The Standard Model

Part III Lent 2020

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1 Introduction and History

Prerequisites

It is necessary to have attended the *Quantum Field Theory* and the *Symmetries, Fields, and Particles* courses, or to be familiar with the material covered in them. It would be advantageous to attend the *Advanced Quantum Field Theory* course during the same term as this course, or to study renormalisation and non-abelian gauge fixing.

1.1 Introduction

Definition 1 (standard model)**:** A theoretical physics construction (theory, model) that describes all known elementary particles and their interactions based on relativistic quantum field theory (QFT).

The Standard Model of particle physics is the most successful application of QFT we currently have. Based on the gauge group $SU(3)\times SU(2)\times U(1)$, it accurately describes, at the time of writing, all experimental measurements involving strong, weak, and electromagnetic interactions.

Ingredients

(i) spacetime: $3+1$ dimensional Minkowski space

symmetry: Poincaré group

(ii) particles:

spin $s=0$ Higgs

spin $s=1/2$ three families of quarks and leptons

(iii) interactions:

$s = 1$ three gauge interactions

$s = 1$ gravity[1]

Gauge (local) symmetry: $SU(3)_C \times SU(2)_L \times U(1)_Y \xrightarrow[Breaking]{Symmetry} SU(3)_C \times U(1)_{EM}$

C color: strong

L left: electroweak

Y hypercharge

These are related via $Q = T_3 + Y$.

Particle representations[2]:

- Quarks and Leptons: $\overbrace{3}^{\text{families (flavour)}} [\underbrace{(3,2;\frac{1}{6})}_{Q_L} + \underbrace{(\bar{3},1;-\frac{2}{3})}_{U_R} + \underbrace{(\bar{3},1;\frac{1}{3})}_{d_R} + \underbrace{(1,2;-\frac{1}{2})}_{L_L} + \underbrace{(1,1;1)}_{e_R} + \underbrace{(1,1;0)}_{\nu_R}]$
- Higgs: $(1,2;-\frac{1}{2})$
- Gauge: $\underbrace{(8,1;0)}_{gluons} + \underbrace{(1,3;0)}_{W^{\pm},\mathbb{Z}} + \underbrace{(1,1;0)}_{\gamma}$

Comments

- interactions given by QFT
- main tool: symmetry
- total symmetry: spacetime $\otimes$ internal (gauge)[3]
- also accidental (global) symmetries $\sim$ baryon + lepton number
- plus approximate (flavour) symmetries:
- very rigid: $\sum Y = \sum Y^3 = 0$[4], $\#3 = \#\bar{3}$, $\#2$ even
- rich structure (3 phases: Coulomb, Higgs, confining)

[1] as important as it is, we will not be concerned with gravity for most of this course
[2] numbers tell us representations under $(C, L; Y)$
[3] Theorem: cannot mix these two symmetries. Supersymmetry provides a way around this.
[4] gravitational anomaly

Motivation

Why to learn about the SM?

- It is fundamental.
- It is based on elegant principles of symmetry.
- It is true!
  - outstanding predictions: ($\mathbb{Z}^0, W^\pm$, Higgs, $\ldots$)
  - precision tests:
    anomalous magnetic dipole moment of the electron:
    $$a = \frac{g-2}{2} = (1159.65218091 \pm 0.00000026) \times 10^{-6} \tag{1.1}$$
    fine structure constant (at $E \ll 10^3$GeV):
    $$\alpha^{-1} = \frac{\hbar c}{e^2} = 137.035999084(21) \tag{1.2}$$
- It is the best test of QFT.
- It is incomplete!

Take another look at the quarks and leptons. For $Q_L : (3, 2, \frac{1}{6})$ the second entry tells us that these are doublets under $SU(2)$. This means that we have $Q_L = \begin{pmatrix} v_L \\ d_L \end{pmatrix}$.

1.2 History

Weinberg has a good paper on advice he gives to young researchers. One of the advices is to study history and the history of physics in particular. This is because it gives us some sense of how physical theories developed and also that it makes us feel part of a bigger development in the pursuit of knowledge, no matter how small our contributions may seem.

t < 20$^{\text{th}}$ century: only two interactions (gravity and electromagnetism)

discreteness of matter not established

1896 Radioactivity (Becquerel, Pierre and Marie Curie, Rutherford) α, β, γ rays

This was a big discovery at the time; there is no inherent stability in nature! This was a manifestation that pointed to the existence of other interactions.

1897 J. J. Thomson: discovered the *electron* (e^-) and measured e/m a few hundred meters from where we are right now in Cambridge (close to The Eagle so you can enjoy that on your visit too). This was the first particle ever discovered, marking the beginning of particle physics.

1900's **1900-1930** Quantum mechanics developed. Probably the biggest revolution ever in science.

1905 Special relativity. These two still are the two basic theories to study in nature. The nature of quantum mechanics also implies that light behaves as a particle, which we now know as the photon.

In the same decade, Rutherford's group also discovered the atom.

1910's Francis Aston (1919) defined the 'whole number rule', for the ratio of different atomic nuclei to the hydrogen mass. This led to the discovery of the proton.

Cosmic rays were studied, in particular by using cloud chambers.

Einstein's theory of general relativity.

1920's Bose, Fermi statistics.

Beginning of QFT (Jordan, Heisenberg, Dirac, ...)

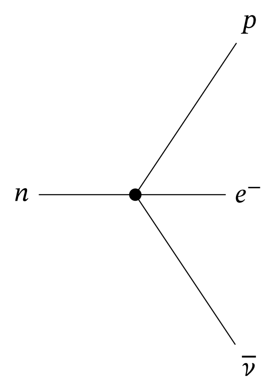
Dirac equation. This predicted a positive particle, which he thought could be the proton ...

1930's …but then came to predict that this is the *positron* (e^+), which he just squeezed into the introduction of his paper on magnetic monopoles (1931).

1932 Anderson[1] discovered it.

1932 Chadwick discovered the *neutron*.

1930 Pauli predicted the *neutrino*: β-decay: $n \to p + e^- + \overline{\nu}$. Fermi described this in terms of a four-point field theory



(1.3)

1934 Yukawa theory of strong interactions: scalar mediators of strong interactions (Pions). Potential $V(r) \sim e^{-mr}/r$, where $m \sim 100$MeV is the pion mass. This explained the short range and kept field theory going, so people started to search for this new particle.

1936 Anderson discovered the *muon* ($m \sim 100$MeV).

1932 Heisenberg, 1936 (Condon et al.) introduced isospin $n \leftrightarrow p$. He thought that in the same way that the electron has spin-up and spin-down, the proton and neutron have such similar properties that they also have an internal symmetry.

1940's The history of physics is different to the history you learn at school; at much happens in physics during war-time.

1947 Lamb shift, QED (Schwinger, Feynman + Tomonaga + Dyson)

Pions π were discovered, explaining why the naive picture of Yukawa made sense.

1950's

- A time of great optimism. Particle accelerators and bubble chambers were built (E > MeV). People say that the 50's were a decade of wealth; the numbers of particles were also very rich. Dozens of new particles were discovered (kaons, hyperons, …), mostly strongly interacting (*hadrons*), which are now classified into mesons (bosons) and baryons (fermions).
- Strangeness (Gell-Mann, Nishijima, Pais)
- Parity Violation (Lee and Yang) in 1936, Wu discovered it in 1957

[1] There were multiple people who arguably should get some more credit for this. Blackett discovered the positron but did not publish it fast enough. There were also a Russian scientist and a graduate student at CalTech who did similar discoveries.

- Discovery of (anti-)neutrino (Cowan- Reines, 1956)
- $V - A$ property of weak interactions (Marshak, Sudarshan, 1957)[1]
- Pentecorvo; neutrino oscialltions
- Yang–Mills 1954, non-Abelian gauge theory. In QED, there is a $U(1)$ gauge theory giving a massless photon. In Yang–Mills theory, with a greater group such as $SU(2)$, that this should give further massless / long-range particles. But these have never been seen so Pauli predicted correctly that this theory was not relevant to nature.

[1] Four experiements seemed to deny their theory. However, the theory was so strong that they were convinced that these experiments must have been wrong. All four actually turned out to be wrong.

1960's **1961** Eightfold way (Gell-Mann, Neemann)

Put order to zoo of discovered particles by considering representations of $SU(3)_{\text{flavour}}$. By con-

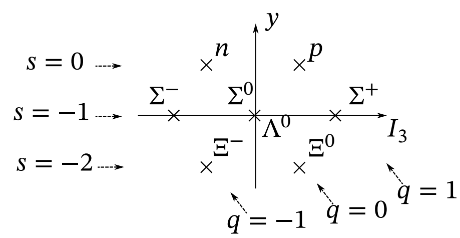


Figure 1.1: The eightfold way is the 8-dimensional representation of $SU(3)$.

sidering the 10-dimensional representation of $SU(3)$, depicted in Fig. 1.2 the Ω^- was predicted.

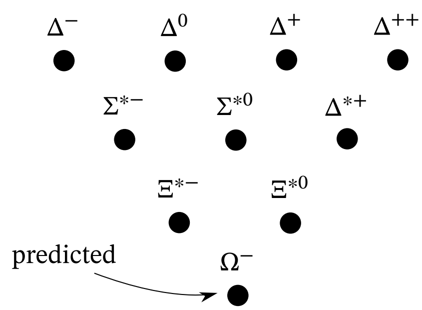


Figure 1.2: The 10-dimensional representation of $SU(3)$.

1964 Gell-Mann, Zweig came up with the theory of *quarks*. This theory was not accepted at the time since three quarks needed to be in the same state for some particles, violating Pauli's exclusion principle.

- $3 \oplus \overline{3} \to$ Mesons $s = 0$; $3 \otimes \overline{3} = 8 + 1$
- $3 \oplus 3 \otimes 3 \to$ baryons $s = \frac{1}{2}$; $3 \otimes 3 \otimes 3 = 10 + 8 + 8 + 1$

1964 Greenberg, 1965 (Nambu and Han) → colour

1967 Deep inelastic scattering. Evidence for substructure in the proton nucleus.

1961 Symmetry breaking (Nambu, Goldstone, Salam, Weinberg), Goldstone bosons (massless)

1964 Higgs Mechanism (Higgs, Brout, Englert, Kibble, Guralnik, Haden)

If the broken symmetry is local, then

- the gauge field is massive
- the Goldstone boson is is eaten and leaves behind a physical massive particle (Higgs)

The problem that Pauli pointed out to Yang and Mills is solved! Now you can have non-Abelian gauge symmetries, and broken symmetries.

