Ergodic Theory of Kusuoka Measure

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Abstract

In analysis on fractal sets, the Kusuoka measure has a prominent role, as it is used, together with a bilinear energy form, to define a Laplacian. Little is known about the measure itself, however, except from Kusuoka's intriguing paper [10], Kajino's [4] and the recent result by Strichartz and his collaborators [2]. Here we turn on an investigation of the Kusuoka measure from an ergodic theoretic viewpoint. We prove exponential rate of convergence for the iterates of the transfer operator for Kusuoka measure defined on post-critically finite fractal sets. We obtain specific results in the case of some specific examples, including the Sierpinski Gasket. In addition, we prove that the Kusuoka measure satisfies a functional central limit theorem for all post-critically finite fractals.

1 The Kusuoka measure

1.1 Preliminaries

1.1.1 The Hilbert-schmidt norm and SVD

Given a separable Hilbert-space $\mathcal{H}$ with Hermitian inner product

$$(u, v) \mapsto \langle u, v\rangle = u^* v.$$

An operator $\mathsf{W} : \mathcal{H} \to \mathcal{H}$ is a *Hilbert-Schmidt* operator if it is bounded in the Frobenius norm

$$\|\mathsf{W}\|^2 := \mathrm{Tr}(\mathsf{W}^*\mathsf{W})$$

where trace is defined as

$$\mathrm{Tr}(\mathsf{W}) = \sum_i u_i^* \mathsf{W} u_i,$$

for any ON-basis $\{u_1, u_2, \dots\}$. Such operators W are compact, i.e. the image $\mathsf{W}(\mathcal{U})$ of the unit ball in $\mathcal{H}$ is compact. Moreover, for any bounded linear operator A, the compositions $A\mathsf{W}$ and $\mathsf{W}A$ remain Hilbert-Schmidt.

For Hilbert-Schmidt operators $B : \mathcal{H} \to \mathcal{H}$, we use the (Frobenius-) norm

$$\|B\|_{HS} = (\mathrm{Tr}(B^* B))^{1/2}.$$

We will also use the *singular value decomposition* (SVD). That is, for a operator $B : \mathcal{H} \to \mathcal{H}$, we have a factorisation

$$B = VDU^*,$$

where U and V are two unitary operators and D is a diagonal operator. The diagonal of D is a unique sequence of positive and decreasing *singular values* of B. The subspaces spanned by the columns in V and U corresponding to the same singular value $\sigma_i(B)$; these subspaces constitute the eigenspaces of the operators B^*B and BB^*, respectively, corresponding to the common eigenvalue $\lambda_i = \sigma_i^2(B)$. We can recover the singulaar values using the variational definition

$$\sigma_k(B) = \min \{\|B - F\|_{\mathcal{H}} : \mathrm{rank}(F) = k - 1\}$$

where $\|\cdot\|_{\mathcal{H}}$ denotes the operator norm

$$\|A\|_{\mathcal{H}} = \sup_{u \in \mathcal{H}} \frac{\|Au\|}{\|u\|}.$$

The operator norm i given by the largest singular value σ_1.

Note that for an Hilbert-Schmidt operator B with SVD $B = VDU^*$, we have

$$\|B\|_{HS} = \|D\|_{HS} = \sqrt{\sigma_1^2(B) + \sigma_2^2(B) + \ldots},$$

since it holds in general that

$$\|UAV\|_{HS} = \|A\|_{HS}$$

for any pair U and V of unitary operators.

1.1.2 The induced norm $\|\cdot\| = \|\cdot\|_{\mathsf{W}}$

Let W be a fixed *symmetric* Hilbert-Schmidt operator W such that $\mathrm{Tr}(\mathsf{W}^*\mathsf{W}) = 1$. For bounded operators $A : \mathcal{H} \to \mathcal{H}$, we define the norm

$$\|A\| = \|A\mathsf{W}\|_{HS} = \mathrm{Tr}(\mathsf{W}A^*A\mathsf{W}) = \mathrm{Tr}(A\mathsf{W}^2A^*), \tag{1}$$

where the last equality is due to the fact that $\mathrm{Tr}(R^*R) = \mathrm{Tr}(RR^*)$ and W is symmetric. We also define $\mathbf{s}(A) = (s_1, s_2, \dots)$ as the singular values of the operator AW, so that the singular value decomposition of AW is given by

$$AW = V\ \mathrm{diag}(\mathbf{s}(A))\, U^* \tag{2}$$

The following properties are easily established for the norm $\|A\|$.

