PROGRAM "EM1DTM"

Version 1.0 a, 12 March 2009.

Contents

FS. Input & Output Files

FS.1 Input files

FS.1.1 Main input file (Required, and called "`em1dtm.in`")

This is the main input file containing the parameters specifying the names of the file containing the observations and the files containing the starting and reference models, the type of the inversion algorithm, and the various parameters for the inversion and for the forward modelling.

The structure of the file "`em1dtm.in`" is:

rootname	← line 1: root for names of output files;
obsfname	← line 2: name of file containing the observations;
stconfname	← line 3: starting conductivity model file, or layer thicknesses file;
rsconfname	← line 4: reference (smallest) conductivity model file;
rzconfname	← line 5: reference (flattest) conductivity model file;
`NONE`	← line 6: information about additional model weights;
hc eps ees epz eez	← line 7: parameters for Huber & Ekblom measures;
acs acz	← line 8: coefficients of model norm components;
iatype	← line 9: type of inversion algorithm;
iapara(s)	← line 10: additional inversion algorithm parameter(s);
maxniters	← line 11: maximum number of iterations in an inversion;
`DEFAULT`	← line 12: small number for convergence tests;
`DEFAULT`	← line 13: number of explicit evaluations of Hankel transform kernels;
`DEFAULT`	← line 14: information for explicit evaluations of Fourier transform kernels;
outflag	← line 15: flag indicating amount of output.

where, on ...

line 1, *rootname* is the root for the names of all output files (character string of length less than or equal to 20 characters);

line 2, *obsfname* is the name of the file containing the observations (see section FS.1.2) (character string of length less that or equal to 99 characters);

line 3, *stconfname* is the name of the file containing the starting conductivity model, or the name of the file containing the layer thicknesses if best-fitting halfspaces are to be used as the starting models (see section FS.1.3 for the format of this file), required (character string of length less than or equal to 99 characters);

line 4, *rsconfname* is the name of the file containing the reference conductivity model for the smallest component of the model norm (see section FS.1.5), required if $asc > 0$ (if $asc = 0$, "`NONE`" can be entered on this line) (character string of length less than or equal to 99 characters), or "`DEFAULT`" if the best-fitting homogeneous halfspace is to be used, or the conductivity of thte homogeneous halfspace to use;

line 5, *rzconfname* is the name of the file containing the reference conductivity model for the flattest component of the model norm (see section FS.1.5), completely optional – if such a model is supplied in a file whose name is given here then it will be used in the inversion, otherwise, if "`NONE`" is given on this line of the input file, there will be no reference conductivity model in the flattest component of the model norm (character string of length less than or equal to 99 characters if a file name is being given, or the string "`NONE`" if no reference model is being supplied), or "`DEFAULT`" if the best-fitting homogeneous halfspace is to be used, or the value of the halfspace to be used;

line 6, either "`NONE`" is specified to indicate that no additional user-supplied weights are to be provided for use in the model norm, or the name of the file containing the additional weighting (see section FS.1.9 for the format of this file) (character string of length less than or equal to 99 characters);

line 7, the parameters *hc*, *eps*, *ees*, *epz* and *eez*, where *hc* is the parameter c in the Huber measure for the misfit:

$$\rho(x_j) \;=\; \begin{cases} x_j^2 & |x_j| \le c, \\ 2c|x_j| - c^2 & |x_j| > c, \end{cases}$$

and *eps* and *ees* are the parameters p and ε in Ekblom's measure for the smallest component of the model norm:

$$\rho(x_j) \;=\; \big(x_j^2+\varepsilon^2\big)^{p/2},$$

and *epz* and *eez* are the parameters p and ε in Ekblom's measure for the flattest component of the model norm;

line 8, the two parameters *acs* and *acz*, where the value of *acs* is α_s in the expression for the model norm below, and the value of *acz* is α_z:

$$\phi_m \;=\; \alpha_s\,\phi_s\big(W_s(m-m_s^{\rm ref})\big)+\alpha_z\,\phi_z\big(W_z(m-m_z^{\rm ref})\big)$$

(the two parameters that are read in are real numbers greater than or equal to zero);

line 9, *iatype* indicates the type of inversion algorithm to be used, *iatype* = 1 implies a fixed, user-supplied value for the trade-off parameter, *iatype* = 2 implies that the trade-off parameter will be chosen by means of a line search so that a target misfit is achieved (or, if this is not possible, then the smallest misfit), *iatype*= 3 implies the trade-off parameter will be chosen using the GCV criterion, and *iatype*= 4 implies that the trade-off parameter will be chosen using the L-curve criterion;

line 10, if *iatype* = 1, the value of the trade-off parameter is expected (and optionally the starting value of the trade-off parameter, and the factor by which the trade-off parameter is to be decreased at each iteration, down to the specified value), or if *iatype* = 2, the target misfit (in terms of the factor *chifac* where the target misfit is *chifac* times the total number of observations for the sounding) and the greatest allowed decrease in the misfit (in terms of *decr* where $\phi_d^{n+1,\rm tar} = \max(\mathit{chifac}\times N, \mathit{decr}\times\phi_d^n))$ at any one iteration (and optionally the starting value of the trade-off parameter) are expected , or if *iatype* =3 or 4, the greatest allowed decrease in the trade-off parameter at any one iteration (in terms of *decr* where $\min(\beta^{n+1}) = \mathit{decr}\times\beta^n$) (and optionally the starting value of the trade-off parameter);

line 11, *maxniters* is the maximum number of iterations to be carried out in an inversion (a strictly positive integer);

line 12, either "`DEFAULT`" can be entered to indicate that the default value of 10^{-4} is to be used in the tests of convergence for an inversion, or, if another value is desired, it can be entered on this line (a strictly positive real number);

line 13, either "`DEFAULT`" can be entered to indicate the kernel of the Hankel transforms is to be explicitly evaluated the default number of times (=50), or, if there are concerns about the accuracy of the Hankel transform computations, a number greater than 50 can be entered on this line;

line 14, either "`DEFAULT`" to indicate the kernel of the Fourier transforms is to be explicitly evaluated at the default number of frequencies (=50), or, if there are concerns about the accuracy of the Fourier transform computations, a number greater than 50 can be entered on this line, or the number of frequencies and the minimum and maximum frequencies can be supplied;

line 15, *outflg* is the flag indicating the amount of output from the program (*outflg*= 1 implies the output of a brief convergence/termination report for each sounding plus the final two-dimensional composite model for all the soundings and the corresponding forward-modelled data, *outflg*= 2 implies the aforementioned output plus the final one-dimensional model and corresponding forward-modelled data for each sounding, *outflg* = 3 implies the aforementioned output plus the values of the various components of the objective function at each iteration in the inversion for each sounding, and *outflg* = 4 implies the aforementioned output plus an additional diagnostics file for each sounding which records the progress of the inversion for the sounding, a record of misfit, GCV function or L-curve curvature versus trade-off parameter, and a diagnostics file for the LSQR solution routine if it is used).

FS.1.2 Observations file (Required)

The file that contains the observations and all the survey parameters (except for any user-supplied transmitter current waveform):

- the number of soundings;
- the (absolute) x- and y-coordinates and elevation of each sounding (and any other information that is to pass throught EM1DTM to the output files, such as a fiducial and/or line number);

– the number of segments in each transmitter loop, the (relative) x- & y-coordinates of the start of each segment, the z-coordinate of the plane of the transmitter loop;
– the name of the file containing the transmitter current waveform information (flag indicating what the transmitter current waveform is, and relevant parameters such as current as a function of time);
– the number of receivers, and units for all times;
– the dipole moment of each receiver, the (relative) x-, y- & (abssolute) z-coordinates, and the orientation of the receiver, the number of measurement times, flag for units of time, flag for units/normalization of data;
– time, sweep index, datum, flag for type of uncertainty, uncertainty.

The structure of the observations file is:

nsounds ← line A;
soundx_a(i_s) *soundy_a*(i_s) *soundz_a*(i_s) ← line B;
ntsegs_a(i_s) (*xt_a*(j, i_s), *yt_a*(j, i_s), $j = 1, \ldots,$ *ntsegs_a*(i_s)) *zt_a*(i_s) ← line C;
tcwfn_a(i_s) ← line D;
nr_a(i_s) *tu_a*(i_s) ← line E;
momr_a(i_r,i_s) *xr_a*(i_r,i_s) *yr_a*(i_r,i_s) *zr_a*(i_r,i_s) *or_a*(i_r,i_s) *nt_a*(i_r,i_s) …
… *ontype_a*(i_r,i_s) ← line F;

t_a(i_t,i_r,i_s) *tf_a*(i_t,i_r,i_s) *obs_a*(i_t,i_r,i_s) *utype_a*(i_t,i_r,i_s) *uncert_a*(i_t,i_r,i_s) ← line Ga,
OR
t1_a(i_t,i_r,i_s) *t2_a*(i_t,i_r,i_s) *tf_a*(i_t,i_r,i_s) *obs_a*(i_t,i_r,i_s) *utype_a*(i_t,i_r,i_s) …
… *uncert_a*(i_t,i_r,i_s) ← line Gb.

where:

nsounds is the number of soundings;

soundx_a(i_s) is the x-coordinate (m) of the i_sth sounding;

soundy_a(i_s) is the y-coordinate (m) of the i_sth sounding;

soundz_a(i_s) is the elevation (m) above sea level (or some other datum) of the i_sth sounding;

ntsegs_a(i_s) is the number of linear segments in the transmitter loop for the i_sth sounding (i.e., one transmitter per sounding, but can be different for different soundings);

xt_a(j, i_s) the (relative) x-coordinate (m) of the start of the jth segment of the i_sth transmitter loop (the end of the jth segment is assumed to be joined to the start of the $(j + 1)$-th segment, with the end of the *ntsegs*-th segment joined to the start of the first segment);

yt_a(j, i_s) the (relative) y-coordinate (m) of the start of the jth segment of the transmitter loop for the i_sth sounding;

zt_a(i_s) the z-coordinate (m) of the plane (which is assumed to be horizontal) of the transmitter loop for the i_sth sounding (with z positive downwards, $z = 0$ being at the Earth's surface, and *zt_a*≤ 0 for all soundings);

tcwfn_a(i_s) the name of the file with the transmitter current waveform information (either the code and required parameters for specific waveforms, or the current as a function of time) for the i_sth sounding;

nr_a(i_s) the number of receivers for the i_sth sounding, with each different component being considered to be from a distinct receiver;

tu_a(i_s) a flag to indicate the units of the measurement times for the i_sth sounding (= 1 implies microseconds, = 2 means milliseconds, = 3 implies seconds);

momr_a(i_r,i_s) the dipole moment ($\mathrm{A\,m^2}$) of the i_rth receiver for the i_sth sounding;

