EXISTENCE OF CONTINUOUS EIGENFUNCTIONS FOR THE DYSON MODEL IN THE CRITICAL PHASE

ANDERS JOHANSSON, ANDERS ÖBERG, AND MARK POLLICOTT

Abstract. We prove that there exists a continuous eigenfunction of the transfer operator defined by potentials for the so-called Dyson model for all inverse critical temperatures that are strictly less than the critical inverse temperature. This includes all cases when the potential does not have summable variations, the classical condition that ensures the existence of a continuous eigenfunction of the transfer operator. As a consequence the inverse critical temperatures for the one-sided and the two-sided models are the same. We use the random cluster model for our method of proof, so the results also hold in this wider context. In particular, we base our conclusions of this paper by recent results of Duminil-Copin, Garban, and Tassion [9].

1. Introduction

It is well-known [18] that there exists a continuous and strictly positive eigenfunction h for any transfer operator defined on a symbolic shift space with a finite number of symbols such that the potential has summable variations. Here we prove the existence of a continuous eigenfunction for the important special class of Dyson potentials up to the critical phase, when the potential does not satisfy the condition of summable variations. We stress that it is the continuity that is the main difficulty. To see that there is a measurable eigenfunction follows from a simple application of the martingale convergence theorem.

More precisely, let T be the left shift on the space $X = S^{\mathbb{Z}_+}$, where S is a finite set. Let $\phi : X \to \mathbb{R}$ be a continuous function defined up to an additive constant. We refer to ϕ as a one-point *potential*. A *Gibbs measure* ν for ϕ is one that minimises the free energy $P(\nu; \phi) = \nu(\phi) - H(\nu)$, where $H(\nu)$ denotes the entropy $\lim_{n\to\infty} H(\nu|_{\mathcal{F}_n})/n$ per time unit. This measure ν can also be obtained as a Gibbs measure on X with respect to the potential $\sum_{k=0}^{\infty} \phi(T^k x)$ on X. The *equilibrium measure* μ is a minimiser of $P(\mu; \phi)$ among all translation invariant measures $\mu \in \mathcal{M}_T(X)$. Taking the natural extension of μ, we can represent μ as a translation invariant measure on the *two-sided* space $X_0 = S^{\mathbb{Z}}$.

In addition, we can define the transfer operator $\mathcal{L} = \mathcal{L}_\phi$ on continuous functions $f(x)$ by

$$\mathcal{L}f(x) = \sum_{y \in T^{-1}x} e^{\phi(y)} f(y), \tag{1}$$

Since X is a compact space, it follows automatically from the Schauder–Tychonoff theorem that the dual $\mathcal{L}^*$, restricted to the probability measures, has an eigenmeasure ν: $\mathcal{L}^*\mu = \lambda\nu$, for some maximal $\lambda > 0$. This eigenmeasure ν is also the Gibbs measure with respect to ν constructed above. The existence of a continuous eigenfunction h such that $\mathcal{L}h = \lambda h$, $\lambda > 0$, is however not automatic.

If one assumes summable variations of ϕ,

$$\sum_{n=1}^{\infty} \mathrm{var}_n(\phi) < \infty, \tag{2}$$

where $\mathrm{var}_n(\phi) = \sup_{x\sim_n y} |\phi(x) - \phi(y)|$ ($x \sim_n y$ means that x and y coincide in the first n entries), then the existence of a continuous eigenfunction follows from the typical "cone-argument" used in *e.g.*, Walters [18], which to date is the only known method for providing the existence of a continuous eigenfunction. (??)

