Applications of Differential Geometry to Physics

Part III Lent 2020

Lectures by Maciej Dunajski

Report typos to: uco21@cam.ac.uk

More notes at: uco21.user.srcf.net

February 20, 2020

Contents

0.1 Kepler / Newton Orbits

$$\ddot{\mathbf{r}} = -\frac{GMv}{r^3}\mathbf{r} \quad \leftrightarrow \quad \text{conic sections} \tag{1}$$

General conic section is

$$ax^2 + by^2 + cxy + dx + ey + f = 0 \tag{2}$$

This is nowadays more generally studied in what we now call *algebraic geometry* rather than differential geometry.

Apolonius of Penge (?) asked 'what is the unique conic thorugh five points, no three of which are co-linear?'

The space of conics is $\mathbb{R}^6 - \{0\}/n = \mathbb{RP}^5$ (projective 5-space).

$$[a, b, c, d, e, f] \sim [\gamma a, \gamma b, \gamma c, \gamma d, \gamma c, \gamma f], \gamma \in \mathbb{R}^* \tag{3}$$

This is an application of geometry, rather than an application of differential geometry.

Remark: Apolonius proved this geometrically.

In this course however, we will look at the following.

1) Hamiltonian mechanics (mid 19$^{\text{th}}$). This is an elegant way of reformulating Newton's mechanics, turning second order differential equations into first order differential equations with the use of a function $H(p, q)$. The system of ODEs is

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{q} = -\frac{\partial H}{\partial q} \tag{4}$$

This led to the development of symplectic geometry (1960s). The connection is that the phase-space to which p and q belong has a 2 -form $dp \wedge dq$. Using the Hamiltonian function, one can find a vector field

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} \tag{5}$$

and looks for a one-parameter group of transformations, called symplectomorphisms, generated by this vector field. Under these symplectomorphisms, the 2 -form is unchanged meaning that the area illustrated in F2 is preserved. Details of this are going to come within the course.

2) General Relativity (1915) ← Riemannian Geometry (1850)

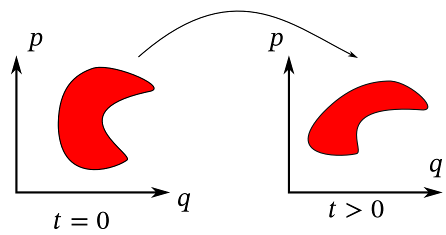

Figure 1

3) Gauge theory (Maxwell, Yang Mills) ↔ Connection on Principal Bundle (U(1) (Maxwell), SU(2), SU(3))

$$A_+ = A_- + dg \qquad g = \psi_+ - \psi_- \qquad \omega = \begin{cases} A_+ + d\psi_+ \\ A_- + d\psi_- \end{cases} \tag{6}$$

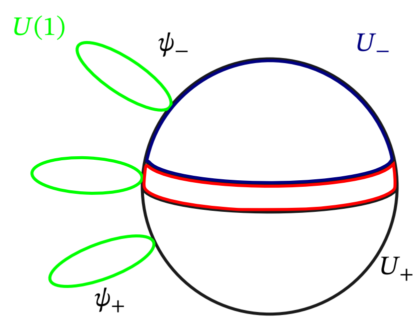

Figure 2

This course: cover 1, 2, 3 in some detail. Unifying feature: Lie groups.

- Prove some theorems, *lots of* examples (often instead of proofs)
- Want to be able to do calculations; compute characteristic classes etc.

We will assume that you took either Part III General Relativity, or Part III Differential Geometry, or some equivalent course.

1 Manifolds

Definition 1 (manifold)**:** An n -dimensional *smooth manifold* is a set M and a collection[2] of open sets U_α , labelled by $\alpha = 1, 2, 3, \dots$, called *charts* such that

- U_α cover M
- $\exists$ 1-1 maps $\phi_\alpha :\ U_\alpha \to V_\alpha \in \mathbb{R}^n$ such that

$$\phi_\beta \circ \phi_\alpha^{-1} :\ \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{1.1}$$

  is a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

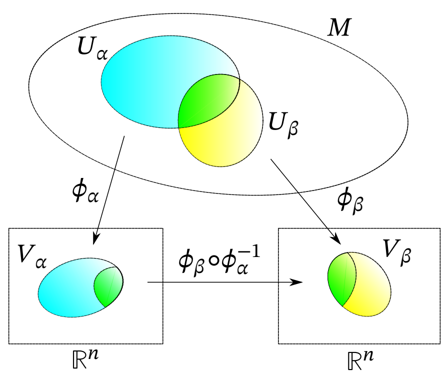

Figure 1.1: Manifold

As such, manifolds are topological spaces with additional structure, allowing us to do calculus.

Example 1.0.1 ($M = \mathbb{R}^n$)**:** There is the *trivial manifold*, which can be covered by only one open set. There are other possibilities. In fact, there are infinitely many smooth structures on $\mathbb{R}^4$ (Proof by Donaldson in 1984 in his PhD. He used Gauge theory).

[2]In all examples that we will look at, there will be finitely α.

Example 1.0.2 (sphere $S^n = \{\mathbf{r} \in \mathbb{R}^{n+1}, |\mathbf{r}| = 1\}$ **):** Intuitively, the n-sphere S^n is an n-dimensional manifold. To show this, we will construct a map $\phi : S^n \to \mathbb{R}^n$ by projecting the north pole $N : (0, 0 \ldots, 0, 1) \in \mathbb{R}^{n+1}$, through the point $p \in S^n$ onto the hyperplane defined by $r_{n+1} = 0$. This is illustrated for S^2 in Fig. 1.2. From this figure, we can already see that this projection map is ill-

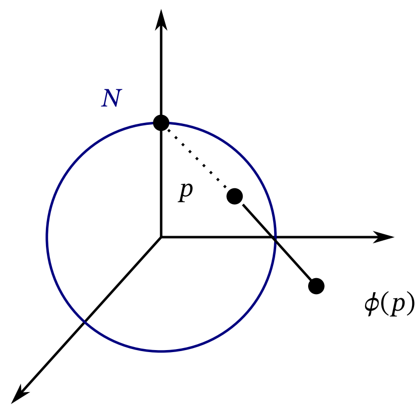

Figure 1.2: The map ϕ projects the north pole N through a point $p \in S^2$ onto $\phi(p)$, which lies in the R^2-plane defined by $z = 0$. This map is not defined for the north-pole itself.

defined at the north pole itself. However, the definition of a manifold allows us to work with multiple charts. Let us cover S^n with the following two open sets

$$U = S^n / \{0, 0, 0, , \ldots, 0, 1\}, \qquad \widetilde{U} = S^n / \{0, 0, 0, , \ldots, 0, -1\}, \tag{1.2}$$

so U does not include the north pole and $\widetilde{U}$ does not include the south pole. We can then define two maps, ϕ and $\widetilde{\phi}$, which project from the north pole and the south pole respectively. Using x_i to denote coordinates in $\mathbb{R}^n$ and r_i to denote coordinates in $\mathbb{R}^{n+1}$ we take maps

$$\text{defined on } U: \qquad \phi(r_1, \ldots, r_{n+1}) = \left(\frac{r_1}{1 - r_{n+1}}, \ldots, \frac{r_n}{1 - r_{n+1}}\right) = (x_1, \ldots, x_n) \tag{1.3}$$

$$\text{defined on } \widetilde{U}: \qquad \widetilde{\phi}(r_1, \ldots, r_{n+1}) = \left(\frac{r_1}{1 + r_{n+1}}, \ldots, \frac{r_n}{1 + r_{n+1}}\right) = (\widetilde{x}_1, \ldots, \widetilde{x}_n). \tag{1.4}$$

On the overlap $U \cap \widetilde{U}$, the coordinates are related

$$\overbrace{\frac{r_k}{1 + r_{n+1}}}^{\widetilde{x}_k} = \frac{1 - r_{n+1}}{1 + r_{n+1}} \overbrace{\frac{r_k}{1 - r_{n+1}}}^{x_k}, \qquad k = 1, \ldots, n, \tag{1.5}$$

where we can write the transition factor on the right in terms of the coordinates x_i:

$$\frac{1 - r_{n+1}}{1 + r_{n+1}} = \frac{(1 - r_{n+1})^2}{r_1^2 + r_2^2 + \cdots + r_n^2} = \frac{1}{x_1^2 + x_2^2 + \cdots + x_n^2}. \tag{1.6}$$

So on $U \cap \widetilde{U}$, the transition function is

$$\widetilde{\phi} \circ \phi^{-1} : (x_1, \ldots, x_n) \longmapsto (\widetilde{x}_1, \ldots, \widetilde{x}_n) = \left(\frac{x_1}{x_1^2 + \cdots + x_n^2}, \ldots, \frac{x_n}{x_1^2 + \cdots + x_n^2}\right), \tag{1.7}$$

which is a smooth map from $\mathbb{R}^n \to \mathbb{R}^n$.

Example 1.0.3: A Cartesian product of manifolds is a manifold, for example we have the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$.

