

Markov modelling for random dynamical systems

Gary Froyland

Department of Mathematics and Computer Science

University of Paderborn

Warburger Str. 100, Paderborn 33098, GERMANY.

Email: froyland@math.uni-paderborn.de

URL: <http://www-math.uni-paderborn.de/~froyland>

Abstract

Techniques for approximating the dynamics of deterministic systems based on a discrete Markov chain are well known and have been successfully used in the past. We now extend these techniques to random dynamical systems by defining a suitably averaged Markov model. These constructions are often numerically superior to iterative orbit based methods and provide greater theoretical control.

1 Introduction and Motivation

By a random dynamical system, we mean a random composition of mappings taken from a collection $\{T_k\}_{k=1}^r$. For the most part, we will assume that each T_k is a continuous mapping on a compact subset M of \mathbb{R}^d . One defines a random orbit $\{x_N\}_{N=0}^\infty$ by $x_N = x_N(k_{N-1}, \dots, k_1, k_0, x_0) := T_{k_{N-1}} \circ \dots \circ T_{k_1} \circ T_{k_0} x_0$, where the sequence k_0, k_1, \dots is produced by some stationary stochastic process, with $k_i \in \{1, 2, \dots, r\}$ for all $i \geq 0$. We will be concerned with the situations where the random sequence of indices arise as either an iid or Markov process. This amounts to our random system being defined by an iid or Markov random composition of mappings.

Example 1.1:

- (i) Define $T_k x = Tx + \epsilon_k$, where $\epsilon_k \in \mathbb{R}^d$ is small, so that each T_k is a small perturbation of some fixed map T . A random iid composition of the $\{T_k\}$ models a deterministic system subjected to small iid perturbations.
- (ii) A dynamical system may be subjected to external random inputs or stimulations. Different inputs produce different dynamics, so that effectively a different map is applied for each possible input. Very general input structures may be modelled using Markov compositions of maps.

Our goal is to perform some ergodic-theoretical analysis on such random dynamical systems. The central object of such a study is a suitably defined *invariant measure* for the random system; this invariant measure should describe the distribution of typical¹ orbits in phase space. By using this reference measure, one may define several important dynamical indicators such as Lyapunov exponents, entropy and the rate of decay of correlations, and talk about things such as the statistics of return times.

¹It is sometimes the case that trajectories $\{x_i\}_{i=0}^N$ starting in different regions of phase space will have different asymptotic behaviours, with each of these limiting behaviours corresponding to a different invariant measure. In such a case, we will restrict ourselves to a region of phase space for which almost all points have the same long term behaviour

We approach our analysis from the following viewpoint. The most common method of analysing random systems is to simulate random trajectories on a computer, and to perform analyses directly from these (finite) orbits. Under the (often implicit) assumption that almost all random orbits distribute themselves according to some distinguished invariant measure, it seems reasonable to (again, often implicitly) define an approximate invariant measure as $\mu_N := \frac{1}{N} \sum_{i=0}^{N-1} \delta_{x_i}$, and calculate the various dynamical properties with respect to this approximate measure (δ_x denotes the unit point mass at $x \in M$). Drawbacks with this approach include:

- (i) Statistical fluctuations in the *finite time* distribution μ_N (away from μ).
- (ii) Statistical fluctuations in the *finite time* distribution of the indices k_0, k_1, \dots, k_{N-1} (away from the invariant distribution of the underlying stochastic process).
- (iii) No information on the rate of convergence of $\mu_N \rightarrow \mu$ (or convergence of other mathematical objects).
- (iv) No information on the error $\text{dist}(\mu_N, \mu)$ (or errors for other mathematical objects)

The latter two points are essentially a result of not being able to well enough control the two types of statistical fluctuations.

To overcome these problems, we encode all of the information about how the random system evolves into a single linear operator on the space of probability measures. This evolution operator acts on probability measures in the same way that the maps T_k act on individual points $x \in M$. The fact that our underlying stochastic processes have very short memories (no memory for iid processes, and one-step memory for Markovian processes) allow us “average out” all or most of the “past”, and to produce a very concise representation of this evolution operator. This averaging process also immediately removes the problem (ii) alluded to above. Of course, for numerical calculations, we cannot work with the exact evolution operator, and instead produce a finite-dimensional approximation (a matrix). In a sense, this approximate evolution is a trade-off with problem (i). However, we have better control of the approximate evolution, and in some cases are able to provide convergence rates and error bounds, to solve problems (iii) and (iv) (see [4, 6]).

