EXISTENCE OF CONTINUOUS EIGENFUNCTIONS FOR THE DYSON MODEL IN THE CRITICAL PHASE

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ABSTRACT. We prove that there exists a continuous eigenfunction of the transfer operator defined by potentials for the so-called Dyson model for all inverse critical temperatures that are strictly less than the critical inverse temperature. This includes all cases when the potential does not have summable variations, the classical condition that ensures the existence of a continuous eigenfunction of the transfer operator. As a consequence the inverse critical temperatures for the one-sided and the two-sided models are the same. We use the Bernoulli random model for our method of proof, so the results also hold in this wider context. We also derive corresponding results for the Random cluster model under assumptions of Duminil-Copin, Garban, and Tassion [9].

1. INTRODUCTION AND RESULTS

1.1. **The problem.** It is well-known [19] that there exists a continuous and strictly positive eigenfunction h for any transfer operator defined on a symbolic shift space with a finite number of symbols such that the potential has summable variations. Here we prove the existence of a continuous eigenfunction for the important special class of Dyson potentials up to the critical phase, when the potential does not satisfy the condition of summable variations. We stress that it is the continuity that is the main difficulty.

More precisely, let T be the left shift on the space $X = S^{\mathbb{Z}_+}$, where S is a finite set. Let $\phi : X \to \mathbb{R}$ be a continuous function defined up to an additive constant. We refer to ϕ as the *one-point potential.* A *Gibbs measure* $\nu \in \mathcal{M}(X)$ for ϕ is one that minimises the free energy $P(\nu;\phi) = \nu(\phi) - H(\nu)$, where $H(\nu)$ denotes the entropy $\lim_{n\to\infty} H(\nu|_{\mathcal{F}_n})/n$ per time unit. This measure ν can also be obtained as the Gibbs measure on X obtained from the full potential

$$\Phi(x) = \sum_{k=0}^{\infty} \phi(T^k x)$$

on X. A Gibbs measure ν is also obtained as an eigenmeasure of the dual of the *transfer operator* $\mathcal{L} = \mathcal{L}_\phi$ defined on continuous functions by

$$\mathcal{L}f(x) = \sum_{y\in T^{-1}x} e^{\phi(y)} f(y). \tag{1}$$

Such an eigenmeasure ν satisfies $\mathcal{L}^*\mu = \lambda\nu$, for some maximal positive eigenvalue $\lambda > 0$.

The *equilibrium measure* μ is a minimiser of $P(\mu;\phi)$ among all *translation invariant* measures $\mu \in \mathcal{M}_T(X)$. Taking the natural extension of μ, we can also represent μ as a translation invariant measure on the *two-sided* space $\overline{X} = S^{\mathbb{Z}}$ and we can alternatively construct the measure $\mu \in \mathcal{M}_T(\overline{X})$ as the Gibbs measure to the full two-sided potential

$$\overline{\Phi}(x) = \sum_{k=-\infty}^{\infty} \phi(T^k x)$$

defined on $x \in \overline{X}$. If there exists a continuous eigenfunction $h(x)$ to the transfer operator, the measure μ can be recovered as the Doeblin measure [5] (a.k.a. g-measure in the terminology of Keane [16]) corresponding to the *Doeblin function* $g(x)$ where

$$g(x) = \frac{h(x)e^{\phi(y)}}{\lambda h(Tx)}. \tag{2}$$

The existence of a continuous eigenfunction h to $\mathcal{L}$, such that $\mathcal{L}h = \lambda h$, is however not automatic in same way the existence of the eigenmeasure ν is. If one assumes summable variations of ϕ,

$$\sum_{n=1}^{\infty} \mathrm{var}_n(\phi) < \infty, \tag{3}$$

where $\mathrm{var}_n(\phi) = \sup_{x\sim_n y} |\phi(x) - \phi(y)|$ ($x \sim_n y$ means that x and y coincide in the first n entries), then the existence of a continuous eigenfunction $h(x)$ follows from a "cone-argument" used in *e.g.*, Walters [19].

In this paper, we use that if there is a translation invariant measure μ which is absolutely continuous with respect to the Gibbs measure μ the Radon–Nikodym derivative $h(x) = \dfrac{d\mu}{d\nu}$ is an eigenfunction of $\mathcal{L}$.