Lemma 1. *The norm of the identity operator is one*

$$\|I\| = 1. \tag{3}$$

For a bounded operator A it holds that

$$\|A\| = \|A^*\| \tag{4}$$

and for any pair of unitary operators U and V

$$\|UAV\| = \|A\| \tag{5}$$

Proof. □

Let $\mathcal{P}_k$ denote the set of orthogonal projections $P : \mathcal{H} \to \mathcal{H}'$, where $\dim \mathcal{H}' = k$ and let $\mathsf{W}\mathcal{P}_k$ be the image of $\mathcal{P}_k$ under the compact map W.

Lemma 2. *For a operator B, we have*

$$s_1(B)^2 + s_2(B)^2 + \dots s_k(B)^2 = \max_{R \in \mathsf{W}(\mathcal{P}_k)} \operatorname{Tr}(R^* B^* B R)$$

and

$$s_{k+1}(B)^2 + s_{k+2}(B)^2 + \doteq \min_{R \in \mathsf{W}(\mathcal{P}_k)} \operatorname{Tr}((W-R)^* B^* B (W-R)).$$

1.1.3 Random products

Consider a finite set $\mathcal{S} = \{A_1, \dots, A_N\}$ of invertible symmetric operators such that

$$A_1^2 + A_2^2 + \dots + A_N^2 = I. \tag{6}$$

We also assume that the there is no common invariant subspace of $\mathcal{H}$ to the A_is except $\mathcal{H}$ and $\{0\}$.

We consider a random sequence $\{X_n : n \geq 0\}$ of bounded operators such that for $n \geq 1$

$$X_n = X_n' X_{n-1}.$$

where $X_n' \in \mathcal{S}$ has distribution

$$\mathsf{P}(X_n' = A_i \mid X_n, X_{n-1}, \dots) = \frac{\|A_i X_n\|^2}{\|X_n\|}$$

It follow that $\{X_n\}$ is a Markov process.

Let

$$Q_n = \frac{X_n^* X_n}{\|X_n\|^2}.$$

Then Q_n is symmetric and $\|Q_n^{1/2}\| = 1$. The following lemma states that Q_n converge to some rank-one operator UU^* at an exponential rate.

Lemma 3. *There is an η, $0 < \eta < 1$, such that some random unit vector $U \in \mathcal{H}$, we have*

$$\|Q_n^{1/2} - (UU*)^{1/2}\| = O(\eta^n) \tag{7}$$

almost surely. It follows that

$$\left\|\mathbf{s}(\hat{X}_n) - (1, 0, 0, \dots)\right\|_{\ell^2} = O(\eta^n),$$

where $\hat{X}_n = \frac{X_n}{\|X_n\|}$.

Let

Lemma 4. *There is an η, $0 < \eta < 1$, such that*

$$\mathsf{E}(X_n X_n^*) = \|X_0\| I + O(\eta^n) \tag{8}$$

1.1.4 The symbolic space and elementary cylinders

Let $X = S^{\mathbb{Z}}$ be the two-sided symbolic space and let $X_+ = S^{\mathbb{Z}_{\geq 0}}$ be the corresponding one-sided space. Both spaces are equipped with the product topology and the corresponding borel sigma-algebra $\mathcal{F}$ ($\mathcal{F}_+$). For $n, k \in \mathbb{Z}$, $k \leq 0$, we have a map $\pi_n^k : X \to S^k$; $x \mapsto (x_n, x_{n+1}, \dots, x_{n+k-1})$. An elementary cylinder of length k is a subset of X (or X_+) of the form

$$[w]_n^k = (\pi_n^k)^{-1}(w),$$

where w is a fixed word $w = (w_1, \dots, w_k) \in S^k$. Elementary cylinders make up an $\mathcal{I}$-system that generates $\mathcal{F}$, and it is thus enough to define measures on cylinders in a consistent way. The set of probability measures μ on $\mathcal{F}$ is denoted $\mathcal{M}(X, \mathcal{F})$.