xr_a(i_r,i_s) the (relative) x-coordinate (m) of the i_rth receiver for the i_sth sounding;

yr_a(i_r,i_s) the (relative) y-coordinate (m) of the i_rth receiver for the i_sth sounding;

zr_a(i_r,i_s) the (absolute) z-coordinate (m) of the i_rth receiver for the i_sth sounding;

or_a(i_r,i_s) the orientation (x, y, or z) of the i_rth receiver for the i_sth sounding (and something else if extra data types such as a total field amplitude are to be considered);

nt_a(i_r,i_s) the number of measurement times for the i_rth receiver for the i_sth sounding;

ontype_a(i_r,i_s) some sort of info about normalisation/scaling/units of the observations (for the i_t) for the i_r receiver for the i_s sounding (= 1 implies voltages in microVolts, = 2 implies voltages in milliVolts, = 3 means voltages in Volts, = 4 implies B-field in nano-Tesla, = 5 implies B-field in micro-Tesla, = 6 implies B-field in milli-Tesla);

t_a(i_t,i_r,i_s) the i_tth measurement time (in the units indicated by *tu_a*(i_r,i_s)) for the i_rth receiver for the i_sth sounding;

t1_a(i_t,i_r,i_s) the start time for the window for the i_tth measurement (in the units indicated by *tu_a*(i_r,i_s)) for the i_rth receiver for the i_sth sounding;

t2_a(i_t,i_r,i_s) the end time for the window for the i_tth measurement (in the units indicated by *tu_a*(i_r,i_s)) for the i_rth receiver for the i_sth sounding;

tf_a(i_t,i_r,i_s) some extra info such as to which sweep the i_tth time for the i_rth receiver for the i_sth sounding belongs;

obs_a(i_t,i_r,i_s) the observed data at the i_tth measurement time for the i_rth receiver for the i_sth sounding in the units specified by *ontype_a*(i_r,i_s);

utype_a(i_t,i_r,i_s) the form of the uncertainty (relative in % or absolute in the same units as the observations) in the i_tth observation for the i_rth receiver for the i_sth sounding;

uncert_a(i_t,i_r,i_s) the uncertainty in the i_tth observation for the i_rth receiver for the i_sth sounding.

This structure of the observations file is designed to be as general as possible, enabling the program to handle any conceivable survey configuration. The lines in the file form a series of nested loops. Repetition occurs over the indices: $i_t = 1, \ldots, nt\_a(i_r,i_s)$; $i_r = 1, \ldots, nr\_a(i_s)$; and $i_s = 1, \ldots, nsounds$. In other words, line G is repeated for each time for each receiver for each particular sounding; line F and the associated line(s) G are repeated for each receiver for each sounding; lines B, C, D & E, and the associated line(s) F and line(s) G are repeated for each sounding. All the information for a particular measurement time is expected on the same single line in the file.

For example:

FS.1.3 Starting conductivity model file (Required)

The file containing the starting conductivity model for all soundings, or, if (crude) best-fitting halfspace(s) are to be used as the starting model(s), then this is the file containing the number of layers and the layer thicknesses. If this file is the starting model, the relevant quantities are the number of layers, and the thickness (m) and conductivity (S/m) of each layer. A dummy value for the thickness of the basement halfspace is required in this file, but nothing is ever done with it after it is read in. It is from this file that the program gets the number of layers and their thicknesses, which then must be the same for all other models read in by the program. This file is therefore required. The structure of this file is as follows (just as for all 1d model files):

nlayers
thicks_a(1) *con_a*(1)
thicks_a(2) *con_a*(2)
⋮
thicks_a(*nlayers*−1) *con_a*(*nlayers*−1)
`0.` *con_a*(*nlayers*)

where *nlayers* is the number of layers in the model, $thicks\_a(j)$ is the thickness in metres of the jth layer and $con\_a(j)$ is the conductivity in S/m of the jth layer, or:

$$\begin{array}{l} nlayers \\ thicks\_a(1) \\ thicks\_a(2) \\ \quad\vdots \\ thicks\_a(nlayers-1) \\ \quad \texttt{0.} \end{array}$$

if best-fitting halfspace(s) are to be used as the starting model(s).

FS.1.4 File for reference conductivity model (for smallest model component) (Optional)

The file containing the reference conductivity model for the smallest component of the model norm if one is required for the inversion. This file is in the same format as the starting conductivity model file (see section FS.1.3), and must have the same number of layers with exactly the same thicknesses as the starting model.

FS.1.5 File for reference conductivity model (for flattest model component) (Optional)

The file containing the reference conductivity model for the flattest component of the model norm if one is required for the inversion. Whether or not this file is specified in "`em1dtm.in`" determines whether or not such a reference model plays a part in the inversion. This file is in the same format as the starting conductivity model file (see section FS.1.3), and must have the same number of layers with exactly the same thicknesses as the starting model.

FS.1.6 File for additional model-norm weights (Optional)

The file containing any additional weighting of the layers for one or both of the two components of the model norm. The first line of this file must contain the number of layers in the model. The order of the two possibilities must be the same as shown below, with any set of weights that is not needed by the program simply omitted.

$$\begin{array}{l} nlayers \\ (uswcs\_a(j), \quad j = 1, \ldots, nlayers) \\ (uswcz\_a(j), \quad j = 1, \ldots, nlayers-1) \end{array}$$

where *nlayers* is the number of layers in the model, $uswcs\_a(j)$ is the weight for the jth layer in the smallest component of the model norm, and $uswcz\_a(j)$ is the weight for the difference between the jth and $(j+1)$th layers in the flattest component of the model norm. The supplied weights must be greater than zero. A weight greater than one increases the weight relative to the default setting, and a weight less than one decreases the weight relative to the default setting.

FS.1.7 Transmitter current waveform for each/all transmitters (Required)

The file containing the particulars of the transmitter current waveform for each sounding. There can be one file for each sounding (one transmitter per "sounding"), or one file for all soundings if they all have the same particulars, or any mix 'n match. The first line of a transmitter current waveform file conatins either the code for a special waveform (followed by the necessary parameters), or the number of times in a discretized, user-supplied waveform. If it's a discretized waveform, the subsequent lines in the file contain the times (in the same units as the measurement times for the sounding: see tu_a) and values of current (in Ampères) at the discrete points at which the waveform is supplied.

If the special waveform is a *step*, the code on the first line of the file is "ste" (or "STE"). If the effects of previous negative & positive step-offs are to be taken into account, then the number of previous step-offs to take into account and the time (in the same units as for the observation times: see *tu_a*) are expected on this line. (If one previous step-off is asked for, it's the negative one (i.e., of opposite sense to the main, first one) at the specified time before. If two previous step-offs are asked for, it's the negative one at the specified time before, and the positive one at twice the specified time. Etc.)

If the special waveform is a *linear ramp turn-off*, the code on the first line of the file is "ram" (or "RAM"). Also on this first line of the file, the number of different ramp turn-off times (between 1 & 6 inclusive), and the values of the ramp turn-off times (in the same units as for the observations times: see *tu_a*) are expected. For this waveform, all observations times are measured from the end of the ramp.

FS.1.8 File for frequency-domain filter (Optional)

The file containing the frequency-domain filter, if one is to be applied. This file has to be called "`freqfilter`". If it exists, it will be read in: if not, then no frequency-domain filter will be applied. The first line contains the number of points in the filter (which must be greater than 4). The other lines in this file each contain the frequency in Hertz, and the real and imaginary values of the filter function at each point that it is specified.

$$
\begin{array}{lll}
\textit{nfltrpts} & & \\
\textit{freq\_a}(1) & \textit{rfltr\_a}(1) & \textit{ifltr\_a}(1) \\
\textit{freq\_a}(2) & \textit{rfltr\_a}(2) & \textit{ifltr\_a}(2) \\
\vdots & \vdots & \vdots \\
\textit{freq\_a}(\textit{nfltrpts}) & \textit{rfltr\_a}(\textit{nfltrpts}) & \textit{ifltr\_a}(\textit{nfltrpts})
\end{array}
$$

Appendices

A. Mathematics for Forward-Modelling Computations

This appendix covers the aspects of the forward-modelling procedure for the three components of the magnetic field above a horizontally-layered Earth model for a horizontal, many-sided transmitter loop also above the surface of the model that are not covered in Appendix A of the notes for program `EM1DFM`. The field for the transmitter loop is computed by the superposition of the fields due to horizontal electric dipoles (see eqs. 4.134–4.152 of Ward & Hohmann). Because the loop is closed, the contributions from the ends of the electric dipole are ignored, and the superposition is carried out only over the "TE"-mode component. This TE-mode only involves the z-component of the Schelkunoff $\mathbf{F}$ potential, just as for program `EM1DFM`. The propagation of F through the stack of layers therefore happens in exactly the same way, and so is not repeated here (see eqs. A–1 to A–36 of Appendix A for `EM1DFM`).

Assumptions: $e^{i\omega t}$ time-dependence (just as in W & H); quasi-static approximation throughout; z positive downwards; air halfspace ($\sigma = 0$) for $z < 0$, piecewise constant model ($\sigma > 0$) of N layers for $z \geq 0$, Nth layer being the basement (i.e. homogeneous) halfspace; magnetic permeability everywhere equal to that of free space.

From the propagation of F through the layers gives the following expression for the kernel of the Hankel transform, $\tilde{F}$, in the air halfspace ($z < 0$):

$$\tilde{F}_0 \;= D_0^S \Big(e^{-u_0 z} \;+\; \frac{P_{21}}{P_{11}} e^{u_0 z} \Big) \tag{A–1}$$

(eq. A–36 from Appendix A for `EM1DFM`).