1967-8 Weinberg, Salam, (Ward) tried non-Abelian gauge theory for the strong interaction, which failed. Trying it for the weak interaction gave Electroweak unification

$$\underbrace{SU(2)}_{L}\times\underbrace{U(1)}_{Y} \to \underbrace{U(1)}_{\text{EM}} \tag{1.4}$$

(Glashow 1962 identified $SU(2)\times U(1)$)

1964 experimental discovery of CP violation (Cronin, Fitsch) $\Rightarrow$ particle $\leftrightarrow$ antiparticle

1970's Glashow–Illoporlos–Maiani (GIM) mechanism. Explain no FCNC $\Rightarrow$ new quark: *charm* c. As such, the magic number of three, leading Gell-Mann to quarks, is not magic at all. The previous symmetry was only approximate, which was not noticed since c is very massive. In hindsight it was obvious that we needed a fourth quark:

1969 Jackiw–Bell–Adler; Anomalies. Need partner of s: $\begin{pmatrix} c \\ s \end{pmatrix}_L$.

1973 • weak neutral currents discovered

- Asymptotic Freedom (Gross, Wilczek, Politzer)

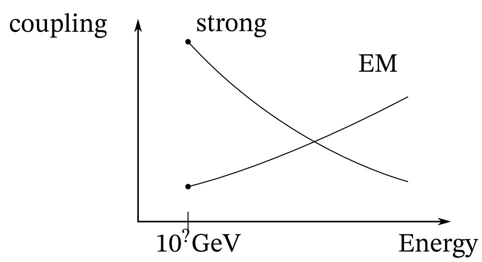


Figure 1.3: The running coupling gives hope for unification.

1974 J/ψ discovered → *charm*

1975-9 jets (quarks, gluons), for instance $e^+e^- \to qq$ gives 2 jets, but $e^+e^- \to qgq$ gives 3 jets.

$$R = \frac{e^+e^- \to \text{hadrons}}{e^+e^- \to \text{muons}} = \frac{33}{9} \tag{1.5}$$

depends on the number of colours. This gave evidence for 3 colours, confirming the idea of quarks.

1980's **1983** $Z^0, W^\pm$ discovered

1990's **1995** *top quark* discovered. This was not a surprise since people already knew about the bottom quark, which needed a partner. We end up with three families

$$\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix} \tag{1.6}$$

2000's *Tau neutrino*

2012 Higgs!

We are lucky to be taking this course in a time where the standard model is essentially solved. In this course we will see that this structure is essentially forced on us. The structure is very rigid.

2 Spacetime Symmetries

The symmetries we have are spacetime $\otimes$ internal (1967 Coleman–Mandula). In particular, the spacetime symmetry is the Poincaré symmetry.

2.1 Poincaré Symmetries and Spinors

A general transformation of the Poincaré group acts on spacetime x^μ as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \qquad \mu = 0,1,2,3 \tag{2.1}$$

where $\Lambda^\mu{}_\nu$ are the Lorentz transformations and a^μ translations. We write the Poincaré group therefore as $O(3,1) \rtimes \mathbb{R}^4$, where $\rtimes$ denotes the semi-direct product, which does not commute.

These transformations leave the Minkowski metric $ds^2 = dx^\mu \eta_{\mu\nu} dx^\nu$ invariant, where $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$: For $\Lambda \in O(3,1)$, we have

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \qquad \text{or} \qquad \Lambda^T \eta \Lambda = \eta \tag{2.2}$$

This means that we have a choice of either $\det \Lambda = \pm 1$.

$$(\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2 = 1. \tag{2.3}$$

Then $\left|\Lambda^0{}_0\right| \geq 1$ implies that for each of the two choices of determinant, we can have either $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$. Therefore, $O(3,1)$ has 4 disconnected components. The element continuously connected to the identity, $SO(3,1)^\uparrow$, is the proper orthochronous Lorentz group with $\det\Lambda = 1$ and $\Lambda^0{}_0 \geq 1$. Any other element in $O(3,1)$ can be obtained by combining elements of $SO(3,1)^\uparrow$ with

$$\{\mathbb{1}, \Lambda_P, \Lambda_T, \Lambda_{PT}\}, \qquad \text{Klein Group} \tag{2.4}$$

where $\Lambda_P = \mathrm{diag}(+1,-1,-1,-1)$ are the parity transformations and $\Lambda_T = \mathrm{diag}(-1,+1,+1,+1)$ time reversal.

From now on we work with $SO(3,1)^\uparrow \to SO(3,1)$.

2.1.1 Poincaré Algebra

As usual to derive the algebra, we consider the infinitesimal transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}; \qquad a^{\mu} = \epsilon^{\mu}; \qquad \omega^{\mu}{}_{\nu}, \epsilon^{\mu} \ll 1. \tag{2.5}$$

The invariance (2.2) of the metric then gives

$$(\delta^{\mu}_{\rho} + \omega^{\mu}{}_{\rho})\eta_{\mu\nu}(\delta^{\nu}_{\sigma} + \omega^{\nu}{}_{\sigma}) = \eta_{\rho\sigma}. \tag{2.6}$$

This implies that $\omega_{\sigma\rho} = -\omega_{\rho\sigma}$ is antisymmetric. As such, we have 6 parameters $\omega_{\mu\nu}$ for the Lorentz transformations. In addition to this, we have 4 parameters ϵ_{μ} for translations. In total, the Poincaré group has 10 dimensions.

To study the algebra of the Poincaré group, we will look at its representation on a Hilbert space, since we are interested in quantum theory. We are working with a state $|\psi\rangle$ and consider transformations enacted by unitary operators $U(\Lambda, a) = \exp\big(i[\omega_{\mu\nu}M^{\mu\nu} + \epsilon_{\mu}P^{\mu}]\big)$ as

$$|\psi\rangle \to U(\Lambda, a)\,|\psi\rangle\,, \tag{2.7}$$

where $U(\Lambda, a)$ form a representation of the Poincaré group and the generators $M^{\mu\nu}$ and P^{μ} of the Poincaré algebra are Hermitian. Near the identity, we can expand the exponential

$$U(1+\omega, \epsilon) = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + i\epsilon_{\mu}P^{\mu}, \tag{2.8}$$

Since $\omega_{\mu\nu}$ is antisymmetric, so is $M^{\mu\nu}$.

To determine the algebra, we also need to find the Lie brackets. Since the translations commute, $[P^{\mu}, P^{\nu}] = 0$. More complicated is the bracket $P^{\sigma}, M^{\mu\nu}$.

P^{μ} by itself has a double personality. It is a vector, which means that under infinitesimal Lorentz transformations it transforms as

$$\begin{aligned} P^{\sigma} \to \Lambda^{\sigma}{}_{\rho}P^{\rho} &\simeq (\delta^{\sigma}{}_{\rho} + \omega^{\sigma}{}_{\rho})P^{\rho} && (2.9)\\ &= P^{\sigma} + \frac{1}{2}(\omega_{\alpha\rho} - \omega_{\rho\alpha})\eta^{\sigma\alpha}P^{\rho} && (2.10)\\ &= P^{\sigma} + \frac{1}{2}\omega_{\alpha\rho}(\eta^{\rho\alpha}P^{\rho} - \eta^{\sigma\rho}P^{\alpha}). && (2.11)\end{aligned}$$

However, it is also an operator, which acts on the Hilbert space. Therefore, it transforms as

$$\begin{aligned} P^{\sigma} \to U^{\dagger}P^{\sigma}U &= (\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})P^{\sigma}(\mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}) && (2.12)\\ &= P^{\sigma} + \frac{i}{2}\omega_{\mu\nu}[M^{\mu\nu}, P^{\sigma}] + O(\omega^2). && (2.13)\end{aligned}$$

Comparing the above two expressions, we have

$$\boxed{[P^{\sigma}, M^{\mu\nu}] = -i(P^{\mu}\eta^{\nu\sigma} - P^{\nu}\eta^{\mu\sigma})} \tag{2.14}$$

As such, whenever we see an algebra $[X^{\mu_1\cdots}, M^{\rho\sigma}]$, then we should know that the right hand side actually tells us how $X^{\mu_1\cdots}$ transforms under Lorentz transformations!

Similarly,

$$\boxed{[M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho})}. \tag{2.15}$$

Again, this tells us that $M^{\mu\nu}$ transforms under Lorentz transformations as a tensor.

Example 2.1.1: A 4-dimensional matrix representation of $M^{\mu\nu}$ is given by

$$(M^{\rho\sigma})^{\mu}{}_{\nu} = -i(\eta^{\mu\sigma}\delta^{\rho}_{\nu} - \eta^{\rho\mu}\delta^{\sigma}_{\nu}) \tag{2.16}$$

Comment 1

Since $P^0 = H$ is the Hamiltonian, we find that the commutation relation have some physical meaning:

$$[P^0, P^\mu] = 0 \implies \text{Energy-Momentum conservation} \tag{2.17}$$

Comment 2

The algebra of $SO(3,1)$ is determined by the algebra of $SU(2)\times SU(2)$. Define Hermitian operators $J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$ and $K_i = M_{0i}$. Their algebra arises directly from the commutators (2.15) of the M_{ij}:

$$[J_i, J_j] = i\epsilon_{ijk}, \qquad [J_i, K_k] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{2.18}$$

We can also define $A_i = \frac{1}{2}(J_i + iK_i)$ and $B_i = \frac{1}{2}(J_i - iK_i)$. These are not Hermitian. However, this gives a nice separation between the algebras

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \qquad [B_i, B_j] = i\epsilon_{ijk}B_k, \qquad [A_i, B_j] = 0, \tag{2.19}$$

and in each case the A- and B-subalgebra look like the algebra of the J's: These are like $SU(2)$ algebras, except they are not Hermitian.

Representations of $SU(2)\times SU(2)$

For representations of $SU(2)\times SU(2)$, recall that that the $SU(2)$ states are labelled by half-integers $j = 0, \frac{1}{2}, \dots$. Then the A_i and B_i algebra states are labelled by $A, B = 0, \frac{1}{2}, \dots$ respectively. Therefore, the representations of $SO(3,1)$ can be labelled by specifying (A, B).

Remark: Under parity

$$P: \qquad J_i \to J_i \tag{2.20}$$

$$K_i \to -K_i \tag{2.21}$$

$$A_i \leftrightarrow B_i \tag{2.22}$$

$$(A, B) \leftrightarrow (B, A) \tag{2.23}$$

Therefore, we can denote either one of these, say (A, B), as 'left'. Then (B, A) is 'right' and vice-versa.

Comment 3

We have the homomorphism

$$SO(3,1) \simeq SL(2,\mathbb{C}). \tag{2.24}$$

Consider first $SO(3,1)$. Let $X = X_\mu e^\mu = (X_0, X_1, X_2, X_3)$ denote a 4 -vector. Under Lorentz transformation $X \to \Lambda X$, where $\Lambda \in SO(3,1)$, the modulus squared $|X|^2 = X_0^2 - X_1^2 - X_2^2 - X_3^2$ remains invariant.

