Existence for an eigenfunction for the critical phase of the Dyson model

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1 Introduction

It is well-known that there exists a continuous and strictly positive eigenfunction h for any transfer operator defined on a symbolic shift space with finitely many symbols such that the potential has summable variations. Here we prove the existence of an eigenfunction for the important special class of Dyson potentials close to the critical phase, when the potential does not satisfy the condition of summable variations.

More precisely, let T be the left shift on the space $X_+ = S^{\mathbb{Z}_+}$, where S is a finite set. Define a transfer operator $\mathcal{L}$ on continuous functions f by

$$\mathcal{L}f(x) = \sum_{y \in T^{-1}x} e^{\phi(y)} f(y), \tag{1}$$

where ϕ is a continuous potential. Since X_+ is a compact space, it follows automatically from the Schauder–Tychonoff theorem that the dual $\mathcal{L}^*$, restricted to the probability measures, has an eigenmeasure ν: $\mathcal{L}^*\mu = \lambda\nu$, for some $\lambda > 0$. The existence of a continuous eigenfunction h such that $\mathcal{L}h = \lambda h$, $\lambda > 0$, is however not automatic for any continuous potential ϕ. If one assumes summable variations of ϕ,

$$\sum_{n=1}^{\infty} \mathrm{var}_n(\phi) < \infty, \tag{2}$$

where $\mathrm{var}_n(\phi) = \sup_{x \sim_n y} |\phi(x) - \phi(y)|$ ($x \sim_n y$ means that x and y coincide in the first n entries), then existence of a continuous eigenfunction follows for a typical "cone-argument", such as in, *e.g.*, Walters [**?**].

If μ is the equilibrium measure (translation invariant Gibbs measure) for the continuous potential $\psi : X \to X$, where $X = S^{\mathbb{Z}}$, then (with slight abuse of notation) μ is recovered on X_+ as the g-measure for the probability potential

$$g(x) = \frac{h(x)e^{\phi(y)}}{\lambda h(Tx)}. \tag{3}$$

For a transfer operator $\mathcal{L}_g$, defined on continuous functions f by

$$\mathcal{L}_g f(x) = \sum_{y \in T^{-1}x} g(y) f(y) \tag{4}$$

we have $\mathcal{L}_g^* \mu = \mu$, and also the invariance $\mu \circ T^{-1} = \mu$.

More precisely, if

$$\theta(x_0, x_1, \ldots) = \beta \sum_{j=1}^{\infty} \frac{x_j}{j^\alpha},$$

then we may define the two-sided Dyson potential ψ as

$$\psi(x) = \sum_{k=-\infty}^{\infty} \theta(T^k x).$$

A one-sided version can be defined as

$$\phi(x) = \sum_{k=0}^{\infty} \theta(T^k x).$$

We then have for $\alpha > 2$

$$\mu_{|\mathcal{F}_{[0,\infty)}} = h\nu,$$

where μ is the two-sided translation invariant Gibbs measure (the equilibrium measure) and where $\mathcal{L}^* \nu = \lambda \nu$, and $h > 0$ is a Hölder continuous eigenfunction. We are interested in the boundary case $\alpha = 2$, when there exists a unique equilibrium measure for ψ for $\beta < \beta_c$. In this case the summable variations condition is not satisfied for neither ψ nor ϕ; we may have multiple eigenmeasures for $\mathcal{L}*$. In this context we have $\mathrm{var}_n(\phi) = O(\frac{1}{n})$ and it is not clear if there exists an eigenfunction even in the cases we have a unique equilibrium measure, such as in the case of the Berbee condition [**?**]. Berbee proved that there exists a unique equlibrium measure whenever

$$\sum_{n=1}^{\infty} e^{-r_1 - r_2 - \cdots - r_n} = \infty, \tag{5}$$

where $r_n = \mathrm{var}_n \log \psi$, or if $r_n = \mathrm{var}_n \log phi$. Berbee's condition is very robust in the sense that it gives uniqueness for both the two-sided and one-sided potentials, whereas square summable variations (see [**?**]) only gives a unique g-measure, $\mathcal{L}_g^* \mu = \mu$. Since we have proved in [**?**] that for all $epsilon > 0$ we can find a one-sided potential ϕ with

$$\sum_{n=1}^{\infty} (\mathrm{var}_n \phi)^{1+\epsilon} < \infty$$

such that $\mathcal{L}^* \nu = \lambda \nu$ has multiple solutions ν, it seems that the existence of a at least a very regular eigenfunction of $\mathcal{L}$ would be in doubt in view of uniqueness of a g-measure corresponding to multiple one-sided Gibbs measures.

