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Statistical dynamics of wealth inequality in stochastic models of growth

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ABSTRACT

Understanding the statistical dynamics of growth and inequality is a fundamental challenge to ecology and society. Recent analyses of wealth and income in contemporary societies show that economic inequality is very dynamic and that individuals experience substantially different wealth growth rates over time. However, despite a fast-growing body of evidence for the importance of fluctuations, we still lack a general statistical theory for understanding the dynamical effects of heterogeneous growth across a population. Here we derive the statistical dynamics of correlated wealth growth rates in heterogeneous populations. We show that correlations between growth rate fluctuations at the individual level influence aggregate population growth, while only driving inequality on short time scales. We also find that growth rate fluctuations are a much stronger driver of long-term inequality than income volatility. Our findings show that the dynamical effects of statistical fluctuations in growth rates are critical for understanding the emergence of inequality over time and motivate a greater focus on the properties and endogenous origins of growth rates in stochastic environments.

1. Introduction

Exponential growth and broad inequality are general features of population dynamics in biology and society [1]. In biology, the maintenance of diverse populations is associated with greater biodiversity and with larger pools of variability enabling faster processes of evolution by natural selection [2]. In human societies, rates of long-term (economic) growth are much higher than in most ecosystems, often generating widening inequalities and leading to familiar stresses of social justice and equity [3,4].

Researchers have recently taken a special interest in the dynamics of economic inequality [1], applying multidisciplinary approaches that include historical data analyses [5] and searches for social, political, and economic mechanisms that generate, distribute, and rebalance wealth within populations [3,6]. Research in economics has emphasized the importance of heterogeneity in populations [4] as a source of widening wealth inequality. These heterogeneities are associated with a number of different population features such as the divergence of incomes between capital and labor [3], between management and workers within firms [7,8], and between people with different educational attainment [9]. These primarily empirical approaches point to the fundamental importance of diverging income growth for different

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subpopulations and to the need for rebalancing mechanisms if inequality is to remain controlled. However, we still do not have statistical theories of wealth dynamics applicable in general circumstances and, as a consequence, we lack a clear picture of how to concurrently manage growth and inequality.

To this end, research in physics has derived a number of results on the role of stochasticity as a source of inequality, on inter-agent exchanges (particularly in the mean-field limit), and proposed redistribution mechanisms in the context of stochastic geometric growth models [10–17]. The main focus of this work has been the design of wealth redistribution schemes towards creating long-time stationary limits [18–20] with parametrically controlled levels of inequality and whether such stationary solutions exist [21,22].

While these models have been used to fit data on wealth distributions [20,22–24], their specialized approaches have left open questions on the fundamental statistical dynamics underpinning the generation and allocation of wealth. For example, evidence suggests models following strict application of Gibrat's law (individual growth rates independent of wealth) cannot characterize the rapid emergence of inequality experienced in recent years [11,25], motivating explicit analysis of open-ended dynamical effects due to various sources of fluctuations and correlations. These developments support multiplicative stochastic growth as a starting point for modeling [26], but call for the consideration of statistical effects due to fluctuations and correlations in model parameters, especially growth rates.

To address these issues, we derive the time evolution of the distribution of resources, a proxy for wealth, in a statistical population of heterogeneous agents experiencing geometric random growth. We focus on the dynamic effects of growth rate fluctuations and their correlations with resources across agents and over time. We show that these two effects lead to two different dynamical time scales, which require different control measures so that inequality does not explode in a population over time. Specifically, we show that natural schemes to reduce inequality are subject to reversal over longer time scales because of the variability of growth rates in the population. We end by discussing the mechanisms that may simultaneously lead to sustained exponential growth and control of long-term sources of inequality and propose redistributing growth opportunities through agent-based learning in correlated stochastic environments.

2. Theory: Geometric Brownian motion in correlated populations

The fundamental model for the dynamics of resources in populations relies on stochastic exponential growth [25,26]. In its simplest form, known as geometric Brownian motion, agents generate wealth by (re)investing incomes net of costs. Crucially these models, unlike additive stochastic growth, generate population lognormal and power-law statistics, which characterize observed wealth and income statistics [27,28]

Specifically, the dynamics of stochastic multiplicative growth start by tracking the resources (wealth), $r(t)$, of a specific agent at time t. This quantity changes in time via the difference between an income, $y(t)$, and costs $c(t)$. The difference, $y - c$ (net income), is then defined to be proportional to resources, expressed as a (potentially r dependent) stochastic growth rate, $\eta(t) = [(y(t) - c(t)]/r(t)$, for $r > 0$. It follows that the time evolution of resources obeys a simple Gaussian multiplicative stochastic process: $\frac{dr}{dt} = \eta(t)r$, where $\eta(t)$ is a stochastic growth rate with temporal mean $\bar{\eta}$, and temporal variance σ^2, where the standard deviation, σ, is known as the volatility. When these two time averages are independent of r and t, this stochastic equation can be integrated via Itō calculus [26] to give

$$\ln \frac{r(t)}{r(0)} = \left(\bar{\eta} - \frac{\sigma^2}{2}\right)t + \sigma W(t), \tag{1}$$

where $\gamma \equiv \bar{\eta} - \sigma^2/2$ is the effective growth rate and r_0 the agent's resources at $t = 0$, $r_0 = r(0)$. The decrease in the mean growth rate $\bar{\eta}$ due to finite volatility is an important feature of multiplicative growth. The quantity $\sigma W(t)$ is a Wiener process with magnitude proportional to the volatility and units of $t^{-1/2}$.

