturbed dynamical system satisfies the cycle condition (3.1). Then the purpose is to identify the components X_j. By Proposition 3.7 we know that the approximating operator P_d^ε has the eigenvalues ω_r^k, $k = 0, 1, \ldots, r-1$, and we now show that the cyclic components can be approximated by certain linear combinations of the corresponding eigenvectors.

It is instructive to consider the simplest case where $r = 2$ before we turn our attention to the general situation. Suppose that we have two components X_0 and X_1 which are cyclically permuted by our process. Then the aim is to find approximations of eigenmeasures μ_0 and $\mu_1 = P_\varepsilon \mu_0$ of P_ε^2 with support on X_0 and X_1, respectively; see Proposition 3.3. By the same proposition we know that $\omega^0 = 1$ and $\omega^1 = -1$ are eigenvalues of P_ε. Let ν_0 and ν_1 be corresponding (real) eigenmeasures. Then, by (3.2), there are $\alpha_0, \alpha_1 \in \mathbb{R}$ such that

$$\nu_0 = \alpha_0(\mu_0 + P_\varepsilon \mu_0) \quad \text{and} \quad \nu_1 = \alpha_1(\mu_0 - P_\varepsilon \mu_0).$$

Rescaling ν_0 and ν_1 so that $\nu_0(X_0) = \nu_1(X_0) = 1$ we can compute μ_0 and μ_1 by

$$\mu_0 = \frac{1}{2}(\nu_0 + \nu_1) \quad \text{and} \quad \mu_1 = \frac{1}{2}(\nu_0 - \nu_1).$$

This process also shows how to construct eigenvectors of the Galerkin approximation for which linear combinations are appropriate approximations of the probability measures μ_0 and μ_1.

We now consider the general case. For $\ell = 0, 1, \ldots, r-1$ denote by $\mu_\ell = P_\varepsilon^\ell \mu_0$ the invariant measure of P_ε^r with support on X_ℓ (see Proposition 3.3).

LEMMA 5.1. *For $s \in \{0, 1, \ldots, r-1\}$ let*

$$\nu_k^s = \sum_{j=0}^{r-1} \omega_r^{-kj} P_\varepsilon^j \mu_s \tag{5.1}$$

be a specific choice for the eigenmeasures of P_ε corresponding to the eigenvalues ω_r^k, $k = 0, 1, \ldots, r-1$ (see (3.2)). Then

$$\frac{1}{r} \sum_{k=0}^{r-1} \omega_r^{\ell k} \nu_k^s = \mu_{\ell+s \bmod r}.$$

Proof. We compute

$$\frac{1}{r} \sum_{k=0}^{r-1} \omega_r^{\ell k} \nu_k^s = \sum_{j=0}^{r-1} \left(\frac{1}{r} \sum_{k=0}^{r-1} \omega_r^{(\ell-j)k} \right) P_\varepsilon^j \mu_s = P_\varepsilon^\ell \mu_s = \mu_{\ell+s \bmod r}.$$

Here we have used the identity

$$\frac{1}{r} \sum_{k=0}^{r-1} \omega_r^{(\ell-j)k} = \delta_{\ell j},$$

where $\delta_{\ell j}$ is the Kronecker symbol. □

The previous lemma indicates how to approximate the cyclic components of X: we have to find eigenvectors $v_0^s, \ldots, v_{r-1}^s$ of the matrix $M_2^{-1} M_1$ (see (3.5)) which are

approximations of the eigenmeasures ν_k^s in (5.1) for an $s \in \{0, 1, \ldots, r-1\}$. Then we can compute

$$u_{\ell+s \bmod r} = \frac{1}{r} \sum_{k=0}^{r-1} \omega_r^{\ell k} v_k^s$$

for $\ell = 0, 1, \ldots, r-1$, and the positive components of u_j provide the desired information about the support of μ_j on X_j $(j = 0, 1, \ldots, r-1)$.

Hence it remains to describe how to construct eigenvectors $v_0^s, \ldots, v_{r-1}^s$ approximating the ν_k^s in (5.1) for an $s \in \{0, 1, \ldots, r-1\}$. We do this for the case where the eigenvalues ω_r^k are simple—the case of several coexisting cycles will be treated in the following section.

Suppose that we have a set of eigenmeasures ρ_k corresponding to the eigenvalues ω_r^k, $k = 0, 1, \ldots, r-1$. Since the eigenvalues are simple we know that for each $s \in \{0, 1, \ldots, r-1\}$ there is a constant $\alpha_k^s \in \mathbb{C}$ such that ρ_k can be written as

$$\rho_k = \alpha_k^s \nu_k^s.$$

Hence the task is to rescale ρ_k so that $\alpha_k^s = 1$ for all k. By (3.2) it is easy to see that for each $s \in \{0, 1, \ldots, r-1\}$

$$\rho_k(X_s) \neq 0.$$

We choose a particular s and rescale the ρ_k's by (complex) factors so that

$$\rho_k(X_s) = 1 \quad \text{for all } k = 0, 1, \ldots, r-1.$$

With this choice it follows that $\rho_k = \nu_k^s$.

