The Standard Model

Part III Lent 2019

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Contents

1 Introduction and History

1.1 Introduction

Definition 1 (standard model)**:** A theoretical physics construction (theory, model) that describes all known elementary particles and their interactions based on relativistic quantum field theory (QFT).

Ingredients

(i) spacetime: $3+1$ dimensional Minkowski space

symmetry: Poincaré group

(ii) particles:

spin $s = 0$ Higgs

spin $s = 1/2$ three families of quarks and leptons

(iii) interactions:

$s = 1$ three gauge interactions

$s = 1$ gravity[2]

Gauge (local) symmetry: $SU(3)_C \times SU(2)_L \times U(1)_Y \xrightarrow[Breaking]{Symmetry} SU(3)_C \times U(1)_{EM}$

C color: strong

L left: electroweak

Y hypercharge

These are related via $Q = T_3 + Y$.

Particle representations[3]:

[2]as important as it is, we will not be concerned with gravity for most of this course

[3]numbers tell us representations under $(C, L; Y)$

- Quarks and Leptons: $\overbrace{3}^{\text{families (flavour)}}[\underbrace{(3,2;\frac{1}{6})}_{Q_L}+\underbrace{(\bar{3},1;-\frac{2}{3})}_{U_R}+\underbrace{(\bar{3},1;\frac{1}{3})}_{d_R}+\underbrace{(1,2;-\frac{1}{2})}_{L_L}+\underbrace{(1,1;1)}_{e_R}+\underbrace{(1,1;0)}_{\nu_R}]$
- Higgs: $(1,2;-\frac{1}{2})$
- Gauge: $\underbrace{(8,1;0)}_{gluons}+\underbrace{(1,3;0)}_{W^\pm,\mathbb{Z}}+\underbrace{(1,1;0)}_{\gamma}$

Comments

- interactions given by QFT
- main tool: symmetry
- total symmetry: spacetime $\otimes$ internal (gauge)[1]
- also accidental (global) symmetries $\sim$ baryon + lepton number
- plus approximate (flavour) symmetries:
- very rigid: $\sum Y = \sum Y^3 = 0$[2], $\#3 = \#\bar{3}$, $\#2$ even
- rich structure (3 phases: Coulomb, Higgs, confining)

Motivation

Why to learn about the SM?

- It is fundamental.
- It is based on elegant principles of symmetry.
- It is true!
  - outstanding predictions: $(\mathbb{Z}^0, W^\pm, \text{Higgs}, \dots)$
  - precision tests:

    anomalous magnetic dipole moment of the electron:

$$a = \frac{g-2}{2} = (1159.65218091 \pm 0.00000026) \times 10^{-6} \tag{1.1}$$

[1] Theorem: cannot mix these two symmetries. Supersymmetry provides a way around this.
[2] gravitational anomaly

fine structure constant (at $E \ll 10^3$GeV):

$$\alpha^{-1} = \frac{\hbar c}{e^2} = 137.035999084(21) \tag{1.2}$$

- It is the best test of QFT.
- It is incomplete!

Take another look at the quarks and leptons. For $Q_L : (3, 2, \frac{1}{6})$ the second entry tells us that these are doublets under $SU(2)$. This means that we have $Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$.

1.2 History

Weinberg has a good paper on advice he gives to young researchers. One of the advices is to study history and the history of physics in particular. This is because it gives us some sense of how physical theories developed and also that it makes us feel part of a bigger development in the pursuit of knowledge, no matter how small our contributions may seem.

t < 20^th^ century: only two interactions (gravity and electromagnetism)

discreteness of matter not established

1896 Radioactivity (Becquerel, Pierre and Marie Curie, Rutherford) α, β, γ rays

This was a big discovery at the time; there is no inherent stability in nature! This was a manifestation that pointed to the existence of other interactions.

1897 J. J. Thomson: discovered the *electron* (e^-) and measured e/m a few hundred meters from where we are right now in Cambridge (close to The Eagle so you can enjoy that on your visit too). This was the first particle ever discovered, marking the beginning of particle physics.

1900's **1900-1930** Quantum mechanics developed. Probably the biggest revolution ever in science.

1905 Special relativity. These two still are the two basic theories to study in nature. The nature of quantum mechanics also implies that light behaves as a particle, which we now know as the photon.