For $x \in X$ (or $x \in X_+$), we use the notation $[x]_n^k$ for the cylinder $[\pi_n^k(x)]_n^k$. Often, we assume $n = 1$ and we can unburden the notation by writing $[w]$ or $[x]^k$ instead of $[w]_1$ and $[x]_1^n$.

1.2 The Kusuoka measure on a general Hilbert space

1.2.1 The Hilbert space and the energy form

We are given a separable Hilbert-space $\mathcal{H}$ with Hermitian inner product $(u, v) \mapsto \langle u, v \rangle = u^* v$. On $\mathcal{H}$ we have a continuous positive definite *energy-form* $(u, v) \mapsto \mathcal{E}(u, v)$ defined.[1] We assume that the form $\mathcal{E}$ is normalised, i.e.

$$\mathrm{Tr}(\mathcal{E}) := \sum_i \mathcal{E}(u_i, u_i) = 1 \tag{9}$$

for some (and then every) ON-basis $\{u_1, u_2, \dots\}$ of $\mathcal{H}$.

[1] In our applications, the space $\mathcal{H}$ is the space of inverse limits of resistive tableaux of finite energy on graph sequences $\mathbf{G} = \{G_m\}$. Alternatively, one can start with the subspace $\mathcal{H}' \subset \mathcal{H}$ as the space of limits of harmonic tableaux. For a p.c.f. fractals, the harmonic space is finite-dimensional.

It follows from (9) that $\mathcal{E}$ is obtained from some Hilbert-Schmidt [2] operator $\mathsf{W} : \mathcal{H} \to \mathcal{H}$ such that $\mathcal{E}(u,v) = u^*\mathsf{W}^*\mathsf{W}v$. The Frobenius norm of W is one, on account of (9).

1.2.2 Restriction operators

In addition to the energy form, we have a family $\{A_s : s \in S\}$ of *restriction operators*, $A_s : \mathcal{H} \to \mathcal{H}$ indexed by a finite set of symbols $S = \{s_1, \dots, s_k\}$ with the property[3] that

$$\mathcal{E}(u,v) = \sum_s \mathcal{E}(A_s u, A_s v).$$

It follows that

$$\sum_s A_s A_s^* = \sum_s A_s^* A_s = 1 \tag{10}$$

where 1 denotes the identity operator.

If $\mathcal{H}' \subset \mathcal{H}$ is a common invariant subspace for the restriction operators[4], i.e. $A_s\mathcal{H}' \subset \mathcal{H}'$, then we can carry over all our arguments to $\mathcal{H}'$.

1.3 Definition of the Kusuoka measure and energy measure

We associate to each word $w = (w_1, w_2, \dots, w_k) \in S^k$ the operator

$$A_w = A_{w_k} A_{w_{k-1}} \dots A_{w_1}.$$

The Kusuoka measure $\nu \in \mathcal{M}(X, \mathcal{B})$ is defined on X by setting

$$\nu([w]_m) = \mathrm{Tr}\,(\mathcal{E} \circ (A_w \times A_w)) = \mathrm{Tr}(A_w^* \mathsf{W}^* \mathsf{W} A_w) = \|\mathsf{W}A_w\|^2. \tag{11}$$

The Kusuoka measure ν is clearly invariant on X, since $\nu([w]_m)$ does not depend on m, and it is enough to define it on cylinders $[w] = [w]_0$. The consistency is due to (10), i.e. we have

$$\sum_s \nu([ws]) = \sum_s \mathrm{Tr}(\mathsf{W}A_w A_s A_s^* A_w^* \mathsf{W}^*) = \mathrm{Tr}(\mathsf{W}A_w 1 A_w^* \mathsf{W}^*) = \nu([w]),$$

and hence it extends to a probability measure on $\mathcal{B}$. That ν is normalised follows from (9).

For a fixed non-zero $h \in \mathcal{H}$, we can also define the corresponding *energy-measure* ν_h on X by setting

$$\nu_h([w]) = \mathcal{E}(A_w h, A_w h) h^* A_w^* \mathsf{W}^* \mathsf{W} A_w h, .$$

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[2]That the energy form $\mathcal{E}_{\mathbf{G}}$ has this property follows from the fact that $\mathcal{E}$ is obtained as the limit of $\mathcal{E}_m$, where $\mathcal{E}_m$ is the energy form on the finite graph G_m. We can factor $\mathcal{H}$ into orthogonal subspaces as $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$, where $u \in \mathcal{H}_n$ for $n > m$ has zero $\mathcal{E}_m$-energy.