If μ is the corresponding equilibrium measure (translation invariant Gibbs measure) for the continuous potential $\psi : X \to X$, where $X = S^{\mathbb{Z}}$, then (with slight abuse of notation) μ is recovered on X as an eigenmeasure for the probability potential

$$g(x) = \frac{h(x)e^{\phi(y)}}{\lambda h(Tx)}. \tag{3}$$

More precisely, for a transfer operator $\mathcal{L}_g$, defined on continuous functions f by

$$\mathcal{L}_g f(x) = \sum_{y\in T^{-1}x} g(y)f(y) \tag{4}$$

we have $\mathcal{L}_g^*\mu = \mu$, and also the invariance $\mu \circ T^{-1} = \mu$. In [12] it was proved that uniqueness of such g-measures (in the terminology of Keane [16]) follows if we have

$$\sum_{n=1}^{\infty} (\mathrm{var}_n\, g)^2 < \infty,$$

and more recently Berger et al. [5] proved that for such *Doeblin measures* uniqueness follows if $\mathrm{var}_n \log g < 2/\sqrt{n}$ (see also [14] for a slightly stronger condition). This is in contrast to Dyson's counterexample in [10] where it is shown that there are examples of multiple equilibrium measures for ϕ whenever

$$\sum_{n=1}^{\infty} (\mathrm{var}_n\, \phi)^{1+\epsilon} < \infty,$$

where $\epsilon > 0$. Dyson's example is for a two-sided model, but we showed in [15] that the inverse critical temperature β_c^+ satisfies $\beta_c^+ \leq 8\beta_c$, where β_c is the inverse critical temperature for the two-sided model, and this show that Dyson's example of multiple equilibrium measures can be formulated for a one-sided model as above.

Since we have the "translation" via (3) between the case for general potentials and for Doeblin measures, we may guess that the existence of a continuous eigenfunction cannot be moved very far away from the summability of variations condition for a potential.

One possibility for the existence of a continuous eigenfunction in general could be the Berbee condition. Berbee proved ([4]) that there exists a unique equilibrium measure whenever

$$\sum_{n=1}^{\infty} e^{-r_1-r_2-\cdots-r_n} = \infty, \tag{5}$$

where $r_n = \mathrm{var}_n \log \phi$. Berbee's condition gives uniqueness for both the two-sided and one-sided potentials

Here, however, we study only the Dyson potential: Fix $\alpha > 1$ and $\beta > 0$. Let the one-point potential ϕ be given by

$$\phi(x_0, x_1, \ldots) = x_0 \cdot \beta \sum_{j=1}^{\infty} \frac{x_j}{j^{\alpha}},$$

and define the one-sided and two-sided Dyson potentials, $\phi : X \to \mathbb{R}$ and $\psi : X \to \mathbb{R}$, respectively, as

$$\phi(x) = \sum_{k=0}^{\infty} x_k \theta(T^k x) \quad \psi(x) = \sum_{k=-\infty}^{\infty} x_k \theta(T^k x).$$

Let μ and ν be the Gibbs measures on $\mathcal{M}(X)$ and $\mathcal{M}(X)$, corresponding to ψ and ϕ, respectively.

We then have for $\alpha > 2$

$$\mu|_{\mathcal{F}_{[0,\infty)}} = h\nu,$$

where μ is the two-sided translation invariant Gibbs measure (the equilibrium measure) and where $\mathcal{L}^*\nu = \lambda\nu$, and $h > 0$ is a Hölder continuous eigenfunction. We are interested in the boundary case $\alpha = 2$, when there exists a unique equilibrium measure for ψ for $\beta < \beta_c$ [1]. In this case the summable variations condition is not satisfied for neither ψ nor ϕ; hence we may have multiple eigenmeasures for $\mathcal{L}^*$. In this context we have $\mathrm{var}_n(\phi) = O(\frac{1}{n})$, but as we noted earlier, in the general setting above, is not clear that there exists a continuous eigenfunction even in the cases we have a unique equilibrium measure

We prove that if the inverse critical temperature β is strictly smaller than the critical inverse temperature β_c, which will be seen to be the same for the two-sided and one-sided models, then we have a continuous eigenfunction of $\mathcal{L}$.

Theorem 1. *Let μ be the Gibbs equilibrium measure with respect to the Dyson potential ϕ and let ν be the one-sided Gibbs measure, i.e., $\mathcal{L}^*\nu = \lambda\nu$. Define*

$$h_n(x) = \frac{\mu[x_0, \ldots x_n]}{\nu[x_0, \ldots, x_n]},$$

and consider the measurable function $h(x) = \lim_{n\to\infty} h_n(x)$ (that exists by virtue of the martingale convergence theorem). If $\beta < \beta_c$, then h is a continuous function on X, and which is also an eigenfunction of $\mathcal{L}$.