In the realization of the approximation of the ν_k^s's we proceed with the eigenvectors of $M_2^{-1} M_1$ in an analogous way: by our choice of the Galerkin approximation we just need to find an index such that the corresponding components of the eigenvectors do not vanish. The scaling as described above can then be done in a similar way: find complex multiples of the eigenvectors so that they possess (real) positive components which add up to one for each vector. We will illustrate the method by examples in section 6.

Identification of several coexisting cycles or invariant sets. In applications it may occur that there exist several different cycles in the dynamical system under consideration. We now show how to identify these sets numerically. Replacing P_ε by an appropriate power P_ε^r if necessary we can, without loss of generality, restrict our attention to the case where there are different invariant sets.

Again we begin with the simplest case and assume that P_ε has two linearly independent invariant probability measures ν_1 and ν_2. Then, by Theorem 2.6, there are constants α_j^i $(i, j = 1, 2)$ with

$$\alpha_1^1 \nu_1 + \alpha_2^1 \nu_2 = \pi_1 \quad \text{and} \quad \alpha_1^2 \nu_1 + \alpha_2^2 \nu_2 = \pi_2,$$

where π_1 and π_2 are probability measures with $\pi_i(E_i) = 1$ for invariant sets E_1 and E_2. Let $B_i \subset E_i$ be subsets with $\nu_i(B_i) \neq 0$. Then the coefficients α_j^i can be found as the solutions of the equations

$$\alpha_1^1 \nu_1(B_2) + \alpha_2^1 \nu_2(B_2) = 0,$$
$$\alpha_1^2 \nu_1(B_1) + \alpha_2^2 \nu_2(B_1) = 0,$$

with the additional requirement that $\pi_i = \alpha_1^i \nu_1 + \alpha_2^i \nu_2$ are probability measures. Indeed, let β_j^i be constants such that

$$\alpha_1^1 \nu_1 + \alpha_2^1 \nu_2 = \beta_1^1 \pi_1 + \beta_2^1 \pi_2,$$
$$\alpha_1^2 \nu_1 + \alpha_2^2 \nu_2 = \beta_1^2 \pi_1 + \beta_2^2 \pi_2.$$

Then, in particular,

$$0 = \alpha_1^1 \nu_1(B_2) + \alpha_2^1 \nu_2(B_2) = \beta_1^1 \pi_1(B_2) + \beta_2^1 \pi_2(B_2) = \beta_2^1 \pi_2(B_2),$$
$$0 = \alpha_1^2 \nu_1(B_1) + \alpha_2^2 \nu_2(B_1) = \beta_1^2 \pi_1(B_1) + \beta_2^2 \pi_2(B_1) = \beta_1^2 \pi_1(B_1).$$

Since $\nu_i(B_i) \neq 0$, it follows that $\pi_i(B_i) \neq 0$ and therefore $\beta_2^1 = \beta_1^2 = 0$ as desired.

We now generalize this observation. Suppose that the eigenspace of the Frobenius–Perron operator corresponding to the eigenvalue 1 is e-dimensional. Then, by Theorem 2.6, there are e distinct invariant sets $E_1, E_2, \ldots, E_e$ and invariant measures $\pi_1, \pi_2, \ldots, \pi_e$ such that $\pi_j(E_j) = 1$.

LEMMA 5.2. *Let ν_j, $j = 1, \ldots, e$, be invariant measures spanning the e-dimensional eigenspace of the Frobenius–Perron operator corresponding to the eigenvalue 1. Let $B_i \subset E_{k_i}$ be subsets such that $\nu_i(B_i) \neq 0$, $i = 1, 2, \ldots, e$. We have that*

(a) *the matrix $M = (\nu_j(B_i))_{i,j=1,\ldots,e}$ has full rank if and only if*

$$\{k_1, k_2, \ldots, k_e\} = \{1, 2, \ldots, e\}; \tag{5.2}$$

(b) *if* $\operatorname{rank}(M) = e$, *then the invariant measures π_j, $j = 1, \ldots, e$, are given by*

$$\pi_{k_\ell} = \sum_{j=1}^{e} \alpha_j^\ell \nu_j,$$

where $\alpha^\ell = (\alpha_j^\ell)$ is the (rescaled) null vector of $M_\ell = (\nu_j(B_i))_{i,j=1,\ldots,e,\ i \neq \ell}$.

Proof. Suppose that (5.2) holds. Let $\alpha \in \mathbb{R}^e$ be an element of the kernel of the matrix $M = (\nu_j(B_i))$, that is,

$$\sum_{j=1}^{e} \alpha_j \nu_j(B_i) = 0, \quad i = 1, 2, \ldots, e.$$

Using the fact that there are constants β_j, $j = 1, \ldots, e$, such that

$$\sum_{j=1}^{e} \alpha_j \nu_j = \sum_{j=1}^{e} \beta_j \pi_j,$$

we obtain

$$0 = \sum_{j=1}^{e} \beta_j \pi_j(B_i) = \beta_{k_i} \pi_{k_i}(B_i), \quad i = 1, 2, \ldots, e.$$

Since $0 \neq \nu_i(B_i) = \gamma_i \pi_{k_i}(B_i)$ for appropriate constants γ_i, it follows that $\beta_i = 0$ for $i = 1, 2, \ldots, e$ (here we have used (5.2)). Hence $\sum_{j=1}^{e} \alpha_j \nu_j = 0$, and since the ν_j's span the e-dimensional eigenspace of the Frobenius–Perron operator corresponding to the eigenvalue 1, we may conclude that $\alpha = 0$.

