

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/1919638>

Pressure and Equilibrium States in Ergodic Theory

Article · May 2008

Source: arXiv

CITATIONS

30

READS

170

2 authors, including:



Jean-René Chazottes

French National Centre for Scientific Research

104 PUBLICATIONS 1,290 CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:



Quantitative recurrence, hitting time, extremes and logarithm laws. [View project](#)

PRESSURE AND EQUILIBRIUM STATES IN ERGODIC THEORY

J-R CHAZOTTES, G KELLER

1. Definition of the Subject and Its Importance	2
2. Introduction	2
3. Warming Up: Thermodynamic Formalism for Finite Systems	4
4. Shift spaces, Invariant Measures and Entropy	7
5. The Variational Principle: a Global Characterisation of Equilibrium	9
6. The Gibbs Property: a Local Characterisation of Equilibrium	13
7. Examples on Shift Spaces	16
8. Examples from Differentiable Dynamics	18
9. Nonequilibrium Steady States and Entropy Production	22
10. Some Ongoing Developments and Future Directions	23
References	23

Contents

Glossary

Dynamical System In this article: a continuous transformation T of a compact metric space X . For each $x \in X$, the transformation T generates a trajectory (x, Tx, T^2x, \dots) .

Invariant measure In this article: a probability measure μ on X which is invariant under the transformation T , *i.e.*, for which $\langle f \circ T, \mu \rangle = \langle f, \mu \rangle$ for each continuous $f : X \rightarrow \mathbb{R}$. Here $\langle f, \mu \rangle$ is a short-hand notation for $\int_X f d\mu$. The triple (X, T, μ) is called a measure-preserving dynamical system.

Ergodic theory Ergodic theory is the mathematical theory of measure-preserving dynamical systems.

Entropy In this article: the maximal rate of information gain per time that can be achieved by coarse grained observations on a measure-preserving dynamical system. This quantity is often denoted $h(\mu)$.

Equilibrium State In general, a given dynamical system $T : X \rightarrow X$ admits a huge number of invariant measures. Given some continuous $\phi : X \rightarrow \mathbb{R}$ (“potential”), those invariant measures which maximise a functional of the form $F(\mu) = h(\mu) + \langle \phi, \mu \rangle$ are called “equilibrium states” for ϕ .

Pressure The maximum of the functional $F(\mu)$ is denoted by $P(\phi)$ and called the “topological pressure” of ϕ , or simply the “pressure” of ϕ .

Gibbs State In many cases, equilibrium states have a local structure that is determined by the local properties of the potential ϕ . They are called “Gibbs states”.

Sinai-Ruelle-Bowen measure Special equilibrium or Gibbs states that describe the statistics of the attractor of certain smooth dynamical systems.

1. DEFINITION OF THE SUBJECT AND ITS IMPORTANCE

Gibbs and equilibrium states of one-dimensional lattice models in statistical physics play a prominent role in the statistical theory of chaotic dynamics. They first appear in the ergodic theory of certain differentiable dynamical systems, called “uniformly hyperbolic systems”, mainly Anosov and Axiom A diffeomorphisms (and flows). The central idea is to “code” the orbits of these systems into (infinite) symbolic sequences of symbols by following their history on a finite partition of their phase space. This defines a nice shift dynamical system called a subshift of finite type or a topological Markov chain. Then the construction of their “natural” invariant measures and the study of their properties are carried out at the symbolic level by constructing certain equilibrium states in the sense of statistical mechanics which turn out to be also Gibbs states. The study of uniformly hyperbolic systems brought out several ideas and techniques which turned out to be extremely fruitful for the study of more general systems. Let us mention the concept of Markov partition and its avatars, the very important notion of SRB measure (after Sinai, Ruelle and Bowen) and transfer operators. Recently, there was a revival interest in Axiom A systems as models to understand nonequilibrium statistical mechanics.

2. INTRODUCTION

Our goal is to present the basic results on one-dimensional Gibbs and equilibrium states viewed as special invariant measures on symbolic dynamical systems, and then to describe without technicalities a sample of results they allowed to obtain for certain differentiable dynamical systems. We hope that this contribution will illustrate the symbiotic relationship between ergodic theory and statistical mechanics, and also information theory.

We start by putting Gibbs and equilibrium states in a general perspective. The theory of Gibbs states and equilibrium states, or Thermodynamic Formalism, is a branch of rigorous Statistical Physics. The notion of a Gibbs state dates back to R.L. Dobrushin (1968-1969) [17, 18, 19, 20] and O.E. Lanford and D. Ruelle (1969) [41] who proposed it as a mathematical idealisation of an equilibrium state of a physical system which consists of a very large number of interacting components. For a finite number of components, the foundations of statistical mechanics were already laid in the nineteenth century. There was the well-known Maxwell-Boltzmann-Gibbs formula for the equilibrium distribution of a physical system with given energy function. From the mathematical point of view, the intrinsic properties of very large objects can be made manifest by performing suitable limiting procedures. Indeed, the crucial step made in the 1960’s was to define the notion of a Gibbs measure or Gibbs state for a system with an infinite number of interacting components. This was done by the familiar probabilistic idea of specifying the interdependence structure by means of a suitable class of conditional probabilities built up according to the Maxwell-Boltzmann-Gibbs formula [29]. Notice that Gibbs

states are often called “DLR states” in honour of Dobrushin, Lanford and Ruelle. The remarkable aspect of this construction is the fact that a Gibbs state for a given type of interaction may fail to be unique. In physical terms, this means that a system with this interaction can take several distinct equilibria. The phenomenon of nonuniqueness of a Gibbs measure can thus be interpreted as a phase transition. Therefore, the conditions under which an interaction leads to a unique or to several Gibbs measures turns out to be of central importance. While Gibbs states are defined locally by specifying certain conditional probabilities, equilibrium states are defined globally by a *variational principle*: they maximise the entropy of the system under the (linear) constraint that the mean energy is fixed. Gibbs states are always equilibrium states, but the two notions do not coincide in general. However, for a class of sufficiently regular interactions, equilibrium states are also Gibbs states.

In the effort of trying to understand phase transitions, simplified mathematical models were proposed, the most famous one being undoubtedly the Ising model. This is an example of a lattice model. The set of configurations of a lattice model is $X := A^{\mathbb{Z}^d}$, where A is a finite set, which is invariant by “spatial” translations. For the physical interpretation, X can be thought, for instance, as the set of infinite configurations of a system of spins on a crystal lattice \mathbb{Z}^d and A may be taken as $= \{+1, -1\}$, *i.e.*, spins can take two orientations, “up” and “down”. The Ising model is defined by specifying an interaction (or potential) between spins and then study the corresponding (translation-invariant) Gibbs states. The striking phenomenon is that for $d = 1$ there is a unique Gibbs state (in fact a Markov measure) whereas if $d \geq 2$, there may be several Gibbs states although the interaction is very simple [29].

Equilibrium states and Gibbs states of one-dimensional lattice models ($d = 1$) played a prominent role in understanding the ergodic properties of certain types of differentiable dynamical systems, namely uniformly hyperbolic systems, Axiom A diffeomorphisms in particular. The link between one-dimensional lattice systems and dynamical systems is made by *symbolic dynamics*. Informally, symbolic dynamics consists in replacing the orbits of the original system by its history on a finite partition of its phase space labelled by the elements of the “alphabet” A . Therefore, each orbit of the original system is replaced by an infinite sequence of symbols, *i.e.*, by an element of the set $A^{\mathbb{Z}}$ or $A^{\mathbb{N}}$, depending on the fact that the map describing the dynamics is invertible or not. The action of the map on an initial condition is then easily seen to correspond to the translation (or shift) of its associated symbolic sequence. In general there is no reason to get all sequences of $A^{\mathbb{Z}}$ or $A^{\mathbb{N}}$. Instead one gets a closed invariant subset X (a subshift) which can be very complicated. For a certain class of dynamical systems the partition can be successfully chosen so as to form a *Markov partition*. In this case, the dynamical system under consideration can be coded by a *subshift of finite type* (also called a *topological Markov chain*) which is a very nice symbolic dynamical system. Then one can play the game of statistical physics: for a given continuous, real-valued function (a “potential”) on X , construct the corresponding Gibbs states and equilibrium states. If the potential is regular enough, one expects uniqueness of the Gibbs state and that it is also the unique equilibrium state for this potential. This circle of ideas - ranging from Gibbs states on finite systems over invariant measures on symbolic systems and their

(Shannon-)entropy with a digression to Kolmogorov-Chaitin complexity to equilibrium states and Gibbs states on subshifts of finite type - is presented in Sections 3 - 6.

At this point it should be remembered that the objects which can actually be observed are not equilibrium states (they are measures on X) but individual symbol sequences in X , which reflect more or less the statistical properties of an equilibrium state. Indeed, most sequences reflect these properties very well, but there are also rare sequences that look quite different. Their properties are described by *large deviations principles* which are not discussed in the present article. We shall indicate some references along the way.

In Sections 7 and 8 we present a selection of important examples: measure of maximal entropy, Markov measures and Hofbauer's example of nonuniqueness of equilibrium state; uniformly expanding Markov maps of the interval, interval maps with an indifferent fixed point, Anosov diffeomorphisms and Axiom A attractors with Sinai-Ruelle-Bowen measures, and Bowen's formula for the Hausdorff dimension of conformal repellers. As we shall see, Sinai-Ruelle-Bowen measures are the only physically observable measures and they appear naturally in the context of nonuniformly hyperbolic diffeomorphisms [71].

A revival of the interest to Anosov and Axiom A systems occurred in statistical mechanics in the 1990's. Several physical phenomena of nonequilibrium origin, like entropy production and chaotic scattering, were modelled with the help of those systems (by G. Gallavotti, P. Gaspard, D. Ruelle, and others). This new interest led to new results about old Anosov and Axiom A systems, see, *e.g.*, [15] for a survey and references. In Section 9, we give a very brief account on *entropy production* in the context of Anosov systems which highlights the role of *relative entropy*.

This article is a little introduction to a vast subject in which we have tried to put forward some aspects not previously described in other expository texts. For people willing to deepen their understanding of equilibrium and Gibbs states, there are the classic monographs by Bowen [5] and by Ruelle [58], the monograph by one of us [38], and the survey article by Chernov [15] (where Anosov and Axiom A flows are reviewed). Those texts are really complementary.

3. WARMING UP: THERMODYNAMIC FORMALISM FOR FINITE SYSTEMS

We introduce the thermodynamic formalism in an elementary context, following Jaynes [34]. In this view, entropy, in the sense of information theory, is the central concept.

