MATH 110 –Spring 2016 — Homework 1 Solutions

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1.2 #1, 2, 7, 8, 10, 13, 17
1.3 #1, 3, 8, 12, 19, 25, 29

1.2

1. <u>Problem:</u> *Label the following statements as True or False*

(a) *Every vector space contains a zero vector.*

$\boxed{TRUE.}$ By the definition of the properties of vector sets, there must be a zero vector or else the set is not a vector set.

(b) *A vector space may have more than one zero vector.*

$\boxed{FALSE.}$ By contradiction, Suppose there are multiple values for zero, u and u'. $u+u' = u$ and $u'+u = u'$ therefore $u = u'$ and our assumption that there are more than one zero is false.

(c) *In any vector space, $ax = bx$ implies that $a = b$.*

$\boxed{FALSE.}$ When $x = \vec{0}$ then $ax = bx\ \forall a, b \in \mathbb{F}$

(d) *In any vector space, $ax = ay$ implies that $x = y$.*

$\boxed{FALSE.}$ If $a = 0$, then $ax = ay\ \forall x, y \in V$

(e) *A vector in $\mathbb{F}^n$ may be regarded as a matrix in $M_{n\times 1}(\mathbb{F})$.*

$\boxed{TRUE.}$ The single column of the matrix can be interpreted as a vector and vise versa because there is a one to one correspondence and both are members of a vector space.

(f) *An $m \times n$ matrix has m columns and n rows.*

$\boxed{FALSE.}$ It has m rows and n columns

(g) *In $P(\mathbb{F})$ only polynomials of the same degree may be added.*

$\boxed{FALSE.}$ Any degree polynomial may be added.

(h) *If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.*

$\boxed{FALSE.}$ Counter example: $f = -2x^n$ and $g = 2x^n$

(i) *If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n.*

$\boxed{TRUE.}$ f has term ax^n, where $a \neq 0$ so cf will have term $(c \cdot a)x^n$ where $ca \neq 0$

(j) *A nonzero scalar of $\mathbb{F}$ may be considered to be a polynomial in $P(\mathbb{F})$ having degree zero.*

$\boxed{TRUE.}$ Any polynomial, $f \in P(\mathbb{F})$, with degree 0 is written as $f = a_0$, where $a_0 \in \mathbb{F}$. Thus, any non-zero scalar $k \in \mathbb{F}$ maps to the zero-degree polynomial $f = k$ and any zero-degree polynomial $f = k' : k' \in \mathbb{F}$ maps to the non-zero scalar k' Establishing a bijective correspondence

(k) *Two functions in $F(S, \mathbb{F})$ are equal if and only if they have the same value at each element of S.*

$\boxed{TRUE.}$ That is the definition of equals.

2. Problem: *Write the zero vector of $M_{3\times 4}(\mathbb{F})$*

$$\begin{bmatrix} \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} \\ \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} \\ \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} \end{bmatrix}$$

7. Problem: *Let $S = \{0, 1\}$ and $\mathbb{F} = \mathbb{R}$. In $F(S, \mathbb{R})$, show that $f = g$ and $f + g = h$, where $f(t) = 2t + 1$, $g(t) = 1 + 4t - 2t^2$, and $h(t) = 5^t + 1$.*

Solution:
F is the set of all functions that map the set $\{0, 1\}$ to the set $\mathbb{R}$ So the functions $f, g, h \in F(S, \mathbb{R})$ given to us can be thought of as any function that is defined for the input values of 0 and 1 To show that $f = g$ we show that f and g map the values 0 and 1 to the same values in $\mathbb{R}$, respectively:

$$\begin{array}{ll} f(0) = 2 \cdot (0) + 1 = \mathbf{1}, & f(1) = 2 \cdot (1) + 1 = \mathbf{3} \\ g(0) = 1 + 4 \cdot (0) - 2 \cdot (0)^2 = \mathbf{1}, & g(1) = 1 + 4 \cdot (1) - 2 \cdot (1)^2 = \mathbf{3} \\ f(0) = g(0) \wedge f(1) = g(1) & \therefore \boxed{f(t) = g(t) \text{ in } F(S, \mathbb{R})} \end{array} \quad \blacksquare$$

