Existence for an eigenfunction for the sub-critical phase of the Dyson Ising model

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1 Introduction

1.1 The random cluster model and potentials

We consider measures $\alpha \in \mathcal{M}(X)$ on configuration spaces $X = A^S$, where A is a finite set (the "alphabet") and S (the "sites") is a countable and usually infinite. A *potential* $\phi(x)$ on X is a limit $\phi(x) = \lim_{n\to\infty} \phi_n(x)$ of functions such that the difference $\phi(x) - \phi(y)$ is finite and well defined for any pair of configurations x and y that coincide outside any finite set $\Lambda \subset S$. Given a probability measure α on X and a potential ϕ on X, we denote by

$$e^{s\phi(x)} \ltimes \, d\alpha(x)$$

the family of *Gibbs measure*, i.e. the weak limits $\Lambda \nearrow S$ of

$$p_\Lambda(x|_\Lambda \mid x|_{\Lambda^c}) = \frac{e^{s\phi(x)}\alpha(x|_\Lambda \mid x|_{\Lambda^c})}{Z_\Lambda(x|_{\Lambda^c})}$$

Examples of unique Gibbs measures are Bernoulli measures.

1.2 The one-dimensional Random cluster model and the Ising-Dyson model

For a finite graph, let $\omega(G)$ denote the number of connected components ("clusters") in the graph G. For simple graphs $G \subset \binom{V}{2}$ on an countably infinite set V of vertices, we consider the number of clusters $\omega(G)$ as a *potential*. This means that the difference $\Delta\omega(G, F) = \omega(G) - \omega(F)$ is defined for any two graphs F and G that coincide outside a *finite* subset $\Lambda \subset \binom{V}{2}$.

A random graph $G \sim \alpha$ on a set of vertices V is a probability distribution α on the set $\{0,1\}^{\binom{V}{2}}$. The random cluster models $\mathcal{R}(V,p)$, we consider are specified by a set of vertices V and a weight function p, $ij \mapsto p(ij) \in [0,1]$, defined on the set of pairs $ij \in \binom{V}{2}$. The model $\mathcal{R}(V,p)$ is a Gibbs distribution on random graphs, i.e. configurations i $\{0,1\}^{\binom{V}{2}}$, such that a

In the one-dimensional Ising-Dyson model let $V = \mathbb{Z}$ (or $V = V_+ = \mathbb{Z}_{\geq 0}$ for the one-sided case) and for $i,j \in V$ let

$$J(ij) = \frac{\beta}{|i-j|^\alpha}. \tag{1}$$

We will consider the case $\alpha = 2$ and $0 < \beta < \beta_c$. For $\beta > 0$, let $\gamma \sim \eta$ be the Bernoulli random graph $\eta \in \mathcal{G}(V)$ with

$$\mathsf{P}(ij \in \gamma) = 1 - e^{-\beta J(ij)}.$$

The extended *random cluster model* can be obtained by considering the productmeasure $d\eta(\gamma) \otimes d1(x)$ between an independent bernoulli-distributed $\gamma \sim \eta_p$ random graph on $\mathcal{G}(V)$ and the uniform measure $x \sim 1$ on the spin sequences $x \in \{-1,+1\}^V$. The extended random cluster model $d\mu(x,\gamma)$ is the joint distribution of γ and x obtained by conditioning on the on the event that x and γ are compatible: That is, the event $C(x,\gamma)$ that no spins $x_i = +1$ and $x_j = -1$ in x are connected by a path (edge) in γ. One obtain, simultaneously, the Ising model $d\mu(x)$ and the random cluster model $d\mu(\gamma)$ as the marginal distributions of x and γ, respectively.

We can also introduce the random cluster model μ as the Gibbs measure on $\{0,1\}^{\binom{V}{2}}$

$$d\mu = 2^{\omega(\gamma)} \ltimes d\eta(\gamma),$$

where $\omega(\gamma)$ is the potential counting then number of connected components ("clusters") in γ. For the values of β we consider the Gibbs measure is unique. Thus can we parameterise the random clusters models as $\mu = \mathcal{R}(V,J)$ where $J(ij)$ is a given weighting on $\binom{V}{2}$ such as (1).