Incomplete knowledge about a system is conveniently described in terms of probability distributions on the set of its possible states. This is particularly simple if the set of states, call it X , is finite. Then the equidistribution on X describes complete lack of knowledge, whereas a probability vector that assigns probability 1 to one single state and probability 0 to all others represents maximal information about the system. A well established measure of the amount of uncertainty represented by a probability distribution

$\nu = (\nu(x))_{x \in X}$ is its *entropy*

$$H(\nu) := - \sum_{x \in X} \nu(x) \log \nu(x),$$

which is zero if the probability is concentrated in one state and which attains its maximum value $\log |X|$ if ν is the equidistribution on X , *i.e.*, if $\nu(x) = |X|^{-1}$ for all $x \in X$. In this completely elementary context we will explore two concepts whose generalisations are central to the theory of equilibrium states in ergodic theory:

- equilibrium distributions - defined in terms of a variational problem,
- the Gibbs property of equilibrium distributions,

The only mathematical prerequisite for this section are calculus and some elements from probability theory.

3.1. Equilibrium Distributions and the Gibbs Property. Suppose that a finite system can be observed through a function $U : X \rightarrow \mathbb{R}$ (an “observable”), and that we are looking for a probability distribution μ which maximises entropy among all distributions ν with a prescribed expected value $\langle U, \nu \rangle := \sum_{x \in X} \nu(x) U(x)$ for the observable U . This means we have to solve a variational problem under constraints:

$$(1) \quad H(\mu) = \max\{H(\nu) : \langle U, \nu \rangle = E\}$$

As the function $\nu \mapsto H(\nu)$ is strictly concave, there is a unique maximising probability distribution μ provided the value E can be attained at all by some $\langle U, \nu \rangle$. In order to derive an explicit formula for this μ we introduce a Lagrange multiplier $\beta \in \mathbb{R}$ and study, for each β , the unconstrained problem

$$(2) \quad H(\mu_\beta) + \langle \beta U, \mu_\beta \rangle = p(\beta U) := \max_{\nu} (H(\nu) + \langle \beta U, \nu \rangle)$$

In analogy to the convention in ergodic theory we call $p(\beta \phi)$ the *pressure* of $\beta \phi$ and the maximiser μ_β the corresponding *equilibrium distribution* (synonymously *equilibrium state*).

The equilibrium distribution μ_β satisfies

$$(3) \quad \mu_\beta(x) = \exp(-p(\beta U) + \beta U(x)) \quad \text{for all } x \in X$$

as an elementary calculation using Jensen’s inequality for the strictly convex function $t \mapsto -t \log t$ shows:

$$H(\nu) + \langle \beta U, \nu \rangle = \sum_{x \in X} \nu(x) \log \frac{e^{\beta U(x)}}{\nu(x)} \leq \log \sum_{x \in X} \nu(x) \frac{e^{\beta U(x)}}{\nu(x)} = \log \sum_{x \in X} e^{\beta U(x)},$$

with equality if and only if $e^{\beta U}$ is a constant multiple of ν . The observation that $\nu = \mu_\beta$ is a maximiser proves at the same time that $p(\beta U) = \log \sum_{x \in X} e^{\beta U(x)}$.

The equality expressed in (3) is called the *Gibbs property* of μ_β , and we say that μ_β is a Gibbs distribution if we want to stress this property.

In order to solve the constrained problem (1) it remains to show that there is a unique multiplier $\beta = \beta(E)$ such that $\langle U, \mu_\beta \rangle = E$. This follows from the fact that the map

$\beta \mapsto \langle U, \mu_\beta \rangle$ maps the real line monotonically onto the interval $(\min U, \max U)$ which, in turn, is a direct consequence of the formulas for the first and second derivative of $p(\beta U)$ w.r.t β :

$$(4) \quad \frac{dp}{d\beta} = \langle U, \mu_\beta \rangle, \quad \frac{d^2p}{d\beta^2} = \langle U^2, \mu_\beta \rangle - \langle U, \mu_\beta \rangle^2.$$

As the second derivative is nothing but the variance of U under μ_β , it is strictly positive (except when U is a constant function), so that $\beta \mapsto \langle U, \mu_\beta \rangle$ is indeed strictly increasing. Observe also that $\frac{dp}{d\beta}$ is indeed the directional derivative of $p : \mathbb{R}^{|A|} \rightarrow \mathbb{R}$ in direction U . Hence the first identity in (4) can be rephrased as: μ_β is the gradient at βU of the function p .

A similar analysis can be performed for an \mathbb{R}^d -valued observable ϕ . In that case a vector $\beta \in \mathbb{R}^d$ of Lagrange multipliers is needed to satisfy the d linear constraints.

3.2. Systems on a Finite Lattice. We now assume that the system has a lattice structure, modelling its extension in space, for instance. The system can be in different states at different positions. More specifically, let $L_n = \{0, 1, \dots, n-1\}$ be a set of n positions in space, let A be a finite set of states that can be attained by the system at each of its sites, and denote by $X := A^{L_n}$ the set of all configurations of states from A at positions of L_n . It is helpful to think of X as the set of all words of length N over the alphabet A . We focus on observables U_n which are sums of many local contributions in the sense that $U_n(a_0 \dots a_{n-1}) = \sum_{i=0}^{n-1} \phi(a_i \dots a_{i+r-1})$ for some “local observable” $\phi : A^r \rightarrow \mathbb{R}$. (The index $i+r-1$ has to be taken modulo n .) In terms of ϕ the maximising measure can be written as

$$(5) \quad \mu_\beta(a_0 \dots a_{n-1}) = \exp \left(-nP(\beta\phi) + \beta \sum_{i=0}^{n-1} \phi(a_i \dots a_{i+r-1}) \right)$$

where $P(\beta\phi) := n^{-1}p(\beta U_n)$. A first immediate consequence of (5) is the invariance of μ_β under a cyclic shift of its argument, namely $\mu_\beta(a_1 \dots a_{n-1} a_0) = \mu_\beta(a_0 \dots a_{n-1})$. Therefore we can restrict the maximisations in (1) and (2) to probability distributions ν which are invariant under cyclic translations which yields

$$(6) \quad P(\beta\phi) = \max_{\nu} (n^{-1}H(\nu) + \langle \beta\phi, \nu \rangle) = n^{-1}H(\mu_\beta) + \langle \beta\phi, \mu_\beta \rangle.$$

If the local observable ϕ depends only on one coordinate, μ_β turns out to be a product measure:

$$\mu_\beta(a_0 \dots a_{n-1}) = \prod_{i=0}^{n-1} \exp(-P(\beta\phi) + \beta\phi(a_i)).$$

Indeed, comparison with (3) shows that μ_β is the n -fold product of the probability distribution μ_β^{loc} on A that maximises $H(\nu) + \beta\nu(\phi)$ among all distributions ν on A . It follows that $n^{-1}H(\mu_\beta) = H(\mu_\beta^{loc})$ so that (6) implies $P(\beta\phi) = p(\beta\phi)$ for observables ϕ that depend only on one coordinate.

4. SHIFT SPACES, INVARIANT MEASURES AND ENTROPY

We now turn to *shift dynamical systems* over a finite alphabet A .

4.1. Symbolic Dynamics. We start by fixing some notation. Let \mathbb{N} denote the set $\{0, 1, 2, \dots\}$. In the sequel we need

- a finite set A (the “alphabet”),
- the set $A^{\mathbb{N}}$ of all infinite sequences over A , *i.e.*, the set of all $\underline{x} = x_0x_1\dots$ with $x_n \in A$ for all $n \in \mathbb{N}$.
- the translation (or shift) $\sigma : A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$, $(\sigma \underline{x})_n = x_{n+1}$, for all $n \in \mathbb{N}$,
- a shift invariant subset $X = \sigma(X)$ of $A^{\mathbb{N}}$. With a slight abuse of notation we denote the restriction of σ to X by σ again.

We mention two interpretations of the dynamics of σ : it can describe the evolution of a system with state space X in discrete time steps (this is the prevalent interpretation if $\sigma : X \rightarrow X$ is obtained as a symbolic representation of another dynamical system), or it can be the spatial translation of the configuration of a system on an infinite lattice (generalising the point of view from Subsection 3.2). In the latter case one usually looks at the shift on the two-sided shift space $A^{\mathbb{Z}}$, for which the theory is nearly identical.

On $A^{\mathbb{N}}$ one can define a metric d by

$$(7) \quad d(\underline{x}, \underline{y}) := 2^{-N(\underline{x}, \underline{y})} \quad \text{where} \quad N(\underline{x}, \underline{y}) := \min\{k \in \mathbb{N} : x_k \neq y_k\}.$$

Hence $d(\underline{x}, \underline{y}) = 1$ if and only if $x_0 \neq y_0$, and $d(\underline{x}, \underline{x}) = 0$ upon agreeing that $N(\underline{x}, \underline{x}) = \infty$ and $2^{-\infty} = 0$. Equipped with this metric, $A^{\mathbb{N}}$ becomes a compact metric space and σ is easily seen to be a continuous surjection of $A^{\mathbb{N}}$. Finally, if X is a closed subset of $A^{\mathbb{N}}$, we call the restriction $\sigma : X \rightarrow X$, which is again a continuous surjection, a shift dynamical system. We remark that d generates on $A^{\mathbb{N}}$ the product topology of the discrete topology on A , just as many variants of d do. For more details see [[Marcus]]. As usual, $C(X)$ denotes the space of real-valued continuous functions on X equipped with the supremum norm $\|\cdot\|_{\infty}$.

4.2. Invariant Measures. A probability distribution ν (or simply distribution) on X is a Borel probability measure on X . It is unambiguously specified by its values $\nu[a_0 \dots a_{n-1}]$ ($n \in \mathbb{N}$, $a_i \in A$) on *cylinder sets*

$$[a_0 \dots a_{n-1}] := \{\underline{x} \in X : x_i = a_i \text{ for all } i = 0, \dots, n-1\}.$$

Any bounded and measurable $f : X \rightarrow \mathbb{R}$ (in particular any $f \in C(X)$) can be integrated by any distribution ν . To stress the linearity of the integral in both, the integrand and the integrator, we use the notation

$$\langle f, \nu \rangle := \int_X f \, d\nu.$$

In probabilistic terms, $\langle f, \nu \rangle$ is the expectation of the observable f under ν . The set $\mathcal{M}(X)$ of all probability distributions is compact in the weak topology, the coarsest

topology on $\mathcal{M}(X)$ for which $\nu \mapsto \langle f, \nu \rangle$ is continuous for all $f \in C(X)$, see [[Petersen, 6.1]]. (Note that in functional analysis this is called the weak-* topology.) Henceforth we will use both terms, “measure” and “distribution”, if we talk about probability distributions.