Here we prove that if the inverse critical temperature is small enough, close to the critical phase, then we still have an eigenfunction of $\mathcal{L}$.

Theorem 1. *Let μ be the Gibbs equilibrium measure with respect to the Dyson potential ϕ and let ν be the one-sided Gibbs measure, i.e., $\mathcal{L}^*\mu = \lambda\mu$. Define*

$$h_n(x) = \frac{\mu[x_0, \dots x_n]}{\nu[x_0, \dots, x_n},$$

and consider the measurable function $h(x) = \lim_{n\to\infty} h_n(x)$. If $\beta < \beta_c$, then h is a continuous function on X_+, and which is also an eigenfunction of $\mathcal{L}$.

We conjecture that there exists a continuous eigenfunction for a potential ϕ that satisfies Berbee's condition (5).

2 Proof of the main results

Lemma 2. *There exists a continuous eigenfunction h of $\mathcal{L}$, if*

$$\frac{\mu[x_0, \dots x_N, \dots, x_n]}{\nu[x_0, \dots, x_N, \dots, x_n]} = (1 + o(1))\frac{\mu[x_0, \dots, x_N]}{\nu[x_0, \dots, x_N]},$$

as $N \to \infty$.

Proof. Let $\lambda = 1$. By the assumption, the measure $\mu = h\nu$ is translation invariant, i.e., $\mu \circ T^{-1} = \mu$. Hence we have $(h\nu) \circ T^{-1} = h\nu$ and it suffices to show that $(h\nu) \circ T^{-1} = (\mathcal{L}h)\nu$. Let A be any Borel subset of X_+. Then

$$(h\nu) \circ T^{-1}(A) = \int_A \sum_{y:Ty=x} h(y)e^{\phi(y)}\, d\nu(x) = \int_A h(x)\, d\nu(x).$$

$\square$

Lemma 3. *If*

$$\frac{\mu[x_0, \dots X_N, \dots, x_n]}{\nu[x_0, \dots, X_N, \dots, x_n]} = e^{xi}\ \frac{\mu[x_0, \dots, x_N]}{\nu[x_0, \dots, x_N]}$$

then $|\xi| = O(P(A_N))$, where A_N is the event that there exists a cluster $C \subset [(-\infty, 0]$ with two edges that goes from C into $[N, \infty)$.

Proof. For the random cluster model, we have that

$$\mu[x_0, \dots, x_N] \propto P((t_{ij} \sim [x_0, \dots, x_N]),$$

where $\sim$ means that the graph t_{ij} is compatible with the cylinder $[x_0, \dots, x_N]$.

We will study conditional probabilities $P(B_M|B_N)$, where B_M means $t_{ij} \sim [x_0, \ldots, x_M]$.

We have

$$e^{\xi} = \frac{K(x_0, \ldots, x_N)P(B_M|B_N}{\tilde{K}(x_0, \ldots, x_N)P(\tilde{B}_M|\tilde{B}_N)}.$$

We make a coupling with the one-sided system $\tilde{t} = t[0, \infty)$ and use the corresponding events $\tilde{B}_M : \tilde{t} \sim [x_0, \ldots, x_M]$. We have $B_M \subset \tilde{B}_M$.

We have

$$\frac{P(B_M|B_N)}{P(\tilde{B}_M|\tilde{B}_N)} = \frac{P(B_M|\tilde{B}_M, B_N)\mathsf{P}(\tilde{B}_M|B_N)}{P(\tilde{B}_M|\tilde{B}_N)} = P(B_M|\tilde{B}_M, B_N) = e^{o(1)},$$

as $N \to \infty$, independently of M. $\square$

Lemma 4.

$$\lim_{N\to\infty} P(A_N) = 0.$$

Proof. Let $M(C) = \inf\{i \in C\}$ and let $S = P(A_N|t[(-\infty, 0]])$.

We have

$$P(A_N|t[(-\infty, 0]]) \leq K \sum_{k=1}^{\infty} \frac{C_k}{N + M(C_k)}.$$

The conclusion follows from $E(S) < \infty$.

Let $X(C)$ be the number of clusters C that have edges between C and $[N, \infty)$. By assuming that $X(C) \sim P_0(\lambda)$, we can make the estimate

$$P(X(C) \geq 2) \leq c \cdot \lambda^2 \leq c \cdot \frac{|C|^2}{(N + M)^2} \cdot \beta^2,$$

where $c = \sum_{k=2}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!}$.