For a horizontal x-directed electric dipole at a height h (i.e., $z = -h$, $h > 0$) above the surface of the layered Earth, the downward-decaying part of the primary solution of $\tilde{F}$ (and the only downward-decaying part of the solution in the air halfspace) at the surface of the Earth ($z = 0$) is given by

$$D_0^S \;=\; -\,\frac{i\omega\mu_0}{2u_0}\,\frac{ik_y}{k_x^2 + k_y^2}\, e^{-u_0 h} \tag{A–2}$$

(Ward & Hohmann, eq. 4.137). Substituting this into eq. (A–1) gives

$$\tilde{F}_0 \;= -\,\frac{i\omega\mu_0}{2u_0}\,\frac{ik_y}{k_x^2 + k_y^2}\,\Big(e^{-u_0 (z+h)} \;+\; \frac{P_{21}}{P_{11}} e^{u_0 (z-h)} \Big). \tag{A–3}$$

Generalizing this expression for z above ($z < -h$) as well as below the source ($z > -h$):

$$\tilde{F}_0 \;= -\,\frac{i\omega\mu_0}{2u_0}\,\frac{ik_y}{k_x^2 + k_y^2}\,\Big(e^{-u_0 |z+h|} \;+\; \frac{P_{21}}{P_{11}} e^{u_0 (z-h)} \Big). \tag{A–4}$$

Applying the inverse Fourier transform (see eq. A–4 in Appendix A for `EM1DFM`) to eq. (A–4) gives

$$F_0(x,y,z,\omega) \;=\; -\,\frac{1}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{i\omega\mu_0}{2u_0}\,\frac{ik_y}{k_x^2 + k_y^2}\,\Big(e^{-u_0 |z+h|} \;+\; \frac{P_{21}}{P_{11}} e^{u_0 (z-h)} \Big)\, e^{i(k_x x + k_y y)}\, dk_x\, dk_y \tag{A–5}$$

(cf. eq. 4.139 of Ward & Hohmann). Using the identity

$$\int_{\infty}^{\infty}\int_{\infty}^{\infty} \tilde{F}(k_x^2 + k_y^2)\, dk_x\, dk_y \;=\; 2\pi \int_0^{\infty} \tilde{F}(\lambda)\,\lambda\, J_0(\lambda r)\, d\lambda, \tag{A–6}$$

(Ward & Hohmann, eq.2.10) where $\lambda^2 = k_x^2 + k_y^2$ and $r^2 = x^2 + y^2$, eq. (A–5) can be rewritten as

$$F_0(x,y,z,\omega) \;=\; -\,\frac{1}{2\pi}\,\frac{\partial}{\partial y}\int_0^\infty \frac{i\omega\mu_0}{2u_0}\,\frac{1}{\lambda^2}\left(e^{-u_0|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\right)\lambda\,J_0(\lambda r)\,d\lambda, \tag{A–7}$$

$$= \;-\,\frac{i\omega\mu_0}{4\pi}\,\frac{\partial}{\partial y}\int_0^\infty \left(e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\right)\frac{1}{\lambda^2}\,J_0(\lambda r)\,d\lambda, \tag{A–8}$$

$$= \;\frac{i\omega\mu_0}{4\pi}\,\frac{y}{r}\int_0^\infty \left(e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\right)\frac{1}{\lambda}\,J_1(\lambda r)\,d\lambda, \tag{A–9}$$

since

$$\frac{\partial J_0(\lambda r)}{\partial y} \;=\; -\,\lambda\frac{y}{r}\,J_1(\lambda r) \tag{A–10}$$

(Ward & Hohmann, eq. 4.44 (almost)).

The H-field in the air halfspace can be obtained from eq. (A–9) (or eq. A–8) by using eq. (1.130) of Ward & Hohmann:

$$H_x \;=\; \frac{1}{i\omega\mu_0}\,\frac{\partial^2 F_0}{\partial x\partial z}, \tag{A–11}$$

$$H_y \;=\; \frac{1}{i\omega\mu_0}\,\frac{\partial^2 F_0}{\partial y\partial z}, \tag{A–12}$$

$$H_z \;=\; \frac{1}{i\omega\mu_0}\left(\frac{\partial^2}{\partial z^2} \;+\; \kappa_0^2\right)F_0 \tag{A–13}$$

$$= \;\frac{1}{i\omega\mu_0}\,\frac{\partial^2 F_0}{\partial z^2}. \tag{A–14}$$

since $\kappa_0^2 = 0$. Applying eq. (A–11) to eq. (A–9) gives

$$H_x(x,y,z,\omega) \;=\; \frac{1}{4\pi}\,\frac{\partial}{\partial x}\,\frac{y}{r}\int_0^\infty \left(\pm\, e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\right)J_1(\lambda r)\,d\lambda. \tag{A–15}$$

(The plus/minus is to do with whether or not the observation location is above or below the source. In the program, *perhaps* it is only the secondary fields that are computed using the above expressions: the primary field, which corresponds to the first term in each Hankel transform kernel above is computed using its for in (x,y,z)-space.) When the above expression for a horizontal electric dipole is integrated along a wire all that is left is the effects of the endpoints. These will cancel when integrating around the closed loop. So as far as the part of H_x that contributes to the file due to a closed loop:

$$H_x(x,y,z,\omega) \;=\; 0. \tag{A–16}$$

For the y-component of the H-field, first consider differentiating the expression for F_0 in eq. (A–5) with respect to y:

$$\frac{\partial F_0}{\partial y} \;=\; -\,\frac{1}{4\pi^2}\,\frac{\partial}{\partial y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\omega\mu_0}{2u_0}\,\frac{ik_y}{k_x^2+k_y^2}\left(e^{-u_0|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\right)e^{i(k_x x+k_y y)}\,dk_x\,dk_y, \tag{A–17}$$

$$= \;\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\omega\mu_0}{2u_0}\,\frac{k_y^2}{k_x^2+k_y^2}\left(e^{-u_0|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\right)e^{i(k_x x+k_y y)}\,dk_x\,dk_y, \tag{A–18}$$

$$\begin{aligned} = \;&\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\omega\mu_0}{2u_0}\left(e^{-u_0|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\right)e^{i(k_x x+k_y y)}\,dk_x\,dk_y \\ &-\;\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\omega\mu_0}{2u_0}\,\frac{k_x^2}{k_x^2+k_y^2}\left(e^{-u_0|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\right)e^{i(k_x x+k_y y)}\,dk_x\,dk_y, \end{aligned} \tag{A–19}$$

since

$$\frac{k_y^2}{k_x^2 + k_y^2} \;=\; 1 \;-\; \frac{k_x^2}{k_x^2 + k_y^2}. \tag{A–20}$$

Converting the k_x^2 into derivatives with respect to x, and converting the two-dimensional Fourier transforms to Hankel transforms gives

$$\begin{aligned}\frac{\partial F_0}{\partial y} \;&=\; \frac{i\omega\mu_0}{4\pi}\int_0^\infty \Big(e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}} e^{\lambda(z-h)} \Big)\, J_0(\lambda r)\, d\lambda \\ &\quad +\; \frac{i\omega\mu_0}{4\pi}\,\frac{\partial^2}{\partial x^2}\int_0^\infty \Big(e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}} e^{\lambda(z-h)} \Big)\,\frac{1}{\lambda^2}\, J_0(\lambda r)\, d\lambda, \qquad \text{(A–21)} \\ &=\; \frac{i\omega\mu_0}{4\pi}\int_0^\infty \Big(e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}} e^{\lambda(z-h)} \Big)\, J_0(\lambda r)\, d\lambda \\ &\quad -\; \frac{i\omega\mu_0}{4\pi}\,\frac{\partial}{\partial x}\,\frac{x}{r}\int_0^\infty \Big(e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}} e^{\lambda(z-h)} \Big)\,\frac{1}{\lambda}\, J_1(\lambda r)\, d\lambda, \qquad \text{(A–22)}\end{aligned}$$

using eq. (4.144) & (4.117) of Ward & Hohmann. Differentiating eq. (A-22) with respect to z and scaling by $i\omega\mu_0$ (see eq. A–12) gives

$$\begin{aligned}H_y(x,y,z,\omega) \;&=\; \frac{1}{4\pi}\int_0^\infty \Big(\pm\, e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}} e^{\lambda(z-h)} \Big)\, \lambda\, J_0(\lambda r)\, d\lambda \\ &\quad -\; \frac{1}{4\pi}\,\frac{\partial}{\partial x}\,\frac{x}{r}\int_0^\infty \Big(\pm\, e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}} e^{\lambda(z-h)} \Big)\, J_1(\lambda r)\, d\lambda \qquad \text{(A–23)}\end{aligned}$$

(cf. eq. 4.150 of Ward & Hohmann). The second integral in the above expression only contributes at the ends of the dipole. So the only part of H_y required to compute the field due to the closed loop is

$$H_y(x,y,z,\omega) \;=\; \frac{1}{4\pi}\int_0^\infty \Big(\pm\, e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}} e^{\lambda(z-h)} \Big)\, \lambda\, J_0(\lambda r)\, d\lambda. \tag{A–24}$$

Finally, applying eq. (A–14) to eq. (A–9) gives the z-component of the H-field:

$$H_z(x,y,z,\omega) \;=\; \frac{1}{4\pi}\,\frac{y}{r}\int_0^\infty \Big(e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}} e^{\lambda(z-h)} \Big)\, \lambda\, J_1(\lambda r)\, d\lambda \tag{A–25}$$

(cf. eq. 4.152 of Ward & Hohmann).

Equations (A–24) & (A–25) are for the total H-field ($H_x = 0$ from eq. A–16) for an x-directed electric dipole excluding the effects at the end-points, that is, the wholespace field up in the air plus the field due to currents induced in the layered Earth. In eqs. (A–24) & (A–25), the first part of the kernel of the Hankel transform corresponds to the primary wholespace field:

$$\begin{aligned}H_y(x,y,z,\omega) \;&=\; \frac{1}{4\pi}\int_0^\infty \pm\, e^{-\lambda|z+h|}\, \lambda\, J_0(\lambda r)\, d\lambda, \qquad \text{(A–26)} \\ &=\; \frac{1}{4\pi}\,\frac{\partial}{\partial z}\int_0^\infty e^{-\lambda|z+h|}\, J_0(\lambda r)\, d\lambda, \qquad \text{(A–27)}\end{aligned}$$

and

$$\begin{aligned}H_z(x,y,z,\omega) \;&=\; \frac{1}{4\pi}\,\frac{y}{r}\int_0^\infty e^{-\lambda|z+h|}\, \lambda\, J_1(\lambda r)\, d\lambda \qquad \text{(A–28)} \\ &=\; -\,\frac{1}{4\pi}\,\frac{y}{r}\,\frac{\partial}{\partial r}\int_0^\infty e^{-\lambda|z+h|}\, J_0(\lambda r)\, d\lambda. \qquad \text{(A–29)}\end{aligned}$$

From Ward & Hohmann eq. (4.53), the integral in the above two expressions can be done:

$$\int_0^\infty e^{-\lambda|z+h|}\, J_0(\lambda r)\, d\lambda \;=\; \frac{1}{\big(r^2+(z+h)^2\big)^{1/2}}. \tag{A–30}$$

So

$$H_y(x,y,z,\omega) \;=\; \frac{1}{4\pi}\,\frac{\partial}{\partial z}\,\frac{1}{\big(r^2+(z+h)^2\big)^{1/2}}, \tag{A–31}$$

$$=\; -\,\frac{1}{4\pi}\,\frac{z}{\big(r^2+(z+h)^2\big)^{3/2}} \tag{A–32}$$

(cf. eq. 2.42 of Ward & Hohmann for $\sigma = 0$), and

$$H_z(x,y,z,\omega) \;=\; -\,\frac{1}{4\pi}\,\frac{y}{r}\,\frac{\partial}{\partial r}\,\frac{1}{\big(r^2+(z+h)^2\big)^{1/2}}, \tag{A–33}$$

$$=\; \frac{1}{4\pi}\,\frac{y}{r}\,\frac{r}{\big(r^2+(z+h)^2\big)^{3/2}}, \tag{A–34}$$

$$=\; \frac{1}{4\pi}\,\frac{y}{\big(r^2+(z+h)^2\big)^{3/2}} \tag{A–35}$$

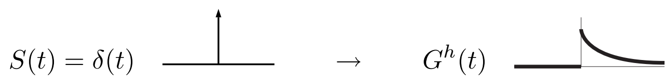
(cf. eq. 2.42 of Ward & Hohmann for $\sigma = 0$).