A measure ν on X is *invariant* if expectations of observables are unchanged under the shift, *i.e.*, if

$$\langle f \circ \sigma, \nu \rangle = \langle f, \nu \rangle \quad \text{for all bounded measurable } f : X \rightarrow \mathbb{R}.$$

The set of all invariant measures is denoted by $\mathcal{M}_\sigma(X)$. As a closed subset of $\mathcal{M}(X)$ it is compact in the weak topology. Of special importance among all invariant measures ν are the ergodic ones which can be characterised by the property that, for all bounded measurable $f : X \rightarrow \mathbb{R}$,

$$(8) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\sigma^k \underline{x}) = \langle f, \nu \rangle \quad \text{for } \nu\text{-}\mathfrak{a}(\text{almost every}) \underline{x},$$

i.e., for a set of \underline{x} of ν -measure one. They are the indecomposable “building blocks” of all other measures in $\mathcal{M}_\sigma(X)$, see [[Petersen, 6.2]] or [[del Junco]]. The almost everywhere convergence in (8) is Birkhoff’s ergodic Theorem [[del Junco]], the constant limit characterises the ergodicity of ν .

4.3. Entropy of Invariant Measures. We give a brief account of the definition and basic properties of the entropy of the shift under an invariant measure ν . For details and the generalisation of this concept to general dynamical systems we refer to [[King]] or [37], and to [36] for an historical account.

Let $\nu \in \mathcal{M}_\sigma(X)$. For each $n > 0$ the cylinder probabilities $\nu[a_0 \dots a_{n-1}]$ give rise to a probability distribution on the finite set $A^{\mathbb{L}_n}$, see section 3, so

$$H_n(\nu) := - \sum_{a_0, \dots, a_{n-1} \in A} \nu[a_0 \dots a_{n-1}] \log \nu[a_0 \dots a_{n-1}]$$

is well defined. Invariance of ν guarantees that the sequence $(H_n(\nu))_{n>0}$ is *subadditive*, *i.e.*, $H_{k+n}(\nu) \leq H_k(\nu) + H_n(\nu)$, and an elementary argument shows that the limit

$$(9) \quad h(\nu) := \lim_{n \rightarrow \infty} \frac{1}{n} H_n(\nu) \in [0, \log |A|]$$

exists and equals the infimum of the sequence. We simply call it the *entropy* of ν . (Note that for general subshifts X many of the cylinder sets $[a_0 \dots a_{n-1}] \subseteq X$ are empty. But, because of the continuity of the function $t \mapsto t \log t$ at $t = 0$, we set $0 \log 0 = 0$, and, hence, this does not affect the definition of $H_n(\nu)$.)

The entropy $h(\nu)$ of an ergodic measure ν can be obtained along a “typical” trajectory. That is the content of the following theorem, sometimes called the “ergodic theorem of information theory”.

Shannon-McMillan-Breiman Theorem:

$$(10) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \nu[x_0 \dots x_{n-1}] = -h(\nu) \quad \text{for } \nu\text{-}\mathfrak{a}\underline{x}.$$

Observe that (9) is just the integrated version of this statement. A slightly weaker reformulation of this theorem (again for ergodic ν) is known as the “asymptotic equipartition property”.

Asymptotic Equipartition Property:

Given (arbitrarily small) $\epsilon > 0$ and $\alpha > 0$, one can, for each sufficiently large n , partition the set A^n into a set \mathcal{T}_n of typical words and a set \mathcal{E}_n of exceptional words such that each $a_0 \dots a_{n-1} \in \mathcal{T}_n$ satisfies

$$(11) \quad e^{-n(h(\nu)+\alpha)} \leq \nu[a_0 \dots a_{n-1}] \leq e^{-n(h(\nu)-\alpha)}$$

and the total probability $\sum_{a_0 \dots a_{n-1} \in \mathcal{E}_n} \nu[a_0 \dots a_{n-1}]$ of the exceptional words is at most ϵ .

4.4. A Short Digression on Complexity. Kolmogorov [40] and Chaitin [14] introduced the concept of complexity of an infinite sequence of symbols. Very roughly it is defined as follows: First, the complexity $K(x_0 \dots x_{n-1})$ of a finite word in A^n is defined as the bit length of the shortest program that causes a suitable general purpose computer (say a PC or, for the mathematically minded reader, a Turing machine) to print out this word. Then the complexity of an infinite sequence is defined as $K(\underline{x}) := \limsup_{n \rightarrow \infty} \frac{1}{n} K(x_0 \dots x_{n-1})$. Of course, the definition of $K(x_0 \dots x_{n-1})$ depends on the particular computer, but as any two general purpose computers can be programmed to simulate each other (by some finite piece of software), the limit $K(\underline{x})$ is machine independent. It is the optimal compression factor for long initial pieces of a sequence \underline{x} that still allows complete reconstruction of \underline{x} by an algorithm. Brudno [8] showed:

If $X \subseteq A^{\mathbb{N}}$ and $\nu \in \mathcal{M}_\sigma(X)$ is ergodic, then $K(\underline{x}) = \frac{1}{\log 2} h(\nu)$ for ν -a.e. $\underline{x} \in X$.

4.5. Entropy as a Function of the Measure. An important technical remark for the further development of the theory is that the entropy function $h : \mathcal{M}_\sigma(X) \rightarrow [0, \infty)$ is *upper semicontinuous*. This means that all sets $\{\nu : h(\nu) \geq t\}$ with $t \in \mathbb{R}$ are closed and hence compact. In particular, upper semicontinuous functions attain their supremum. Indeed, suppose a sequence $\nu_k \in \mathcal{M}_\sigma(X)$ converges weakly to some $\nu \in \mathcal{M}_\sigma(X)$ and $h(\nu_k) \geq t$ for all k so that also $\frac{1}{n} H_n(\nu_k) \geq t$ for all n and k . As $H_n(\nu)$ is an expression that depends continuously on the probabilities of the finitely many cylinders $[a_0 \dots a_{n-1}]$ and as the indicator functions of these sets are continuous, $\frac{1}{n} H_n(\nu) = \lim_{k \rightarrow \infty} \frac{1}{n} H_n(\nu_k) \geq t$, hence $h(\nu) \geq t$ in the limit $n \rightarrow \infty$.

A word of caution seems in order: the entropy function is rarely continuous. For example, on the full shift $X = A^{\mathbb{N}}$ each invariant measure, whatever its entropy is, can be approximated in the weak topology by equidistributions on periodic orbits. But all these equidistributions have entropy zero.

5. THE VARIATIONAL PRINCIPLE: A GLOBAL CHARACTERISATION OF EQUILIBRIUM

Usually, a dynamical systems model of a “physical” system consists of a state space and a map (or a differential equation) describing the dynamics. An invariant measure

for the system is rarely given *a priori*. Indeed, many (if not most) dynamical systems arising in this way have uncountably many ergodic invariant measures. This limits considerably the “practical value” of Birkhoff’s ergodic theorem (8) or the Shannon-McMillan-Breiman theorem (10): not only do the limits in these theorems depend on the invariant measure ν , but also the sets of points for which the theorems guarantee almost everywhere convergence are practically disjoint for different ν and ν' in $\mathcal{M}_\sigma(X)$. Therefore a choice of ν has to be made which reflects the original modelling intentions. We will argue in this and the next sections that a variational principle with a judiciously chosen “observable” may be a useful guideline - generalising the observations for finite systems collected in Section 3. As announced earlier we restrict again to shift dynamical systems, because they are rather universal models for many other systems.

5.1. Equilibrium States. We define the *pressure* of an observable $\phi \in C(X)$ as

$$(12) \quad P(\phi) := \sup\{h(\nu) + \langle \phi, \nu \rangle : \nu \in \mathcal{M}_\sigma(X)\}.$$

Since $\mathcal{M}_\sigma(X)$ is compact and the functional $\nu \mapsto h(\nu) + \langle \phi, \nu \rangle$ is upper semicontinuous, the supremum is attained - not necessarily at a unique measure as we will see (which is a remarkable difference to what happens in finite systems). Each measure ν for which the supremum is attained is called an *equilibrium state* for ϕ . Here the word “state” is used synonymously with “distribution” or “measure” - a reflection of the fact that in “well-behaved cases”, as we will see in the next section, this measure is uniquely determined by the constraint(s) under which it maximises entropy, and that means by the *macroscopic state* of the system. (In contrast, the word “state” was used in Section 3 to designate microscopic states.)

As, for each $\nu \in \mathcal{M}_\sigma(X)$, the functional $\phi \mapsto h(\nu) + \langle \phi, \nu \rangle$ is affine on $C(X)$, the pressure functional $P : C(X) \rightarrow \mathbb{R}$, which, by definition, is the pointwise supremum of these functionals, is convex. It is therefore instructive to fit equilibrium states into the abstract framework of convex analysis [32, 38, 45, 68]. To this end recall the identities in (4) that identify, for finite systems, equilibrium states as gradients of the pressure function $p : \mathbb{R}^{|A|} \rightarrow \mathbb{R}$ and guarantee that p is twice differentiable and strictly convex. In the present setting where P is defined on the Banach space $C(X)$, differentiability and strict convexity are no more guaranteed, but one can show:

Equilibrium states as (sub)-gradients:

- (13) $\mu \in \mathcal{M}_\sigma(X)$ is an equilibrium state for ϕ if and only if μ is a subgradient (or tangent functional) for P at ϕ , *i.e.*, if $P(\phi + \psi) - P(\phi) \geq \langle \psi, \mu \rangle$ for all $\psi \in C(X)$. In particular, ϕ has a unique equilibrium state μ if and only if P is differentiable at ϕ with gradient μ , *i.e.*, if $\lim_{t \rightarrow 0} \frac{1}{t} (P(\phi + t\psi) - P(\phi)) = \langle \psi, \mu \rangle$ for all $\psi \in C(X)$.

Let us see how equilibrium states on $X = A^\mathbb{N}$ can directly be obtained from the corresponding equilibrium distributions on finite sets A^n of Subsection 3.2. Define $\phi^{(n)} : A^n \rightarrow \mathbb{R}$ by $\phi^{(n)}(a_0 \dots a_{n-1}) := \phi(a_0 \dots a_{n-1} a_0 \dots a_{n-1} \dots)$, denote by U_n the corresponding global observable on A^n , and let μ_n be the equilibrium distribution on

A^n that maximises $H(\mu) + \langle U_n, \mu \rangle$. Then all weak limit points of the “approximative equilibrium distributions” μ_n on A^n are equilibrium states on $A^\mathbb{N}$.

