Symmetries, Fields and Particles

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Contents

1 Introduction

In this course, we will cover Lie groups G, Lie algebras $\mathscr{L}(G)$, and their Representations.

Resources

- notes (online):
  - Martan
  - Osborn
  - Gutowski (Cartan classification)
- Books:
  - Fuchs & Schwegert (Ch 1-7) "Symmetries …"

1.1 Introduction

Definition 1 (Symmetry)**:** Transformation of dynamical variables leaves the form of physical laws invariant.

Example (Rotation)**:**

$$\mathbf{x} \in \mathbb{R}^3 \longrightarrow \mathbf{x}' = M \cdot \mathbf{x} \in \mathbb{R}^3, \tag{1.1}$$

where M is a 3×3 matrix, which is orthogonal ($MM^T = 1_3$) and "special" ($\det M = 1$).

We can do this with Newton's laws:

$$\mathbf{F} = m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} \to \mathbf{F}' = M \cdot \mathbf{F}' = m\frac{\mathrm{d}^2\mathbf{x}'}{\mathrm{d}t^2}. \tag{1.2}$$

1.2 Groups and Lie Groups

Definition 2 (Group)**:** A group is a (finite or infinite) set G with the following properties:

1. Closure: $g_1, g_2 \in G \implies g_1 g_2 \in G$
2. Unit: $\exists e \in G : eg = ge = g \forall g \in G$
3. Inverse: $\forall g \in G$, $\exists g^{-1} \in G : g^{-1}g = g^{-1} = e$
4. Associativity: $\forall g_1, g_2, g_3 \in G : (g_1 g_2) g_3 = g_1 (g_2 g_3)$

In physics, we will in general deal with unconstrained, infinite groups.

Definition 3 (Abelian Groups)**:** A commutative group G, where $g_1 g_2 = g_2 g_1$, is called *Abelian*.

The set of all 3×3 special orthogonal matrices, representing rotations, form a group under matrix multiplication.

Exercise 1.1: Check this!

A rotation in $\mathbb{R}^3$ depends continuously on 3 parameters $\hat{n} \in S^2, \theta \in [0, \pi]$.

Definition 4 (Lie Group)**:** A Lie group G is a group which is also a *smooth manifold*.

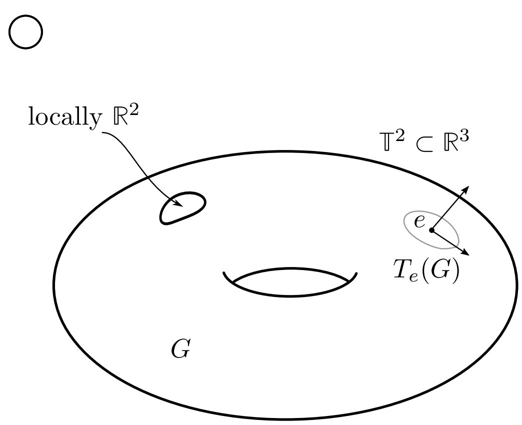


Figure 1.1: Locally, the torus $\mathbb{T}^2$ looks like $\mathbb{R}^2$

Due to the algebraic properties of the group we have $\forall g \in G$ a map from the group to itself

$$\begin{aligned} \mathcal{L}_g \colon G &\to G \\ h &\mapsto gh \end{aligned} \tag{1.3}$$

Roughly speaking, this definition will require that group and manifold structures must be compatible (e.g. $\mathcal{L}_g$ must be smooth). The existence of these smooth maps, as we will see later, allow us to move around the manifold. As a result, G is almost completely determined by its behaviour "near" the identity e. In other words, the Lie group is almost completely defined by infinitesimal symmetry transformations.

Rather than the whole of the manifold, we will only think about the tangent space $\mathcal{T}_e(G)$ of the identity.

Tangent vectors at e, $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{T}_e(G)$, equipped with a bracket $[,] : \mathcal{T}_e(G) \times \mathcal{T}_e(G) \to \mathcal{T}_e(G)$, define a *Lie algebra* $\mathscr{L}(G)$. Lie groups are (almost) determined by their Lie algebra.

1.3 Key Result: Cartan Classification (1895)

This arises from the question of whether we can have a finite dimensional Lie algebra. The problem reduces to analysing *simple* Lie algebras, which all Lie algebras can be built from.

Theorem 1 (Cartan Classification)**:** All finite-dimensional semi-simple Lie algebras (over $\mathbb{C}$) belong to one of

1. four infinite families A_n, B_n, C_n, D_n, where $n \in \mathbb{N}$
2. five exceptional cases E_6, E_7, E_8, G_2, F_4

This basically lists the allowed gauge theories, since the gauge groups must correspond to the Lie algebras in this list. All physical groups come from the low-n values of the infinite families. In more modern theoretical physics, exceptional Lie groups also show up. For example, in String theory, certain anomalies only cancelled with combinations like $E_8 \oplus E_8$ or $\mathfrak{so}(32) = D_8$.

1.4 Classical Physics

By Noether's theorem, symmetries $\implies$ conserved quantities.

Example: invariance under rotations in $\mathbb{R}^3 \implies$ conserved angular momentum $\mathbf{L} = (L_1, L_2, L_3)$

1.5 Quantum Physics

Instead of phase space,

- states are now vectors $|\psi\rangle$ in a (potentially infinite dimensional) Hilbert space $\mathcal{H}$
- observables are linear operators $\hat{O} : \mathcal{H} \to \mathcal{H}$. These do not commute $\hat{O}_1\hat{O}_2 \neq \hat{O}_2\hat{O}_1$

The angular momentum operators $\hat{L}_1, \hat{L}_2\hat{L}_3$ obey the $\mathscr{L}(SO(3))$ lie algebra, lie bracket commutator:

$$[\hat{L}_i, \hat{L}_j] = i\hbar\varepsilon_{ijk}\hat{L}_k. \tag{1.4}$$

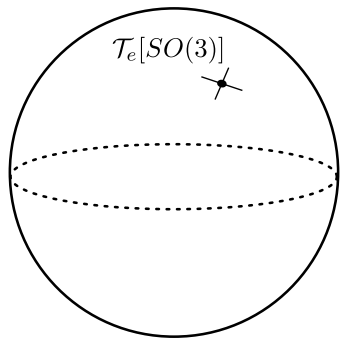


Figure 1.2: The angular momentum operators form a basis of the tangent space of $SO(3)$.

These operators form a basis of the tangent space $\mathcal{T}_e(SO(3))$ where the commutator is the lie bracket.

A particle with a definite spin defines a vector space with a certain dimension. The angular momentum operators in QM often act on finite dimensional vector spaces.

Example ($\mathbb{C}^2$)**:** The Hilbert space $\mathbb{C}^2$ is spanned by

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.5}$$

They correspond to the $2d$ representation $\mathscr{L}(SO(3))$. In place of the angular momentum operators, we have 3 2×2 matrices Σ_i, $i = 1, 2, 3$, which obey the same commutation relations

$$\langle \Sigma_i, \Sigma_i \rangle = i\hbar \Sigma_k. \tag{1.6}$$

The Lie algebra is appearing again in a way in which its generators / basis vectors are represented by the Pauli matrices σ_i:

$$\Sigma_i = \frac{1}{2}\hbar\sigma_i. \tag{1.7}$$

Recall that the Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.8}$$

Definition 5 (Representation)**:** A *representation* of a Lie algebra $\mathscr{L}(G)$ is a map

$$R \colon \mathscr{L}(G) \to \mathrm{Mat}_n(\mathbb{C}) \tag{1.9}$$

which preserves the bracket of the Lie algebra.

> **Remark:** In the context of QM systems, we need to know both about the Lie algebra, but also about its representation in terms of finite dimensional matrices. Lie groups are largely determined by their Lie algebra. As seen in the last lecture, Lie algebras itself can be classified. One of the goals of this course will also be the classification of their representations.

In Quantum mechanics, a rotational symmetry manifests itself in the commutation relation

$$\left\langle \hat{\mathcal{H}}, \hat{L}_i \right\rangle = 0 \tag{1.10}$$

with the Hamiltonian $\mathcal{H}$. The fact that the angular momentum generators, which move you around in the Hilbert space, commute with the Hamiltonian means that states in any representation of $\mathscr{L}(SO(3))$ have the same energy.

Example: The spin vectors $|\uparrow\rangle$ and $|\downarrow\rangle$ have the same energy in a rotationally invariant system.

1.6 Key Idea

The degeneracies in the spectrum of a quantum system are effectively determined by the representations of the (global) symmetry group. This can be seen as a tool for which we do not yet know the underlying symmetry and the Lagrangian.

Example (Approximate Symmetry of Hadrons)**:** Strongly interacting particles have an observed degeneracy which led Gell-Mann to postulate the approximate symmetry $G = SU(3)$ of 3×3 complex, unitary matrices with unit determinant. This is where group theory really took off in physics.

It was observed that there were sets of particles in the accelerators with approximately the same energy. Their interaction was characterised by conserved charges which could be assigned to integer values.

The particular pattern is illustrated in Fig 1.3.

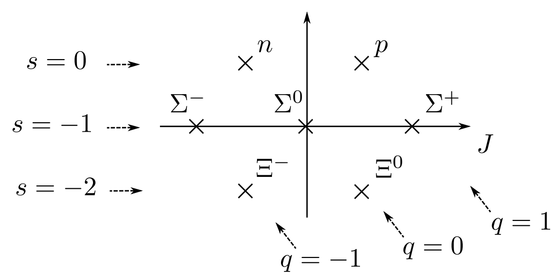


Figure 1.3: The "eightfold way" showing the approximate octet formed by the spin-$\frac{1}{2}$ baryons.

This turned out to be precisely explained by the mathematics of Lie groups and Lie algebras, and gives us the main tool of organising structures in particle physics.

Definition 6 (Global symmetries)**:** *Global* symmetries can be understood as operators in some Hilbert space which commute with the Hamiltonian. These include the spacetime symmtries, which fit into the pattern of non-Abelian Lie-groups:

- Rotations $SO(3)$
- Lorentz transformations $SO(3, 1)$
- Poincaré group (+ translations)

However, the Poincaré group is not a simple Lie group.

Moreover, we also have the *internal* symmetries

- flavour symmetry (approximate symmetry of strong interactions)
- baryon number
- lepton number

Advancements in particle physics have often hinged on the idea of enlarging the symmetry group. This is where the interaction between mathematics and physics also really took off: There are powerful theorems which prevent the combination of the global and internal symmetry groups, in the context of ordinary Lie algebras. Physicists then started to relax the constraints of Lie algebras, which led to *supersymmetry*.

Gauge Symmetries

The other topic which we will talk about is *Gauge symmetry*. This is not really a symmetry since it does not obey the definition in the first lecture. It is actually a *redundancy* in the mathematical description of the physics. Examples of gauge symmetries are

- phase of the wavefunction: $\psi \to e^{i\delta}\psi$
- electromagnetism: $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi$, where χ is some arbitrary scalar function.

These transformations, constituting the Gauge group, do not affect any physical quantities.

> **Remark:** In QFT, as far as we know, only Gauge theories can describe in a consistent, renormalisable way the interaction of spin-1 particles.

The Standard Model is a particular type of Gauge theory with $G_{SM} = SO(3) \times SU(2) \times U(1)$.

2 Lie Groups

2.1 Manifold Structure and Coordinates

Definition 7 (Manifold)**:** A manifold $\mathcal{M}$ is a space that locally looks like Euclidean space. For each coordinate patch $\mathcal{P}$, there is a bijective map $\phi_{\mathcal{P}} : \mathcal{P} \leftrightarrow \mathbb{R}^n$. Moreover, the transition

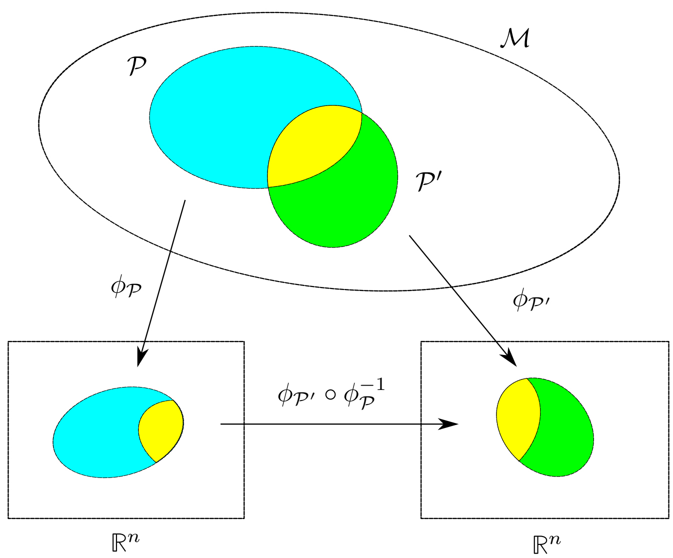


Figure 2.1: An illustration of the concept of a manifold.

functions between the coordinates have to be *smooth*.