In the same decade, Rutherford's group also discovered the atom.

1910's Francis Aston (1919) defined the 'whole number rule', for the ratio of different atomic nuclei to the hydrogen mass. This led to the discovery of the proton.

Cosmic rays were studied, in particular by using cloud chambers.

Einstein's theory of general relativity.

1920's Bose, Fermi statistics.

Beginning of QFT (Jordan, Heisenberg, Dirac, ...)

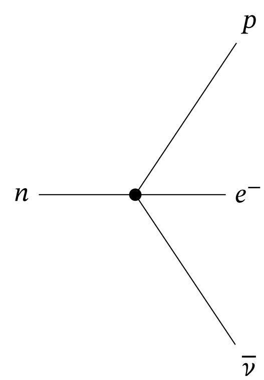
Dirac equation. This predicted a positive particle, which he thought could be the proton ...

1930's …but then came to predict that this is the *positron* (e^+), which he just squeezed into the introduction of his paper on magnetic monopoles (1931).

1932 Anderson[1] discovered it.

1932 Chadwick discovered the *neutron*.

1930 Pauli predicted the *neutrino*: β-decay: $n \to p + e^- + \overline{\nu}$. Fermi described this in terms of a four-point field theory



(1.3)

1934 Yukawa theory of strong interactions: scalar mediators of strong interactions (Pions). Potential $V(r) \sim e^{-mr}/r$, where $m \sim$ 100MeV is the pion mass. This explained the short range and kept field theory going, so people started to search for this new particle.

1936 Anderson discovered the *muon* ($m \sim$ 100MeV).

1932 Heisenberg, 1936 (Condon et al.) introduced isospin $n \leftrightarrow p$. He thought that in the same way that the electron has spin-up and spin-down, the proton and neutron have such similar properties that they also have an internal symmetry.

1940's The history of physics is different to the history you learn at school; at much happens in physics during war-time.

1947 Lamb shift, QED (Schwinger, Feynman + Tomonaga + Dyson)

Pions π were discovered, explaining why the naive picture of Yukawa made sense.

1950's
- A time of great optimism. Particle accelerators and bubble chambers were built (E > MeV). People say that the 50's were a decade of wealth; the numbers of particles were also very rich. Dozens of new particles were discovered (kaons, hyperons, …), mostly strongly interacting (*hadrons*), which are now classified into mesons (bosons) and baryons (fermions).
- Strangeness (Gell-Mann, Nishijima, Pais)
- Parity Violation (Lee and Yang) in 1936, Wu discovered it in 1957

[1] There were multiple people who arguably should get some more credit for this. Blackett discovered the positron but did not publish it fast enough. There were also a Russian scientist and a graduate student at CalTech who did similar discoveries.

- Discovery of (anti-)neutrino (Cowan- Reines, 1956)
- $V - A$ property of weak interactions (Marshak, Sudarshan, 1957)[1]
- Pentecorvo; neutrino oscialltions
- Yang–Mills 1954, non-Abelian gauge theory. In QED, there is a $U(1)$ gauge theory giving a massless photon. In Yang–Mills theory, with a greater group such as $SU(2)$, that this should give further massless / long-range particles. But these have never been seen so Pauli predicted correctly that this theory was not relevant to nature.

[1] Four experiements seemed to deny their theory. However, the theory was so strong that they were convinced that these experiments must have been wrong. All four actually turned out to be wrong.

1960's **1961** Eightfold way (Gell-Mann, Neemann)

Put order to zoo of discovered particles by considering representations of $SU(3)_{\text{flavour}}$. By con-

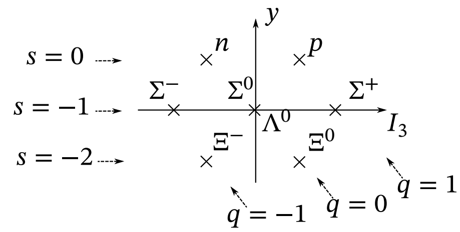


Figure 1.1: The eightfold way is the 8-dimensional representation of $SU(3)$.

sidering the 10-dimensional representation of $SU(3)$, depicted in Fig. 1.2 the Ω^- was predicted.