► This can be seen as follows: Let the measure μ on $A^\mathbb{N}$ be any weak limit point of the μ_n . Then, given $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$h(\mu) + \langle \phi, \mu \rangle \geq \frac{1}{k} H_k(\mu) + \langle \phi, \mu \rangle - \epsilon \geq \frac{1}{k} H_k(\mu_n) + \langle \phi^{(n)}, \mu_n \rangle - 2\epsilon$$

for arbitrarily large n , because $\|\phi - \phi^{(n)}\|_\infty \rightarrow 0$ as $n \rightarrow \infty$ by construction of the $\phi^{(n)}$. As the μ_n are invariant under cyclic coordinate shifts (see Subsection 3.2), it follows from the subadditivity of the entropy that

$$h(\mu) + \langle \phi, \mu \rangle \geq \frac{1}{n} (H_n(\mu_n) + \langle U_n, \mu_n \rangle) - 2\epsilon - \frac{k}{n} \log |A|.$$

Hence, for each $\nu \in \mathcal{M}_\sigma(X)$,

$$h(\mu) + \langle \phi, \mu \rangle \geq \frac{1}{n} (H_n(\nu) + \langle U_n, \nu \rangle) - 2\epsilon - \frac{k}{n} \log |A| \rightarrow h(\nu) + \langle \phi, \nu \rangle - 2\epsilon$$

as $n \rightarrow \infty$, and we see that μ is indeed an equilibrium state on $A^\mathbb{N}$. ◀

5.2. The Variational Principle. In Subsection 3.1, the pressure of a finite system was defined as a certain supremum and then identified as the logarithm of the normalising constant for the Gibbsian representation of the corresponding equilibrium distribution. We are now going to approximate equilibrium states by suitable Gibbs distributions on finite subsets of X . As a by-product the pressure $P(\phi)$ is characterised in terms of the logarithms of the normalising constants of these approximating distributions. Let $S_n \phi(\underline{x}) := \phi(\underline{x}) + \phi(\sigma \underline{x}) + \dots + \phi(\sigma^{n-1} \underline{x})$. From each cylinder set $[a_0 \dots a_{n-1}]$ we can pick a point \underline{z} such that $S_n \phi(\underline{z})$ is the maximal value of $S_n \phi$ on this set. We denote the collection of the $|A|^n$ points we obtain in this way by E_n . Observe that E_n is not unambiguously defined, but any choice we make will do.

Variational principle for the pressure:

$$(14) \quad P(\phi) = \limsup_{n \rightarrow \infty} \frac{1}{n} P_n(\phi) \quad \text{where} \quad P_n(\phi) := \log \sum_{\underline{z} \in E_n} e^{S_n \phi(\underline{z})}$$

► To prove the “ \leq ” direction of this identity we just have to show that $\frac{1}{n} H_n(\nu) + \langle \phi, \nu \rangle \leq \frac{1}{n} P_n(\phi)$ for each $\nu \in \mathcal{M}_\sigma(X)$ or, after multiplying by n , $H_n(\nu) + \langle S_n \phi, \nu \rangle \leq P_n(\phi)$. But Jensen’s inequality implies:

$$\begin{aligned} H_n(\nu) + \langle S_n \phi, \nu \rangle &\leq \sum_{a_0, \dots, a_{n-1} \in A} \nu[a_0 \dots a_{n-1}] \log \left(\frac{\sup \{ e^{S_n \phi(\underline{x})} : \underline{x} \in [a_0 \dots a_{n-1}] \}}{\nu[a_0 \dots a_{n-1}]} \right) \\ &\leq \log \sum_{a_0, \dots, a_{n-1} \in A} \sup \left\{ e^{S_n \phi(\underline{x})} : \underline{x} \in [a_0 \dots a_{n-1}] \right\} \\ &= \log \sum_{\underline{z} \in E_n} e^{S_n \phi(\underline{z})} = P_n(\phi). \end{aligned}$$

For the reverse inequality consider the discrete Gibbs distributions

$$\pi_n := \sum_{\underline{z} \in E_n} \delta_{\underline{z}} \exp(-P_n(\phi) + S_n\phi(\underline{z}))$$

on the finite sets E_n , where $\delta_{\underline{z}}$ denotes the unit point mass in \underline{z} . One might be tempted to think that all weak limit points of the measures π_n are already equilibrium states. But this need not be the case because there is no good reason that these limits are shift invariant. Therefore one forces invariance of the limits by passing to measures μ_n defined by $\langle f, \mu_n \rangle := \langle \frac{1}{n} \sum_{k=0}^{n-1} f \circ \sigma^k, \pi_n \rangle$. Weak limits of these measures are obviously shift invariant, and a more involved estimate we do not present here shows that each such weak limit μ satisfies $h(\mu) + \langle \phi, \mu \rangle \geq P(\phi)$. \blacktriangleleft

We note that the same arguments work for any other sequence of sets E_n which contain exactly one point from each cylinder. So there are many ways to approximate equilibrium states, and if there are more than one equilibrium state, there is generally no guarantee that the limit is always the same.

5.3. Nonuniqueness of Equilibrium States: an Example. Before we turn to sufficient conditions for the uniqueness of equilibrium states in the next section, we present one of the simplest nontrivial examples for nonuniqueness of equilibrium states. Motivated by the so-called Fisher-Felderhof droplet model of condensation in statistical mechanics [24, 25], Hofbauer [31] studies an observable ϕ on $X = \{0, 1\}^{\mathbb{N}}$ defined as follows: Let (a_k) be a sequence of negative real numbers with $\lim_{k \rightarrow \infty} a_k = 0$. Set $s_k := a_0 + \dots + a_k$. For $k \geq 1$ denote $M_k := \{\underline{x} \in X : x_0 = \dots = x_{k-1} = 1, x_k = 0\}$ and $M_0 := \{\underline{x} \in X : x_0 = 0\}$, and define

$$\phi(\underline{x}) := a_k \quad \text{for } \underline{x} \in M_k \quad \text{and} \quad \phi(11\dots) = 0.$$

Then $\phi : X \rightarrow \mathbb{R}$ is continuous, so that there exists at least one equilibrium state for ϕ . Hofbauer proves that there is more than one equilibrium state if and only if $\sum_{k=0}^{\infty} e^{s_k} = 1$ and $\sum_{k=0}^{\infty} (k+1)e^{s_k} < \infty$. In that case $P(\phi) = 0$, so one of these equilibrium states is the unit mass $\delta_{11\dots}$, and we denote the other equilibrium state by μ_1 , so $h(\mu_1) + \langle \phi, \mu_1 \rangle = 0$. In view of (13) the pressure function is not differentiable at ϕ .

How does the pressure function $\beta \mapsto P(\beta\phi)$ look like? As $h(\delta_{11\dots}) + \langle \beta\phi, \delta_{11\dots} \rangle = 0$ for all β , $P(\beta\phi) \geq 0$ for all β . Observe now that $\phi(\underline{x}) \leq 0$ with equality only for $\underline{x} = 11\dots$. This implies that $\langle \phi, \mu \rangle < 0$ for all $\mu \in \mathcal{M}_\sigma(X)$ different from $\delta_{11\dots}$. From this we can conclude:

- $P(\beta\phi) \leq P(\phi) = 0$ for $\beta > 1$, so $P(\beta\phi) = 0$ for $\beta \geq 1$.
- $P(\beta\phi) \geq h(\mu_1) + \langle \beta\phi, \mu_1 \rangle = h(\mu_1) + \langle \phi, \mu_1 \rangle - (1 - \beta)\langle \phi, \mu_1 \rangle = -(1 - \beta)\langle \phi, \mu_1 \rangle$.

It follows that, at $\beta = 1$, the derivative from the right of $P(\beta\phi)$ is zero, whereas the derivative from the left is at least $-\langle \phi, \mu_1 \rangle > 0$.

5.4. More on Equilibrium States. In more general dynamical systems the entropy function is not necessarily upper semicontinuous and hence equilibrium states need not exist, *i.e.*, the supremum in (12) need not be attained by any invariant measure. A well known sufficient property that guarantees the upper semicontinuity of the entropy function is the *expansiveness* of the system, see, *e.g.*, [53]: a continuous transformation

T of a compact metric space is *positively expansive*, if there is a constant $\gamma > 0$ such that for any two points x and y from the space there is some $n \in \mathbb{N}$ such that $T^n x$ and $T^n y$ are at least a distance γ apart. If T is a homeomorphism one says it is *expansive*, if the same holds for some $n \in \mathbb{Z}$. The previous results carry over without changes (although at the expense of more complicated proofs) to general expansive systems. The variational principle (14) holds in the very general context where T is a continuous action of \mathbb{Z}_+^d on a compact Hausdorff space X . This was proved in [44] in a simple and elegant way. In the monograph [45] it is extended to amenable group actions.

6. THE GIBBS PROPERTY: A LOCAL CHARACTERISATION OF EQUILIBRIUM

In this section we are going to see that, for a sufficiently regular potential ϕ on a topologically mixing subshift of finite type, one has a unique equilibrium state which has the ‘‘Gibbs property’’. This property generalises formula (5) that we derived for finite lattices. Subshifts of finite type are the symbolic models for Axiom A diffeomorphisms, as we shall see later on.

6.1. Subshifts of Finite Type. We start by recalling what is a subshift of finite type and refer the reader to [[Marcus]] or [43] for more details. Given a ‘‘transition matrix’’ $M = (M_{ab})_{a,b \in A}$ whose entries are 0’s or 1’s, one can define a subshift X_M as the set of all sequences $\underline{x} \in A^{\mathbb{N}}$ such that $M_{x_i x_{i+1}} = 1$ for all $i \in \mathbb{N}$. This is called a subshift of finite type or a topological Markov chain. We assume that there exists some integer p_0 such that M^p has strictly positive entries for all $p \geq p_0$. This means that M is irreducible and aperiodic. This property is equivalent to the property that the subshift of finite type is topologically mixing. A general subshift of finite type admits a decomposition into a finite union of transitive sets, each of which being a union of cyclically permuted sets on which the appropriate iterate is topologically mixing. In other words, topologically mixing subshifts of finite type are the building blocks of subshifts of finite type.

6.2. The Gibbs Property for a Class of Regular Potentials. The class of regular potentials we consider is that of ‘‘summable variations’’. We denote by $\text{var}_k(\phi)$ the modulus of continuity of ϕ on cylinders of length $k \geq 1$, that is,

$$\text{var}_k(\phi) := \sup\{|\phi(\underline{x}) - \phi(\underline{y})| : \underline{x} \in [y_0 \dots y_{k-1}]\}.$$

If $\text{var}_k(\phi) \rightarrow 0$ as $k \rightarrow \infty$, this means that ϕ is (uniformly) continuous with respect to the distance (7). We impose the stronger condition

$$(15) \quad \sum_{k=1}^{\infty} \text{var}_k(\phi) < \infty.$$

We can now state the main result of this section.

The Gibbs state of a summable potential. Let X_M be a topologically mixing subshift of finite type. Given a potential $\phi : X_M \rightarrow \mathbb{R}$ satisfying the summability condition (15), there is a (probability) measure μ_ϕ supported on X_M , that we call a Gibbs state. It is the unique σ -invariant measure which satisfies the property:

There exists a constant $C > 0$ such that, for all $\underline{x} \in X_M$ and for all $n \geq 0$,

$$(16) \quad C^{-1} \leq \frac{\mu_\phi[x_0 \dots x_{n-1}]}{\exp(S_n \phi(\underline{x}) - nP(\phi))} \leq C. \quad (\text{“Gibbs property”})$$

Moreover, the Gibbs state μ_ϕ is ergodic and is also the unique equilibrium state of ϕ , *i.e.*, the unique invariant measure for which the supremum in (12) is attained.