We limit our study to the Dyson–Ising potential, since we exploit a random cluster formalization of the problem. Fix $\alpha > 1$ and $\beta > 0$. Let the one-point long-range Ising potential $\phi = \phi_{\alpha,\beta}$ be given by

$$\phi(x_0, x_1, \ldots) = x_0 \cdot \beta \sum_{j=1}^{\infty} \frac{x_j}{j^\alpha},$$

and define the one-sided and two-sided Dyson potentials, $\Phi : X \to \mathbb{R}$ and $\bar{\Phi} : \overline{X} \to \mathbb{R}$ as above. Let μ and ν be the Gibbs measures on $\mathcal{M}(X)$ and $\mathcal{M}(\overline{X})$, corresponding to Φ and $\overline{\Phi}$, respectively.

1.2. **The result.** For $\alpha > 2$ we then have summable variations of the one-sided potential Φ, in fact it is Hölder continuous. We then have

$$\mu = h\nu,$$

where $h > 0$ is a Hölder continuous eigenfunction. We are interested in the boundary cases where $1 < \alpha \leq 2$, when there exists a unique equilibrium measure for $\overline{\Phi}$ for $\beta < \beta_c$ [1] and multiple equilibrium measures when $\beta > \beta_c$, where the critical parameter $\beta_c = \beta_c(\alpha) > 0$ for $1 < \alpha \leq 2$. We show here that the same uniqueness properties holds for the "one-sided" Gibbs measure ν.

In this case the summable variations condition is not satisfied for neither $\overline{\Phi}$ nor Φ; hence we may have multiple eigen-measures for $\mathcal{L}^*$. In this context we have $\mathrm{var}_n(\phi) = O(\frac{1}{n})$.

Crucial to our proof of the continuity of an eigenfunction is the existence of a certain dominating random variable whose boundedness in L^1 (Lemma 6) follows from known estimates of the exponential decay of the upper tail cluster size distribution (Lemma 2). Since this is known better for certain models than others, we also obtain slightly different results for these different models. We obtain the existence of a continuous eigenfunction when $\beta(\alpha) < \beta_c(\alpha)$, whenever $\alpha > 1$, for the Bernoulli model ($q = 1$ in the random cluster model), but for other models ($q > 1$), we obtain these results for some $\beta^*(\alpha) \leq \beta^*(\alpha) < \beta_c(\alpha)$, $\alpha > 1$. In all cases we have $\alpha \leq 2$, summable a variations condition on the potential is not satisfied.

Let μ be the Gibbs equilibrium measure with respect to the Dyson potential ϕ and let ν be the one-sided Gibbs measure. Define

$$h_n(x) = \frac{\mu[x_0, \dots x_n]}{\nu[x_0, \dots, x_n]},$$

and consider the measurable function $h(x) = \limsup_{n\to\infty} h_n(x)$.

Theorem 1. *If $\alpha > 1$ and $\beta(\alpha)$ is sufficiently small, then $h(x)$ is a continuous function on X, $h > 0$, and it is also an eigenfunction of $\mathcal{L}$. In the Bernoulli case we assume only that $\beta(\alpha) < \beta_c$, that is for all subcritical inverse temperatures.*

1.3. **A conjecture.** We conjecture that there exists a continuous eigenfunction for a potential ϕ whenever that potential satisfies Berbee's condition [4]. Berbee proved that equilibrium measures are unique if we have

$$\sum_{n=1}^{\infty} e^{-r_1 - r_2 - \cdots - r_n} = \infty, \tag{4}$$

where $r_n = \mathrm{var}_n\,\Phi$ or $r_n = \mathrm{var}_n\,\overline{\Phi}$, that is for both the one-sided and two-sided models.

In [12] it was proved that uniqueness g-measures follows whenever

$$\sum_{n=1}^{\infty} (\mathrm{var}_n\, g)^2 < \infty,$$

and more recently Berger et al. [5] proved that for such *Doeblin measures* uniqueness follows if $\mathrm{var}_n \log g < 2/\sqrt{n}$ (see also [14] for a slightly stronger condition).

This connects to Dyson's counterexample in [10] where it is shown that there are examples of multiple equilibrium measures for ϕ whenever

$$\sum_{n=1}^{\infty}(\mathrm{var}_n\,\phi)^{1+\epsilon} < \infty,$$

where $\epsilon > 0$.