2.1. Average growth rate in correlated populations

Under these circumstances, single-agent dynamics produce the population dynamics of a noninteracting community of agents with uniform growth rates and volatilities. That is, for homogeneous wealth value r and growth rate $\bar{\eta}$, the sample average growth over a finite population of size N is given by $\langle \bar{\eta} r\rangle_N \equiv \frac{1}{N}\sum_i \bar{\eta}_i r_i = \bar{\eta} r$. This basic case is unrealistic as wealth and income vary across real populations. To deal with this more general situation, we introduce heterogeneity to our population by sampling agents from distributions of $\bar{\eta}$, $\langle \bar{\eta}\rangle_N = G$, and r, $\langle r_0\rangle_N = \mu_0$. When the parameters are uncorrelated and sampled from independent distributions, the population average growth is the growth averaged over the population, $\langle \bar{\eta} r\rangle_N = G\mu_0$. However, when initial wealth r and growth rates $\bar{\eta}$ are correlated as historical wealth distribution analysis would suggest [5], growth of the average separates into the population mean term and a covariance term given by (Appendix)

$$\langle \bar{\eta} r\rangle_N = G\mu_0 + \mathrm{cov}_N(\bar{\eta}_i, r_i) \tag{2}$$

for parameter covariance cov_N.

We call the growth term of the correlated expectation value, $G' \equiv G + \mathrm{cov}_N\left(\bar{\eta}_i, \frac{r_i}{\mu_0}\right)$ the population averaged resource growth rate. This quantity can be calculated analytically for Gaussian distributed growth rates and log resources, $\ln r_0$, with

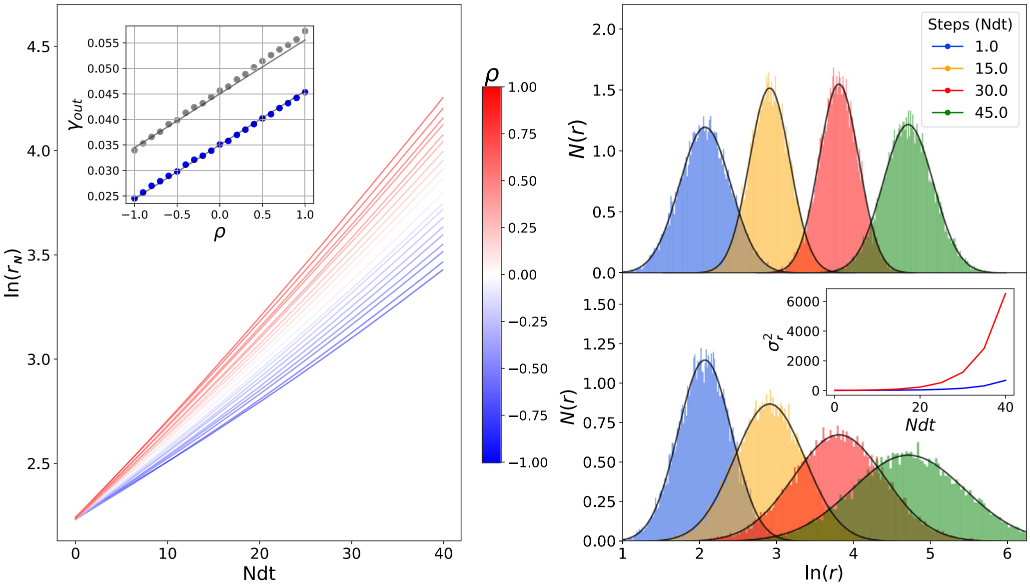

Fig. 1. Monte Carlo simulations of growth dynamics with correlated growth rates and initial resources. *Left:* Aggregate resource trajectories of populations with ρ ranging from -1 (blue) to 1 (red). *Inset*: The population-averaged effective growth rate (blue) scales linearly in ρ, following Eq. (4). The temporal averaged growth rate (gray) scales similarly. *Right:* Distributions of log wealth at different time steps for a regressive, $\rho = .75$ (bottom), and progressive, $\rho = -.75$ (top) population. Black curves represent normal distribution fits. The regressive assignments broaden more quickly, indicating that the populations are becoming more rapidly unequal. *Inset:* Variance in log resources of the progressive (blue) and regressive (red) populations over many time steps. Data are simulated from populations of 10^5 agents with Gaussian distributed initial resources of mean $\ln \mu_0 = 2$, mean growth rate $\bar{\eta} = .04$, and standard deviations $\sigma_{\bar{\eta}} = .015$, $\sigma_r = .682$.