We differ between the one-sided random cluster model $\nu = \mathcal{R}(V_+,J)$ and the usual two-sided model $\mu = \mathcal{R}(V,J)$. We will use that a configuration γ can be factored as $\gamma = (\gamma_+, \varepsilon, \gamma_-)$, where γ_- is the induced graph $\gamma[V_-]$ on vertices $-j \in V_- = \mathbb{Z}_{<0}$ and $\gamma_+ = \gamma[V_+]$ is the graph induced on vertices $i \in V_+ = \mathbb{Z}_{\geq 0}$. The graph $\varepsilon = \gamma \cap E(V_+, V-)$ consists of edges ji, $i \geq 0$ and $j \geq 1$, connecting vertices $-j \in V_-$ with vertices $i \in V_+$. Note that we often use positive indices i,j, $i \geq 0$ and $j > 0$, as labels for edges in ε. Thus $J(ij) = \beta/(i+j)^\alpha$ with this labelling.

We extend the one-sided model ν to a probability distribution on V by identifying ν with the product measure

$$d\nu(\gamma) = d\nu(\gamma_-) \otimes d\tilde{\eta}(\varepsilon) \otimes d\nu(\gamma_+)$$

where $d\tilde{\eta}(\varepsilon)$ is the Bernoulli measure $2^{-|\varepsilon|} \ltimes d\eta(\eta)$. Since

$$\omega(\gamma) = \omega(\gamma_+) + \omega(g_-) - |\varepsilon| + R(\gamma)$$

where the correction R is defined as the co-rank of ε in γ, i.e. $R(\gamma)$ counts the number of edges that can be removed from ε without increasing $\omega(\gamma)$.

Let

$$\varepsilon_{ij} = \varepsilon \cap \{i'j' \mid i' < i \text{ or } i' = i \text{ and } j' < j\}$$

and $\gamma_{ij} = (\gamma_-, \varepsilon_{ij}, \gamma_+)$. By the greedy property of matroids, it follows that $R(\gamma) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} R_{ij}(\gamma)$ where

$$R_{ij}(\gamma) = \begin{cases} 1 & \omega(\gamma_{ij}) = \omega(g_{ij} \setminus ij) \\ 0 & \text{otherwise.} \end{cases}$$

It follows that we can write

$$d\nu(\gamma) = 2^{\omega(\gamma)+\omega(\gamma_+)-|\varepsilon|} \ltimes d\eta(\gamma) \tag{2}$$

$$d\mu(\gamma) = 2^{\omega(\gamma)+\omega(\gamma_+)-|\varepsilon|+R(\gamma)} \ltimes d\eta(\gamma) = 2^{R(\gamma)} \ltimes d\nu(\gamma). \tag{3}$$

Lemma 1. *We have that*

$$\int 2^{R(\gamma)} d\nu(\gamma) < \infty$$

and thus $\nu(\gamma)$ and $\mu(\gamma)$ are absolutely continuous as random cluster models.

Proof of Lemma 1. We have since $\beta < \beta_c$ that

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n} < \infty.$$

□

For any fixed $x \in X = \{-1, +1\}^V$ and an integer $N \geq 0$, let $x_N = x|_{[0,N)} = (x_0, \ldots, x_{N-1})$ and let $[x]_N = \{y \in X \mid y_N = x_N\}$. Furthermore, for any $x_N \in \{-1, +1\}^{[0,N)}$ let $[x_N] = \{y \mid y_N = x_N\}$.

Let also $\mathcal{R}_N(V, J)$ denote $\mathcal{R}(V + \{s\}, J_N)$ where

$$J_N(ij) = \begin{cases} J(ij) & i, j \neq s \\ \infty & (i,j), (j,i) \in [0, N) \times \{s\} \\ 0 & \text{otherwise} \end{cases}$$

The model $\mathcal{R}_N(V)$ is the random cluster model on V, where all vertices in $[0,N)$ have been contracted to one. We have that

$$d\mu_N = 2^{-\omega_N(\gamma)} \ltimes \, d\mu(\gamma) = 2^{\overline{\omega}_N(\gamma)} \ltimes \, d\eta(\gamma)$$

where $\omega_N(\gamma)$ denotes the number of clusters in γ that intersect $[0,N)$ and the potential $\overline{\omega}_N(\gamma) = \omega(\gamma) - \omega_N(\gamma)$ counts the number of clusters that do not.