2 Averaged evolution operators

In the case of a single deterministic mapping T , the relevant evolution operator on the space of Borel (signed) measures is $T^* : \mathcal{S}(M) \rightrightarrows$, defined by $T^*\nu = \nu \circ T^{-1}$. The action of T^* is completely analogous to that of the *Perron-Frobenius operator*² \mathcal{P} for T , where the latter acts on integrable functions (the “densities” of signed measures), rather than the signed measures themselves. What we need is an averaged random analogue of T^* and/or \mathcal{P} .

iid case Suppose that the probability of a map T_k being selected is w_k . Then it turns out [12, 4] that the appropriate averaged evolution operator $T_{\text{iid}}^* : \mathcal{S}(M) \rightrightarrows$ is defined by

$$T_{\text{iid}}^*\nu = \sum_{k=1}^r w_k T_k^* \nu \tag{1}$$

Markov case Suppose that the probability of a map T_l being selected, given that the map T_k was selected at the previous time step, is W_{kl} . We assume that the Markov chain governed by W is ergodic (has a single invariant density defined by the fixed left eigenvector w of W). Since our underlying random process now has some memory, we must keep track of what happened one step

²see [9] for details.

into the past. To do this, we define our averaged evolution operator $T_{\text{Markov}}^* : \mathcal{S}(M) \times \cdots \times \mathcal{S}(M) \rightarrow$ on the r -fold product of $\mathcal{S}(M)$. The action of T_{Markov}^* is given by

$$T_{\text{Markov}}^*(\nu^{(1)}, \dots, \nu^{(r)}) = \left(\sum_{k=1}^r W_{1k}^* T_k^* \nu^{(k)}, \sum_{k=1}^r W_{2k}^* T_k^* \nu^{(k)}, \dots, \sum_{k=1}^r W_{rk}^* T_k^* \nu^{(k)} \right) \quad (2)$$

where $W_{lk}^* = W_{kl} w_k / w_l$; see [4] for details.

3 Constructing approximate averaged evolution operators

We now outline the numerical method by which we construct an approximate averaged evolution operator. Partition the state space M (or the region of M for which orbits display the same asymptotic behaviour) into a finite number of connected sets with non-empty interior, and denote this partition as $\mathfrak{P}_n := \{A_1, A_2, \dots, A_n\}$. Later we intend to refine this partition so as to decrease the diameter of the partition sets, and study the quality of the resulting approximate evolution operators as $n \rightarrow \infty$.

For each map T_k , construct a stochastic $n \times n$ matrix:

$$P_{n,ij}(k) = \frac{m(A_i \cap T_k^{-1} A_j)}{m(A_i)} \quad (3)$$

The matrix $P_n(k)$ is a finite dimensional approximation of the action of T_k^* ; see [10]. One may think of $P_{n,ij}(k)$ as the probability that a typical point in A_i moves into A_j under the action of T_k . Numerical approaches for computing the $P_n(k)$ are summarised in [3]. How the matrices $P_n(1), \dots, P_n(r)$ are put together depends on whether the underlying stochastic process is iid or Markov.

iid case By (1), the natural finite-dimensional approximation of T_{iid}^* is

$$P_n = \sum_{k=1}^r w_k P_n(k) \quad (4)$$

The value $P_{n,ij}$ may be thought of as the probability that a typical point in A_i moves into A_j “on average” under one iteration of the random system.

Markov case Similarly, using (2), we approximate T_{Markov}^* by

$$S_n = \begin{pmatrix} W_{11}^* P_n(1) & W_{21}^* P_n(1) & \cdots & W_{r1}^* P_n(1) \\ W_{12}^* P_n(2) & W_{22}^* P_n(2) & \cdots & W_{r2}^* P_n(2) \\ \vdots & \vdots & \ddots & \vdots \\ W_{1r}^* P_n(r) & W_{2r}^* P_n(r) & \cdots & W_{rr}^* P_n(r) \end{pmatrix}, \quad (5)$$

4 What can be done? Algorithms and theory

4.1 Invariant measures

An invariant measure³ of our random system will be derived from fixed points of our averaged evolution operators. An obvious choice for approximate invariant measures are fixed points of our approximate averaged evolution operators.