To complete the proof of part (a) it remains to show that $M = (\nu_j(B_i))$ is singular if (5.2) is not satisfied, that is, if there is a $k \in \{1, 2, \dots, e\}$ which is not in $\{k_1, k_2, \dots, k_e\}$. Writing π_k as

$$\pi_k = \sum_{j=1}^{e} \alpha_j \nu_j$$

and using $B_i \not\subset E_k$ we obtain for $i = 1, \dots, e$

$$0 = \pi_k(B_i) = \sum_{j=1}^{e} \alpha_j \nu_j(B_i).$$

Hence $\alpha = (\alpha_j)$ is a nontrivial null vector of M. The statement in part (b) is an immediate consequence. □

For the numerical identification of the sets $E_1, \dots, E_e$ we choose nonvanishing components of the e approximations $v^1, \dots, v^e$ of eigenmeasures $\nu_1, \dots, \nu_e$ in such a way that the matrix $M = (v_i^j)_{i,j=1,\dots,e}$ is nonsingular. Then we identify the distinct invariant components as the support of the eigenmeasures approximated by the scaled null vectors of the matrices $M_\ell = (v_i^j)_{i,j=1,\dots,e,\ i \neq \ell}$.

Extraction of almost cyclic behavior. We distinguish two different scenarios by which an *almost cyclic* behavior may occur in a dynamical system:

(i) The first scenario we have in mind is that cyclic components $X_0, X_1, \dots, X_{r-1}$ merge while a control parameter is varied in the system. If this has happened, then the cyclic behavior can frequently be observed although it is strictly no longer present.

(ii) Second, it may happen that two *different* cycles merge while a control parameter is varied. For instance, if two invariant sets "collide," then immediately after the collision there are still two subsets in state space which are *almost invariant.*

In section 6 we will illustrate both scenarios by numerical examples.

We know that the rth roots of unity are eigenvalues of P_ε if there are r cyclic components. If these components merge, then these eigenvalues leave the unit circle. The main purpose of this section is to relate the modulus of these eigenvalues to the probability that the cyclic behavior is still observed. As in the previous subsection, we may just consider *almost invariant* sets by replacing P_ε by P_ε^r if necessary. In this case a cluster of eigenvalues moves away from one along the real line while several (precisely) invariant sets disappear.

DEFINITION 5.3. *A subset $A \subset X$ is* δ-almost invariant with respect to $\rho \in \mathcal{M}$ *if* $\rho(A) \neq 0$ *and*

$$\int_A p_\varepsilon(x, A)\, d\rho(x) = \delta \rho(A).$$

Remark 5.4.

(a) Using the definition of p_ε we compute for a subset $A \subset X$

$$p_\varepsilon(x, A) = \frac{m(A \cap B_{f(x)}(\varepsilon))}{m(B_0(\varepsilon))}.$$

Hence

$$\delta = \frac{1}{\rho(A)} \int_A \frac{m(A \cap B_{f(x)}(\varepsilon))}{m(B_0(\varepsilon))}\, d\rho(x).$$

(b) We have seen that $p_\varepsilon(x,\cdot) \to \delta_{f(x)}$ for $\varepsilon \to 0$. Thus, we obtain in the deterministic limit

$$\int_A p_0(x,A)\, d\rho(x) = \int_A \delta_{f(x)}(A)\, d\rho(x) = \rho(f^{-1}(A) \cap A).$$

Therefore in this case δ is the relative ρ-measure of the subset of points in A which are mapped into A.

According to the classification of the occurrence of almost cyclic behavior given at the beginning of this section we identify almost cyclic behavior in the numerical realization as follows:

(i) If cyclic components $X_0, X_1, \ldots, X_{r-1}$ merge while a control parameter is varied in the dynamical system, then we use the same linear combinations as in the unperturbed case to identify the components of the "almost-cycle." The overlap of the different components indicates the subset of points which no longer follow the cyclic behavior.

(ii) If two invariant sets merge, then we use the results obtained for the identification of coexisting invariant sets. However, observe that in the perturbed case we can no longer expect that an approximation of the matrix M in Lemma 5.2 will be singular although the chosen subsets do not properly represent the (almost) cyclic behavior. In the numerical realization we take this into account and use as a criterion the condition number of these matrices: in the construction of the matrix M we choose the components of the approximating vectors in such a way that the condition number of M is as small as possible.

From now on we assume that $\lambda \neq 1$ is an eigenvalue of P_ε with corresponding real valued eigenmeasure $\nu \in \mathcal{M}_{\mathbb{C}}$, that is,

$$P_\varepsilon \nu = \lambda \nu.$$

Recall that in this case $\nu(X) = 0$ (see Remark 3.2 (a)). The aim is to relate the value of this eigenvalue to the probability δ in Definition 5.3. We begin with the following elementary observation.