Definition 8 (Lie groups)**:** A *Lie group* G is a group that is also a manifold. The group operations must define *smooth maps* on the manifold. The *dimension* of the Lie group is the dimension of the manifold.

The definition of the manifold allows us to introduce coordinates $\{\theta^i\}$, $i = 1, \ldots, D = \dim(G)$. In the patch $\mathcal{P}$, the group elements depend continuously on the coordinates $\{\theta^i\}$. WLOG, we can choose coordinates in which identity element lies at the origin: $g(0) = e$.

2.2 Compatibility of Group and Manifold Structures

2.2.1 Multiplication

The group operation of multiplication will define a map on the manifold. Since this map has to be smooth, it will have to be composed of continuous differentiable functions in the coordinate systems.

By closure of the group, and assuming that multiplication gives us another element in the same patch:

$$g(\theta)g(\theta') = g(\varphi) \in G \tag{2.1}$$

> **Remark:** In general it is not necessarily the case that the new group element will be in the same coordinate patch $\mathcal{P}$. In that case, we will have to make use of the transition functions.

This defines a map $G \times G \to G$ from a pair of group elements to a third. We can express this map in coordinates as $\varphi^i = \varphi^i(\theta, \theta') : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. In terms of these coordinates, the condition of *compatibility* between a group operation and a manifold structure simply means that these maps φ^i have to be continuous and differentiable.

2.2.2 Inversion

Group inversion also defines a smooth map, this time from G to itself. For all elements $g(\theta) \in G$, where θ describes the position in the coordinate patch $\mathcal{P}$, the group axioms imply that there exists an inverse element $g^{-1}(\theta)$. Assuming that this is still in the same coordinate patch, we can write this as $g^{-1}(\theta) = g(\widetilde{\theta})$. If it is not, we simply apply the relevant transition function.

$$g(\theta)g(\widetilde{\theta}) = g(\widetilde{\theta})g(\theta) = e \tag{2.2}$$

In coordinates, $\widetilde{\theta}^i = \widetilde{\theta}^i(\theta)$ have to be continuous and differentiable.

Example: The simplest possible example of a Lie group is $G = (\mathbb{R}^D, +)$.

- operation: $\mathbf{x}'' = \mathbf{x} + \mathbf{x}'$ for all $\mathbf{x}$, $\mathbf{x}' \in \mathbb{R}^D$
- inversion: $\mathbf{x}^{-1} = -\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^D$

This is an *Abelian* group.

2.3 Embedded Submanifolds

We can define properties of manifold in an intrinsic way. However, for some of these properties it is significantly easier to define manifolds as subspaces embedded in real space. In general, we can achieve this if we write

$$\mathcal{M} = \{\mathbf{x} \in \mathbb{R}^m \mid \mathcal{F}_\alpha(\mathbf{x}) = 0\} \tag{2.3}$$

where $\mathcal{F}_\alpha : \mathbb{R}^m \to \mathbb{R}$, with $\alpha = 1, \cdots, l$ is a smooth map.

Theorem 2 (Embedding Theorem)**:** $\mathcal{M}$ is a manifold of dimension $D = m - l$ if and only if the Jacobian matrix

$$(\mathcal{J})_{\alpha,i} = \frac{\partial \mathcal{F}_\alpha}{\partial x_i} \tag{2.4}$$

has maximal rank l everywhere on the manifold $\mathcal{M}$.

Example (S^2)**:** We can realise the two-sphere S^2 as a manifold embedded in three dimensional Euclidean space

$$\mathcal{M} = \left\{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| = R\right\} \tag{2.5}$$

The solution space of $x^2 + y^2 + z^2 - R^2 = F$ defines a manifold. In this case, the Jacobian

$$\mathcal{J} = \left(\frac{\partial \mathcal{F}}{\partial x}, \frac{\partial \mathcal{F}}{\partial y}, \frac{\partial \mathcal{F}}{\partial z}\right) = 2(x, y, z) \tag{2.6}$$

has rank 1 except at $x = y = z = 0$, but that point is not on the manifold, so that is allowed by the theorem.

Definition 9 (connected)**:** A manifold is said to be *connected* if there is a smooth path between any two paths on the manifold.

Definition 10 (simply connected)**:** A manifold is said to be *simply connected* if all loops are "trivial", in the sense that they can be continuously be contracted to a point.

Example: The spherical surface S^2 is simply connected, while the torus T^2 is not.

Definition 11 (compact)**:** A manifold is said to be a *compact space* if any *closed* and *bounded* subset of $\mathbb{R}^n$ is *compact*.

Example: The sphere is a compact space, while the hyperboloid is not.

2.4 Matrix Lie Groups

Matrix multiplication is *closed* and *associative* and there exists a unit element

$$e = 1_n \in \mathrm{Mat}_n(F). \tag{2.7}$$

Here, F is some field. We will mostly be working with $F = \mathbb{R}$ or $\mathbb{C}$. However, the set of matrices $\mathrm{Mat}_n(F)$ is not a group under matrix multiplication since not all matrices are *invertible*.

Definition 12 (general linear group)**:** The general linear group of dimension n is the set of matrices with non-vanishing determinant

$$GL(n, F) = \{M \in \mathrm{Mat}_n(F) \mid \det M \neq 0\} \tag{2.8}$$

guaranteeing invertibility.

Definition 13 (special linear group)**:** The special linear group has unit determinant

$$SL(n, F) = \{M \in GL(n, F) \mid \det; = 1\} \tag{2.9}$$

These are enough to guarantee that these are groups. In particular, the closure property follows from

$$\det(M_1 M_2) = \det(M_1)\det(M_2) \qquad \forall M_1, M_2 \in \mathrm{Mat}_n(F). \tag{2.10}$$

This is connected to the embedding theorem in the following way. Taking $SL(n, \mathbb{R})$ we apply the embedding theorem with $m = n^2$. The number of constraints is $l = 1$ due to the determinant being constraint to unity:

$$F_1(M) = \det M - 1 \tag{2.11}$$

To apply the embedding theorem we have to calculate the Jacobian. It is useful to recall the definition of a minor:

Definition 14 (minor)**:** Let $M \in \mathrm{Mat}_n(\mathbb{R})$ be an $n \times n$ matrix with real entries. We define the *minor* $\hat{M}^{(ij)}$ of each element of the matrix as the $(n-1) \times (n-1)$ matrix with ith row and jth column deleted.

Using this definition, we can differentiate as follows

$$\frac{\partial F}{\partial M_{ij}} = \pm \det\left(\hat{M}^{(ij)}\right). \tag{2.12}$$

$\frac{\partial F}{\partial M_{ij}}$ has rank 1, unless all the determinants of the minors vanish. This is equivalent to the determinant of the matrix vanishing

$$\det\left(\hat{M}^{(ij)}\right) = 0 \iff \det(M) = 0 \neq 1 \tag{2.13}$$

This shows that $SL(n, \mathbb{R})$ is a smooth manifold of dimension $n^2 - 1$. We could do this with $SL(n, \mathbb{C})$ by splitting the coordinates and then the detminant condition in its real and imaginary parts. For the general linear groups, we define them by removing a condition.

Exercise 2.1: Complete the proof that $SL(n, \mathbb{R})$ is a Lie group. For this, you have to convince yourself that matrix multiplication, considered element by element, provides a smooth map.

This gives us four families of matrix Lie groups. We have established that

$$\dim(SL(n, \mathbb{R})) = n^2 - 1. \tag{2.14}$$

A similar argument allows us to find that

$$\dim(SL(n, \mathbb{C})) = 2n^2 - 2. \tag{2.15}$$

Moreover, for the general linear group we can find that

$$\dim(GL(n, \mathbb{R})) = n^2 \qquad \dim(GL(2, \mathbb{C})) = 2n^2. \tag{2.16}$$

Definition 15 (submanifold)**:** A sub manifold is a subspace of a manifold that is also a manifold.

Definition 16 (Lie subgroup)**:** A *Lie subgroup* is a subset of a Lie group which is also a Lie group.

Definition 17 (orthogonal groups)**:** The *orthogonal groups* are defined to be those elements of the general linear groups which satisfy the orthogonality condition

$$O(n) = \left\{ M \in GL(n, \mathbb{R}) \mid MM^T = 1_n \right\}. \tag{2.17}$$

Definition 18 (orthogonal transformations)**:** Orthogonal transformations are of the form

$$\mathbf{v} \in \mathbb{R}^n \to \mathbf{v}' = M \cdot \mathbf{v} \in \mathbb{R}^n \tag{2.18}$$

where $M \in O(n)$ is an orthogonal matrix.

Orthogonal transformations can be thought of as linear transformations which preserve the length of vectors since

$$|\mathbf{b}'| = \mathbf{v}'^T \cdot \mathbf{v}' = \mathbf{v} \cdot M^T M \cdot \mathbf{v} = \mathbf{v}^T \cdot \mathbf{v} = |\mathbf{v}|^2. \tag{2.19}$$

If we have an orthogonal real matrix, we have

$$\det\left(MM^T\right) = \det(M)^2 = 1, \tag{2.20}$$

so its determinant is $\det(M) = \pm 1$.

From this, by continuity, we can tell that $O(n)$ is not a connected manifold. Indeed, $O(n)$ must have 2 *connected components*. Moreover, only one of these can contain the identity element. As such, we can consider the space which only includes the connected components of the identity:

Definition 19 (special orthogonal group)**:**

$$SO(n) = \{M \in O(n) \mid \det(M) = 1\} \tag{2.21}$$

How do we distinguish between matrices which have $\det(M) = \pm 1$? Given a frame $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ in $\mathbb{R}^n$ and an orthogonal transformation

$$\mathbf{v}_a \in \mathbb{R}^n \to \mathbf{v}'_a = M \cdot \mathbf{v}_a \in \mathbb{R}^n \tag{2.22}$$

with $M \in O(n)$, preserves the *orientation* of a frame, i.e. the sign of the volume element

$$\Omega = \varepsilon^{i_1,\ldots,i_n} v_1^{i_1} v_2^{i_2} \cdots v_n^{i_n}, \tag{2.23}$$

where ε is the n-dimensional alternating tensor, only if $M \in SO(n)$. Elements of $SO(n)$ correspond to *rotations*, whereas elements of $O(n)$ with $\det(M) = -1$ correspond to some mixture of *rotation* and *reflection*.

Exercise 2.2: Use the embedding theorem to check that $O(n)$ is a manifold and that its dimension is

$$\dim(O(n)) = \dim SO(n) = \frac{1}{2}n(n-1). \tag{2.24}$$

Remember to show that the Jacobian matrix has maximal rank.

Recall the two defining properties of the orthogonal group $O(n)$ with respect to the matrix eigenvalue equation

$$M\mathbf{v}_\lambda = \lambda \mathbf{v}_\lambda, \qquad M \in O(n): \tag{2.25}$$

1. Complex conjugate: If λ is an eigenvalue, then λ^* is an eigenvalue as well.
2. Normalisation: $|\lambda|^2 = 1$.

Proof. 1. Complex conjugating both sides of (2.25) gives

$$M\mathbf{v}_\lambda^* = \lambda^* \mathbf{v}_\lambda^* \tag{2.26}$$

2. First, note that we have $(M\mathbf{v}^*)^T \cdot M\mathbf{v} = \mathbf{v}^\dagger M^T M \mathbf{v} = \mathbf{v}^\dagger \cdot \mathbf{v}$. Then, if $\mathbf{v} = \mathbf{v}_\lambda$:

$$(M\mathbf{v}_\lambda^*)^T \cdot M\mathbf{v}_\lambda = |\lambda|^2 \mathbf{v}_\lambda^\dagger \mathbf{v}_\lambda = \mathbf{v}_\lambda^\dagger \mathbf{v}_\lambda \implies |\lambda|^2 = 1. \tag{2.27}$$

□

Example (G = SO(2))**:** Let M be a matrix in $SO(2)$. Then M has eigenvalues $\lambda = e^{i\theta}, e^{-i\theta}$ for small $\theta \in \mathbb{R}$, with the identification $\theta \sim \theta + 2\pi$. In a matrix representation, we write

$$M = M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2.28}$$

Although this is a real matrix, its eigenvalues are complex. Provided that we made the identification $\theta \sim \theta + 2\pi$, the matrix is uniquely specified by θ. Therefore, the manifold of this Lie group is $M(SO(2)) \cong S^1$. Moreover, since the matrices are commutative, $M(\theta_1)M(\theta_2) = M(\theta_2)M(\theta_1) = M(\theta_1 + \theta_2)$, this is an Abelian Lie group.

> **Remark:** This is in fact the simplest compact Lie group.

