EXISTENCE OF CONTINUOUS EIGENFUNCTIONS FOR THE DYSON MODEL IN THE CRITICAL PHASE

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Abstract. We prove that there exists a continuous eigenfunction of the transfer operator defined by potentials for the so-called Dyson model for all inverse critical temperatures that are strictly less than the critical inverse temperature. This includes all cases when the potential does not have summable variations, the classical condition that ensures the existence of a continuous eigenfunction of the transfer operator. As a consequence the inverse critical temperatures for the one-sided and the two-sided models are the same. We use the random cluster model for our method of proof, so the results also hold in this wider context. In particular, we base our conclusions of this paper by recent results of Duminil-Copin, Garban, and Tassion [9].

1. Introduction

It is well-known [18] that there exists a continuous and strictly positive eigenfunction h for any transfer operator defined on a symbolic shift space with a finite number of symbols such that the potential has summable variations. Here we prove the existence of a continuous eigenfunction for the important special class of Dyson potentials up to the critical phase, when the potential does not satisfy the condition of summable variations. We stress that it is the continuity that is the main difficulty. To see that there is a measurable eigenfunction follows from a simple application of the martingale convergence theorem.

More precisely, let T be the left shift on the space $X = S^{\mathbb{Z}_+}$, where S is a finite set. Let $\phi : X \to \mathbb{R}$ be a continuous function defined up to an additive constant. We refer to ϕ as the *one-point potential.* A *Gibbs measure* $\nu \in \mathcal{M}(X)$ for ϕ is one that minimises the free energy $P(\nu;\phi) = \nu(\phi) - H(\nu)$, where $H(\nu)$ denotes the entropy $\lim_{n\to\infty} H(\nu|_{\mathcal{F}_n})/n$ per time unit. This measure ν can also be obtained as the Gibbs measure on X obtained from the full potential

$$\Phi(x) = \sum_{k=0}^{\infty} \phi(T^k x)$$

on X. A Gibbs measure ν is also obtained as an eigenmeasure of the dual of the *transfer operator* $\mathcal{L} = \mathcal{L}_\phi$ defined on continuous functions by

$$\mathcal{L} f(x) = \sum_{y \in T^{-1}x} e^{\phi(y)} f(y). \tag{1}$$

Such an eigenmeasure ν satisfies $\mathcal{L}^*\mu = \lambda\nu$, for some maximal positive eigenvalue $\lambda > 0$.

The *equilibrium measure* μ is a minimiser of $P(\mu;\phi)$ among all *translation invariant* measures $\mu \in \mathcal{M}_T(X)$. Taking the natural extension of μ, we can also represent μ as a translation invariant measure on the *two-sided* space $\overline{X} = S^{\mathbb{Z}}$ and we can alternatively construct the measure $\mu \in \mathcal{M}_T(\overline{X})$ as the Gibbs measure to the full two-sided potential

$$\overline{\Phi}(x) = \sum_{k=-\infty}^{\infty} \phi(T^k x)$$

defined on $x \in \overline{X}$. If there exists a continuous eigenfunction $h(x)$ to the transfer operator, the measure μ can be recovered as the Doeblin measure [5] (a.k.a. g-measure in the terminology of Keane [16]) corresponding to the *Doeblin function* $g(x)$ where

$$g(x) = \frac{h(x)e^{\phi(y)}}{\lambda h(Tx)}. \tag{2}$$

The existence of a continuous eigenfunction h to $\mathcal{L}$, such that $\mathcal{L}h = \lambda h$, is however not automatic in same way the existence of the eigenmeasure ν is. If one assumes summable variations of ϕ,

$$\sum_{n=1}^{\infty} \mathrm{var}_n(\phi) < \infty, \tag{3}$$

where $\mathrm{var}_n(\phi) = \sup_{x\sim_n y} |\phi(x) - \phi(y)|$ ($x \sim_n y$ means that x and y coincide in the first n entries), then the existence of a continuous eigenfunction $h(x)$ follows from the typical "cone-argument" used in *e.g.*, Walters [18], which to date is the only known method for providing the existence of a continuous eigenfunction. (??) In this paper, we prove in a lemma that if there is a translation invariant measure μ which is absolutely continuous with respect to the Gibbs measure μ the Radon–Nikodym derivative $h(x) = \dfrac{\partial \mu}{\partial \nu}$ is an eigenfunction of $\mathcal{L}$.