Dyson's example is for a two-sided model, but we showed in [15] that the inverse critical temperature β_c^+ satisfies $\beta_c^+ \leq 8\beta_c$, where β_c is the inverse critical temperature for the two-sided model, and this show that Dyson's example of multiple equilibrium measures can be formulated for a one-sided model as above.

Since we have the "translation" via (2) between the case for general potentials and for Doeblin measures, we may guess that the existence of a continuous eigenfunction cannot be moved very far away from the summability of variations condition for a potential.

2. The one-dimensional random cluster model and the Ising–Dyson model

For a finite graph, let $\omega(G)$ denote the number of connected components ("clusters") in the graph G. For simple graphs $G \subset \binom{V}{2}$ on an countably infinite set V of vertices, we consider the number of clusters $\omega(G)$ as a *potential.* This means that the difference $\Delta\omega(G, F) = \omega(G) - \omega(F)$ is defined for any two graphs F and G that coincide outside a *finite* subset $\Lambda \subset \binom{V}{2}$.

A random graph $G \sim \alpha$ on a set of vertices V is a probability distribution α on the set $\{0, 1\}^{\binom{V}{2}}$. The random cluster models $\mathcal{R}(V, p, q)$ (FK-model [**?**]), we consider are specified by a set of vertices V and a probabilities $p(ij) \in [0, 1]$, defined on the set of pairs $ij \in \binom{V}{2}$. The model $\mathcal{R}(V, p, q)$ is a Gibbs distribution on random graphs, i.e. configurations i $\gamma \in \{0, 1\}^{\binom{V}{2}}$, with non-continuous potential

$$\phi(\gamma) = -\log q \cdot \omega(\gamma) + \sum_{ij} \gamma(ij) \cdot \log p(ij) + (1 - \gamma(ij)) \cdot \log(1 - p(ij)).$$

The models $\mathcal{R}(V, p, 1) = \eta(\binom{V}{2}, p)$ are referred to as "Bernoulli percolation". In the one-dimensional Potts–Dyson random cluster model, $\mathcal{R}(V, \beta, \alpha, q)$, we let $V = \mathbb{Z}$ (or $V = \mathbb{Z}_{\geq 0}$ and consider the random cluster model $\mathcal{R}(V, p_J, q)$ where

$$p_J(ij) = 1 - e^{-J(ij)} \quad \text{where} \quad J(ij) = \frac{\beta}{|i-j|^{\alpha}}. \tag{5}$$

When q

Percolation is the event that the random graph contains an infinite cluster. It is well-known [1] that for all $\alpha \in (1, 2]$ there exists a critical $\beta_c = \beta_c(\alpha)$ such that, percolation does not occur with probability 1 for $0 < \beta < \beta_c$, while for $\beta \in (\beta_c, \infty)$ there is with probability 1 no infinite cluster.

We will mainly consider the subcritical case $1 < \alpha \leq 2$ and $0 < \beta < \beta_c(\alpha)$ and use the following lemma.

Lemma 2. *Let $X = X(C_s)$ denote the size of the cluster C_s containing a specified vertex $s \in V$. Then there are constants K and r, $0 < r < 1$, such that*

$$\mathsf{P}(X \geq m) \leq K r^m.$$

Proof. As surveyed by Panagiotis [17] (Theorem 1.2.1), we know that it holds for Bernoulli percolation ($q = 1$) when $0 < \beta < \beta_c$. Since $\mathcal{R}(V, p, 1)$ stochastically dominates $\mathcal{R}(V, p, q)$, $q > 1$, it also holds for the FK-model. Finally, Duminil-Copine ([9]) proves that $\beta_c(\alpha, q)$ are the same for the case $q = 1$ and $q > 1$. $\square$

2.0.1. *The extended random cluster model.* The extended *random cluster model* with $\mathcal{R}_e(V, p, q)$ can be obtained as the joint distribution of the spin sequences $x \in \{1, 2, \ldots, q\}^V$ and a random graph $\gamma \in \{0, 1\}^{\binom{V}{2}}$. The distribution of (x, γ) is obtained by first considering the pair chosen independently: The spin sequence $x \in \{0, 1, \ldots, q\}^V$ according to the uniform Bernoulli measure $x \sim \eta(V, p = \frac{1}{q})$ on the spin sequences and the random graph γ according to the Bernoulli measure $\eta(\binom{V}{2}, p) = \mathcal{R}(V, p, 1)$. Then $\mathcal{R}_e(V, p, q)$ is the distribution of (x, γ) *conditioned on* x and γ being *compatible*: That is, the event $C(x, \gamma)$ that no spins $x_i = +1$ and $x_j = -1$ in x are connected by a path (edge) in γ.