[3]In our applications, this follows from the fact that we can partition $\mathbf{G}$ into $|S|$ closed subsequences $\mathbf{G}[s]$, $s \in S$, that are isomorphic to $\mathbf{G}$; that is, the self-similarity property.

[4]Our example is the subspace of harmonic tableaux $\mathcal{H}' = \mathcal{H}(\mathbf{G})$ of the space $\mathcal{H}'$ of resistive tableaux $\mathcal{R}(\mathbf{G})$.

In this way, we may write the Kusuoka measure as the sum

$$\nu(A) = \sum_i \nu_{u_i}(A)$$

for any ON-basis u_i of $\mathcal{H}$. [5]

1.4 The Kusuoka measure as a g-measure

As usual, we use $\nu(A|B)$ to denote the conditional measure and for the Kusuoka measure we have

$$\nu(w|u) = \frac{\nu(wu)}{\nu(u)} = \frac{\|A_u A_w\|^2}{\|A_u\|^2}.$$

The process

$$x \mapsto (n \mapsto w \mapsto \nu(w \mid [x]_n)), \quad x \in X_+, n \in \mathbb{Z}_+, w \in S^k$$

is a ν-martingale with values $w \mapsto \nu(w \mid [x]_n)$ in the compact space of distributions on S^k.

By the Martingale convergence theorem (MCT), $\nu(w|[x]_n)$ converges ν-almost everywhere. Note that, with $w = s \in S$, this means that the "g-function"

$$g(sx) = \lim_{n\to\infty} \nu(s|[x]_n),$$

which defines the transfer operator L, is defined ν-almost everywhere. In our case, the g-function is not continuous, in fact it is not continuous anywhere.

Let $\hat{A}$ denote the normalisation $A/\|A\|$ of A with respect to the Frobenius norm. Then

$$\nu(ws|w) = \|A_s \hat{A}_w\|^2 = \|A_s A_w\|^2/\|A_w\|^2.$$

It means that the sequence

$$\hat{A}_n(x) := \hat{A}_{[x]_n}$$

is a Markov chain on the unit sphere $\mathcal{U} = \mathcal{U}(\mathbb{R}^{d\times d}$ with respect to the Frobenius norm. If $\mathcal{H}$ is finite-dimensional this is a compact space.

We will use the *singular value decomposition (SVD):* That is, for a matrix $d \times d$, C we have a (more or less) unique factorisation

$$C = VDU^*$$

where D is a positive diagonal matrix with the singular values $\sigma_i(C)$,

$$\sigma_1(C) \geq \sigma_2(C) \geq \cdots \geq \sigma_n(C)$$

decreasing along the diagonal and U and V are both orthogonal matrices. The Frobenius norm of C is given by

$$\|C\| = \|D\| = \sum_{i=1}^n \sigma_i(C)^2.$$

[5]In our applications, we can interpret $\nu_h([w])$ as the power that dissipates on the induced sub-graph $\mathbf{G}[w]$.

Let $M \to \kappa(M) \geq 1$ denote the *condition number* of the matrix M, i.e. the ratio between the largest and the smallest singular values.

Let $A_n(x) = V_n(x)D_n(x)U_n(x)^*$ be the SVD of $A_n(x)$. Define also $E_n(x) = \hat{D_n}(x)$, $C_n(x) = V_n(x)E_n(x)$ and $Q_n(x) = E_n(x)U_n(x)^*$. Note that, for any $w \in S^k$, we have

$$\nu(w \mid [x]_n) = \|\hat{A}_n(x)A_w\| = \|Q_n(x)A_w\|^2.$$

Thus the set of x for which $\nu(w \mid [x]_n)$ converges is equal to, independently of w, the set of x for which $Q_n(x)$ converges to some limit $Q(x)$. We can for these x and only these x define the g-function $g(sx) = \nu(s|x)$.

We now obtain information about the rate of ν-a.e. convergence to the g-function from its local approximants.

