Existence for an eigenfunction for the sub-critical phase of the Dyson Ising model

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1 Introduction

1.1 The random cluster model and potentials

We consider measures $\alpha(x) \in \mathcal{M}(X)$ on configurations $x \in \mathcal{X} = A^V$, where A is a finite set (the "alphabet") and V (the "sites") is a countable. For a subset $\Lambda \subset V$, we write x_Λ for the restriction $x|_\Lambda$ of the configuration x to Λ.

A *potential* $\phi(x)$ on X is a limit $\phi(x) = \lim_{n\to\infty} \phi_n(x)$ of functions such that, for any pair of configurations $x, y \in \mathcal{X}$ that coincide outside some finite set, the difference is well defined. That is,

$$\phi(x) - \phi(y) = \lim_{n\to\infty} \phi_n(x) - \phi_n(y) < \infty \tag{1}$$

if $x_{\Lambda^c} = y_{\Lambda^c}$ and $\Lambda \Subset V$.

Given a positive measure $\alpha(x)$ on $x \in \mathcal{X}$ and a potential ϕ on $\mathcal{X}$, we denote by

$$e^{\phi(x)} \ltimes \alpha(x)$$

the set of corresponding *Gibbs measures.* That is, $e^{\phi} \ltimes \alpha$ is the set of probability measures obtained as weak limits of

$$\mathsf{P}(x|_\Lambda \mid x|_{\Lambda^c}) = \frac{e^{\phi(x_\Lambda, x_{\Lambda^c})}\alpha(x_\Lambda, x_{\Lambda^c})}{\int_{x_\Lambda} e^{\phi(x_\Lambda, x_{\Lambda^c})}\, d\alpha(x_\Lambda, x_{\Lambda^c})}$$

where $\Lambda \nearrow V$ is a sequence of finite subsets $\Lambda \Subset V$. In our applications we can assume that $e^{\phi(x)} \ltimes \alpha$ is unique.

One defines the *Potts* model with parameters α, β and q as follows: Let $\mathcal{X} = A^{\mathbb{Z}}$, $A = \{1, \ldots, q\}$, and Potts model is the following Gibbs measure

$e^{\phi(x)} \ltimes \kappa$ with and

$$\phi(x) = \beta \sum_{i \neq j} J(ij) \delta_x(i) \bullet \delta_x(j) \tag{2}$$

where the interaction strengths $J(ij)$ are

$$J(ij) = \frac{1}{|i-j|^\alpha}. \tag{3}$$

1.1.1 Rules

We use the following rules pertaining to the construction of Gibbs measures: First note that

$$e^{\phi(x)} \ltimes (e^{\psi(x)} \ltimes \alpha(x)) = e^{\phi(x)+\psi(x)} \ltimes \alpha(x). \tag{4}$$

Secondly, if $\{S, T\}$ is a partition of V and the distribution $\alpha(x)$ of $x = (x_S, x_T)$ is a product measure $\alpha(x) = \beta(x_S) \otimes \gamma(x_T)$ then

$$e^{\phi(x_S)+\psi(x_T)} \ltimes (\beta(x_S) \otimes \gamma(x_T)) = (e^{\phi(x_S)} \ltimes (\beta(x_S)) \otimes (e^{\psi(x_T)} \ltimes \gamma(x_T)) \tag{5}$$

Finally, if $e^\psi \in L^1(\alpha)$ then

$$e^\psi \ltimes \alpha = e^{\psi(x)-\langle\psi\rangle} \cdot \alpha(x) \tag{6}$$

where $\langle\psi\rangle = \log \int e^\psi \, d\alpha$.

