Käenmäki example

This example provides a g-measure μ such that the ratio

$$\frac{\mu([x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_m])}{\mu([x_1, x_2, \dots, x_m]) \cdot \mu([y_1, y_2, \dots, y_m])}$$

is uniformly bounded from above but not from below. A restatement says that the likelihood ratios between any conditioning of the measure and measure itself, i.e.

$$\frac{\mu([x_1, x_2, \dots, x_m] \mid y)}{\mu([x_1, x_2, \dots, x_m])} = \frac{\mu([x_1, x_2, \dots, x_m] \mid y)}{\int \mu([x_1, x_2, \dots, x_m \mid z])\, d\mu(z)},$$

are bounded from above but not from below.

Let $X = \{0, 1\}^S$, $S = \{0, 1, 2, \dots\}$, and let $\mu \in \mathcal{M}(X)$ be the unique Hofbauer-type g-measure with

$$g(0x) = e^{-s_{W(x)}}, \quad g(1x) = 1 - e^{-s_{W(x)}}$$

where $W(x)$ is the number of initial zeros in $x \in X$ and

$$s_k = 2 - \frac{1}{1+k}.$$

Note that μ is sandwiched (in stochastic dominance ordering) between two iid Bernouilli sequences with $p = 1 - e^{-1}$ and $p = 1 - e^{-2}$.

For the Hofbauer-type measure a likelihood ratio only depends on the span of the zero-sequence, i.e., if we set $V = W(y)$ then

$$\frac{\mu([x_1, x_2, \dots, x_m]|y)}{\mu([x_1, x_2, \dots, x_m])} = \frac{\mu\left([0_k] \mid [0_V, 1]\right)}{\mu([0]_k)}$$

where 0_k denotes a sequence of k zeros and where k is given by the condition that either $k = m$ or else

$$x_{m-k} = 1, \quad x_{m-k+1} = x_{m-k+2} = \cdots = x_m = 0.$$

Hence, it is enough to consider the case $(x_1, \dots, x_m) = 0_m$.

Let $V = W(y)$ be the waiting time for the sequence y. Note that the ratio

$$\begin{aligned}
\frac{\mu(0_m)}{\mu(0_m \mid y)} &= \int \exp\left\{(s_V - s_W) + (s_{V+1} - s_{W+1}) + \cdots + (s_{V+m-1} - s_{W+m-1})\right\} d\mu(z) \\
&= \int \exp\left\{ \sum_{k=W+1}^{W+m} \frac{1}{k} - \sum_{k=V+1}^{V+m} \frac{1}{k} \right\} d\mu(z),
\end{aligned}$$

where $W = W(z)$.

Since $\sum_{k=1}^{N} \frac{1}{k} = \ln N + \gamma + o(1)$ as $N \to \infty$, we can deduce that

$$\frac{\mu([0_m])}{\mu([0_m] \mid y)} = \int \exp\left\{ \sum_{k=W+1}^{W+m} \frac{1}{k} - \sum_{k=V+1}^{V+m} \frac{1}{k} \right\} d\mu(z)$$
$$\simeq \frac{V+1}{V/m+1} \cdot \int \frac{W/m+1}{W+1} \, d\mu(z).$$

where $f \simeq g$ here means that $|f/g|$ is bounded away from 0 and ∞ as for all m.

Since $W(z)$ is sandwiched between two geometric distributions with $p = 1 - e^{-2}$ and $p = 1 - e^{-1}$, it is clear that the integral

$$I = \int \frac{W/m+1}{W+1} \, d\mu(z) = \frac{1}{m} + (1 - \frac{1}{m}) \cdot \mathsf{E}\left(\frac{1}{W+1}\right)$$

satisfies

$$(1 - e^{-1}) < I < 1.$$

Thus, we obtain the following expression for the likelihood ratios

$$\frac{\mu([0_m] \mid y)}{\mu([0_m])} \simeq \frac{1}{I} \cdot \frac{V/m+1}{V+1}.$$

The right hand side is less than $1/I$. Hence the likelihood ratios are bounded from above. On the other hand, if we let $V \to \infty$ then

$$\frac{\mu([0_m] \mid y)}{\mu([0_m])} \to \frac{1}{I} \cdot \Theta(\frac{1}{m})$$

and the likelihood-ratios are thus not bounded away from zero since m is arbitrary.