Frequency- to time-domain transformation – part I

The solution for the H-field in the frequency domain for a delta-function source in time (and hence a flat, constant, real source term in the frequency domain) is, for example,

$$H_z(x, y, z, \omega) \;=\; \frac{1}{4\pi}\,\frac{y}{r} \int_0^\infty \left(e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}} e^{\lambda(z-h)} \right) \lambda\, J_1(\lambda r)\, d\lambda. \tag{A–25}$$

Doing the inverse Fourier transform of these kinds of expressions does not encounter any subtleties, I think, and gives an H-field as a function of time that, schematically, looks like:



This is the basic "response" that program EM1DTM computes. Notation of $G^h(t)$ because this is the Green's function for convolution with the transmitter current waveform $S(t)$ to give the H-field:

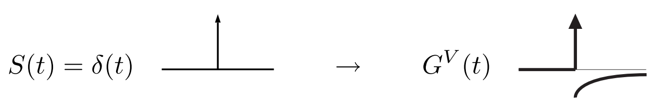
$$h(t) \;=\; \int_{t'=-\infty}^{\infty} G^h(t-t')\, S(t')\, dt'. \tag{AA–1}$$

The H-field for the delta-function source, that is, G^h certainly exists for $t > 0$. Also, it is certainly zero for $t < 0$. And at $t = 0$, it certainly is not infinite (not physical). Let's re-describe the function G^h (shown in the diagram above) as

$$G^h(t) \;=\; X(t)\, \tilde{G}^h(t), \tag{AA–2}$$

where $\tilde{G}^h(t)$ is equal to G^h for $t > 0$, $\tilde{G}^h(0) = \lim_{t\to 0+} G^h$, and does anything it wants for $t < 0$. And $X(t)$ is the Heaviside function. This moves all issues about what is happening at $t = 0$ into the Heaviside function. (I think Christophe does this, or something like it.)

For measurements of voltage, the Green's function ("impulse response") that is required is the time derivative of G^h (and for all t, not just $t > 0$). Schematically:



In terms of math:

$$V(t) \;=\; \int_{t'=-\infty}^{\infty} G^V(t-t')\, S(t')\, dt'. \tag{AA–3}$$

Let's take the time derivative of eq. (AA-2) to get the full expression for G^V:

$$\begin{aligned} G^V(t) \;&=\; \frac{dG^h}{dt}, \\ &=\; \frac{d}{dt}\big(X\, \tilde{G}^h\big), \\ &=\; X\, \frac{d\tilde{G}^h}{dt} \;+\; \delta\, \tilde{G}^h, \end{aligned} \tag{AA–4}$$

where δ is the delta function. Now, this is not a time derivative that should be happening numerically. So, given the basic $G^h(t)$ and some representation of the transmitter current waveform $S(t)$, program EM1DTM currently uses the re-arrangement of eq. (AA–3) given by the substitution of eq. (AA-4) into eq. (AA-3) followed by some integration by parts:

$$\begin{aligned} V(t) \;&=\; \int_{t'=-\infty}^{\infty} \left\{ X(t-t')\, \frac{d\tilde{G}^h}{dt'}(t-t') \;+\; \delta(t-t')\, \tilde{G}^h(t-t') \right\} S(t')\, dt', \\ &=\; \tilde{G}^h(0)\, S(t) \;+\; \int_{t'=-\infty}^{t} \frac{d\tilde{G}^h}{dt'}(t-t')\, S(t')\, dt', \end{aligned} \tag{AA–5}$$

where the Heaviside function has been used to restrict the limits of the integration. Now doing the integration by parts:

$$\begin{aligned} V(t) \;&=\; \tilde{G}^h(0)\,S(t) \;+\; \Big[\tilde{G}^h(t-t')\,S(t')\Big]_{t'=-\infty}^{t} \;-\; \int_{t'=-\infty}^{t} \tilde{G}^h(t-t')\,\frac{dS}{dt'}(t')\,dt', \\ &=\; \tilde{G}^h(0)\,S(t) \;+\; \tilde{G}^h(0)\,S(t) \;-\; \int_{t'=-\infty}^{t} \tilde{G}^h(t-t')\,\frac{dS}{dt'}(t')\,dt'. \end{aligned} \qquad \text{(AA–6)}$$

Which looks as though it has the “expected” additional non-convolution-integral term.

However, perhaps there should be an additional minus sign in going from eq. (AA–4) to the one before eq. (AA-5) because the derivative has changed from d/dt to d/dt'. But perhaps not.

Frequency- to time-domain transformation

The Fourier transform that was applied to Maxwell's equations to get the frequency-domain equations was (see Ward & Hohmann, eq. 1.1)

$$F(\omega) \;=\; \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt,$$

and the corresponding inverse transform is

$$f(t) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} d\omega.$$

For the frequency domain computations, it is assumed that the source term is the same for all frequencies. In other words, a flat spectrum, which corresponds to a delta-function time-dependence of the source.

Consider at the moment a causal signal, that is, one for which $f(t) = 0$ for $t < 0$. The Fourier transform of this signal is then

$$\begin{aligned} F(\omega) \;&=\; \int_{0}^{\infty} f(t)\, e^{-i\omega t} dt, \\ &=\; \int_{0}^{\infty} f(t)\; \cos\omega t\; dt \;-\; i \int_{0}^{\infty} f(t)\; \sin\omega t\; dt. \end{aligned}$$

Note that because of the dependence of the real part of $F(\omega)$ on cos ωt and of the imaginary part on sin ωt, the real part of $F(\omega)$ is even and the imaginary part of $F(\omega)$ is odd. Hence, $f(t)$ can be obtained from either the real or imaginary part of its Fourier transform via the inverse cosine or sine transform:

$$\begin{aligned} f(t) \;&=\; \frac{2}{\pi} \int_{0}^{\infty} \mathrm{Re}\, F(\omega)\; \cos\omega t\; d\omega, \quad \text{or} \\ f(t) \;&=\; -\frac{2}{\pi} \int_{0}^{\infty} \mathrm{Im}\, F(\omega)\; \sin\omega t\; d\omega. \end{aligned}$$

(For factor of $2/\pi$ see, for example, Arfken.)

Now consider that we've computed the H-field in the frequency domain for a uniform source spectrum. Then from the above expressions, the time-domain H-field for a *positive delta-function* source time-dependence is

$$\begin{aligned} h_{\delta+}(t) \;&=\; \frac{2}{\pi} \int_{0}^{\infty} \mathrm{Re}\, H(\omega)\; \cos\omega t\; d\omega, \quad \text{or} \\ h_{\delta+}(t) \;&=\; -\frac{2}{\pi} \int_{0}^{\infty} \mathrm{Im}\, H(\omega)\; \sin\omega t\; d\omega, \end{aligned}$$

where $H(\omega)$ is the frequency-domain H-field for the uniform source spectrum. For a *negative delta-function* source:

$$\begin{aligned} h_{\delta-}(t) \;&=\; -\frac{2}{\pi} \int_{0}^{\infty} \mathrm{Re}\, H(\omega)\; \cos\omega t\; d\omega, \quad \text{or} \\ h_{\delta-}(t) \;&=\; \frac{2}{\pi} \int_{0}^{\infty} \mathrm{Im}\, H(\omega)\; \sin\omega t\; d\omega. \end{aligned}$$

The negative delta-function source dependence is the derivative with respect to time of a step turn-off source dependence. Hence, the *derivative* of the time-domain H-field due to a *step turn-off* is also given by the above expressions:

$$\begin{aligned} \frac{\partial h_{\mathrm{s}}}{\partial t}(t) \;&=\; -\frac{2}{\pi} \int_{0}^{\infty} \mathrm{Re}\, H(\omega)\; \cos\omega t\; d\omega, \quad \text{or} \\ \frac{\partial h_{\mathrm{s}}}{\partial t}(t) \;&=\; \frac{2}{\pi} \int_{0}^{\infty} \mathrm{Im}\, H(\omega)\; \sin\omega t\; d\omega. \end{aligned}$$

Integrating the above two expressions gives the H-field for a *step turn-off* source:

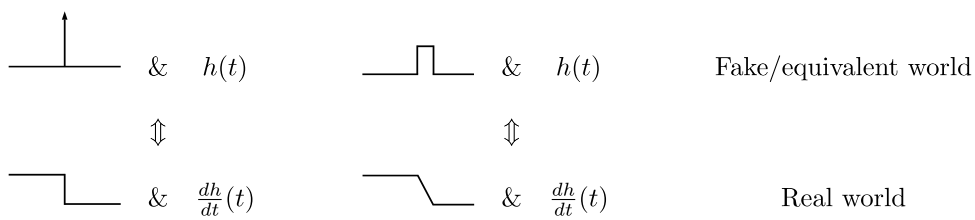
$$h_{\rm s}(t) \;=\; h(0) \;-\; \frac{2}{\pi}\int_0^\infty {\rm Re}\,H(\omega)\,\frac{1}{\omega}\,\sin\omega t\,d\omega, \quad \text{or}$$

$$h_{\rm s}(t) \;=\; -\,\frac{2}{\pi}\int_0^\infty {\rm Im}\,H(\omega)\,\frac{1}{\omega}\,\cos\omega t\,d\omega.$$

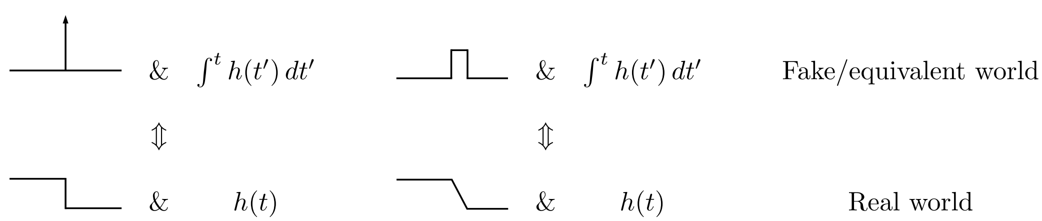
(See also Newman, Hohmann & Anderson, and Kaufman & Keller for all this.)