Definition 2 (surface)**:** Let $f_1, \ldots, f_k : \mathbb{R}^N \to \mathbb{R}$ be smooth functions. A surface $f_1 = 0, \ldots, f_k = 0$ is a manifold of dimension dim $n = N - k$ if the rank of the matrix $\frac{\partial f_\alpha}{\partial x^i}$, $\alpha = 1, \ldots, k$ and $i = 1, \ldots, N$ is maximal and equal to k at all points of $\mathbb{R}^N$..

Example 1.0.4: The n-sphere S^n is a surface in $\mathbb{R}^{n+1}$ with $f_1 = 1 - |\mathbf{r}|^2$.

Theorem 1 (Whitney)**:** Every smooth manifold of dimension n is an embedded surface in $\mathbb{R}^N$, where $N \leq 2n$.

> If you enjoy using geometrical intuition and looking at surfaces, this theorem ensures that you can always do that and not loose generality.

Definition 3 (real projective space)**:** The n-dimensional *real projective space* is defined as the quotient

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim, \tag{1.8}$$

where we quotient out the equivalence classes $[X_1, \ldots, X_n + 1] \sim [cX_1, \ldots, cX_{n+1}]$ for all $c \in \mathbb{R}^*$. The $[X_1, \ldots, X_{n+1}]$ are called *homogeneous coordinates*.

In other words, this is the space of all lines through the origin in $\mathbb{R}^{n+1}$.

Claim 1: $\mathbb{RP}^n$ is a smooth manifold of dimension n with $(n + 1)$ open sets.

Proof. Let us define our open sets with respect to the homogeneous coordinates. We define the set $U_\alpha : [X] \in \mathbb{RP}^n$ such that $X_\alpha \neq 0 \ \alpha = 1, \ldots, n + 1$. We can now find local coordinates on $\phi_\alpha : U_\alpha \to V_\alpha \in \mathbb{R}^n$

$$x_1 = \frac{X_1}{X_\alpha} \quad \ldots \quad x_{\alpha-1} = \frac{X_{\alpha-1}}{X_\alpha} \quad x_{\alpha+1} = \frac{X_{\alpha+1}}{X_\alpha} \quad \ldots \quad x_n = \frac{X_n}{X_\alpha}. \tag{1.9}$$

$\square$

Exercise 1.1: Prove smoothness of $\phi_\beta \circ \phi_\alpha^{-1}$.

Now it turns out that this manifold is equivalent to $\mathbb{RP}^n = S^n / \mathbb{Z}^2$. From quantum mechanics, we know that this means in particular $\mathbb{RP}^3 = SO(3)$. This is illustrated in 1.3.

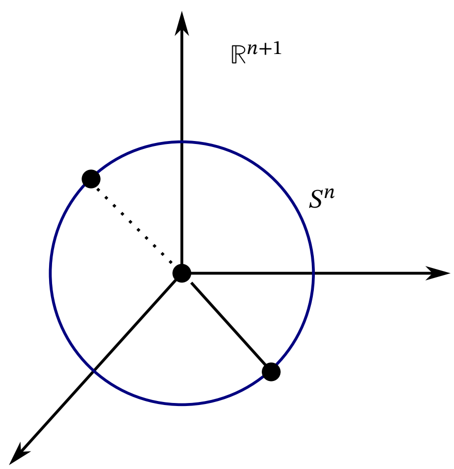

Figure 1.3: Real projective space $\mathbb{RP}^n$ is isomorphic to $S^n/\mathbb{Z}^n$, identifying antipodal points.

2 Vector Fields

Let $M, \widetilde{M}$ be smooth manifolds of dimension $n, \tilde{n}$.

Definition 4 (smooth map)**:** A map $f : M \to \widetilde{M}$ is *smooth* if $\widetilde{\phi}_\beta \circ f \circ \phi_\alpha^{-1}$ is a smooth map from $\mathbb{R}^n$ to $\widetilde{\mathbb{R}}^n$ for all α, β. We call $f : M \to \mathbb{R}$ a *function*, whereas we call $f : \mathbb{R} \to M$ a *curve*.

Let $\gamma : \mathbb{R} \to M$ be a curve. For some $U \in M$, $U \simeq \mathbb{R}^n$, we can define local coordinates $(x^1, \dots, x^n)$

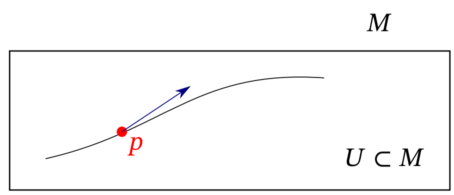

Figure 2.1

.

Definition 5 (tangent vector)**:** A *tangent vector* V to γ at p is

$$V|_p = \left.\frac{\mathrm{d}\psi}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \in T_pM, \tag{2.1}$$

where T_pM is the *tangent space* to M at p.

Definition 6 (tangent bundle)**:** We define the *tangent bundle* as $TM := \bigcup_{p\in M} T_pM$.

Definition 7 (vector field)**:** A *vector field* assigns a tangent vector to all $p \in M$.

Let $f : M \to \mathbb{R}$. The rate of change of f along γ is

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(x^a(\epsilon))|_{\epsilon=0} = \sum_a \dot{x}^a \frac{\partial f}{\partial x^a} \tag{2.2}$$

$$= \sum_a V^a \left.\frac{\partial f}{\partial x^a}\right|_{\epsilon=0}, \tag{2.3}$$

where $V^a := \dot{x}^a|_{\epsilon=0,\ldots,x_n}$.

Vector fields are first order differential operators

$$V = \sum_a V^a(\mathbf{x}) \frac{\partial}{\partial x^a}. \tag{2.4}$$

The derivatives $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}\Big|_p$ form a basis of T_pM.

2.1 Integral curves

Definition 8 (integral curve)**:** An *integral curve* (a *flow*) of a vector field is defined by

$$\dot{\gamma}(\epsilon) = V|_{\gamma(\epsilon)}, \tag{2.5}$$

where the dot denotes differentiation with respect to ϵ.

On n first order ODEs: $\dot{x}^a = V^a(x)$.

There exists a unique solution given initial data $X^a(0)$. Given a solution $X^a(\epsilon)$, we can expand it in a Taylor series as

$$X^a(\epsilon) = X^a(0) + V^a \cdot \epsilon + O(\epsilon^2). \tag{2.6}$$

Up to first order in ϵ, the vector field determines the flow. We call V a *generator* of its flow.

The following example illustrates how you get from a vector field to its flow.

Example 2.1.1 ($M = \mathbb{R}^2, x^a = (x, y)$)**:** Consider the vector field $V = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. The system of ODEs we solve is $\dot{x} = x$ and $\dot{y} = 1$. This gives us the integral curve $(x(\epsilon), y(\epsilon)) = (x(0)e^\epsilon, y(0) + \epsilon)$. From this we can see that $x(\epsilon)\cdot\exp(-y(\epsilon))$ is constant along γ. Using this we can draw the unparametrised integral curve in Fig. 2.2.

This example motivates the following definition.

Definition 9 (invariant)**:** An *invariant* of a vector field V is a function f constant along the flow of V.

$$f(x^a(0)) = f(x^a(\epsilon)) \qquad \forall\epsilon. \tag{2.7}$$

Equivalently, $V(f) = 0$.

Let us now consider an example that goes the other way: from flow to vector field.

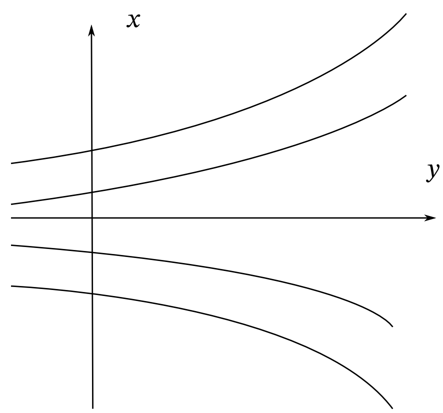

Figure 2.2

Example 2.1.2: Consider the 1 -parameter group of rotations of a plane.

$$(x(\epsilon), y(\epsilon)) = (x_0 \cos\epsilon - y_0 \sin\epsilon, x_0 \sin\epsilon + y_0 \cos\epsilon). \tag{2.8}$$

The associated vector field is

$$V = \left(\frac{\partial y(\epsilon)}{\partial\epsilon}\frac{\partial}{\partial y} + \frac{\partial x(\epsilon)}{\partial\epsilon}\frac{\partial}{\partial x}\right)\bigg|_{\epsilon=0} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial y}. \tag{2.9}$$

Now you can add vector fields, but there is also another operation.

Definition 10 (Lie bracket)**:** A *Lie bracket* $[V, W]$ of two vector fields V, W is a vector field defined by

$$[V, W](f) = V(W(f)) - W(V(f)) \qquad \forall f. \tag{2.10}$$

This is indeed another vector field since the commutator of two first order operators is another first order operator.

Example 2.1.3: Let $V = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $W = \frac{\partial}{\partial x}$. We then have $[V, W] = -W$.