Assume $(x, \mu) \sim \mu = \mathcal{R}_e(V, p, q)$. Then marginal distribution $\mu \circ x^{-1}$ of x is the Potts model with interactions given by $J(ij) = \log(1 - p(ij))$ and the marginal distribution $\mu \circ \gamma^{-1}$ of γ is the random cluster model $\mathcal{R}(V, p, q)$ described above.

One obtains the long range Ising model with the Dyson potential $\phi_{\beta,\alpha}(x)$ as the marginal distribution of x and the random cluster model $\mathcal{R}(V, p, 2)$ as the marginal distributions γ.

If μ is a random cluster measure, we have

Let $S \subset V$ be a finite subset. We need to establish the probability $\mu([x]_S)$ of a cylinder $[x]_S = \{(y, \gamma) \mid y|_S = x|_S\}$ under $\mu = \mathcal{R}_e(V, p, q)$. Let $B([x]_S, \gamma)$ be the indicator for the event that γ is compatible with the cylinder $[x]_S$. Let also $\omega_S(\gamma)$ denote the number of clusters in γ that intersects S and $\bar{\omega}_S(\gamma)$ the number of components in the graph $\gamma \triangleright S$ obtained from joining all vertices in S in a cluster, e.g. by adding a vertex s connected to the elements of S.

In our application, when S are the intervals $[0, n)$ on the integer line, we write $[x]_n$, ω_n, $\bar{\omega}_n$, $\mu_{\bar{n}}$, etc.

Lemma 3. *Under an extended random cluster model then for any $x \in X$*

$$\mu([x]_S) = \int 2^{-\omega_S(\gamma)} B([x]_S, \gamma)\, d\mu(\gamma) \tag{6}$$

$$= 2^{\langle -\omega_S(\gamma)\rangle} \cdot \int B([x]_S, \gamma)\, d\mu_{\bar{S}}(\gamma) \tag{7}$$

where $\mu_{\bar{S}}$ denote the random cluster model and we define the normalising constant by

$$\langle -\omega_S(\gamma)\rangle = \log_2 \int 2^{-\omega_S(\gamma)}\, d\mu(\gamma, x)$$

We now observe that a configuration γ in the two-sided model $\mathcal{R}(\mathbb{Z}, \alpha, \beta)$ difference between the one-sided random cluster model $\nu = \mathcal{R}(V_+, J)$ and the usual two-sided model $\mu = \mathcal{R}(V, J)$. We will use that a configuration γ can be factored as $\gamma = (\gamma_+, \epsilon, \gamma_-)$, where γ_- is the induced graph $\gamma[V_-]$ on vertices $-j \in V_- = \mathbb{Z}_{<0}$ and $\gamma_+ = \gamma[V_+]$ is the graph induced on vertices $i \in V_+ = \mathbb{Z}_{\geq 0}$.

Let

$$R(\gamma) = |\epsilon| - (\omega(\gamma \setminus \epsilon) - \omega(\gamma)).$$

Note that

$$0 \leq R(\gamma) \leq Q(\gamma_-, \epsilon) = \sum_{C \in \mathcal{C}(\gamma_-)} (d(C, \epsilon) - 1)_+ \tag{8}$$

Let also

$$\tilde{\eta}(\epsilon) = 2^{-|\epsilon|} \ltimes \eta(\epsilon).$$

Note that both $\eta(\cdot)$ and $\tilde{\eta}(\cdot)$ are Bernoulli measures. Let also

Lemma 4. *We have*

$$\mu(\gamma) = q^{R(\gamma) - \langle R(\gamma)\rangle} \cdot \mu(\gamma_+) \otimes \tilde{\eta}(\epsilon) \otimes \mu(\gamma_-) \tag{9}$$

$$\mu_{\bar{S}}(\gamma) = 2^{R(\gamma \triangleright S) - \langle R(\gamma \triangleright S)\rangle} \cdot \mu_{\bar{S}}(\gamma_+) \otimes \tilde{\eta}(\epsilon) \otimes \mu(\gamma_-). \tag{10}$$