Theorem 1. *Let*

$$g_n(x) = \frac{\nu([x]_n)}{\nu([Tx]_{n-1})}.$$

For ν-a.e. x, we have, wherever $g(x)$ is defined, that for some $\theta < 1$

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{g_n(x)}{g(x)} = \theta.$$

Proof. For a fixed $w \in S^k$, we define also $\tilde{A}_n(x) = \tilde{V}_n(x)\tilde{D}_n(x)\tilde{U}_n(x)$ as the SVD of the matrix $\tilde{A}_n(x) = A_n(x)A_w$ and $\tilde{C}_n(x) = \tilde{V}_n(x)\tilde{E}_n(x)$, where $\tilde{E}_n(x) = \widehat{\tilde{D}_n(x)}$. Note that C_n and $\tilde{C}_n$ are processes with statespace also equal to $\mathfrak{U}$. Moreover, the distribution of $C_n(x)$ is Markovian with respect to ν, since

$$\nu([x]_{n+1} \mid [x]_n) = \frac{\|A_{x_n}A_n(x)\|^2}{\|A_n(x)\|^2} = \|A_{x_n}C_n(x)U_n(x)^*\|^2 = \|A_{x_n}C_n(x)\|^2.$$

With $m \leq n$, we also have

$$\begin{aligned}\nu(w \mid [x]_m) - \nu(w \mid [x]_n) &= \frac{\|A_m(x)A_w\|^2}{\|A_m(x)\|^2} - \frac{\|A_n(x)A_w\|^2}{\|A_n(x)\|^2} \\ &= \left(1 - \prod_{k=m+1}^{n} \frac{\|A_{x_k}\tilde{C}_k(x)\|^2}{\|A_{x_k}C_k(x)\|^2}\right) \cdot \|C_m(x)A_w\|^2\end{aligned}$$

We will show that

Lemma 5.

$$\sum_{k=m}^{\infty} \int \|C_n(x) - \tilde{C}_n(x)\|^2 \, d\nu(x) = O(\theta^m).$$

Proof. To prove this we use contractivity in mean for the map $\mathfrak{U} \to \mathfrak{U}$ $C \mapsto C'$ where $C' = V'E'$, where E' is the normalisation of D' and $V'D'U'^*$ is the SVD of the matrix A_sC and A_s is the random restriction.

For the SG, we know that A_s takes the form

$$A_s = \text{constant} \times R^s \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} R^{-s},$$

and $s = 0, 1, 2$ for a fixed rotation R by $120°$. We can think of C as an ellipsis and it is clear that the map is mean contractive. $\square$

From this lemma it follows that

$$\int \left(\prod_{k=m+1}^{n} \frac{\|A_{x_k} \tilde{C}_k(x)\|^2}{\|A_{x_k} C_k(x)\|^2}) \right) d\nu(x) = 1 + O(\theta^m)$$

which proves the sought statement.

$\square$

2 Exponential convergence

Let X_c be the set of points x where the g-function is defined.

Theorem 2. *For any pcf fractal, we have for the associated Kusuoka measure ν and for any fixed Hölder continuous function f*

$$\sup_{x \in X_c} |\mathcal{L}^m f(x) - \int f \, d\nu| \leq C\beta^m \int |f| \, d\nu,$$

where C is a uniform constant (depending only on f) and $0 < \beta < 1$.

Proof. The result follows if we can establish that for a fixed $x \in X_+$ and cylinder sets $[u] = [T^{-m}x]_m$ and $[w] = [T^{-m-k}x]_k$ we have for some $0 < \alpha < 1$ (note that $\alpha \leq \beta$ in the formulation of the theorem; which $\beta < 1$ we get depends on the Hölder space the test function f belongs to)

$$\nu(w|x) = \mathcal{L}^m 1_{[w]} = \sum_u \nu(wu|x) = (1 + O(\alpha^m))\, \nu(w). \tag{12}$$

In matrix form, (12) says that

$$\sum_u \|Q(x) A_u A_w\|^2 \approx \|\sum_u A_u A_w\|^2 = \|A_w\|^2. \tag{13}$$

Here $Q(x)$ is a normalized matrix in the unit ball $\mathcal{U}$ given by the limit $Q(x) = \lim_{n\to\infty} Q_n(x)$ where

$$Q_n(x) = E_n(x) U_n^*(x)$$

and $U_n(x)$ and $E_n(x) = \hat{D}_n(x)$ are derived from the SVD $A_n = V_n D_n U_n^*$ of $A_n(x) = A([x]_n)$.