population variances $\langle(\bar{\eta} - G)^2\rangle_N = \sigma_{\bar{\eta}}^2$ and $\langle(\ln r_0 - \ln \mu_0)^2\rangle_N = \sigma_r^2$. The result involves the variances of both quantities as well as the Pearson correlation coefficient between them, as (Appendix)

$$\mathrm{cov}_N\left(\bar{\eta}_i, \frac{r_i}{\mu_0}\right) = \rho\sigma_{\bar{\eta}}\sigma_r \exp[\sigma_r^2/2]. \tag{3}$$

The transformation to lognormal distributed resources is performed for convenience in later calculations, and introduces the exponential term to the resources in the covariance (if instead resources were normally distributed, the exponential term becomes unity). Thus, the population average effective resource growth rate, Γ', is expressed in terms of the model parameters as

$$\Gamma' = G - \frac{\sigma^2}{2} + \rho\sigma_{\bar{\eta}}\sigma_r \exp[\sigma_r^2/2], \tag{4}$$

where Γ is reserved for the mean effective growth rate $\Gamma \equiv G - \sigma^2/2$.

This expression shows that a positive covariance between initial log resources and growth rates results in a higher population averaged effective growth rate and vice versa. Naturally, associating higher-growth opportunities with wealthier individuals will exacerbate inequalities in the population over time, whereas the opposite would reduce inequality, at least over short time scales. As such, we denote population dynamics with positive correlation as *regressive*, and with negative correlation as *progressive*. We visualize the tradeoff between short-term change in inequality and growth through numerical simulations in Fig. 1.

2.1.1. Conditions for progressive population dynamics

We see that these simple considerations present an apparent paradox for any attempt to simultaneously maximize total wealth, a measure of average social welfare, and reduce inequality. Decreasing ρ, thereby making the societal dynamics more *progressive* results in a social opportunity cost in terms of a decline in average growth. This effect may even lead to negative exponential growth, and thus to a decline in societies starting out with low average growth and high volatility. One way out of this dilemma is for a progressive policy assignment to create, in ways to be specified, higher average growth rates than the regressive case.

To analyze this possibility, Eq. (4) introduces the threshold correlation below which the average effective resource growth rate becomes negative. Computed under the condition $\Gamma' = 0$, this critical correlation value (denoted with