Now consider the space of 2×2 matrices with basis given by the Pauli matrices

$$\sigma^\mu := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{2.25}$$

We can then write any matrix $\widetilde{X}$ as a linear combination of these

$$\widetilde{X} = X_\mu \sigma^\mu = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & X_0 - X_3 \end{pmatrix}. \tag{2.26}$$

Taking the components (X_0, X_1, X_2, X_3) to be the same as the 4-vector above, this is just another way of representing the same information. Furthermore, the action of $SL(2,\mathbb{C})$ on $\widetilde{X}$ is

$$\widetilde{X} \to N\widetilde{X}N^\dagger, \qquad N \in SL(2,\mathbb{C}). \tag{2.27}$$

Since $N \in SL(2,\mathbb{C})$, we have $\det N = 1$. The determinant has the form $\det \widetilde{X} = X_0^2 - X_1^2 - X_2^2 - X_3^2$. Exactly as in the case of $SO(3,1)$, this quantity is kept invariant.

This is the defining feature. As such, the map from $SL(2,\mathbb{C}) \to SO(3,1)$ defined by (2.26) is homomorphic. This map is 2 to 1 since it maps $N = \pm\mathbb{1} \to \Lambda = \mathbb{1}$.

Claim 1: But $SL(2,\mathbb{C})$ is *simply connected.*

Proof. Polar decomposition: We can write $N = e^h U$, where $h = h^\dagger$ is hermitian and $U = (U^\dagger)^{-1}$ unitary. Since the eigenvalues of a hermitian matrix are positive, the trace of h is positive. Then, using that $\det e^h = e^{\operatorname{tr} h}$, we find that $\det N = 1$ implies that $\operatorname{tr} h = 0$ and $\det U = 1$.

$$h = \begin{pmatrix} a & b+ic \\ b-ic & -a \end{pmatrix} \qquad U = \begin{pmatrix} x+iy & z+iw \\ -z+iw & x-iy \end{pmatrix}. \tag{2.28}$$

For h, the variables $a, b, c \in \mathbb{R}$ define the manifold $\mathbb{R}^3$. Similarly the components of U have the condition $x^2 + y^2 + z^2 + w^2 = 1$, defining the manifold S^3. Therefore, we have that $SL(2,\mathbb{C})$ has the manifold structure $\mathbb{R}^3 \times S^3$, which is simply connected. □

Corollary: Thus, the $SO(3,1)$ manifold, which is obtained from a 2 to 1 map from $SL(2,\mathbb{C})$, is $\mathbb{R}^3 \times S^3/\mathbb{Z}_2$.

Representations of $SL(2,\mathbb{C})$

Definition 2 (fundamental)**:** The *fundamental representation* ψ_α, $\alpha = 1, 2$ is given by

$$\psi'_\alpha = N_\alpha{}^\beta \psi_\beta. \tag{2.29}$$

The ψ_α transforming in this way are called *left-handed Weyl spinors*.

Definition 3 (conjugate)**:** The *antifundamental* or *conjugate representation* is given by *right-handed Weyl spinors* $\overline{\chi}_{\dot{\alpha}}$, $\dot{\alpha} = 1, 2$ transforming as

$$\overline{\chi}'_{\dot{\alpha}} = (N^*)_{\dot{\alpha}}{}^{\dot{\beta}} \overline{\chi}_{\dot{\beta}}. \tag{2.30}$$

Definition 4 (contravariant)**:** The *contravariant representation* is

$$\psi'^{\alpha} = \psi^{\beta}(N^{-1})_{\beta}{}^{\alpha}, \qquad \overline{\chi}'^{\dot{\alpha}} = \overline{\chi}^{\dot{\beta}}(N^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}}. \tag{2.31}$$

To raise and lower indices, we need *invariant tensors*:

SO(3, 1) $\eta^{\mu\nu} = (\eta_{\mu\nu})^{-1}$

SL(2, $\mathbb{C}$) $\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}}$

Invariance means that $\epsilon^{\alpha\beta}$ transforms as

$$\epsilon'^{\alpha\beta} = \epsilon^{\rho\sigma} N_{\rho}{}^{\alpha} N_{\sigma}{}^{\beta} = \epsilon^{\alpha\beta} \det N = \epsilon^{\alpha\beta} \tag{2.32}$$

Mixed $SO(3,1)$ and $SL(2,\mathbb{C})$

$$\widetilde{X}_{\alpha\dot{\alpha}} = (X_\mu \sigma^\mu)_{\alpha\dot{\alpha}} \to N^{\beta}{}_{\alpha}(X_\nu\sigma^\nu)_{\beta\dot{\gamma}}(N^*)^{\dot{\gamma}}{}_{\alpha} = 1 \tag{2.33}$$

$$\Rightarrow \ \sigma^{\mu}{}_{\alpha\dot{\alpha}} = N_{\alpha}{}^{\beta}(\sigma^\nu)_{\beta\dot{\gamma}}(\Lambda^{-1})^{\mu}{}_{\nu}(N^*)^{\dot{\gamma}}{}_{\dot{\alpha}} \tag{2.34}$$

and similarly

$$(\overline{\sigma}^{\mu})^{\alpha\dot{\alpha}} := \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^\mu)_{\beta\dot{\beta}} \tag{2.35}$$

Can choose

$$\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu = 2\eta^{\mu\nu}. \tag{2.36}$$

These are the *Dirac generators* of $SL(2,\mathbb{C})$.

Definition 5: From these, we can define new matrices

$$(\sigma^{\mu\nu})_{\alpha}{}^{\beta} := \frac{i}{4}\left(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu\right)_{\alpha}{}^{\beta}, \tag{2.37}$$

$$(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{4}\left(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu\right)^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{2.38}$$

Claim 2: The $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ are generators of the Lorentz group in spinor representations, meaning that they obey the commutation relations

$$[\sigma^{\mu\nu},\sigma^{\lambda\rho}] = i\left(\eta^{\mu\rho}\sigma^{\nu\lambda} + \eta^{\nu\lambda}\sigma^{\mu\rho} - \eta^{\mu\lambda}\sigma^{\nu\rho} - \eta^{\nu\rho}\sigma^{\mu\lambda}\right) \tag{2.39}$$

and similar for $\overline{\sigma}$.

For $SL(2,\mathbb{C})$: Left handed (fundamental)

$$\psi_\alpha \to (e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}})_{\alpha}{}^{\beta}\psi. \tag{2.40}$$

Right handed (conjugate)

$$\overline{\chi}^{\dot{\alpha}} \to \left(e^{-\frac{i}{2}m_{\mu\nu}\overline{\sigma}^{\mu\nu}}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\overline{\chi}^{\beta}. \tag{2.41}$$

Fundamental

$$J_i = \frac{1}{2}\epsilon_{ijk}\sigma_{jk} = \frac{1}{2}\sigma_i, \quad K_i = \sigma_{0i} = -\frac{i}{2}\sigma_i. \tag{2.42}$$

Exercise 2.1: Check $[\sigma_i,\sigma_j] = 2i\epsilon^{ijk}\sigma_k$.

$$A_i = \frac{1}{2}(J_i + iK_i) = \frac{\sigma_i}{2} \qquad B_i = \frac{1}{2}(J_i - iK_i) = 0 \tag{2.43}$$

$$(A,B) = (\frac{1}{2},0) \qquad \text{representation} \tag{2.44}$$

Conjugate:

$$(A,B) = (0,\frac{1}{2}) \tag{2.45}$$

Parity:

$$(A,B) \xrightarrow{P} (B,A) \tag{2.46}$$

Definition 6 (product)**:** Product of Weyl spinors:

$$\chi\psi := \chi^{\alpha}\psi_{\alpha} = -\chi_{\alpha}\psi^{\alpha} \tag{2.47}$$

$$\overline{\chi}\overline{\psi} := \overline{\chi}_{\dot{\alpha}}\overline{\psi}^{\dot{\alpha}} = -\overline{\chi}^{\dot{\alpha}}\overline{\psi}_{\dot{\alpha}} \tag{2.48}$$

In particular,

$$\psi\psi = \psi^{\alpha}\psi_{\alpha} = \epsilon^{\alpha\beta}\psi_{\beta}\psi_{\alpha} = \psi_2\psi_1 - \psi_1\psi_2. \tag{2.49}$$

Choose ψ_{α} to be Grassmann numbers

$$\psi_1\psi_2 = -\psi_2\psi_1. \tag{2.50}$$

Then $\psi\psi = 2\psi_2\psi_1$.

All representations of Lorentz algebra can be obtained from products of $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$:

$$(\frac{1}{2}, 0)^m \otimes (0, \frac{1}{2})^n \quad \text{use } j_1 \otimes j_2 = |j_1 - j_2| \oplus \cdots \oplus j_1 + j_2. \tag{2.51}$$

Example 2.1.2:

$$(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}) \tag{2.52}$$

$$\psi_{\alpha}\overline{\chi}_{\dot{\alpha}} = \frac{1}{2}(\psi\sigma_{\mu}\overline{\chi})\sigma^{\mu}{}_{\alpha\dot{\alpha}}. \tag{2.53}$$

Example 2.1.3:

$$(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) = (1, 0) \oplus (0, 0) \tag{2.54}$$

$$\psi_{\alpha}\psi_{\beta} = \frac{1}{2}\epsilon_{\alpha\beta}\underbrace{(\psi\chi)}_{\text{scalar}} + \frac{1}{2}(\sigma^{\mu\nu}\epsilon^{T})_{\alpha\beta}\underbrace{(\psi\sigma_{\mu\nu}\chi)}_{(1,0)} \tag{2.55}$$

Connection to Dirac Matrices and Spinors

Definition 7 (Dirac spinor)**:** We define the *Dirac spinor* to be

$$\psi_D := \begin{pmatrix} \psi_{\alpha} \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{2.56}$$

Definition 8 (gamma matrices)**:** The Dirac γ-matrices are defined such that $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}\mathbb{1}$, for example

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \overline{\sigma}^{\mu} & 0 \end{pmatrix} \tag{2.57}$$

Lorentz generators:

$$\Sigma^{\mu\nu} = \frac{i}{4}\gamma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{pmatrix} \tag{2.58}$$

Definition 9: We define a fifth gamma matrix

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{2.59}$$

Claim 3: This acts on Dirac spinors as

$$\gamma^5\psi_D = \begin{pmatrix} -\psi_\alpha \\ +\overline{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{2.60}$$

Definition 10 (projection operators)**:** We define *projection operators*

$$P_L := \frac{1}{2}(\mathbb{1} - \gamma^5), \qquad P_R := \frac{1}{2}(\mathbb{1} + \gamma^5) \tag{2.61}$$

Definition 11 (Dirac conjugation)**:** We define the *Dirac conjugate* of a Dirac spinor as

$$\overline{\psi}_D := (\chi^\alpha, \overline{\chi}_{\dot{\alpha}}). \tag{2.62}$$

Definition 12:

$$\psi_D^C := \begin{pmatrix} \chi_\alpha \\ \overline{\psi}^{\dot{\alpha}} \end{pmatrix} \tag{2.63}$$

Claim 4:

$$\psi_D^C = C\overline{\psi}_D^T, \qquad C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \tag{2.64}$$

Definition 13 (Majorano spinor)**:**

$$\psi_M := \begin{pmatrix} \psi_\alpha \\ \overline{\psi}^{\dot{\alpha}} \end{pmatrix} = \psi_M^C \tag{2.65}$$

We have

$$\psi_D = \psi_M^1 + i\psi_M^2. \tag{2.66}$$

No Majorana + Weyl spinor.

