MATH 110 –Spring 2016 — Homework 1 Solutions

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1.2 #1, 2, 7, 8, 10, 13, 17
1.3 #1, 3, 8, 12, 19, 25, 29

1.2

1. _Problem:_ *Label the following statements as True or False*

(a) *Every vector space contains a zero vector.*

$\boxed{TRUE.}$ By the definition of the properties of vector spaces, there must be a zero vector or else the set is not a vector space.

(b) *A vector space may have more than one zero vector.*

$\boxed{FALSE.}$ By contradiction, Suppose there are multiple values for zero, u and u'. $u+u' = u$ and $u' + u = u'$ therefore $u = u'$ and our assumption that there are more than one zero is false.

(c) *In any vector space, $ax = bx$ implies that $a = b$.*

$\boxed{FALSE.}$ When $x = \vec{0}$ then $ax = bx \ \forall a, b \in \mathbb{F}$

(d) *In any vector space, $ax = ay$ implies that $x = y$.*

$\boxed{FALSE.}$ If $a = 0$, then $ax = ay \ \forall x, y \in V$

(e) *A vector in $\mathbb{F}^n$ may be regarded as a matrix in $M_{n\times1}(\mathbb{F})$.*

$\boxed{TRUE.}$ The single column of the matrix can be interpreted as a vector and vise versa because there is a one to one correspondence and both are members of a vector space.

(f) *An $m \times n$ matrix has m columns and n rows.*

$\boxed{FALSE.}$ It has m rows and n columns

(g) *In $P(\mathbb{F})$ only polynomials of the same degree may be added.*

$\boxed{FALSE.}$ Any degree polynomial may be added.

(h) *If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.*

$\boxed{FALSE.}$ Counter example: $f = -2x^n$ and $g = 2x^n$

(i) *If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n.*

$\boxed{TRUE.}$ f has term ax^n, where $a \neq 0$ so cf will have term $(c \cdot a)x^n$ where $ca \neq 0$

(j) *A nonzero scalar of $\mathbb{F}$ may be considered to be a polynomial in $P(\mathbb{F})$ having degree zero.*

$\boxed{TRUE.}$ Any polynomial, $f \in P(\mathbb{F})$, with degree 0 is written as $f = a_0$, where $a_0 \in \mathbb{F}$. Thus, any non-zero scalar $k \in \mathbb{F}$ maps to the zero-degree polynomial $f = k$ and any zero-degree polynomial $f = k' : k' \in \mathbb{F}$ maps to the non-zero scalar k' Establishing a bijective correspondence

(k) *Two functions in $F(S, \mathbb{F})$ are equal if and only if they have the same value at each element of S.*

$\boxed{TRUE.}$ That is the definition of equals.

2. Problem: *Write the zero vector of $M_{3\times4}(\mathbb{F})$*

$$\begin{bmatrix} \mathbf{0}_\mathbb{F} & \mathbf{0}_\mathbb{F} & \mathbf{0}_\mathbb{F} & \mathbf{0}_\mathbb{F} \\ \mathbf{0}_\mathbb{F} & \mathbf{0}_\mathbb{F} & \mathbf{0}_\mathbb{F} & \mathbf{0}_\mathbb{F} \\ \mathbf{0}_\mathbb{F} & \mathbf{0}_\mathbb{F} & \mathbf{0}_\mathbb{F} & \mathbf{0}_\mathbb{F} \end{bmatrix}$$

7. Problem: *Let $S = \{0, 1\}$ and $\mathbb{F} = \mathbb{R}$. In $F(S, \mathbb{R})$, show that $f = g$ and $f + g = h$, where $f(t) = 2t + 1$, $g(t) = 1 + 4t - 2t^2$, and $h(t) = 5^t + 1$.*

