

# MATH 110 –Spring 2016 — Homework 1 Solutions

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**1.2** #1, 2, 7, 8, 10, 13, 17

**1.3** #1, 3, 8, 12, 19, 25, 29

## 1.2

1. Problem: Label the following statements as True or False

(a) *Every vector space contains a zero vector.*

**TRUE.** By the definition of the properties of vector sets, there must be a zero vector or else the set is not a vector set.

(b) *A vector space may have more than one zero vector.*

**FALSE.** By contradiction, Suppose there are multiple values for zero,  $u$  and  $u'$ .  $u + u' = u$  and  $u' + u = u'$  therefore  $u = u'$  and our assumption that there are more than one zero is false.

(c) *In any vector space,  $ax = bx$  implies that  $a = b$ .*

**FALSE.** When  $x = \vec{0}$  then  $ax = bx \forall a, b \in \mathbb{F}$

(d) *In any vector space,  $ax = ay$  implies that  $x = y$ .*

**FALSE.** If  $a = 0$ , then  $ax = ay \forall x, y \in V$

(e) *A vector in  $\mathbb{F}^n$  may be regarded as a matrix in  $M_{n \times 1}(\mathbb{F})$ .*

**TRUE.** The single column of the matrix can be interpreted as a vector and vice versa because there is a one to one correspondence and both are members of a vector space.

(f) *An  $m \times n$  matrix has  $m$  columns and  $n$  rows.*

**FALSE.** It has  $m$  rows and  $n$  columns

(g) *In  $P(\mathbb{F})$  only polynomials of the same degree may be added.*

**FALSE.** Any degree polynomial may be added.

(h) *If  $f$  and  $g$  are polynomials of degree  $n$ , then  $f + g$  is a polynomial of degree  $n$ .*

**FALSE.** Counter example:  $f = -2x^n$  and  $g = 2x^n$

- (i) If  $f$  is a polynomial of degree  $n$  and  $c$  is a nonzero scalar, then  $cf$  is a polynomial of degree  $n$ .

TRUE.  $f$  has term  $ax^n$ , where  $a \neq 0$  so  $cf$  will have term  $(c \cdot a)x^n$  where  $ca \neq 0$

- (j) A nonzero scalar of  $\mathbb{F}$  may be considered to be a polynomial in  $P(\mathbb{F})$  having degree zero.

TRUE. Any polynomial,  $f \in P(\mathbb{F})$ , with degree 0 is written as  $f = a_0$ , where  $a_0 \in \mathbb{F}$ . Thus, any non-zero scalar  $k \in \mathbb{F}$  maps to the zero-degree polynomial  $f = k$  and any zero-degree polynomial  $f = k' : k' \in \mathbb{F}$  maps to the non-zero scalar  $k'$ . Establishing a bijective correspondence

- (k) Two functions in  $F(S, \mathbb{F})$  are equal if and only if they have the same value at each element of  $S$ .

TRUE. That is the definition of equals.

2. Problem: Write the zero vector of  $M_{3 \times 4}(\mathbb{F})$

$$\begin{bmatrix} \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} \\ \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} \\ \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} & \mathbf{0}_{\mathbb{F}} \end{bmatrix}$$

7. Problem: Let  $S = \{0, 1\}$  and  $\mathbb{F} = \mathbb{R}$ . In  $F(S, \mathbb{R})$ , show that  $f = g$  and  $f + g = h$ , where  $f(t) = 2t + 1$ ,  $g(t) = 1 + 4t - 2t^2$ , and  $h(t) = 5t + 1$ .

Solution:

$F$  is the set of all functions that map the set  $\{0, 1\}$  to the set  $\mathbb{R}$ . So the functions  $f, g, h \in F(S, \mathbb{R})$  given to us can be thought of as any function that is defined for the input values of 0 and 1. To show that  $f = g$  we show that  $f$  and  $g$  map the values 0 and 1 to the same values in  $\mathbb{R}$ , respectively:

$$\begin{aligned} f(0) &= 2 \cdot (0) + 1 = \mathbf{1}, & f(1) &= 2 \cdot (1) + 1 = \mathbf{3} \\ g(0) &= 1 + 4 \cdot (0) - 2 \cdot (0)^2 = \mathbf{1}, & g(1) &= 1 + 4 \cdot (1) - 2 \cdot (1)^2 = \mathbf{3} \\ f(0) &= g(0) \wedge f(1) = g(1) & \therefore & \boxed{f(t) = g(t) \text{ in } F(S, \mathbb{R})} \quad \blacksquare \end{aligned}$$

Now to show that  $f + g = h$  in  $F(S, \mathbb{R})$ , we repeat the same process as above for  $(f + g)(t)$  and  $h(t)$ :

$$\begin{aligned} (f + g)(0) &= 2, & h(0) &= 2 \\ (f + g)(1) &= 6, & h(1) &= 6 \\ (f + g)(0) &= h(0) \wedge (f + g)(1) = h(1) & \therefore & \boxed{(f + g)(t) = h(t) \text{ in } F(S, \mathbb{R})} \quad \blacksquare \end{aligned}$$

8. Problem: In any vector space  $\mathbf{V}$ , show that,  $(a + b)(x + y) = ax + ay + bx + by$  for any  $x, y \in V$  and any  $a, b \in \mathbb{F}$ .

Solution:

Starting with  $(a + b)(x + y)$ , we first let  $z = x + y$ ; we know that  $z \in \mathbf{V}$  by the 1st property of vector spaces. We can rewrite our equation as  $(a + b)(z)$ . But by **Axiom 8**, we can rewrite this as  $az + bz$ . Replacing  $z$  with our original relation, we transform our expression to  $a(x + y) + b(x + y)$ . And by **Axiom 7** this is the same as  $ax + ay + bx + by$ . Thus completing our proof. ■

10. Problem: Let  $\mathbf{V}$  denote the set of all differentiable real-valued functions defined on the real line. Prove that  $\mathbf{V}$  is a vector space with the operations of addition and scalar multiplication defined in Example 3.

Solution:

Let  $f$  be a function  $\in \mathbf{V}$

13. Problem: Let  $\mathbf{V}$  denote the set of ordered pairs of real numbers. If  $(a_1, a_2)$  and  $(b_1, b_2)$  are elements of  $\mathbf{V}$  and  $c \in \mathbb{R}$ , define  $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$  and  $c(a_1, a_2) = (ca_1, a_2)$ . Is  $\mathbf{V}$  a vector space over  $\mathbb{R}$  with these operations? Justify your answer.

17. Problem:

## 1.3

1.

3.

8.

12.

19.

25.

29.