³see [4] for a rigorous definition of invariant measures for random dynamical systems.

iid case A fixed point of the stochastic matrix P_n is simply a vector p_n satisfying $p_n P_n = p_n$; we always normalise p_n so that $\sum_{i=1}^r p_{n,i} = 1$. The i^{th} entry of p_n corresponds to the set A_i , and we define an approximate invariant measure on M by

$$\mu_n(A_i) = p_{n,i}. \quad (6)$$

How the mass contained within A_i is distributed is relatively unimportant.

Markov case We again find a fixed left eigenvector s_n of S_n , and write this as $[s^{(1)}|s^{(2)}|\cdots|s^{(r)}]$, where each $s^{(k)}$ is a $1 \times n$ vector, normalised so that $\sum_{i=1}^n s_{n,i}^{(k)} = 1$ for $k = 1, \dots, r$. We define an approximate invariant measure on M by

$$\mu_n(A_i) = \sum_{k=1}^r w_k s_{n,i}^{(k)}, \quad (7)$$

where w is the fixed $1 \times r$ left eigenvector of W .

Our random system may have infinitely many invariant measures, but we are interested in the measure of physical significance, in the sense that it describes the distribution of almost all random trajectories. It is important to know that the measures μ_n approximate this distinguished physical measure, and not some other non-physical measure.

We give a rigorous definition of a physical measure, and then present a partial, but relatively general result in this direction.

Definition 4.1: We will say that a probability measure μ on M is a *physical invariant measure* if for every continuous function $g : M \rightarrow \mathbb{R}$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} g(T_{k_j} \circ \cdots \circ T_{k_0} x) \rightarrow \int_M g \, d\mu \quad (8)$$

for Lebesgue almost all $x \in M$ and almost all random sequences of maps.

With this definition, one may set $g(x) = \chi_A(x)$, where $A \subset M$ is such that $\mu(\partial A) = 0$, to obtain

$$\frac{1}{N} \text{card}\{0 \leq j \leq N-1 : T_{k_j} \circ \cdots \circ T_{k_1} \circ T_{k_0} x \in A\} \rightarrow \mu(A),$$

for Lebesgue almost all $x \in M$ and almost all random sequences. In this sense, the probability measure μ is of physical significance.

By refining our partition, we may produce a sequence of approximate invariant measures $\{\mu_n\}_{n=n_0}^\infty$. Compactness of M allows us to always find at least one (weak) limit of this sequence, taking convergent subsequences if necessary. We denote this limiting measure by μ^* .

Theorem 4.2: Suppose that each T_k is continuous, $k = 1, \dots, r$, and let μ be a physical measure for our random dynamical system. Set $S_\infty = \cap_{n=n_0}^\infty \text{supp } \mu_n$, where $\text{supp } \mu_n$ is the support of μ_n (the smallest closed set with full μ_n measure). Then

- (i) μ^* is an invariant measure for the random dynamical system.
- (ii) $\text{supp } \mu^* \subset \text{supp } \mu_n$ for all $n \geq 0$, and so $\text{supp } \mu^* \subset S_\infty$,
- (iii) $\text{supp } \mu \subset \text{supp } \mu_n$ for all $n \geq 0$, and so $\text{supp } \mu \subset S_\infty$,

PROOF: The proofs of (i) and (iii) are relatively simple extensions of ideas from [8] and [11], respectively. Part (ii) is a straightforward consequence of weak convergence. \square

Example 4.3: Consider the map $T_\alpha : S^1 \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $T_\alpha(\varphi, v) = (\varphi + v, \alpha v - 10 \cos(\varphi + v))$. This map describes the evolution of a ball bouncing⁴ on a sinusoidally forced table, where $\varphi \in S^1$ denotes the phase of the table at impact, $v \in \mathbb{R}$ denotes the exit velocity of the ball after impact, and $\alpha \leq 1$ is the coefficient of restitution of the ball. To create a random system, we assume that our ball has a hard side and a soft side, with coefficients $\alpha = 0.5$ and $\alpha = 0.1$ respectively. We suppose that once landing on the hard side, there is a 50/50 chance of which side is next impacted upon, and after landing on the soft side, the next impact is surely on the hard side. This leads to an underlying 2-state Markov process governed by the transition matrix $W = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$. A simulated orbit is shown in Figure 1 and a fixed point of an approximate averaged evolution operator in Figure 2. There is good agreement in the mass distributions of the two figures.