Solution:
F is the set of all functions that map the set $\{0, 1\}$ to the set $\mathbb{R}$ So the functions $f, g, h \in F(S, \mathbb{R})$ given to us can be thought of as any function that is defined for the input values of 0 and 1 To show that $f = g$ we show that f and g map the values 0 and 1 to the same values in $\mathbb{R}$, respectively:

$$\begin{aligned} &f(0) = 2 \cdot (0) + 1 = \mathbf{1}, && f(1) = 2 \cdot (1) + 1 = \mathbf{3} \\ &g(0) = 1 + 4 \cdot (0) - 2 \cdot (0)^2 = \mathbf{1}, && g(1) = 1 + 4 \cdot (1) - 2 \cdot (1)^2 = \mathbf{3} \\ &f(0) = g(0) \wedge f(1) = g(1) && \therefore \boxed{f(t) = g(t) \text{ in } F(S, \mathbb{R})} \end{aligned} \quad \blacksquare$$

Now to show that $f + g = h$ in $F(S, \mathbb{R})$, we repeat the same process as above for $(f + g)(t)$ and $h(t)$:

$$\begin{aligned} &(f + g)(0) = 2, \quad h(0) = 2 \\ &(f + g)(1) = 6, \quad h(1) = 6 \end{aligned}$$

$$(f + g)(0) = h(0) \wedge (f + g)(1) = h(1) \quad \therefore \boxed{(f + g)(t) = h(t) \text{ in } F(S, \mathbb{R})} \quad \blacksquare$$

8. Problem: *In any vector space $\mathbf{V}$, show that, $(a+b)(x+y) = ax+ay+bx+by$ for any $x, y \in V$ and any $a, b \in \mathbb{F}$.*

Solution:
Starting with $(a+b)(x+y)$, we first let $z = x+y$; we know that $z \in \mathbf{V}$ by the 1st property of vector spaces. We can rewrite our equation as $(a+b)(z)$. But by **Axiom 8**, we can rewrite this as $az + bz$. Replacing z with our original relation, we transform our expression to $a(x+y)+b(x+y)$. And by **Axiom 7** this is the same as $ax+ay+bx+by$. Thus completing our proof. ■

10. Problem: *Let* $\mathbf{V}$ *denote the set of all differentiable real-valued functions defined on the real line. Prove that* $\mathbf{V}$ *is a vector space with the operations of addition and scalar multiplication defined in Example 3.*

Solution:
We know that the addition and scalar multiplication operations defined in Example 3 satisfy the 8 Axioms for Vector Spaces. So we just have to show that applying either operation on the vectors results in another vector also in $\mathbf{V}$.

For the addition property, we have that $(f+g)(x) = f(x)+g(x)$ and we know that $f(x)$ and $g(x)$ are both differntiable $\forall x \in \mathbb{R}$. For $(f+g)(x)$ to be in $\mathbf{V}$, it must also be differentiable on the real line and map $\mathbb{R} \mapsto \mathbb{R}$. But this is definitely the case because $(f+g)'(x) = f'(x)+g'(x)$ and we are guaranteed f' and g' exist and are in $\mathbb{R}$.

For multiplication, we have that $(cf)(x) = c \cdot f(x)$. And again, we have that $(cf)'(x) = c \cdot f'(x)$ which means, by the same justification as above, that $(cf)(x) \in \mathbf{V}$ as well. ■

13. Problem: *Let* $\mathbf{V}$ *denote the set of ordered pairs of real numbers. If* $(a1, a2)$ *and* $(b1, b2)$ *are elements of* $\mathbf{V}$ *and* $c \in \mathbb{R}$*, define* $(a1, a2) + (b1, b2) = (a1 + b1, a2b2)$ *and* $c(a1, a2) = (ca1, a2)$*. Is* $\mathbf{V}$ *a vector space over* $\mathbb{R}$ *with these operations? Justify your answer.*