2.2 Unitary Representations of the Poincaré Group

This is due to Wigner in 1939.

2.2.1 Rotation Group

We already know how to find the representations of the rotation group in $3D$ $SO(3) \simeq SU(2)$. Its generators J_i obey $[J_i, J_j] = i\epsilon_{ijk} J_k$.

Definition 14 (Casimir operator)**:** The *Casimir operator* is

$$J^2 = J_1^2 + J_2^2 + J_3^2, \qquad [J^2, J_i] = 0. \tag{2.67}$$

The representations or multiplets of the rotation group, which are a collection of states, are labelled by the eigenvalues of the Casimir operators

$$J^2 |j\rangle = j(j+1)|j\rangle, \qquad j = 0, \frac{1}{2}, 1, \ldots \tag{2.68}$$

Within one representation, we can diagonalise one of the J_i generators, for instance J_3 . Its eigenvalues are j_3. Acting on any given state, labelled by j_3, in the representation that is labelled by j, we have

$$J_3 |j, j_3\rangle = j_3 |j, j_3\rangle, \qquad j_3 = -j, -j+1, \ldots, +j. \tag{2.69}$$

Remark: For a compact group, the number of Casimir operators is the rank of the group, but for non-compact group we have to try by hand.

2.2.2 Poincaré Group

We follow exactly the same steps for the Poincaré group. The generators are P^μ and $M^{\mu\nu}$, obeying the commutation relations (2.14) and (2.15).

Definition 15 (Casimir operator)**:** The *Casimir operators* for the Poincaré group are

$$C_1 = P^\mu P_\mu \qquad C_2 = W^\mu W_\mu. \tag{2.70}$$

Remark: Note that we now have two Casimirs, so the representations will be indicated by two labels.

Claim 5: The Casimir operators commute with the Poincaré algebra's generators:

$$[C_{1,2}, P^\mu] = 0 = [C_{1,2}, M^{\mu\nu}]. \tag{2.71}$$

Exercise 2.2: Prove this!

Definition 16 (Pauli–Ljubanski vector)**:** The *Pauli–Ljubanski vector* is

$$W_M := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}. \tag{2.72}$$

Claim 6: The Pauli–Ljubanski vector and the Poincaré generators have the following commutation relations

$$[W_\mu, P_\nu] = 0, \qquad [W_\mu, M_{\rho\sigma}] = i(\eta_{\mu\rho}W_\sigma - \eta_{\mu\sigma}W_\rho). \tag{2.73}$$

Claim 7: Using Claim 6, one can show

$$[W_\mu, W_\nu] = -i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma, \tag{2.74}$$

so the W_μ do not form an algebra, except if the P^μ are fixed!

As before, we label representations of the Poincaré algebra by eigenvalues C_1 and C_2

$$|C_1, C_2\rangle. \tag{2.75}$$

Within an irreducible representation (irrep), we now need to pick a subset of generators that can be simultaneously diagonalised. Let us pick the P^μ, with eigenvalue p^μ, since they already commute with each other. Since $C_1 = P^\mu P_\mu$, the eigenvalues $p^\mu p_\mu$ can be bigger than, equal to, or less than zero.

$p^\mu p_\mu > 0$ In other words, p^μ is timelike. We can choose coordinates in which $p^\mu = (m, 0, 0, 0)$. It is immediately obvious that this vector is invariant under rotations J_i. We say that $SO(3)$ is the *little group*. Now $C_1 = P^\mu P_\mu = m^2$ and since $W_\mu = (0, -mJ_i)$, we have $C_2 = m^2 J^2$. The multiplet is then labelled by

$$|c_1, c_2; p^\mu, j_3\rangle = |m, j; p^\mu, j_3\rangle. \tag{2.76}$$

These are *massive one-particle states*!

Remark: We cannot overemphasise the importance of this result: elementary particles are irreducible representations of the Poincaré algebra.

$P^\mu P_\mu = 0$ We can find a frame in which $p^\mu = (E, 0, 0, E)$ and have $C_1 = 0$. The W_μ are given by

$$(W_0, W_1, W_2, W_3) = E(J_3, -J_1 + K^2, -J_2 - K_1, -J_3). \tag{2.77}$$

The commutation relations are

$$[W_1, W_2] = 0, \qquad [W_3, W_1] = -iEW_2, \qquad [W_3, W_2] = iEW_1. \tag{2.78}$$

The little group is the $2D$ Euclidean group E_2, which has infinite-dimensional representations. However, the corresponding particles have never been observed.

Remark: So far, we do not have a satisfactory explanation of why this particle should not be there. [Weinberg] gives a phenomenological explanation, whereas Wigner gave an explanation in terms of the heat capacity. The lecturer proposed an explanation in terms of string theory.

Set $W_1 = W_2 = 0$. Then W_3 is generating $SO(2)$, rotations around x_3. We have

$$W_\mu = EJ_3(1,0,0,-1) \propto p_\mu. \tag{2.79}$$

Finally, the second Casimir operator is $C_2 = W^\mu W_\mu = 0$. The irreducible representation is labelled

$$|0,0;p^\mu,\lambda\rangle := |p^\mu,\lambda\rangle\,, \tag{2.80}$$

where $\lambda = 0, \pm\frac{1}{2}, \pm 1, \dots$ is the eigenvalue of J_3, which is called *helicity*

$$e^{2\pi i\lambda}\,|p^\mu,\lambda\rangle = \pm\,|p^\mu,\lambda\rangle\,. \tag{2.81}$$

Remark: We will identify these with the Higgs ($\lambda = 0$), quarks and leptons ($\pm\frac{1}{2}$), gauge bosons (± 1) and the graviton (± 2). There are no higher spin particles! The $\lambda = \pm\frac{3}{2}$ particle, dubbed the gravitino, has not yet been found.

Remark: There is another state with $P^\mu P_\mu = 0$, which is simply $p^\mu = (0,0,0,0)$. This is what people call the vacuum; a state without any particles.

$P^\mu P_\mu < 0$ These are tachyonic states.

Remark: [Weinberg, Chapter 2] goes into further detail on this.

2.3 Discrete Spacetime Symmetries

We know of two discrete symmetries already. We have seen parity P, with $\Lambda_P = \mathrm{diag}(+1, -1, -1, -1)$ and time reversal T with $\Lambda_T = \mathrm{diag}(-1, +1, +1, +1)$. Previously we ignored them because these are not continuously connected to the identity. These transformations move us between the various disconnected pieces of the Lorentz group.

We can represent these as operators acting on a Hilbert space

$$P = U(\Lambda_P, 0) \qquad T = U(\Lambda_T, 0). \tag{2.82}$$

It will turn out that P is unitary while T is not.

Let us first consider how P and T act on operators of the Hilbert space. For any $U(\Lambda, a)$, these act as

$$PUP^{-1} = U(\Lambda_P, \Lambda, \Lambda_P^{-1}, \Lambda_P a), \qquad TUT^{-1} = U(\Lambda_T, \Lambda, \Lambda_T^{-1}, \Lambda_T a). \tag{2.83}$$

Infinitesimally, where we take the Lorentz transformations to be $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ and $\alpha^\mu = \epsilon^\mu$, with $\omega^\mu{}_\nu, \epsilon^\mu \ll 1$. We also write its unitary representation as $U(\Lambda, a) = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + i\epsilon_\mu P^\mu$. Then

$$PJ_iP^{-1} = J_i, \qquad PK_iP^{-1} = -K_i, \qquad PP_iP^{-1} = -P_i, \qquad PP_0P^{-1} = P_0. \tag{2.84}$$

Naively, we expect $TP_0T = -P_0$, but this would imply negative energy.

Theorem 1 (Wigner)**:** Transformations on a Hilbert space preserving probabilities are either unitary and linear or antiunitary and antilinear.

Proof. See [Weinberg, Vol. 1]. □

Unitary and Linear: We have two states $|\phi\rangle, |\psi\rangle$. Then

$$\langle U\phi | U\psi\rangle = \langle \phi|\psi\rangle \qquad U(\alpha\phi + \beta\psi) = \alpha U\phi + \beta U\psi. \tag{2.85}$$

Antiunitary and Antilinear: With similar states, we have instead

$$\langle U\phi | U\psi\rangle = \langle \phi|\psi\rangle^* \qquad U(\alpha\phi + \beta\psi) = \alpha^* U\phi + \beta^* U\psi. \tag{2.86}$$

In particular, if α is imaginary, then it changes sign, so 'U does not commute with i'.

Then pick T to be antiunitary and antilinear ($Ti = -iT$).

$$TJ_iT^{-1} = -J_i, \qquad TK_iT^{-1} = K_i, \qquad TP_iT^{-1} = -P_i, \qquad TP_0T^{-1} = P_0. \tag{2.87}$$

Now we want to know how these act on particle states.

Claim 8: For massive particles, we have

$$|m, i; p^\mu, j_3\rangle \xrightarrow{P} \eta_P\, |m, j; -p^\mu, j_3\rangle\,, \tag{2.88}$$

$$\xrightarrow{T} \eta_T(-1)^{j-j_3}\, |m, j; -p^\mu, -j_3\rangle\,, \tag{2.89}$$

where η_P and η_T represent some phase. For massless particles,

$$|p^\mu, \lambda\rangle \xrightarrow{P} \eta_P e^{\mp i\pi\lambda}\, |-p^\mu, -\lambda\rangle\,, \tag{2.90}$$

$$\xrightarrow{T} \eta_T e^{\pm i\pi\lambda}\, |-p^\mu, -\lambda\rangle\,. \tag{2.91}$$

Proof. See [Weinberg, Vol. 1]. □

Comments

- If $\lambda \neq 0$, then $|p^\mu, \lambda\rangle \to |-p^\mu, -\lambda\rangle$, so the states come in two polarisations $\pm\lambda$, so $\lambda = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$ This happens whenever parity is conserved. For most of the interactions, such as the photon or the graviton, this is true. However, for the weak interaction, parity is not conserved. We interpret the second helicity eigenvalue to belong to its antiparticle.
- For a massive particle of spin j, there are $2j + 1$ polarisation states, since $j_3 = -j, \ldots, +j$. For instance, a massive spin-1 particle has 3 polarisation states. However, the massless object of helicity λ has always only two states $\pm\lambda$. This difference between becomes even more pronounced for higher spins.
- We have not yet talked about any fields here. Special relativity and quantum mechanics is enough to imply the existence of particles. Fields will be introduced as a way to describe interactions.