Thinking in terms of the time-domain inhomogeneous differential equation:

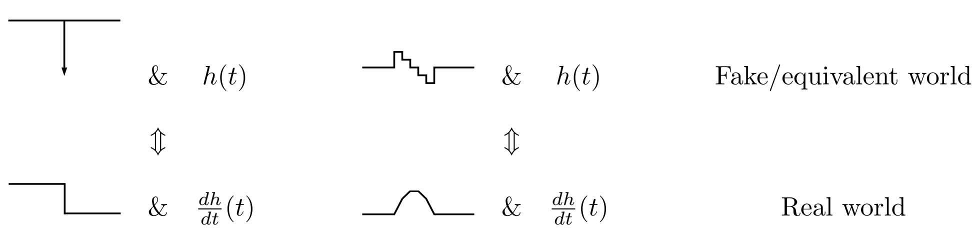
$$\begin{aligned} L\,h_{\delta-} &= \delta_-, \\ \Rightarrow\quad L\,h_{\delta-} &= \frac{\partial}{\partial t}H_{\rm o}, \\ \Rightarrow\quad L\,\frac{\partial h_s}{\partial t} &= \frac{\partial}{\partial t}H_{\rm o}. \end{aligned}$$



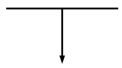
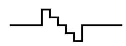
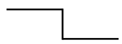
Top left is what we know (flat frequency spectrum for the source & sine transform of the imaginary part of the field), and bottom left is what we're after. Also, top right is obtained from top left by convolution with the box-car, and bottom right is what we're considering it to be. Note that there should really be some minus signs in the above diagram.



Again, top left is what we have (flat frequency spectrum for the source and cosine transform of the imaginary part divided by frequency), and bottom left is what we're thinking it is. And top right is the convolution with a box-car, and bottom right is what we're considering it to be: the H-field for a ramp turn-off.



Top left is what we have, and bottom is what we're thinking it is. And top right is the convolution with a discretized half-sine, and bottom right is what we're considering it to be: the time-derivative of the H-field for a half-sine waveform.

	& $\int^t h(t')\,dt'$		& $\int^t h(t')\,dt'$	Fake/equivalent world
	$\Updownarrow$		$\Updownarrow$	
	& $h(t)$		& $h(t)$	Real world

Top left is what we have, and bottom is what we're thinking it is. And top right is the convolution with a discretized half-sine, and bottom right is what we're considering it to be: the H-field for a half-sine waveform.

Integration of cubic splined function

The time-domain voltage or magnetic field ends up being known at a number of discrete, logarithmically/exponentially-spaced times as a result of Anderson's cosine/sine digital transform. This time-domain function is cubic splined in terms of the logarithms of the times. Hence, between any two discrete times, the time-domain function is approximated by the cubic spline

$$y(h) \;=\; y_0 \;+\; q_1\,h \;+\; q_2\,h^2 +\; q_3\,h^3,$$

(see routines `RSPLN` & `RSPLE`) where $h = \log x - \log t_i$, x is the time at which the function y is required, t_i is the ith time at which y is known ($t_i \le x \le t_{i+1}$), $y_0 = y(\log t_i)$, and q_1, q_2 & q_3 are the spline coefficients. The required integral is

$$\begin{aligned}
\int_{x=a}^{b} y(\log x)\,dx \;&=\; \int_{\log x=\log a}^{\log b} y(\log x)\,x\,d(\log x),\\
&=\; \int_{\log x=\log a}^{\log b} y(\log x)\,e^{\log x}\,d(\log x),\\
&=\; \int_{h=\log a-\log t_i}^{\log b-\log t_i} y(h)\,e^{(h+\log t_i)}\,dh,\\
&=\; t_i \int_{h=\log a-\log t_i}^{\log b-\log t_i} y(h)\,e^{h}\,dh.
\end{aligned}$$

Substituting the polynomial expression for $y(h)$ into the above integral and worrying about each term individually gives:

$$\int y_0\,e^h\,dh \;=\; y_0\,e^h,$$

$$\int q_1 h\,e^h\,dh \;=\; q_1 e^h(h-1)$$

(G & R 2.322.1),

$$\int q_2 h^2 e^h\,dh \;=\; q_2 e^h(h^2-2h+2)$$

(G & R 2.322.2), and

$$\int q_3 h^3 e^h\,dh \;=\; q_3 e^h(h^3-3h^2+6h-6)$$

(G & R 2.322.3). Hence, summing the integrals above,

$$\begin{aligned}
\int_{x=a}^{b} y(\log x)\,dx \;&= t_i\,y_0\Big(\frac{b}{t_i}\;-\;\frac{a}{t_i}\Big) \\
&+\;t_i\,q_1\Big(\frac{b}{t_i}(\log b-\log t_i-1)\;-\;\frac{a}{t_i}(\log a-\log t_i-1)\Big) \\
&+\;t_i\,q_2\Big(\frac{b}{t_i}\big((\log b-\log t_i)^2-2(\log b-\log t_i)+2\big)\;- \\
&\qquad\qquad\frac{a}{t_i}\big((\log a-\log t_i)^2-2(\log a-\log t_i)+2\big)\Big) \\
&+\;t_i\,q_3\Big(\frac{b}{t_i}\big((\log b-\log t_i)^3-3(\log b-\log t_i)^2+6(\log b-\log t_i)-6\big)\;- \\
&\qquad\qquad\frac{a}{t_i}\big((\log a-\log t_i)^3-3(\log a-\log t_i)^2+6(\log a-\log t_i)-6\big)\Big).
\end{aligned}$$

The original plan was to treat a discretised transmitter current waveform as a piecewise linear function (*i.e.*, straight line segments between the provided sampled points), which meant that the response coming out of Anderson's filtering routine was convolved with the piecewise constant time-derivative of the transmitter current waveform to give voltages. This proved to be not good enough for on-time calculations (the step-y nature of the approximation of the time derivative of the transmitter current waveform could be seen in the computed voltages). So it was decided to cubic spline the transmitter current waveform, which gives a piecewise quadratic approximation to the time derivative of the waveform. And so the convolution of the stuff coming out of Anderson's routine is now with a constant, a linear time term and a quadratic term. The involved integral above is still required, along with:

$$\begin{aligned}
\int_{x=a}^{b} x\,y(\log x)\,dx \;&=\; \int_{\log x=\log a}^{\log b} y(\log x)\,x^2\,d(\log x), \\
&=\; \int_{\log x=\log a}^{\log b} y(\log x)\,e^{2\log x}\,d(\log x), \\
&=\; \int_{h=\log a-\log t_i}^{\log b-\log t_i} y(h)\,e^{(2h+2\log t_i)}\,dh, \\
&=\; t_i^2\int_{h=\log a-\log t_i}^{\log b-\log t_i} y(h)\,e^{2h}\,dh.
\end{aligned}$$

Using the integrals above for the various powers of h times e^h, the relevant integrals for the various parts of the cubic spline representation of $y(h)$ are:

$$\int y_0\,e^{2h}\,dh \;=\; y_0\,\frac{1}{2}\,e^{2h},$$

$$\int q_1 h\,e^{2h}\,dh \;=\; q_1\frac{1}{4}e^{2h}(2h-1),$$

$$\int q_2 h^2 e^{2h}\,dh \;=\; q_2\frac{1}{8}e^{2h}(4h^2-4h+2),$$

$$\int q_3 h^3 e^{2h}\,dh \;=\; q_3\frac{1}{16}e^{2h}(8h^3-12h^2+12h-6).$$

The limits for the integral are $h = \log a - \log t_i$ and $h = \log b - \log t_i$. The term e^{2h} becomes:

$$
\begin{aligned}
e^{2(\log X - \log t_i)} &= \left\{ e^{(\log X - \log t_i)} \right\}^2, \\
&= \left\{ \frac{e^{\log X}}{e^{\log t_i}} \right\}^2, \\
&= \left(\frac{X}{t_i} \right)^2, \\
&= \frac{X^2}{t_i^2},
\end{aligned}
$$

where X is either a or b. Hence,

$$
\begin{aligned}
\int_{x=a}^{b} x\, y(\log x)\, dx \;=\; & t_i^2\, y_0 \Big(\frac{b^2}{t_i^2} \;-\; \frac{a^2}{t_i^2} \Big) \\
&+\; t_i^2\, q_1\, \frac{1}{4} \Big(\frac{b^2}{t_i^2} (2\log b - 2\log t_i - 1) \;-\; \frac{a^2}{t_i^2} (2\log a - 2\log t_i - 1) \Big) \\
&+\; t_i^2\, q_2\, \frac{1}{8} \Big(\frac{b^2}{t_i^2} \big(4(\log b - \log t_i)^2 - 4(\log b - \log t_i) + 2 \big) - \\
&\qquad \frac{a^2}{t_i^2} \big(4(\log a - \log t_i)^2 - 4(\log a - \log t_i) + 2 \big) \Big) \\
&+\; t_i^2\, q_3\, \frac{1}{16} \Big(\frac{b^2}{t_i^2} \big(8(\log b - \log t_i)^3 - 12(\log b - \log t_i)^2 + 12(\log b - \log t_i) - 6 \big) - \\
&\qquad \frac{a^2}{t_i^2} \big(8(\log a - \log t_i)^3 - 12(\log a - \log t_i)^2 + 12(\log a - \log t_i) - 6 \big) \Big).
\end{aligned}
$$

And

$$
\begin{aligned}
\int_{x=a}^{b} x^2\, y(\log x)\, dx \;&=\; \int_{\log x = \log a}^{\log b} y(\log x)\, x^3\, d(\log x), \\
&= \int_{\log x = \log a}^{\log b} y(\log x)\, e^{3\log x}\, d(\log x), \\
&= \int_{h = \log a - \log t_i}^{\log b - \log t_i} y(h)\, e^{(3h + 3\log t_i)}\, dh, \\
&= t_i^3 \int_{h = \log a - \log t_i}^{\log b - \log t_i} y(h)\, e^{3h}\, dh.
\end{aligned}
$$

And

$$\int y_0\, e^{3h}\, dh \;=\; y_0\, \frac{1}{3}\, e^{3h},$$

$$\int q_1 h\, e^{3h}\, dh \;=\; q_1 \frac{1}{9} e^{3h} (3h - 1),$$

$$\int q_2 h^2 e^{3h}\, dh \;=\; q_2 \frac{1}{27} e^{3h} (9h^2 - 6h + 2),$$

$$\int q_3 h^3 e^{3h}\, dh \;=\; q_3 \frac{1}{81} e^{3h} (27h^3 - 27h^2 + 18h - 6).$$