Example (G = SO(3))**:** We consider now matrices M in the three-dimensional special orthogonal group $SO(3)$. The eigenvalues are $\lambda = e^{+i\theta}, e^{-i\theta}, +1$, where we again have made the identification $\theta \sim \theta + 2\pi$. To parametrise a rotation matrix in three dimensions, consider the normalised eigenvector corresponding to the $\lambda = +1$ eigenvalue:

$$\hat{\mathbf{n}} \in \mathbb{R}^3, \qquad M\hat{\mathbf{n}} = \hat{\mathbf{n}}, \qquad \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1. \tag{2.29}$$

The direction of $\hat{\mathbf{n}}$ parametrises the axis, and θ parametrises the angle of rotation.

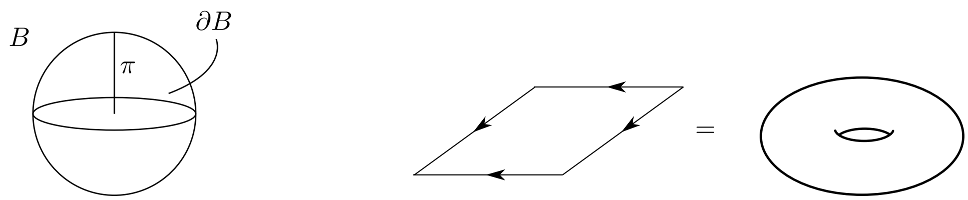
Exercise 2.3: One can write a general group element of SO(3) as

$$M(\hat{\mathbf{n}}, \theta)_{ij} = \cos\theta\delta_{ij} + (1 - \cos\theta)n_i n_j - \sin\theta\varepsilon_{ijk}n_k. \tag{2.30}$$

We want to specify the elements uniquely. Above, one needs to be careful about the uniqueness, due to two issues:

1. Identification: $M(\hat{\mathbf{n}}, 2\pi - \theta) = M(-\hat{\mathbf{n}}, \theta)$
2. If $\theta = 0$, then for all directions $\hat{\mathbf{n}}$, we have $M(\hat{\mathbf{n}}, 0) = I_3$,

To be precise, we need to identify these rotations. To get a better parametrisation, define the parameter $\boldsymbol{\omega} = \theta\hat{\mathbf{n}}$. Consider the ball $B_3 \subset \mathbb{R}^3 = \left\{\boldsymbol{\omega} \in \mathbb{R}^3 \mid |\boldsymbol{\omega}| \leqslant \pi\right\}$. The group manifold associated with SO(3) is obtained by taking B_3 and identifying antipodal points on the boundary.



(a) The ball B_3 with radius π.

(b) Identifying opposing edges of a sheet of paper gives a space topologically equivalent to a torus.

Figure 2.2: The group manifold associated with SO(3) is obtained by identifying antipodal points on the boundary ∂B of the ball B_3.

> **Remark:** In general, freely acting quotients give a manifold. Here, the group we quotiented out is the group of inversion $\mathbb{R}_2$.

The resulting manifold is connected, but not simply connected. This is because loops that come out and back via the identification cannot be contracted to a point; antipodal points are always antipodal. This is illustrated in Figure 2.3. As such, we have

$$\pi_1(SO(3)) \neq \{0\}\,, \qquad \pi_1(SO(3)) \simeq \mathbb{Z}_2 = \{+1, -1\}\,. \tag{2.31}$$

2.4.1 Non-compact subgroups of $GL(n, \mathbb{R})$

Orthogonal matrices obey $MM^T = I_n$. We can also read this as $MI_nM^T = I_n$; orthogonal transformations preserve the Euclidean metric $g = \text{diag}\,\underbrace{(1, \cdots, 1)}_{n \text{ times}}$ on $\mathbb{R}^n$. Similarly, we can generalise to say that $O(p, q)$ transformations preserve the metric of signature p, q :

$$O(p, q) = \left\{M \in GL(n, \mathbb{R}) \mid M^T\eta M = \eta, \text{where } \eta = \text{diag}(\underbrace{-1, \ldots, -1}_{p \text{ times}}, \underbrace{+1, \ldots, +1}_{q \text{ times}})\right\} \tag{2.32}$$

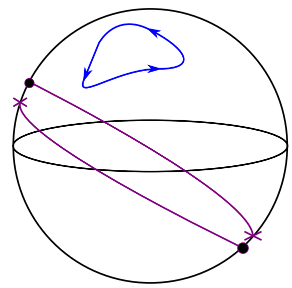


Figure 2.3: Loops passing through the identification of antipodal points cannot be contracted to a point. Note that the purple loop is constructible. This can be seen by “rotating” one half of the loop until it matches up with the other.

SO(2) $\sim \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, with $\theta \sim \theta + 2\pi$

SO(1, 1) $\sim \begin{pmatrix} \cosh\varphi & -\sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix}$, with $\varphi \in \mathbb{R}$, $M\big(SO(1,1)\big) \simeq \mathbb{R}$.

2.4.2 Subgroups of $GL(n, \mathbb{C})$

Claim 1: Let $U \in U(n) = \{U \in GL(n,\mathbb{C}) \mid U^\dagger U = I_n\}$ be a matrix in the unitary group. Under such a unitary transformation, the length of a vector $\mathbf{v}$ is unchanged.

Proof. The vector $\mathbf{v}$ transforms as $\mathbf{v} \in \mathbb{C}^n \to \mathbf{v}' = U\mathbf{v} \in \mathbb{C}^n$. Using the property $UU^\dagger = 1_n$ of unitary matrices, we have

$$|\mathbf{v}'|^2 = \mathbf{v}'^\dagger \cdot \mathbf{v}' = (\mathbf{v}^\dagger U^\dagger) \cdot (U\mathbf{v}) = \mathbf{v}^\dagger \cdot \mathbf{v} = |\mathbf{v}|^2 \tag{2.33}$$

□

Claim 2: Let $U \in U(n)$ be an element of the group of unitary $n \times n$ matrices. Then $\det U = e^{i\delta}$, where $\delta \in \mathbb{R}$.

Proof. $U^\dagger U = 1_n \implies |\det U|^2 = 1 \implies \det U = e^{i\delta}$, $\delta \in \mathbb{R}$. □

> **Remark:** Since $\delta \in \mathbb{R}$ is able to vary continuously, $U(n)$ is connected, whereas $O(n)$ was not.

2.4.3 Special Unitary Groups

$$SU(n) = \{U \in U(n) \mid \det U = 1\} \tag{2.34}$$

The groups $U(n)$ and $SU(n)$ are indeed Lie groups $\subset GL(n, \mathbb{C})$.

Claim 3: Their dimensions are given by

$$\dim(U(n)) = 2n^2 - n^2 = n^2, \qquad \dim(SU(n)) = n^2 - 1. \tag{2.35}$$

Proof. To see this, take any matrix $M \in \text{Mat}(n, \mathbb{C}) \leftrightarrow \mathbb{R}^{2n^2}$ (real and complex components) and apply the embedding theorem. The constraint for $U(n)$ is $\mathcal{F} = UU^\dagger - I = 0$. This gives quadratic constraints in the matrix elements. Since $H = UU^\dagger$ is Hermitian, these are actually n^2 constraints instead of $2n^2$. For $SU(n)$, we require $\mathcal{F} = \det U - 1 = 0$. Since $\det U = e^{i\varphi}$, this is only one additional constraint. $\square$

Span of Vectors

To calculate the dimension of $O(n)$ and $U(n)$, it is useful to consider the span of a set of vectors. This is because the condition $O^T O = I$ implies that columns of O are unit vectors, each successively orthogonal to the span of the preceding ones.

Definition 20 (span)**:** Let $\{v_i\}$, $i = 1, \ldots, n$ be a set of vectors in a vector space V. The *span of the set* $\{v_i\}$, denoted $\text{span}(v_1, \ldots, v_n)$ is the et of all linear combinations of these vectors.

Example: Let $V = \{v_1, v_2\}$. Then $\text{span}(V) = \{c_1 v_1 + c_2 v_2\}$, where c_1, c_2 are elements of the field over which the vector space is defined.

In this course, we are interested in classifying Lie groups and Lie algebras. We have isomorphisms and homeomorphisms, maps which preserve group and manifold structure respectively.

Definition 21 (isomorphism)**:** Two Lie groups G and G' are *isomorphic* ($G \simeq G'$) if there exists a one-to-one smooth map $J : G \to G'$ such that for all $g_1, g_2 \in G$, we have $J(g_1 g_2) = J(g_1)J(g_2)$.

Let us look at some low-dimensional examples of unitary groups:

Example ($G = U(1)$)**:** Let $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. A general element $z = e^{i\theta}$ of $G = U(1)$, parametrised by $\theta \in \mathbb{R}$ with identification $\theta \sim \theta + 2\pi$, corresponds to a unique element

$$M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.36}$$

of $G' = SO(2)$ via the map

$$\begin{aligned} J \colon U(1) &\to SO(2) \\ z(\theta) = e^{i\theta} &\mapsto M(\theta). \end{aligned} \tag{2.37}$$

The map J is one-to-one and

$$J(z(\theta_1)z(\theta_2)) = M(\theta_1 + \theta_2) = M(\theta_1)M(\theta_2) = J(z(\theta_1))J(z(\theta_2)). \tag{2.38}$$

This implies that $U(1) \simeq SO(2)$.

Example ($G = SU(2)$)**:** This is a three-dimensional group. The matrix parametrised as $U = a_0 I_2 + i\mathbf{a}\cdot\boldsymbol{\sigma}$, where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $a_0 \in \mathbb{R}$, $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$, is an element of $SU(2)$ provided that

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1. \tag{2.39}$$

This implies that $M(SU(2)) \simeq S^3 \subset \mathbb{R}^4$. Since

$$\pi_1(SU(2)) \simeq \{1\}, \qquad \pi_1(SO(3)) = \mathbb{R}_2, \tag{2.40}$$

this means that $SU(2) \not\simeq SO(3)$.

3 Lie Algebras

Definition 22 (Lie algebra)**:** A *Lie algebra* $\mathfrak{g}$ is a vector space over a field $F = \mathbb{R}, \mathbb{C}$ with a bracket,

$$[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{3.1}$$

with the following properties for all $X, Y, Z \in \mathfrak{g}$:

1. Anti-symmetry: $[X, Y] = -[Y, X]$.
2. (Bi-)linearity: $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$, for all coefficients $\alpha, \beta \in F$.
3. Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Let V be a vector space that has a product $* : V \times V \to V$, which is associative, meaning that for all $X, Y, Z \in V$,

$$(X * Y) * Z = X * (Y * Z). \tag{3.2}$$

Moreover, the product is distributive over the field

$$Z * (\alpha X + \beta Y) = (\alpha Z * X + \beta Z * Y). \tag{3.3}$$

Then, we obtain a Lie algebra from the following definition of a Lie bracket

$$[X, Y] = X * Y - Y * X. \tag{3.4}$$

One example we have in mind is the case where V is the vector space of matrices and $*$ is matrix multiplication.

> **Remark:** Compare this to the Lie algebra of differential operators/vectors in differential geometry.

Definition 23 (dimension)**:** The dimension of a Lie algebra $\mathfrak{g}$ is the dimension of its underlying vector space.

Choose a basis B for $\mathfrak{g}$

$$B = \{T^a, a = 1, \ldots, n = \dim(\mathfrak{g})\}\,. \tag{3.5}$$

Then any $X \in \mathfrak{g}$ can be written as

$$X = X_a T^a :- \sum_{a=1}^{n} X_a T^a \tag{3.6}$$

where $X_a \in F$. Bracket of elements $X, Y \in \mathfrak{g}$ can then be written as

$$[X, Y] = X_a Y_b [T^a, T^b]. \tag{3.7}$$

Therefore, knowing the brackets of basis elements allows us to construct the full Lie algebra:

$$[T^a, T^b] = f^{ab}{}_c T^c. \tag{3.8}$$

The *structure constants* $f^{ab}{}_c$ therefore define the Lie algebra, and two Lie algebras are isomorphic if they have the same structure constants. Note however, that structure constants are basis dependent. We will want to find a way to classify Lie algebras that is independent of our choice of basis.

3.1 Structure Constants

Let $f^{ab}{}_c \in F,\ a, b, c = 1, \ldots, \dim(\mathfrak{g})$ be structure constants of a Lie algebra $\mathfrak{g}$. The axioms of Lie algebras then imply

1. $\implies f^{ab}{}_c = -f^{ba}{}_c$
2. $\implies f^{ab}{}_c f^{cd}{}_e + f^{da}{}_c f^{cb}{}_e + f^{bd}{}_c f^{ca}{}_e = 0$

Definition 24 (isomorphism)**:** Two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are said to be *isomorphic*, $\mathfrak{g} \simeq \mathfrak{g}'$ if there exists a linear, one-to-one map $f : \mathfrak{g} \to \mathfrak{g}'$ such that

$$[f(X), f(Y)] = f([X, Y]), \qquad \forall X, Y \in \mathfrak{g}. \tag{3.9}$$

Definition 25 (subalgebra)**:** A *subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ which is also a Lie algebra.