Now to show that $f + g = h$ in $F(S, \mathbb{R})$, we repeat the same process as above for $(f + g)(t)$ and $h(t)$:

$$(f + g)(0) = 2, \quad h(0) = 2$$
$$(f + g)(1) = 6, \quad h(1) = 6$$

$$(f + g)(0) = h(0) \wedge (f + g)(1) = h(1) \quad \therefore \boxed{(f + g)(t) = h(t) \text{ in } F(S, \mathbb{R})} \quad \blacksquare$$

8. Problem: *In any vector space $\mathbf{V}$, show that, $(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in V$ and any $a, b \in \mathbb{F}$.*

<u>Solution:</u>
Starting with $(a+b)(x+y)$, we first let $z = x+y$; we know that $z \in \mathbf{V}$ by the 1st property of vector spaces. We can rewrite our equation as $(a+b)(z)$. But by **Axiom 8**, we can rewrite this as $az + bz$. Replacing z with our original relation, we transform our expression to $a(x+y)+b(x+y)$. And by **Axiom 7** this is the same as $ax+ay+bx+by$. Thus completing our proof. ■

10. <u>Problem:</u> *Let* $\mathbf{V}$ *denote the set of all differentiable real-valued functions defined on the real line. Prove that* $\mathbf{V}$ *is a vector space with the operations of addition and scalar multiplication defined in Example 3.*

<u>Solution:</u>
We know that the addition and scalar multiplication operations defined in Example 3 satisfy the 8 Axioms for Vector Spaces. So we just have to show that applying either operation on the vectors results in another vector also in $\mathbf{V}$.

For the addition property, we have that $(f+g)(x) = f(x)+g(x)$ and we know that $f(x)$ and $g(x)$ are both differntiable $\forall x \in \mathbb{R}$. For $(f+g)(x)$ to be in $\mathbf{V}$, it must also be differentiable on the real line and map $\mathbb{R} \mapsto \mathbb{R}$. But this is definitely the case because $(f+g)'(x) = f'(x)+g'(x)$ and we are guaranteed f' and g' exist and are in $\mathbb{R}$.

For multiplication, we have that $(cf)(x) = c \cdot f(x)$. And again, we have that $(cf)'(x) = c \cdot f'(x)$ which means, by the same justification as above, that $(cf)(x) \in \mathbf{V}$ as well. ■

13. <u>Problem:</u> Let $\mathbf{V}$ denote the set of ordered pairs of real numbers. If $(a1, a2)$ and $(b1, b2)$ are elements of $\mathbf{V}$ and $c \in \mathbb{R}$, define $(a1, a2) + (b1, b2) = (a1 + b1, a2b2)$ and $c(a1, a2) = (ca1, a2)$. Is $\mathbf{V}$ a vector space over $\mathbb{R}$ with these operations? Justify your answer.

<u>Solution:</u>
No, it is not a vector space because **Axiom 3** and **Axiom 4** contradict each other. **Axiom 3** states that: $\exists\, \vec{0} \in \mathbf{V} : \forall\, \vec{v} \in \mathbf{V}\ (\vec{v} + \vec{0} = \vec{v})$ which implies that the $\vec{0} = (0, 1)$. **Axiom 4** states that: $\forall \vec{x} \in \mathbf{V}, \exists \vec{y} \in \mathbf{V} : (\vec{x} + \vec{y} = \vec{0})$. However, with the vector $\vec{u} = (0, 0)$, there is no other vector $\vec{u'} \in \mathbf{V}$ such that $\vec{u} + \vec{u'} = \vec{0}$, becuase such a vector would have to satisfy the equation $u'_2 \cdot 0 = 1$ where $u'_2 \in \mathbb{R}$

17. <u>Problem:</u> *Let* $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{F}\}$, *where* $\mathbb{F}$ *is a field. Define addition of elements of* $\mathbf{V}$ *coordinatewise, and for* $c \in \mathbb{F}$ *and* $(a_1, a_2) \in \mathbf{V}$, *define* $c(a_1, a_2) = (a_1, 0)$. *Is* $\mathbf{V}$ *a vector space over* $\mathbb{F}$ *with these operations? Justify your answer.*

<u>Solution:</u>
$\boxed{NO}$. Because the multiplication operation violates **Axiom 5**. Which is that $1 \cdot x = x,\ \forall\, x \in \mathbf{V}$. Which would mean that $1 \cdot (a1, a2) = (a1, a2),\ \forall\ a_1, a_2 \in \mathbb{F}$ But in fact $1 \cdot (a1, a2) \neq (a1, a2)\ \forall\ a_2 \neq 0$

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