This is not always the case but sometimes the Lie bracket reproduces some of the vector fields. There is an interesting algebraic structure to this.

Definition 11 (Lie algebra)**:** A *Lie algebra* is a vector space $\mathfrak{g}$ with an anti-symmetric, bilinear operation $[\ ,\ ]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, called a *Lie bracket*, which satisfies the *Jacobi identity*

$$[V,[U,W]] + [W,[V,U]] + [U,[W,V]] = 0 \qquad \forall U, V, W \in \mathfrak{g}. \tag{2.11}$$

We will spend some time discussing this abstractly, but then focus on the Lie algebras of vector fields in the main part of this course.

Any two vector spaces of a given dimension are isomorphic; there is nothing special other than the dimension distinguishing vector spaces. For Lie algebras this is not so.

Example 2.1.4: Even in dimension 2, which is the lowest non-trivial dimension, there are two Lie algebras (up to isomorphism)

$$a) \quad [V,W] = -W, \qquad b) \quad [V,W] = 0. \tag{2.12}$$

If the vector space underlying $\mathfrak{g}$ is finite-dimensional, and V_α, $\alpha = 1, \dots, \dim \mathfrak{g}$ is a basis of $\mathfrak{g}$, we can define the Lie algebra by specifying the brackets

$$[V_\alpha, V_\beta] = \sum_\gamma f_{\alpha\beta}{}^\gamma V_\gamma, \tag{2.13}$$

where $f_{\alpha\beta}{}^\gamma$ are the *structure constants.*

Example 2.1.5 ($\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R})$)**:** The vector space is given by $n \times n$ real matrices, and the Lie bracket is the matrix commutator. The dimension of this Lie algebra is $\dim \mathfrak{g} = n^2$.

Example 2.1.6 (Vector fields)**:** The set of all vector fields on a manifold M form an infinite-dimensional Lie algebra.

Example 2.1.7: Consider diff($\mathbb{R}$) or diff(S^1), vector fields on a line or on a circle respectively.

$$\text{diff}(\mathbb{R}), \quad x \in \mathbb{R}, \quad V_\alpha = -x^{\alpha+1}\frac{\partial}{\partial x} \tag{2.14}$$

$$\text{diff}(S^1), \quad \theta \in S^1, \quad V_\alpha = ie^{i\alpha\theta}\frac{\partial}{\partial\theta} \tag{2.15}$$

$$[V_\alpha, V_\beta] = (\alpha-\beta)V_{\alpha+\beta}. \tag{2.16}$$

Example 2.1.8 (Virasoro algebra)**:** The *Virasoro algebra* Vir $=$ diff(S^1) $\oplus\, \mathbb{R}$ is the central extension[1] of diff(S^1), with *central charge* $c = \mathbb{R}$.

$$\begin{cases} [V_\alpha, c] = 0 \\ [V_\alpha, V_\beta]_{\text{vir}} = (\alpha-\beta)V_{\alpha+\beta} + \dfrac{c}{12}(\alpha^3-\alpha)\delta_{\alpha+\beta,0} \end{cases} \tag{2.17}$$

Remark:

$$[f(\theta)\frac{\partial}{\partial\theta}, g(\theta)\frac{\partial}{\partial\theta}] = \underbrace{(fg'-gf')}_{\text{Wronskian}}\frac{\partial}{\partial\theta} \tag{2.18}$$

'After Witten'.

$$[f\frac{\partial}{\partial\theta}, g\frac{\partial}{\partial\theta}]_{\text{vir}} = [f\frac{\partial}{\partial\theta}, g\frac{\partial}{\partial\theta}] + \frac{ic}{48\pi}\int_0^{2\pi}(f'''g - g'''f)\,\mathrm{d}\theta \tag{2.19}$$

[1] We will meet the concept of central extension and central charge in this term's *String Theory* course.

Theorem 2 (Ado)**:** Every finite-dimensional Lie algebra is isomorphic to some matrix Lie algebra, a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.

> **Remark:** n is not necessarily the dimension of the Lie algebra.

3 Lie Groups

Definition 12 (Lie group)**:** A *Lie group* is a smooth manifold G, which is also a group, such that the group operations

$$\text{multiplication} \qquad G \times G \to G, \qquad (g_1, g_2) \longmapsto g_1 \cdot g_2 \tag{3.1}$$

$$\text{inverse} \qquad G \to G, \qquad g \longmapsto g^{-1} \tag{3.2}$$

are smooth maps between manifolds.

Example 3.0.1 ($G = GL(n, \mathbb{R}) \in \mathbb{R}^{n^2}$)**:** The general linear group $GL(n, \mathbb{R})$ is defined as the set of invertible matrices $\{g \in G \mid \det g \neq 0\}$. The dimension is $\dim(G) = n^2$.

Example 3.0.2 ($G = O(n, \mathbb{R})$)**:** This is the group of orthogonal matrices, defined by $\frac{1}{2}n(n+1)$ conditions $g^T g = \mathbb{1}$. The dimension is then $\dim O(n, \mathbb{R}) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$. We also have to check that these conditions define a manifold in the sense that the associated Jacobian has maximal rank.

Definition 13 (group action)**:** A *group action* on a manifold M is a map $G \times M \to M$ mapping $(g, p) \to g(p)$ such that

$$e(p) = p, \qquad g_1(g_2(p)) = (g_1 \cdot g_2)(p) \tag{3.3}$$

for all $p \in M$ and al g_1, g_2 on G.

Definition 14 (transformation group)**:** If we have a group action, we refer to G as a group of *transformations*.

Example 3.0.3: Take $M = \mathbb{R}^2$ and $G = E(2)$, the three-dimensional Euclidean group.

$$g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \tag{3.4}$$

Take $G_E \in G$ to be a one-parameter subgroup of G. There are three such subgroups

$$G_\theta : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos\theta \cdot x - \sin\theta \cdot y \\ \sin\theta \cdot x + \cos\theta \cdot y \end{pmatrix} \tag{3.5}$$

$$G_a : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x + a \\ y \end{pmatrix} \tag{3.6}$$

$$G_b : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x \\ y + b \end{pmatrix}. \tag{3.7}$$

Each of these one-parameter subgroups generates a flow. We can think of this flow as being gener-

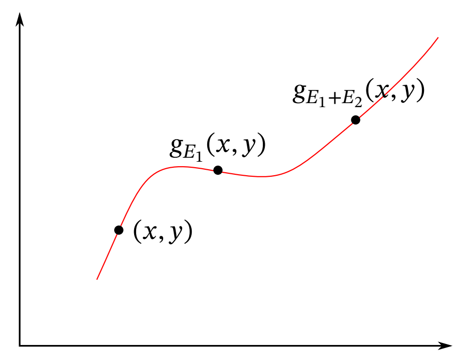

Figure 3.1

ated by a vector field $V|_p = \frac{\mathrm{d}}{\mathrm{d}E} g_E(p)\Big|_{E=0}$.

$$V_\theta = d\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{3.8}$$

$$V_a = \left(\frac{\mathrm{d}\tilde{x}}{\mathrm{d}a}\frac{\partial}{\partial \tilde{x}} + \frac{\mathrm{d}\tilde{y}}{\mathrm{d}a}\frac{\partial}{\partial \tilde{y}}\right)\Big|_{a=0} = \frac{\partial}{\partial x} \tag{3.9}$$

$$V_b = \frac{\partial}{\partial y}. \tag{3.10}$$

We define a 3-dimensional Lie algebra of $E(2)$ as

$$[V_a, V_\theta] = V_b \qquad [V_b, V_\theta] = -V_a \qquad [V_a, V_b] = 0, \tag{3.11}$$

represented by vector fields on M.

3.1 Geometry on Lie Groups

Definition 15 (tangent map)**:** Let $f: M \to \widetilde{M}$ be a smooth map between manifolds. We define the *tangent map* or *push forward* to be

$$\begin{aligned} f_*: T_p(M) &\to T_{f(p)}(\widetilde{M}) \\ V &\longmapsto f_*(V) = \frac{\mathrm{d}}{\mathrm{d}E} f(\gamma(E))\bigg|_{E=0}. \end{aligned} \tag{3.12}$$

This extends to the tangent bundle $T(M)$. If x^α are coordinates of $\mathcal{M} \supset M$, $(y^{\alpha'})$ coordinates on $\widetilde{\mathcal{M}} \subset M$, then

$$V = V^\alpha \frac{\partial}{\partial x^a} \qquad f_*(V) = V^\alpha \frac{\partial f^i}{\partial x^a} \frac{\partial}{\partial y^i}. \tag{3.13}$$

Definition 16 (Lie derivative)**:** Let V, W be vector fields, where V generates a flow $V = \dot{\gamma}$. The *Lie derivative* is

$$\mathcal{L}_V W|_p := \lim_{\epsilon \to 0} \frac{W(p) - \gamma(\epsilon)_* W(p)}{\epsilon} \tag{3.14}$$

We can extend this definition over the whole manifold.

Exercise 3.1: Show that $L_V W = [V, W]$.