We have

$$E(S) \leq K \sum_{k=1}^{\infty} \frac{E(C_k^2)}{(N + kE(C_k))^2},$$

where C_k are clusters in $t[(-\infty, 0])$ ordered after $M(C_k)$, $M(C_k) \geq k$. We can make an estimate

$$E(S) \leq E(C_k^2) \sum_{k=1}^{\infty} \frac{1}{(N + k)^2},$$

which proves the lemma, since $E(C_k^2) < \infty$ by [**?**].

$\square$

3 The random cluster model for ferromagnetic Ising spin models

Given a finite graph (V, E) and edge probabilities $p : E \to [0,1]$, $ij \mapsto p_{ij}$, the Bernoulli graph model $\xi = \xi_p$ is a probability on the set of spanning subgraphs $G \subset E$ viewed as functions $G : E \to \{0,1\}$ such that the probability of each $G \in \{0,1\}^E$ is given by

$$\xi(G) = p^G (1-G)^{1-G} := \prod_{ij} p_{ij}^{G_{ij}} \cdot (1-p_{ij})^{1-G_{ij}}.$$

The expectation of a function $f(G)$, $G \in \{\pm 1\}^V$, with respect to ξ is written $\xi\left(()\, f(G)\right)$, i.e. $\xi\left(()\, f\right) = \sum_G \xi(G) \cdot f(G)$.

An Ising spin vector on G is a vector $x \in \{\pm 1\}^V$ indexed by vertices in G. Consider a potential of the form

$$H(x) = \beta \sum_{ij \in E} J_{ij} x_i x_j,$$

where $J_{ij} \geq 0$ gives the interaction strength along edge $ij \in E$. The corresponding (ferromagnetic) Ising model on G is the probability measure μ where the probability of $x \in \{\pm 1\}^V$ is given by

$$\mu(x) = \frac{1}{Z} e^{-H(x)}.$$

The *random cluster model* relates the Ising model with a Bernoulli random graph model as follows: Consider a G having Bernoulli distribution $\xi = \xi_p$, where the edge probabilities are

$$p_{ij} = 1 - e^{-\beta J_{ij}} \quad \text{for } ij \in E$$

and a uniformly chosen spin configuration $x \in \{\pm 1\}^V$, i.e. where the spins x_i are chosen from $\{\pm 1\}$ according to a fair coin. We say that $x \in \{\pm 1\}^V$ and $G \in \{0,1\}^E$ are *compatible* if no path in the graph G connects a spin $x_i = +1$ with a spin $x_j = -1$, $i \neq j$. We write $b(G, x)$ to indicate the event that G and x are compatible. The joint distribution of x and G *conditioned* on the event that they are compatible is what is named the random cluster model $\rho(x, G)$. One readily verify that the marginal distribution of x equals the Ising model on $\{\pm 1\}^V$ with potential H as above. The marginal distribution of G is given by

$$\tilde{\xi}(G) = \frac{2^{\omega(G)} \xi(G)}{\xi(2^{\omega(G)})}$$

where $\omega(G)$ denotes the number of connected components in the graph G.

In fact, the probability of a spin configuration $x \in \mathcal{X}_V$ is proportional to the probability that a $G \sim \xi(p)$ satisfies $B(G, x) = 1$, which is proportional to $e^{-H(x)}$. Moreover, the distribution of G is given by

$$\xi'(G) = \xi(G)\frac{2^{\omega(G)}}{\xi(2^{\omega(G)})},$$

where $\omega(G)$ denotes the number of components (a.k.a. "clusters") in G.

For a subset $S \subset V$ of vertices, consider the spin cylinder $[x]_S$. The conditional distribution of $[x]_S$, given the graph G, is then clearly

$$\mu(x_S|G) = 2^{-\tilde{\omega}_S(G)} \cdot b(G, [x]_S)$$

where $\tilde{\omega}_S(G)$ denote the number of clusters in G that intersects S. Thus $\omega(G) = \overline{\omega}_S(G) + \tilde{\omega}_S(G)$ where $\overline{\omega}_S(G)$ gives the number of clusters in G that are disjoint with S. We obtain that the marginal distribution of the cylinder $[x]_S$ is

$$\mu([x]_S) = \frac{\xi\left(2^{\overline{\omega}_S(G)} b(G, [x]_S)\right)}{\xi\left(2^{\omega(G)}\right)}.$$

If $R \subset S$ then we have

$$\mu([x]_S \mid [x]_R) = \frac{\xi\left(2^{\overline{\omega}_S(G)} \cdot b(G, [x]_S)\right)}{\xi\left(2^{\overline{\omega}_R(G)} \cdot b(G, [x]_R)\right)}.$$