We conjecture that there exists a continuous eigenfunction for a potential ϕ that satisfies Berbee's condition (5).

The next result is a corollary of Theorem 1:

Theorem 2. *The critical β_c for the two-sided model is the same as for the one-sided model, i.e., $\beta_c^+ = \beta_c$.*

This improves our result of Theorem 1 in [15] that $\beta_c^+ \leq 8\beta_c$.

2. The one-dimensional random cluster model and the Ising–Dyson model

For a finite graph, let $\omega(G)$ denote the number of connected components ("clusters") in the graph G. For simple graphs $G \subset \binom{V}{2}$ on an countably infinite set V of vertices, we consider the number of clusters $\omega(G)$ as a *potential*. This means that the difference $\Delta\omega(G, F) = \omega(G) - \omega(F)$ is defined for any two graphs F and G that coincide outside a *finite* subset $\Lambda \subset \binom{V}{2}$.

A random graph $G \sim \alpha$ on a set of vertices V is a probability distribution α on the set $\{0,1\}^{\binom{V}{2}}$. The random cluster models $\mathcal{R}(V, p)$, we consider are specified by a set of vertices V and a weight function p, $ij \mapsto p(ij) \in [0, 1]$, defined on the set of pairs $ij \in \binom{V}{2}$. The model $\mathcal{R}(V, p)$ is a Gibbs distribution on random graphs, i.e. configurations i $\{0,1\}^{\binom{V}{2}}$, such that a

In the one-dimensional Ising–Dyson model we let $V = \mathbb{Z}$ (or $V = V_+ = \mathbb{Z}_{\geq 0}$ for the one-sided case) and for $i, j \in V$ let

$$J(ij) = \frac{\beta}{|i-j|^\alpha}. \tag{6}$$

We will consider the case $\alpha = 2$ and $0 < \beta < \beta_c$. For $\beta > 0$, let $\gamma \sim \eta$ be the Bernoulli random graph $\eta \in \mathcal{G}(V)$ with

$$\mathsf{P}(ij \in \gamma) = 1 - e^{-\beta J(ij)}.$$

The extended *random cluster model* can be obtained by considering the product measure $d\eta(\gamma) \otimes d1(x)$ between an independent Bernoulli distributed $\gamma \sim \eta_p$ random graph on $\mathcal{G}(V)$ and the uniform measure $x \sim 1$ on the spin sequences $x \in \{-1,+1\}^V$. The extended random cluster model $d\mu(x,\gamma)$ is the joint distribution of γ and x obtained by conditioning on the on the event that x and γ are compatible: That is, the event $C(x,\gamma)$ that no spins $x_i = +1$ and $x_j = -1$ in x are connected by a path (edge) in γ. One obtains the Ising model $d\mu(x)$ and the random cluster model $d\mu(\gamma)$ as the marginal distributions of x and γ, respectively.

We can also introduce the random cluster model μ as the Gibbs measure on $\{0,1\}^{\binom{V}{2}}$

$$d\mu = 2^{\omega(\gamma)} \ltimes d\eta(\gamma),$$

where $\omega(\gamma)$ is the potential counting then number of connected components ("clusters") in γ. For the values of β we consider the Gibbs measure is unique. Thus can we parameterise the random clusters models as $\mu = \mathcal{R}(V, J)$ where $J(ij)$ is a given weighting on $\binom{V}{2}$ such as (6).