Definition 17: On functions $f : M \to \mathbb{R}$, we define the Lie derivative as $\mathcal{L}_V(f) = V(f)$.

Claim 2 (Cartan's Magic Formula)**:** On differential forms, we can use the Leibniz rule to show that

$$\mathcal{L}_V \Omega = d(\iota_V \Omega) + \iota_V(d\Omega) = d(V \rightharpoonup \Omega) + V \rightharpoonup d\Omega. \tag{3.15}$$

Definition 18: We define the cotangent space $T_p^* M = \operatorname{Span}\{dx^1, \dots, dx^n\}$ as the space of one-forms. The cotangent bundle is then

$$\bigcup_{p \in M} T_p^* M = T^* M. \tag{3.16}$$

Definition 19 (r-form)**:** Using the wedge product, which is anti-commutative on one-forms $dx^i \wedge dx^j = -dx^j \wedge dx^i$, we can define an r-form

$$\Omega = \frac{1}{r!} \Omega_{ij\dots k} dx^i \wedge dx^j \wedge \cdots \wedge dx^k. \tag{3.17}$$

Definition 20 (contraction)**:** We write a *contraction* as

$$\frac{\partial}{\partial x^i} \rightharpoonup dx^j = \iota_{\frac{\partial}{\partial x^i}} dx^j = \delta_i^{\ j}. \tag{3.18}$$

For a general vector field V and one-form Ω, we have

$$\iota_V \Omega = V \rightharpoonup \Omega = V^i \frac{\partial}{\partial x^i} \rightharpoonup \Omega_j dx^j = V^i \Omega_j \delta^j_{\ i} = V^i \Omega_i. \tag{3.19}$$

Remark: No metric is needed to define contraction.

Definition 21: A *Lie algebra* $\mathfrak{g}$ of a Lie group G is the tangent space $T_e G$ to G at the identity. The Lie bracket on $\mathfrak{g}$ is the commutator of vector fields on G.

Definition 22 (Left translations)**:** For all $\mathfrak{g} \in G$, we define the *left translations*

$$\begin{aligned} L_g : G &\to G \\ h &\longmapsto g \cdot h \end{aligned} \tag{3.20}$$

Definition 23 (left invariant vector fields)**:** Using the left translation maps, we define their push forward $(L_g)_* : \mathfrak{g} \equiv T_e G \to T_g G$, which maps $V \in \mathfrak{g}$ to vector fields $(L_g)_*(V)$ on G. This defines *left invariant vector fields* as sections $V \in TG$ such that $(L_g)_* V = V$ for all $g \in G$. Therefore,

$$[(L_g)_* V, (L_g)_* W] = (L_g)_* [V, W]_{\mathfrak{g}}. \tag{3.21}$$

Remark: It is important to understand the notation!

Left-invariant vector fields form a basis of $\mathfrak{g}$, meaning that $\dim(G)$ is the number of global, non-vanishing vector fields on G.

Definition 24 (parallelisable)**:** A manifold M is *parallelisable* if there exists a set of vector fields $\{V_i\}$, where $i = 1, \dots, \dim M$, such that $\forall p \in M$, the tangent vectors $\{V_i(p)\}$ form a basis for the tangent space $T_p M$.

Claim 3: Lie groups are parallelisable manifolds.

Claim 4: The converse is not true.

Proof. S^1, S^3, S^7 are the only parallelisable spheres.

The first two are indeed manifolds:

$$S^1 = U(1), \qquad S^3 = SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| \, |a|^2 + |b|^2 = 1 \right\}. \tag{3.22}$$

However, S^7 is not a Lie group. □

This has introduced the field of K-theory.

Claim 5: Let $\{L_\alpha\}$, $\alpha = 1, \dots, \dim \mathfrak{g}$ be a basis of left invariant vector fields with $[L_\alpha, L_\beta] = \sum_\gamma f_{\alpha\beta}{}^{\gamma} L_\gamma$. Let σ^α be a dual basis of left-invariant one-forms, meaning that $L_\alpha \rightharpoonup \sigma^\beta = \delta_\alpha{}^\beta$. Then

$$d\sigma^\alpha + \frac{1}{2} f^\alpha{}_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma = 0. \tag{3.23}$$

Proof (Sheet 1). Use the identity

$$d\Omega(V, W) = V(\Omega(W)) - W(\Omega(V)) - \Omega([V, W]). \tag{3.24}$$

□

Watch out for signs and factors in the upcoming derivations! Things can easily go wrong.

Definition 25 (Maurer–Cartan 1-form)**:** Assume that G is a matrix Lie group. The *Maurer–Cartan one-form* on G is then defined as

$$\rho := g^{-1} dg. \tag{3.25}$$

Claim 6: The Maurer–Cartan 1-form

1. is left invariant,
2. takes values in the Lie algebra,
3. obeys the *Maurer–Cartan equation*

$$d\rho + \rho \wedge \rho = 0. \tag{3.26}$$

Proof. 1. With $g_0 \in G$ we have

$$(g_0 g)^{-1} d(g_0 g) = g^{-1} dg. \tag{3.27}$$

2. Take C a smooth curve $g(s) \subset G$.

$$g^{-1}(s)g(s+\epsilon) = \underbrace{\epsilon}_{\mathbb{1}} + \epsilon \underbrace{g^{-1}\frac{\mathrm{d}g}{\mathrm{d}\epsilon}}_{\in T_e G \simeq \mathfrak{g}}|_{\epsilon=0} + O(\epsilon^2). \tag{3.28}$$

So $g^{-1}dg = \sum_\alpha \sigma^\alpha \otimes T_\alpha$, where T_α are matrices with $[T_\alpha, T_\beta] = \sum_\gamma f_{\alpha\beta}{}^\gamma T_\gamma$.

3. Consider first the exterior derivative term

$$d\rho = \sum_\alpha d\sigma^\alpha \cdot T_\alpha = -\frac{1}{2} f^\alpha{}_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma \cdot T_\alpha. \tag{3.29}$$

The wedge product term is

$$\rho \wedge \rho = \sigma^\alpha T_\alpha \wedge \sigma^\beta T_\beta = \frac{1}{2}\sigma^\alpha \wedge \sigma^\beta [T_\alpha, T_\beta] = \frac{1}{2}\sigma^\alpha \wedge \sigma^\beta f_{\alpha\beta}{}^\gamma T_\gamma. \tag{3.30}$$

□

Example 3.1.1 (Heisenberg group)**:** The *Heisenberg group* (sometimes just called *Nil*) is the group of upper-triangular matrices

$$g = \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} = \mathbb{1} + xT_1 + yT_2 + zT_3, \tag{3.31}$$

where $\{T_i\}$ are also the generators of the Lie algebra. Explicitly, we have

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.32}$$

and so their commutation relations are

$$[T_1, T_2] = T_3, \quad [T_1, T_3] = 0 = [T_2, T_3]. \tag{3.33}$$

We can interpret $T_1 = \hat{x}$ as position, $T_2 = \hat{p}$ as momentum, and $T_3 = i\hbar\hat{\mathbb{1}}$ as the identity.

Remark: Examples such as the lecture today will be important for the exam! If someone gives you a matrix Lie group, you will proceed in this order.

Taking the inverse of (3.31), we construct the Maurer–Cartan 1-form

$$\rho = g^{-1}dg = \begin{pmatrix} 1 & -x & -z+xy \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} & dx & dz \\ & & dy \\ & & \end{pmatrix} \tag{3.34}$$

$$= T_1 dx + T_2 dy + T_3(dz - xdy). \tag{3.35}$$

We define the dual basis of left-invariant one-forms

$$\sigma^1 = dx, \qquad \sigma^2 = dy, \qquad \sigma^3 = dz - xdy \tag{3.36}$$

$$d\sigma^1 = 0 \qquad d\sigma^2 = 0 \qquad d\sigma^3 = -dx \wedge dy = d\sigma^1 \wedge \sigma^2. \tag{3.37}$$

From these we find the left-invariant vector fields

$$L_1 = \frac{\partial}{\partial x}, \qquad L_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad L_3 = \frac{\partial}{\partial z}. \tag{3.38}$$

We find the Lie algebra by computing the brackets

$$[L_1, L_2] = L_3, \qquad [L_1, L_3] = 0, \qquad [L_2, L_3] = 0. \tag{3.39}$$

This is the same as (3.33), the Lie algebra of the Heisenberg group. We have a choice of representing the Lie algebra either in terms of matrices, T_i or left-invariant vector fields L_i.

3.1.1 Right-Invariance

We could have defined the *right translations* $R_g(h) = h \cdot g$, and associated right-invariant vector fields and one-forms.

Claim 7: The commutation relations for right-invariant vector fields differs by a minus sign from left-invariant vector fields and they commute

$$[R_\alpha, R_\beta] = -f_{\alpha\beta}{}^{\gamma} R_\gamma, \qquad [R_\alpha, L_\beta] = 0. \tag{3.40}$$

Example 3.1.2 (Nil)**:** For the Heisenberg group, using $dg \cdot g^{-1}$

$$R_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \qquad R_2 = \frac{\partial}{\partial y}, \qquad R_3 = \frac{\partial}{\partial z}. \tag{3.41}$$

Notation (2)**:** Right-invariant vector fields are said to generate left-translations and vice-versa.