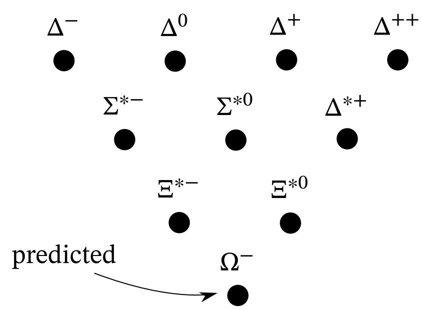


Figure 1.2: The 10-dimensional representation of $SU(3)$.

1964 Gell-Mann, Zweig came up with the theory of *quarks*. This theory was not accepted at the time since three quarks needed to be in the same state for some particles, violating Pauli's exclusion principle.

- $3 \oplus \overline{3} \to$ Mesons $s = 0$; $3 \otimes \overline{3} = 8 + 1$
- $3 \oplus 3 \otimes 3 \to$ baryons $s = \frac{1}{2}$; $3 \otimes 3 \otimes 3 = 10 + 8 + 8 + 1$

1964 Greenberg, 1965 (Nambu and Han) → colour

1967 Deep inelastic scattering. Evidence for substructure in the proton nucleus.

1961 Symmetry breaking (Nambu, Goldstone, Salam, Weinberg), Goldstone bosons (massless)

1964 Higgs Mechanism (Higgs, Brout, Englert, Kibble, Guralnik, Haden)

If the broken symmetry is local, then

- the gauge field is massive
- the Goldstone boson is is eaten and leaves behind a physical massive particle (Higgs)

The problem that Pauli pointed out to Yang and Mills is solved! Now you can have non-Abelian gauge symmetries, and broken symmetries.

1967-8 Weinberg, Salam, (Ward) tried non-Abelian gauge theory for the strong interaction, which failed. Trying it for the weak interaction gave Electroweak unification

$$\underbrace{SU(2)}_{L} \times \underbrace{U(1)}_{Y} \to \underbrace{U(1)}_{\text{EM}} \tag{1.4}$$

(Glashow 1962 identified $SU(2) \times U(1)$)

1964 experimental discovery of CP violation (Cronin, Fitsch) $\Rightarrow$ particle $\leftrightarrow$ antiparticle

1970's Glashow–Illoporlos–Maiani (GIM) mechanism. Explain no FCNC $\Rightarrow$ new quark: *charm* c. As such, the magic number of three, leading Gell-Mann to quarks, is not magic at all. The previous symmetry was only approximate, which was not noticed since c is very massive. In hindsight it was obvious that we needed a fourth quark:

1969 Jackiw–Bell–Adler; Anomalies. Need partner of s: $\begin{pmatrix} c \\ s \end{pmatrix}_L$.

1973 • weak neutral currents discovered

- Asymptotic Freedom (Gross, Wilczek, Politzer)

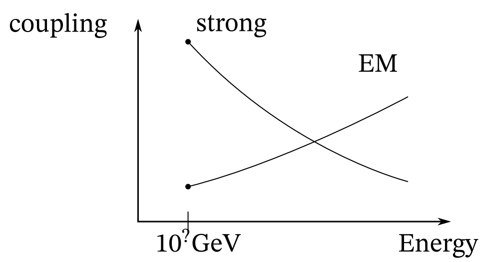


Figure 1.3: The running coupling gives hope for unification.

1974 J/ψ discovered $\to$ *charm*

1975-9 jets (quarks, gluons), for instance $e^+e^- \to qq$ gives 2 jets, but $e^+e^- \to qgq$ gives 3 jets.

$$R = \frac{e^+e^- \to \text{hadrons}}{e^+e^- \to \text{muons}} = \frac{33}{9} \tag{1.5}$$

depends on the number of colours. This gave evidence for 3 colours, confirming the idea of quarks.

1980's **1983** $Z^0, W^\pm$ discovered

1990's **1995** *top quark* discovered. This was not a surprise since people already knew about the bottom quark, which needed a partner. We end up with three families

$$\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix} \tag{1.6}$$

2000's *Tau neutrino*

2012 Higgs!

We are lucky to be taking this course in a time where the standard model is essentially solved. In this course we will see that this structure is essentially forced on us. The structure is very rigid.

2 Spacetime Symmetries

The symmetries we have are spacetime $\otimes$ internal (1967 Coleman–Mandula). In particular, the spacetime symmetry is the Poincaré symmetry.