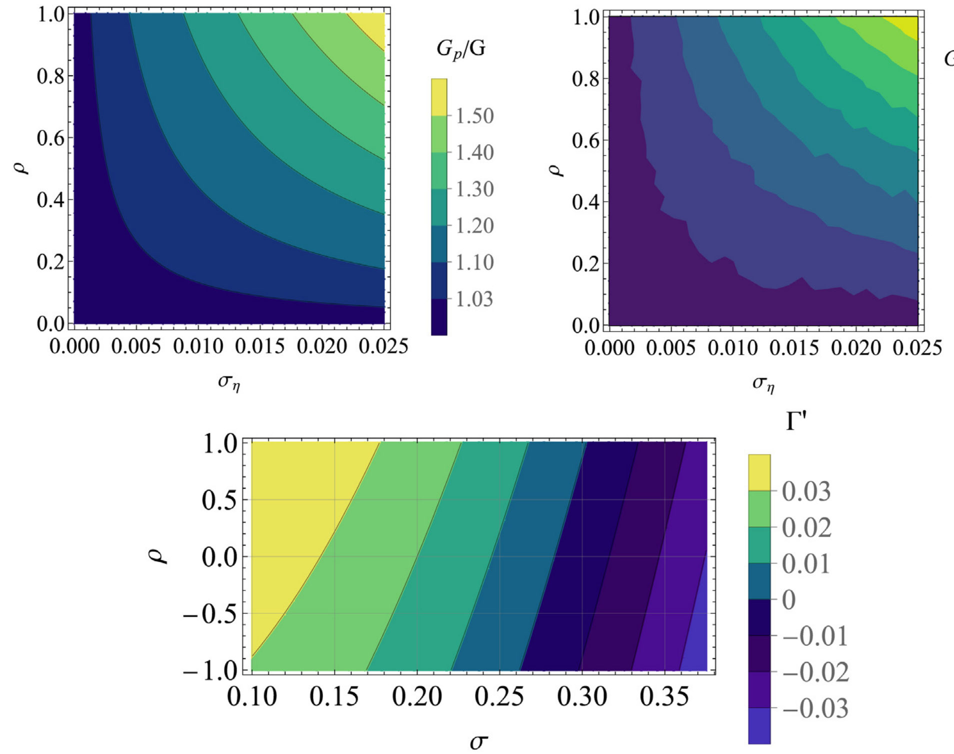

Fig. 2. Parameter spaces for progressive/regressive growth rate ratios across variances (top) and growth rate regimes across correlation and volatility values (bottom) *Upper Left:* Contour plot of G^p/G for correlations and growth rate variances with condition $\sigma_{\bar{\eta}} = \sigma^p_{\bar{\eta}}$. *Upper Right:* A Monte-Carlo simulation comparing the correlated average resource growth rates of populations with identical parameters up to a change in sign of ρ agrees with theory. To stay competitive, the progressive society requires a larger growth rate as either parameter increases. *Lower:* The effective growth rate for $\mu_0 = .04, \sigma_r = 1, \sigma_{\bar{\eta}} = .025$ transitions from positive to negative along the correlation axis at high volatility ($\sigma \approx .275$).

subscript c) is

$$\rho_c = \frac{\frac{\sigma^2}{2} - G}{\sigma_r \sigma_{\bar{\eta}} \exp[\sigma_r^2/2]}. \tag{5}$$

We see that the threshold value increases with volatility, decreases with average resource growth rate, and decreases in magnitude with variance in either growth rate or resource variance. Progressive assignments are manifestly less feasible in more volatile and heterogeneous societies. Similarly, the critical volatility marking the crossover from positive to negative average growth, denoted σ_c, can be determined by rearranging Eq. (5) for σ. This relationship is plotted in Fig. 2, and gives a theoretical benchmark volatility for sustainable growth in a population for a set of distribution and growth parameters. Critical volatility decreases as a society becomes more progressive.

We can directly compare progressive and regressive assignments by computing the ratio of growth rates between a population with population effective growth rate Γ', and its progressive counterpart, Γ'_p with distribution parameters with superscript p. We set $\Gamma' = \Gamma'_p$ and rearrange for the ratio of mean growth rates between the population G^p/G. In the simplest case, where we assume identical initial population conditions up to the variance in growth rates with opposite correlation coefficients, $\rho^p = -\rho > 0$, the volatilities cancel out and the ratio is given by

$$\frac{G^p}{G} = 1 + \frac{1}{G}\rho\sigma_r(\sigma_{\bar{\eta}} + \sigma^p_{\bar{\eta}})\exp[\sigma_r^2/2]. \tag{6}$$

Thus, a progressive arrangement must have a larger average growth rate in order to achieve a population effective growth rate equal to its regressive counterpart. This difference is greater in populations with stronger initial inequality, σ_r, and means that a higher progressive growth rate, G^p, is required to overcome larger growth rate variances, $\sigma_{\bar{\eta}}+\sigma^p_{\bar{\eta}}$. Fig. 2 demonstrates this equation's agreement with population Monte-Carlo simulations in which r_0 is normally distributed.

2.2. Fokker Planck solution

So far, we have shown that the instantaneous population average growth rate collects a correction from covariances between resources and growth rates. This introduces a trade-off between managing inequality in the short run and maximizing overall growth which can only be resolved, at each time interval, if progressive assignments also result in higher average growth rates.

But what happens over the longer term, as growth rate fluctuations persist? To answer this question, we consider the solution to the Fokker–Planck equation (FPE) for the time-dependent probability of resources for each agent. The FPE for the wealth distribution $P(r,t)$ for geometric Brownian motion is given by

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial r}\left[\gamma r P(r,t)\right] + \frac{\partial^2}{\partial r^2}\left[\frac{\sigma^2}{2} r^2 P(r,t)\right]. \tag{7}$$

2.2.1. Dynamics of a homogeneous population

The solution with initial conditions $r(0) = r_0$ (a delta function at the individual level) is a lognormal distribution [29] with time-dependent mean and variance. This can be re-written as a Gaussian distribution of $\ln r$ as

$$P(\ln r, t|\gamma, \ln r_0) = \frac{d\ln r}{\sqrt{2\pi\sigma^2 t}} \exp\left[-\frac{(\ln r - \ln r_0 - \gamma t)^2}{2\sigma^2 t}\right], \tag{8}$$

where at early times, the expected resources are given by the linear equation $\ln \bar{r}(t) = \ln r_0 + \gamma t$, which asymptotically approach γt. The variance of log resources increases linearly in time, $\sigma^2 t$, which shows that individual wealth also becomes more uncertain, and population wealth becomes more unequal as a function of time. The critical volatility at which growth ceases is, as before, given by $\sigma_c = \sqrt{2\bar{\eta}}$. We call Eq. (8) the *homogeneous* population solution.

Changing variables and working explicitly with time-averaged growth rates yields further insight into the dynamics. Dividing each factor of the exponential by t, and performing the variable transformation $\frac{1}{t}\ln\frac{r}{r_0} \to \phi$, $td\phi = d\ln r$ yields the distribution of growth rates

$$P(\phi, t|\gamma, \sigma) = \frac{d\phi}{\sqrt{2\pi\sigma^2/t}} \exp\left[-\frac{(\phi - \gamma)^2}{2\sigma^2/t}\right]. \tag{9}$$

This shows explicitly that the statistics of the time-averaged growth rates are simpler than those of resources. The average growth rate becomes stationary at long times as the mean remains constant, while growth rate fluctuations over the population decrease in time as t^{-1}, converging in distribution to a delta function on a time scale $t \gg \sigma^2$.