Lemma 2. *For any given $x \in X$ we have*

$$\mu([x]_N) \propto \int B(x_N,\gamma)\, d\mu_N(\gamma)$$

and

$$\nu([x]_N) \propto \int B(x_N,\gamma)\, d\nu_N(\gamma).$$

Furthermore,

$$\frac{d\mu_N}{d\nu_N}(\gamma) \propto 2^{Q_N(\gamma)} = 2^{Q(\gamma)-\tilde{R}_N(\gamma)}$$

where $Q(\gamma) = \lim_N Q_N(\gamma)$ and $Q_N(\gamma)$ is the number of edges ij, $i \le n$, such that $\omega_N(\gamma_{ij} + \varepsilon_{ij}) = \omega_N(\varepsilon_{ij})$. Note that $\tilde{R}_N(\gamma) = Q(\gamma) - Q_N(\gamma) \ge 0$.

Note that, $Q(\gamma)$ is *independent* of γ_+.

Proof of Lemma 2. Then the probability that a random spin-assignment to clusters of γ should give an element in $[x_N]$ is $\mathsf{P}(x_N|\gamma) = B(x_N,\gamma)\cdot 2^{-\omega_N(\gamma)}$. Thus

$$\mu([x_N]) = \mathsf{E}\left[\mathsf{E}(x_N|\gamma)\right] = \int B(x_N,\gamma) 2^{-\omega_N(\gamma)}\, d\mu(\gamma).$$

We obtain

$$1 = \sum_{x_N} \mu([x]_N) = \int \sum_{X_N} B(x_N,\gamma)\, d\mu_N(\gamma) = \int 2^{\omega_N(\gamma)}\, d\mu_N(\gamma).$$

$\square$

We obtain that

$$f_N(x) = \frac{\mu([x_N])}{\nu([x_N])} \propto \int C(x_N,\gamma) 2^{-Q_N(\gamma)}\, d\nu_N^{\pm}(\gamma)$$

where

$$C(x_N,\gamma) = \frac{B(x_N,\gamma)}{B(x_N,\gamma_+)}$$

Lemma 3. *We have*

$$\int 2^{Q(\gamma_-,\varepsilon)}\,d\mu(\gamma_-,\varepsilon) < \infty.$$

Proof of Lemma 3. Let $W(\gamma) \geq 0$ be the rightmost element in the cluster containing 0. We need to show that

$$\int Q(\gamma)\,d\mu(\gamma) \leq \sum_{n=1}^{\infty} \frac{1}{n} \mathsf{P}(W \geq n) \propto \mathsf{E}(\ln W) < \infty.$$

This should be obvious since $\mathsf{E}(\ln W|X)$ cannot grow superlinearly in the size of the cluster $X = |C(0)|$.

Van der Berg och Kesten showed that

$$f(\beta) = \sum_n \frac{\mathsf{P}_\beta(X = n)}{n}$$

was continuous for all β.

$\square$

The two-sided random cluster model is denoted by μ, i.e.

$$d\mu(\gamma) = 2^{\omega(\gamma)} \ltimes\ d\eta(\gamma),$$

where $\omega(G)$ is the potential $\omega(\gamma) = \lim_{\Lambda_n \nearrow V} \omega(\gamma[\Lambda_n])$ and η denote the bernoulli measure.

The one-sided model can be captured as the marginal distribution of γ_+ in the product measure

$$d\nu(G) = (2^{\omega(\gamma_+)} \ltimes\ d\eta(\gamma_-)) \otimes (2^{-|\epsilon|} \ltimes\ d\eta(\epsilon)) \otimes (2^{\omega(\gamma_+)} \ltimes\ d\eta(\gamma_+)).$$

Since

$$\omega(G) = \omega(\gamma_-) + \omega(\gamma_+) - |\epsilon| + R(\gamma_+, \epsilon, \gamma_+)$$

it follows that the two-sided Ising model can be obtained as

$$d\mu(\gamma) = 2^{\omega(\gamma)} \ltimes\ d\eta(\gamma) = 2^{R(\gamma)} \ltimes\ d\nu(\gamma).$$