LEMMA 5.5. *Suppose that ν is scaled so that $|\nu| \in \mathcal{M}$, and let $A \subset X$ be a set with $\nu(A) = \frac{1}{2}$. Then $\nu = |\nu|$ on A.*

Proof. Obviously, $\nu(B) \leq |\nu|(B)$ for all measurable B. For contradiction let $B \subset A$ be a measurable set with $\nu(B) < |\nu|(B)$. It follows that there is a $C \subset A$ with $\nu(C) < 0$. Hence we have for $E = A - C$

$$\nu(E) > \nu(A) = \frac{1}{2}.$$

Using $\nu(X) = 0$ and $|\nu|(X) = 1$, this leads to a contradiction

$$1 = |\nu|(E) + |\nu|(X - E) \geq |\nu(E)| + |\nu(X - E)| > \frac{1}{2} + \frac{1}{2} = 1. \qquad \square$$

Remark 5.6. Observe that by the *Hahn decomposition* (see, e.g., [28]) the existence of a set A with $\nu(A) = \frac{1}{2}$ is guaranteed.

PROPOSITION 5.7. *Suppose that ν is scaled so that $|\nu| \in \mathcal{M}$, and let $A \subset X$ be a set with $\nu(A) = \frac{1}{2}$. Then*

$$\delta + \sigma = \lambda + 1, \tag{5.3}$$

if A is δ-almost invariant and $X - A$ is σ-almost invariant with respect to $|\nu|$.

Proof. By Lemma 5.5 we have

$$\int_A p_\varepsilon(x, A)\, d\nu(x) = \int_A p_\varepsilon(x, A)\, d|\nu|(x) = \delta\nu(A)$$

since $\nu(A) = \frac{1}{2}$. Similarly,

$$\begin{aligned}\int_{X-A} p_\varepsilon(x, A)\, d\nu(x) &= \int_{X-A} 1 - p_\varepsilon(x, X-A)\, d\nu(x)\\ &= \nu(X-A) - \sigma\nu(X-A)\\ &= (\sigma - 1)\nu(A),\end{aligned}$$

since $\nu(X-A) = -\nu(A)$. Finally, using the fact that ν is an eigenmeasure we compute

$$\begin{aligned}\lambda\nu(A) &= \int_A p_\varepsilon(x, A)\, d\nu(x) + \int_{X-A} p_\varepsilon(x, A)\, d\nu(x)\\ &= \delta\nu(A) + (\sigma - 1)\nu(A) = (\delta + \sigma - 1)\nu(A),\end{aligned}$$

yielding (5.3). □

Remark 5.8.

(a) Observe that in the case where λ is close to one we may assume that the probability measure $|\nu|$ is close to the invariant measure μ of the system. In this sense we have derived the desired relation between the eigenvalue λ and the probability that the system is still behaving in a cyclic way.

(b) In our numerical computations we will work with the unperturbed equations rather than introduce noise artificially. Thus, it would be important to know whether the eigenvalues of P_0 and P_ε are close to each other for small ε. First results concerning the stochastic stability of the spectrum of the Frobenius–Perron operator are obtained in [2].

In (5.3) both δ and σ occur, and in general there will be no relation between these constants. However, if the underlying system possesses an additional symmetry, then we can express one of them in terms of the other one.

To illustrate this fact let us consider the simplest case where we have a symmetry transformation κ in the problem with $\kappa^2 = id$. In that case

$$p_\varepsilon(x, B) = p_\varepsilon(\kappa x, \kappa B) \quad \text{for all measurable } B \subset X, \tag{5.4}$$

which implies that for any $\rho \in \mathcal{M}$

$$\int_B p_\varepsilon(x, B)\, d\rho = \int_{\kappa B} p(x, \kappa B)\, d\kappa^*\rho.$$

Hence we have the following corollary.

COROLLARY 5.9. *Suppose in addition to the assumptions in Proposition* 5.7 *that*

(i) p_ε *is symmetric, that is,* (5.4) *holds,*

(ii) *the set* A *satisfies* $\kappa A = X - A$, *and*

(iii) *the measure* $|\nu|$ *is* κ*-symmetric, that is,* $\kappa^*|\nu| = |\nu|$.

Then $X - A$ *is* δ*-almost invariant with respect to* $|\nu|$ *if and only if* A *is* δ*-almost invariant, and in particular*

$$\delta = \frac{\lambda + 1}{2}. \tag{5.5}$$

Numerical solution of the eigenvalue problems. Since in this article we were mainly interested in the description of how to extract numerically information on the dynamical behavior from the spectrum of P_ε, we just outline the algorithmic steps which are necessary for the numerical approximation and solution of the eigenvalue problem (3.5). For the details concerning the implementation the reader is asked to consult the references listed below.

All the algorithms described in the following are integrated into the software package

GAIO (**G**lobal **A**nalysis of **I**nvariant **O**bjects),

which can be obtained online via http. (See the first page of this paper for the authors' homepages.)

(i) *Construction of a box-covering.* We begin with the construction of a box-covering of the interesting (randomly perturbed) dynamics in state space. This can be done either by a subdivision technique (see [12, 6, 5, 7]) or by a cell-mapping approach (see, e.g., [20, 14]). This way we obtain a collection of boxes B_k, $k = 1, 2, \ldots, N$, such that the part of state space containing the interesting dynamics is covered by their union.