Let n stand for the integer interval $[0, n-1]$. Define the likelihoods

$$h_n(x) = \frac{\mu([x]_n)}{\mu_+([x]_n)}$$

$$h_n(x) = \frac{\int B([x]_n, \gamma) \cdot 2^{R(\gamma) - K + \omega_n(\gamma_+) - \omega_n(\gamma_-)}\, d\nu(\gamma_+)\, d\tilde{\eta}(\epsilon)\, d\nu(\gamma_-)}{\int B([x]_n, \gamma_+) \cdot 2^{-\omega_n(\gamma_+)} d\nu(\gamma_+)}, \tag{11}$$

where $\omega_n(\gamma)$ is the number of clusters in γ intersecting $[0, n)$. We define

$$R_n = R(\gamma) - K + \omega_n(\gamma_+) - \omega_n(\gamma_-),$$

and

$$B'_n(x, \gamma) = \frac{B([x]_n, \gamma)}{B([x]_n, \gamma_+)},$$

and write

$$h_n(x) = \int B'_n(x, \gamma) 2^{R_n(\gamma)}\, d\tilde{\nu}_n(\gamma_+)\, d\tilde{\eta}(\epsilon)\, d\nu(\gamma_-),$$

where ν_n is normalised. We now study the sequence of functions

$$g_n(\epsilon, \gamma_-) = \int B'_n(x, \gamma) 2^{R_n(\gamma)} \, d\tilde{\nu}_n(\gamma_+).$$

Crucial to our proof of the continuity of h is the L^1-bound of 2^Q, in order to prove the continuity of h.

3. Proof of Theorem 1

Lemma 5. *If $\mu \in \mathcal{M}_T(X)$ is a translation invariant measure which is absolutely continuous with the Gibbs measure ν for ϕ then the Radon–Nikodym derivative $h(x) = \dfrac{d\mu}{d\nu}(x)$ is an eigenfunction to the transfer operator $\mathcal{L} = \mathcal{L}_\phi$.*

Proof. We can assume $\lambda = 1$. By assumption, the measure $\mu = h\nu$ is translation invariant, i.e., $\mu \circ T^{-1} = \mu$. Hence we have $(h\nu) \circ T^{-1} = h\nu$ and it suffices to show that $(h\nu) \circ T^{-1} = (\mathcal{L}h)\nu$.

Let A be any Borel subset of X. Then

$$(h\nu) \circ T^{-1}(A) = \int_A \sum_{y:Ty=x} h(y) e^{\phi(y)} \, d\nu(x) = \int_A h(x) \, d\nu(x).$$

□

We need to prove that the $h_n(x)$ converge to a continuous function $h(x)$ as $n \to \infty$. It is enough to show that

$$\lim_{m \geq n \to \infty} \sup_x |h_m(x) - h_n(x)| = 0.$$

We are then certain to have a continuous eigenfunction h that is strictly positive, since if it had zeroes it would be identically zero, since if x_0 is a zero of h, then we have (with the eigenvalue λ normalised to 1):

$$0 = h(x_0) = \sum_{y \in T^{-n} x_0} e^{\Phi(y)} h(y).$$

That h is not identically zero follows because it must integrate to 1 with respect to ν, since $\int h_n \, d\nu = 1$ for all n.

Recall that

$$h_n(x) = \int B'_n 2^{R_n} \, d\nu(\gamma_-) \, d\tilde{\eta}(\epsilon) \, d\nu_n(\gamma_+ | B_n^+). \tag{12}$$

Let

$$g_n(x, \gamma_-, \epsilon) = \int B'_n(\gamma_+, \epsilon, \gamma_-) 2^{R_n(\gamma_+, \epsilon, \gamma_-)} \, d\nu_n(\gamma_+ | B_n^+),$$

so that

$$h_n(x) = \int g_n(x, \gamma_-, \epsilon) \, d\nu_n(\gamma_-) \, d\tilde{\eta}(\epsilon).$$

We also notice that since R_n is the number of edges in ϵ that do not reduce (with respect to some order) the number of components in $\gamma \triangleright [0, n]$, it is clear that

$$R_n \leq Q_n \tag{13}$$

where Q_n is the number of edges $ij \in \epsilon$ where $i > n$ and $-j$ belongs to a cluster C in γ_- that sends at least one more edge to $[0, \infty)$.