2.4 From Particles to Fields

We do not only want to see that single particle states exist, but it is also extremely important to be able to describe *interactions* among many particles. Such interactions can be described by a Hamiltonian

$$H = H_0 + H_{\text{interaction}}, \tag{2.92}$$

where H_0 is the Hamiltonian of the free, non-interacting theory.

There are some conditions on the observables:

(i) Lorentz and translation invariance

(ii) Unitarity (guarantees that probabilities are conserved). Unitary evolution $U = e^{-iHt}$, where H is Hermitian.

(iii) Locality. This is where fields enter. We want to describe interactions at spacetime points, so we need some local structure there. The interactions are described as functions of x and t and those functions are the fields. The principle of *Cluster decomposition* means that things that are far away do not interact.

If all of these conditions are satisfied, then we are necessarily driven towards the introduction of fields.

We describe interactions between initial states at the infinite past $t \to -\infty$ to final states in the infinite future $t \to +\infty$, where both are assumed to act just like free states. This is described by the S-matrix:

$$S_{\beta\alpha} := \left\langle \beta^{\text{out}} \middle| \alpha^{\text{in}} \right\rangle = \delta_{\beta\alpha} + \delta(p^2 - m^2) M_{\beta\alpha}. \tag{2.93}$$

The $|\alpha\rangle$ and $|\beta\rangle$ are many particle states.

We can write the Hilbert space as

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots, \tag{2.94}$$

where H_n is the space of n -particle states. Notably, the vacuum $|0\rangle$ is the only state in $\mathcal{H}_0$, whereas $\mathcal{H}_1$ are the massless states $|p^\mu, \lambda\rangle = a^\dagger(p^\mu, \lambda)\,|0\rangle$. Furthermore, two particle states in $\mathcal{H}_2$ are then obtained from states in $\mathcal{H}_1$ as

$$\begin{aligned} \left| p_1^\mu, \lambda_1; p_2^\mu, \lambda_2 \right\rangle &= a^\dagger(p_2, \lambda_2) \left| p_1^\mu, \lambda \right\rangle & (2.95)\\ &= a^\dagger(p_2, \lambda_2) a^\dagger(p_1, \lambda_1)\,|0\rangle & (2.96)\\ &= \pm a^\dagger(p_1, \lambda) a^\dagger(p_2, \lambda_2)\,|0\rangle, & (2.97) \end{aligned}$$

where the sign depends on the type of particle. *Bosons* commute and have integer spin (helicity), whereas *fermions*, which have half-integer spin (helicity), anticommute.

2.4.1 General Conditions on Interactions

The conditions that we outlined earlier can now be written slightly differently:

(i) Unitarity (probabilities add up to 1) preserve by unitary time evolution $U = e^{-iHt}$. The S-matrix is unitary $S_{\beta\alpha} = \langle \beta | S\alpha \rangle$, $S^\dagger S = 1$.

(ii) Amplitudes (S-matrix) invariant under Poincaré transformations.

(iii) Locality (Cluster decomposition)

- The Hamiltonian has to be a local function $H = \int \mathrm{d}^3x\, \mathscr{H}(x,t)$; it is the sum of energy densities at each point. Similarly with the Lagrangian $L = \int \mathrm{d}^3x\, \mathscr{L}(x,t)$ and the Action $S = \int \mathrm{d}^4x\, \mathscr{L}(x^\mu)$.
- $\mathscr{H}$ and $\mathscr{L}$ are operators in position space x^μ, but particle states are defined in momentum space p^μ. Therefore, we need to Fourier transform to move to x-space. This leads us to defining a field. For instance, for a massless particle we can write down

$$A_\alpha(x) := \int đp\, e^{ipx} u_\alpha(p, \lambda) a(p, \lambda). \tag{2.98}$$

- Causality: operators at different spatial locations have to commute at the same time

$$[\phi_\alpha(x,t), \phi_\alpha^\dagger(y,t)] = 0. \tag{2.99}$$

The field A_α itself is local but not causal. To make it causal we introduce another field

$$B_\alpha(x) := \int đp\, e^{ipx} v_\alpha(p,\lambda) b(p,\lambda), \tag{2.100}$$

so that the total field, which satisfies (2.99) is

$$\phi_\alpha = A_\alpha + \xi B_\alpha^\dagger. \tag{2.101}$$

Therefore, field theory requires the existence of *antiparticles*!

(iv) Stability: Energy bounded from below ($|0\rangle$)

(v) Effective Field Theories ($\supset$ renormalisability and non-renormalisability)

- Physics is organised by scales. For instance, in QED we talk about electrons and photons. This is okay until we hit the energy threshold $E = 2m_\mu$ of the muon mass.
- The Lagrangian density can be written $\mathscr{L} = c_i \mathcal{O}_i(\phi_\alpha)$, where $\mathcal{O}_i$ are operators. We know $[\mathscr{L}] = 4$. The theory is said to be *renormalisable* if $[c_i] \geq 0$. This is very restrictive since $[\mathcal{O}_i] \geq 0$.

$$[c_i] + [\mathcal{O}_i] = 4. \tag{2.102}$$

This is very predictive since it allows only a few c_i.

The theory is non-renormalisable if there is a c_i with $[c_i] \leq 0$. For example, consider the scalar field with Lagrangian density

$$\mathscr{L} = \underbrace{\overbrace{\partial^\mu\phi\partial_\mu\phi - m^2\phi^2 - \lambda\phi^4}^{\text{renormalisable}} + \frac{\alpha_1}{M}\phi^5 + \frac{\alpha_2}{M^2}\phi^6 + \ldots}_{\text{non-renormalisable}}. \tag{2.103}$$

There is nothing wrong with this theory *as long as* we are working on energies much smaller than M, so that the expansion in (E/M) is under control. However, this fails for $E \sim M$. We require UV completion.

In a renormalisable theory like the Standard Model we cannot predict when the theory will break down, since we do not know of the scale of M. For gravity, which is non-renormalisable, we can be much more certain about the energy scales on which our predictions should be valid.

This is why in the standard model we will only tend to draw quadratic and quartic potentials; the requirement of renormalisability means that we cannot go beyond quartic and these are the only symmetric choices we have.

3 Internal Symmetries

The most general symmetries of the S-matrix are of the form

$$\text{spacetime} \otimes \text{internal}. \tag{3.1}$$

We have already seen the spacetime symmetries P^μ and $M^{\mu\nu}$. In general, the spacetime symmetries include supersymmetries Q^I_α , which anticommute $\{Q^J_\alpha, Q^{-J}_\alpha\} = 2\sigma??_{\alpha\alpha'}P_\mu$. These imply the representation $n_{\text{boson}} = n_{\text{fermions}}$, which is not observed in nature so far. We will therefore not think about these further in this course, other than saying that the *gravitino* with $\lambda = \frac{3}{2}$ can only exist with supersymmetry.

Internal symmetries are transforming the operators and fields but leave them at the same spacetime position x

$$|\psi\rangle \to U|\psi\rangle, \qquad \mathcal{O}(x) \to \mathcal{O}'(x) = U^\dagger \mathcal{O} U. \tag{3.2}$$

These are symmetries if $[U, H] = 0$ and the action S is left invariant under these transformations.

3.0.1 Types of Symmetries

(i) Spacetime or internal

(ii) Continuous or discrete

For example, a continuous $U(1)$ symmetry is $\phi \to e^{i\alpha}\phi$, whereas an example of a discrete symmetry is $\phi \to -\phi$. Imposing this latter symmetry, odd terms ϕ^{2n+1} cannot exist in the Lagrangian.

(iii) Global or local

For example, the transformation $\phi \to e^{i\alpha\phi}$ is global if α is constant, and is local if $\alpha = \alpha(x)$ depends on position. In the latter case, we would need to adjust the Lagrangian to be invariant under this by replaying the partial with the covariant derivative $D_\mu\phi = \partial_\mu\phi - ieA_\mu\phi$. The symmetry not only transforms ϕ but also $A_\mu \to A_\mu + \partial_\mu\alpha$. This A_μ is called the *gauge field* and we add to the Lagrangian a kinetic term $F^{\mu\nu}F_{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The interactions between A_μ and ϕ are hidden in $D^\mu\phi D_\mu\phi^*$. This is a nice story: the interaction between electrons and photons comes from local symmetry. The beauty of this story will be questioned in Sec. 3.2.

Also for Dirac fields, the Lagrangian is

$$\mathscr{L} = \overline{\psi}\partial\!\!\!/\psi = \overline{\psi}_L i\partial\!\!\!/\psi_L + \overline{\psi}_R i\partial\!\!\!/\psi_R - m(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R). \tag{3.3}$$

These fields transform as $\psi_{L,R} \to e^{i\alpha_{L,R}}\psi_{L,R}$. The kinetic terms have $U(1)_L \otimes U(1)_R$, which is a chiral symmetry. The mass terms $m \neq 0$ break this since $\alpha_R = \alpha_L$ and we only have $U(1)$. If $\alpha_L(x), \alpha_R(x)$ are local, then

$$\overline{\psi}\partial\!\!\!/\psi \to \overline{\psi}D\!\!\!\!/\psi; \qquad D_\mu = \partial_\mu - ieA_\mu. \tag{3.4}$$

We have made a separation between left- and right-handed since we already know that some interactions treat them differently.

(iv) Manifest or hidden

For example, in a spontaneous symmetry broken sombrero potential, the local vacuum does not see the overall symmetry, even though it is still there.

(v) Anomalous or non-anomalous

An anomalous symmetry is broken by quantum corrections, whereas non-anomalous symmetries are exact.

(vi) Real or accidental

(vii) Compact or non-compact

We will reserve the non-compact class only for the spacetime symmetries, since we know that the Poincaré group is non-compact. However, all the internal symmetries will be compact, since we want to have finite-dimensional representations.

(viii) Abelian or non-Abelian

An Abelian group is $U(1)$, whereas the non-Abelian ones are $SU(N), SO(N), Sp(2N), G_2, F_4$, E_6, and E_8, as we have seen from the Cartan classification in the *Symmetries, Particles, and Fields* course in Michaelmas term.

3.1 Noether's Theorem

If a Lagrangian $\mathscr{L}(\phi_\alpha)$ has a continuous symmetry for $\phi_a \to \phi'_\alpha$, then there exists a current j^μ that is conserved, $\partial_\mu j^\mu = 0$, when the field equations are satisfied. This implies that there exists a conserved charge

$$Q = \int \mathrm{d}^3x\, j^0, \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = \int \mathrm{d}^3x\, \partial_0 j^0 = -\int \mathrm{d}^3x\, \nabla \cdot j = 0. \tag{3.5}$$

Example 3.1.1 (Poincaré group)**:** We have already seen that translations $x^\mu \to x^\mu + a^\mu$ have an associated current $T^\mu{}_\nu$ with charges $P^0 = E = \int \mathrm{d}^3x\, T^{00}$ and $P^i = \int \mathrm{d}^3x\, T^{0i}$.