Hence,

$$
\begin{aligned}
\int_{x=a}^{b} x^2\, y(\log x)\, dx \; &= t_i^3\, y_0 \Big(\frac{b^3}{t_i^3} \; - \; \frac{a^3}{t_i^3} \Big) \\
&+ \; t_i^3\, q_1\, \frac{1}{9} \Big(\frac{b^3}{t_i^3} (3\log b - 3\log t_i - 1) \; - \; \frac{a^3}{t_i^3} (3\log a - 3\log t_i - 1) \Big) \\
&+ \; t_i^3\, q_2\, \frac{1}{27} \Big(\frac{b^3}{t_i^3} \big(9(\log b - \log t_i)^2 - 6(\log b - \log t_i) + 2 \big) - \\
&\qquad\qquad \frac{a^3}{t_i^3} \big(9(\log a - \log t_i)^2 - 6(\log a - \log t_i) + 2 \big) \Big) \\
&+ \; t_i^3\, q_3\, \frac{1}{81} \Big(\frac{b^3}{t_i^3} \big(27(\log b - \log t_i)^3 - 27(\log b - \log t_i)^2 + 18(\log b - \log t_i) - 6 \big) - \\
&\qquad\qquad \frac{a^3}{t_i^3} \big(27(\log a - \log t_i)^3 - 27(\log a - \log t_i)^2 + 18(\log a - \log t_i) - 6 \big) \Big).
\end{aligned}
$$

In the previous two integrals of the product of x & x^2 with the function splined in terms of $\log x$, the x & x^2 should really be $(B - x)$ & $(B - x)^2$, where B is the end of the relevant interval of the splined transmitter current waveform (because it's convolution that's happening):

$$
\begin{aligned}
I_1 \; &= \; \int_{x=a}^{b} \big(B - x \big)\, y(\log x)\, dx, \quad \text{and} \\
I_2 \; &= \; \int_{x=a}^{b} \big(B - x \big)^2\, y(\log x)\, dx.
\end{aligned}
$$

Also, it was not really x & x^2 in those integrals because these terms are coming from the cubic splining of the transmitter current waveform, which means that in each interval between discretization points, it should be $(x - A)$ & $(x - A)^2$ that are involved, where A is the start of the relevant interval for the transmitter current waveform. Because

$$
\begin{aligned}
\int_{x=a}^{b} \big(x - A \big)\, y(\log x)\, dx \; &= \; -\, A \int_{x=a}^{b} y(\log x)\, dx \; + \; \int_{x=a}^{b} x\, y(\log x)\, dx, \quad \text{and} \\
\int_{x=a}^{b} \big(x - A \big)^2\, y(\log x)\, dx \; &= \; A^2 \int_{x=a}^{b} y(\log x)\, dx \; - \; 2A \int_{x=a}^{b} x\, y(\log x)\, dx \; + \; \int_{x=a}^{b} x^2\, y(\log x)\, dx,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{x=a}^{b} \big(B - x \big)\, y(\log x)\, dx \; &= \; B \int_{x=a}^{b} y(\log x)\, dx \; - \; \int_{x=a}^{b} x\, y(\log x)\, dx, \quad \text{and} \\
\int_{x=a}^{b} \big(B - x \big)^2\, y(\log x)\, dx \; &= \; B^2 \int_{x=a}^{b} y(\log x)\, dx \; - \; 2B \int_{x=a}^{b} x\, y(\log x)\, dx \; + \; \int_{x=a}^{b} x^2\, y(\log x)\, dx,
\end{aligned}
$$

then

$$
\begin{aligned}
I_1 \; &= \; \big(B - A \big) \int_{x=a}^{b} y(\log x)\, dx \; - \; \int_{x=a}^{b} \big(x - A \big)\, y(\log x)\, dx, \qquad \text{and} \\
I_2 \; &= \; \big(B - A \big)^2 \int_{x=a}^{b} y(\log x)\, dx \; - \; 2\big(B - A \big) \int_{x=a}^{b} \big(x - A \big)\, y(\log x)\, dx \; + \; \int_{x=a}^{b} \big(x - A \big)^2\, y(\log x)\, dx.
\end{aligned}
$$

A. Mathematics for Forward-Modelling Computations

This appendix contains the full and explicit description of the mathematics for the forward-modelling computations. The z-component of the Schelkunoff **F**-potential (Ward & Hohmann, 1988, *in* "Electromagnetic Methods in Applied Geophysics", v1, pp131–311, SEG) is used for all computations. This single component of the vector potential is all that is needed for computing all components of the magnetic field above the Earth's surface for all orientations of a magnetic dipole source also above the surface for the quasi-static assumption (see top of page 225 of Ward & Hohmann). The computations are carried out by propagating through the layered model the matrices formed by applying the conditions on the component of **F** at each interface between layers (see CGF's thesis and Farquharson & Oldenburg, 1996, GJI, v126, pp235–252; and Partha's thesis).

Assumptions: $e^{i\omega t}$ time-dependence; quasi-static approximation throughout; z positive downwards; air halfspace ($\sigma = 0$, $\chi = 0$) for $z < 0$, piecewise constant model ($\sigma > 0$, $\chi \geq 0$) of N layers for $z \geq 0$, Nth layer being the basement (i.e. homogeneous) halfspace.

Consider the Schelkunoff **F**-potential:

$$\mathbf{F}(x, y, z, \omega) \;=\; F(x, y, z, \omega)\,\hat{\mathbf{z}} \tag{A–1}$$

where $\hat{\mathbf{z}}$ is the unit vector in the z-direction. In each layer layer of the model the conductivity and susceptibility are constant. In the jth layer of conductivity σ_j and susceptibility $\mu_j = \mu_0(1 + \chi_j)$ not containing a source, F satisfies the homogeneous differential equation:

$$\nabla^2 F_j \;+\; \kappa_j^2 F_j \;=\; 0 \tag{A–2}$$

where $\kappa_j^2 = -i\omega\mu_j\sigma_j$ (Ward & Hohmann, eqs. 1.124 & 4.6). Consider the 2D Fourier transform such that

$$\tilde{F}(k_x, k_y, z, \omega) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x, y, z, \omega)\, e^{-i(k_x x + k_y y)}\; dx\, dy\;, \tag{A–3}$$

$$F(x, y, z, \omega) \;=\; \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{F}(k_x, k_y, z, \omega)\, e^{i(k_x x + k_y y)}\; dk_x\, dk_y \tag{A–4}$$

(Ward & Hohmann eqs. 4.7 & 4.8). Applying this Fourier transform to eq. (A–2) gives

$$\frac{d^2\tilde{F}_j}{dz^2} \;-\; u_j^2\,\tilde{F}_j \;=\; 0, \tag{A–5}$$

where $u_j^2 = k_x^2 + k_y^2 - \kappa_j^2 = k_x^2 + k_y^2 + i\omega\mu_j\sigma_j$. The solution to this equation is:

$$\tilde{F}_j(k_x, k_y, z, \omega) \;= D_j(k_x, k_y, \omega)\, e^{-u_j(z - z_j)} \;+\; U_j(k_x, k_y, \omega)\, e^{u_j(z - z_j)}, \tag{A–6}$$

where D_j and U_j correspond to the coefficients of the downward- and upward-decaying parts of the solution in this the jth layer (Ward & Hohmann, eq. 4.13; Partha's thesis). (Note that the real part of u_j is positive.)

At the interface between layer $j-1$ and layer j, which is at a depth of z_j, the conditions on $\tilde{F}$ are:

$$\tilde{F}_{j-1}\Big|_{z=z_j} \;=\; \tilde{F}_j\Big|_{z=z_j}\,, \tag{A–7}$$

$$\frac{1}{\mu_{j-1}}\,\frac{d\tilde{F}_{j-1}}{dz}\bigg|_{z=z_j} \;=\; \frac{1}{\mu_j}\,\frac{d\tilde{F}_j}{dz}\bigg|_{z=z_j} \tag{A–8}$$

(Ward & Hohmann eqs. 1.152 and 1.153). Applying the first of these conditions to the solution in layer j ($j \geq 2$) given by eq. (A–6) and the corresponding expression for the solution in layer $j-1$ gives:

$$D_{j-1}\, e^{-u_{j-1}(z_j - z_{j-1})} \;+\; U_{j-1}\, e^{u_{j-1}(z_j - z_{j-1})} \;=\; D_j \;+\; U_j. \tag{A–9}$$

Applying the condition given in eq. (A–8) at $z = z_j$ gives:

$$\frac{1}{\mu_{j-1}} \Big\{ -\, D_{j-1}\, u_{j-1}\, e^{-u_{j-1}(z_j - z_{j-1})} \;+\; U_{j-1}\, u_{j-1}\, e^{u_{j-1}(z_j - z_{j-1})} \Big\} \;=\; \frac{1}{\mu_j} \Big\{ -\, D_j\, u_j \;+\; U_j\, u_j \Big\}. \qquad \text{(A–10)}$$

The last two equations can be combined and expressed in matrix notation:

$$\begin{pmatrix} e^{-u_{j-1}t_{j-1}} & e^{u_{j-1}t_{j-1}} \\ -\frac{u_{j-1}}{\mu_{j-1}} e^{-u_{j-1}t_{j-1}} & \frac{u_{j-1}}{\mu_{j-1}} e^{u_{j-1}t_{j-1}} \end{pmatrix} \begin{pmatrix} D_{j-1} \\ U_{j-1} \end{pmatrix} \;=\; \begin{pmatrix} 1 & 1 \\ -\frac{u_j}{\mu_j} & \frac{u_j}{\mu_j} \end{pmatrix} \begin{pmatrix} D_j \\ U_j \end{pmatrix}, \qquad \text{(A–11)}$$

where $t_{j-1} = z_j - z_{j-1}$ is the thickness of layer $j-1$. Rearranging gives:

$$\begin{pmatrix} D_{j-1} \\ U_{j-1} \end{pmatrix} \;=\; \begin{pmatrix} e^{-u_{j-1}t_{j-1}} & e^{u_{j-1}t_{j-1}} \\ -\frac{u_{j-1}}{\mu_{j-1}} e^{-u_{j-1}t_{j-1}} & \frac{u_{j-1}}{\mu_{j-1}} e^{u_{j-1}t_{j-1}} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ -\frac{u_j}{\mu_j} & \frac{u_j}{\mu_j} \end{pmatrix} \begin{pmatrix} D_j \\ U_j \end{pmatrix} \qquad \text{(A–12)}$$

$$= \frac{\mu_{j-1}}{2u_{j-1}} \begin{pmatrix} \frac{u_{j-1}}{\mu_{j-1}} e^{u_{j-1}t_{j-1}} & -e^{u_{j-1}t_{j-1}} \\ \frac{u_{j-1}}{\mu_{j-1}} e^{-u_{j-1}t_{j-1}} & e^{-u_{j-1}t_{j-1}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\frac{u_j}{\mu_j} & \frac{u_j}{\mu_j} \end{pmatrix} \begin{pmatrix} D_j \\ U_j \end{pmatrix} \qquad \text{(A–13)}$$