Both the question of uniqueness and the question about the a continuous eigenfunction seems related to the smoothness of the potential. One possibility for the existence of a continuous eigenfunction in general could be the *Berbee condition.* Berbee proved ([4]) that the Gibbs measure ν and the equilibrium measure μ are unique if we have

$$\sum_{n=1}^{\infty} e^{-r_1-r_2-\cdots-r_n} = \infty, \tag{4}$$

where $r_n = \mathrm{var}_n\,\phi$.

In [12] it was proved that uniqueness of Doeblin measures follows whenever

$$\sum_{n=1}^{\infty} (\mathrm{var}_n\, g)^2 < \infty,$$

and more recently Berger et al. [5] proved that for such Doeblin measures uniqueness follows if $\mathrm{var}_n \log g < 2/\sqrt{n}$ (see also [14] for a slightly stronger condition). This connects to Dyson's counterexample in [10] where it is shown that there are examples of multiple equilibrium measures for ϕ whenever

$$\sum_{n=1}^{\infty}(\operatorname{var}_n \phi)^{1+\epsilon} < \infty,$$

where $\epsilon > 0$.

Dyson's example is for a two-sided model, but we showed in [15] that the inverse critical temperature β_c^+ satisfies $\beta_c^+ \leq 8\beta_c$, where β_c is the inverse critical temperature for the two-sided model, and this show that Dyson's example of multiple equilibrium measures can be formulated for a one-sided model as above.

Since we have the "translation" via (2) between the case for general potentials and for Doeblin measures, we may guess that the existence of a continuous eigenfunction cannot be moved very far away from the summability of variations condition for a potential.

Here, however, we study only the Dyson potential: Fix $\alpha > 1$ and $\beta > 0$. Let the one-point long-range Ising potential $\phi = \phi_{\alpha,\beta}$ be given by

$$\phi(x_0, x_1, \ldots) = x_0 \cdot \beta \sum_{j=1}^{\infty} \frac{x_j}{j^\alpha},$$

and define the one-sided and two-sided Dyson potentials, $\Phi : X \to \mathbb{R}$ and $\bar{\Phi} : \overline{X} \to \mathbb{R}$ as above. Let μ and ν be the Gibbs measures on $\mathcal{M}(X)$ and $\mathcal{M}(\overline{X})$, corresponding to Φ and $\overline{\Phi}$, respectively.

We then have for $\alpha > 2$

$$\mu = h\nu,$$

where $h > 0$ is a Hölder continuous eigenfunction. We are interested in the boundary cases where $1 < \alpha \leq 2$, when there exists a unique equilibrium measure for $\overline{\Phi}$ for $\beta < \beta_c$ [1] and multiple equilibrium measures when $\beta > \beta_c$, where the critical parameter $\beta_c = \beta_c(\alpha) > 0$ for $1 \leq \alpha \leq 2$. We show here that the same uniqueness properties holds for the "one-sided" Gibbs measure ν.

In this case the summable variations condition is not satisfied for neither $\overline{\Phi}$ nor Φ; hence we may have multiple eigenmeasures for $\mathcal{L}^*$. In this context we have $\operatorname{var}_n(\phi) = O(\frac{1}{n})$, but as we noted earlier, in the general setting above, is not clear that there exists a continuous eigenfunction even in the cases we have a unique equilibrium measure

We prove that if the inverse critical temperature β is strictly smaller than the critical inverse temperature β_c, which will be seen to be the same for the two-sided and one-sided models, then we have a continuous eigenfunction of $\mathcal{L}$.

Let μ be the Gibbs equilibrium measure with respect to the Dyson potential ϕ and let ν be the one-sided Gibbs measure. Define

$$h_n(x) = \frac{\mu[x_0, \ldots x_n]}{\nu[x_0, \ldots, x_n]},$$

and consider the measurable function $h(x) = \limsup_{n\to\infty} h_n(x)$.

Theorem 1. *If $\beta < \beta_c$, i.e. in the subcritical phase, then $h(x)$ is a continuous function on X and it is also an eigenfunction of $\mathcal{L}$.*

We conjecture that there exists a continuous eigenfunction for a potential ϕ whenever that potential satisfies Berbee's condition (4).

The next result is a corollary of Theorem 1:

Theorem 2. *The critical β_c for the two-sided model is the same as for the one-sided model, i.e., $\beta_c^+ = \beta_c$.*

This improves our result of Theorem 1 in [15] that $\beta_c^+ \leq 8\beta_c$.