For rotations, $M^{ij} = \int \mathrm{d}^3x\, \left(x^i T^{0j} - x^j T^{0i}\right)$.

This is true for classical mechanics as well. What is different in QFT is that each of these conserved charges are also the generators of their corresponding group.

Example 3.1.2 (Internal symmetry)**:** If you have an internal symmetry generated by a non-Abelian group G, then

$$G:\, \psi^i \to \psi^i + i\alpha^a (T_a)^i{}_j \psi^j, \tag{3.6}$$

where the conserved charge is $T_a = \int \mathrm{d}^3x\, J^0{}_a$. Again, the conserved charge in the Noether theorem is the generator of the corresponding theory:

$$[T_a, \psi^i] = -(T_a)^i{}_j \psi_j. \tag{3.7}$$

For $U(1)$, the conserved charge Q is the electric charge.

3.2 Origin of Gauge (Local) Symmetries

Consider a massless helicity-1 field.

$$A_\mu(x) = \sum_{\lambda=\pm 1} \int \mathrm{d}p \left(\epsilon_\mu(p^\mu, \lambda) a_{p\lambda} e^{ipx} + \epsilon^*_\mu(p^\mu, \lambda) a^\dagger_{p\lambda} e^{-ipx} \right), \tag{3.8}$$

where ϵ_μ is the polarisation vector. As it is written, A_μ has four degrees of freedom $\mu = 0, 1, 2, 3$. However, we know that the massless helicity-1 field only has two degrees of freedom $\lambda = \pm 1$, so we need some (Lorentz invariant) constraints. We can impose

$$p^\mu \epsilon_\mu = 0, \tag{3.9}$$

which leads us from 4 to 3 degrees of freedom; this would be enough for a massive vector with $j_3 = -1, 0, 1$, but it is not satisfactory for the massless particle. In fact, there are no other Lorentz invariant constraints. Thus we know that the ϵ_μ have some extra degree of freedom. The constraint (3.9) leaves open a redundancy

$$\epsilon_\mu \equiv \epsilon_\mu + \alpha p_\mu, \tag{3.10}$$

which defines an equivalence class for the ϵ_μ . Transforming back to position space, the redundancy (3.10) becomes the gauge invariance condition

$$A_\mu \equiv A_\mu + \partial_\mu \alpha. \tag{3.11}$$

The origin of gauge invariance lies in the Lorentz invariant description of a massless helicity-1 field. One might say that Lorentz invariance implies gauge invariance!

Remark: The polarisation vector ϵ_μ is not a Lorentz covariant object despite carrying a vector index. This is because it transforms as $\epsilon_\mu \to \Lambda_\mu{}^\nu \epsilon_\nu + \alpha p_\mu$.

Similarly, for helicity $\lambda = 2$, we work with a field $h_{\mu\nu}$ and polarisations $\epsilon_{\mu\nu}$. We end up with a similar reundancy

$$\epsilon_{\mu\nu} \equiv \epsilon_{\mu\nu} + \alpha_\mu p_\nu + p_\mu \alpha_\nu \;\Rightarrow\; h_{\mu\nu} \equiv h_{\mu\nu} + \partial_\mu \alpha_\nu + \partial_\nu \alpha_\mu. \tag{3.12}$$

Again, the diffeomorphism invariance of (linearised) general relativity is implied by the Lorentz invariance of a massless helicity-2 particle.

All the beauty of the geometry of general relativity, or the gauge symmetry, is gone. It is all really Lorentz invariance.

Any amplitude (recall: $S_{\alpha\beta} = \delta_{\beta\alpha} + (2\pi)\delta(p_\alpha - p_\beta) M_{\alpha\beta}$) will be of the form

$$M_{\alpha\beta}(p_i^\mu, \lambda_i) = \epsilon^\mu (M_\mu)_{\alpha\beta}. \tag{3.13}$$

The redundancy (3.10) due to Lorentz invariance implies the *Ward identity*

$$\boxed{p^\mu M_\mu = 0}. \tag{3.14}$$

3.2.1 Charge Conservation

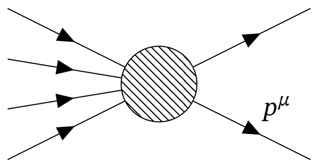


Figure 3.1

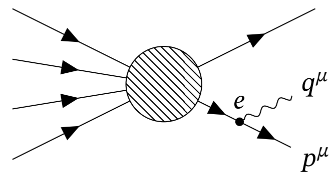


Figure 3.2

Consider the interaction illustrated in Fig. 3.1, which is described by some matrix element M_0. Fig. 3.2 depicts the same interaction, except that we added a 'soft photon' with momentum $q^\mu \ll p^\mu$ to the line with momentum p^μ. This interaction has matrix element $\mathcal{M}$. The most general vertex is $\Gamma^\mu = Fp^\mu + Qq^\mu$, where F, Q are functions of p^2, q^2, and $p \cdot q$. Computing $\epsilon^\mu \Gamma_\mu$ and $q^\mu q_\mu = 0$, we can forget the Q term. If $p^2 = m^2$ and $q^2 = 0$, then $F = F(\frac{p \cdot q}{m^2}) \approx F(0)$ without loss of generality. Thus we have

$$M = M_0 \times \left(\frac{\epsilon^\mu F p_\mu}{(p+q)^2 - m^2} \right) \simeq M_0 \times \left(\frac{F \epsilon \cdot p}{2p \cdot q} \right) \tag{3.15}$$

Adding soft photons to all external lines gives the Ward identity

$$M = M_0 \left(- \sum_{\text{incoming}} \frac{p_i \cdot \epsilon}{2p_i \cdot q} F_i(0) + \sum_{\text{outgoing}} \frac{p_i \cdot \epsilon}{2p_i \cdot q} F_i(0) \right). \tag{3.16}$$

This gives *charge conservation*

$$\sum_{\text{ingoing}} F_i(0) - \sum_{\text{outgoing}} F_i(0) = 0, \tag{3.17}$$

where $F_i(0) := Q_i$.

Let us do the same for $\lambda = 2$. This time we add a soft graviton instead of a soft photon. After going through the same considerations, one arrives at the Ward identity

$$\sum_{\text{in}} k_i p_i^\nu - \sum_{\text{out}} k_i p_i^\nu = 0. \tag{3.18}$$

The extra factors of the p_i^ν complicate this. How can you have an interaction where all the momenta are conserved and then this new combination of momenta is also conserved? This is only possible if all k_i are equal! This is the *principle of equivalence*: all particles interact gravitationally with the same strength.

What happens for $\lambda = 3$? We find that $\sum g_i p_i^\mu p_j^\nu$ is conserved. This implies that all $g_i = 0$. This is an extremely important result: there are no interacting theories for massless particles with helicity higher than $\lambda = 2$. Detailed proofs are given in [Weinberg, Vol. 1, Chapter 13] and in [Schwartz, Chapter 9].

3.2.2 Non Abelian

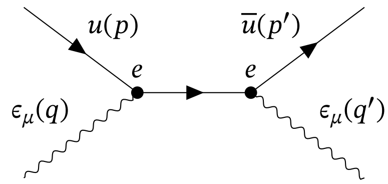


Figure 3.3: Compton scattering.

First, consider Compton scattering $e^+\gamma \to e^+\gamma$ in QED, illustrated by the Feynman diagram in Fig. 3.3. The matrix element can be written $M = M_{\mu\nu}\epsilon^\nu_{\text{in}}\epsilon^\mu_{\text{out}}$ where

$$M_{\mu\nu} = i(-ie)^2\overline{u}(p',\sigma')\left(\frac{\gamma_\mu(\not{p}+\not{q}+m)\gamma_\nu}{(p+q)^2-m^2}+\frac{\gamma_\nu(\not{p}-\not{q}'+m)\gamma_\mu}{(p-q')^2-m^2}\right)u(\mathbf{p},\sigma), \tag{3.19}$$

where $\sigma = g_s$. Consider the Ward identity $M_{\mu\nu}q^\nu\epsilon^\mu_{\text{out}} = 0$. Let us check whether this holds by considering the left hand side

$$M_{\mu\nu}q^\nu\epsilon^\mu_{\text{out}} = i(-ie)^2\overline{u}(p',\sigma')\left(\frac{\not{\epsilon}_{\text{out}}(\not{p}+\not{q}+m)\not{q}}{(p+q)^2-m^2}+\frac{\not{q}(\not{p}-\not{q}'+m)\not{\epsilon}_{\text{out}}}{(p'-q)^2-m^2}\right)u(\mathbf{p},\sigma), \tag{3.20}$$

where we used momentum conservation $p+q = p'+q'$. Using another trick: writing $\not{q} = \not{q}+\not{p}-m-(\not{p}-m)$ and using the Dirac equation

$$(\not{p}-m)u = 0 \qquad \overline{u}(\not{p}'-m) = 0, \tag{3.21}$$

we have

$$M_{\mu\nu}q^\nu\epsilon^\mu_{\text{out}} = i(-ie)^2\overline{u}(p',\sigma')\not{\epsilon}_{\text{out}}u(p,\sigma)\left(\frac{2p\cdot q}{\underbrace{(p+q)^2-m^2}_{2p\cdot q}}+\frac{2p'\cdot q}{\underbrace{(p'-q)^2-m^2}_{-2p'\cdot q}}\right) = 0. \tag{3.22}$$

Thus, the Ward identity holds as expected.

However, this is not the most general case we could have considered. Imagine that the couplings e_1 and e_2 at the two vertices would have been different. The terms in the big parenthesis in (3.22) would be $e_1e_2 - e_2e_1$ and, assuming these couplings do not commute, we could not have factored them out as we did. For the most general possible couplings T^a_{ij} between two fermions and a gauge boson, illustrated in Fig. 3.4, the ward identity implies

$$T^a_{ik}T^b_{kj} - T^b_{ik}T^a_{kj} = 0 \tag{3.23}$$

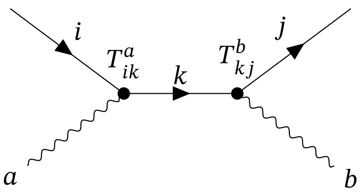


Figure 3.4: General couplings T^a_{ik} between two fermions i and k and a boson a.

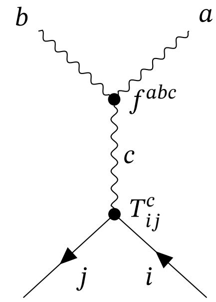


Figure 3.5: Allowing a general coupling f^{abc} between three bosons a, b, and c.

unless there is a coupling f^{abc} between the photons themselves. In that case, which is depicted in diagram 3.5, we have instead

$$T^a_{ik}T^b_{kj} - T^b_{ik}T^a_{kj} = f^{abc}T^c_{ij}. \tag{3.24}$$

$$[T^a, T^b] = if^{abc}T^c, \tag{3.25}$$

where the factor i in the second line is a matter of normalisation of the structure constants f^{abc}. This is the Lie algebra of a non-Abelian gauge (Yang–Mills) theory!