We observe the difference between the one-sided random cluster model $\nu = \mathcal{R}(V_+, J)$ and the usual two-sided model $\mu = \mathcal{R}(V, J)$. We will use that a configuration γ can be factored as $\gamma = (\gamma_+, \epsilon, \gamma_-)$, where γ_- is the induced graph $\gamma[V_-]$ on vertices $-j \in V_- = \mathbb{Z}_{<0}$ and $\gamma_+ = \gamma[V_+]$ is the graph induced on vertices $i \in V_+ = \mathbb{Z}_{\geq 0}$. The graph $\epsilon = \gamma \cap E(V_+, V-)$ consists of edges ji, $i \geq 0$ and $j \geq 1$, connecting vertices $-j \in V_-$ with vertices $i \in V_+$. Note that we often use positive indices i, j, $i \geq 0$ and $j > 0$, as labels for edges in ϵ. Thus $J(ij) = \beta/(i + j)^\alpha$ with this labelling.

Let

$$\tilde{\eta}(\epsilon) = 2^{-|\epsilon|} \ltimes \eta(\epsilon).$$

Note that both $\eta(\cdot)$ and $\tilde{\eta}(\cdot)$ are Bernoulli measures. Let also

$$d\tilde{\nu}_n(\gamma) = \frac{d\nu(\gamma_-) \otimes \tilde{\eta}(\epsilon) \otimes \nu(\gamma_+)}{\nu(B([x]_n, \gamma_+))}$$

Let R_n be the number of correcting edges, i.e.,

$$R_n = \#\{ij \in \epsilon : \omega(\gamma_{<ij} + ij) = \omega(\gamma_{ij}\}, \quad j > n.$$

Let $B_n(\gamma) = C([x]_n, \gamma)$ be the indicator of the event that cylinder $[x]_n$ is compatible with graph $\gamma = (\gamma_-, \epsilon, \gamma_+)$. Let also $B'_n(\gamma) = B_n(\gamma) + (1 - B_n(\gamma_+))$ indicate compatibility of $[x]_n$ with γ or not compatible with γ_+. We then have

$$h_n(x) = \frac{\mu[x]_n}{\nu[x]_n} \propto \int B_n(\gamma) \cdot 2^{R_n(\gamma)}\, d\tilde{\nu}_n(\gamma)$$

Note that

$$R_n(\gamma) \leq Q_n(\gamma_-, \epsilon)$$

where Q_n denotes the number of edges in ϵ that connects a vertex $j \geq n$ to a cluster in γ_- with at least one more edge from the cluster to $[0, n-1]$. That is,

$$Q_n = \#\{ij \in \epsilon \mid \exists k\, \exists l\; kl \in \epsilon, i \sim_{\gamma_-} k, k > i, j > n\}.$$

Notice that Q_n only depends on (γ_-, ϵ).

3. Proofs of the main results

Lemma 3. *There exists a continuous eigenfunction h of $\mathcal{L}$, if,*

$$\frac{\mu[x_0, \ldots x_n, \ldots, x_m]}{\nu[x_0, \ldots, x_n, \ldots, x_m]} = (1 + o(1)) \frac{\mu[x_0, \ldots, x_n]}{\nu[x_0, \ldots, x_n]},$$

for all $m \geq n$, as $n \to \infty$.

Proof. Let $\lambda = 1$. By the assumption, the measure $\mu = h\nu$ is translation invariant, i.e., $\mu \circ T^{-1} = \mu$. Hence we have $(h\nu) \circ T^{-1} = h\nu$ and it suffices to show that $(h\nu) \circ T^{-1} = (\mathcal{L}h)\nu$.

Let A be any Borel subset of X. Then

$$(h\nu) \circ T^{-1}(A) = \int_A \sum_{y:Ty=x} h(y) e^{\phi(y)}\, d\nu(x) = \int_A h(x)\, d\nu(x).$$

□

We now fix some notation before proceeding to prove that we have a continuous eigenfunction.