Definition 26 (ideal)**:** An *ideal* of $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with

$$[X, Y] \in \mathfrak{h}, \qquad \forall X \in \mathfrak{g},\ Y \in \mathfrak{h} \tag{3.10}$$

The notion of ideal roughly corresponds to the concept of a normal subgroup.

Example (trivial algebras)**:** Every Lie algebra $\mathfrak{g}$ has two ‘trivial’ ideals:

$$\mathfrak{h} = \{0\} \qquad \text{and} \qquad \mathfrak{h} = \mathfrak{g}. \tag{3.11}$$

Example (derived algebra)**:** The *derived algebra*

$$i = [\mathfrak{g}, \mathfrak{g}] :-= \text{Span}_F \{[X, Y] \mid X, Y \in \mathfrak{g}\} \tag{3.12}$$

is an ideal of $\mathfrak{g}$.

Example (centre)**:** The *centre* of $\mathfrak{g}$

$$\xi(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \quad \forall Y \in \mathfrak{g}\}. \tag{3.13}$$

Definition 27 (abelian)**:** An Abelian Lie algebra is such that all brackets vanish:

$$[X, Y] = 0 \qquad \forall X, Y \in \mathfrak{g} \tag{3.14}$$

For an Abelian Lie algebra, the Lie algebra is equal to its own centre $\mathfrak{g} = \xi(\mathfrak{g})$ and the ideal is trivial $i(g) = \{0\}$.

Definition 28 (simple)**:** A Lie algebra $\mathfrak{g}$ is said to be *simple* if it is non-Abelian and it has no non-trivial ideals.

This implies that for simple Lie algebras, $\xi(\mathfrak{g}) = \{0\}$ and $i(\mathfrak{g}) = \mathfrak{g}$.

The main theorem that we will work up to is the *Cartan classification*, which will allow us to classify all finite-dimensional, simple, complex Lie algebras $\mathfrak{g}$.

4 Lie Algebras from Lie Groups

So far, we have introduced the concepts of Lie group and Lie algebras. These appear to be very different objects. We will now relate these two concepts by showing that every Lie group gives rise to a Lie algebra, by considering the tangent space near its identity element.

4.1 Preliminaries

Let $\mathcal{M}$ be a smooth manifold of dimension D. Pick a point $p \in \mathcal{M}$. Because of the manifold structure, we can introduce a coordinate chart $\left\{x^i \in \mathbb{R}\right\}$, $i = i, \ldots, D$ in some region $\mathcal{P} \subset \mathcal{M}$. Without loss of generality, we can choose the coordinates such that the point p corresponds to the origin $x^i = 0$, $\forall i$.

Definition 29 (tangent space)**:** The *tangent space* $\mathcal{T}_p(\mathcal{M})$ to $\mathcal{M}$ at p is a D-dimensional vector space spanned by differential operators $\{\frac{\partial}{\partial x^j}\}$, $i = 1, \ldots, D$ acting on functions $f : \mathcal{M} \to \mathbb{R}$.

Definition 30 (tangent vector)**:** An element of the tangent space is called a *tangent vector*. We can expand every tangent vector V as a linear combination of the differential operators

$$V = v^i \frac{\partial}{\partial x^i} \in \mathcal{T}_p(\mathcal{M}), \tag{4.1}$$

where the real coefficients $v^i \in \mathbb{R}$ are called the *components* of V. This decomposition tells us that tangent vectors act on a function $f = f(x)$ as

$$V \cdot f = v^i \left. \frac{\partial f(x)}{\partial x^i} \right|_{x=0} \tag{4.2}$$

where the evaluation at $x = 0$ signifies that we are dealing with a tangent space at point p, which is at the origin of our coordinates.

Definition 31 (smooth curve)**:** A *smooth curve* C on a manifold $\mathcal{M}$ is a smooth map from an interval $I \in \mathbb{R}$ to $\mathcal{M}$. In coordinates, we can use a parameter $t \in I$ to parametrise the curve as

$$C : t \mapsto x^i(t). \tag{4.3}$$

We will work with curves in which the coordinates $\{x^i(t)\}$ are continuous and differentiable at least once. Moreover, we choose coordinates in which $x^i(0) = 0$, $\forall i = 1, \ldots, D$.

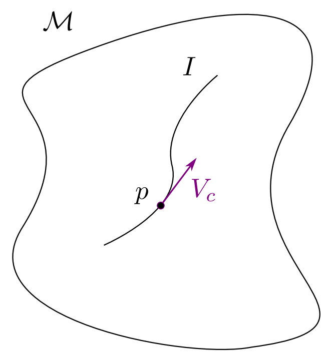


Figure 4.1: Tangent vector V_c to curve $C : I \to \mathcal{M}$ at point $p \in \mathcal{M}$.

Now consider a curve $C : I \to \mathcal{M}$ passing through the point p at $x^i(0) = 0$. The tangent vector to C at p is

$$V_c = \dot{x}^i(0)\frac{\partial}{\partial x^i} \in \mathcal{T}_p(\mathcal{M}), \tag{4.4}$$

where the derivatives with respect to the parameter t are denoted $\dot{x}^i(t) = \frac{\mathrm{d}x^i(t)}{\mathrm{d}t}$. Intuitively, in a physical picture where C is the trajectory of the particle, parametrised by a time coordinate t, the tangent vector $V_c \in \mathcal{T}_p(\mathcal{M})$ corresponds to the velocity vector at $t = 0$. Every smooth curve has a tangent vector at every point it passes through. This construction will also work for the end points of the parameter interval.

4.2 The Lie Algebra $\mathscr{L}(G)$

Let G be a Lie group of dimension D. Recall that Lie groups are also manfolds; we will apply the definitions above to examine certain tangent spaces on the manifold structure of G. Let $\{\theta^i\}$, $i = 1, \ldots, D$, be a set of coordinates in some coordinate patch $\mathcal{P}$ containing the identity $e \in \mathcal{P} \subset G$. These coordinates allow us to parametrise any element $g \in \mathcal{P}$ as $g = g(\theta) \in G$. The statement that the coordinates are taken to be centered at the identity e translates to $g(0) = e$.

Claim 4: Let G be a Lie group and consider the tangent space $\mathcal{T}_e(G)$ at the identity element $e \in G$. We can define a Lie bracket operation

$$[\cdot,\cdot] : \mathcal{T}_e(G) \times \mathcal{T}_e(G) \to \mathcal{T}_e(G) \tag{4.5}$$

so that the tangent space becomes a Lie algebra $\mathscr{L}(G) = \{\mathcal{T}_e(G), [\cdot,\cdot]\}$ when we equip it with this bracket structure.

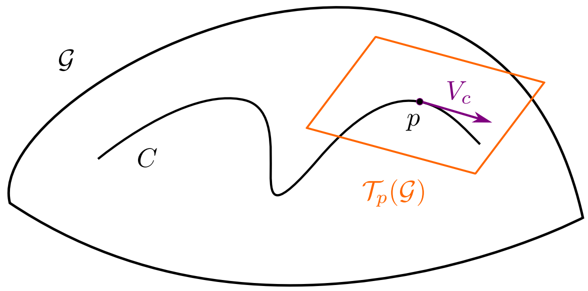


Figure 4.2: Tangent space $\mathcal{T}_p(\mathcal{M})$ at point $p \in \mathcal{G}$.

Proof. It is evident that the tangent space $\mathcal{T}_e(G)$ at the identity is a D-dimensional vector space. We have to find a suitable bracket that satisfies Definition 22 and makes this tangent space into a Lie algebra. Showing this is easiest for matrix Lie groups. Let $G \subset \mathrm{Mat}_n(F)$ be a subspace of $n \in \mathbb{N}$ dimensional matrices over the field $F = \mathbb{R}$ or $\mathbb{C}$. We can map tangent vectors to matrices by constructing the following map:

$$\begin{aligned} f\colon \mathcal{T}_e(G) &\to \mathrm{Mat}_n(F) \\ v^i \frac{\partial}{\partial \theta^i} &\mapsto v^i \left.\frac{\partial g(\theta)}{\partial \theta^i}\right|_{\theta=0}. \end{aligned} \tag{4.6}$$

The map f is injective and linear. This allows us to identify the tangent space $\mathcal{T}_e(G)$ with the subspace of $\mathrm{Mat}_n(F)$ spanned by

$$\left\{ \left.\frac{\partial g(\theta)}{\partial \theta^i}\right|_{\theta=0}, \quad i = 1, \dots, D \right\}. \tag{4.7}$$

Due to this identification with matrices, the *matrix commutator*, defined for any two $X, Y \in \mathcal{T}_e(G)$ as

$$[X, Y] \coloneqq XY - YX, \tag{4.8}$$

provides an obvious candidate for the bracket. It is easy to check that the defining properties of Definition 22 of the Lie algebra hold. Note in particular that the Jacobi identity follows from the associativity of matrix multiplication.

It remains to show *closure*: For any two tangent vectors X, Y, we need to show that their commutator $[X, Y]$ is itself also a tangent vector. To achieve this, we use the correspondence between tangent vectors and *curves* on a manifold. Let $C : I \to G$ be a smooth curve passing through the identity e.

$$\begin{aligned} C\colon I \subset \mathbb{R} &\to G \\ t &\mapsto g(t) \in G. \end{aligned} \tag{4.9}$$

The parameter is chosen in such a way that $t = 0$ parametrises the identity matrix: $g(0) = I_N$.

By the chain rule, we can differentiate a group element g as

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} = \frac{\mathrm{d}\theta^i(t)}{\mathrm{d}t}\frac{\partial g(\theta)}{\partial \theta^i}. \tag{4.10}$$

Consider the derivative at the origin:

$$\dot{g}(0) = \left.\frac{\mathrm{d}g(t)}{\mathrm{d}t}\right|_{t=0} = \dot{\theta}^i(0)\left.\frac{\partial g(\theta)}{\partial \theta^i}\right|_{\theta=0} \in \mathcal{T}_e(G). \tag{4.11}$$

This is the tangent vector to the curve C at e. We also have an explicit representation $\dot{g}(\theta) \in \mathrm{Mat}_n(F)$ for the matrix Lie group. However, they are not in general elements of the Lie group G.

Near $t = 0$, we have a Taylor expansion

$$g(t) = I_n + Xt + O(t^2) \tag{4.12}$$

where the term X appearing in the first order expansion is a tangent vector $X = \dot{g}(0) \in \mathscr{L}(G)$.

Given two elements $X_1, X_2 \in \mathscr{L}(G)$, we can find smooth curves $C_1 : t \mapsto g_1(t) \in G$ and $C_2 : t \mapsto g_2(t) \in G$, passing through the origin $g_1(0) = g_2(0) = I_n$, such that $\dot{g}_1(0) = X_1$ and $\dot{g}_2(0) = X_2$.

Near $t = 0$, another way of saying what we just said is that

$$g_1(t) = I_n + X_1 t + W_1 t^2 + O(t^3), \qquad g_2(t) = I_n + X_2 t + W_2 t^2 + O(t^3) \tag{4.13}$$

for some $W_1, W_2 \in \mathrm{Mat}_n(F)$.

Define a new curve

$$h(t) = g_1^{-1}(t) g_2^{-1}(t) g_1(t) g_2(t) \in G. \tag{4.14}$$

Since this is a composition of smooth maps, h is itself smooth. Equivalently, we have $\forall t \in I$

$$g_1(t) g_2(t) = g_2(t) g_1(t) h(t). \tag{4.15}$$

Expanding this near $t = 0$ in steps,

$$g_1(t) g_2(t) = I_n + (X_1 + X_2)t + (X_1 X_2 + W_1 + W_2)t^2 + O(t^3) \tag{4.16}$$
$$g_2(t) g_1(t) = I_n + (X_1 + X_2)t + (X_2 X_1 + W_1 + W_2)t^2 + O(t^3). \tag{4.17}$$

Here we can already see the terms $X_1 X_2$ and $X_2 X_1$ that we will want to isolate for our Lie bracket.

Now $h(t)$ has a Taylor series of the form

$$h(t) = I_n + h_1 t + h_2 t^2 + O(t^3). \tag{4.18}$$

Plugging this into (4.15) and using the above expansions for g_1g_2 and g_2g_1, we find that the coefficients in the Taylor series expansion are given by

$$h_1 = 0 \qquad h_2 = (X_1X_2 - X_2X_1) = [X_1, X_2]. \tag{4.19}$$

Thus, the curve h_3 is written in its Taylor series expansion as

$$h(t) = I_n + t^2[X_1, X_2] + O(t^3). \tag{4.20}$$

However, a curve should have its tangent vector in the linear term of its parameter. To have a sneeky way around this issue, we redefine our parameter $s = t^2 \geq 0$, where e lies at the end point. This defines a new curve $C_3 : s \mapsto g_3(s) = h(\sqrt{s})$, which near $s = 0$ has the form

$$g_3(s) = I_n + s[X_1, X_2] + O(s^{3/2}). \tag{4.21}$$

This is a good curve in C^1, provided that $s \geq 0$. Finally, we can isolate the commutator by taking the derivative and evaluating it at $s = 0$:

$$[X_1, X_2] = \left.\frac{\mathrm{d}g_3(s)}{\mathrm{d}s}\right|_{s=0} = \dot{g}_3(0) \in \mathscr{L}(G) \tag{4.22}$$

This demonstrates the closure property of the Lie algebra, since the bracket of two tangent vectors gives another. □

Note that the second derivative of g_3 will give us a negative power, and as $s \to 0$, the second derivative would blow up. So $\ddot{g}_3(0)$ does not exist. As slightly longer proof can construct good C^n curves for any n, but $n = 1$ is sufficient for this proof.