Since $\|B\|^2 = \operatorname{Tr} BB^* = \operatorname{Tr} B^*B$ we can use linearity of trace to obtain the equality

$$\nu(w|x) = \sum_u \operatorname{Tr} Q(x)A_uA_wA_w^*A_u^*Q(x)^* = \operatorname{Tr} Q(x)R_m(A_w)Q(x)^*. \tag{14}$$

where

$$R_m(B) = \sum_u A_uBB^*A_u^*.$$

We are able to establish (12) if we can show the following two facts:
(1) The limit $Q(x) = \lim_{n\to\infty} Q_n(x)$ exists ν-a.e., where

$$\|Q_n(x)A_u\|^2 = \frac{\|A_n(x)A_u\|^2}{\|A_n(x)\|^2} = \frac{\nu(u[x]_n)}{\nu([x]_n)} = \nu(u|[x]_n).$$

This follows from the discussion in subsection 3.2 by the martingale convergence theorem in the same way as for the existence ν-a.e. of the g-function. Hence we may assume that the limit $Q(x)$ exists for $x \in X_c$.
(2) The bound

$$\frac{\nu(w|x)}{\nu(w)} = \frac{\operatorname{Tr} Q(x)R_m(A_w)Q^*(x)}{\operatorname{Tr} R_m(A_w)} = 1 + O(\alpha^m). \tag{15}$$

We obtain (15) if $R_m(A_w)$ satisfies

$$R_m(A_w) = \sum_u A_uA_wA_w^*A_u^* = \frac{1}{d} \cdot \|A_w\|^2(I + O(\alpha^m)). \tag{16}$$

It is easily checked, that (16) holds, given the matrices A_s. The map

$$H \mapsto \mathsf{M}H := \sum_{\sigma\in S} A_\sigma H A_\sigma^*$$

takes positive definite symmetric matrices in $\mathbb{R}^{d\times d}$ to positive definite symmetric matrices. Note that we can write

$$R_m(A_w) = \mathsf{M}^m(A_wA_w^*).$$

Since

$$\operatorname{Tr} A_wA_w^* = d \cdot \nu(w) = d \cdot \sum_\sigma \nu(w\sigma) = \mathsf{M}(A_wA_w^*),$$

we also know it preserves the trace, i.e.

$$\operatorname{Tr} \mathsf{M}H = \operatorname{Tr} H.$$

It is easy to see that any constant times the identity matrix I is a fixed-point for M and it remains to show that $\frac{1}{d}I$ is an attracting fixed-point on $\mathcal{U}'$, where $\mathcal{U}'$ denotes the set of symmetric positive definite $d \times d$-matrices of trace 1. An estimate of the corresponding Lyapunov-exponent gives the α. Since M is linear it is a matter of finding eigenvalues and eigenvectors of the corresponding matrix. $\square$

3 Examples

For simplicity, we now let $f \in m\mathfrak{F}_k$.

3.1 The Sierpinski Gasket

For the Sierpinski Gasket, SG, we obtain the following rate of convergence.

Theorem 3.

$$\sup_{x \in X_c} |\mathcal{L}^{m+k} f(x) - \int f\, d\nu| \leq C \left(\frac{4}{5}\right)^m \int |f|\, d\nu,$$

where C is a uniform constant (depending only on f).

Proof. For SG we have $S = \mathbb{Z}_3 = \{0, 1, 2\}$, $d = 2$ and corresponding matrices

$$A_s = R^s D R^{-s}$$

where

$$D = \begin{bmatrix} 3/\sqrt{15} & 0 \\ 0 & 1/\sqrt{15} \end{bmatrix}$$

and R is the rotation-matrix $R = \mathrm{Rot}(2\pi/3)$. The map M takes the form

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \mapsto \begin{bmatrix} \frac{1}{10}(9a + c) & \frac{4b}{5} \\ \frac{4b}{5} & \frac{1}{10}(a + 9c) \end{bmatrix}$$

and it is easy to see that

$$\mathsf{M}^m H = I + O(\left(\frac{4}{5}\right)^m)$$

for any symmetric matrix H with trace 2. That the off-diagonal elements converge with rate $(4/5)^m$ to 0 is immediate from inspection. On the diagonal, we see that the difference between the two entries in $\mathsf{M}H$ is $\frac{4}{5}(a - c)$. It follows that the diagonal entries of $\mathsf{M}H$ converge to 1 with rate $(4/5)^m$, since M preserves trace. The result follows. $\square$