Solution:
No, it is not a vector space because **Axiom 3** and **Axiom 4** contradict each other. **Axiom 3** states that: $\exists\, \vec{0} \in \mathbf{V} : \forall\, \vec{v} \in \mathbf{V}\ (\vec{v} + \vec{0} = \vec{v})$ which implies that the $\vec{0} = (0, 1)$. **Axiom 4** states that: $\forall \vec{x} \in \mathbf{V}, \exists \vec{y} \in \mathbf{V} : (\vec{x} + \vec{y} = \vec{0})$. However, with the vector $\vec{u} = (0, 0)$, there is no other vector $\vec{u'} \in \mathbf{V}$ such that $\vec{u} + \vec{u'} = \vec{0}$, becuase such a vector would have to satisfy the equation $u'_2 \cdot 0 = 1$ where $u'_2 \in \mathbb{R}$

17. Problem: *Let* $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{F}\}$*, where* $\mathbb{F}$ *is a field. Define addition of elements of* $\mathbf{V}$ *coordinatewise, and for* $c \in \mathbb{F}$ *and* $(a_1, a_2) \in \mathbf{V}$*, define* $c(a_1, a_2) = (a_1, 0)$*. Is* $\mathbf{V}$ *a vector space over* $\mathbb{F}$ *with these operations? Justify your answer.*

Solution:
$\boxed{NO}$. Because the multiplication operation violates **Axiom 5**. Which is that $1 \cdot x = x,\ \forall\, x \in \mathbf{V}$. Which would mean that $1 \cdot (a1, a2) = (a1, a2),\ \forall\ a_1, a_2 \in \mathbb{F}$ But in fact $1 \cdot (a1, a2) \neq (a1, a2)\ \forall\ a_2 \neq 0$

1.3

1. Problem: *Label the following statements as true or false.*

(a) *If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.*

$\boxed{FALSE}$. Because the subset has to be closed under addition and scalar multiplication as well.

(b) *The empty set is a subspace of every vector space.*

$\boxed{FALSE}$. It needs a zero vector in the set, but there are no vectors in the set.

(c) *If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.*

$\boxed{TRUE}$. The zero-vector space is a subspace itself.

(d) *The intersection of any two subsets of V is a subspace of V.*

$\boxed{FALSE}$. The zero vector may not neccesarily be in the intersection of two arbitrary subsets.

(e) *An $n \times n$ diagonal matrix can never have more than n nonzero entries.*

$\boxed{TRUE}$. A diagonal matrix is by definition the matrix with all zeroes except the diagonal which may or may not be zero, so it can have at most n n nonzero entries.

(f) *The trace of a square matrix is the product of its diagonal entries.*

$\boxed{FALSE}$. It is the sum.

(g) *Let W be the $xy-plane$ in $\mathbb{R}^3$ that is, $W = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$. Then $W = \mathbb{R}^2$.*

$\boxed{FALSE}$. vectors in W are $\mathbb{R}^3$ which is not comparable with vectors in $\mathbb{R}^2$

3. Problem: *Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in M_{m\times n}(F)$ and any $a, b \in F$*

Solution:

$$(\mathbf{aA} + \mathbf{bB})^{\mathbf{T}} =$$

$$= \left(\begin{bmatrix} aA_{11} & aA_{12} & \dots & aA_{1n} \\ aA_{21} & aA_{22} & \dots & aA_{2n} \\ \vdots & \vdots & & \vdots \\ aA_{m1} & aA_{m2} & \dots & aA_{mn} \end{bmatrix} + \begin{bmatrix} bB_{11} & bB_{12} & \dots & bB_{1n} \\ bB_{21} & bB_{22} & \dots & bB_{2n} \\ \vdots & \vdots & & \vdots \\ bB_{m1} & bB_{m2} & \dots & bB_{mn} \end{bmatrix} \right)^T$$

$$= \begin{bmatrix} aA_{11} + bB_{11} & aA_{12} + bB_{12} & \dots & aA_{1n} + bB_{1n} \\ aA_{21} + bB_{21} & aA_{22} + bB_{22} & \dots & aA_{2n} + bB_{2n} \\ \vdots & \vdots & & \vdots \\ aA_{m1} + bB_{m1} & aA_{m2} + bB_{m2} & \dots & aA_{mn} + bB_{mn} \end{bmatrix}^T$$