$d\tilde{\eta}(\epsilon)$	The Bernoulli measure $2^{-\lvert\epsilon\rvert} \ltimes d\eta(\epsilon)$
B_n	The indicator of the event $[x]_n$ compatible with γ.
B_n^+	The indicator of the event $[x]_n$ compatible with γ_+.
$B_n'(\gamma)$	The indicator of the event $[x]_n$ is compatible with γ or not compatible with γ_+.
$\hat{B}_n(\gamma_-, \epsilon)$	The indicator of the event that there are no two edges from some cluster C in γ_- to a pair $i, j \in [0, n)$ having opposite spins, i.e. such that $x_i x_j = -1$.
$\hat{X}_n(\gamma_-, \epsilon)$	The indicator of the event that $Q_n = 0$.
$R_n(\gamma)$	Correction term so that $$d\mu(\gamma \vert B_n^+) = B_n' \cdot 2^{R_n(\gamma)} \ltimes d\nu(\gamma_-)\, d\tilde{\eta}(\epsilon)\, d\nu(\gamma_+ \vert B_n)$$
$Q_{>n}(\gamma_-, \epsilon)$	Number of edges in ϵ from clusters of γ_- to vertices in (n, ∞) such that there is at least one more edge in ϵ from the same cluster to $[0, \infty)$ preceding it in some order
$Q(\gamma_-, \epsilon)$	Number of edges in ϵ from clusters of γ_- to vertices in $[0, \infty)$ such that there is at least one more edge in ϵ from the same cluster to $[0, \infty)$ preceding it in some order
$X(C)$	For a cluster $C \subset \gamma_-$ it is the number of edges in ϵ to $[0, \infty]$.
$i(C)$	For a cluster $C \subset \gamma_-$ it is the rightmost vertex, i.e. $i(C) = \max\{j \in C\}$.
$\lambda(C)$	The sum $\lambda(C) = \frac{\beta}{2} \sum_{j \in C} \frac{1}{j}$.

Lemma 4. *The limit $h(x) = \lim_{n \to \infty} h_n(x)$ is continuous where, form $m \geq 1$ $h_m(x) = \dfrac{\mu([x]_m)}{\nu([x]_m)}$.*

Proof

Recall that

$$h_m(x) = \int B_m' 2^{R_m}\, d\nu(\gamma_-)\, d\tilde{\eta}(\epsilon)\, d\nu(\gamma_+ \vert B_m^+). \tag{7}$$

Since R_m is the number of edges in ϵ that do not reduce (with respect to some order) the number of components in $\gamma \triangleright [0, n]$, it is clear that

$$R_m \leq Q_{>m} \tag{8}$$

where $Q_{>m}$ is the number of edges $ij \in \epsilon$ where $i > n$ and $-j$ belongs to a cluster C in γ_- that sends at least one more edge to $[0, \infty)$.

For $n \leq m$, we have

$$\hat{B}_n \hat{X}_n \leq B'_m \leq \hat{B}_n$$

and, on account of (8), it follows that

$$\int \hat{B}_n \hat{X}_n \, d\nu(\gamma_-) \, d\tilde{\eta}(\epsilon) \leq h_m(x) \leq \int \hat{B}_n 2^{Q_{>n}} \, d\nu(\gamma_-) \, d\tilde{\eta}(\epsilon). \tag{9}$$

We have used that $\hat{B}_n$, $\hat{X}_n$ and $Q_{>m}$ are independent of γ_+ and that

$$\int d\nu(\gamma_+ | B_m^+) = 1.$$

Since both $\hat{B}_n$ and $\hat{X}_n$ are decreasing in (γ_-, ϵ) it follows from the FKG inequality that

$$I_n \cdot \int \hat{X}_n \, d\nu(\gamma_-) \, d\tilde{\eta}(\epsilon) \leq h_m(x) \leq I_n \cdot \int 2^{Q_{>n}} \, d\nu(\gamma_-) \, d\tilde{\eta}(\epsilon). \tag{10}$$

where

$$I_n = \int \hat{B}_n \, d\nu(\gamma_-) \, d\tilde{\eta}(\epsilon). \tag{11}$$

We prove the following lemma.

Lemma 5. *The integral*

$$\int 2^{Q_{>n}} \, d\nu(\gamma_-) \, d\tilde{\eta}(\epsilon) = 1 + o(1).$$

as $n \to \infty$.