2. The one-dimensional random cluster model and the Ising–Dyson model

For a finite graph, let $\omega(G)$ denote the number of connected components ("clusters") in the graph G. For simple graphs $G \subset \binom{V}{2}$ on an countably infinite set V of vertices, we consider the number of clusters $\omega(G)$ as a *potential*. This means that the difference $\Delta\omega(G,F) = \omega(G) - \omega(F)$ is defined for any two graphs F and G that coincide outside a *finite* subset $\Lambda \subset \binom{V}{2}$.

A random graph $G \sim \alpha$ on a set of vertices V is a probability distribution α on the set $\{0,1\}^{\binom{V}{2}}$. The random cluster models $\mathcal{R}(V,p,q)$ (FK-model [**?**]), we consider are specified by a set of vertices V and a probabilities $p(ij) \in [0,1]$, defined on the set of pairs $ij \in \binom{V}{2}$. The model $\mathcal{R}(V,p,q)$ is a Gibbs distribution on random graphs, i.e. configurations i $\gamma \in \{0,1\}^{\binom{V}{2}}$, with non-continuous potential

$$\phi(\gamma) = -\log q \cdot \omega(\gamma) + \sum_{ij} \gamma(ij) \cdot \log p(ij) + (1-\gamma(ij)) \cdot \log(1-p(ij)).$$

The models $\mathcal{R}(V,p,1) = \eta(\binom{V}{2}, p)$ are referred to as "Bernoulli percolation". In the one-dimensional Potts–Dyson random cluster model, $\mathcal{R}(V,\beta,\alpha,q)$, we let $V = \mathbb{Z}$ (or $V = \mathbb{Z}_{\geq 0}$ and consider the random cluster model $\mathcal{R}(V,p_J,q)$ where

$$p_J(ij) = 1 - e^{-J(ij)} \quad \text{where} \quad J(ij) = \frac{\beta}{|i-j|^\alpha}. \tag{5}$$

When q

Percolation is the event that the random graph contains an infinite cluster. It is well-known [1] that for all $\alpha \in (1,2]$ there exists a critical $\beta_c = \beta_c(\alpha)$ such that, percolation does not occur with probability 1 for $0 < \beta < \beta_c$, while for $\beta \in (\beta_c, \infty)$ there is with probability 1 no infinite cluster.

We will mainly consider the subcritical case $1 < \alpha \leq 2$ and $0 < \beta < \beta_c(\alpha)$ and use the following lemma.

Lemma 3. *Let $X = X(C_s)$ denote the size of the cluster C_s containing a specified vertex $s \in V$. Then there are constants K and r, $0 < r < 1$, such that*

$$\mathsf{P}(X \geq m) \leq K r^m.$$

Proof. As surveyed by Panagiotis [**?**] (Theorem 1.2.1), we know that it holds for Bernoulli percolation ($q = 1$) when $0 < \beta < \beta_c$. Since $\mathcal{R}(V, p, 1)$ stochastically dominates $\mathcal{R}(V, p, q)$, $q > 1$, it also holds for the FK-model. Finally, Duminil-Copine ([9]) proves that $\beta_c(\alpha, q)$ are the same for the case $q = 1$ and $q > 1$. □

2.0.1. *The extended random cluster model.* The extended *random cluster model* with $\mathcal{R}_e(V, p, q)$ can be obtained as the joint distribution of the spin sequences $x \in \{1, 2, \dots, q\}^V$ and a random graph $\gamma \in \{0, 1\}^{\binom{V}{2}}$. The distribution of (x, γ) is obtained by first considering the pair chosen independently: The spin sequence $x \in \{0, 1, \dots, q\}^V$ according to the uniform Bernoulli measure $x \sim \eta(V, p = \frac{1}{q})$ on the spin sequences and the random graph γ according to the Bernoulli measure $\eta(\binom{V}{2}, p) = \mathcal{R}(V, p, 1)$. Then $\mathcal{R}_e(V, p, q)$ is the distribution of (x, γ) *conditioned on* x and γ being *compatible*: That is, the event $C(x, \gamma)$ that no spins $x_i = +1$ and $x_j = -1$ in x are connected by a path (edge) in γ.