Example (G = SO(n))**:** Let $g(t) = R(t) \in SO(n)$ for all $t \in \mathcal{I} \subset \mathbb{R}$, with $R(0) = I_n$ be a curve on the group manifold $G = SO(n)$. This implies that

$$R^T(t)R(t) = I_n \quad \forall t \in \mathcal{I}. \tag{4.23}$$

Differentiating with respect to t, we have

$$\dot{R}^T(t)R(t) + R^T(t)\dot{R}(t) = 0, \qquad \forall t \in \mathcal{I}. \tag{4.24}$$

If $x_i = \dot{R}(0)$, then at $t = 0$ we get $x^T + x = 0$. Since continuity automatically implies $\det R(t) = +1$, there is no further constraint from imposing this. Therefore,

$$\mathscr{L}(O(n)) \simeq \mathscr{L}(SO(n)) = \{X \in \mathrm{Mat}_n(\mathbb{R}) \mid X^T = -X\}. \tag{4.25}$$

Counting the parameters in these matrices, we then have

$$\dim(\mathscr{L}(SO(n))) = \frac{1}{2}n(n-1) = \dim(SO(n)). \tag{4.26}$$

Example (G = SU(n))**:** Let $g(t) = U(t) \in SU(n)$ be a curve passing through the origin, with $U(0) = I_n$. As in the previous example, we differentiate the unitarity property $U^\dagger(t)U(t) = I_n$ with respect to t and evaluate at $t = 0$ to give

$$Z^\dagger + Z = 0 \tag{4.27}$$

for the derivatives $Z = \dot{U}(0) = \mathscr{L}(SU(n))$. Here, in contrast to the previous example, we *will* get an additional constraint from $\det U(t) = 1\ \forall t \in \mathbb{R}$.

Exercise 4.1: Show that $\det U(t) = 1 + t \operatorname{Tr} Z + O(t^2)$

Therefore, the constraint on the determinant imposes that the traces $\operatorname{Tr} Z$ vanish. The higher order terms in the Taylor series will not impose any additional constraints. Hence, we can use the same Lie algebra as $SO(n)$, except that we have to constrain the matrices be traceless:

$$\mathscr{L}(SU(n)) = \{Z \in \mathrm{Mat}_n(\mathbb{C}) \mid Z^\dagger = -Z,\ \operatorname{Tr} Z = 0\} \tag{4.28}$$

From this, we find that $\dim(\mathscr{L}(SU(n))) = 2n^2 - n^2 - 1 = n^2 - 1$; the dimension of the Lie algebra is the same as the dimension of the Lie group $\dim(SU(n))$.

Example:

$$\dim SU(2) = \dim SO(3) = 3 \tag{4.29}$$

Let $G = SU(2)$, then from our considerations above we know that the Lie algebra $\mathscr{L}(SU(2))$ is the set of 2×2 traceless anti-hermitian matrices. The basis for the Lie algebra of $SU(2)$ can be built from the hermitian Pauli matrices: for $a = 1, 2, 3$, they satisfy

$$\sigma_a = \alpha_a^\dagger \qquad \text{and} \qquad \operatorname{Tr} \sigma_a = 0. \tag{4.30}$$

A basis element in the Lie algebra of $SU(2)$ can then be written

$$T^a = -\frac{1}{2} i\alpha_a, \tag{4.31}$$

where the i guarantees that T^a is anti-hermitian. At the moment, the upper or lower placement of indices is purely notational at the moment, but will gain significance later. In order to compute the structure constants, we recall that the Pauli matrices obey the following identity:

$$\sigma_a \sigma_b = \delta_{ab} I_2 + i\varepsilon_{abc}\sigma_c. \tag{4.32}$$

Using this, the structure constants $f^{ab}{}_c$ are, according to their definition, obtained from the commutator

$$[T^a, T^b] = -\frac{1}{4}[\sigma_a, \sigma_b] = -\frac{1}{2} i\varepsilon_{abc}\sigma_c = f^{ab}{}_c T^c. \tag{4.33}$$

Therefore, we simply read off

$$\boxed{f^{ab}{}_c = \varepsilon_{abc}} \qquad a, b, c = 1, 2, 3. \tag{4.34}$$

Example (G = SO(3))**:** From our previous analysis, we know that the Lie algebra of $SO(3)$ consists of

$$\mathscr{L}(SO(3)) = \{3 \times 3 \text{ real anti-symmetric matrices}\}. \tag{4.35}$$

We can write down a convenient basis as

$$\widetilde{T}^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \widetilde{T}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \widetilde{T}^3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.36}$$

This can be used to find the structure constants as

$$[\widetilde{T}^a, \widetilde{T}^b] = f^{ab}{}_c \widetilde{T}^c \tag{4.37}$$

with $f^{ab}{}_c = \varepsilon_{abc}$ with $a, b, c = 1, 2, 3$. We find that the Lie algebras are the same

$$\mathscr{L}(SO(3)) \cong \mathscr{L}(SU(2)). \tag{4.38}$$

However, the Lie groups themselves are not isomorphic:

$$SO(3) \ncong SU(2). \tag{4.39}$$

We learn that different Lie groups can lead to the same Lie algebra. However, we will see later that this degeneracy will be easy to deal with. In fact, there is a relation between these two Lie groups:

$$SO(3) \simeq \frac{SU(2)}{\mathbb{Z}_2}. \tag{4.40}$$

Remark: This is because there is a *double cover*, or two-to-one, surjective homomorphism

$$\phi\colon\ SU(2) \to SO(3), \qquad \ker(\phi) = \mathbb{Z}_2 = \{\pm 1\}. \tag{4.41}$$

5 Lie Algebras from Lie Groups II

5.1 Diffeomorphisms

A Lie group is a very special type of manifold. In particular, for each element h in the Lie group G, we have two smooth maps from the group to itself that come from left and right group multiplication,

$$L_h : G \to G \qquad g \in G \mapsto hg \in G, \tag{5.1}$$

$$R_h : G \to G \qquad g \in G \mapsto gh \in G, \tag{5.2}$$

known as *left-* and *right-translations* respectively.

Claim 5: These maps are *surjective*, which means that for every $g' \in G$, there is an element $g \in G$, such that $L_h(g) = g'$.

Proof. To see this, set $g = h^{-1}g'$. □

Claim 6: These maps are also *injective*, which means that for all $g, g' \in G$, we have that if $L_h(g) = L_h(g)$, then we must have $g = g'$.

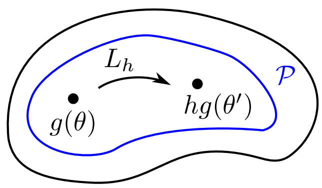
Proof. $L_h(g) = L_h(g') \implies gh = gh' \implies g = g'$. □

Definition 32: Since L_h and R_h are both surjective and injective—they are said to be *one-to-one*, or *diffeomorphisms* of $\mathcal{M}(G)$—the *inverse map*$(L_h)^{-1} = L_{h^{-1}}$ exists and is smooth.

Now introduce coordinates $\{\theta^i\}$, $i = 1, \ldots, D$ in some region containing the identity element. We can then parametrise every $g = g(\theta) \in G$, and we can choose the parametrisation such that $g(0) = e$. Assuming that we map to an element g' that is in the same coordinate patch as g, let $g' = g(\theta') = L_h(g) = hg(\theta)$. In coordinates, L_h is specified by D real functions $\theta'^i = \theta'^i(\theta)$, $i = 1, \ldots, D$. Since L_h is a diffeomorphism, the Jacobian matrix

$$J^i{}_j(\theta) = \frac{\partial \theta'^i}{\partial \theta^j} \tag{5.3}$$

exists and is invertible, which is equivalent to saying that $\det J \neq 0$.

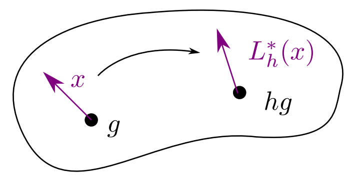


We have this family of invertible diffeomorphisms. Such maps would not normally exist on a generic manifold. Let us see what the consequences of the existence of these diffeomorphisms are.

5.1.1 Consequences

The map $L_h\colon G \to G$ induces a map L_h^* from T_gG to $T_{hg}G$. This map, called the *differential* of L_h, is in coordinate $\{\theta^i\}$ defined as

$$\begin{aligned} L_h^*\colon T_g(G) \quad &\to \quad T_{hg}(G) \\ v = v^i \frac{\partial}{\partial \theta^i} \quad &\mapsto \quad v' = v'^i \frac{\partial}{\partial \theta'^i} = J^i_j(\theta) v^j \frac{\partial}{\partial \theta'^i}, \end{aligned} \tag{5.4}$$



Definition 33: A *vector field* V on G specifies a tangent vector $V(g) \in T_g(G)$ at each point. This choice of tangent vector has to vary smoothly. In coordinates, $g = g(\theta)$,

$$V(\theta) = v^i(\theta) \frac{\partial}{\partial \theta^i} \in T_{g(\theta)}(G), \qquad i = 1, \dots, D. \tag{5.5}$$

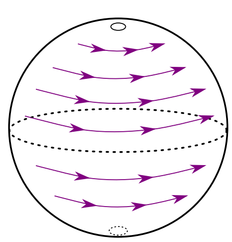
A vector field V is *smooth* if $v^i(\theta)$ are continuous and differentiable.

Starting from a tangent vector at the identity $\omega \in T_e(G)$, we define a vector field $V(g)$ for each element $g \in G$ as

$$V(g) = L_g^*(\omega) \in T_g(G). \tag{5.6}$$

As soon as we have a left multiplication map, we can shift tangent vectors across the manifold and define a smooth tangent vector field! However, we also have some more restrictive properties that come with this map. As L_g^* is smooth and invertible, $V(g)$ is smooth and *non-vanishing*. Starting from a basis $\{\omega_a\}$, $a = 1, \dots, D$, for $T_e(G)$, we actually get D independent, non-vanishing vector fields

$$V_g(g) = L_g^*(\omega_a). \tag{5.7}$$



These are called the *left-invariant vector fields*; these exist on any Lie group manifold.

Example (hairy ball theorem)**:** On a two sphere $\mathcal{M}(G) \simeq S^2$, you can never have a smooth, non-vanishing vector field. Informally, you can never comb flat the hair on a spherical doll; there will always be at least two zeros where the hair parts.

In general, the number of zeros a manifold has is related to the Euler character. Out of the two dimensional compact manifolds, the only allowed manifold is the torus $\mathcal{M}(G) = T^2$, where $G = U(1) \times U(1)$.

We can also use the technology of left-invariant vector fields to define the Lie algebra of vector fields without resorting to matrix representations of Lie groups.

5.2 Matrix Lie Groups

Let us go back to the more restricted context of matrix Lie groups. This means that we can realise things more explicitly since we know how to multiply matrices.

Claim 7: Let $G = \mathrm{Mat}_N(F)$, $n \in \mathbb{N}$, $F = \mathbb{R}$ or $\mathbb{C}$. Then, $\forall h \in G$, and $\forall X \in \mathscr{L}(G)$, we can define the left multiplication map $L_h^*(X) = hX \in T_h(G)$ simply as the multiplication of two matrices.

> **Remark:** It is highly non-obvious that we can do this, since h and X are different objects; one is an element of the Lie group, the other of the Lie algebra.

Proof. Consider the Taylor series of a point g lying on a curve C in the Lie group G, as depicted in Figure 5.1.

$$g(t) = g(0) + \dot{g}(0)t + O(t^2). \tag{5.8}$$

Then $\dot{g}(0) \in T_{g(0)}(G)$ Consider the curve $C\colon t \in I \subset R \mapsto g(t) \in G$. Let $g(0) = e = \mathbb{1}_n$ and $\dot{g}(0) = X \in \mathscr{L}(G) \simeq T_e(G)$. We can then define a new curve, $C'\colon t \mapsto h(t) = h \cdot g(t) \in G$. Near $t = 0$,

$$h(t) \simeq h + thX + O(t^2). \tag{5.9}$$

Therefore, $hX \in T_h(G)$. □

Equivalently, consider the smooth curve $C\colon t \mapsto g(t) \in G$. Then

$$\dot{g}(t)T_{g(t)}(G) \implies g^{-1}(t)\dot{g}(t) = L^*_{g^{-1}(t)}(\dot{g}(t)) \in T_e(G) = \mathscr{L}(G). \tag{5.10}$$

This is a very general story that we will meet again when studying gauge theories. For matrix Lie groups, we can always define an element of the Lie algebra by a multiplication of the form $g^{-1}\dot{g}$.