We now make several comments on this theorem.

- The Gibbs property (16) gives a *uniform control* of the measure of *all cylinders* in terms of their “energy”. This strengthens considerably the asymptotic equipartition property (11) that we recover if we restrict (16) to the set of μ_ϕ measure one where Birkhoff’s ergodic Theorem (8) applies, and use the identity $\langle \phi, \mu_\phi \rangle - P(\phi) = -h(\mu_\phi)$.
- Gibbs measures on topologically mixing subshifts of finite type are ergodic (and actually mixing in a strong sense) as can be inferred from Ruelle’s Perron-Frobenius Theorem (see Subsection 6.3).
- Suppose that there is another invariant measure μ' satisfying (16), possibly with a constant C' different from C . It is easy to verify that $\mu' = f\mu$ for some μ -integrable function f by using (16) and the Radon-Nikodym theorem. Shift invariance imposes that, μ -a.e., $f = f \circ \sigma$. Then the ergodicity of μ implies that f is a constant μ -a.e., thus $\mu' = \mu$; see [5].
- One could define a Gibbs state by saying that it is an invariant measure μ satisfying (16) for a given continuous potential ϕ . If one does so, it is simple to verify that such a μ must also be an equilibrium state. Indeed, using (16), one can deduce that $\langle \phi, \mu \rangle + h(\mu) \geq P(\phi)$. The converse need not be true in general, see Subsection 7.4. But the summability condition (15) is indeed sufficient for the coincidence of Gibbs and equilibrium states. A proof of this fact can be found in [58] or [38].

6.3. Ruelle’s Perron-Frobenius Theorem. The powerful tool behind the theorem in the previous subsection is a far-reaching generalisation of the classical Perron-Frobenius theorem for irreducible matrices. Instead of a matrix, one introduces the so-called transfer operator, also called the “Perron-Frobenius operator” or “Ruelle’s operator”, which acts on a suitable Banach space of observables. It is D. Ruelle [52] who first introduced this operator in the context of one-dimensional lattice gases with exponentially decaying interactions. In our context, this corresponds to Hölder continuous potentials: these are potentials satisfying $\text{var}_k(\phi) \leq c\theta^k$ for some $c > 0$ and $\theta \in (0, 1)$. A proof of “Ruelle’s Perron-Frobenius Theorem” can be found in [4, 5]. It was then extended to potentials with summable variations in [67]. We refer to the book of V. Baladi [1] for a comprehensive account on transfer operators in dynamical systems.

We content ourselves to define the transfer operator and state Ruelle's Perron-Frobenius Theorem. Let $\mathcal{L} : C(X_M) \rightarrow C(X_M)$ be defined by

$$(\mathcal{L}f)(\underline{x}) := \sum_{\underline{y} \in \sigma^{-1}\underline{x}} e^{\phi(\underline{y})} f(\underline{y}) = \sum_{a \in A: M(a, x_0)=1} e^{\phi(a\underline{x})} f(a\underline{x}).$$

(Obviously, $a\underline{x} := ax_0x_1 \dots$)

Ruelle's Perron-Frobenius Theorem. Let X_M be a topologically mixing subshift of finite type. Let ϕ satisfy condition (15). There exist a number $\lambda > 0$, $h \in C(X_M)$, and $\nu \in \mathcal{M}(X)$ such that $h > 0$, $\langle h, \nu \rangle = 1$, $\mathcal{L}h = \lambda h$, $\mathcal{L}^*\nu = \lambda\nu$, where \mathcal{L}^* is the dual of \mathcal{L} . Moreover, for all $f \in C(X_M)$,

$$\|\lambda^{-n} \mathcal{L}^n f - \langle f, \nu \rangle \cdot h\|_\infty \rightarrow 0, \text{ as } n \rightarrow \infty.$$

By using this theorem, one can show that $\mu_\phi := h\nu$ satisfies (16) and $\lambda = e^{P(\phi)}$.

Let us remark that for potentials which are such that $\phi(\underline{x}) = \phi(x_0, x_1)$ (i.e., potentials constant on cylinders of length 2), \mathcal{L} can be identified with a $|A| \times |A|$ matrix and the previous theorem boils down to the classical Perron-Frobenius theorem for irreducible aperiodic matrices [63]. The corresponding Gibbs states are nothing but Markov chains with state space A [29, Chapter 3]. We shall take another point of view below (Subsection 7.2).

6.4. Relative Entropy. We now define the relative entropy of an invariant measure $\nu \in \mathcal{M}_\sigma(X_M)$ given a Gibbs state μ_ϕ as follows. We first define

$$(17) \quad H_n(\nu|\mu_\phi) := \sum_{a_0, \dots, a_{n-1} \in A} \nu[a_0 \dots a_{n-1}] \log \frac{\nu[a_0 \dots a_{n-1}]}{\mu_\phi[a_0 \dots a_{n-1}]}$$

with the convention $0 \log(0/0) = 0$. Now the relative entropy of ν given μ_ϕ is defined as

$$h(\nu|\mu_\phi) := \limsup_{n \rightarrow \infty} \frac{1}{n} H_n(\nu|\mu_\phi).$$

(By applying Jensen's inequality, one verifies that $h(\nu|\mu_\phi) \geq 0$.) In fact the limit exists and can be computed quite easily using (16):

$$(18) \quad h(\nu|\mu_\phi) = P(\phi) - \langle \phi, \nu \rangle - h(\nu).$$

► To prove this formula, we first make the following observation. It can be easily verified that the inequalities in (16) remain the same when $S_n\phi$ is replaced by the "locally averaged" energy $\tilde{\phi}_n := (\nu[x_0 \dots x_{n-1}])^{-1} \int_{[x_0 \dots x_{n-1}]} S_n\phi(\underline{y}) d\nu(\underline{y})$ for any cylinder with $\nu[x_0 \dots x_{n-1}] > 0$. Cylinders with ν measure zero does not contribute to the sum in (17).

We can now write that

$$-\frac{1}{n} \log C \leq -\frac{1}{n} H_n(\nu|\mu_\phi) + \left(P(\phi) - \frac{1}{n} \langle S_n\phi, \nu \rangle - \frac{1}{n} H_n(\nu) \right) \leq \frac{1}{n} \log C.$$

To finish we use that $\langle S_n\phi, \nu \rangle = n\langle \phi, \nu \rangle$ (by the invariance of ν) and we apply (9) to obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n} H_n(\nu|\mu_\phi) = P(\phi) - \langle \phi, \nu \rangle - \lim_{n \rightarrow \infty} \frac{1}{n} H_n(\nu) = P(\phi) - \langle \phi, \nu \rangle - h(\nu)$$

which proves (18). ◀

The variational principle revisited. We can reformulate the variational principle in the case of a potential satisfying the summability condition (15):

$$(19) \quad h(\nu|\mu_\phi) = 0 \quad \text{if and only if} \quad \nu = \mu_\phi,$$

i.e., given μ_ϕ , the relative entropy $h(\cdot|\mu_\phi)$, as a function on $\mathcal{M}_\sigma(X_M)$, attains its minimum only at μ_ϕ .

Indeed, by (18) we have $h(\nu|\mu_\phi) = P(\phi) - \langle \phi, \nu \rangle - h(\nu)$. We now use (12) and the fact that μ_ϕ is the unique equilibrium state of ϕ to conclude.

6.5. More Properties of Gibbs States. Gibbs states enjoy very good statistical properties. Let us mention only a few. They satisfy the “Bernoulli property”, a very strong qualitative mixing condition [4, 5, 67]. The sequence of random variables $(f \circ \sigma^n)_n$ satisfies the central limit theorem [15, 16, 49] and a large deviation principle if f is Hölder continuous [21, 38, 39, 70]. Let us emphasise the central role played by relative entropy in large deviations. (The deep link between thermodynamics and large deviations is described in [42] in a much more general context.) Finally, the so-called “multifractal analysis” can be performed for Gibbs states, see, e.g., [48].

7. EXAMPLES ON SHIFT SPACES

7.1. Measure of Maximal Entropy and Periodic Points. If the observable ϕ is constant zero, an equilibrium state simply maximises the entropy. It is called *measure of maximal entropy*. The quantity $P(0) = \sup\{h(\nu) : \nu \in \mathcal{M}_\sigma(X)\}$ is called the *topological entropy* of the subshift $\sigma : X \rightarrow X$. When X is a subshift of finite type X_M with irreducible and aperiodic transition matrix M , there is a unique measure of maximal entropy, see, e.g., [43]. As a Gibbs state it satisfies (16). By summing over all cylinders $[x_0 \dots x_{n-1}]$ allowed by M , it is easy to see that the topological entropy $P(0)$ is the asymptotic exponential growth rate of the number of sequences of length n that can occur as initial segments of points in X_M . This is obviously identical to the logarithm of the largest eigenvalue of the transition matrix M .

It is not difficult to verify that the total number of periodic sequences of period n equals the trace of the matrix M^n , i.e., we have the formula

$$\text{Card}\{\underline{x} \in X_M : \sigma^n \underline{x} = \underline{x}\} = \text{tr}(M^n) = \sum_{i=1}^m \lambda_i^n,$$

where $\lambda_1, \dots, \lambda_m$ are all the eigenvalues of M . Asymptotically, of course, $\text{Card}\{\underline{x} \in X_M : \sigma^n \underline{x} = \underline{x}\} = e^{nP(0)} + O(|\lambda'|^n)$, where λ' is the second largest (in absolute value) eigenvalue of M .

The measure of maximal entropy, call it μ_0 , describes the distribution of periodic points in X_M : one can prove [3, 37] that for any cylinder $B \subset X_M$

$$\lim_{n \rightarrow \infty} \frac{\text{Card}\{\underline{x} \in B : \sigma^n \underline{x} = \underline{x}\}}{\text{Card}\{\underline{x} \in X_M : \sigma^n \underline{x} = \underline{x}\}} = \mu_0(B).$$

In other words, the finite atomic measure that assigns equal weights $1/\text{Card}\{\underline{x} \in X_M : \sigma^n \underline{x} = \underline{x}\}$ to each periodic point in X_M with period n weakly converges to μ_0 , as $n \rightarrow \infty$. Each such measure has zero entropy while $h(\mu_0) = P(0) > 0$, so the entropy is not continuous on the space of invariant measures. It is, however, upper-semicontinuous (see Subsection 4.5).

In fact, it is possible to approximate any Gibbs state μ_ϕ on X_M in a similar way, by finite atomic measures on periodic orbits, by assigning weights properly (see, *e.g.*, [37, Theorem 20.3.7]).