Assume $(x, \mu) \sim \mu = \mathcal{R}_e(V, p, q)$. Then marginal distribution $\mu \circ x^{-1}$ of x is the Potts model with interactions given by $J(ij) = \log(1 - p(ij))$ and the marginal distribution $\mu \circ \gamma^{-1}$ of γ is the random cluster model $\mathcal{R}(V, p, q)$ described above.

Let $S \subset V$ be a finite subset. We need to establish the probability $\mu([x]_S)$ of a cylinder $[x]_S$ under $\mu = \mathcal{R}_e(V, p, q)$. We have the following lemma.

Lemma 4 (Probability of cylinder)**.** *We have*

$$\mu([x]_S) = \int q^{-\omega_S(\gamma)} B(x_S, \gamma)\, d\mu(\gamma) = \int q^{-\tilde{\omega}_S(\gamma)}$$

(***)

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2.0.2. *The interface perturbation.* We now fix some notation before proceeding to prove that we have a continuous eigenfunction.

$d\tilde{\eta}(\epsilon)$	The Bernoulli measure $2^{-\lvert\epsilon\rvert} \ltimes d\eta(\epsilon)$
B_n	The indicator of the event $[x]_n$ compatible with γ.
B_n^+	The indicator of the event $[x]_n$ compatible with γ_+.
$B_n'(\gamma)$	The indicator of the event $[x]_n$ is compatible with γ or not compatible with γ_+.
$\hat{B}_n(\gamma_-, \epsilon)$	The indicator of the event that there are no two edges from some cluster C in γ_- to a pair $i, j \in [0, n)$ having opposite spins, i.e. such that $x_i x_j = -1$.
$\hat{X}_n(\gamma_-, \epsilon)$	The indicator of the event that $Q_n = 0$.
$R_n(\gamma)$	Correction term so that $$d\mu(\gamma \vert B_n^+) = B_n' \cdot 2^{R_n(\gamma)} \ltimes d\nu(\gamma_-)\, d\tilde{\eta}(\epsilon)\, d\nu(\gamma_+ \vert B_n)$$
$Q_{>n}(\gamma_-, \epsilon)$	Number of edges in ϵ from clusters of γ_- to vertices in (n, ∞) such that there is at least one more edge in ϵ from the same cluster to $[0, \infty)$ preceding it in some order
$Q(\gamma_-, \epsilon)$	Number of edges in ϵ from clusters of γ_- to vertices in $[0, \infty)$ such that there is at least one more edge in ϵ from the same cluster to $[0, \infty)$ preceding it in some order
$X(C)$	For a cluster $C \subset \gamma_-$ it is the number of edges in ϵ to $[0, \infty]$.
$i(C)$	For a cluster $C \subset \gamma_-$ it is the rightmost vertex, i.e. $i(C) = \max\{j \in C\}$.
$\lambda(C)$	The sum $\lambda(C) = \frac{\beta}{2} \sum_{j \in C} \frac{1}{j}$.

One obtains the long range Ising model with the Dyson potential $\phi_{\beta,\alpha}(x)$ as the marginal distribution of x and the random cluster model $\mathcal{R}(V, p, 2)$ as the marginal distributions γ.

We now observe that a configuration γ in the two-sided model $\mathcal{R}(\mathbb{Z}, \alpha, \beta)$ difference between the one-sided random cluster model $\nu = \mathcal{R}(V_+, J)$ and the usual two-sided model $\mu = \mathcal{R}(V, J)$. We will use that a configuration γ can be factored as $\gamma = (\gamma_+, \epsilon, \gamma_-)$, where γ_- is the induced graph $\gamma[V_-]$ on vertices $-j \in V_- = \mathbb{Z}_{<0}$ and $\gamma_+ = \gamma[V_+]$ is the graph induced on vertices $i \in V_+ = \mathbb{Z}_{\geq 0}$. The graph $\epsilon = \gamma \cap E(V_+, V-)$ consists of edges ji, $i \geq 0$ and $j \geq 1$, connecting vertices $-j \in V_-$ with vertices $i \in V_+$. Note that we often use positive indices i, j, $i \geq 0$ and $j > 0$, as labels for edges in ϵ. Thus $J(ij) = \beta/(i+j)^\alpha$ with this labelling.