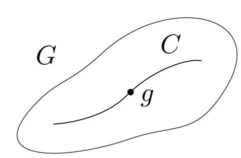


Figure 5.1

5.2.1 The exponential map

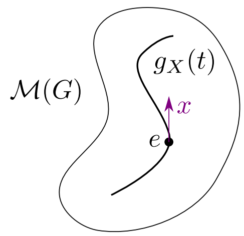


Figure 5.2

Let us now work backwards; imagine that we are given an element $X \in \mathscr{L}(G)$ of the Lie algebra. Since there is a correspondence between curves and tangent vectors, we can explicitly reconstruct a curve $C\colon I \in \mathbb{R} \to G$ on the Lie group by solving the ODE

$$g^{-1}(t)\frac{\mathrm{d}g(t)}{\mathrm{d}t} = X. \tag{5.11}$$

This is a curve whose tangent vector at the identity is X. We can enforce this to have a unique solution by specifying the boundary condition $g(0) = \mathbb{1}_n$. To solve (5.11), we define the exponential of a matrix:

Definition 34: The exponential of a matrix $M \in \mathrm{Mat}_n(F)$ is defined to be

$$\mathrm{Exp}(M) \coloneqq \sum_{l=0}^{\infty} \frac{1}{l!} M^l. \tag{5.12}$$

Claim 8: We solve (5.11) by setting $g(t) = \mathrm{Exp}(tX)$, $\forall t \in \mathbb{R}$.

Proof. Note that $g(0) = \mathrm{Exp}(0) = \mathbb{1}_n$.

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} = \sum_{l=1}^{\infty} \frac{1}{(l-1)!} t^{l-1} X^l = \mathrm{Exp}(tX) \cdot X = g(t)X. \tag{5.13}$$

□

The exponential map takes us from the Lie algebra to the Lie group. It must be the case that

$$\mathrm{Exp}(tX) \in G, \qquad \forall t \in \mathbb{R},\ \forall X \in \mathscr{L}(G). \tag{5.14}$$

> **Remark:** As previously mentioned, we can also define the exponential map without reference to the matrix representation of Lie groups.

Exercise 5.1: If $X \in \mathscr{L}(SU(N))$, check that $\mathrm{Exp}(tX) \in SU(N)$, $\forall t \in \mathbb{R}$.

5.3 Reconstructing G from $\mathscr{L}(G)$

$$\mathrm{Exp}\colon \mathscr{L}(G) \to G \tag{5.15}$$

We have the exponential map

$$\mathrm{Exp}(\mathcal{M}) \coloneqq \sum_{l=0}^{\infty} \frac{1}{l!} M^l \in \mathrm{Mat}_n(F) \tag{5.16}$$

The exponential map is one-to-one in some neighbourhood of the identity $e \in G$. Given $X, Y \in \mathscr{L}(G)$, we can construct $g_X = \mathrm{Exp}(X)$ and $g_Y = \mathrm{Exp}(Y) \in G$ such that

$$g_X g_Y = g_Z = \mathrm{Exp}(Z) \in G \tag{5.17}$$

By comparing the left and hight hand side, we get the *Baker-Campbell-Hausdorff* (BCH) formula

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] - [Y, [X, Y]) + \dots, \tag{5.18}$$

where $\dots$ reflects the terms that are quartic and higher in the matrices. This series allows us to reconstruct from the bracket of the Lie algebra close to the identity the multiplication law of the group.

> **Remark:** The formula is given purely in terms of sums of nested commutators of matrices. This reflects the fact that this is a formula in the Lie algebra. In fact, the same formula holds and makes sense even if we do not deal with matrix Lie groups. Moreover, this series is not unique since we can always shift around the higher order terms with the Jacobi identity.

Therefore, $\mathscr{L}(G)$ determines G in some neighbourhood of the identity (where the BCH series converges). More generally, Exp is not globally one-to-one.

Claim 9: The map Exp is not surjective when G is not connected.

Example ($G = O(n)$)**:** The Lie algebra of $O(n)$ is the set of real anti-symmetric $n \times n$ matrices:

$$\mathscr{L}(O(n)) = \left\{x \in \mathrm{Mat}_n(\mathbb{R}) \mid X + X^T = 0\right\} \tag{5.19}$$

Recall that the matrices $X \in \mathscr{L}(O(n))$ are traceless $\mathrm{Tr}X = 0$. To find out which connected component $\mathrm{Exp}(X)$ is in, we use the identity

$$\det(\mathrm{Exp}(X)) = \exp(\mathrm{Tr}X) = +1. \tag{5.20}$$

Therefore, for any element X of the Lie algebra, $\mathrm{Exp}(X) \in SO(n)$. Since the image of the exponential map is not the whole of $O(n)$, but only $SO(n) \subset O(n)$, this is not surjective.

Claim 10: In general, for compact Lie groups G, the image of $\mathscr{L}(G)$ under Exp is the connected component of the identity.

Claim 11: The exponential map is not injective when G has subgroup $U(1)$.

Example ($G = U(1)$)**:** The Lie algebra of $U(1)$ is $\mathscr{L}(U(1)) = \{X \in ix \in \mathbb{C} \mid x \in \mathbb{R}\}$. Then $g = \mathrm{Exp}(X) = \exp(ix) \in U(1)$. Lie algebra elements ix and $ix + 2\pi i$ yield the same group element. The inverse of Exp is multi-valued so it is not injective.

5.4 SU(2) vs SO(3)

We know that the Lie algebras have the same structure constants, meaning that they are isomorphic:

$$\mathscr{L}(SU(2)) \simeq \mathscr{L}(SO(3)). \tag{5.21}$$

Although we cannot construct an isomorphism $SU(2) \not\simeq SO(3)$, we can construct a *double-covering*, that is a globally 2-to-one map

$$\begin{aligned} d\colon SU(2) &\to SO(3) \\ A &\mapsto d(A), \end{aligned} \tag{5.22}$$

where $d(A)_{ij} = \frac{1}{2}\operatorname{tr}_2(\sigma_i A \sigma_j A^\dagger)$. Note that $d(A) = d(-A)$, $\forall A \in SU(2)$. This map provides an isomorphism between groups

$$SO(3) \simeq SU(2)/\mathbb{Z}_2, \qquad \mathbb{Z}_2 = \{\mathbb{1}_2, -\mathbb{1}_2\} \text{ centre of SU(2).} \tag{5.23}$$

However, in this case this is more than merely an isomorphism between groups: Lie groups are also manifolds. The manifold of $SU(2)$ is $\mathcal{M}(SU(2)) \simeq S^3$.

$$S^3 \simeq \left\{\mathbf{x} \in \mathbb{R}^4 \mid |\mathbf{x}|^2 = 1\right\}. \tag{5.24}$$

Moreover, the manifold of $SO(3)$ is S^3 with antipodal points identified. Quotienting out $\mathbb{Z}_2$ is the same as taking S^3 and identifying antipodal points $\mathbf{x} \sim -\mathbf{x}$. This is written as

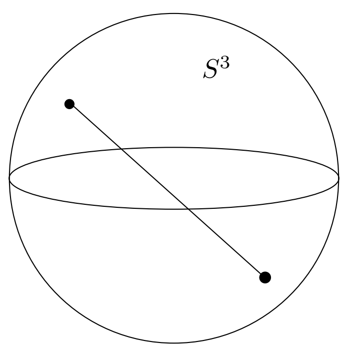


Figure 5.3: S^3 with antipodal points identified.

$$\mathcal{M}(SO(3)) \simeq S^3_+ \cup \{\text{equator with antipodal points identified}\}, \tag{5.25}$$

where S^3_+ is the upper hemisphere $x_3 \geq 0$. Note that $S_3^+ \simeq B_3$.

5.5 Representations

Definition 35: A *representation* of a group G is a map $D\colon G \to \mathrm{Mat}_n(F)$, $n \in \mathbb{N}$, $F = \mathbb{R}$ or $\mathbb{C}$, that preserves the structure of the group. In particular, for all $g_1, g_2 \in G$ we have

$$D(g_1)D(g_2) = D(g_1 g_2). \tag{5.26}$$

Definition 36: A representation D is *faithful* if it is injective; in other words, distinct group elements have distinct representations.

Definition 37: For a *Lie group* G, a representation D is a group representation in the sense above, but the map D must be smooth.

Definition 38: For a *Lie algebra* $\mathfrak{g}$, a *representation* d is a map $d : \mathfrak{g} \to \mathrm{Mat}_n(F)$ that preserves the structure of the Lie algebra. In particular, for all $X_1, X_2 \in \mathfrak{g}$ and $\alpha, \beta \in F$ we have

1. $[d(X_1), d(X_2)] = d([X_1, X_2])$
2. $d(\alpha X_1 + \beta X_2) = \alpha d(X_1) + \beta d(X_2)$

> **Remark:** Again, representations of Lie groups need not be faithful.

Definition 39: The *dimension* of a representation is the dimension n of the matrices.

Definition 40: Matrices act on a vector space $V \simeq F^n$ known as *representation space*.

> **Remark:** In a physical context, this will be the Hilbert space of states on which the matrix operator act.

Claim 12: There is a direct relation between representations of a Lie group G and of elements in the Lie algebra $\mathscr{L}(G)$. Take a representation D of a matrix Lie group G. Note that in general, $n = \dim D \neq \dim G = m$. We will construct a corresponding representation of the Lie algebra. To do this, we use again the correspondence between tangent vectors and curves. For $X \in \mathscr{L}(G)$, define a curve $C_X\colon I \in \mathbb{R} \to G$ that maps $t \mapsto g_X(t)$. We then expand $g_X(t) \simeq \mathbb{1}_n + tX + \ldots$ and define the representation d of the Lie algebra as

$$d(X) = \frac{\mathrm{d}}{\mathrm{d}t}\left(D(g(t))\right)\big|_{t=0} \in \mathrm{Mat}_n(F). \tag{5.27}$$

Map the curve onto $D(g(t)) \in GL(n, F) \subset \mathrm{Mat}_n(F)$ in the space of matrices. We then Taylor expand this around the origin to give

$$D(g(t)) = D(\mathbb{1}_m) + \left.\frac{\mathrm{d}D(g(t))}{\mathrm{d}t}\right|_{t=0} t + O(t^2). \tag{5.28}$$

We have of course $D(\mathbb{1}_m) = \mathbb{1}_n$. Moreover, we define

$$d(X) \coloneqq \left.\frac{\mathrm{d}D(g(t))}{\mathrm{d}t}\right|_{t=0} \qquad \forall X \in \mathscr{L}(G). \tag{5.29}$$

This provdides a representation of $\mathscr{L}(G)$.

Proof. For any $X_1, X_2 \in \mathscr{L}(G)$, we construct two curves $C_i \colon t \mapsto g_i(t) \in G$, $i = 1, 2$, with starting point $g_1(0) = g_2(0) = \mathbb{1}_m$ and $\dot{g}_1(0) = X_1$, $\dot{g}_2(0) = X_2$. As on page 41, define

$$h(t) = g_1^{-1}(t) g_2^{-1}(t) g_1(t) g_2(t) \in G \tag{5.30}$$

$$= \mathbb{1}_m + t^2 [X_1, X_2] + O(t^3). \tag{5.31}$$

As D is a representation of G, we have for all t that

$$D(h) = D(g_1^{-1} g_2^{-1} g_1 g_2) = D(g_1)^{-1} D(g_2)^{-1} D(g_1) D(g_2) \tag{5.32}$$

$$D(g_1(t)) = D(\mathbb{1}_m + tX_1 + \dots) \tag{5.33}$$

$$= D(\mathbb{1}_m) + t d(X_1) + O(t^2) \tag{5.34}$$

$$D(g_2(t)) = D(\mathbb{1}_n) + t d(X_2) + O(t^2) \tag{5.35}$$

Therefore, we have

$$D(h(t)) = D(\mathbb{1}_m + t^2 [X_1, X_2] + O(t^3)) \tag{5.36}$$

$$= D(\mathbb{1}_m) + t^2 \left. \frac{\mathrm{d}^2 D(h(t))}{\mathrm{d}t^2} \right|_{t=0} + \dots \tag{5.37}$$

$$= \mathbb{1}_n + d([X_1, X_2]) t^2 + \dots \tag{5.38}$$

$$D(h) = D(g_1)^{-1} D(g_2)^{-1} D(g_1) D(g_2) \tag{5.39}$$

We then plug in Taylor lens (?) and copare terms at $O(t^2)$ to give

$$d([X_1, X_2]) = [d(X_1), d(X_2)] \tag{5.40}$$

The second property of Lie brackets follows automatically from the definition. □

Exercise 5.2: Let d be a representation of $\mathscr{L}(G)$. For all elements of G of the form $g = \mathrm{Exp}(X)$, with $X \in \mathscr{L}(G)$. Then define

$$D(g) = D(\mathrm{Exp} X) \coloneqq \mathrm{Exp}(d(X)). \tag{5.41}$$

Under what circumstances does this give a good representation of G?