7.2. Markov Chains over Finite Alphabets. Let $Q = (q_{a,b})_{a,b \in A}$ be an irreducible stochastic matrix over the finite alphabet A . It is well known (see, *e.g.*, [63]) that there exists a unique probability vector π on A that defines a stationary Markov measure ν_Q on $X = A^\mathbb{N}$ by $\nu_Q[a_0 \dots a_{n-1}] = \pi_{a_0} q_{a_0 a_1} \dots q_{a_{n-2} a_{n-1}}$. We are going to identify ν_Q as the *unique Gibbs distribution* $\mu \in \mathcal{M}_\sigma(X)$ that maximises entropy under the constraints $\mu[ab] = \mu[a]q_{ab}$, *i.e.*, $\langle \phi^{ab}, \mu \rangle = 0$ ($a, b \in A$), where $\phi^{ab} := \mathbb{1}_{[ab]} - q_{ab}\mathbb{1}_{[a]}$. Indeed, as μ is a Gibbs measure, there are $\beta_{ab} \in \mathbb{R}$ ($a, b \in A$) and constants $P \in \mathbb{R}$, $C > 0$ such that

$$(20) \quad C^{-1} \leq \frac{\mu[x_0 \dots x_{n-1}]}{\exp(\sum_{a,b \in A} \beta_{ab} \phi_n^{ab}(\underline{x}) - nP)} \leq C$$

for all $\underline{x} \in A^\mathbb{N}$ and all $n \in \mathbb{N}$. Let $r_{ab} := \exp(\beta_{ab} - \sum_{b' \in A} \beta_{ab'} q_{ab'} - P)$. Then the denominator in (20) equals $r_{x_0 x_1} \dots r_{x_{n-2} x_{n-1}}$, and it follows that μ is equivalent to the stationary Markov measure defined by the (non-stochastic) matrix $(r_{ab})_{a,b \in A}$. As μ is ergodic, μ is this Markov measure, and as μ satisfies the linear constraints $\mu[ab] = \mu[a]q_{ab}$, we conclude that $\mu = \nu_Q$.

7.3. The Ising Chain. Here the task is to characterise all “spin chains” in $\underline{x} \in \{-1, +1\}^\mathbb{N}$ (or, more commonly, $\{-1, +1\}^\mathbb{Z}$) which are as random as possible with the constraint that two adjacent spins have a prescribed probability $p \neq \frac{1}{2}$ to be identical. With $\phi(\underline{x}) := x_0 x_1$ this is equivalent to requiring that \underline{x} is typical for a Gibbs distribution $\mu_{\beta\phi}$ where $\beta = \beta(p)$ is such that $\langle \phi, \mu_{\beta\phi} \rangle = 2p - 1$. It follows that there is a constant $C > 0$ such that for each $n \in \mathbb{N}$ and any two “spin patterns” $\underline{a} = a_0 \dots a_{n-1}$ and $\underline{b} = b_0 \dots b_{n-1}$

$$\left| \log \frac{\mu_{\beta\phi}[a_0 \dots a_{n-1}]}{\mu_{\beta\phi}[b_0 \dots b_{n-1}]} - \beta(N_{\underline{a}} - N_{\underline{b}}) \right| \leq C$$

where $N_{\underline{a}}$ and $N_{\underline{b}}$ are the numbers of identical adjacent spins in \underline{a} and \underline{b} , respectively.

7.4. More on Hofbauer's Example. We can come back to the example described in Subsection 5.3. It is easy to verify that in that example $\text{var}_{k+1}(\phi) = |a_k|$. For instance, if $a_k = -\frac{1}{(k+1)^2}$ there is a unique Gibbs/equilibrium state. If $a_k = -3 \log\left(\frac{k+1}{k}\right)$ for $k \geq 1$ and $a_0 = -\log \sum_{j=1}^{\infty} j^{-3}$, then from [31] we know that ϕ admits more than one equilibrium state, one of them being $\delta_{11\dots}$, which cannot be a Gibbs state for any continuous ϕ .

8. EXAMPLES FROM DIFFERENTIABLE DYNAMICS

In this section we present a number of examples to which the general theory developed above does not apply directly but only after a transfer of the theory from a symbolic space to a manifold. We restrict to examples where the *results* can be transferred because those aspects of the smooth dynamics we focus on can be studied as well on a shift dynamical systems that is obtained from the original one via symbolic coding. (We do not discuss the coding process itself which is sometimes far from trivial, but we focus on the application of the Gibbs and equilibrium theory.) There are alternative approaches where instead of the results the *concepts and (partly) the strategies of proofs* are transferred to the smooth dynamical systems. This has lead both to an extension of the range of possible applications of the theory and to a number of refined results (because some special features of smooth systems necessarily get lost by transferring the analysis to a completely disconnected metric space).

In the following examples, T denotes a (possibly piecewise) differentiable map of a compact smooth manifold M . Points on the manifold are denoted by u and v . In all examples there is a Hölder continuous coding map $\pi : X \rightarrow M$ from a subshift of finite type X onto the manifold which respects the dynamics, *i.e.*, $T \circ \pi = \pi \circ \sigma$. This *factor map* π is “nearly” invertible in the sense that the set of points for in M with more than one preimage under π has measure zero for all T -invariant measures we are interested in. Hence such measures $\tilde{\mu}$ on M correspond unambiguously to shift invariant measures $\mu = \tilde{\mu} \circ \pi^{-1}$. Similarly observables $\tilde{\phi}$ on M and $\phi = \tilde{\phi} \circ \pi$ on X are related.

8.1. Uniformly Expanding Markov Maps of the Interval. A transformation T on $M := [0, 1]$ is called an *Markov map*, if there are $0 = u_0 < u_1 < \dots < u_N = 1$ such that each restriction $T|_{(u_{i-1}, u_i)}$ is strictly monotone, C^{1+r} for some $r > 0$, and maps (u_{i-1}, u_i) onto a union of some of these N monotonicity intervals. It is called *uniformly expanding* if there is some $k \in \mathbb{N}$ such that $\lambda := \inf_x |(T^k)'(x)| > 1$. It is not difficult to verify that the symbolic coding of such a system leads to a topological Markov chain over the alphabet $A = \{1, \dots, N\}$. To simplify the discussion we assume that the transition matrix M of this topological Markov chain is irreducible and aperiodic.

Our goal is to find a T -invariant measure $\tilde{\mu}$ represented by $\mu \in \mathcal{M}_\sigma(X_M)$ which minimises the relative entropy to Lebesgue measure on $[0, 1]$

$$h(\tilde{\mu}|m) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{a_0, \dots, a_{n-1} \in \{1, \dots, N\}} \mu[a_0 \dots a_{n-1}] \log \frac{\mu[a_0 \dots a_{n-1}]}{\nu_n[a_0 \dots a_{n-1}]}$$

where $\nu_n[a_0 \dots a_{n-1}] := |I_{a_0 \dots a_{n-1}}|$. (Recall that, without insisting on invariance, this would just be Lebesgue measure itself.) The existence of the limit will be justified below - observe that m is not a Gibbs state as in Section 6. The argument rests on the simple observation (implied by the uniform expansion and the piecewise Hölder-continuity of T') that T has *bounded distortion*, *i.e.*, that there is a constant $C > 0$ such that for all $n \in \mathbb{N}$, $a_0 \dots a_{n-1} \in \{1, \dots, N\}^n$ and $u \in I_{a_0 \dots a_{n-1}}$ holds

$$(21) \quad C^{-1} \leq |I_{a_0 \dots a_{n-1}}| \cdot |(T^n)'(u)| \leq C, \text{ or, equivalently, } C^{-1} \leq \frac{|I_{a_0 \dots a_{n-1}}|}{\exp(S_n \tilde{\phi}(u))} \leq C$$

where $\tilde{\phi}(u) := -\log |T'(u)|$. (Observe the similarity between this property and the Gibbs property (16).) Assuming bounded distortion we have at once

$$h(\tilde{\mu}|m) = \lim_{n \rightarrow \infty} \frac{1}{n} \left(-H_n(\mu) - \sum_{k=0}^{n-1} \langle \phi \circ \sigma^k, \mu \rangle \right) = -h(\mu) - \langle \phi, \mu \rangle,$$

and minimising this relative entropy just amounts to maximising $h(\mu) + \langle \phi, \mu \rangle$ for $\phi = -\log |T'| \circ \pi$. As the results on Gibbs distributions from Section 6 apply, we conclude that

$$C^{-1} \leq \frac{\mu[a_0 \dots a_{n-1}]}{|I_{a_0 \dots a_{n-1}}|} \leq C$$

for some $C > 0$. So the unique T -invariant measure $\tilde{\mu}$ that minimises the relative entropy $h(\tilde{\mu}|m)$ is equivalent to Lebesgue measure m . (The existence of an invariant probability measure equivalent to m is well-known, also without invoking entropy theory. It is guaranteed by a ‘‘Folklore Theorem’’ [33].)

8.2. Interval Maps with an Indifferent Fixed Point. The presence of just one point $x \in [0, 1]$ such that $T'(x) = 1$ dramatically changes the properties of the system. A canonical example is the map $T_\alpha : x \mapsto x(1 + 2^\alpha x^\alpha)$ if $x \in [0, 1/2[$ and $x \mapsto 2x - 1$ if $x \in [1/2, 1]$. We have $T'(0) = 1$, *i.e.*, 0 is an indifferent fixed point. For $\alpha \in [0, 1[$ this map admits an absolutely continuous invariant probability measure $d\mu(x) = h(x)dx$, where $h(x) \sim x^{-\alpha}$ when $x \rightarrow 0$ [66]. In the physics literature, this type of map is known as ‘‘Manneville-Pomeau’’ map. It was introduced as a model of transition from laminar to intermittent behaviour [50]. In [28] the authors construct a piecewise affine version of this map to study the complexity of trajectories (in the sense of Subsection 4.4). This gives rise to a countable state Markov chain. In [69] the close connection to the Fisher-Felderhof model and Hofbauer’s example (see Subsection 5.3) was realised. We refer to [61] for recent developments and a list of references.

8.3. Axiom A Diffeomorphisms, Anosov Diffeomorphisms, Sinai-Ruelle-Bowen Measures. The first spectacular application of the theory of Gibbs measures to differentiable dynamical systems was Sinai’s approach to Anosov diffeomorphism via Markov partitions [64] that allowed to code the dynamics of these maps into a subshift of finite type and to study their invariant measures by methods from equilibrium statistical mechanics [65] that had been developed previously by Dobrushin, Lanford and Ruelle [17, 18, 19, 20, 41]. Not much later this approach was extended by Bowen [2] to Smale’s Axiom A diffeomorphisms (and to Axiom A flows by Bowen and Ruelle [7]); see also

[54]. The interested reader can consult, *e.g.*, [71] for a survey, and either [5] or [15] for details.