Let
$$\tilde{\eta}(\epsilon) = 2^{-\lvert\epsilon\rvert} \ltimes \eta(\epsilon).$$
Note that both $\eta(\cdot)$ and $\tilde{\eta}(\cdot)$ are Bernoulli measures. Let also
$$d\tilde{\nu}_n(\gamma) = \frac{d\nu(\gamma_-) \otimes \tilde{\eta}(\epsilon) \otimes \nu(\gamma_+)}{\nu(B([x]_n, \gamma_+))}.$$

Let R_n be the number of correcting edges, i.e.,
$$R_n = \#\{ij \in \epsilon : \omega(\gamma_{<ij} + ij) = \omega(\gamma_{<ij})\}, \quad j > n.$$
Let $B_n(\gamma) = C([x]_n, \gamma)$ be the indicator of the event that cylinder $[x]_n$ is compatible with graph $\gamma = (\gamma_-, \epsilon, \gamma_+)$. Let also $B_n'(\gamma) = B_n(\gamma) + (1 -$

$B_n(\gamma_+)$) indicate compatibility of $[x]_n$ with γ or not compatible with γ_+. We then have

$$h_n(x) = \frac{\mu[x]_n}{\nu[x]_n} \propto \int B_n(\gamma) \cdot 2^{R_n(\gamma)}\, d\tilde{\nu}_n(\gamma)$$

Note that

$$R_n(\gamma) \le Q_n(\gamma_-, \epsilon)$$

where Q_n denotes the number of edges in ϵ that connects a vertex $j \ge n$ to a cluster in γ_- with at least one more edge from the cluster to $[0, n-1]$. That is,

$$Q_n = \#\{ij \in \epsilon \mid \exists k\, \exists l\; kl \in \epsilon, i \sim_{\gamma_-} k, k > i, j > n\}.$$

Notice that Q_n only depends on (γ_-, ϵ).

3. Proof of Theorem 1

Lemma 5. *If $\mu \in \mathcal{M}_T(X)$ is a translation invariant measure which is absolutely continuous with the Gibbs measure ν for ϕ then the Radon–Nikodym derivative $h(x) = \dfrac{\partial \mu}{\partial \nu}(x)$ is an eigenfunction to the transfer operator $\mathcal{L} = \mathcal{L}_\phi$.*

Proof. We can assume $\lambda = 1$. By assumption, the measure $\mu = h\nu$ is translation invariant, i.e., $\mu \circ T^{-1} = \mu$. Hence we have $(h\nu) \circ T^{-1} = h\nu$ and it suffices to show that $(h\nu) \circ T^{-1} = (\mathcal{L}h)\nu$.

Let A be any Borel subset of X. Then

$$(h\nu) \circ T^{-1}(A) = \int_A \sum_{y:Ty=x} h(y) e^{\phi(y)}\, d\nu(x) = \int_A h(x)\, d\nu(x).$$

□

We need to prove that the $h_n(x)$ converge to a continuous function $h(x)$ as $n \to \infty$. It is enough to show that

$$\sup_{x, m \ge n} \frac{h_m(x)}{h_n(x)} = 1 + o(1) \quad \text{as } n \to \infty$$

since this shows that $\log h_n$ is a Cauchy-sequence.

Recall that

$$h_m(x) = \int B'_m 2^{R_m}\, d\nu(\gamma_-)\, d\tilde{\eta}(\epsilon)\, d\nu(\gamma_+ | B^+_m). \tag{6}$$

Since R_m is the number of edges in ϵ that do not reduce (with respect to some order) the number of components in $\gamma \triangleright [0, n]$, it is clear that

$$R_m \le Q_{>m} \tag{7}$$

where $Q_{>m}$ is the number of edges $ij \in \epsilon$ where $i > n$ and $-j$ belongs to a cluster C in γ_- that sends at least one more edge to $[0, \infty)$.

For $n \le m$, we have

$$\hat{B}_n \hat{X}_n \le B'_m \le \hat{B}_n$$

and, on account of (7), it follows that

$$\int \hat{B}_n \hat{X}_n \, d\nu(\gamma_-)\, d\tilde{\eta}(\epsilon) \leq h_m(x) \leq \int \hat{B}_n 2^{Q_{>n}} \, d\nu(\gamma_-)\, d\tilde{\eta}(\epsilon). \tag{8}$$

We have used that $\hat{B}_n$, $\hat{X}_n$ and $Q_{>m}$ are independent of γ_+ and that

$$\int d\nu(\gamma_+ | B_m^+) = 1.$$

Since both $\hat{B}_n$ and $\hat{X}_n$ are decreasing in (γ_-, ϵ) it follows from the FKG inequality that

$$I_n \cdot \int \hat{X}_n \, d\nu(\gamma_-)\, d\tilde{\eta}(\epsilon) \leq h_m(x) \leq I_n \cdot \int 2^{Q_{>n}} \, d\nu(\gamma_-)\, d\tilde{\eta}(\epsilon). \tag{9}$$

where

$$I_n = \int \hat{B}_n \, d\nu(\gamma_-)\, d\tilde{\eta}(\epsilon). \tag{10}$$

We prove the following lemma.