1.1.2 Bernoulli measures

Examples of unique Gibbs measures are *Bernoulli measures*: Consider the Bernoulli measure $\eta(x)$ where the marginal distribution of $x_i \in A$ is $p(i)$, i.e. $p(i)(a) = \mathsf{P}(x_i = a)$, $a \in A$. We can obtain this as the Gibbs measure $\eta(x) = e^{\phi(x)} \ltimes \kappa(x)$ where κ is the counting measure and the potential used is the linear function

$$\phi(x) = \sum_{i \in V} \delta_x(i) \bullet \log p(i) \sum_{a \in A} \delta_x(i)(a) \, \log p_i(a)$$

in the representation $\delta_x \in (\{0,1\}^A)^V$ of x. In general, any gibbs measure $e^{\phi(x)} \ltimes \eta(x)$ with a linear potential $\phi(x)$ and a Bernoulli measure η results in a new unique Bernoulli measure.

1.2 The one-dimensional random cluster model and the Ising-Dyson model

For a finite graph, let $\omega(G)$ denote the number of connected components ("clusters") in the graph G. For simple graphs $\gamma \in \{0,1\}^{\binom{V}{2}}$ on an countably infinite set V of vertices, the number of clusters $\omega(\gamma)$ is well defined as a *potential* since the difference $\omega(\gamma) - \omega(\gamma')$ is well defined and finite for any two graphs γ and γ' that coincide on $\binom{\Lambda^c}{2}$ for a *finite* subset $\Lambda \Subset V$.

Starting with $p : \binom{V}{2} \to [0,1]$, we can consider the Bernoulli graph model as the Bernoulli measure $\eta(\gamma; p)$ with marginals $\mathsf{P}(\gamma(ij) = 1) = p(ij)$ and $\mathsf{P}(\gamma(ij) = 0) = 1 - p(ij)$. The Random Cluster Model $\mathcal{RC}(\gamma; p, q)$, corresponding to p and $q \geq 1$, is the Gibbs measure $q^{\omega(\gamma)} \ltimes \eta(p)$.

One obtains the extended *random cluster model* by considering the product measure $d\eta(\gamma; p) \otimes d1(\frac{1}{q})$ between an independent Bernoulli distributed $\gamma \sim \eta(p)$ random graph and the uniform Bernoulli measure $\eta(\frac{1}{q})$ on the spin (or "color") sequences $x \in \mathcal{X} = A^V$, where $A = \{1, \ldots, q\}$. The extended random cluster model $\pi(x, \gamma)$ is the joint distribution of γ and x obtained by conditioning on the event that x and γ are *compatible*: That is, the event $B(x, \gamma)$ that no path in γ join two sites of different spins or, equivalently, all clusters $C \in \mathcal{C}(\gamma)$ are monochromatic under the colouring x. One obtain, simultaneously, the Potts model as the distrubution $\pi(x)$ of x and the random cluster model $\pi(g) = \mathcal{RC}(\gamma; ,)$ as the marginal distributions of x and γ, respectively. See [?].

In the one-dimensional Potts-Dyson model, let $V = \mathbb{Z}$ (or $V = \mathbb{Z}_+$). For $i, j \in V$ let

$$p(ij) = 1 - \exp(-\beta J(ij)) \quad \text{where} \quad J(ij) = \frac{1}{|i-j|^\alpha}. \tag{7}$$

and the Dyson model we consider is random cluster model $\pi(x, \gamma; \beta, q) = \mathcal{RC}(x, \gamma; p, q)$.

We can also introduce the random cluster model μ as the Gibbs measure on $\{0,1\}^{\binom{V}{2}}$

$$d\mu = 2^{\omega(\gamma)} \ltimes d\eta(\gamma),$$

where $\omega(\gamma)$ is the potential counting then number of connected components ("clusters") in γ. For the values of β we consider the Gibbs measure is unique. Thus can we parameterise the random clusters models as $\mu = \mathcal{R}(V, J)$ where $J(ij)$ is a given weighting on $\binom{V}{2}$ such as (7).