6 Representations of Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra of dimension D and let $X \in \mathfrak{g}$. For any matrix Lie algebra $\mathfrak{g} = \mathscr{L}(G)$ of some matrix Lie group $G \subset \mathrm{Mat}_n(F)$, we can always write down three canonical representations: For all $X \in \mathfrak{g}$, we define

the trivial representation d_0 with dimension $\dim(d_0) = 1$ by $d_0(X) = 0$.

the fundamental representation d_f with dimension $\dim(d_f) = n$ by $d_f(X) = X$.

the adjoint representation $d_{\mathbf{Adj}}$ with dimension $\dim(d_{\mathrm{Adj}}) = \dim(\mathfrak{g}) = D$. To define the adjoint representation $d_{\mathrm{Adj}}(X)$, we first consider the linear map ad_X defined by

$$\begin{aligned} ad_X \colon \mathfrak{g} &\to \mathfrak{g} \\ Y &\mapsto [X, Y] \end{aligned} \tag{6.1}$$

ad_X is equivalent to a $D \times D$ matrix. For a choice of basis $B = \{T^a\}$, for $a = 1, \ldots, D$, we can expand two elements $X, Y \in \mathfrak{g}$ as

$$X = X_a T^a \qquad Y = Y_a T^a. \tag{6.2}$$

In particular, as we have seen before, the bracket of X, Y is determined by the bracket of the generators T^a

$$[X, Y] = X_a Y_b [T^a, T^b] = X_a Y_b f^{ab}{}_c T^c. \tag{6.3}$$

Similarly, the c^{th} component of $ad_X(Y)$ is

$$[ad_X(Y)]_c = (R_X)^b_c Y_b \quad \implies \quad \boxed{(R_X)^b{}_c = X_a f^{ab}{}_c}. \tag{6.4}$$

The adjoint representation is then defined for all $X \in \mathfrak{g}$ by

$$d_{\mathrm{Adj}}(X) = ad_X. \tag{6.5}$$

More concretely, we can define it in a particular basis as $[d_{\mathrm{Adj}}(X)]^b{}_c = (R_X)^b{}_c$.

Remark: The trivial and adjoint representations exist for all Lie algebras.

Claim 13: We want to show that the adjoint representation is indeed a representation. Let us check the defining properties of a representation. For all $X, Y \in \mathfrak{g}$, we must have

$$[d_{\text{Adj}}(X), d_{\text{Adj}}(Y)] = d_{\text{Adj}}([X, Y]) \tag{6.6}$$

Proof. We have $d_{\text{Adj}}(X) = ad_X$ and $d_{\text{Adj}}(Y) = ad_Y$. Hence, for all elements $Z \in \mathfrak{g}$ of the Lie algebra, we have

$$\big(d_{\text{Adj}}(X) \circ d_{\text{Adj}}(Y)\big)(Z) = [X, [Y, Z]] \tag{6.7}$$

$$\big(d_{\text{Adj}}(Z) \circ d_{\text{Adj}}(Y)\big)(X) = [Z, [Y, X]] \tag{6.8}$$

$$\tag{6.9}$$

Subtracting these two, we get

$$[d_{\text{Adj}}(X), d_{\text{Adj}}(Y)](Z) = [X, [Y, Z]] - [Y, [X, Z]] \tag{6.10}$$

This is the left hand side of (6.6). Subtracting the right hand side we get

$$(\text{LHS} - \text{RHS})(Z) = [X, [Y, Z]] - [Y, [X, Z]] - [[X, Y], Z] \tag{6.11}$$

$$= [X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \tag{6.12}$$

where we used the Jacobi identity. We see that Jacobi identity of the Lie algebra therefore ensures that the Lie algebra has an adjoint representation on itself. Property two of the Lie algebra is satisfied since ad_X is linear in X. □

If we change the basis of the representation, the matrices change, but we still have the same representation. This motivates the following definition:

Definition 41: Two representations R_1 and R_2 of $\mathfrak{g}$ are *isomorphic*, $R_1 \simeq R_2$, if there exists an invertible matrix S such that for all $X \in \mathfrak{g}$.

$$R_2(X) = SR_1(X)S^{-1} \tag{6.13}$$

Definition 42: A representation R over a vector space V has an *invariant subspace* $U \in V$ if $\forall X \in \mathfrak{g}, u \in U$,

$$R(X)u \in U. \tag{6.14}$$

Example: Any representation has two 'trivial' invariant subspaces

$$U = \{0\} \quad \text{and} \quad U = V. \tag{6.15}$$

Definition 43: An *irreducible* representation R of $\mathfrak{g}$ has no non-trivial invariant subspaces.

6.1 Representation theory of $\mathscr{L}(SU(2))$

6.1.1 Cartan Basis

Take the real basis $T^a = -\frac{1}{2}i\sigma^a$ with $a = 1, 2, 3$, which we have already met before. Then

$$\mathscr{L}(SU(2)) = \mathrm{Span}_{\mathbb{R}} \{T^a \mid a = 1, 2, 3\}. \tag{6.16}$$

Now, consider what happens when we change to a new (complex) basis:

$$H = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.17}$$

$$E_+ == \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{6.18}$$

$$E_- = \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{6.19}$$

Note that for $X \in \mathscr{L}(SU(2))$, we have $X = -X^\dagger$. Thus if $X = X_H H + H_+ E^+ + X_- E^-$, then $X_H = i\mathbb{R}$ and $X_+ = -(X_-)^*$.

This is called the *Cartan basis*, which defines a *complexification* of the Lie algebra:

$$\mathscr{L}_{\mathbb{C}}(SU(2)) = \mathrm{Span}_{\mathbb{C}} \{T^a \mid a = 1, 2, 3\}. \tag{6.20}$$

The $\{H, E^+, E^-\}$ are a basis for $\mathscr{L}_{\mathbb{C}}(SU(2))$. Often we refer to H as the *Cartan element* of the basis, while $E_\pm$ are the step operators. These matrices satisfy the same commutation relations as we are used to from angular momentum:

$$[H, E_\pm] = \pm 2E_\pm \qquad [E_+, E_-] = H. \tag{6.21}$$

The relation $[H, E_\pm]$ implies that

$$ad_H(E_\pm) = \pm 2E_\pm, \tag{6.22}$$

while $ad_H(H) = 0$ corresponds to the statement that $[H, H] = 0$. Thus, in the Cartan basis, $\{H, E_+, E_-\}$ are eigenvectors of

$$ad_H : \mathscr{L}(SU(2)) \to \mathscr{L}(SU(2)) \tag{6.23}$$

with eigenvectors $\{0, +2, -2\}$. We sometimes refer to these as *roots* of the Lie algebra $\mathscr{L}(SU(2))$.

> **Remark:** From now on, we will write $\mathscr{L}(su(2)) = \mathfrak{SU}(2)$.

Consider a representation R of $\mathfrak{su}(2)$ with representation space V. H is diagonal, assume $R(H)$ is diagonalisable. This is equivalent to saying that the representation space V is spanned by the eigenvectors of $R(H)$. We will introduce those eigenvectors by their eigenvectors $\lambda \in \mathbb{C}$:

$$R(H)v_\lambda = \lambda v_\lambda. \tag{6.24}$$

These eigenvalues $\{\lambda\}$ of $R(H)$ are called the *weights* of the representation R.

> **Remark:** Weights belong to a representation, whereas roots belong to the Lie algebra itself.

6.1.2 Step Operators

As we have casually mentioned before, $E_\pm$ are known as *step operators*. In the following discussion, the reasons for this will become clear. We can change the order of two representation matrices by introducing the commutator:

$$R(H)R(E_\pm)v_\lambda = (R(E_\pm)R(H) + [R(H), R(E_\pm)])v_\lambda \tag{6.25}$$

By definition, the commutator is $[R(H), R(E_\pm)] = \pm 2R(E_\pm)$. Therefore, we get

$$\cdots = (\lambda \pm 2)R(E_\pm)v_\lambda. \tag{6.26}$$

Therefore, when we act on the basis with a representation of the step operator, we will move up or down to different values of the weight. In a finite-dimensional representation, we need to have a finite basis, and therefore a finite number of different weights. A finite-dimensional representation R of $\mathfrak{su}(2)$ must therefore have a highest weight $\Lambda \in \mathbb{C}$ and a highest weight vector v_Λ with

$$R(H)v_\Lambda = \Lambda v_\Lambda, \qquad R(E_+)v_\Lambda = 0. \tag{6.27}$$

Later on, we will also use the fact there needs to be a lowest weight as well for a finite-dimensional representation.

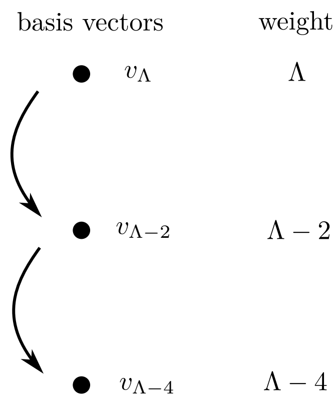


Figure 6.1

Starting from v_Λ, we can generate a new basis vector with weight $\Lambda - 2$ by acting on it with $R(E_-)$. If we repeat this process, which is illustrated in 6.1, do we eventually get all of the basis vectors of the representation?

Claim 14: If R is irreducible, then this is indeed the case; the remaining basis must be generated by arbitrary strings of $R(H)$, $R(E_+)$, and $R(E_-)$ acting on v_Λ.

> **Remark:** These strings can also be reordered using the commutators.

Proof. Let us define $v_{\Lambda-2n} = (R(E_-))^n v_\Lambda$ for $n \in \mathbb{N}$. Consider the action of $R(E_+)$ on $v_{\Lambda-2n}$. Then, we recall that we can obtain this by acting on the step down operator

$$R(E_+)v_{\Lambda-2n} = R(E_+)R(E_-)\Lambda_{\Lambda-2n+2}. \tag{6.28}$$

Using the commutator, we can then reverse the order to give

$$\begin{aligned} \ldots &= \big(R(E_-)R(E_+) + [R(E_+), R(E_-)]\big)v_{\Lambda-2n+2} && (6.29) \\ &= R(E_-)R(E_+)v_{\Lambda-2n+2} + (\Lambda - 2n + 2)v_{\Lambda-2n+2} && (6.30) \end{aligned}$$

Let us now set $n = 1$ in equation (6.30). The relation then boils down to

$$R(E_+)v_{\Lambda-2} = \Lambda v_\Lambda. \tag{6.31}$$

Thus, we do not get a new linearly independent eigenvector, but instead a multiple of the highest weight vector. Similarly, considering the $n = 2$ case yields

$$R(E_+)v_{\Lambda-4} = R(E_-)R(E_+)v_{\Lambda-2} + (\Lambda - 2)v_{\Lambda-2}. \tag{6.32}$$

Using the result (6.31), we get

$$\cdots = \Lambda R(E_-)v_\Lambda + (\Lambda - 2)v_{\Lambda-2} = (2\Lambda - 2)v_{\Lambda-2}. \tag{6.33}$$

Proceeding by induction, we find that for all n,

$$R(E_+)v_{\Lambda-2n} \propto v_{\Lambda-2n+2}. \tag{6.34}$$

The constants of proportionality can be obtained by substitution into (6.30). In particular, we will set the constants of proportionality to

$$R(E_+)v_{\Lambda-2n} = r_n v_{\Lambda-2n+2}. \tag{6.35}$$

Equation (6.30) implies the following first order recurrence relation for these coefficients

$$r_n = r_{n-1} + \Lambda - 2n + 2 \tag{6.36}$$

In addition to this, we know that $R(E_+)v_\Lambda = 0$. This sets the boundary condition $r_0 = 0$. We can solve (6.36) to get

$$r_n = (\Lambda + 1 - n)n. \tag{6.37}$$

Finally, we consider the fact that the finite dimension of the representation R implies the existence of a lowest weight $\Lambda - 2N$ for some $N \in \mathbb{N}_0$. By definition there exists some lowest weight vector $v_{\Lambda-2N} \neq 0$, which is annihilated by the lowering operator

$$R(E_-)v_{\Lambda-2N} = 0 \implies v_{\Lambda-2N-2} = 0 \tag{6.38}$$

Using $n = N + 1$ in Equation (6.35), we deduce that

$$R(E_+)v_{\Lambda-2N-2} = r_{N+1}v_{\Lambda-2N} = 0. \tag{6.39}$$

Therefore, since $v_{\Lambda-2N} \neq 0$, we must have $r_{N+1} = 0$. In particular,

$$r_{N+1} = (\Lambda - N)(N + 1) = 0. \tag{6.40}$$

Therefore, we find that the highest weight, rather than being an arbitrary number, is equal to an integer $\Lambda = N$.