3.2 SG3

4 The Central Limit Theorem

Let $\mathcal{B}$ be the usual Borel sigma algebra for the shift space X. Let $\mathcal{F}_n$ be the sub-sigma-algebra defined by $\mathcal{F}_n = \sigma^{-n}\mathcal{B}$ such that $\mathcal{E}(X_n|\mathcal{F}_n) = 0$ and $\mathcal{E}(X_{n-1}|\mathcal{F}_n) = 0$, i.e., a reverse Martingale difference.

There is a standard statement of the Martingale Central Limit Theorem for independent random variables X_n with mean $\mathbb{E}(X_n) = 0$ and the same variance $\sigma^2 = [X_n^2]$.

Theorem 4 (Revere Martingale Central Limit Theorem)**.** *Consider a reverse Martingale difference. For $S_N = \sum_{n=0} X_n$, $N \geq 1$, the values $\frac{S_N}{\sqrt{N}}$ converges in distribution to the normal distribution centred at 0 and with variance σ^2.*

Proof. The statement appears, for example. J. Neveu, Mathematical foundations of the calculus of probability, Holden, Day, San Fran cisco, 1965 $\square$

There is actually an even stronger statement, of the Functional Central Limit Theorem (or Weak invariance Principle) which holds under the same hypotheses. [6]

In order to apply this to the g-measure ν we want to show the corresponding result with $X_n = f \circ \sigma^n$, where f is a Hölder continuous function satisfying $\int f d\nu = 0$.

We follow a classical approach of Gordin, described in Liverani, Carlangelo. Central limit theorem for deterministic systems. International Conference on Dynamical Systems (Montevideo, 1995), 56–75, Pitman Res. Notes Math. Ser., 362, Longman, Harlow, 1996. In fact it is convenient to replace f by $\overline{f} = f + u \circ \sigma - u$, for some suitable continuous function $u : \Sigma \to \mathbb{R}$. In particular, we can define

$$u := \sum_{n=0}^{\infty} \mathcal{L}^n f \in C^0(\Sigma)$$

where convergence is guaranteed since $\|\mathcal{L}^n f\|_\infty \leq C\beta^n$, for $n \geq 1$. We now see that

$$\begin{aligned}\mathcal{L}(\overline{f}) &= \mathcal{L}\left(f + u \circ \sigma - u\right) \\ &= f + \sum_{n=1}^{\infty} \mathcal{L}^n f - \sum_{n=0}^{\infty} \mathcal{L}^n f = 0.\end{aligned}$$

Since $\mathcal{L}1 = 1$ we see that $\mathbb{E}(\overline{f}|\mathcal{B}_1) = (\mathcal{L}\overline{f}) \circ \sigma = 0$. In particular, $S_n\overline{f} = \sum_{k=0}^{n-1} \overline{f} \circ \sigma^k$ is a Martingale and so by the Theorem we have that $\frac{S_N\overline{f}}{\sqrt{N}}$ converges in distribution to the normal distribution centred at 0 and with variance $\sigma^2 = \int (\overline{f})^2 d\mu$.

To relate this back to f by observing that

$$\begin{aligned}S_n\overline{f} &= \sum_{k=0}^{n-1} \overline{f} \circ \sigma^k \\ &= g \circ \sigma^n - g + \sum_{k=0}^{n-1} f \circ \sigma^k\end{aligned}$$

and thus

$$\frac{S_n\overline{f}}{\sqrt{n}} = \frac{S_n f}{\sqrt{n}} + \frac{g \circ \sigma^n - g}{\sqrt{n}}$$

In particular, since the last term tends to zero in distribution we see from the theorem that $\frac{S_N f}{\sqrt{N}}$ also converges in distribution to the normal distribution centred at 0 and with variance σ^2.

[6] It may well be the case that strong ASIP also hold following M. Denker and W. Philipp, Approximation by Brownian motion for Gibbs measures and flows under a function. Ergodic Theory Dynam. Systems 4 (1984), no. 4, 541–552.

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