2.2.2. Dynamics of a heterogeneous population

This suggests a simple picture emerging at the population level; however, covarying the growth rates among agents in the population will introduce some complications. The most direct route to assess these effects follows by positing Gaussian distributions on growth rates, from the asymptotic behavior of growth rate distributions [26,30,31], and initial log resources across the population, from the solution to the FPE.

To obtain the dynamical solution for the distribution of resources in the population, we convolve Eq. (8) with a bivariate static Gaussian as described in 2.1.1. If we hold volatilities constant such that the marginal $P(\gamma)d\gamma = P(\bar{\eta} - \sigma^2/2)d\bar{\eta}$, we can perform the convolution over effective growth rates via the equation

$$P(\ln r, t) = \int P(\ln r, t|\gamma, \ln r_0) P(\ln r_0, \gamma)\, d\ln r_0\, d\gamma. \tag{10}$$

The calculation is a straightforward Gaussian. The growth rate terms of the bivariate Gaussian distribution equation can be factored out of the resource integral by completing the square. The resource and growth rate integrals are then successively evaluated using Gaussian integration, leading to

$$P(\ln r, t) = \frac{d\ln r}{\sqrt{2\pi\,\Sigma^2}} \exp\left[-\frac{(\ln r - \ln \mu_0 - \Gamma t)^2}{2\Sigma^2}\right], \tag{11}$$

where $\Sigma^2 = \sigma_r^2 + (\sigma^2 + 2\sigma_r\sigma_{\bar{\eta}}\rho)t + \sigma_{\bar{\eta}}^2 t^2$. The covariance matrix is thus deduced as

$$K_\Sigma = \begin{pmatrix} \sigma_r^2 & 0 & \sigma_r\sigma_{\bar{\eta}}\rho t \\ 0 & \sigma^2 t & 0 \\ \sigma_{\bar{\eta}}\sigma_r\rho t & 0 & \sigma_{\bar{\eta}}^2 t^2 \end{pmatrix}. \tag{12}$$

We call Eq. (11) the *heterogeneous* population solution. The quadratic expression in time for variance is characterized by two different timescales, $t_{c1} = \sigma_r^2/(\sigma^2 + 2\rho\sigma_r\sigma_{\bar{\eta}})$ and $t_{c2} = (\sigma^2 + 2\rho\sigma_r\sigma_{\bar{\eta}})/\sigma_{\bar{\eta}}^2$. The natural timescale in this analysis is years [3], such that *early* times are on the order of a few years, *intermediate* over a few decades, and *long* timescales are several decades to centuries. Effects over long timescales can thus be thought of as generational, although we do not incorporate life cycle dynamics in this analysis. Annual growth rates are typically on the order of a few percent a year ($\gamma \simeq 10^{-2}$/yr), whereas log resources vary in the population on a typical scale of $\delta \ln r \simeq 10^1$, so the magnitude of $\sigma_{\bar{\eta}}$ is naturally an order of magnitude smaller than σ_r. This results in $t_{c1} < t_{c2}$, producing three distinct dynamical regimes for the population variance and hence inequality dynamics. For early times $t < t_{c1}$, the variance of resources in the population is given approximately by its initial condition $\Sigma^2 \simeq \sigma_r^2$. For intermediate times $t_{c1} < t < t_{c2}$, $\Sigma^2 \simeq (\sigma^2 + 2\rho\sigma_r\sigma_{\bar{\eta}})t$. The intermediate regime introduces the explicit decrease in inequality from a progressive assignment, so long as $\sigma^2 < 2\rho\sigma_r\sigma_{\bar{\eta}}$. However, this benefit is short-lived, as for later times $t > t_{c2}$, $\Sigma^2 \simeq \sigma_{\bar{\eta}}^2 t^2$, and variance invariably explodes due to fluctuations in growth rates. Note that the long-time regime does not require biases between resources and growth rates; it persists under the weaker conditions of a finite variance in growth rates and appears at earlier times for larger growth rate variances.

Finally, we observe that the asymptotic population averaged resource growth rate distribution simplifies to a Gaussian

$$\lim_{t\to\infty} P(\phi, t) = \frac{d\phi}{\sqrt{2\pi\sigma_{\bar{\eta}}^2}} \exp\left[-\frac{(\phi - \Gamma)^2}{2\sigma_{\bar{\eta}}^2}\right], \tag{13}$$

in contrast to the case of a homogeneous population, reflecting the Gaussian distribution of growth rates in the population. Over the long term, population dynamics are thus dominated by fluctuations in growth rates, leading to wider inequality the larger the growth rate variance.

