AN EXAMPLE

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1. Reminder - The old example

Recall the example from [1]: The alphabet in this example is $A = \{0,1\} \times \{-1,1\}$, and we consider the input to the g-function as two sequences: $(X_{-n})_{n\geq 1}$, with entries in $\{-1,1\}$, and $(Y_{-n})_{n\geq 1}$, with entries in $\{0,1\}$. It is most convenient to define the g-function as a mechanism for sampling X_0 and Y_0 given $(X_{-n})_{n\geq 1}$ and $(Y_{-n})_{n\geq 1}$, and this is done as follows (many crucial details omitted):

(1) Sample Y_0 as a $1/2$-Bernoulli variable, completely independent of $(X_{-n})_{n\geq 1}$ and $(Y_{-n})_{n\geq 1}$.
(2) Using $(Y_{-n})_{n\geq 0}$ choose a finite set $S \subseteq \{-n : n \in \mathbb{N}\}$. Either $|S|$ is odd, or $S = \emptyset$.
(3) Using $(Y_{-n})_{n\geq 0}$ choose a number ξ between 0.5 and 0.75.
(4) If $S \neq \emptyset$, then take $X_0 = \operatorname{sgn} \sum_{k\in S} X_k$ with probability ξ, and $X_0 = -\operatorname{sgn} \sum_{k\in S} X_k$ with probability $1-\xi$. If $S = \emptyset$, choose X_0 to be 1 w.p. 0.5 and -1 w.p. 0.5.

Then, if one chooses S and ξ wisely enough, then g is continuous (in fact the variance of g is in L^p for p arbitrarily close to 2), and g has multiple g-measures. In order to show that g has multiple g-measures, we do the following: Let μ be a measure in the set K_g.

(1) First note that the μ-distribution of the sequence (Y_n) is just i.i.d. Bernoulli $1/2$.
(2) Using $(Y_{-n})_{n\geq 0}$, we define a sequence (S_k) of finite subsets of $\{-n, n \in \mathbb{N}\}$. μ-a.s, all but finitely many of them are of odd size.
(3) Write $W_n = \operatorname{sgn} \sum_{k\in S_n} X_k$. Then a.s. $W = \lim_{n\to\infty} W_n$ exists, and is in $\{-1,1\}$.
(4) Then one can see that μ is completely determined by $\mu(W=1)$. In particular, μ is shift invariant, and must be a convex combination of the two extremal measures μ_+ and μ_-, which are defined in a guessable way.

2. Construction of the new example

Let g be the function from [1], we then define $\tilde{g}$ as follows. Define the inversion $F : \{-1,1\}^E \to \{-1,1\}^E$, where E is a set of numbers, typically $-\mathbb{N}$ or $\mathbb{Z}$, by

$$\big(F\big((X_n)\big)\big)_n = (-1)^n X_n.$$

For convenience we also write $F\big((X_n),(Y_n)\big) := \big(F\big((X_n)\big),(Y_n)\big)$.

Then we take $\tilde{g}(\bar{X},\bar{Y}) = g(F(\bar{X},\bar{Y})) = g(F(\bar{X}),\bar{Y})$. Let μ be extremal in $K_{\tilde{g}}$. Then $\mu \circ F^{-1}$ is extremal in K_g, and the push-forward by F is a bijection between the sets of extremal measures in $K_{\tilde{g}}$ and K_g.

Also note that $\mu_- \circ F^{-1} = T(\mu_+ \circ F^{-1})$ where T is the shift operator (since F is an inversion, we immediately get also $\mu_+ \circ F^{-1} = T(\mu_- \circ F^{-1})$). Thus the unique stationary element of $K_{\tilde{g}}$ is $\big(\mu_+ \circ F^{-1} + \mu_- \circ F^{-1}\big)/2$, but the set $K_{\tilde{g}}$ also contains elements that are not stationary, which implies that the convergence in your original problem statement does not hold.

References

[1] Noam Berger, Christopher Hoffman, and Vladas Sidoravicius. Non-uniqueness for specifications in. *Ergodic Theory and Dynamical Systems*, 38(4):1342–1352, 2018.