Remark 5.10. To simplify the description we assume that all the boxes have the same volume. However, it turned out that a box-covering can be constructed in an even more efficient way if the size of the boxes is chosen in an adaptive way in each step of the subdivision procedure. This fact is explored in [9].

(ii) *Galerkin approximation.* The basis functions we have chosen are the characteristic functions of the B_k's,

$$\varphi_k = \chi_{B_k}.$$

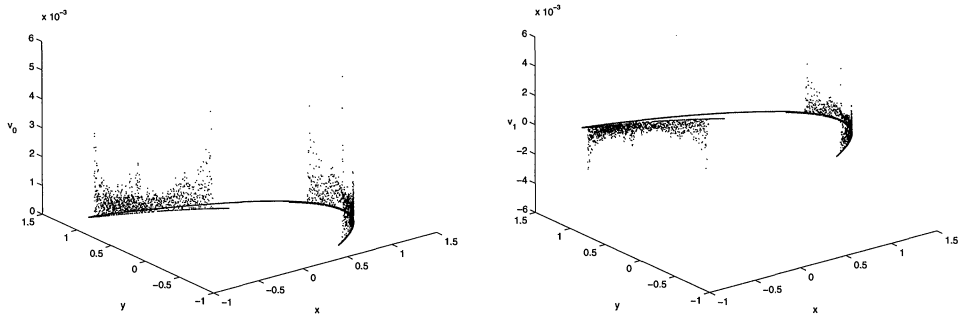
We assume that the Hausdorff distance between X and the covering $\cup B_k$ is small enough so that the assumption (3.4) on the φ_k's is satisfied.

Remark 5.11. In practice we use the subdivision technique from [6] to construct a box-covering for which (3.4) holds. In fact, we simply neglect those boxes which have measure zero once a certain number of steps in the subdivision algorithm has been performed.

(iii) *Approximation and solution of the eigenvalue problem.* We approximate the coefficients of the matrices M_1 and M_2 in (3.5) by a numerical evaluation of the integrals which are involved. Observe that by our specific choice of the boxes and the φ_k's the matrix M_2 is a multiple of the identity. Hence the main numerical effort lies in the computation of the inner products to approximate the operator P_ε. This is done either by a Monte Carlo method or by an exhaustion technique as described in [13].

For the computation of eigenvalues and corresponding eigenvectors we use ARPACK which provides an iterative eigenvalue solver for sparse matrices (see [22]). Alternatively we use an approach based on *bordered matrices* for finding specific single eigenvectors. For the solution of the corresponding system of linear equations we use an iterative method taking into account the fact that the matrix M_1 is extremely sparse.

6. Examples. In this section we illustrate our numerical methods by three examples in which the transition from a true cyclic behavior to an almost cycling one becomes apparent. First we use the well-known Hénon map as an example to analyze a 2-cycle and an almost 2-cycle. The second example is a $\mathbb{Z}_3$-equivariant mapping in the complex plane which shows a cycling behavior of period six. Finally we investi-



(a) v_0 (b) v_1

FIG. 6.1. *Eigenvectors of the approximation of the Frobenius–Perron operator for the Hénon map* ($a = 1.2$, $b = 0.2$).

gate numerically the *Chua circuit*. Since this is an ordinary differential equation, no nontrivial cycling is expected to be seen. However, in this case we will identify two sets in phase space which are almost invariant with respect to the flow.

All the computations were done without an artificial introduction of noise. Rather it turned out to be sufficient for our purposes to interpret the round off error as a small random perturbation.

A 2-cycle in the Hénon map. We consider a scaled version of the well-known *Hénon map*,

$$f(x, y) = (1 - ax^2 + y/5,\ 5bx),$$

where we fix $b = 0.2$ and vary a. For $a = 1.2$ this map possesses a 2-cycle, and we have used the approximation procedure described in section 5 to identify the two components X_0 and X_1. In Figure 6.1 we show the approximations v_0 and v_1 of the two eigenmeasures of the Frobenius–Perron operator corresponding to the eigenvalues $\lambda_0 = 1$ and $\lambda_1 = -1$.

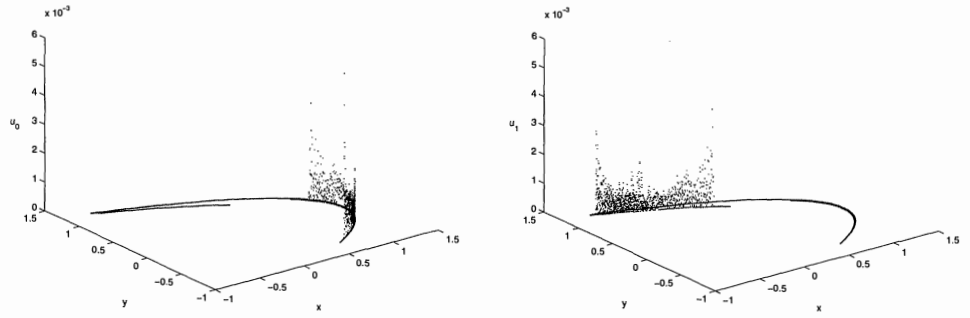
By Lemma 5.1

$$u_0 = \frac{1}{2}(v_0 + v_1) \quad \text{and} \quad u_1 = \frac{1}{2}(v_0 - v_1)$$

are approximations of probability measures μ_0 and μ_1 which have support on X_0 and X_1, respectively. These are shown in Figure 6.2.