$$= \begin{bmatrix} aA_{11}+bB_{11} & aA_{21}+bB_{21} & \dots & aA_{n1}+bB_{n1} \\ aA_{12}+bB_{12} & aA_{22}+bB_{22} & \dots & aA_{n2}+bB_{n2} \\ \vdots & \vdots & & \vdots \\ aA_{1m}+bB_{1m} & aA_{2m}+bB_{2m} & \dots & aA_{nm}+bB_{nm} \end{bmatrix}$$

$$= \begin{bmatrix} aA_{11} & aA_{21} & \dots & aA_{m1} \\ aA_{12} & aA_{22} & \dots & aA_{m2} \\ \vdots & \vdots & & \vdots \\ aA_{1n} & aA_{2m} & \dots & aA_{nm} \end{bmatrix} + \begin{bmatrix} bB_{11} & bB_{21} & \dots & bB_{m1} \\ bB_{12} & bB_{22} & \dots & bB_{m2} \\ \vdots & \vdots & & \vdots \\ bB_{1n} & bB_{2n} & \dots & bB_{nm} \end{bmatrix}$$

$$= \mathbf{aA^T} + \mathbf{bB^T} \quad \blacksquare$$

8. Problem *Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.*

(a) $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$

$\boxed{\text{YES, it is a subspace}}$. Because the variables can be thought of as coordinates in 3D and with the given constraints, give us a line in 3 dimension, which is indeed a subspace of $\mathbb{R}$.

(b) $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$

$\boxed{\text{YES, it's a subspace}}$. Because this is a plane in 3 dimensions, which is a subspace.

(c) $W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$

$\boxed{\text{YES, it's a subspaces}}$. Again, this is just a plane in $\mathbb{R}^3$, thus it is a subspace.

(d) $W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$.

$\boxed{\text{YES. a subspace}}$ Again this is a plane, so the vectors belonging to this set will form a subspace.

(e) $W_5 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$

$\boxed{\text{NO, Not a subspace}}$. Becuase the zero vector is not in the set.

(f) $W_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a2^2 + 6a_3^2 = 0\}$

$\boxed{NO}$. This is a cone in 3 dimension, so you could have two vectors along the side that, for example $\nwarrow$ and $\nearrow$ that sums to $\uparrow$, but $\vec{v} = (\uparrow) \notin W_6$

12. Problem: *An $m \times n$ matrix A is called upper triangular if all entries lying below the diagonal entries are zero, that is, if $A_{ij} = 0$ whenever $i > j$. Prove that the upper triangular matrices form a subspace of $M_{m\times n}(F)$.*

Solution:
Let's denote the subset of upper triangular matrices in $M_{m\times n}$ as M_{UT}. we know that $\vec{0} \in M_{UT}$ since the matrix with all zeroes is upper triangular. Let A and B be matrices in M_{UT}, $A+B$ is also guranteed to be upper triangular, becuase $0_\mathbb{F} + 0_\mathbb{F} = 0_\mathbb{F}$ and thus in M_{UT} as well. Scalar multiplication holds as well because any $c \in \mathbb{F}$ times $0_\mathbb{F} = 0_\mathbb{F}$. And the axiomss for

the operations hold as well because we are dealing with a subset of an already proven Vector Space. ■

19. <u>Problem:</u> *Let W_1 and W_2 be subspaces of a vector space $\boldsymbol{V}$ and $w_1, w_2, ..., w_n$ are in W. Prove that $W_1 \cup W_2$ is a subspace of V i.f.f $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$*

<u>Solution:</u>
We will first proove the back direction, that is: $W_1 \subseteq W_2$ or $W_2 \subseteq W_1 \implies W_1 \cup W_2$ is a subspace of $\mathbf{V}$. There are two cases: if $W_1 \subseteq W_2$, then $W_1 \cup W_2 = W_2$ or if $W_2 \subseteq W_1$, in which case $W_1 \cup W_2 = W_1$ and we are given that W_1 and $W_2$are subspaces, so we're done. ■