We find that a system with many massless helicity-1 fields comprises

- either photon-like particles, which do not interacting with themselves,
- or non-Abelian Yang–Mills gauge bosons with algebra $[T^a, T^b] = if^{abc}T^c$.

3.3 Yang–Mills Theory

Yang–Mills theory is the theory of a non-Abelian gauge group G.

(i) group elements $U = e^{i\theta^a T^a} \simeq \mathbb{1} + i\theta^a T^a$. The T^a are the generators and θ^a are the parameters or coordinates in the group manifold.

(ii) U acting on matter field $iU\psi$.

- We can then introduce the covariant derivative $D_\mu := \partial_\mu - igA_\mu$, which acts on ψ covariantly

$$D_\mu\psi \to UD_\mu\psi. \tag{3.26}$$

Requiring this, one can check that A_μ has to transform as

$$A_\mu \to A'_\mu = U A_\mu U^{-1} - \frac{i}{g}\partial_\mu U U^{-1}. \tag{3.27}$$

Infinitesimally, we may expand U in terms of the generators to give

$$A'^{\,a}_\mu = A^a_\mu + \frac{i}{g}\partial_\mu \alpha^a - f^{abc}\alpha^b A^c_\mu \tag{3.28}$$

- From the commutator

$$[D_\mu, D_\nu] = \left[-ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2[A_\mu, A_\nu]\right]\psi, \tag{3.29}$$

we obtain the gauge (Yang–Mills) field strength

$$F_{\mu\nu} := \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{3.30}$$

It transforms covariantly as

$$F_{\mu\nu} \to F'_{\mu\nu} = U F_{\mu\nu} U^{-1} \tag{3.31}$$

$$\Rightarrow \; F'^{\,a}_{\mu\nu} = F^a_{\mu\nu} - f^{abc}\alpha^b F^c_{\mu\nu}. \tag{3.32}$$

We can build a gauge invariant (renormalisable) Lagrangian as

$$\mathscr{L} = -\frac{1}{4} g_{ab} F^a_{\mu\nu} F^{b\mu\nu} + \mathscr{L}_M(\psi, D_M\psi) + \theta F^a_{\mu\nu} \widetilde{F}^{a\mu\nu}, \tag{3.33}$$

where $\widetilde{F}^{a\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. Usually people do not write the (topological) θ-term since it is a total derivative and does not influence the physical classically. However, quantum mechanically it does! Non-perturbative features of Yang–Mills theories like instantons come from this term. Moreover, it is related to Chern–Simons theory and has some amazing mathematics behind it.

The second term $\mathscr{L}_M$ is the matter Lagrangian, coupling the gauge field to the matter fields ψ.

What about the first term? In order to behave like a real propagating field, we want positive kinetic energy. This implies that g_{ab} is constant, invariant $\mathscr{L}$, and positive definite. This implies that G cannot just be any group; it has to be compact, simple, or semi-simple. This is where it pays off that we studied the Cartan classification of these groups! We are restricted to work with the groups listed in Table 3.1.

G	rank	dimension
$SU(N)$	$N-1$	N^2-1
$SO(N)$	$\frac{N}{2}, \frac{N-1}{2}$	$\frac{N(N-1)}{2}$
$Sp(N)$	N	$N(2N+1)$
G_2	2	14
F_4	4	52
E_6	6	78
E_7	7	133
E_8	8	248

Table 3.1: Cartan Classification

Compactness implies $\mathrm{Tr}(T^a T^a) >= 0$, which means that we have finite-dimensional representations with hermitian generators. For $SU(N)$, we have the representations:

fundamental: $\phi_i \to \phi_i + i\alpha^a (T^a_F)_{ij} \phi_j$, where T^a_F are hermitian.

antifundamental: Generators are $T^a_{AF} = -(T^a_F)^*$, so the corresponding elements of the representation $\phi^*_i \to \phi^*_i + i\alpha^a (T^a_{AF})_{ij} \phi^*_j = \phi^*_i - i\alpha^a (T^a_F)_{ji} \phi^*_j$.

adjoint: $(T^a_A)^{bc} := -i f^{abc}$. This is an (N^2-1)-dimensional representation.

These are normalised as

$$\mathrm{Tr}\big(T^a T^b\big) = \frac{1}{2}\delta^{ab} \tag{3.34}$$

$$T^a T^b = \frac{1}{2N}\delta^{ab} + \frac{1}{2} d^{abc} T^c + \frac{1}{2} i f^{abc} T^c, \tag{3.35}$$

where $d^{abc} = 2\,\mathrm{Tr}\big[T^a, \{T^b, T^c\}\big]$ is symmetric. The quadratic Casimir of the representation R is

$$C(R) = T_R^a T_R^a. \tag{3.36}$$

The index $T(R)$ is obtained as the trace of the product

$$\mathrm{Tr}\big[T_R^a T_R^b\big] = T(R)\delta^{ab}. \tag{3.37}$$

With the Lagrangian (3.33) we can look for the field equations. We take $g_{ab} = \delta_{ab}$ and ignore the θ-term since it is a total derivative, so that the Lagrangian is

$$\mathscr{L} = -\frac{1}{4}(F_{\mu\nu}F^{\mu\nu}) + \mathscr{L}_M(\psi, D\psi). \tag{3.38}$$

The Euler–Lagrange equations are

$$\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu A_\nu^a)}\right) = \frac{\partial\mathscr{L}}{\partial A_\nu^a}, \tag{3.39}$$

which give the field equations

$$-\,\partial_\mu F^{a\mu\nu} = -gF^{c\nu\mu}f^{abc}A_\mu^b - i\frac{\partial\mathscr{L}_\mu}{\partial(D_\nu\psi)}T^a\psi. \tag{3.40}$$

These field equations can also be written

$$\partial_\mu F_a^{\mu\nu} = -J_a^\nu, \tag{3.41}$$

with the current J_a^ν being defined as

$$J_a^\nu = -gf_{abc}F_c^{\nu\mu}A_{b\mu} - i\frac{\partial\mathscr{L}_\mu}{\partial(D_\nu\psi)}T_a\psi. \tag{3.42}$$

This current is conserved (by Noether's theorem), meaning that $\partial_\nu J_a^\nu = 0$. However, it is not gauge-covariant!

On the other hand, the field equations can be rewritten

$$\boxed{D_\mu F_a^{\mu\nu} = -j_a^\nu} \tag{3.43}$$

in terms of the covariant derivative and another current

$$j_a^\nu = -i\frac{\partial\mathscr{L}_\mu}{\partial(D_\nu\psi)}T_a\psi. \tag{3.44}$$

This current is not conserved, but it is covariantly conserved, meaning $D_\nu j_a^\nu = 0$!

This is very reminiscent of gravity, where the stress tensor is covariantly conserved.

We also have the Bianchi identity

$$D_\mu F^a_{\nu\lambda} + D_\nu F^a_{\lambda\mu} + D_\lambda F^a_{\mu\nu} = 0. \tag{3.45}$$

This gives

$$\boxed{D_\mu \widetilde{F}^{\mu\nu} = 0} \tag{3.46}$$

Equations (3.43) and (3.46) generalise Maxwell's equations.

This is very similar but not identical to electromagnetism. It generalises it in several ways. The term $F^a_{\mu\nu}F^{a\mu\nu}$ has terms ∂AAA and $AAAA$, which are self-interactions between the field that were absent for the classical photon! On the other hand, the similarity with gravity is compelling. There is a geometrical interpretation of definition (3.30) of the Maxwell tensor F as the curvature of the internal gauge space with A being the connection on the fibre bundle.

4 Broken Symmetries

4.1 Motivation

So far spin/helicity $0, \pm\frac{1}{2}$ OK

massless helicity $\pm 1 \Rightarrow$ QED, Yang–Mills

massless helicity $\pm 2 \Rightarrow$ gravity

No more interactions.

But massless Yang–Mills fields have not been observed.

What about massive $s = 1$? A massive spin-1 field $A_\mu(x, t)$ is associated with a polarisation vector $\epsilon_\mu(p)$. It has three polarisation states $j_z = -1, 0, 1$. Since $A_\mu(x, t)$ has 4 degrees of freedom, we impose the condition $p^\mu \epsilon_\mu = 0$ to get three polarisations. There is no gauge redundancy.

Pick $p^\mu = (E, 0, 0, p_z)$ with the condition that $E^2 - p_z^2 = m^2$. We have two transverse polarisations $\epsilon_1^\mu = (0, 1, 0, 0)$ and $\epsilon_2^\mu = (0, 0, 1, 0)$ and one longitudinal polarisation. With the normalisation condition $\epsilon_\mu^2 = -1$, there is only one possibility: $\epsilon_L^\mu = (\frac{p_z}{m}, 0, 0, \frac{E}{m})$.

Suppose we are at high energies, where $E \gg m$. We then have $p \approx E$. Then the longitudinal polarisation becomes $\epsilon_L^\mu \approx \frac{E}{m}(1, 0, 0, 1)$. The amplitudes will have a term $M \sim g^2 \epsilon_L^0 \epsilon_L^z \sim g^2 \frac{E^2}{m^2}$. This blows up! However, the amplitudes are probabilities, which have to be less than one; so this blowing-up breaks unitarity. The theory of massless spin-1 particles must have a cutoff; it cannot be valid until high energies since it breaks perturbative unitarity.

Example 4.1.1: For a mass $m \simeq 100$GeV and coupling $g \sim 0.1$, the energy has to be less than $E \lesssim 1$Tev.

Massive spin-1 particles lead to interactions that can be valid only at small energies and have to be superseded by a consistent UV completion.

To search for a UV completion?

4.2 Spontaneously Broken Discrete Symmetries

Let us warm up to this concept slowly by considering discrete symmetries. Take the theory of a real scalar field with symmetry $\phi \to -\phi$. The most general renormalisable Lagrangian is

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V_\pm(\phi). \tag{4.1}$$

There are two possible scalar potentials

$$V_\pm(\phi) = \pm\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 + \kappa_\pm. \tag{4.2}$$

Here, $\lambda > 0$ for stability and $\kappa_\pm$ is introduced to be able to tune the value of V at the minimum.

Remark: Gauge invariance is imposed only for massless particles. For massive particles, the most general theory involves a mass-term, which breaks gauge invariance. This is totally okay, since gauge invariance is only secondary, coming from our requirement of Lorentz invariance.