3.2 Metrics on Lie Groups

We can do some local differential geometry by defining a left-invariant metric on G.

Definition 26 (left-invariant metric)**:** A *left-invariant metric* h on a Lie group G is of the form

$$h = g_{\alpha\beta}\sigma^\alpha \odot \sigma^\beta, \qquad \alpha, \beta = 1, \dots, \dim G, \tag{3.42}$$

where $g_{\alpha\beta}$ is a non-degenerate constant matrix.

Given the group and left-invariant metric, the right-invariant vector fields are generating isometries for this metric, which means that they are Killing vectors. Recall that the Maurer–Cartan 1-form is $\rho = g^{-1}dg = \sigma^\alpha \otimes T_\alpha$, where σ^α are invariant under $g \to g_0 g$. Therefore

$$\mathcal{L}_{R_\alpha}\sigma^\beta = 0 \qquad \forall \alpha, \beta. \tag{3.43}$$

Therefore, right invariant vector fields are Killing vectors for h, meaning that

$$\mathcal{L}_{R_\alpha}(h) = 0. \tag{3.44}$$

Example 3.2.1 (Nil)**:** The metric is defined as

$$h = \delta_{\alpha\beta}\sigma^\alpha \cdot \sigma^\beta = dx^2 + dy^2 + (dz - xdy)^2. \tag{3.45}$$

How do we find the isometries?

- We can see that the metric components do not involve z , so it is invariant under $z \to z + \omega$, which is generated by $\frac{\partial}{\partial z} = R_3$.
- Similarly, we can see the same for $y \to y + \epsilon$, which is generated by $\frac{\partial}{\partial y} = R_2$.
- Finally, let us consider what happens for $x \to x + \epsilon$. As it stands, this is not an isometry. The parenthesis includes a term δdy, which we can get rid off by introducing another transformation $z \to z + \epsilon y$. This is generated by $\frac{\partial}{\partial x} + y\frac{\partial}{\partial z} = R_1$.

These agree with the right-invariant vector fields of Eq. (3.41).

3.2.1 Kaluza–Klein Interpretation

Consider the motion of a charged particle on the space of orbits of $R_3 = \frac{\partial}{\partial z}$.

This was introduced as a way to combine the two known forces at the time: gravity and electromagnetism. This is done by introducing extra dimensions, by looking at gravity in dimension $d = 5$, which reduces to gravity and electromagnetism in dimension $d = 4$. The higher dimension is taken to be compactified into a circle of very large radius, so it is not detected by experiment. This idea is still stuck with us today in *String Theory*.

Example 3.2.2: For the Heisenberg group, the metric is independent of z, so we can take our manifold to be periodic in z.

To find the equations of motion, it is useful to write down the geodesic Lagrangian

$$\mathscr{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + (\dot{z} - x\dot{y})^2), \tag{3.46}$$

where the dot denotes $\bullet = \frac{\mathrm{d}}{\mathrm{d}s}$. The Euler–Lagrange equations are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial \mathscr{L}}{\partial \dot{x}^i} &= \frac{\partial \mathscr{L}}{\partial x^i} && (3.47)\\ \ddot{x} &= -\dot{y}(\dot{z} - x\dot{y}) && (3.48)\\ \frac{\mathrm{d}}{\mathrm{d}s}\left(\dot{y} - x(\dot{z} - x\dot{y})\right) &= 0 && (3.49)\\ \frac{\mathrm{d}}{\mathrm{d}s}(\dot{z} - x\dot{y}) &= 0. && (3.50) \end{aligned}$$

Using the final equation to introduce a constant $c = \dot{z} - x\dot{y}$, the first two equations reduce to

$$\ddot{x} = -c\dot{y} \qquad \& \qquad \ddot{y} = c\dot{x}. \tag{3.51}$$

Compare this with geodesic motion in a magnetic field. Let the spacetime be the Riemannian manifold $(M, g = g_{ij}dx^i \odot dx^j)$ and the magnetic field be the closed 2-form $F = \frac{1}{2}F_{ij}dx^i \wedge dx^j$. The components of the Levi–Civita connection associated to g are

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right). \tag{3.52}$$

The geodesic equation of motion is then given by

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = cF^i{}_j\dot{x}^j, \tag{3.53}$$

where we see $F^i{}_j$ as an endomorphism rather than a 2-form. This is the general form of geodesic motion in a magnetic field. We want to compare this to (3.51), so we take $M = \mathbb{R}^2$ and $g_{ij} = \delta_{ij}$, which gives $\Gamma^k_{ij} = 0$. Moreover, we take $F = -dx \wedge dy$, meaning that $F_{ij} = -\epsilon_{ij}$ is the volume-form. So geodesics of the left-invariant metric h on $G = \mathrm{Nil}$ projects to the trajectories of a charged particle in a constant magnetic field. We think of this as in Fig. 3.2.

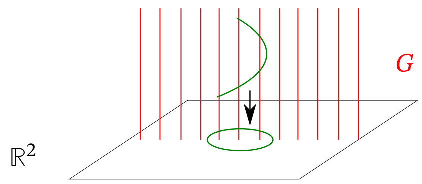

Figure 3.2: Kaluza–Klein reduction.

For the general case of Kaluza–Klein reduction, we have

$$h = (dz + A)^2 + g_{ij} dx^i \odot dx^j. \tag{3.54}$$

Take coordinates (z, x^i), where z is the extra dimension. Moreover, A is a one-form on M and we define $F = dA$. The geodesic equation is

$$\dot{z} + A_i \dot{x}^i = c \qquad \text{(conserved charge)} \tag{3.55}$$

Assume z is periodic. On G, the Laplacian is

$$\nabla^2 = L_1^2 + L_2^2 + L_3^2. \tag{3.56}$$

The Schrödinger equation is then $-\nabla^2\phi = E\phi$.

$$\phi_{xx} + \phi_{zz} + (\partial_y + x d_z)^2\phi = -E\phi. \tag{3.57}$$

Changing variables to $\phi = \Psi(x,y)e^{iez}$, we have

$$\Psi_{xx} - e^2\Psi + (\partial_y + ixe)^2\Psi = -E\Psi. \tag{3.58}$$

We can package some of the terms together to obtain a Maxwell potential. Recall that $F = dy \wedge dx = dA$. The equation for a charged particle moving in a magnetic field is

$$(\partial_x - ieA_x)^2\Psi + (\partial_y - ieA_y)^2\Psi = -(E - e^2)\Psi. \tag{3.59}$$

Comparing these equations, find that $A_x = 0$ and $A_y = -x$, which means that we have

$$A = -xdy, \qquad dA = -dx \wedge dy = F. \tag{3.60}$$

Landau problem; $0 \leq z < 2\pi L$. Charge quantisation $e \cdot L \in \mathbb{Z}$.

3.2.2 Killing Metric

Recall the Maurer–Cartan one-form $\rho = g^{-1}dg = \sigma^\alpha \otimes T_\alpha$. Define a metric to be

$$h := -\operatorname{Tr}\bigl(g^{-1}dg \odot g^{-1}dg\bigr) \tag{3.61}$$

$$= -\operatorname{Tr}\bigl(T_\alpha \cdot T_\beta\bigr)\sigma^\alpha \odot \sigma^\beta \tag{3.62}$$

$$= h_{\alpha\beta}\sigma^\alpha \odot \sigma^\beta \tag{3.63}$$

$$= -\operatorname{Tr}\bigl(dg \cdot g^{-1} \odot dgg^{-1}\bigr). \tag{3.64}$$

This is both left-invariant and right-invariant. We say that the metric is *bi-invariant*.

Example 3.2.3 ($G = SU(2)$)**:** As a manifold, $SU(2)$ is the three-dimensional sphere S^3. The Killing metric will be the round metric on S^3. Its isometry group is $SO(4)$. It fits into $SO(3) \rtimes SO(3)$; one of these generates the left-invariant vector fields and the other the right-invariant ones.

4 Hamiltonian Mechanics and Symplectic Geometry

Let M be a $2n$-dimensional manifold, which we refer to as the *phase space*. It does not come equipped with a metric, but there is another structure on it.

Definition 27 (Poisson bracket)**:** If $f, g : M \to \mathbb{R}$ are functions on the phase space, then their *Poisson bracket* is

$$\left\{f, g\right\}_{\mathrm{PB}} = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a}, \tag{4.1}$$

where (q^a, p_a), with $a = 1, \dots, n$ is a local coordinate system on M.