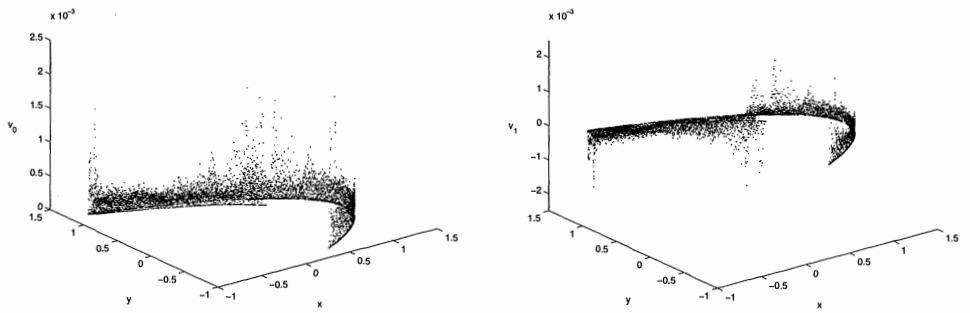
Remark 6.1. In the computation the box-covering was obtained by the continuation algorithm described in [5]. The boxes were of size $1/2^{10}$ in each coordinate direction and the continuation was restricted to the square $Q = [-2, 2]^2 \subset \mathbb{R}^2$. This way we have produced a covering of the closure of the one-dimensional unstable manifold of the hyperbolic fixed point in the first quadrant by 2525 boxes.

Next we set $a = 1.272$. For this parameter value the 2-cycle has disappeared, but in simulations the cycling behavior can still be observed for most iterates. Correspondingly we find that $\lambda_1 = -0.9944$ is an eigenvalue of the approximation of the Frobenius–Perron operator. Using the same notation as before we show in Figures 6.3



(a) u_0 (b) u_1

FIG. 6.2. *Approximations of probability measures with support on the two components of the 2-cycle* ($a = 1.2$, $b = 0.2$).



(a) v_0 (b) v_1

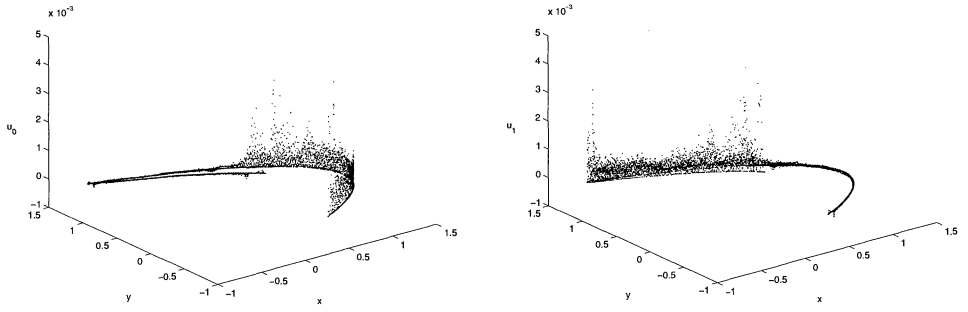
FIG. 6.3. *Eigenvectors of the approximation of the Frobenius–Perron operator for the Hénon map* ($a = 1.272$, $b = 0.2$).

and 6.4 the approximations of the eigenmeasures. In this case the box-covering has 3101 elements. Note that the supports of u_0 and u_1 have a nonempty intersection. This fact is illustrated in Figure 6.5 where we have marked by black circles all boxes on which $u_0 > 10$ and $u_1 > 10$.

A period six cycle. As the second example we slightly modify a mapping from [4] and consider the dynamical system $f : \mathbb{C} \to \mathbb{C}$,

$$f(z) = e^{-\frac{2\pi i}{3}} \left((|z|^2 + \alpha)z + \frac{1}{2}\bar{z}^2 \right),$$

for the parameter value $\alpha = -1.7$. For the computation of the box-covering we have used the subdivision algorithm described in [6]. Starting with the square $Q = [-1.5, 1.5]^2$ we have subdivided Q seven times by bisection in each coordinate direction



(a) u_0 (b) u_1

FIG. 6.4. *Approximations of probability measures which correspond to the two components of the almost 2-cycle* ($a = 1.272$, $b = 0.2$).

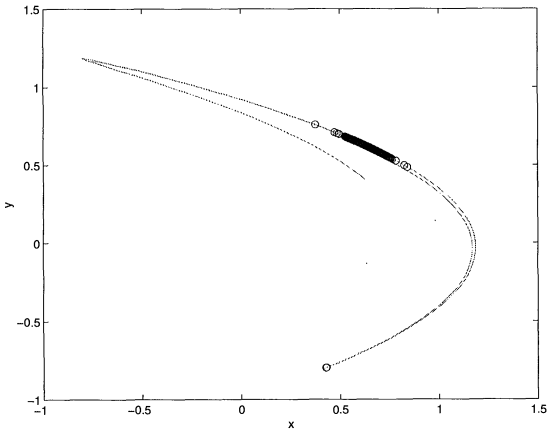


FIG. 6.5. *Approximation of a subset which is in the supports of both probability measures corresponding to the almost 2-cycle.*

which leads to a box-covering by 3606 boxes. In Figure 6.6 we show the approximation of the invariant measure, that is, the eigenvector v_0 corresponding to the eigenvalue $\lambda_0 = 1$ of the discretized Frobenius–Perron operator. In this case this operator additionally has the eigenvalues ω_6^k, $k = 1, \ldots, 5$, and hence we may use Lemma 5.1 to compute approximations $v_0, \ldots, v_5$ of the probability measures with support on the cyclic components $X_0, \ldots, X_5$. These supports are shown in Figure 6.7.