This effect introduces a general mechanism that can account for increases in inequality that occur over intermediate and long timescales [25], and incorporates findings that empirical growth rates are heterogeneous and correlate with wealth over time [32–34]. It also asks that we shift our focus from the dynamics of wealth itself to the variations of growth rates as the drivers of long-term inequality.

3. Results: Dynamics of inequality metrics

We now analyze the detailed dynamics of inequality using two standard metrics. First, the Gini coefficient, G_{ini}, is the most common metric measuring inequality within a population. It ranges from $G_{\mathrm{ini}} = 0$ for equally distributed resources, and trends towards 1 as a society becomes maximally unequal. For lognormally distributed resources in the infinite population limit, $G_{ini} = \mathrm{Erf}\left[\Sigma/2\right]$. Second, we examine the social cost of high growth on inequality, a topic of interest and debate in the social sciences [35,36]. Using the coefficient of variation (CV), denoted c_v, we compare how quickly the standard deviation of a distribution of form (11) increases relative to its mean [37]. The dynamics are given by $c_v = \Sigma/(\ln\mu_0 + \Gamma t)$, and should be governed over long times by the linear time ordering of both the mean and standard deviation terms. A more equitable economy would decrease c_v; growing without increasing inequality. In the egalitarian limit, $c_v = 0$ as every agent has the same resources, and variance is zero. Conversely, an exploding c_v in the positive or negative direction indicates increasing inequality, marked by positive or negative aggregate growth respectively. From Eq. (8), without dynamical redistribution measures, societies will invariably become maximally unequal as multiplicative dynamics are dominated by fluctuations in earnings [1].

We explore the dynamics of homogeneous and heterogeneous populations by comparing time-series trajectories of c_v and G_{ini} across several values of correlation coefficient and volatility. Fig. 3 demonstrates the impact of heterogeneity on inequality (Gini), and its relative effects on growth (CV). In homogeneous populations, volatility spurs inequality from uniform initial conditions causing c_v to initially increase, peak, then decrease as resources increase in $t^{1/2}$ faster than the population standard deviation. The quantity G_{ini} asymptotically approaches 1 in all parameter cases. Heterogeneity causes c_v to increase across all cases with asymptotically constant behavior as the time leading terms of the numerator and denominator of c_v cancel. Heterogeneity causes G_{ini} to generally increase more rapidly. The differences in progressive and regressive G_{ini} are negligible after intermediate times, suggesting that the initial configuration has little effect on its long-term macroscopic dynamics. Volatility strongly determines the value of c_v for homogeneous and heterogeneous populations, while only significantly affecting the behavior of G_{ini} in the homogeneous case.

The dynamics of G_{ini} are thus straightforward, but the dependence of c_v on both growth rates and dynamical variance produces interesting dynamics. To probe the dynamics of c_v, we produce quasi-phase diagrams displaying $\partial_t c_v$ for a continuous spectrum of volatilities over time. Fig. 4 demonstrates these dynamical regimes for the homogeneous and heterogeneous progressive and regressive cases. It shows the magnitude of the initial increase in c_v scales with σ, as does the magnitude of the eventual decrease. The eventual decrease is weaker in regressive populations, occurring only at medium volatility values, and is nonexistent for any volatility value of the progressive population. Furthermore, they reveal a divergent regime past the critical volatility, where resources decrease while inequality continues to increase.

4. Discussion

These results show that growth rate fluctuations pose a general challenge to any proposed mechanism to address widening inequality that must be addressed independently of volatility in single-agent trajectories. Specifically, dynamically balancing growth rates is required to control long-term inequality dynamics. Methods seeking to both reduce growth rate fluctuations and preserve growth require a theory for the origins of wealth growth rate statistics. Extending the stochastic growth models commonly used to include environmental opportunities and agent-based learning, in ways analogous to gambling [38] and portfolio theory [39], naturally produces situations where individual growth rates become distinct and history-dependent but also where average growth rates can be maximized and variances minimized over time [26,38,40]. In this sense, growth rate redistribution takes the form of investment in learning across the population,