Lemma 6. *The integral*

$$\int 2^{Q_{>n}} \, d\nu(\gamma_-) \; d\tilde{\eta}(\epsilon) = 1 + o(1).$$

as $n \to \infty$.

From (9) and this lemma we deduce that

$$h_m(x) = (1 + o(1)) I_n = h_n(x) \cdot (1 + o(1)) \tag{11}$$

if $m \geq n$ as $n \to \infty$. It follows that $\log h_n(x)$ is a Cauchy sequence and hence that the limit $h(x)$ is continuous.

Proof of Lemma 6. We condition on a fixed graph γ_- with distribution ν_-. Let C be a cluster of γ_-. Note that

$$Q = (X(C_1) - 1)_+ + (X(C_2) - 1)_+ + \dots$$

where $X(C)$, is a sum of independent Bernoulli variables

$$X(C) = \sum_{-j \in C} \sum_{i=0}^{\infty} \epsilon_{ji}$$

where

$$\mathsf{P}(\epsilon_{ij} = 1) = \frac{1 - \exp\{-\frac{\beta}{(i+j)^2}\}}{2}$$

It follows that we can approximate $X(C)$ with a Poisson variable (***) $\tilde{X}(C) \sim \mathrm{Po}(\lambda(C))$ with

$$\lambda(C) = \frac{\beta}{2} \sum_{j \in C} \frac{1}{j} \approx \frac{\beta}{2} \sum_{j \in C} \sum_{i=0}^{\infty} \mathsf{P}(\epsilon_{ij} = 1).$$

Note that

$$\lambda(C) \leq \log\left(1 + \frac{|C|}{i(C)}\right) \tag{12}$$

where $-i(C) = \max C$.

Order the clusters of γ_- as $C_1, C_2, \dots$ etc. so that $i(C_1) < i(C_2) < \dots$. For each cluster C_i we can from stochastic dominance construct a random cluster $\tilde{C}_i$ such that (i) $C_i \subset \tilde{C}_i$ and (ii) $i(\tilde{C}_i) = i(C_i)$. We can further assume that the $\tilde{C}_i$s are *independent* with the same distribution.

Let now

$$\tilde{Q} = \sum_{C_i} (\tilde{X}(\tilde{C}_i) - 1)_+.$$

where $\mathsf{P}(\tilde{X}(\tilde{C})|\tilde{C}) = \mathrm{Po}(\lambda(\tilde{C}))$ which stochastically dominates $X(C)$. For a poisson variable $X \sim \mathrm{Po}(\lambda)$ we have

$$\mathsf{E}(2^{(X-1)_+}) = \frac{\exp(\lambda(e^{\ln 2} - 1)) + e^{\lambda}}{2} = \cosh(\lambda)$$

We then have

$$\mathsf{E}(2^{Q}|\gamma_-) \leq \prod \cosh(\lambda(C_i)) \leq \prod \cosh(\lambda(\tilde{C}_i)).$$

We obtain, since $i(C_k) \geq k$ and (12) and the independence of $\tilde{C}_k$, that

$$E(2^Q) \leq \prod_{k=1}^{\infty} \mathsf{E}\left(\frac{1}{2}\left(1 + \frac{Y}{k} + \frac{1}{1 + \frac{Y}{k}}\right)\right) \tag{13}$$

$$\leq \prod_{k=1}^{\infty} \mathsf{E}\left(1 + \frac{Y^2}{k^2} + \frac{Y^3}{k^3} + \dots\right). \tag{14}$$

where Y has the common distribution of $|C_k|$. It is easy to see that this is less than ∞ on account of Lemma 3, which states that the distribution Y has an exponentially decreasing bound for the upper tail.

Since $Q = \lim Q_n$ we obtain from this that

$$\mathsf{E}(2^{Q_{>n}}) = \mathsf{E}(2^{Q - Q_n}) = 1 + o(1) \quad \text{as } n \to \infty. \quad \square$$

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