Definition 28 (Hamiltonian)**:** The *Hamiltonian* is a function $H : M \to \mathbb{R}$ such that Hamilton's equations hold:

$$\dot{p}_a = -\frac{\partial H}{\partial q^a}, \qquad \dot{q}^a = \frac{\partial H}{\partial p_a}. \tag{4.2}$$

Definition 29 (Hamiltonian vector field)**:** The *Hamiltonian vector field* is

$$X_H = \frac{\partial H}{\partial p_a}\frac{\partial}{\partial q^a} - \frac{\partial H}{\partial q^a}\frac{\partial}{\partial p_a}, \tag{4.3}$$

whose integral curves are $t \to (\mathbf{p}(t), \mathbf{q}(t))$.

A more general framework is kicked off by the following definition.

Definition 30 (Poisson manifold)**:** Let M be phase space and $\omega^{ij} = \omega^{[ij]}$, for $i, j = 1, \dots, \dim M = m$, be a tensor field on M. We call $(M, \omega^{ij} = \omega)$ a *Poisson manifold* and ω a *Poisson structure*, if the Poisson bracket

$$\left\{f, g\right\}_{\mathrm{PB}} = \omega^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} \tag{4.4}$$

is such that the Jacobi identity

$$\left\{f, \left\{g, h\right\}_{\mathrm{PB}}\right\}_{\mathrm{PB}} + \left\{h, \left\{f, g\right\}_{\mathrm{PB}}\right\}_{\mathrm{PB}} + \left\{g, \left\{h, f\right\}_{\mathrm{PB}}\right\}_{\mathrm{PB}} = 0, \tag{4.5}$$

holds for all functions f, g, h on M.

Not any anti-symmetric ω^{ij} satisfies this; the Jacobi identity gives conditions on ω^{ij}.

Remark: We do not distinguish between position and momenta x^i.

In the following, we will drop the subscript 'PB' to denote the Poisson bracket.

Example 4.0.1: Let $M = \mathbb{R}^3$ and $\omega^{ij} = \epsilon^{ijk}x^k$. The Poisson brackets are

$$\{x^1, x^2\} = x^3, \qquad \{x^3, x^1\} = x^2, \qquad \{x^2, x^3\} = x^4. \tag{4.6}$$

We can then define the Casimir

$$f(r) = r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{4.7}$$

This Poisson commutes with the x^i, meaning that $\{f(r), x^i\} = 0$. For example, we have

$$\{(x^1)^2 + (x^2)^2 + (x^3)^2, x^1\} = 2x^2 \overbrace{\{x^2, x^1\}}^{-x^3} + 2x^3 \overbrace{\{x^3, x^1\}}^{x^2} = 0. \tag{4.8}$$

Take the Hamiltonian to be

$$H = \frac{1}{2}\left(\frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + \frac{(x^3)^2}{a_3}\right). \tag{4.9}$$

Then the time-evolution is given by

$$\dot{g} = \{g, H\}, \qquad \underbrace{\dot{x}^i = \omega^{ij}\frac{\partial H}{\partial x^j}}_{\text{Hamilton's equations}}. \tag{4.10}$$

Writing Hamilton's equations out explicitly gives Euler's equations for a rigid body

$$\dot{x}^1 = \frac{a_3 - a_2}{a_2 a_3}x^2 x^3, \qquad \dot{x}^2 = \frac{a_1 - a_3}{a_1 a_3}x^1 x^3, \qquad \dot{x}^3 = \frac{a_2 - a_1}{a_2 a_1}x^2 x^1. \tag{4.11}$$

Example 4.0.2: Let us restrict the Poisson structure from the first example to $S^2 \subset \mathbb{R}^3$.

$$x^1 = \sin\theta\cos\phi, \qquad x^2 = \sin\theta\sin\phi, \qquad x^3 = \cos theta. \tag{4.12}$$

So θ, ϕ are functions of $\mathbb{R}^3$. Get

$$\{\theta, \phi\} = \frac{1}{\sin\theta} \tag{4.13}$$

(Exercise) and a Poisson structure on S^2, which is non-degeneate,

$$(\omega^{-1})_{ij}dx^i \wedge dx^j = \underbrace{\sin\theta d\theta \wedge d\phi}_{\text{symplectic structure}}, \tag{4.14}$$

where $x^1 = (\theta, \phi)$.

Last time, we discussd Poisson structures, which we can now specialise.

Definition 31 (symplectic manifold)**:** A *symplectic manifold* is a smooth manifold M of dimension $2n$ with a closed 2-form $\omega \in \Lambda^2(\mathcal{M})$, which is non-degenerate, meaning that

$$\underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n} \neq 0, \tag{4.15}$$

or ω is a 2×2 matrix of maximal rank.

The symplectic form ω provides an isomorphism between TM and T^*M as

$$v \in TM \longmapsto v \rightharpoonup \omega \in T^*M. \tag{4.16}$$

However, we tave to be more careful as this is now antisymmetric. If $f : M \to \mathbb{R}$, then df is a 1-form and is naturally associated to a Hamiltonian vector field X_f,

$$X_f \rightharpoonup \omega = -df, \tag{4.17}$$

where f is the Hamiltonian.

Claim 8: We can define a Poisson bracket by

$$\{f, g\}_{\mathrm{PB}} := X_g(f) = \omega(X_g, X_f) \tag{4.18}$$

Remark: Note that this is antisymmetric since $\omega(X_g, X_f) = -\omega(X_f, X_g)$.

In local coordinates, the Poisson bracket is

$$\{f, g\} = \sum_{i,j=1}^{2n} \omega^{ij} \frac{\partial f}{\partial x^j} \frac{\partial g}{\partial x^i}. \tag{4.19}$$

Exercise 4.1: The Jacobi identity follows from the closure $d\omega = 0$.

If we compute the Lie bracket of two vector fields X_f, X_g, we find the *anti-homomorphism*

$$[X_f, X_g] = -X_{\{f,g\}}. \tag{4.20}$$

Hamiltonian vector fields preserve the symplectic form. The finite way of saying this is that they generate a flow under which the symplectic form is invariant. Infinitesimally this is expressed with the Lie derivative as

$$\mathcal{L}_{X_f}\omega = d(X_f \rightharpoonup \omega) + X_f \rightharpoonup \cancel{d\omega} = -d(df) = 0. \tag{4.21}$$

Theorem 3 (Darboux)**:** Let (M, ω) be a $2n$ -dimensional symplectic manifold. There exist *local* coordinates $x^1 = q^1, \dots, x^n = q^n, x^{n+1} = p_1, \dots, x^{2n} = p_n$ around any point in M, such that

$$\omega = \sum_{a=1}^{n} dp_a \wedge dq^a, \tag{4.22}$$

and the Poisson bracket takes the standard form.

Proof. The proof proceeds by induction with respect to half of the dimension of the symplectic manifold. We first choose an arbitrary function $p_1 : M \to \mathbb{R}$ (which will later become the first momentum coordinate). Given this function, we search for another function $q^1 : M \to \mathbb{R}$ such that

$$X_{p_1}(q^1) = 1. \tag{4.23}$$

We denote this as $\dot{q}^1 = 1$. This is an ordinary differential equation, which will have a solution[1] subject to some initial condition.

The second step is as follows. Consider the surface $M_1 = \{x \in M, p_1 = \text{const.}, q^1 = \text{const.}\}$. This is a submanifold $M_1 \subset M$, which we can prove from the embedding theorem; the maximal rank condition of the Jacobian is encoded in (4.23). This M_1 is locally symplectic with symplectic form $\omega_1 \equiv \omega|_{p_1, q^1 \text{ const.}}$

Now look for p_2, q^2 and so on. A full proof is given in the book by Arnold. □

Claim 9: Let Q, which will play the role of configuration space, be an n-dimensional manifold. The cotangent bundle T^*Q admits a global symplectic structure.

Proof. Our goal is to describe what this symplectic structure is. We have a projection $\pi : T^*Q \to Q$ acting as $\pi(q, p) = q$.

Definition 32 (pull-back)**:** Given a map $f : M \to N$, the *pull-back* $f^* : T^*_{f(p)}N \to T^*_p M$ defined as

$$f^*(p)(V) = p(f_* V). \tag{4.24}$$

In local coordinates, taking x^i, with $i = 1, \dots, \dim M$ coordinates on M and y^a with $a = 1, \dots, \dim N$ coordinates on N, we can use the chain rule to write explicitly

$$f^*(dy^a) = \sum_i \frac{\partial f^a}{\partial x^i} dx^i. \tag{4.25}$$

The pull-back of π is a map

$$\pi^* : T^*(Q) \to T^*(T^*Q). \tag{4.26}$$

[1]The assumption for the existence theorem would be that we work in a Lipshitz-class of functions.