2.1 Poincaré Symmetries and Spinors

A general transformation of the Poincaré group acts on spacetime x^μ as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \qquad \mu = 0, 1, 2, 3 \tag{2.1}$$

where $\Lambda^\mu{}_\nu$ are the Lorentz transformations and a^μ translations. We write the Poincaré group therefore as $O(3,1) \rtimes \mathbb{R}^4$, where $\rtimes$ denotes the semi-direct product, which does not commute.

These transformations leave the Minkowski metric $ds^2 = dx^\mu \eta_{\mu\nu} dx^\nu$ invariant, where $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$: For $\Lambda \in O(3,1)$, we have

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \qquad \text{or} \qquad \Lambda^T \eta \Lambda = \eta \tag{2.2}$$

This means that we have a choice of either $\det \Lambda = \pm 1$.

$$(\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2 = 1. \tag{2.3}$$

Then $\left|\Lambda^0{}_0\right| \geq 1$ implies that for each of the two choices of determinant, we can have either $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$. Therefore, $O(3,1)$ has 4 disconnected components. The element continuously connected to the identity, $SO(3,1)^\uparrow$, is the proper orthochronous Lorentz group with $\det \Lambda = 1$ and $\Lambda^0{}_0 \geq 1$. Any other element in $O(3,1)$ can be obtained by combining elements of $SO(3,1)^\uparrow$ with

$$\{\mathbb{1}, \Lambda_P, \Lambda_T, \Lambda_{PT}\}, \qquad \text{Klein Group} \tag{2.4}$$

where $\Lambda_P = \text{diag}(+1, -1, -1, -1)$ are the parity transformations and $\Lambda_T = \text{diag}(-1, +1, +1, +1)$ time reversal.

From now on we work with $SO(3,1)^\uparrow \to SO(3,1)$.

2.1.1 Poincaré Algebra

As usual to derive the algebra, we consider the infinitesimal transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}; \qquad a^{\mu} = \epsilon^{\mu}; \qquad \omega^{\mu}{}_{\nu}, \epsilon^{\mu} \ll 1. \tag{2.5}$$

The invariance (2.2) of the metric then gives

$$(\delta^{\mu}_{\rho} + \omega^{\mu}{}_{\rho})\eta_{\mu\nu}(\delta^{\nu}_{\sigma} + \omega^{\nu}{}_{\sigma}) = \eta_{\rho\sigma}. \tag{2.6}$$

This implies that $\omega_{\sigma\rho} = -\omega_{\rho\sigma}$ is antisymmetric. As such, we have 6 parameters $\omega_{\mu\nu}$ for the Lorentz transformations. In addition to this, we have 4 parameters ϵ_{μ} for translations. In total, the Poincaré group has 10 dimensions.

To study the algebra of the Poincaré group, we will look at its representation on a Hilbert space, since we are interested in quantum theory. We are working with a state $|\psi\rangle$ and consider transformations enacted by unitary operators $U(\Lambda, a) = \exp\big(i[\omega_{\mu\nu}M^{\mu\nu} + \epsilon_{\mu}P^{\mu}]\big)$ as

$$|\psi\rangle \to U(\Lambda, a)\,|\psi\rangle\,, \tag{2.7}$$

where $U(\Lambda, a)$ form a representation of the Poincaré group and the generators $M^{\mu\nu}$ and P^{μ} of the Poincaré algebra are Hermitian. Near the identity, we can expand the exponential

$$U(1 + \omega, \epsilon) = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + i\epsilon_{\mu}P^{\mu}, \tag{2.8}$$

Since $\omega_{\mu\nu}$ is antisymmetric, so is $M^{\mu\nu}$.

To determine the algebra, we also need to find the Lie brackets. Since the translations commute, $[P^{\mu}, P^{\nu}] = 0$. More complicated is the bracket $P^{\sigma}, M^{\mu\nu}$.