We notice that for $n > N$, where is $N = N(\epsilon, \gamma_-)$ is some random variable, $B'_n(\gamma_+, \epsilon, \gamma_-)2^{R_n(\gamma_+,\epsilon,\gamma_-)}$ is constant (in n). For $n \geq N$, we have $R_n(\gamma_+, \epsilon, \gamma_-) = Q(\epsilon, \gamma_-) = \lim_n Q_n(\epsilon, \gamma_-)$, and $B'_n = B'_N$.

We have that $N < \infty$ a.s., since $N \leq Q \leq 2^Q$ and we have the following lemma.

Lemma 6. *The integral*

$$\int 2^{Q(\gamma_-,\epsilon)}\, d\nu(\gamma_-)\, d\tilde{\eta}(\epsilon) < \infty.$$

We now conclude from dominated convergence (using 2^Q as the dominating function) that since $g_n(\epsilon, \gamma_-) - g_m(\epsilon, \gamma_-) = 0$, if $N \leq n \leq m$, and otherwise less than or equal to 2^Q, we have

$$\lim_{m \geq n \to \infty} \sup_x |h_m(x) - h_n(x)| \leq \lim_{m \geq n \to \infty} \int |g_n(x, \gamma_-, \epsilon) - g_m(x, \gamma_-, \epsilon)|\; d\nu(\gamma_-)\; d\tilde{\eta}(\epsilon) = 0.$$

Proof of Lemma 6. We condition on a fixed graph γ_- with distribution ν_-. Let C be a cluster of γ_-. Note that

$$Q = (X(C_1) - 1)_+ + (X(C_2) - 1)_+ + \dots$$

where $X(C)$, is a sum of independent Bernoulli variables

$$X(C) = \sum_{-j \in C} \sum_{i=0}^{\infty} \epsilon_{ji}$$

where

$$\mathsf{P}(\epsilon_{ij} = 1) = \frac{1 - \exp\{-\frac{\beta}{(i+j)^2}\}}{2}$$

It follows that we can approximate $X(C)$ with a Poisson variable (***) $\tilde{X}(C) \sim \mathrm{Po}(\lambda(C))$ with

$$\lambda(C) = \frac{\beta}{2} \sum_{j \in C} \frac{1}{j} \approx \frac{\beta}{2} \sum_{j \in C} \sum_{i=0}^{\infty} \mathsf{P}(\epsilon_{ij} = 1).$$

Note that

$$\lambda(C) \leq \log\left(1 + \frac{|C|}{i(C)}\right) \tag{14}$$

where $-i(C) = \max C$.

Order the clusters of γ_- as $C_1, C_2, \dots$ etc. so that $i(C_1) < i(C_2) < \dots$. For each cluster C_i we can from stochastic dominance construct a random cluster $\tilde{C}_i$ such that (i) $C_i \subset \tilde{C}_i$ and (ii) $i(\tilde{C}_i) = i(C_i)$. We can further assume that the $\tilde{C}_i$s are *independent* with the same distribution.

Let now

$$\tilde{Q} = \sum_{C_i} (\tilde{X}(\tilde{C}_i) - 1)_+.$$

where $\mathsf{P}(\tilde{X}(\tilde{C})|\tilde{C}) = \mathrm{Po}(\lambda(\tilde{C}))$ which stochastically dominates $X(C)$. For a poisson variable $X \sim \mathrm{Po}(\lambda)$ we have

$$\mathsf{E}(2^{(X-1)_+}) = \frac{\exp(\lambda(e^{\ln 2} - 1)) + e^{\lambda}}{2} = \cosh(\lambda)$$

We then have

$$\mathsf{E}(2^Q|\gamma_-) \le \prod \cosh(\lambda(C_i)) \le \prod \cosh(\lambda(\tilde{C}_i)).$$

We obtain, since $i(C_k) \ge k$ and (14) and the independence of $\tilde{C}_k$, that

$$E(2^Q) \le \prod_{k=1}^{\infty} \mathsf{E}\left(\frac{1}{2}\left(1 + \frac{Y}{k} + \frac{1}{1+\frac{Y}{k}}\right)\right) \tag{15}$$

$$\le \prod_{k=1}^{\infty} \mathsf{E}\left(1 + \frac{Y^2}{k^2} + \frac{Y^3}{k^3} + \dots\right). \tag{16}$$

where Y has the common distribution of $|C_k|$. It is easy to see that this is less than ∞ on account of Lemma 2, which states that the distribution Y has an exponentially decreasing bound for the upper tail.

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