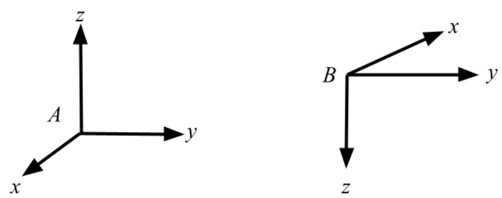
106B Background Assessment

Spring 2024

This short assessment is designed to test your background in some fundamental concepts required for 106B. You may refer to any material posted on the 106A website or any textbooks when completing this assessment, but may not use the open internet.

Problem 1: Rotation Matrices

Find the rotation matrix R_{ab} between the two orthonormal frames below:



Assume any intermediate rotations about the x, y, z axes are of angles that are integer multiples of $\frac{\pi}{2}$.

Solutions

$$R_{ab} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{1}$$

Problem 2: Matrix Exponential

1. If λ_i is an eigenvalue of $A \in \mathbb{R}^{n \times n}$, prove that e^{λ_i} is an eigenvalue of e^A.
2. Prove that if $\lambda_1, \lambda_2, ..., \lambda_n$ are the n potentially repeated eigenvalues of A, it is true that:

$$\det(e^A) = e^{\lambda_1 + \lambda_2 + ... + \lambda_n} \tag{2}$$

Solutions

1. If λ_i is an eigenvalue of A, then for an associated eigenvector x, $Ax = \lambda_i x$. Multiplying the series expansion of e^A by x:

$$\begin{aligned} e^A x &= Ix + Ax + \frac{A^2 x}{2!} + ... && (3) \\ &= x + \lambda_i x + \frac{\lambda_i^2 x}{2!} + ... && (4) \\ &= (1 + \lambda_i + \frac{\lambda_i^2}{2!} + ...)x && (5) \\ &= e^{\lambda_i} x && (6) \end{aligned}$$

Thus, e^{λ_i} is an eigenvalue of e^A. This completes the proof.

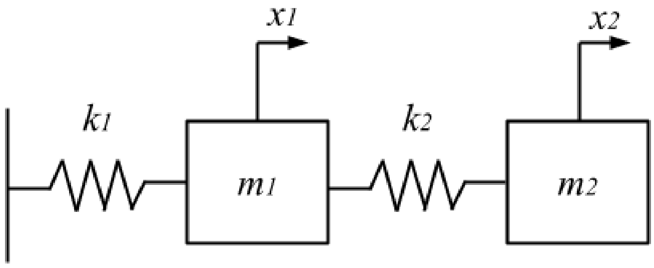
2. The determinant of a matrix is the product of its eigenvalues. Thus, if λ_i, $1 \leq i \leq n$ are the eigenvalues of A:

$$\det(e^A) = e^{\lambda_1} e^{\lambda_2} ... e^{\lambda_n} \tag{7}$$
$$= e^{\lambda_1 + \lambda_2 + ... + \lambda_n} \tag{8}$$

This completes the proof.

Problem 3: Dynamics

A system of two masses, m_1 and m_2 are attached by ideal springs to a rigid wall as follows:



The springs have constants k_1 and k_2 and the masses have positions x_1 and x_2. Assuming x_1 and x_2 are zero when the springs are unstretched, find the set of second order differential equations describing the motion of the masses. You may use either a Newtonian or Lagrangian approach. Assume that there are no frictional or gravitational forces.

Solutions

We can take a Newtonian approach to solve this problem (Lagrangian is just as good but will be quite a bit more work). Evaluating the equations of motion and the forces on the masses:

$$m_1 \ddot{x}_1 = -k_1 x_1 - k_2 x_1 + k_2 x_2 \tag{9}$$
$$m_2 \ddot{x}_2 = -k_2 x_2 + k_2 x_1 \tag{10}$$

Problem 4: Controls

Suppose a physical system is described by the following equations of motion:

$$m\ddot{x} = -mge_3 - \frac{1}{2}\rho C_d A ||\dot{x}||\dot{x} + f \tag{11}$$

Where $x \in \mathbb{R}^3$, $e_3 = [0, 0, 1]^T$, m, g, ρ, C_d, A are constant scalars, and $f \in \mathbb{R}^3$ is an input to the system. If $x_d \in \mathbb{R}^3$ is the desired state of the system, find an expression for f such that the system error $e = x_d - x$ is described by the following differential equation:

$$\ddot{e} + k_d \dot{e} + k_p e = 0 \tag{12}$$

Where $k_p, k_d \in \mathbb{R}$ are arbitrary scalar constants. You may leave your expression for f in terms of any of the variables above and their derivatives.

Solutions

Rearranging the provided equation for convenience:

$$m\ddot{x} + mge_3 + \frac{1}{2}\rho C_d A ||\dot{x}||\dot{x} = f \tag{13}$$

From this equation, we observe that the required feedforward term is:

$$f_{ff} = m\ddot{x}_d + mge_3 + \frac{1}{2}\rho C_d A||\dot{x}||\dot{x} \tag{14}$$

The desired feedback can be given by PD control:

$$f_{fb} = m(k_p e + k_d \dot{e}) \tag{15}$$

The total control input to the system is therefore:

$$f = f_{ff} + f_{fb} = m\ddot{x}_d + mge_3 + \frac{1}{2}\rho C_d A||\dot{x}||\dot{x} + m(k_p e + k_d \dot{e}) \tag{16}$$