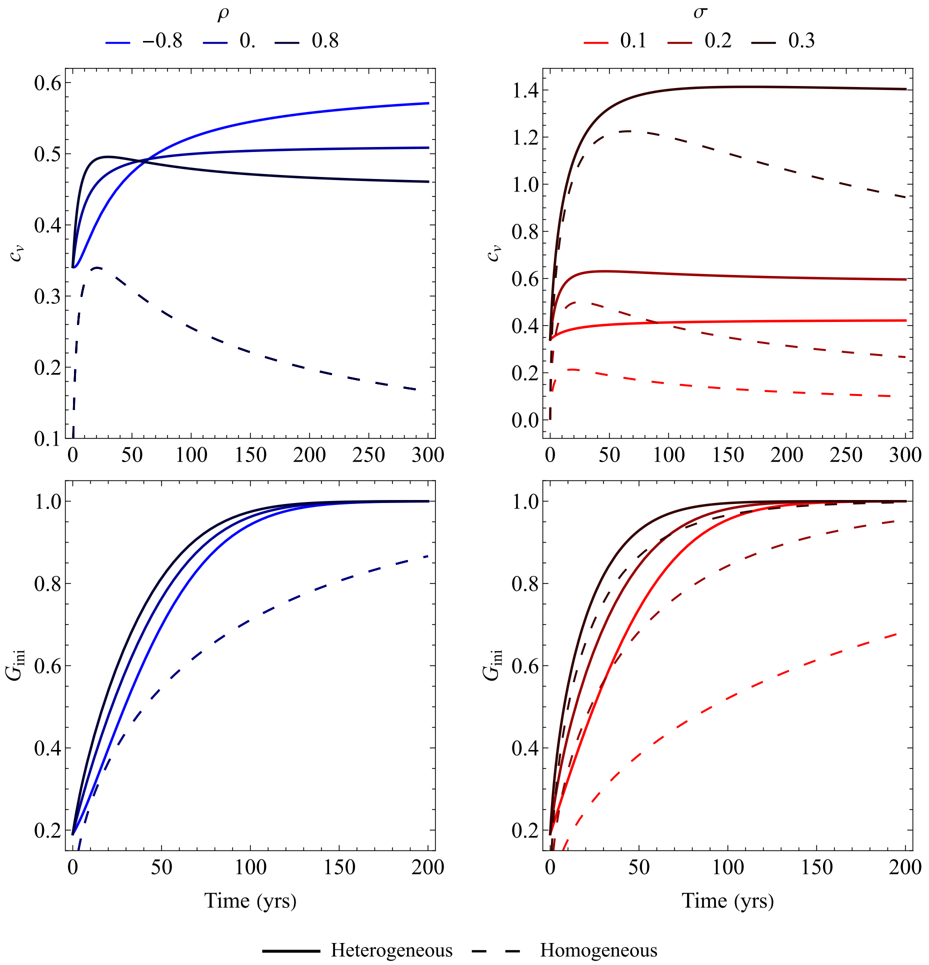

Fig. 3. Trajectories of inequality metrics in various parameter regimes; $\sigma = .15, \sigma_r = .341, \ln \mu_0 = 1, G = .06, \sigma_{\tilde{\eta}} = .025$. See legend for line type mapping. *Left:* Plots for low volatility populations with different correlation coefficient values. For $\rho > 0$, c_v jumps higher than in an uncorrelated society, while c_v temporarily decreases than increases for $\rho < 0$. All heterogeneous trajectories approach similar constant values, while the homogeneous population initially increases, then decreases over long times. At early times, G_{ini} increases more quickly for $\rho > 0$. In all cases, variances in growth rates drive G_{ini} to 1 more rapidly than in homogeneous populations. *Right:* Regressive populations ($\rho = .75$) at different values of volatility. A higher volatility causes a more rapid increase in G_{ini} in both population configurations, and volatility positively influences the peak and asymptotic values of c_v. Progressive assignments hurt long-term growth while marginally affecting long-term inequality dynamics, while heterogeneities play a dominant role in accelerating the emergence of inequality.

through for example education, which over time reduces disparities in growth rates. In such a dynamical picture, initial correlations will also take on dynamic properties in ways that we stated anticipating here. In recognizing recent research on the effects of heterogeneous and dynamical growth on distributions of wealth [29,41–44], we seek a theoretical framework for wealth dynamics that both complements these phenomenological approaches and incorporates strategic agent behavior in statistical environments. These topics will be presented in future work.

Another natural, next step would be to fit this model to data. Data fits require redistribution terms to prevent the distribution from diverging too rapidly and to reflect redistribution in real societies [20,22,23]. Doing so requires a model, if not a theory for how to dynamically redistribute wealth in addition to growth rates in a heterogeneous population. This topic is beyond the scope of this research.

5. Conclusion

In summary, we explored the effects of fluctuations in wealth growth rates and wealth statistics in the temporal development of inequality in a population of heterogeneous agents. We set up the general problem in the context of multiplicative random growth and derived closed-form expressions for how the population inequality changes over time, thus identifying the most important parameters at early and later times. We found that population variances in growth rates, their correlations with resources, and individual temporal volatility are the primary drivers of inequality in general