P^{μ} by itself has a double personality. It is a vector, which means that under infinitesimal Lorentz transformations it transforms as

$$\begin{aligned} P^{\sigma} \to \Lambda^{\sigma}{}_{\rho}P^{\rho} &\simeq (\delta^{\sigma}{}_{\rho} + \omega^{\sigma}{}_{\rho})P^{\rho} && (2.9)\\ &= P^{\sigma} + \frac{1}{2}(\omega_{\alpha\rho} - \omega_{\rho\alpha})\eta^{\sigma\alpha}P^{\rho} && (2.10)\\ &= P^{\sigma} + \frac{1}{2}\omega_{\alpha\rho}(\eta^{\rho\alpha}P^{\rho} - \eta^{\sigma\rho}P^{\alpha}). && (2.11) \end{aligned}$$

However, it is also an operator, which acts on the Hilbert space. Therefore, it transforms as

$$\begin{aligned} P^{\sigma} \to U^{\dagger}P^{\sigma}U &= (\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})P^{\sigma}(\mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}) && (2.12)\\ &= P^{\sigma} + \frac{i}{2}\omega_{\mu\nu}[M^{\mu\nu}, P^{\sigma}] + O(\omega^2). && (2.13) \end{aligned}$$

Comparing the above two expressions, we have

$$\boxed{[P^{\sigma}, M^{\mu\nu}] = -i(P^{\mu}\eta^{\nu\sigma} - P^{\nu}\eta^{\mu\sigma})} \tag{2.14}$$

As such, whenever we see an algebra $[X^{\mu_1\cdots}, M^{\rho\sigma}]$, then we should know that the right hand side actually tells us how $X^{\mu_1\cdots}$ transforms under Lorentz transformations!

Similarly,

$$\boxed{[M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho})}. \tag{2.15}$$

Again, this tells us that $M^{\mu\nu}$ transforms under Lorentz transformations as a tensor.

Example: A 4-dimensional matrix representation of $M^{\mu\nu}$ is given by

$$(M^{\rho\sigma})^{\mu}{}_{\nu} = -i(\eta^{\mu\sigma}\delta^{\rho}_{\nu} - \eta^{\rho\mu}\delta^{\sigma}_{\nu}) \tag{2.16}$$

Comment 1

Since $P^0 = H$ is the Hamiltonian, we find that the commutation relation have some physical meaning:

$$[P^0, P^\mu] = 0 \implies \text{Energy-Momentum conservation} \tag{2.17}$$

Comment 2

The algebra of $SO(3,1)$ is determined by the algebra of $SU(2)\times SU(2)$. Define Hermitian operators $J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$ and $K_i = M_{0i}$. Their algebra arises directly from the commutators (2.15) of the M_{ij}:

$$[J_i, J_j] = i\epsilon_{ijk}, \qquad [J_i, K_k] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{2.18}$$

We can also define $A_i = \frac{1}{2}(J_i + iK_i)$ and $B_i = \frac{1}{2}(J_i - iK_i)$. These are not Hermitian. However, this gives a nice separation between the algebras

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \qquad [B_i, B_j] = i\epsilon_{ijk}B_k, \qquad [A_i, B_j] = 0, \tag{2.19}$$

and in each case the A- and B-subalgebra look like the algebra of the J's: These are like $SU(2)$ algebras, except they are not Hermitian.

Representations of $SU(2)\times SU(2)$

For representations of $SU(2)\times SU(2)$, recall that that the $SU(2)$ states are labelled by half-integers $j = 0, \frac{1}{2}, \ldots$. Then the A_i and B_i algebra states are labelled by $A, B = 0, \frac{1}{2}, \ldots$ respectively. Therefore, the representations of $SO(3,1)$ can be labelled by specifying (A, B).

Remark: Under parity

$$P: \qquad J_i \to J_i \tag{2.20}$$

$$K_i \to -K_i \tag{2.21}$$

$$A_i \leftrightarrow B_i \tag{2.22}$$

$$(A, B) \leftrightarrow (B, A) \tag{2.23}$$

Therefore, we can denote either one of these, say (A, B), as 'left'. Then (B, A) is 'right' and vice-versa.

Comment 3

We have the homomorphism

$$SO(3,1) \simeq SL(2,\mathbb{C}). \tag{2.24}$$

Consider first $SO(3,1)$. Let $X = X_\mu e^\mu = (X_0, X_1, X_2, X_3)$ denote a 4 -vector. Under Lorentz transformation $X \to \Lambda X$, where $\Lambda \in SO(3,1)$, the modulus squared $|X|^2 = X_0^2 - X_1^2 - X_2^2 - X_3^2$ remains invariant.