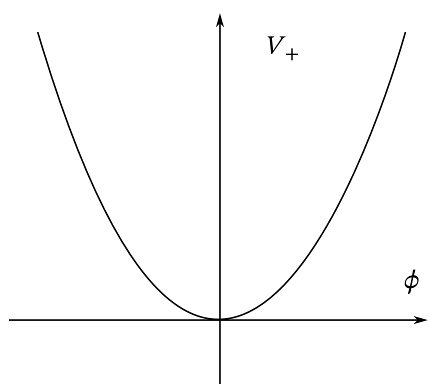


Figure 4.1

The potential V_+ has a minimum at $\phi = \phi_0 = 0$ and is symmetric under $\phi \to -\phi$, as illustrated in Fig. 4.1. We say the "symmetry is manifest".

In the quantum theory, we want to investigate vacuum expectation values (VEV)

$$\langle\phi\rangle := \langle 0|\,\phi\,|0\rangle = \int \mathcal{D}\phi\,\phi e^{\frac{i}{\hbar}\int\mathscr{L}\mathrm{d}^4x}. \tag{4.3}$$

Taking the classical limit $\hbar \to 0$, we recover the minimum of the potential $\langle\phi\rangle \to \phi_0$.

Perturbations around the vacuum

$$\phi = \phi_0 + \sigma(x) = \sigma(x). \tag{4.4}$$

The Lagrangian becomes
$$\mathscr{L} = \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma - \frac{1}{2}m^2\sigma^2 - \frac{\lambda}{4}\sigma^4, \tag{4.5}$$
so σ is a particle of mass m.

We can write the quartic V_- as
$$V_- = \frac{\lambda}{2}\left(\phi^2 - v^2\right)^2, \qquad v := \sqrt{\frac{m^2}{\lambda}}. \tag{4.6}$$
This potential is illustrated in Fig. 4.2. We have two vacua $\langle\phi\rangle = \pm v$ degenerate. Perturbing around

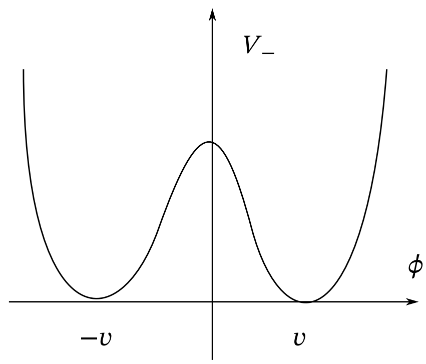


Figure 4.2: If you sit in the vacua $\phi = \pm v$, you do not see the symmetry of the potential.

the vacuum $\phi = \pm v + \sigma(x)$ gives
$$\mathscr{L} = \frac{1}{2}\partial_\mu\sigma\partial_\mu\sigma - \left(\lambda v^2\sigma^2 \pm \lambda v\sigma^3 + \frac{\lambda}{4}\sigma^4\right). \tag{4.7}$$
In this case $\lambda v^2\sigma^2 = m^2\sigma^2 + \dots$. So σ is a particle of squared mass $2m^2 \geq 0$. The second derivative of the potential gives the mass of the particle.
$$m^2 = \left.\frac{\partial^2 V_-}{\partial\phi^2}\right|_{\phi^2=v^2} = 2m^2. \tag{4.8}$$
We got a cubic term in the Lagrangian potential. The symmetry $\phi \to -\phi$ is *hidden* (spontaneously broken). However, the symmetry is of course still there since we have $\sigma \to -\sigma \mp 2v$. Some people say the symmetry is "non-linearly realised", when we have $\phi \to a\phi + X$, where X is something else.

If we would have expanded around $\phi = 0$, we have a tachyonic mass
$$\left.\frac{\partial^2 V}{\partial\phi^2}\right|_{\phi=0} = -m^2. \tag{4.9}$$
In principle, you can consider superpositions of the two vacua, which recovers the symmetry. This is discussed in [Weinberg, Vol. 2]; the upshot is that in large systems you still always have to choose one particular vacuum.

This system is very simple, just possessing a discrete symmetry, but is nonetheless very rich. Consider a thermodynamic system where the coefficient of ϕ^2 depends on the temperature. We have $m^2 \propto (T - T_c)$. For very high temperatures, $T \gg T_c$, we have a potential like V_+. Reducing the temperature to $T < T_c$, the symmetry is broken to the potential V_-. This model explains systems ranging from magnetic materials to cosmological systems. These models can have rich properties such as domain walls and other things we discussed in the *Statistical Field Theory* course.

4.3 Spontaneous Breaking of Continuous Global Symmetries

Let us generalise Sec. 4.2. Instead of a single field, consider an N-component scalar field $\phi = (\phi_1, \phi_2, \dots, \phi_N)^T$ with Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial^\mu \phi \cdot \partial_\mu \phi - V_\pm(\phi).. \tag{4.10}$$

In the case of V_+, the minima are symmetric. We will consider the more interesting case of V_-, which is given by

$$V_-(\phi) = \frac{\lambda}{4}(\phi \cdot \phi - v^2)^2, \qquad v^2 = \frac{m^2}{\lambda}, \quad \lambda > 0. \tag{4.11}$$

This Lagrangian is symmetric under the rotation group $O(N)$. The vacua lie at $\phi \cdot \phi = v^2$. The potential is shown in Fig. 4.3; we have a continuum of vacua.

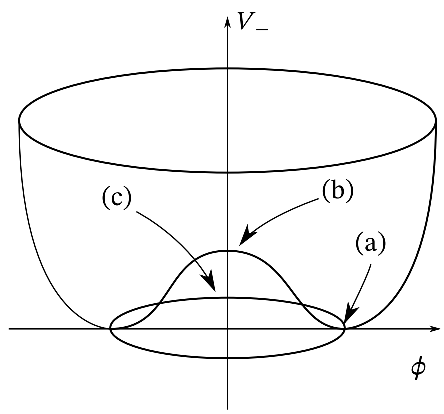


Figure 4.3: The curvature of the vacuum in the (a)-direction gives m_σ^2. The (c)-directions correspond to π's with 0 mass. The curvature at (b) gives the tachyon.

Pick $\langle \phi_0 \rangle = (0, \dots, 0, v)^T$. Fluctuations are then $\phi(x) = (\pi_1(x), \dots, \pi_{N-1}(x), V + \sigma(x))$. Making the substitution, the Lagrangian and potential are

$$\mathscr{L} = \frac{1}{2}\partial_\mu \pi \cdot \partial^\mu \pi + \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma - V_-(\pi_i, \sigma), \tag{4.12}$$

$$V_-(\pi, \sigma) = \frac{1}{2}(2\lambda v^2)\sigma^2 + \lambda(\sigma^2 + \pi^2)\sigma + \frac{\lambda}{4}(\sigma^2 + \pi^2)^2 \tag{4.13}$$

This is essentially the same thing we did for the discrete case. The only field that has a quadratic piece by itself in the potential is σ. There is no mass term for π, only for σ, where $m_\sigma^2 = 2\lambda v^2$. The mass matrix is

$$M_{ij} = \frac{\partial^2 V}{\partial \phi_i \partial \phi_j}|_{\phi = \langle \phi \rangle} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & m_\sigma^2 \end{pmatrix} \tag{4.14}$$

We have $N-1$ massless fields π, called *Goldstone bosons*. The symmetry is broken from $O(N) \to O(N-1)$.

4.3.1 Goldstone's Theorem

In general, if the Lagrangian $\mathscr{L}$ is invariant under a continuous (compact and semisimple) G such that the vacuum expectation value $\langle\phi\rangle \neq 0$ breaks symmetry $G \to H \subset G$. Define the *vacuum manifold* to be the space of all minima

$$\mathcal{M} = \{\phi_0 \mid V(\phi_0) = V_{\text{minimum}}\}, \qquad \phi_0 := \langle\phi\rangle. \tag{4.15}$$

Different vacua are related by group transformations $g \in G$ as $\phi_0' = g\phi_0$. We also define the invariant or stability group $H = \{h \in G \mid h\phi_0 = \phi_0\}$. Observing now that

$$\phi_0' = gh\phi_0 = (ghg^{-1})\phi'. \tag{4.16}$$

Therefore $ghg^{-1} \in H$. As such, $g \in G$, mapping one vacuum to another, defines equivalence classes: $g_1 \sim g_2$ if $\exists h \in H$ such that $g_1 = g_2 h$. We then have

$$\mathcal{M} \simeq \frac{G}{H}. \tag{4.17}$$

Consider now an infinitesimal transformation

$$g\phi = \phi + \delta\phi, \qquad \delta\phi = i\alpha^a T^a \phi, \quad a = 1, \dots, \dim G. \tag{4.18}$$

We can expand in a Taylor expansion and use the symmetry $V(\phi + \delta\phi) = V(\phi)$:

$$V(\phi + \delta\phi) - V(\phi) = i\alpha^a (T^a\phi)_r \frac{\partial V}{\partial \phi_r} = 0. \tag{4.19}$$

This is an important expression, which we will use later. If ϕ_0 is a minimum, then

$$V(\phi) - V(\phi_0) = \frac{1}{2}(\phi - \phi_0)_r \left.\frac{\partial^2 V}{\partial\phi_r \phi_s}\right|_{\phi=\phi_0} (\phi - \phi_0)_s. \tag{4.20}$$

We recognise the mass matrix appearing here. Differentiate (4.19) and evaluate at $\phi = \phi_0$:

$$(T^a\phi_0)_r \left.\frac{\partial^2 V}{\partial\phi_r \phi_r}\right|_{\phi_0} = 0. \tag{4.21}$$

The lessons from this are twofold.

- If the symmetry is unbroken and the vacuum unique, meaning that $g\phi_0 = \phi_0, \forall g \in G$, then $\delta\phi = 0$, then $(T^a\phi_0) = 0$ for all a.

- If $\exists g \in G$ such that there exists $T^a\phi_0 \neq 0$, then $T^a\phi_0$ is an eigenvector of the mass matrix with zero eigenvalue; these are the Goldstone bosons.

$$(T^a\phi_0)_r M^2_{rs} = 0. \tag{4.22}$$

How many massless states do we have? Let us split the generators T^a into two groups. We denote by $\widetilde{T}^i$ the elements of H and by $R^{\hat{a}}$ the remaining ones. The assumptions of compactness and semisimplicity mean that we have the orthogonality condition

$$\operatorname{Tr} \widetilde{T}^i R^{\hat{a}} = 0. \tag{4.23}$$

This means that each vector $R_0^{\hat{a}}$ is a unique eigenvector of M_{rs}. Since $i = 1, \ldots, \dim H$ and $\hat{a} = \dim H, \ldots, \dim G$, we find that the number of Goldstone bosons is

$$\dim \frac{G}{H} = \dim G - \dim H. \tag{4.24}$$

Example 4.3.1 ($O(N)$)**:** Take $G = O(N)$ and $H = O(N-1)$, then the number of Goldstone bosons is

$$\dim O(N) - \dim O(N-1) = \frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N - 1. \tag{4.25}$$

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