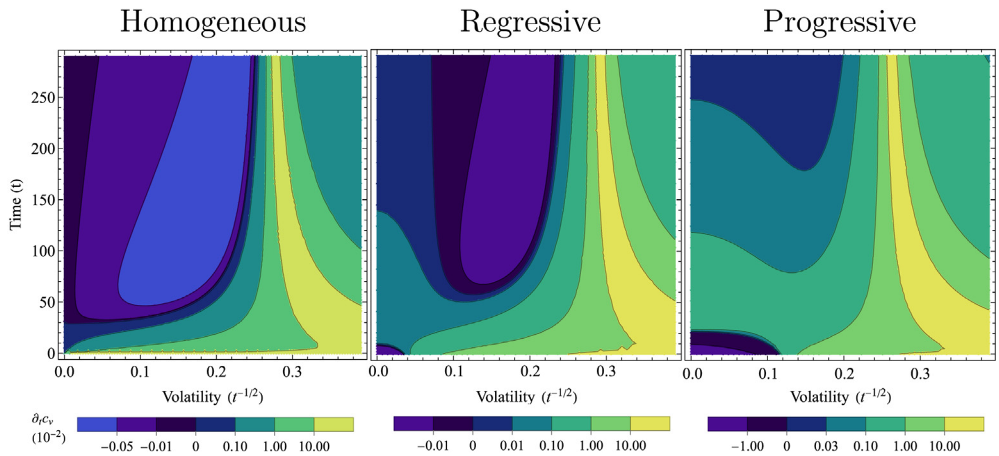

Fig. 4. Contour plots for $\partial_t c_v$ in identical populations to Fig. 3, with $\bar{\eta} = .035$. At low volatilities, homogeneous c_v initially increases, then decreases with magnitude increasing in σ. For $\rho > 0$, this behavior persists following initial times of high increase in c_v. The initial decrease observed for $\rho < 0$ only occurs at low volatility, and c_v does not decrease in later times, as demonstrated in Fig. 3. Past the homogeneous critical volatility, given by $\sigma_c \approx .265$, c_v blows up as the negative growth rate drives average log resources through zero to a negative value, and is seen by the bright yellow bands. In this regime of instability, resources continue to decrease as inequality grows.

situations. The sustained presence of variance in growth rates in a population produces a dominant effect on long-term inequality. While redistribution methods, such as those explored by mean-field or other more complex network exchange models [11,18,19] reduce the dynamical impacts of volatility, we have shown that directly addressing growth rate variance in a population is a more fundamental and general requirement for arresting or reversing increases in wealth inequality.

CRediT authorship contribution statement

Jordan T. Kemp: Methodology, Software, Validation, Investigation, Writing – original draft, Revising, Visualization, Funding. **Luís M.A. Bettencourt:** Conceptualization, Validation, Writing – original draft, Revising, Supervision, Project administration.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Jordan Kemp reports financial support was provided by National Science Foundation.

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Appendix. Parameter covariances

We compute the mean population effective growth rate over a population of size N by separating the expected growth rate into a mean and covariance term

$$\langle \bar{\eta} r \rangle_N = \frac{1}{N}\sum_{j=1}^{N} \bar{\eta}_j r_j, \tag{A.1}$$

$$= \frac{1}{N}\sum_{j=1}^{N} (\bar{\eta}_j - G) r_j + \frac{1}{N}\sum_{j=1}^{N} G r_j, \tag{A.2}$$

$$= G\mu_0 + \mathrm{cov}_N(\bar{\eta}_i, r_i), \tag{A.3}$$

where the population growth rate can be factored out as $G' = G + \mathrm{cov}_N(\bar{\eta}_i, \frac{r_i}{\mu_0})$.

Assuming r is a lognormal distributed quantity, then so is r/μ_0. Define the normally distributed variable $y \equiv \ln r/\mu_0 \sim \mathcal{N}(0, \sigma_r)$, then the covariance is equal to $\mathrm{cov}_N(\bar{\eta}, e^y)$. The first moment is calculated

$$\langle \bar{\eta} e^y \rangle_N = \int d\bar{\eta} dy \bar{\eta} e^y P(\bar{\eta}, y), \tag{A.4}$$

$$= \int dy e^y \langle \bar{\eta} \rangle_{Y=y} = \int dy e^y (G + \rho \frac{\sigma_{\bar{\eta}}}{\sigma_y} y), \tag{A.5}$$

$$= G\langle e^y \rangle_N + \rho \frac{\sigma_{\bar{\eta}}}{\sigma_y} \langle y e^y \rangle_N = (G + \rho \sigma_{\bar{\eta}} \sigma_y) e^{\sigma_y^2/2}. \tag{A.6}$$

where the expectation value of ye^y is calculated in [45]. With an expectation value defined, the covariance becomes

$$\mathrm{cov}(\bar{\eta}, e^y) = \langle \bar{\eta} e^{\bar{\eta}} \rangle_N - \langle \bar{\eta} \rangle_N \langle e^y \rangle_N, \tag{A.7}$$

$$= (G + \rho \sigma_{\bar{\eta}} \sigma_y) e^{\sigma_y^2/2} - G e^{\sigma_y^2/2}, \tag{A.8}$$

$$= \rho \sigma_{\bar{\eta}} \sigma_y e^{\sigma_y^2/2}. \tag{A.9}$$

In the main text, we define σ_y as the variance in lognormal resources, σ_r.

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