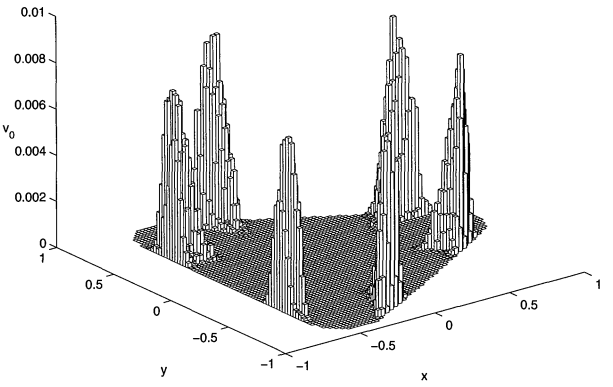


FIG. 6.6. *Approximation of the invariant measure for* $\alpha = -1.7$.

TABLE 1

j	λ_j	$\lvert\lambda_j\rvert$
0	1	1
1,5	$0.4918 \pm 0.8534i$	0.985
2,4	$-0.4880 \pm 0.8437i$	0.9747
3	-0.9709	0.9709

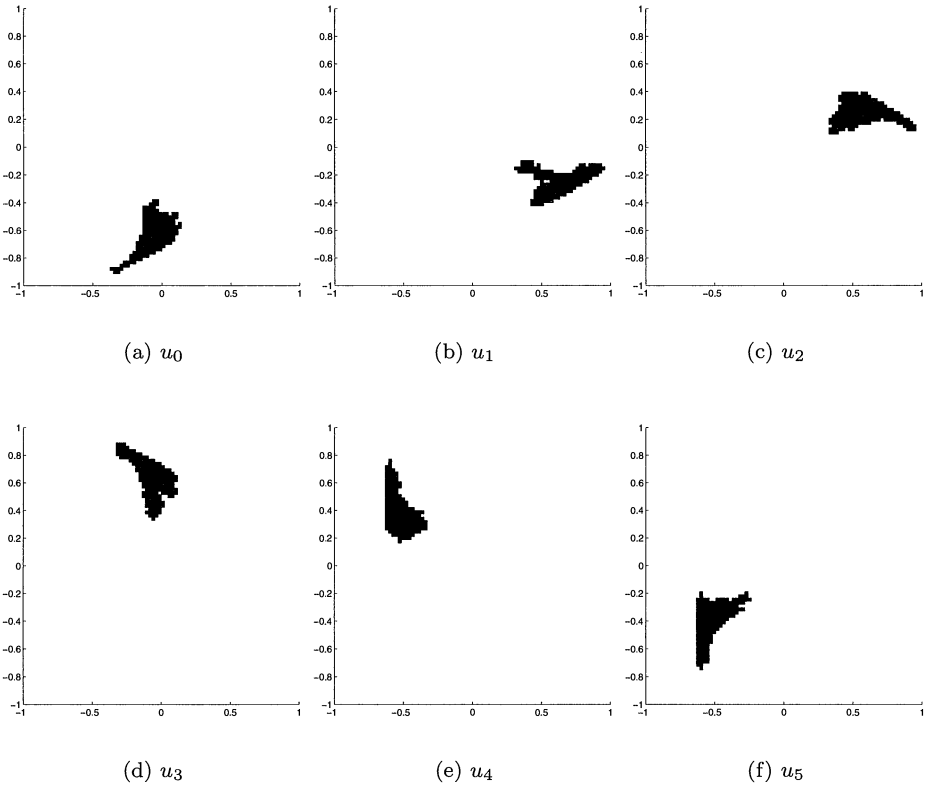
Now we vary the parameter and set $\alpha = -1.8$. For this value of α the strict cyclic behavior disappears and there is an almost 6-cycle. In Figure 6.8(a) we show the "essential" supports of the approximations $v_0, \ldots, v_5$ of the six almost cyclic components. More precisely we show all boxes B_ℓ for which $(v_i)_\ell > 0.1$, $i = 0, \ldots, 5$ (by $(v_i)_\ell$ we denote the ℓth component of the vector v_i). In Figure 6.8(b) we demonstrate that the intersection of these supports is nonempty. We have shown all boxes B_ℓ for which there are at least two indices $i, j \in \{0, \ldots, 5\}$, $i \neq j$, such that $(v_i)_\ell > 0.1$ and $(v_j)_\ell > 0.1$.

In Table 1 we list the corresponding eigenvalues together with their absolute values. We remark that for this parameter value the subdivision algorithm leads to a covering by 4364 boxes.