To complete the whole proof, we now just have to proove that $W_1 \cup W_2$ is a subspace of $\mathbf{V} \implies W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. For contradiction, assume that the latter condition does not hold, that means that there must exist some element $\vec{w} \in W_1$ and some element $\vec{w\prime}$ such that $\vec{w} \notin W_2$ & $\vec{w\prime} \notin W_1$. Under the addition property we know that $(\vec{w} + \vec{w\prime}) \in W_1 \cup W_2$. Which means $(\vec{w} + \vec{w\prime})$ either in W_1 or W_2 or both. But $\vec{w} + \vec{w\prime} \notin W_1$, because that would imply $\vec{w\prime} \in W_1$ by the closure property, and similarly $\vec{w} + \vec{w\prime} \notin W_2$ by the same reasoning. CONTRADICTION! Our initial assumption must have been incorrect $\therefore$ $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$ must hold, thus completing the proof. ■

25. <u>Problem:</u> *Let W_1 denote the set of all polynomials $f(x) \in P(\mathbb{F})$ such that $f(x)$ only consists of terms with even degree on the x term. And let W_2 denote the set of all polynomials $g(x) \in P(\mathbb{F})$ such that $g(x)$ only consists of terms with odd degree on the x term. Proove that $P(\mathbb{F}) = W_1 \bigoplus W_2$*

<u>Solution:</u>
We have to show that **(1)** $W_1 \cap W_2 = \{0\}$ and that **(2)** $W_1 + W_2 = P(\mathbb{F})$.

(1) A polynomial that satisfies the properties of both W_1 and W_2 has coefficients on both even and odd degree terms equal to 0. This means that the only polynomial that satisfies both conditions is the 0 polynomial, thus $W_1 \cap W_2 = \{0\}$ ■

(2) Any $f(x) \in W_1$ can be written as: $a_0x^0 + a_2x^2 + a_4x^4 + ... + a_{2k}x^{2k}$ and likewise any $g(x) \in W_2$ can be written as : $b_1x^1 + b_3x^3 + b_5x^5 + ... + b_{2k+1}x^{2k+1}$. And we know that any polynomial $h(x) \in P(\mathbb{F})$ can be written as: $c_0x^0 + c_1x^1 + c_2x^2 + c_3x^3 + c_4x^4 + ... + c_nx^n$. So for the arbitrary polynomial, we just need to set every $a_i = c_i$, if i is even and every $b_i = c_i$, if i is odd, and we get that $f(x) + g(x) = h(x)$ ■

29. <u>Problem:</u> *Let F be a field that is not of characteristic 2. Define:*

$$W_1 = \{A \in M_{n\times n}(\mathbb{F}) : A_{ij} = 0 \text{ whenever } i \leq j\}$$

and W_2 to be the set of all symmetric$n \times n$ matrices with entries from $\mathbb{F}$. Both W_1 and W_2 are subspaces of $M_{n\times n}(\mathbb{F})$. Prove that $M_{n\times n}(\mathbb{F}) = W_1 \bigoplus W_2$.

<u>Solution:</u>
Again, we must show that **(1)** $W_1 \cap W_2 = \{0\}$ and that **(2)** $W_1 + W_2 = P(\mathbb{F})$.

(1) A polynomial that satisfies the properties of both W_1 and W_2 must be the zero matrix because a lower triangular mtrix which is symmetric will neccesarily mean that every element is 0 obviously.

(2) For any matrix $A \in M_{n\times n}$ we can construct $B \in W_1$ and $D \in W_2$ such that $A = B + D$ like so. $\forall i, j : i \geq j$ set $D_{ij} := A_{ij}$. This also defines the lower half of matrix D by symmetry. Now, $\forall i, j : i < j$ set $B_{ij} := A_{ij} – D_{ij}$ and this completely defines matrix D ∎