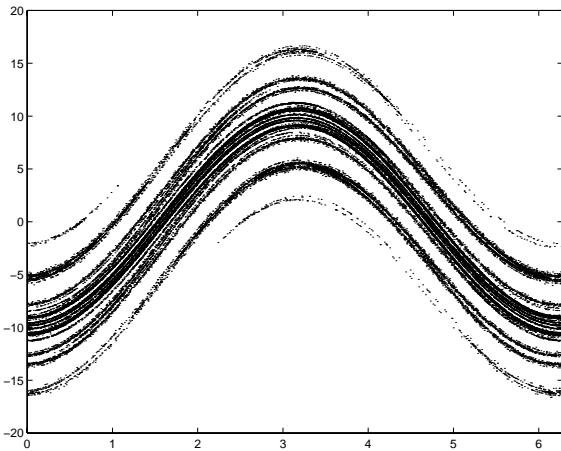


Figure 1: Plot of orbit of length 50000 for the random bouncing ball map

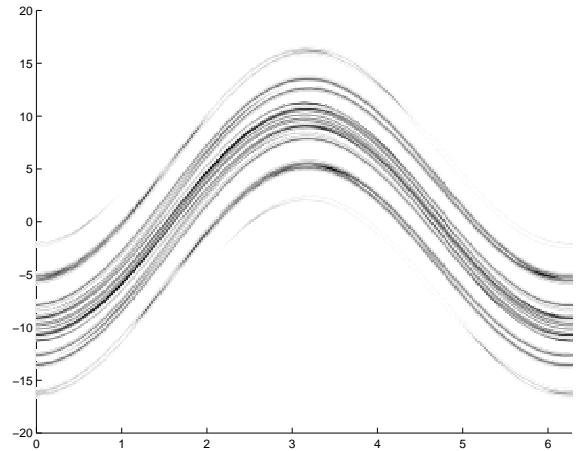


Figure 2: Approximate invariant measure for the random bouncing ball map using 18512 partition sets

4.2 Spectra and eigenfunctions

The spectrum and eigenfunctions of the linear operators T_{iid}^* and T_{Markov}^* can provide useful information in an appropriate setting. Invariant measures μ satisfy $T_{\text{iid}}^*\mu = \mu$, and so are generalised eigenfunctions with eigenvalue 1. We can also search for signed measures ν satisfying $T_{\text{iid}}^*\nu = \lambda\nu$, where $|\lambda| < 1$; such measures provide information about global structures embedded within the dynamics [1].

Naively, we approximate the spectrum and corresponding eigenfunctions of these linear operators by eigenvalues and eigenvectors⁵ of the matrices P_n and S_n . In the iid case, one simply finds an eigenvector q_n such that $q_n P_n = \lambda_n q_n$, and defines a signed measure ν_n as in (6), substituting $\mu_n \leftrightarrow \nu_n$ and $p_{n,i} \leftrightarrow q_{n,i}$. In the Markov case, one finds an eigenvector t_n with $t_n S_n = \lambda_n t_n$ and defines ν_n as in (7) substituting $\mu_n \leftrightarrow \nu_n$ and $s_{n,i}^{(k)} \leftrightarrow t_{n,i}^{(k)}$.

Figures 3 and 4 show the positive and negative parts of the eigenvector corresponding to an eigenvalue $\lambda = -0.5$ for Example 4.3. The negative sign of λ indicates a 2-cycle, and in fact there is an exchange of mass between the high density regions in Figures 3 and 4; note that these regions also correspond to high density areas of Figure 2. The three small high density regions (left, centre, and right) of Figure 3 are approximately mapped to the single large high density region (leftish) of Figure 4 by $T_{\alpha=0.1}$, while $T_{\alpha=0.5}$ roughly reverses this process. These two high density collections have been isolated in the positive and negative parts of the second eigenfunction.

⁴see [7] p.102 for details

⁵Further details in the deterministic setting can be found in [2].