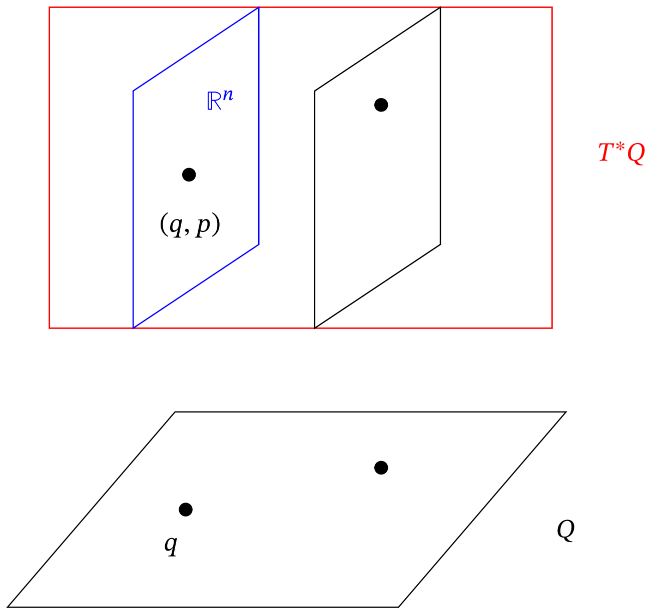

Figure 4.1

Say that $p \in T^*(Q)$ is a one-form on Q. We can define $\theta = \pi^*(p)$ and from this we get our canonical symplectic form

$$\omega = d\theta. \tag{4.27}$$

This is manifestly closed. This construction does not depend on coordinates, but usually in $(p_{\dot{a}}, q^{\dot{a}})$ coordinates we have

$$\theta = p_{\dot{a}} dq^{\dot{a}}, \qquad \omega = d\theta = dp_a \wedge dq^a. \tag{4.28}$$

□

We can define yet another structure that symplectic geometry gives us.

Definition 33 (Canonical transformations)**:** Let (M, ω) be a $2n$-dimensional symplectic manifold. An endomorphism $f : M \to M$ is called *canonical* when $f^*(\omega) = \omega$.

The one-parameter groups of canonical transformations are generated by Hamiltonian vector fields. For dimension $n = 1$, these are area preserving maps.

Consider the canonical transformation $(p_a, q^a) \xrightarrow{f} (P_a, Q^a)$.

$$d(\mathbf{p} \cdot d\mathbf{q}) = -d(\mathbf{Q} \cdot d\mathbf{P}), \tag{4.29}$$

where $\mathbf{P} = P(p, q)$ and $Q = Q(p, q)$. Then

$$d(\mathbf{p} \cdot d\mathbf{q} + \mathbf{Q} d\mathbf{P}) = 0. \tag{4.30}$$

So the one-form $\mathbf{p} \cdot d\mathbf{q} + \mathbf{Q} \cdot d\mathbf{P}$ is a closed-one form. Locally, this implies that it is exact, meaning that there is some *generating function* $S = S(\mathbf{q}, \mathbf{P})$ such that

$$\mathbf{p} \cdot d\mathbf{q} + \mathbf{Q} d\mathbf{P} = dS. \tag{4.31}$$

From this function we can define Q and p as

$$Q^a = \frac{\partial S}{\partial p_a}, \qquad p_a = \frac{\partial S}{\partial q^a}. \tag{4.32}$$

This gives $\mathbf{P}(q, p)$ and $Q(q, p)$.

4.1 Geodesics, Killing vectors, Killing tensors

We will explore the connection between Riemannian and symplectic geometry. Let (M, g) be a (pseudo-)Riemannian manifold of dimension n. In coordinates, we have

$$g = g_{ij}(x) dx^i dx^j. \tag{4.33}$$

Then we know from the *General Relativity* course, that there exists at unique Levi-Civita connection Γ^i_{jk} such that for geodesics $x^i = x^i(\tau)$, we have

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^i \dot{x}^j = 0. \tag{4.34}$$

Geodesics on M are integral curves of a Hamiltonian vector field on (T^*M, ω).

As illustrated in Fig. 4.2, we are looking for a curve in (T^*M, ω) specified by a single point (x^i, p_i) which projects down to the geodesic in M.

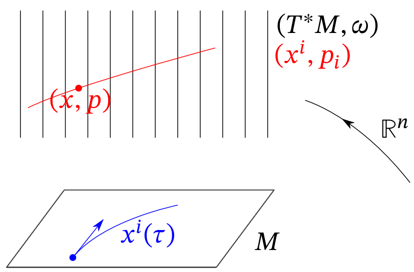

Figure 4.2

$$\dot{x}^i = p^i = g^{ij} p_j = X_H(x^i) \tag{4.35}$$

$$\dot{p}^i = -\Gamma^i_{jk} p^i p^k = X_H(p^i) \tag{4.36}$$

The Hamiltonian vector field is

$$X_H = g^{ij} p_i \frac{\partial}{\partial x^j} - \Gamma^i_{jk} p^j p^k \frac{\partial}{\partial p^i}, \tag{4.37}$$

where $H = \frac{1}{2} g^{ij}(x) p_i p_j$.

Canonical symplectic form $\omega = dp_i \wedge dx^i$.

$$H \to X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial p_i} \tag{4.38}$$

$$= g^{ij} p_j \frac{\partial}{\partial x^i} - \frac{1}{2} \frac{\partial g^{jk}}{\partial x^i} p_j p_k \frac{\partial}{\partial p_i}. \tag{4.39}$$

Now use

$$\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^m_{ki} g_{jm} - \Gamma^m_{kj} g_{im} = 0 \tag{4.40}$$

Definition 34 (Killing vector)**:** *Killing vectors K* satisfy

$$\mathcal{L}_K g = 0 \iff \nabla_{(i}K_{j)} = 0. \tag{4.41}$$

These correspond to first integrals of the Hamiltonian flow which are linear in the momenta. Poisson commuting with H.

$$\{\underbrace{\kappa^i p_i}_{\kappa}, H\}_{\text{PB}} = \frac{\partial \kappa^i}{\partial x^j} p_i g^{jk} p_k - \frac{1}{2}\kappa^i \frac{\partial g_{jk}}{\partial x^i} p^j p^k \tag{4.42}$$

$$\overset{(4.40)}{=} \nabla_{(i}K_{j)} p^i p^j = 0 \tag{4.43}$$

Killing vectors are symmetries of the Hamiltonian flow.

4.1.1 Killing Tensors

Definition 35 (Killing tensor)**:** Write

$$\kappa = K^{ij\dots k} \underbrace{p_i p_j \dots p_k}_{r}. \tag{4.44}$$

We assume nothing about κ except that it Poisson-commutes with the Hamiltonian H. We find that this corresponds to

$$\{H, \kappa\}_{\text{PB}} = 0 \iff \nabla_{(i}\kappa_{jk\dots l)} = 0. \tag{4.45}$$

A κ satisfying this is called a rank-r *Killing tensor*.

We will see that we can think of Killing tensors as higher / hidden symmetries, since we can see them in the cotangent bundle but not in the manifold itself.

The associated Hamiltonian vector field is

$$X_\kappa = \frac{\partial \kappa}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial \kappa}{\partial x^i}\frac{\partial}{\partial p_i} \tag{4.46}$$

$$= rK^{i_1\dots i_r} p_{i_1} \dots p_{i_{r-1}} \frac{\partial}{\partial x^{i_r}} - \frac{\partial K^{i_1\dots i_r}}{\partial x^k} p_{i_1} \dots p_{i_r} \frac{\partial}{\partial p_k}. \tag{4.47}$$

Projecting this down onto the manifold gives

$$\pi_k(X_\kappa) = \begin{cases} 0, & \text{if } r > 1 \\ K^i \frac{\partial}{\partial x^i}, & \text{if } r = 1. \end{cases} \tag{4.48}$$

If $r > 1$, there is no ‘geometric’ symmetry on M, but κ is constant along geodesic.

Example 4.1.1: The Kerr black hole does not have enough Killing vectors to solve the geodesic equations, but using the Killing tensors allows us to find the geodesics.

4.2 Integrability

Intuitively, a Hamiltonian system (e.g. geodesic motion) is *integrable* if there exist sufficiently many first integrals, i.e. functions constant along the flow of the Hamiltonian vector field X_H.

Definition 36 (integrable system)**:** An *integrable system* is a symplectic manifold (M, ω) of dimension $\dim M = 2n$, together with n functions $f_i : M \to \mathbb{R}$, $i = 1, \dots, n$, with the properties of

involution: $\{f_i, f_j\} = 0$ for all i, j,

independence: $df_i \wedge df_j \wedge \cdots \wedge df_n \neq 0$.

This is a strange definition because it does not specify any dynamics, any equations of motion.

The point is that with such a system any f_i in our system can be declared to be a Hamiltonian. The corresponding Hamilton's equations will be solvable!

Theorem 4 (Arnold–Liouville)**:** Let (M, ω, f_i) form an integrable system with the Hamiltonian chosen to be $H = f_1$. Then

- The level set $M_f = \{x \in M \mid f_1 = c_1, \dots, f_n = c_n\}$ (if connected[1]) is diffeomorphic to $\mathbb{R}^k \times T^{n-k}$ for some $0 \leq k \leq n$.[2]
- There exists a canonical transformation to action-angle variables
$$\phi_1, \dots, \phi_n, I_1, \dots, I_n \tag{4.49}$$
such that in a neighbourhood of M_f in M, ϕ_i are the coordinates, called *angles*, on M_f, and I_i are first integrals, called the *actions*.
- Hamilton's equations are solvable by quadratures
$$\dot{I}_i = \frac{\partial H}{\partial \phi_i} = 0 \qquad \dot{\phi}_i = \frac{\partial H}{\partial I_i} = \Omega_i(I_1, \dots, I_n), \tag{4.50}$$
so $I_i(t) = I_i(0)$ and $\phi_i(t) = \phi_i(0) + \Omega_i t$.