We differ between the one-sided random cluster model $\nu = \mathcal{R}(V_+, J)$ and the usual two-sided model $\mu = \mathcal{R}(V, J)$. We will use that a configuration γ can be factored as $\gamma = (\gamma_+, \varepsilon, \gamma_-)$, where γ_- is the induced graph $\gamma[V_-]$ on vertices $-j \in V_- = \mathbb{Z}_{<0}$ and $\gamma_+ = \gamma[V_+]$ is the graph induced on vertices $i \in V_+ = \mathbb{Z}_{\geq 0}$. The graph $\varepsilon = \gamma \cap E(V_+, V-)$ consists of edges ji, $i \geq 0$ and $j \geq 1$, connecting vertices $-j \in V_-$ with vertices $i \in V_+$. Note that we often use positive indices i, j, $i \geq 0$ and $j > 0$, as labels for edges in ε. Thus $J(ij) = \beta/(i+j)^\alpha$ with this labelling.

We extend the one-sided model ν to a probability distribution on V by identifying ν with the product measure

$$d\nu(\gamma) = d\nu(\gamma_-) \otimes d\tilde{\eta}(\varepsilon) \otimes d\nu(\gamma_+)$$

where $d\tilde{\eta}(\varepsilon)$ is the Bernoulli measure $2^{-|\varepsilon|} \ltimes d\eta(\eta)$. Since

$$\omega(\gamma) = \omega(\gamma_+) + \omega(g_-) - |\varepsilon| + R(\gamma)$$

where the correction R is defined as the co-rank of ε in γ, i.e. $R(\gamma)$ counts the number of edges that can be removed from ε without increasing $\omega(\gamma)$.

Let

$$\varepsilon_{ij} = \varepsilon \cap \{i'j' \mid i' < i \text{ or } i' = i \text{ and } j' < j\}$$

and $\gamma_{ij} = (\gamma_-, \varepsilon_{ij}, \gamma_+)$. By the greedy property of matroids, it follows that $R(\gamma) = \sum_{i=0}^\infty \sum_{j=1}^\infty R_{ij}(\gamma)$ where

$$R_{ij}(\gamma) = \begin{cases} 1 & \omega(\gamma_{ij}) = \omega(g_{ij} \setminus ij) \\ 0 & \text{otherwise.} \end{cases}$$

It follows that we can write

$$d\nu(\gamma) = 2^{\omega(\gamma_)+\omega(\gamma_+)-|\varepsilon|} \ltimes d\eta(\gamma) \tag{8}$$

$$d\mu(\gamma) = 2^{\omega(\gamma_)+\omega(\gamma_+)-|\varepsilon|+R(\gamma)} \ltimes d\eta(\gamma) = 2^{R(\gamma)} \ltimes d\nu(\gamma). \tag{9}$$

Lemma 1. *We have that*

$$\int 2^{R(\gamma)} d\nu(\gamma) < \infty$$

and thus $\nu(\gamma)$ and $\mu(\gamma)$ are absolutely continuous as random cluster models.

Proof of Lemma 1. We have since $\beta < \beta_c$ that

$$\sum_{n=1}^\infty \frac{\tau(n)}{n} < \infty.$$

□

For any fixed $x \in X = \{-1, +1\}^V$ and an integer $N \geq 0$, let $x_N = x|_{[0,N)} = (x_0, \ldots, x_{N-1})$ and let $[x]_N = \{y \in X \mid y_N = x_N\}$. Furthermore, for any $x_N \in \{-1, +1\}^{[0,N)}$ let $[x_N] = \{y \mid y_N = x_N\}$.

Let also $\mathcal{R}_N(V, J)$ denote $\mathcal{R}(V + \{s\}, J_N)$ where

$$J_N(ij) = \begin{cases} J(ij) & i, j \neq s \\ \infty & (i,j), (j,i) \in [0, N) \times \{s\} \\ 0 & \text{otherwise} \end{cases}$$

The model $\mathcal{R}_N(V)$ is the random cluster model on V, where all vertices in $[0, N)$ have been contracted to one. We have that

$$d\mu_N = 2^{-\omega_N(\gamma)} \ltimes d\mu(\gamma) = 2^{\overline{\omega}_N(\gamma)} \ltimes d\eta(\gamma)$$

where $\omega_N(\gamma)$ denotes the number of clusters in γ that intersect $[0, N)$ and the potential $\overline{\omega}_N(\gamma) = \omega(\gamma) - \omega_N(\gamma)$ counts the number of clusters that do not.