Both types of diffeomorphisms act on a smooth compact Riemannian manifold M and are characterised by the existence of a compact T -invariant *hyperbolic set* $\Lambda \subseteq M$. Their basic properties are described in detail in the contribution [[Nicol-Petersen]]. Very briefly, the tangent bundle over Λ splits into two invariant subbundles - a stable one and an unstable one. Correspondingly, through each point of Λ there passes a local stable and a local unstable manifold which are both tangent to the respective subspaces of the local tangent space. The unstable derivative of T , *i.e.*, the derivative DT restricted to the unstable subbundle, is uniformly expanding. Its Jacobian determinant, denoted by $J^{(u)}$, is Hölder continuous as a function on Λ . Hence the observable $\phi^{(u)} := -\log |J^{(u)}| \circ \pi$ is Hölder continuous, and the Gibbs and equilibrium theory apply (via the symbolic coding) to the diffeomorphism T (modulo possibly a decomposition of the hyperbolic set into irreducible and aperiodic components, called basic sets, that can be modelled by topologically mixing subshifts of finite type). The main results are:

Characterisation of attractors

The following assertions are equivalent for a basic set $\Omega \subseteq \Lambda$:

- (i) Ω is an attractor, *i.e.*, there are arbitrarily small neighbourhoods $U \subseteq M$ of Ω such that $TU \subset U$.
- (ii) The union of all stable manifolds through points of Ω is a subset of M with positive volume.
- (iii) The pressure $P_{T|_{\Omega}}(\phi^{(u)}) = 0$.

In this case the unique equilibrium and Gibbs state μ^+ of $T|_{\Omega}$ is called the *Sinai-Ruelle-Bowen (SRB) measure* of $T|_{\Omega}$. It is uniquely characterised by the identity $h_{T|_{\Omega}}(\mu^+) = -\langle \phi^{(u)}, \mu^+ \rangle$. (For all other T -invariant measures on Ω one has “ $<$ ” instead of “ $=$ ”.)

Further properties of SRB measures

Suppose $P_{T|_{\Omega}}(\phi^{(u)}) = 0$ and let μ^+ be the SRB measure.

- (a) For a set of points $u \in M$ of positive volume we have:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k u) = \langle f, \mu^+ \rangle.$$

(Indeed, because of (ii) of the above characterisation, this holds for almost all points of the union of the stable manifolds through points of Ω .)

- (b) Conditioned on unstable manifolds, μ^+ is absolutely continuous to the volume measure on unstable manifolds.

In the special case of transitive Anosov diffeomorphisms, the whole manifold is a hyperbolic set and $\Omega = M$. Because of transitivity, property (ii) from the characterisation of attractors is trivially satisfied, so there is always a unique SRB measure μ^+ . As T^{-1} is an Anosov diffeomorphism as well - only the roles of stable and unstable manifolds are

interchanged - T^{-1} has a unique SRB measure μ^- which is the unique equilibrium state of T^{-1} (and hence also of T) for $\phi^{(s)} := \log |J^{(s)}|$. One can show:

SRB measures for Anosov diffeomorphisms

The following assertions are equivalent:

- (i) $\mu^+ = \mu^-$.
- (ii) μ^+ or μ^- is absolutely continuous w.r.t the volume measure on M .
- (iii) For each periodic point $u = T^n u \in M$, $|J(u)| = 1$, where J denotes the determinant of DT .

We remark that, similarly as in the case of Markov interval maps, the unstable Jacobian of T^n at u is asymptotically equivalent to the volume of the “ n -cylinder” of the Markov partition around u . So the maximisation of $h(\mu) + \langle \phi^{(u)}, \mu \rangle$ by the SRB measure μ^+ can again be interpreted as the minimisation of the relative entropy of invariant measures with respect to the normalised volume, and the fact that $P(\phi^{(u)}) = 0$ in the Anosov (or more generally attractor) case means that μ^+ is as close to being absolutely continuous as it is possible for a singular measure. This is reflected by the above properties (a) and (b).

We emphasise the meaning of property (a) above: it tells us that the SRB measure μ^+ is the only *physically observable* measure. Indeed, in numerical experiments with physical models, one picks an initial point $u \in M$ “at random” (*i.e.*, with respect to the volume or Lebesgue measure) and follows its orbit $T^k u$, $k \geq 0$.

8.4. Bowen’s Formula for the Hausdorff Dimension of Conformal Repellers.

Just as nearby orbits converge towards an attractor, they diverge away from a repeller. Conformal repellers form a nice class of systems which can be coded by a subshift of finite type. The construction of their Markov partitions is much simpler than that of Anosov diffeomorphisms, see, *e.g.*, [72].

Let us recall the definition of a conformal repeller before giving a fundamental example. Given a holomorphic map $T : V \rightarrow \mathbb{C}$ where $V \subset \mathbb{C}$ is open and J a compact subset of \mathbb{C} , one says that (J, V, T) is a conformal repeller if

- (i) there exist $C > 0$, $\alpha > 1$ such that $|(T^n)'(z)| \geq C\alpha^n$ for all $z \in J$, $n \geq 1$;
- (ii) $J = \bigcap_{n \geq 1} T^{-n}(V)$ and
- (iii) for any open set U such that $U \cap J \neq \emptyset$, there exists n such that $T^n(U \cap J) \supset J$.

From the definition it follows that $T(J) = J$ and $T^{-1}(J) = J$.

A fundamental example is the map $T : z \rightarrow z^2 + c$, $c \in \mathbb{C}$ being a parameter. It can be shown that for $|c| < \frac{1}{4}$ there exists a compact set J , called a (hyperbolic) *Julia set*, such that $(J, \overline{\mathbb{C}}, T)$ is a conformal repeller. As usual, $\overline{\mathbb{C}}$ denotes the Riemann sphere (the compactification of \mathbb{C}).

Conformal repellers J are in general fractal sets and one can measure their “degree of fractality” by means of their Hausdorff dimension, $\dim_H(J)$. Roughly speaking, one computes this dimension by covering the set J by balls with radius less than or equal to δ . If $N_\delta(J)$ denotes the cardinality of the smallest such covering, then we expect that

$$N_\delta(J) \sim \delta^{-\dim_H(J)}, \text{ as } \delta \rightarrow 0.$$

We refer the reader to [[Schmeling]] or [22, 46] for a rigorous definition (based on Carathéodory’s construction) and for more informations on fractal geometry.

Bowen’s formula relates $\dim_H(J)$ to the unique zero of the pressure function $\beta \mapsto P(\beta\tilde{\phi})$ where $\tilde{\phi} := -(\log |T'|)|J$. It is not difficult to see that indeed this map has a unique zero for some positive β .

► By property (i), $S_n\tilde{\phi} \leq \text{const} - n \log \alpha$, which implies (by (13)) that $\frac{\partial}{\partial \beta} P(\beta\tilde{\phi}) = \langle \tilde{\phi}, \mu_\beta \rangle \leq -\log \alpha < 0$. As $P(0)$ equals the topological entropy of J , *i.e.*, the logarithm of the largest eigenvalue of the matrix M associated to the Markov partition, $P(0)$ is strictly positive. Therefore (recall that the pressure function is continuous) there exists a unique number $\beta_0 > 0$ such that $P(\beta_0\tilde{\phi}) = 0$. ◀

It turns out that this unique zero is precisely $\dim_H(J)$:

Bowen’s formula. The Hausdorff dimension of J is the unique solution of the equation $P(\beta\tilde{\phi}) = 0$, $\beta \in \mathbb{R}$; in particular

$$P(\dim_H(J)\tilde{\phi}) = 0.$$

This formula was proven in [55] for a general class of conformal repellers after the seminal paper [6]. The main tool is a distortion estimate very similar to (21). A simple exposition can be found in [72].

9. NONEQUILIBRIUM STEADY STATES AND ENTROPY PRODUCTION

SRB measures for Anosov diffeomorphisms and Axiom A attractors have been accepted recently as conceptual models for *nonequilibrium steady states* in nonequilibrium statistical mechanics. Let us point out that the word “equilibrium” is used in physics in a much more restricted sense than in ergodic theory. Only diffeomorphisms preserving the natural volume of the manifold (or a measure equivalent to the volume) would be considered as appropriate toy models of physical equilibrium situations. In the case of Anosov diffeomorphisms this is precisely the case if the “forward” and “backward” SRB measures μ^+ and μ^- coincide. Otherwise the diffeomorphism models a situation out of equilibrium, and the difference between μ^+ and μ^- can be related to entropy production and irreversibility.

Gallavotti and Cohen [27, 26] introduced SRB measures as idealised models of nonequilibrium steady states around 1995. In order to have as firm a mathematical basis as possible they made the “*chaotic hypothesis*” that the systems they studied behave like transitive Anosov systems. Ruelle [56] extended their approach to more general (even

nonuniformly) hyperbolic dynamics; see also his reviews [59, 57] for more recent accounts discussing also a number of related problems; see also [51]. The importance of the Gibbs property of SRB measures for the discussion of entropy production was also highlighted in [35], where it is shown that for transitive Anosov diffeomorphisms the relative entropy $h(\mu^+|\mu^-)$ equals the average entropy production rate $\langle \log |J|, \mu^+ \rangle$ of μ^+ where J denotes again the Jacobian determinant of the diffeomorphism. In particular, the entropy production rate is zero if, and only if, $h(\mu^+|\mu^-) = 0$, *i.e.*, using coding and (19), if, and only if, $\mu^+ = \mu^-$. According to Subsection 8.3, this is also equivalent to μ^+ or μ^- being absolutely continuous with respect to the volume measure.

10. SOME ONGOING DEVELOPMENTS AND FUTURE DIRECTIONS

As we saw, many dynamical systems with uniform hyperbolic structure (*e.g.*, Anosov maps, axiom A diffeomorphisms) can be modelled by subshifts of finite type over a finite alphabet. We already mentioned in Subsection 8.2 the typical example of a map of the interval with an indifferent fixed point, whose symbolic model is still a subshift of finite type, but with a countable alphabet. The thermodynamic formalism for such systems is by now well developed [23, 30, 60, 61, 62] and used for multidimensional piecewise expanding maps [13]. An active line of research is related to systems admitting representations by symbolic models called “towers” constructed by using “inducing schemes”. The fundamental example is the class of one-dimensional unimodal maps satisfying the “Collet-Eckmann condition”. A first attempt to develop thermodynamic formalism for such systems was made in [10] where existence and uniqueness of equilibrium measures for the potential function $\tilde{\phi}_\beta(u) = -\beta \log |T'(u)|$ with β close to 1 was established. Very recently, new developments in this direction appeared, see, *e.g.*, [11, 12, 47].

A largely open field of research concerns a new branch of nonequilibrium statistical mechanics, the so-called “chaotic scattering theory”, namely the analysis of chaotic systems with various openings or holes in phase space, and the corresponding repellers on which interesting invariant measures exist. We refer the reader to [15] for a brief account and references to the physics literature. The existence of (generalised) steady states on repellers and the so-called “escape rate formula” have been observed numerically in a number of models. So far, little has been proven mathematically, except for Anosov diffeomorphisms with special holes [15] and for certain nonuniformly hyperbolic systems [9].