$$= \frac{1}{2}\, e^{u_{j-1}t_{j-1}} \begin{pmatrix} 1 & -\frac{\mu_{j-1}}{u_{j-1}} \\ e^{-2u_{j-1}t_{j-1}} & \frac{\mu_{j-1}}{u_{j-1}} e^{-2u_{j-1}t_{j-1}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\frac{u_j}{\mu_j} & \frac{u_j}{\mu_j} \end{pmatrix} \begin{pmatrix} D_j \\ U_j \end{pmatrix} \qquad \text{(A–14)}$$

$$= e^{u_{j-1}t_{j-1}} \begin{pmatrix} \frac{1}{2}\left(1 + \frac{\mu_{j-1}u_j}{\mu_j u_{j-1}}\right) & \frac{1}{2}\left(1 - \frac{\mu_{j-1}u_j}{\mu_j u_{j-1}}\right) \\ \frac{1}{2}\left(1 - \frac{\mu_{j-1}u_j}{\mu_j u_{j-1}}\right) e^{-2u_{j-1}t_{j-1}} & \frac{1}{2}\left(1 + \frac{\mu_{j-1}u_j}{\mu_j u_{j-1}}\right) e^{-2u_{j-1}t_{j-1}} \end{pmatrix} \begin{pmatrix} D_j \\ U_j \end{pmatrix} \qquad \text{(A–15)}$$

$$= e^{u_{j-1}t_{j-1}}\, \underline{\mathbf{M}}_j \begin{pmatrix} D_j \\ U_j \end{pmatrix}, \qquad \text{(A–16)}$$

where

$$\underline{\mathbf{M}}_j \;=\; \begin{pmatrix} \frac{1}{2}\left(1 + \frac{\mu_{j-1}u_j}{\mu_j u_{j-1}}\right) & \frac{1}{2}\left(1 - \frac{\mu_{j-1}u_j}{\mu_j u_{j-1}}\right) \\ \frac{1}{2}\left(1 - \frac{\mu_{j-1}u_j}{\mu_j u_{j-1}}\right) e^{-2u_{j-1}t_{j-1}} & \frac{1}{2}\left(1 + \frac{\mu_{j-1}u_j}{\mu_j u_{j-1}}\right) e^{-2u_{j-1}t_{j-1}} \end{pmatrix}, \qquad \text{(A–17)}$$

for $j \geq 2$. In layer 0 (the air halfspace), $\tilde{F}$ is given by

$$\tilde{F}_0 \;= D_0\, e^{-u_0 z} \;+\; U_0\, e^{u_0 z} \qquad \text{(A–18)}$$

rather than by the expression given in eq. (A–6). The conditions in eqs. (A–7) and (A–8) at $z = z_1 = 0$ therefore give

$$D_0 \;+\; U_0 \;=\; D_1 \;+\; U_1 \qquad \text{(A–19)}$$

and

$$\frac{1}{\mu_0} \Big\{ -\, D_0\, u_0 \;+\; U_0\, u_0 \Big\} \;=\; \frac{1}{\mu_1} \Big\{ -\, D_1\, u_1 \;+\; U_1\, u_1 \Big\}. \qquad \text{(A–20)}$$

In matrix form, these last two equations become

$$\begin{pmatrix} 1 & 1 \\ -\frac{u_0}{\mu_0} & \frac{u_0}{\mu_0} \end{pmatrix} \begin{pmatrix} D_0 \\ U_0 \end{pmatrix} \;=\; \begin{pmatrix} 1 & 1 \\ -\frac{u_1}{\mu_1} & \frac{u_1}{\mu_1} \end{pmatrix} \begin{pmatrix} D_1 \\ U_1 \end{pmatrix} \qquad \text{(A–21)}$$

and hence

$$\begin{pmatrix} D_0 \\ U_0 \end{pmatrix} \;=\; \begin{pmatrix} 1 & 1 \\ -\frac{u_0}{\mu_0} & \frac{u_0}{\mu_0} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ -\frac{u_1}{\mu_1} & \frac{u_1}{\mu_1} \end{pmatrix} \begin{pmatrix} D_1 \\ U_1 \end{pmatrix} \qquad \text{(A–22)}$$

$$= \frac{\mu_0}{2u_0} \begin{pmatrix} \frac{u_0}{\mu_0} & -1 \\ \frac{u_0}{\mu_0} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\frac{u_1}{\mu_1} & \frac{u_1}{\mu_1} \end{pmatrix} \begin{pmatrix} D_1 \\ U_1 \end{pmatrix} \qquad \text{(A–23)}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -\frac{\mu_0}{u_0} \\ 1 & \frac{\mu_0}{u_0} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\frac{u_1}{\mu_1} & \frac{u_1}{\mu_1} \end{pmatrix} \begin{pmatrix} D_1 \\ U_1 \end{pmatrix} \qquad \text{(A–24)}$$

$$= \begin{pmatrix} \frac{1}{2}\left(1 + \frac{\mu_0 u_1}{u_0 \mu_1}\right) & \frac{1}{2}\left(1 - \frac{\mu_0 u_1}{u_0 \mu_1}\right) \\ \frac{1}{2}\left(1 - \frac{\mu_0 u_1}{u_0 \mu_1}\right) & \frac{1}{2}\left(1 + \frac{\mu_0 u_1}{u_0 \mu_1}\right) \end{pmatrix} \begin{pmatrix} D_1 \\ U_1 \end{pmatrix} \qquad \text{(A–25)}$$

$$= \underline{\mathbf{M}}_1 \begin{pmatrix} D_1 \\ U_1 \end{pmatrix} \qquad \text{(A–26)}$$

where

$$\underline{\mathbf{M}}_1 \;=\; \begin{pmatrix} \frac{1}{2}\Big(1+\frac{\mu_0 u_1}{\mu_1 u_0}\Big) & \frac{1}{2}\Big(1-\frac{\mu_0 u_1}{\mu_1 u_0}\Big) \\ \frac{1}{2}\Big(1-\frac{\mu_0 u_1}{\mu_1 u_0}\Big) & \frac{1}{2}\Big(1+\frac{\mu_0 u_1}{\mu_1 u_0}\Big) \end{pmatrix}. \tag{A–27}$$

Equation (A–16) (and A–26) therefore provides a means of relating the solution for $\tilde{F}$ in a layer to the solution in the adjacent layers. Assuming there are N layers in the Earth model with layer 0 being the air halfspace, $z_1 = 0$ (the surface of the Earth), and layer N being the basement halfspace, U_0 and D_0 can be related to U_N and D_N by application of eq. (A–26) and successive applications of eq. (A–16):

$$\begin{pmatrix} D_0 \\ U_0 \end{pmatrix} \;=\; \underline{\mathbf{M}}_1 \;\exp\Big(\sum_{j=2}^{N} u_{j-1} t_{j-1}\Big) \;\prod_{j=2}^{N} \underline{\mathbf{M}}_j \;\begin{pmatrix} D_N \\ U_N \end{pmatrix}. \tag{A–28}$$

It is now assumed that the source is located in the air halfspace above the Earth. The downward-decaying part of the solution for $\tilde{F}$ in layer 0 is therefore due to the source: $D_0 = D_0^S$ where D_0^S is a known quantity once the type of source has been specified. In the basement halfspace there is no upward-decaying part to the solution: $U_N = 0$. Equation (A–28) can therefore be rewritten as

$$\begin{pmatrix} D_0^S \\ U_0 \end{pmatrix} \;=\; E\,\underline{\mathbf{P}}\,\begin{pmatrix} D_N \\ 0 \end{pmatrix}, \tag{A–29}$$

where the matrix $\underline{\mathbf{P}}$ is given by

$$\underline{\mathbf{P}} \;=\; \underline{\mathbf{M}}_1 \prod_{j=2}^{N} \underline{\mathbf{M}}_j, \tag{A–30}$$

and the factor E by

$$E \;=\; \exp\Big(\sum_{j=2}^{N} u_{j-1} t_{j-1}\Big). \tag{A–31}$$

The first of the implied pair of equations in eq. (A–29) gives

$$D_N \;=\; \frac{1}{E}\frac{1}{P_{11}}\,D_0^S, \tag{A–32}$$

and the second gives

$$U_0 \;=\; E\,P_{21}\,D_N. \tag{A–33}$$

Substituting eq. (A–32) into eq. (A–33) gives

$$U_0 \;=\; E\,P_{21}\,\frac{1}{E}\frac{1}{P_{11}}\,D_0^S, \tag{A–34}$$

$$\;=\; \frac{P_{21}}{P_{11}}\,D_0^S. \tag{A–35}$$

The solution for $\tilde{F}$ in the air halfspace ($z < 0$) is now given by

$$\tilde{F}_0 \;=D_0^S\Big(e^{-u_0 z} \;+\; \frac{P_{21}}{P_{11}} e^{u_0 z}\Big). \tag{A–36}$$

For a unit vertical magnetic dipole source at a height h (i.e. $z = -h$ for $h > 0$) above the surface of the Earth, the downward-decaying part of the primary solution for $\tilde{F}$ (and the only downward-decaying part of the solution for $\tilde{F}$ in the air halfspace) at the surface of the Earth ($z = 0$) is given by

$$D_0^S \;=\; \frac{i\omega\mu_0}{2u_0}\,e^{-u_0 h} \tag{A–37}$$

(Ward & Hohmann, eq. 4.40). Substituting this into eq. (A–36) gives

$$\tilde{F}_0 \;=\; \frac{i\omega\mu_0}{2u_0}\Big(e^{-u_0(z+h)} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\Big). \tag{A–38}$$

Generalizing this expression for z above ($z < -h$) as well as below the source ($z > -h$):

$$\tilde{F}_0 \;=\; \frac{i\omega\mu_0}{2u_0}\Big(e^{-u_0|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\Big). \tag{A–39}$$

For a unit x-directed magnetic dipole source at $z = -h$, the downward-decaying part of $\tilde{F}$ in the air halfspace evaluated at the surface of the Earth is

$$D_0^S \;=\; -\frac{i\omega\mu_0}{2}\,\frac{ik_x}{k_x^2+k_y^2}\,e^{-u_0h} \tag{A–40}$$

(Ward & Hohmann, eq. 4.106). Substituting this into eq. (A–36) gives

$$\tilde{F}_0 \;=-\frac{i\omega\mu_0}{2}\;\frac{ik_x}{k_x^2+k_y^2}\Big(e^{-u_0(z+h)} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\Big). \tag{A–41}$$