Now consider the space of 2×2 matrices with basis given by the Pauli matrices

$$\sigma^\mu := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{2.25}$$

We can then write any matrix $\widetilde{X}$ as a linear combination of these

$$\widetilde{X} = X_\mu \sigma^\mu = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & X_0 - X_3 \end{pmatrix}. \tag{2.26}$$

Taking the components (X_0, X_1, X_2, X_3) to be the same as the 4-vector above, this is just another way of representing the same information. Furthermore, the action of $SL(2,\mathbb{C})$ on $\widetilde{X}$ is

$$\widetilde{X} \to N\widetilde{X}N^\dagger, \qquad N \in SL(2,\mathbb{C}). \tag{2.27}$$

Since $N \in SL(2,\mathbb{C})$, we have $\det N = 1$. The determinant has the form $\det \widetilde{X} = X_0^2 - X_1^2 - X_2^2 - X_3^2$. Exactly as in the case of $SO(3,1)$, this quantity is kept invariant.

This is the defining feature. As such, the map from $SL(2,\mathbb{C}) \to SO(3,1)$ defined by (2.26) is homomorphic. This map is 2 to 1 since it maps $N = \pm\mathbb{1} \to \Lambda = \mathbb{1}$.

Claim 1: But $SL(2,\mathbb{C})$ is *simply connected.*

Proof. Polar decomposition: We can write $N = e^h U$, where $h = h^\dagger$ is hermitian and $U = (U^\dagger)^{-1}$ unitary. Since the eigenvalues of a hermitian matrix are positive, the trace of h is positive. Then, using that $\det e^h = e^{\operatorname{tr} h}$, we find that $\det N = 1$ implies that $\operatorname{tr} h = 0$ and $\det U = 1$.

$$h = \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix} \qquad U = \begin{pmatrix} x + iy & z + iw \\ -z + iw & x - iy \end{pmatrix}. \tag{2.28}$$

For h, the variables $a, b, c \in \mathbb{R}$ define the manifold $\mathbb{R}^3$. Similarly the components of U have the condition $x^2 + y^2 + z^2 + w^2 = 1$, defining the manifold S^3. Therefore, we have that $SL(2,\mathbb{C})$ has the manifold structure $\mathbb{R}^3 \times S^3$, which is simply connected. $\square$

Corollary: Thus, the $SO(3,1)$ manifold, which is obtained from a 2 to 1 map from $SL(2,\mathbb{C})$, is $\mathbb{R}^3 \times S^3/\mathbb{Z}_2$.

Representations of $SL(2,\mathbb{C})$

Definition 2 (fundamental)**:** The *fundamental representation* ψ_α, $\alpha = 1, 2$ is given by

$$\psi'_\alpha = N_\alpha{}^\beta \psi_\beta. \tag{2.29}$$

The ψ_α transforming in this way are called *left-handed Weyl spinors*.

Definition 3 (conjugate)**:** The *antifundamental* or *conjugate representation* is given by *right-handed Weyl spinors* $\overline{\chi}_{\dot\alpha}$, $\dot\alpha = 1, 2$ transforming as

$$\overline{\chi}'_{\dot\alpha} = (N^*)_{\dot\alpha}{}^{\dot\beta}\,\overline{\chi}_{\dot\beta}. \tag{2.30}$$

Definition 4 (contravariant)**:** The *contravariant representation* is

$$\psi'^{\alpha} = \psi^{\beta}(N^{-1})_{\beta}{}^{\alpha}, \qquad \overline{\chi}'^{\dot\alpha} = \overline{\chi}^{\dot\beta}(N^{*-1})_{\dot\beta}{}^{\dot\alpha}. \tag{2.31}$$

To raise and lower indices, we need *invariant tensors*:

SO(3, 1) $\eta^{\mu\nu} = (\eta_{\mu\nu})^{-1}$

SL(2, $\mathbb{C}$) $\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot\alpha\dot\beta}$

Invariance means that $\epsilon^{\alpha\beta}$ transforms as

$$\epsilon'^{\alpha\beta} = \epsilon^{\rho\sigma} N_{\rho}{}^{\alpha} N_{\sigma}{}^{\beta} = \epsilon^{\alpha\beta} \det N = \epsilon^{\alpha\beta} \tag{2.32}$$