Two almost invariant sets in the Chua circuit. Finally we present a system of three first order ordinary differential equations in which two almost invariant sets can be identified numerically. The system that we are considering is the *Chua circuit*,

$$\begin{aligned} \dot{x} &= \alpha \left(y - m_0 x - \frac{1}{3} m_1 x^3 \right), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y, \end{aligned}$$

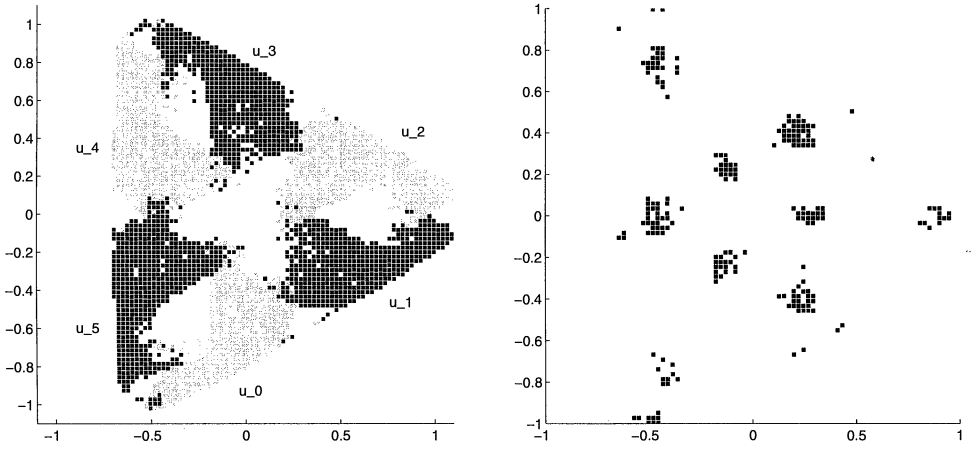
where we have chosen the parameter values $\alpha = 18$, $\beta = 33$, $m_0 = -0.2$, and $m_1 = 0.01$.



(a) u_0 (b) u_1 (c) u_2

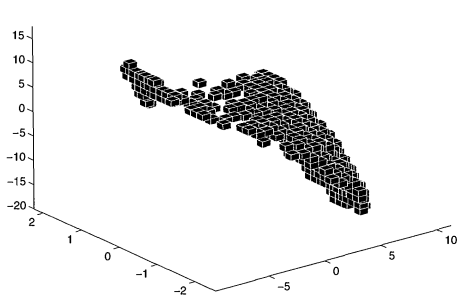
(d) u_3 (e) u_4 (f) u_5

FIG. 6.7. *Approximation of the cyclic components* $X_0, \ldots, X_5$ *for* $\alpha = -1.7$.

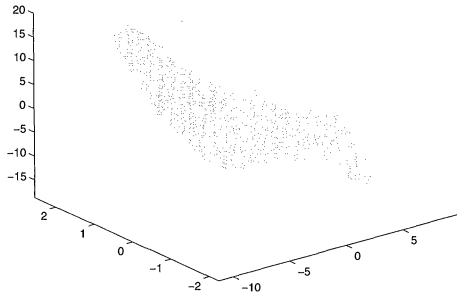


(a) "Essential" supports (b) Intersection of the supports

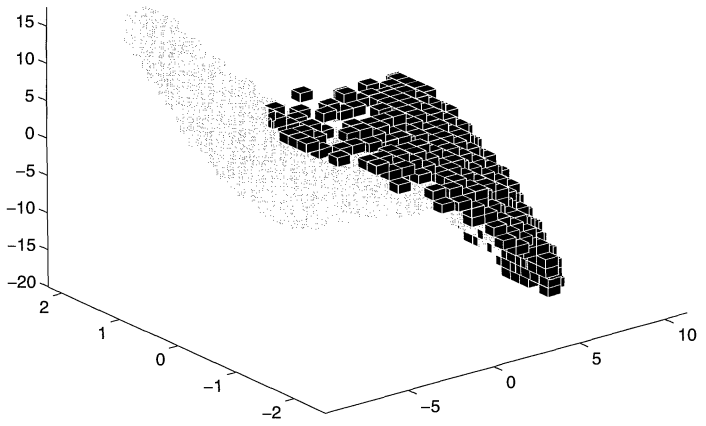
FIG. 6.8. *Approximation of the almost cyclic components for* $\alpha = -1.8$.



(a) u_1



(b) u_2



(c) u_3

FIG. 6.9. *Illustration of the existence of two almost invariant sets in the Chua circuit.* (a) *Boxes corresponding to components of the approximating densities with value bigger than* 10^{-4}; (b) *boxes corresponding to components of the approximating densities with value less than* -10^{-4}; (c) *superposition of the two almost invariant sets.*

A detailed discussion of the dynamical behavior of this system can be found in [16]; see also [23]. We consider the time-0.1 map and—using the continuation method described in [6]—cover the unstable manifold of the origin by 10372 boxes. In addition to the eigenvalue 1 the discretized Frobenius–Perron operator does also possess the eigenvalue $\lambda_1 = 0.9272$. We may conclude from this result that there are two almost invariant sets. Indeed, a numerical approximation of the corresponding regions in phase space leads to the result shown in Figure 6.9. A detailed numerical study of this particular example can be found in [8].

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