For $z < -h$ as well as $-h < z < 0$:

$$\tilde{F}_0 \;=-\frac{i\omega\mu_0}{2}\;\frac{ik_x}{k_x^2+k_y^2}\Big(e^{-u_0|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\Big). \tag{A–42}$$

Applying the inverse Fourier transform given in eq. (A–4) to eq. (A–39) gives

$$F_0(x,y,z,\omega) \;=\; \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\omega\mu_0}{2u_0}\Big(e^{-u_0|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\Big)\,e^{i(k_xx+k_yy)}\;dk_x\,dk_y \tag{A–43}$$

for a vertical magnetic dipole, and applying the inverse Fourier transform to eq. (A–42) gives

$$\begin{aligned} F_0(x,y,z,\omega) \;&=\; -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\omega\mu_0}{2}\;\frac{ik_x}{k_x^2+k_y^2}\Big(e^{-u_0|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\Big)\,e^{i(k_xx+k_yy)}\;dk_x\,dk_y && \text{(A–44)}\\ &=\; -\frac{1}{8\pi^2}\,\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\omega\mu_0}{k_x^2+k_y^2}\Big(e^{-u_0|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\Big)\,e^{i(k_xx+k_yy)}\;dk_x\,dk_y && \text{(A–45)}\end{aligned}$$

for an x-directed magnetic dipole. The kernels in the Fourier integrals in eqs. (A–43) and (A–45) are functions of $k_x^2 + k_y^2$ only. Using the identity

$$\int_{\infty}^{\infty}\int_{\infty}^{\infty}\tilde{F}(k_x^2+k_y^2)\,dk_x\,dk_y \;=\; 2\pi\int_0^{\infty}\tilde{F}(\lambda)\,\lambda\,J_0(\lambda r)\,d\lambda, \tag{A–46}$$

(Ward & Hohmann, eq.2.10) where $\lambda^2 = k_x^2 + k_y^2$ and $r^2 = x^2 + y^2$, eq. (A–43) can be rewritten as

$$\begin{aligned} F_0(x,y,z,\omega) \;&=\; \frac{1}{2\pi}\int_0^{\infty}\frac{i\omega\mu_0}{2u_0}\Big(e^{-u_0|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\Big)\,\lambda\,J_0(\lambda r)\,d\lambda && \text{(A–47)}\\ &=\; \frac{i\omega\mu_0}{4\pi}\int_0^{\infty}\Big(e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\Big)\,J_0(\lambda r)\,d\lambda && \text{(A–48)}\end{aligned}$$

since $u_0 = \lambda$, and eq. (A–45) can be rewritten as

$$F_0(x,y,z,\omega) \;=\; -\,\frac{1}{4\pi}\,\frac{\partial}{\partial x}\int_0^\infty \frac{i\omega\mu_0}{\lambda^2}\Big(e^{-u_0|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\Big)\,\lambda\,J_0(\lambda r)\,d\lambda \qquad \text{(A–49)}$$

$$=\; -\,\frac{i\omega\mu_0}{4\pi}\,\frac{\partial}{\partial x}\int_0^\infty \Big(e^{-u_0|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{u_0(z-h)}\Big)\,\frac{1}{\lambda}\,J_0(\lambda r)\,d\lambda \qquad \text{(A–50)}$$

$$=\; \frac{i\omega\mu_0}{4\pi}\,\frac{x}{r}\int_0^\infty \Big(e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\Big)\,J_1(\lambda r)\,d\lambda, \qquad \text{(A–51)}$$

since

$$\frac{\partial J_0(\lambda r)}{\partial x} \;=\; -\,\lambda\frac{x}{r}\,J_1(\lambda r) \qquad \text{(A–52)}$$

(Ward & Hohmann, eq. 4.44 (almost)), and $u_0 = \lambda$.

The H-field in the air halfspace can be obtained from eqs. (A–48) and (A–51) (well, really eq. A–50) by using eq. (1.130) of Ward & Hohmann:

$$H_x \;=\; \frac{1}{i\omega\mu_0}\,\frac{\partial^2 F_0}{\partial x\partial z}, \qquad \text{(A–53)}$$

$$H_y \;=\; \frac{1}{i\omega\mu_0}\,\frac{\partial^2 F_0}{\partial y\partial z}, \qquad \text{(A–54)}$$

$$H_z \;=\; \frac{1}{i\omega\mu_0}\,\Big(\frac{\partial^2}{\partial z^2} \;+\; \kappa_0^2\Big)\,F_0 \qquad \text{(A–55)}$$

$$=\; \frac{1}{i\omega\mu_0}\,\frac{\partial^2 F_0}{\partial z^2}. \qquad \text{(A–56)}$$

since $\kappa_0^2 = 0$. Applying eqs. (A–53) to (A–56) to eq. (A–48) gives

$$H_x(x,y,z,\omega) \;=\; \frac{1}{4\pi}\,\frac{x}{r}\int_0^\infty \Big(e^{-\lambda|z+h|} \;-\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\Big)\,\lambda^2\,J_1(\lambda r)\,d\lambda, \qquad \text{(A–57)}$$

$$H_y(x,y,z,\omega) \;=\; \frac{1}{4\pi}\,\frac{y}{r}\int_0^\infty \Big(e^{-\lambda|z+h|} \;-\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\Big)\,\lambda^2\,J_1(\lambda r)\,d\lambda, \qquad \text{(A–58)}$$

$$H_z(x,y,z,\omega) \;=\; \frac{1}{4\pi}\int_0^\infty \Big(e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\Big)\,\lambda^2\,J_0(\lambda r)\,d\lambda, \qquad \text{(A–59)}$$

for a z-directed magnetic dipole source at $(0,0,-h)$ (cf. eqs. 4.45 & 4.46 of Ward & Hohmann). Applying eq. (A–53) to eq. (A–50) gives

$$H_x(x,y,z,\omega) \;=\; -\,\frac{1}{4\pi}\,\frac{\partial^2}{\partial x^2}\int_0^\infty \Big(-\,e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\Big)\,J_0(\lambda r)\,d\lambda \qquad \text{(A–60)}$$

$$=\; -\,\frac{1}{4\pi}\Big(\frac{1}{r} - \frac{2x^2}{r^3}\Big)\int_0^\infty \Big(e^{-\lambda|z+h|} \;-\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\Big)\,\lambda\,J_1(\lambda r)\,d\lambda$$
$$-\;\frac{1}{4\pi}\,\frac{x^2}{r^2}\int_0^\infty \Big(e^{-\lambda|z+h|} \;-\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\Big)\,\lambda^2\,J_0(\lambda r)\,d\lambda, \qquad \text{(A–61)}$$

using eqs. (4.114) to (4.118) of Ward & Hohmann. Applying eq. (A–54) to eq. (A–50) gives

$$H_y(x,y,z,\omega) \;=\; -\,\frac{1}{4\pi}\,\frac{\partial^2}{\partial x\partial y}\int_0^\infty \Big(-\,e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\Big)\,J_0(\lambda r)\,d\lambda \qquad \text{(A–62)}$$

$$=\; \frac{1}{2\pi}\,\frac{xy}{r^3}\int_0^\infty \Big(e^{-\lambda|z+h|} \;-\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\Big)\,\lambda\,J_1(\lambda r)\,d\lambda$$
$$-\;\frac{1}{4\pi}\,\frac{xy}{r^2}\int_0^\infty \Big(e^{-\lambda|z+h|} \;-\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\Big)\,\lambda^2\,J_0(\lambda r)\,d\lambda, \qquad \text{(A–63)}$$

again using eqs. (4.114) to (4.118) of Ward & Hohmann. Finally, applying eq. (A–56) to eq. (A–51) gives

$$H_z(x,y,z,\omega) \;=\; \frac{1}{4\pi}\,\frac{x}{r}\int_0^\infty \left(e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\right)\lambda^2\,J_1(\lambda r)\,d\lambda \tag{A–64}$$

(cf. eqs. 4.119 to 4.121 of Ward & Hohmann).

To summarize, for a z-directed magnetic dipole source at $(0,0,-h)$, $h>0$, the three components of the H-field in the air halfspace (i.e. $z<0$) are given by eqs. (A–57) to (A–59):

$$\begin{aligned}
H_x(x,y,z,\omega) \;&=\; \frac{1}{4\pi}\,\frac{x}{r}\int_0^\infty \left(e^{-\lambda|z+h|} \;-\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\right)\lambda^2\,J_1(\lambda r)\,d\lambda,\\
H_y(x,y,z,\omega) \;&=\; \frac{1}{4\pi}\,\frac{y}{r}\int_0^\infty \left(e^{-\lambda|z+h|} \;-\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\right)\lambda^2\,J_1(\lambda r)\,d\lambda,\\
H_z(x,y,z,\omega) \;&=\; \frac{1}{4\pi}\int_0^\infty \left(e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\right)\lambda^2\,J_0(\lambda r)\,d\lambda.
\end{aligned}$$

And for an x-directed magnetic dipole source at $(0,0,-h)$, $h>0$, the three components of the H-field in the air halfspace (i.e. $z<0$) are given by eqs. (A–61), (A–63) and (A–64):

$$\begin{aligned}
H_x(x,y,z,\omega) \;&=\; -\,\frac{1}{4\pi}\left(\frac{1}{r}-\frac{2x^2}{r^3}\right)\int_0^\infty \left(e^{-\lambda|z+h|} \;-\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\right)\lambda\,J_1(\lambda r)\,d\lambda\\
&\qquad -\;\frac{1}{4\pi}\,\frac{x^2}{r^2}\int_0^\infty \left(e^{-\lambda|z+h|} \;-\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\right)\lambda^2\,J_0(\lambda r)\,d\lambda,\\
H_y(x,y,z,\omega) \;&=\; \frac{1}{2\pi}\,\frac{xy}{r^3}\int_0^\infty \left(e^{-\lambda|z+h|} \;-\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\right)\lambda\,J_1(\lambda r)\,d\lambda\\
&\qquad -\;\frac{1}{4\pi}\,\frac{xy}{r^2}\int_0^\infty \left(e^{-\lambda|z+h|} \;-\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\right)\lambda^2\,J_0(\lambda r)\,d\lambda,\\
H_z(x,y,z,\omega) \;&=\; \frac{1}{4\pi}\,\frac{x}{r}\int_0^\infty \left(e^{-\lambda|z+h|} \;+\; \frac{P_{21}}{P_{11}}e^{\lambda(z-h)}\right)\lambda^2\,J_1(\lambda r)\,d\lambda.
\end{aligned}$$

For a y-directed magnetic dipole source, just apply the rotation $(x,y,z)\rightarrow(y,-x,z)$ to the above equations for an x-directed dipole (i.e. replace x everywhere in the equations by y, and replace y everywhere by $-x$).