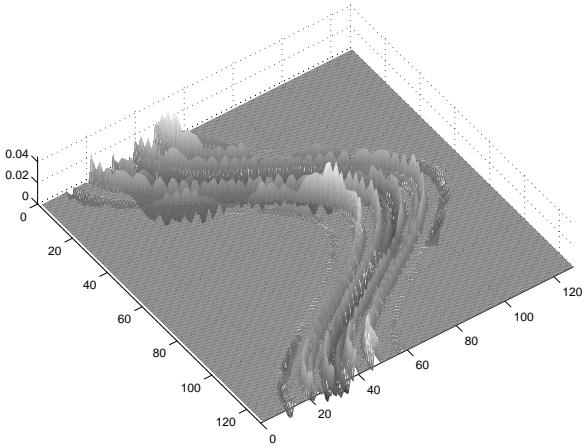


Figure 3: Positive part of the second eigenfunction using 18512 partition sets

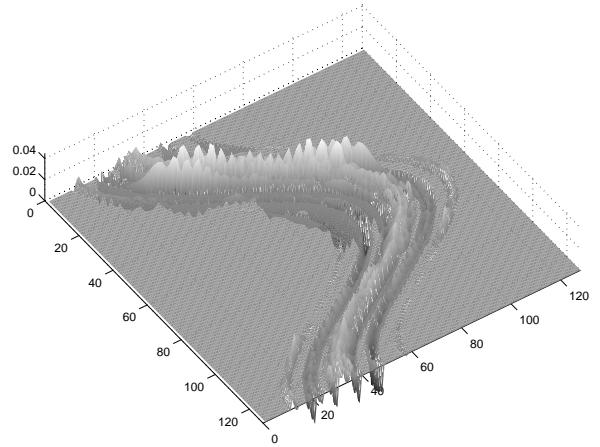


Figure 4: Negative part of the second eigenfunction using 18512 partition sets

5 Other uses

These approximation techniques have been successfully used in the approximation of Lyapunov exponents of random systems [6], and for the estimation of statistics of recurrence times to specified regions of phase space [5] (including extensions to infinite collections of maps $\{T_k\}$).

The evolution operator approach also allows considerable flexibility if the underlying stochastic process is altered while the collection of maps $\{T_k\}$ is retained, as the matrices $P_n(k)$, $k = 1, \dots, r$ need not be recomputed. A new matrix P_n or S_n can be quickly reconstructed allowing for rapid analyses under changing stochastic processes. Iterative orbit-based methods, in contrast, require new simulations for every new random driving process.

References

- [1] Michael Dellnitz, Gary Froyland, and Stefan Sertl. On the isolated spectrum of the Perron-Frobenius operator. Submitted to *Nonlinearity*, 1999.
- [2] Michael Dellnitz and Oliver Junge. On the approximation of complicated dynamical behavior. *SIAM Journal on Numerical Analysis*, 36(2):491–515, 1999.
- [3] Gary Froyland. Extracting dynamical behaviour via Markov models. In *Nonlinear dynamics and statistics*. Birkhäuser. Submitted.
- [4] Gary Froyland. Ulam’s method for random interval maps. *Nonlinearity*, 12(4):1029–1052, 1999.
- [5] Gary Froyland and Kazuyuki Aihara. Estimating the mean and variance of interspike intervals. In preparation.
- [6] Gary Froyland and Kazuyuki Aihara. Rigorous numerical estimation of Lyapunov exponents and invariant measures of iterated function systems and random matrix products. *Int. J. Bifur. Chaos*, 10, 2000. To appear.
- [7] John Guckenheimer and Philip Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, volume 42 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.
- [8] Yuri Kifer. *Random Perturbations of Dynamical Systems*, volume 16 of *Progress in Probability and Statistics*. Birkhäuser, Boston, 1988.
- [9] Andrzej Lasota and Michael C. Mackey. *Chaos, Fractals, and Noise. Stochastic Aspects of Dynamics*, volume 97 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1994.
- [10] Tien-Yien Li. Finite approximation for the Frobenius-Perron operator. A solution to Ulam’s conjecture. *Journal of Approximation Theory*, 17:177–186, 1976.
- [11] Walter M. Miller and Fern Y. Hunt. Approximation of attractors for finite dimensional maps by Ulam’s method. Preprint, May 1996.
- [12] S. Pelikan. Invariant densities for random maps of the interval. *Transactions of the American Mathematical Society*, 281(2):813–825, 1984.