3.1 The one-dimensional Ising-Dyson model

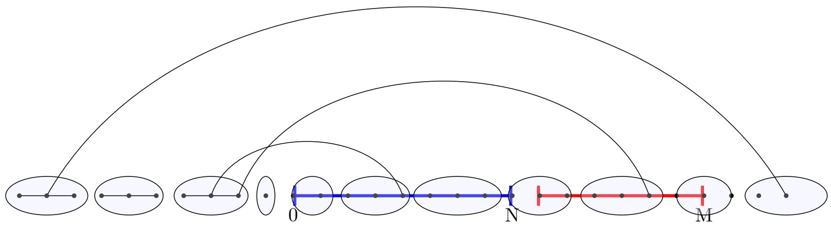
In the one-dimensional Ising-Dyson model let $V = [-L, L] \subset \mathbb{Z}$ and

$$J_{ij} = \frac{\beta}{|i-j|^{\alpha}}.$$

We can partition V as $V_- \cup V_+$, where $V_+ = [0, L]$ and $V_- = [-L, -1]$. The random graph $G^{\pm}$ is the graph G induced on $V_{\pm}$, i.e. $G_{ij} = 1$ only if both ends, i and j, are in the integer interval $V_{\pm}$. Note that the graphs G^+ and G^- are independent under ξ. In fact, under the Bernoulli graph model, the graph G^+ is independent of the graph $G' = G \setminus G^+$.

For $N \geq 0$, let N denote the integer interval $[0, N)$ so that $[x]_M = [x]_{[0,M)}$, and $\overline{\omega}_N = \overline{\omega}_{[0,N)}$, etc. We also use $\overline{N}$ to denote the interval $[N, \infty)$

Let $\mathcal{L}^{\pm} = \{C_1^{\pm}, C_2^{\pm}, \dots\}$ denote the set of clusters of the graphs $G^{\pm}$ induced on the positive (negative) axis, i.e. the components in the graphs $G^{\pm}$ induced by G on V_+ and V_-, respectively. Let $\mathcal{R}_N^{\pm}$ be those clusters in $\mathcal{L}^{\pm}$, respectively, that do not send any edges to the interval $N = [0, N)$. Thus $\overline{\omega}_N(G^{\pm}) = |\mathcal{R}_N^{\pm}|$. Let also $\mathcal{R}''_N$ be the clusters in the graph $G' = G \setminus G^+$ that do not contain any vertex from $[0, N)$.



The one-sided Ising model $\nu(x)$ is obtained from the random cluster model with respect to spins in V_+ and the Bernoulli graph model of G^+. We may write the marginal of the cylinder $[x]_N$ for the *one-sided Ising model* as

$$\nu([x]_N) = \frac{\xi\left(2^{\overline{\omega}_S(G^+)} \cdot b(G^+, [x]_N)\right)}{\xi\left(2^{\omega(G^+)}\right)} = \frac{\xi\left(2^{\overline{\omega}_S(G^+)+f(G')} \cdot b(G^+, [x]_N)\right)}{\xi\left(2^{\omega(G^+)+f(G')}\right)}.$$

where $f(G')$ denote any function of the graph G'.

Our aim is to show that for all $x \in \{\pm 1\}^{[0,\infty)}$

$$\lim_{M,L\to\infty} \frac{\mu([x]_M \mid [x]_N)}{\nu([x]_M \mid [x]_N)} = 1 + o(N) \quad \text{as } N \to \infty.$$

From the computations above we deduce that

$$\frac{\mu([x]_N)}{\nu([x]_N)} = \text{const.} \times \frac{\xi\left(2^{\overline{\omega}_N(G)} \cdot b(G, [x]_N)\right)}{\xi\left(2^{\overline{\omega}_N(G^+)+f(G')} \cdot b(G^+, [x]_N)\right)}$$

where the constant is $\xi\left(2^{\omega(G^+)}\right)/\xi\left(2^{\omega(G)}\right)$.

Then

$$\overline{\omega}_N(G) = \overline{\omega}_N(G^+) + \overline{\omega}_N(G^-) - X_N(G') + Z_N(G) \tag{6}$$

where $X_N(G')$ are the number of edges between clusters in $\mathcal{R}_N^-$ and $\overline{N}$. The term $Z_N(G)$ is a correction term which is not independent of G^+. However, we have

$$Z_N(G) \leq Y_N(G')$$

where $Y_N(G)$ is the number

$$Y_N(G') = \sum_{C \in \mathcal{R}''_N} \max\{|E(C, \overline{N})| - 1, 0\}$$

We claim that $Y_N(G')$ has distribution bounded by a Poisson variable $\mathrm{Po}(\lambda_N)$ where $\lambda_N \to 0$ as $N \to \infty$.