From (10) and this lemma we deduce that

$$h_m(x) = (1 + o(1)) I_n = h_n(x) \cdot (1 + o(1)) \tag{12}$$

if $m \geq n$ as $n \to \infty$. It follows that $\log h_n(x)$ is a Cauchy sequence and hence that the limit $h(x)$ is continuous.

Proof of Lemma 5. We condition on a fixed graph γ_- with distribution ν_-. Let C be a cluster of γ_-. Note that

$$Q = (X(C_1) - 1)_+ + (X(C_2) - 1)_+ + \dots$$

where $X(C_i)$, is a sum of independent Bernoulli variables

$$X(C) = \sum_{-j \in C} \sum_{i=0}^{\infty} \epsilon_{ji}$$

where

$$\mathsf{P}(\epsilon_{ij} = 1) = \frac{1 - \exp\{-\frac{\beta}{(i+j)^2}\}}{2}$$

It follows that we can approximate $X(C)$ with a Poisson variable $\tilde{X}(C) \sim \mathrm{Po}(\lambda(C))$ with

$$\lambda(C) = \frac{\beta}{2} \sum_{j \in C} \frac{1}{j} \approx \frac{\beta}{2} \sum_{j \in C} \sum_{i=0}^{\infty} \mathsf{P}(\epsilon_{ij} = 1).$$

Note that

$$\lambda(C) \leq \log\left(1 + \frac{|C|}{i(C)}\right) \tag{13}$$

where $-i(C) = \max C$.

Order the clusters of γ_- as $C_1, C_2, \ldots$ etc. so that $i(C_1) < i(C_2) < \ldots$. For each clister C_i we can from stochastic dominance construct a random cluster $\tilde{C}_i$ such that (i) $C_i \subset \tilde{C}_i$ and (ii) $i(\tilde{C}_i) = i(C_i)$. We can further assume that the $\tilde{C}_i$s are *independent*.

Let now

$$\tilde{Q} = \sum_{C_i} (\tilde{X}(\tilde{C}_i) - 1)_+.$$

where $\mathsf{P}(\tilde{X}(\tilde{C})|\tilde{C}) = \mathrm{Po}(\lambda(\tilde{C}))$ which stochastically dominates $X(C)$. Note that for a poissonvariable $X \sim \mathrm{Po}(\lambda)$ we have

$$\mathsf{E}(2^{(X-1)_+}) = \frac{\exp(\lambda(e^{\ln 2} - 1)) + e^{\lambda}}{2} = \cosh(\lambda)$$

We then have

$$A := \mathsf{E}(2^Q|\gamma_-) \leq \prod \cosh(\lambda(C_i)) \leq \prod \cosh(\lambda(\tilde{C}_i)).$$

We obtain $(i(C_k) \leq k)$

$$E(A) \leq \prod_{k=1}^{\infty} \left(1 + \frac{E(|\tilde{C}_k|^2)}{k^2}\right) \cdot K,$$

where K is a constant.

If $|C_k| \leq 0.1k$, then

$$\cosh(\lambda(\tilde{C}_k)) \leq 1 + \left(\frac{|\tilde{C}_k|}{i(\tilde{C}_k)}\right)^2,$$

so

$$E[2^Q] \leq \prod_{k=1}^{\infty} E\left(\cosh\left(\log\left(1 + \frac{|\tilde{C}_k|}{k}\right)\right)\right).$$

We have (with $\tilde{C}_k$ independent)

$$E\left(\frac{1}{2}(1 + \frac{|\tilde{C}_k|}{k} + \frac{1}{1 + \frac{|\tilde{C}_k|}{k}})\right)$$

$$= E\left(\frac{1}{2}(1 + \frac{|\tilde{C}_k|^2}{k^2} - \frac{|\tilde{C}_k|^3}{k^3} + \cdots)\right).$$

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Anders Johansson, Department of Mathematics, University of Gävle, 801 76 Gävle, Sweden. Email-address: ajj@hig.se

Anders Öberg, Department of Mathematics, Uppsala University, P.O. Box 480, 751 06 Uppsala, Sweden. E-mail-address: anders@math.uu.se

Mark Pollicott, Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK. Email-address: mpollic@maths.warwick.ac.uk