REFERENCES

- [1] Baladi V (2000) Positive transfer operators and decay of correlations. Advanced Series in Nonlinear Dynamics, 16. World Scientific
- [2] Bowen R (1970) Markov partitions for Axiom A diffeomorphisms. Amer. J. Math. 92, 725–747
- [3] Bowen R (1974/1975) Some systems with unique equilibrium states. Math. Systems Theory 8, 193–202
- [4] Bowen R (1974/75) Bernoulli equilibrium states for Axiom A diffeomorphisms. Math. Systems Theory 8, 289–294
- [5] Bowen R (2008) Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Lecture Notes in Mathematics, Vol. 470. 2nd rev. ed. (1st ed. 1975). Springer

- [6] Bowen R (1979) Hausdorff dimension of quasicircles. *Inst. Hautes Études Sci. Publ. Math.* No. 50 (1979), 11–25
- [7] Bowen R and Ruelle D (1975) The ergodic theory of Axiom A flows. *Invent. Math.* 29, 181–202
- [8] Brudno A A (1983) Entropy and the complexity of the trajectories of a dynamical system. *Trans. Moscow Math. Soc.* 2, 127–151
- [9] Bruin H, Demers M and Melbourne I (2007) Existence and convergence properties of physical measures for certain dynamical systems with holes, preprint
- [10] Bruin H and Keller G (1998) Equilibrium states for S -unimodal maps. *Ergodic Theory Dynam. Systems* 18, no. 4, 765–789
- [11] Bruin H and Todd M (2007) Equilibrium states for interval maps: potentials of bounded range, preprint
- [12] Bruin H and Todd M (2007) Equilibrium states for interval maps: the potential $-t \log |Df|$, preprint
- [13] Buzzi J and Sarig O (2003) Uniqueness of equilibrium measures for countable Markov shifts and multidimensional piecewise expanding maps. *Ergodic Theory Dynam. Systems* 23, no. 5, 1383–1400
- [14] Chaitin G J (1987) Information, randomness & incompleteness. *Papers on algorithmic information theory*. World Scientific Series in Computer Science, 8. World Scientific
- [15] Chernov N (2002) Invariant measures for hyperbolic dynamical systems. *Handbook of dynamical systems*, Vol. 1A, 321–407, North-Holland
- [16] Coelho Z and Parry W (1990) Central limit asymptotics for shifts of finite type. *Israel J. Math.* 69, no. 2, 235–249
- [17] Dobrushin R L (1968) The description of a random field by means of conditional probabilities and conditions of its regularity. *Theory Probab. Appl.* 13, 197–224
- [18] Dobrushin R L (1968) Gibbsian random fields for lattice systems with pairwise interactions. *Functional Anal. Appl.* 2, 292–301
- [19] Dobrushin R L (1968) The problem of uniqueness of a Gibbsian random field and the problem of phase transitions. *Functional Anal. Appl.* 2, 302–312
- [20] Dobrushin R L (1969) Gibbsian random fields. The general case. *Functional Anal. Appl.* 3, 22–28
- [21] Eizenberg A, Kifer Y and Weiss B (1994) Large deviations for \mathbb{Z}^d -actions. *Comm. Math. Phys.* 164, no. 3, 433–454
- [22] Falconer K (2003) *Fractal geometry. Mathematical foundations and applications*. Second edition. John Wiley & Sons
- [23] Fiebig D, Fiebig U-R and Yuri M (2002) Pressure and equilibrium states for countable state Markov shifts. *Israel J. Math.* 131, 221–257
- [24] Fisher M E (1967) The theory of condensation and the critical point. *Physics* 3, 255–283
- [25] Felderhof B and Fisher M (1970) Four articles, *Ann. Phys.* 58, 176, 217, 268, 281
- [26] Gallavotti G (1996) Chaotic hypothesis: Onsager reciprocity and fluctuation-dissipation theorem. *J. Stat. Phys.* 84, 899–925
- [27] Gallavotti G and Cohen E G D (1995) Dynamical ensembles in stationary states. *J. Stat. Phys.* 80, 931–970
- [28] Gaspard P and Wang X-J (1988) Sporadicity: Between periodic and chaotic dynamical behaviors *Proceedings of the National Academy of Sciences U. S. A.* 85, 4591–4595
- [29] Georgii H-O (1988) *Gibbs measures and phase transitions*. de Gruyter Studies in Mathematics, 9. Walter de Gruyter & Co., Berlin
- [30] Gurevich B M and Savchenko S V (1998) Thermodynamic formalism for symbolic Markov chains with a countable number of states. *Russian Math. Surveys* 53, no. 2, 245–344
- [31] Hofbauer F (1977) Examples for the nonuniqueness of the equilibrium state. *Transactions Amer. Math. Soc.* 228, 223–241
- [32] Israel R (1979) *Convexity in the theory of lattice gases*. Princeton Series in Physics. Princeton University Press
- [33] Jakobson M and Świątek (2002) One-dimensional maps. *Handbook of dynamical systems*, Vol. 1A, 321–407, North-Holland
- [34] Jaynes E T (1989) *Papers on probability, statistics and statistical physics*. Kluwer

- [35] Jiang D, Qian M and Qian M-P (2000) Entropy production and information gain in Axiom A systems. *Commun. Math. Phys.* 214, 389–409
- [36] Katok A (2007) Fifty years of entropy in dynamics: 1958–2007. *J. Mod. Dyn.* 1, no. 4, 545–596
- [37] Katok A and Hasselblatt B (1995) Introduction to the modern theory of dynamical systems. *Encyclopaedia of Mathematics and its Applications*, 54. Cambridge University Press
- [38] Keller G (1998) Equilibrium states in ergodic theory. *London Mathematical Society Student Texts*, 42. Cambridge University Press, Cambridge
- [39] Kifer Y (1990) Large deviations in dynamical systems and stochastic processes. *Trans. Amer. Math. Soc.* 321 (1990), 505–524
- [40] Kolmogorov A N (1983) Combinatorial foundations of information theory and the calculus of probabilities. *Uspekhi Mat. Nauk* 38, 27–36
- [41] Lanford O E and Ruelle D (1969) Observables at infinity and states with short range correlations in statistical mechanics. *Commun. Math. Phys.* 13, 194–215
- [42] Lewis J T and Pfister C-E (1995) Thermodynamic probability theory: some aspects of large deviations. *Russian Mathematical Surveys* 50, 279–317
- [43] Lind D and Marcus B (1995) An introduction to symbolic dynamics and coding. Cambridge University Press
- [44] Misiurewicz M (1976) A short proof of the variational principle for a Z_+^N action on a compact space. *International Conference on Dynamical Systems in Mathematical Physics (Rennes, 1975)*, pp. 147–157. *Astérisque*, No. 40, Soc. Math. France
- [45] Moulin Ollagnier J (1985) *Ergodic Theory and Statistical Mechanics*. *Lecture Notes in Mathematics*, Vol. 1115. Springer
- [46] Pesin Y (1997) Dimension theory in dynamical systems. *Contemporary views and applications*. University of Chicago Press
- [47] Pesin Y and Senti S (2007) Equilibrium Measures for Maps with Inducing Schemes, preprint
- [48] Pesin Y and Weiss (1997) The multifractal analysis of Gibbs measures: motivation, mathematical foundation, and examples. *Chaos* 7, no. 1, 89–106
- [49] Pollicott M (2000) Rates of mixing for potentials of summable variation. *Trans. Amer. Math. Soc.* 352, no. 2, 843–853
- [50] Pomeau Y and Manneville P (1980) Intermittent transition to turbulence in dissipative dynamical systems. *Comm. Math. Phys.* 74, no. 2, 189–197
- [51] Rondoni L and Mejía-Monasterio C (2007) Fluctuations in nonequilibrium statistical mechanics: models, mathematical theory, physical mechanisms. *Nonlinearity* 20, no. 10, R1–R37
- [52] Ruelle D (1968) Statistical mechanics of a one-dimensional lattice gas. *Commun. Math. Phys.* 9, 267–278
- [53] Ruelle D (1973) Statistical mechanics on a compact set with Z' action satisfying expansiveness and specification. *Trans. Amer. Math. Soc.* 185, 237–251
- [54] Ruelle D (1976) A measure associated with Axiom A attractors. *Amer. J. Math.* 98, 619–654.
- [55] Ruelle D (1982) Repellers for real analytic maps. *Ergodic Theory Dynamical Systems* 2, no. 1, 99–107
- [56] Ruelle D (1996) Positivity of entropy production in nonequilibrium statistical mechanics. *J. Stat. Phys.* 85, 1–23
- [57] Ruelle D (1998) Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics. *J. Stat. Phys.* 95, 393–468
- [58] Ruelle D (2004) Thermodynamic formalism. The mathematical structures of equilibrium statistical mechanics. Second edition. *Cambridge Mathematical Library*. Cambridge University Press
- [59] Ruelle D (2003) Extending the definition of entropy to nonequilibrium steady states. *Proc. Nat. Acad. Sc.* 100(6), 3054–3058
- [60] Sarig O (1999) Thermodynamic formalism for countable Markov shifts. *Ergodic Theory Dynam. Systems* 19, no. 6, 1565–1593
- [61] Sarig O (2001) Phase transitions for countable Markov shifts. *Comm. Math. Phys.* 217, no. 3, 555–577

- [62] Sarig O (2003) Existence of Gibbs measures for countable Markov shifts. *Proc. Amer. Math. Soc.* 131 (2003), no. 6, 1751–1758
- [63] Seneta E (2006) Non-negative matrices and Markov chains. Springer Series in Statistics. Springer
- [64] Sinai Ja G (1968) Markov partitions and C-diffeomorphisms. *Functional Anal. Appl.* 2, 61-82
- [65] Sinai Ja G (1972) Gibbs measures in ergodic theory. *Russian Math. Surveys* 27(4), 21–69
- [66] Thaler M (1980) Estimates of the invariant densities of endomorphisms with indifferent fixed points. *Israel J. Math.* 37, no. 4, 303–314
- [67] Walters P (1975) Ruelle’s operator theorem and g -measures. *Trans. Amer. Math. Soc.* 214, 375–387
- [68] Walters P (1992) Differentiability properties of the pressure of a continuous transformation on a compact metric space. *J. London Math. Soc.* (2) 46, no. 3, 471–481
- [69] Wang X-J (1989) Statistical physics of temporal intermittency. *Phys. Rev. A* 40, no. 11, 6647–6661
- [70] Young L-S (1990) Large deviations in dynamical systems. *Trans. Amer. Math. Soc.* 318 (1990), 525–543
- [71] Young L-S (2002) What are SRB measures, and which dynamical systems have them? Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays. *J. Statist. Phys.* 108, no. 5-6, 733–754
- [72] Zinsmeister M (2000) Thermodynamic formalism and holomorphic dynamical systems. *SMF/AMS Texts and Monographs*, 2. American Mathematical Society