Proof. The df_k for $k = 1, \dots, n$ are independent, so M_f is a manifold of dimension n. Any function f_k gives rise to a Hamiltonian vector field X_{f_k}. If we contract this with any differential, we have

$$X_{f_k} \lrcorner\, df_j = X_{f_k}(f_j) = -\{f_k, f_j\}_{\text{PB}} = 0, \qquad \forall j, k. \tag{4.51}$$

If df_k are normal to M_f, then the corresponding Hamiltonian vector fields X_{f_j} are tangent to M_f. □

[1] If M_f is not connected, the theorem applies to each connected component.
[2] Usually this theorem is stated with M_f *compact*. In that case, we have $k = 0$ and we just have a torus T^n.

Proof. Then using (4.20) we have the anti-homomorphism $[X_{f_i}, X_{f_j}] = -X_{\{f_i,f_j\}} = 0$. The X_f generate the action of n-dimensional Abelian group $\mathbb{R}^n$ on M and on M_f. Let $p_0 \in M$ and consider the lattice Γ of vectors in $\mathbb{R}^n$, which preserve p_0. This $\Gamma \subset \mathbb{R}^n$ is a discrete subgroup. Therefore, by the orbit-stabiliser theorem,

$$M_f \simeq \mathbb{R}^n/\Gamma = \begin{cases} T^n, & \text{if compact} \\ T^{n-k} \times \mathbb{R}^k, & \text{otherwise} \end{cases}. \tag{4.52}$$

Assume M_f is compact, so that

$$M_f = T^n := \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}. \tag{4.53}$$

Different constants c_i give us different choices. For every choice, we can find a torus. Through every point $p_0 \in M$, there is exactly one torus. We say that M is *foliated* by tori (different choices of c_i).

Moreover, $M_f \subset M$ is a *Lagrangian* submanifold, meaning that $\omega|_{M_f} = 0$. This is because we proved that the n Hamiltonian vector fields are tangent to M_f. These are independent, so at each point $p \in M_f$ they span the tangent space T_pM_f. It is thus sufficient to show that

$$\omega(X_{f_i}, X_{f_j}) = 0, \tag{4.54}$$

which is indeed the case since $\{f_i, f_j\} = 0$.

Now let us construct the action-angle variables (I, ϕ). Local argument in a neighbourhood of M_f in M. $\omega = d\theta$.

Taking any closed contractable curve C, which does not wind around the whole of the torus, as shown in Fig. 4.3, we have

$$\iint_{D \in M_f} \omega = \oint_C \theta = 0. \tag{4.55}$$

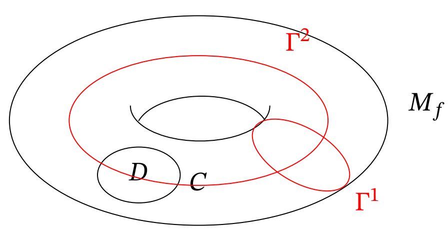

Figure 4.3

missed something

The canonical transformation is

$$(\mathbf{p}, \mathbf{q}) \to (I, \phi) \qquad \omega = d\theta = d(\mathbf{p} \cdot d\mathbf{q}) \tag{4.56}$$

In analogy to last time, we define the generating function to be

$$S(q, I) = \int_{q_0}^{q} \theta. \tag{4.57}$$

The canonical transformations obtained from this generating function are independent of the path taken. This is because we can use Stokes' theorem to integrate to zero around the closed loop obtained by concatenating two different paths. Their difference is thus zero. If the path winds around the whole in the torus, we can add a bit to the action, which will not impact the derivatives with respect to the action, and thus will not impact the canonical transformations.

We define the angles ϕ_k for $k = 1, \dots, n$ as

$$\phi_k = \frac{\partial S}{\partial I_k}. \tag{4.58}$$

These obey

$$\{\phi_k, \phi_j\} = 0, \qquad \{\phi_k, I_j\} = \delta_{kj}. \tag{4.59}$$

The Hamiltonian $H = f_i = H(I_1, \dots, I_n)$ gives time-evolution:

$$\dot{I}_k = -\frac{\partial H}{\partial \phi_k} = 0, \qquad \dot{\phi}_k = \frac{\partial H}{\partial I_k} = \text{const.} \tag{4.60}$$

As in A-L. □

Example 4.2.1 ($M = \mathbb{R}^2$)**:** All Hamiltonian systems are integrable. Take $H = \frac{1}{2}(p^2 + \omega_0^2 q^2)$, where (p, q) are coordinates on $\mathbb{R}^2$ and ω_0 is a constant frequency. Construct

$$M_f = \{(p, q) \in \mathbb{R}^2 \mid H = E\} = \Gamma. \tag{4.61}$$

Imposing one condition on $\mathbb{R}^2$ gives a curve. So M_f is a one-dimensional torus, which has just one cycle Γ. The integral is

$$I = \frac{1}{2\pi} \oint_{\Gamma} \phi d\phi. \tag{4.62}$$

We can solve this in two ways. Either ??. Or we can use Stokes' theorem, by considering Fig. 4.4. We have

$$I = \frac{1}{2\pi} \oint_{\Gamma} \phi d\phi = \frac{1}{2\pi} \iint_{\mathcal{E}} dp dq = \frac{E}{\omega}. \tag{4.63}$$

So $I = \frac{H}{\omega}$ and $\dot{\phi} = \frac{\partial H}{\partial I} = \omega$. Therefore, $\phi(t) = \phi(0) + \omega(t)$.

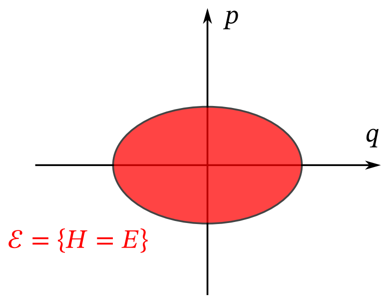

Figure 4.4

4.3 Geodesics in Non-Riemannian Geometries

We want to look at trajectories in Newtonian physics. These are governed by Newton's equations

$$\ddot{\mathbf{x}} = -\boldsymbol{\nabla} V, \qquad V : \mathbb{R}^3 \to \mathbb{R}^3, \tag{4.64}$$

where the dot represents derivative with respect to an absolute time t. These are invariant under the Galilean transformations. We can reformulate (after Cartan) (4.64) as geodesic motion. Newtonian spacetime, like the spacetime of GR, is also four-dimensional. However, the geometry is different. We assemble time and space into a vector $x^a = (t, x^i)$ and look for a connection such that the geodesic equation $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$ is the same as (4.64). The unparametrised solution (i.e. parametrised by a coordinate t) is

$$\Gamma^i_{00} = \delta^{ij}\frac{\partial V}{\partial x^j}, \tag{4.65}$$

and all other components vanishing. Importantly, this is not a Levi-Civita connection, which means that there is no metric underlying it! We call this a *non-Riemannian* connection. Cartan asked what geometry underlies Newtonian physics.

4.4 Newton–Cartan Geometry

Newton–Cartan geometry in n dimensions is a triple (h, θ, ∇), where

- $h = h^{ab}\frac{\partial}{\partial x^a} \odot \frac{\partial}{\partial x^b}$ is a degenerate rank-$(n-1)$ symmetric tensor
- θ is a closed 1-form, called the *clock*, in the kernel of h. This means

$$h^{ab}\theta_a = 0 \qquad d\theta = 0 \to \theta = dt, \tag{4.66}$$

where t is *global time*.

- torsion-free connection ∇ preserving θ and h, meaning that

$$\nabla_a h^{bc} = 0 \qquad \nabla_a \theta_b = 0. \tag{4.67}$$

In our example, $h = \mathrm{diag}(0, 1, \dots, 1)$. The connection ∇ has symbols

$$\Gamma^i_{00} = h^{ij} \frac{\partial V}{\partial x^j}. \tag{4.68}$$

The Newton–Cartan spacetime is drawn in Fig. 4.5.

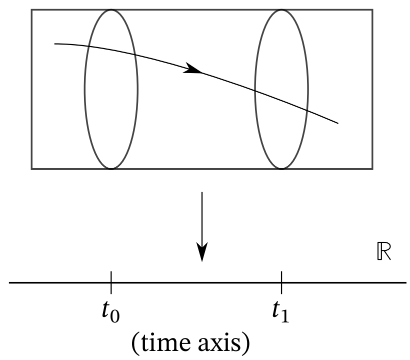

Figure 4.5: Newton–Cartan spacetime. The fibres define a notion of *simultaneity*. A geodesic of ∇ cuts through the fibres. A contravariant metric on the fibre is given by h.