Lemma 2. *For any given $x \in X$ we have*

$$\mu([x]_N) \propto \int B(x_N, \gamma)\, d\mu_N(\gamma)$$

and

$$\nu([x]_N) \propto \int B(x_N, \gamma)\, d\nu_N(\gamma).$$

Furthermore,

$$\frac{d\mu_N}{d\nu_N}(\gamma) \propto 2^{Q_N(\gamma)} = 2^{Q(\gamma) - \tilde{R}_N(\gamma)}$$

where $Q(\gamma) = \lim_N Q_N(\gamma)$ and $Q_N(\gamma)$ is the number of edges ij, $i \leq n$, such that $\omega_N(\gamma_{ij} + \varepsilon_{ij}) = \omega_N(\varepsilon_{ij})$. Note that $\tilde{R}_N(\gamma) = Q(\gamma) - Q_N(\gamma) \geq 0$.

Note that, $Q(\gamma)$ is *independent* of γ_+.

Proof of Lemma 2. Then the probability that a random spin-assignment to clusters of γ should give an element in $[x_N]$ is $\mathsf{P}(x_N|\gamma) = B(x_N, \gamma) \cdot 2^{-\omega_N(\gamma)}$. Thus

$$\mu([x_N]) = \mathsf{E}\left[\mathsf{E}(x_N|\gamma)\right] = \int B(x_N, \gamma) 2^{-\omega_N(\gamma)}\, d\mu(\gamma).$$

We obtain

$$1 = \sum_{x_N} \mu([x]_N) = \int \sum_{X_N} B(x_N, \gamma)\, d\mu_N(\gamma) = \int 2^{\omega_N(\gamma)}\, d\mu_N(\gamma).$$

□

We obtain that

$$f_N(x) = \frac{\mu([x_N])}{\nu([x_N])} \propto \int C(x_N,\gamma) 2^{-Q_N(\gamma)}\, d\nu_N^{\pm}(\gamma)$$

where

$$C(x_N,\gamma) = \frac{B(x_N,\gamma)}{B(x_N,\gamma_+)}$$

Lemma 3. *We have*

$$\int 2^{Q(\gamma_-,\varepsilon)}\, d\mu(\gamma_-,\varepsilon) < \infty.$$

Proof of Lemma 3. Let $W(\gamma) \geq 0$ be the rightmost element in the cluster containing 0. We need to show that

$$\int Q(\gamma)\, d\mu(\gamma) \leq \sum_{n=1}^{\infty} \frac{1}{n} \mathsf{P}(W \geq n) \propto \mathsf{E}(\ln W) < \infty.$$

This should be obvious since $\mathsf{E}(\ln W|X)$ cannot grow superlinearly in the size of the cluster $X = |C(0)|$.

Van der Berg och Kesten showed that

$$f(\beta) = \sum_n \frac{\mathsf{P}_\beta(X = n)}{n}$$

was continuous for all β.

$\square$

The two-sided random cluster model is denoted by μ, i.e.

$$d\mu(\gamma) = 2^{\omega(\gamma)} \ltimes d\eta(\gamma),$$

where $\omega(G)$ is the potential $\omega(\gamma) = \lim_{\Lambda_n \nearrow V} \omega(\gamma[\Lambda_n])$ and η denote the bernoulli measure.

The one-sided model can be captured as the marginal distribution of γ_+ in the product measure

$$d\nu(G) = (2^{\omega(\gamma_+)} \ltimes d\eta(\gamma_-)) \otimes (2^{-|\epsilon|} \ltimes d\eta(\epsilon)) \otimes (2^{\omega(\gamma_+)} \ltimes d\eta(\gamma_+)).$$

Since

$$\omega(G) = \omega(\gamma_-) + \omega(\gamma_+) - |\epsilon| + R(\gamma_+,\epsilon,\gamma_+)$$

it follows that the two-sided Ising model can be obtained as

$$d\mu(\gamma) = 2^{\omega(\gamma)} \ltimes d\eta(\gamma) = 2^{R(\gamma)} \ltimes d\nu(\gamma).$$