Conclusion: The finite dimensional irreps R_Λ of $\mathfrak{su}(2)$ are labeled by the *heighest weight* $\Lambda \in \mathbb{N}_0$. The remaining weights in R_1 are

$$S_\Lambda = \{-\Lambda, -\Lambda + 2, \ldots, \Lambda - 2, \Lambda\} \subset \mathbb{Z}. \tag{6.41}$$

Hence, the dimension of the irrep is $\dim(R_\Lambda) = \Lambda + 1$.

In perfoming this proof, we determined the structure of the irreducible representation of $\mathfrak{su}(2)$.

$\square$

Hence, we have unique candidates for the irreps of dimensions in Table 6.1.

		Dimension
$R_0 = d_0$	trivial representation	1
$R_1 = d_f$	fundamental representation	2
$R_2 = d_{adj}$	adjoint representation	3

Table 6.1

6.1.3 Angular Momentum in Quantum Mechanics

What we have here described is analogous to the theory of angular momentum in quantum mechanics. Recall that the total angular momentum consists of orbital and spin angular momenta. We have a Hermitian operator $\mathbf{J} = (J_1, J_2, J_3)$ of total angular momentum. Its corresponding eigenstates are labelled by $j \in \mathbb{Z}/2$, $j \geq 0$, and $m \in \{-j, -j+1, \ldots, j-1, j\}$:

$$J^2 \left|j,m\right\rangle = \hbar j(j+1) \left|j,m\right\rangle \tag{6.42}$$

$$J_3 \left|j,m\right\rangle = \hbar m \left|j,m\right\rangle, \tag{6.43}$$

where $J^2 = J_1^2 + J_2^2 + J_3^2$. In the Cartan representation, we have the correspondance

$$J_3 = \frac{1}{2} R(H) \tag{6.44}$$

$$J_\pm = J_1 \pm i J_2 = R(E_\pm) \tag{6.45}$$

The highest weight is $\Lambda = 2j \in \mathbb{Z}^+$ and the other weights are $\lambda = 2m \in \mathbb{Z}$. The associated states are

$$v_\Lambda \sim \left|j,j\right\rangle \qquad v_\lambda \sim \left|j,m\right\rangle. \tag{6.46}$$

6.1.4 $SU(2)$ reps from $\mathfrak{su}(2)$ reps

Locally, we can parametrise group elements $A \in SU(2)$ as $A = \mathrm{Exp}(X)$, where X is an element of the corresponding Lie algebra $\mathfrak{su}(2)$. Starting from an irrep R_Λ of $\mathfrak{su}(2)$, we have a representation

$$D_\Lambda(A) = \mathrm{Exp}\big(R_\Lambda(X)\big). \tag{6.47}$$

As before, $\Lambda \in \mathbb{N}_0$. We will now use the relationship between $SU(2)$ and $SO(3)$ that we established. In particular, you can think of $SO(3)$ as the quotient

$$SO(3) \simeq \frac{SU(3)}{\mathbb{Z}_2}, \tag{6.48}$$

corresponding to the identification of antipodal points $A \sim -A$. This condition implies that the representation matrices of A and $-A$ must be the same. In other words, for a representation of $SO(3)$, we require that $\forall A \in SU(3)$

$$D_\Lambda(-A) = D_\Lambda(A). \tag{6.49}$$

Using the representation property, we can see that this is actually equivalent to the simpler condition

$$D_\Lambda(-\mathbb{1}_2) = D_\Lambda(\mathbb{1}_2). \tag{6.50}$$

Now if we represent $-\mathbb{1}_2 = \text{Exp}(i\pi H)$ with

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{Exp}(i\pi H) = \begin{pmatrix} e^{i\pi} & \\ & e^{-i\pi} \end{pmatrix} = -\mathbb{1}. \tag{6.51}$$

Therefore, the representation is

$$D_\Lambda(-\mathbb{1}_2) = \text{Exp}(i\pi R(\Lambda)(H)), \qquad \text{with } R_\Lambda(H) = \begin{pmatrix} \Lambda & & & & \\ & \Lambda-2 & & & \\ & & \ddots & & \\ & & & -\Lambda+2 & \\ & & & & -\Lambda \end{pmatrix} \tag{6.52}$$

$$= \text{Exp}\left(\begin{pmatrix} e^{i\pi\Lambda} & & & \\ & e^{i\pi(\Lambda-2)} & & \\ & & \ddots & \\ & & & e^{-i\pi\Lambda} \end{pmatrix}\right). \tag{6.53}$$

We find that $D_\Lambda(-\mathbb{1}_2)$ has eigenvalue $\exp(i\pi\lambda) = (-1)^\lambda = (-1)^\Lambda$. Hence,

$$D_\Lambda(-\mathbb{1}_2) = D_\Lambda(+\mathbb{1}_2) = \mathbb{1}_{\Lambda+1} \qquad \text{iff} \quad \Lambda \in 2\mathbb{Z}. \tag{6.54}$$

Now we have two cases:

- $\Lambda \in 2\mathbb{Z} \implies D_\Lambda$ is a representation of $SU(2)$ and $SO(3)$.
- $\Lambda \in 2\mathbb{Z}+1 \implies D_\Lambda$ is a representation of $SU(2)$, but *not* of $SO(3)$.

6.2 New representations from old

Definition 44 (conjugate rep)**:** If R is a representation of a *real* Lie algebra $\mathfrak{g}$, we define a *conjugate representation* by $\bar{R}(X) = R(X)^* \quad \forall X \in g$.

Sometimes, we find that $\bar{R} \simeq R$.

> **Remark:** We will find that the if a particle transforms under a representation R, its anti-particles will transform under $\bar{R}$.

Definition 45 (direct sum)**:** Suppose we are given representations R_1 and R_2 of any $\mathfrak{g}$ (not necessarily real) with representation spaces V_1 and V_2 of dimensions d_1 and d_2. We then define a *direct sum* $R_1 \oplus R_2$ as a new representation that acts on the representation space

$$V_1 \oplus V_2 = \{v_1 \oplus v_2 \mid v_1 \in V_1, v_2 \in V_2\} \tag{6.55}$$

as

$$(R_1 \oplus R_2)(X)(v_1 \oplus v_2) = (R_1(X)v_1) \oplus (R_2(X)v_2) \in V_1 \oplus V_2 \tag{6.56}$$

for all $X \in \mathfrak{g}$.

The matrix corresponding to the linear map $(R_1 \oplus R_2)(X)$ is in block matrix notation

$$(R_1 \oplus R_2)(X) = \begin{pmatrix} R_1(X) & 0 \\ 0 & R_2(X) \end{pmatrix} \tag{6.57}$$

6.3 Two particle states

Let us start with some motivation. Assume we have two particles, each occupying a two-particle Hilbert space $\mathcal{H}_i = \{|\uparrow\rangle_i, |\downarrow\rangle_i\}$, $i = 1, 2$. The two particle states are states like $|\uparrow\rangle_1 |\downarrow\rangle_2$ or $|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2 \in \mathcal{H}_{12}$. To analyse this we will have to think more about how to combine two representations.

6.4 Tensor product

Definition 46: Let R_1 and R_2 be two representations of $\mathfrak{g}$ on representations spaces V_1 and V_2 of dimension d_1 and d_2 respectively. We can define the *direct sum* $R_1 \oplus R_2$ to be the new representation acting on $X \in V_1 \oplus V_2$ as

$$(R_1 \oplus R_2)(X) = \begin{pmatrix} R_1(X) & 0 \\ 0 & R_2(X) \end{pmatrix} \tag{6.58}$$

Definition 47: Given two vector spaces V_1 and V_2, we define the *(tensor) product space*

$$V_1 \otimes V_2 = \mathrm{Span}_F \{v_1 \otimes v_2 \mid v_1 \in V_1, v_2 \in V_2\} \tag{6.59}$$

where the tensor product satisfies $\forall v_i \in V_i, \alpha \in F$:

- $(v_1 + w_1) \otimes (v_2 + w_2) = v_1 \otimes v_2 + w_1 \otimes v_2 + v_1 \otimes w_2 + w_1 \otimes w_2$
- $\alpha(v_1 \otimes v_2) = (\alpha v_1) \otimes v_2 = v_1 \otimes (\alpha v_2)$

Definition 48: Given two linear maps $M_i \colon V_i \to V_i$, $i = 1, 2$, we define the *tensor product map*

$$\begin{aligned} (M_1 \otimes M_2) \colon V_1 \otimes V_2 &\to V_1 \otimes V_2 \\ v_1 \otimes v_2 &\mapsto (M_1 v_1) \otimes (M_2 v_2). \end{aligned} \tag{6.60}$$

We extend this action on the basis $v_1 \otimes v_2$ by linearity to the full tensor product space $V_1 \otimes V_2$.

Definition 49: Given representations R_i, $i = 1, 2$ of $\mathfrak{g}$ with corresponding representation spaces V_i, we have, for each element X in the Lie algebra $\mathfrak{g}$, a map $R_i(X) \colon V_i \to V_i$. We now define the *tensor product representation* $R_1 \otimes R_2$ with representation space $V_1 \otimes V_2$ which acts on all $X \in \mathfrak{g}$ as

$$\boxed{(R_1 \otimes R_2)(X) = R_1(X) \otimes I_2 + I_1 \otimes R_2(X)} \tag{6.61}$$

> **Remark:** Note that his is not the same as $R_1(X) \otimes R_2(X)$.

Let us choose a particular set of basis vectors for the two vector spaces:

$$B_1 = \left\{ V_1^j \mid j = 1, \ldots, d_1 \right\} \qquad B_2 = \{ V_2^\alpha \mid \alpha = 1, \ldots, d_2 \} . \tag{6.62}$$

We can then represent $R_1 \otimes R_2$ as a 'matrix' with $i, j = 1, \ldots, d_1$ and $\alpha, \beta = 1, \ldots, d_2$ as

$$(R_1 \otimes R_2)(X) = R_1(X)_{ij} \mathbb{1}_{ij} + \mathbb{1}_{ij} R_2(X)_{\alpha\beta}. \tag{6.63}$$

The representation $R_1 \times R_2$ has dimension $d_1 \times d_2$.

Exercise 6.1: Check that $R_1 \otimes R_2$ as defined is a representation of $\mathfrak{g}$.

> **Remark:** In physics terms, the motivation for this construction come from the laws of quantum mechanics: multi-particle states live in the tensor product space of the individual one-particle states. The definition of the addition of angular momentum is described by this definition of the tensor product representation.

> **Remark:** Up to isomorphism, the tensor product is associative: $(R_1 \otimes R_2) \otimes R_2 = R_1 \otimes (R_2 \otimes R_3)$.

6.5 Reducibility

Definition 50: Let us recall that a representation R with representation space V has an *invariant subspace* $U \subset V$ if $R(X)u \in U$ for all $X \in \mathfrak{g}$ and all $u \in U$.

We already know that an *irreducible* representation (irrep) has no non-trivial invariant subspace.

Definition 51: A *fully reducible* representation can be expressed as a direct sum of irreps.

Let us look at what this means for a particular basis. If R has a non-trivial invariant subspace, then we can find a basis that makes this manifest. In other words, we can find a basis such that $R(X)$ has block diagonal form:

$$R(X) = \begin{pmatrix} A(X) & B(X) \\ 0 & C(X) \end{pmatrix}. \tag{6.64}$$

In this case, elements of U correspond to vectors $\begin{pmatrix} u \\ 0 \end{pmatrix}$ for all elements X of the Lie algebra $\mathfrak{g}$.

If R is fully reducible, then $R = R_1 \oplus R_2 \oplus \cdots \oplus R_l$. Hence, we have a basis, where $R(X)$ is block diagonal, where the blocks are the individual irreps

$$R(X) = \begin{pmatrix} R_1(X) & & & \\ & R_2(X) & & \\ & & \ddots & \\ & & & R_l(X) \end{pmatrix}. \tag{6.65}$$

This is useful for us due to the following theorem, which we will not prove in this course:

Theorem 3: Let R_i, $i = 1, \dots, m$, be finite-dimensional irreps of a simple Lie algebra $\mathfrak{g}$. The tensor product $(R_1 \otimes \cdots \otimes R_m)$ is fully reducible. In other words,

$$R_1 \otimes R_2 \cdots \otimes R_m \simeq \widetilde{R}_1 \oplus \widetilde{R}_2 \oplus \cdots \oplus \widetilde{R}_{m'}. \tag{6.66}$$

> **Remark:** Another more mathematical way to say this is that the tensor representation forms a ring.

6.6 Tensor product of $\mathfrak{su}(2)$ representations

Let R_Λ and $R_{\Lambda'}$ be irreps of $\mathfrak{su}(2)$ with highest weights $\Lambda, \Lambda' \in \mathbb{N}_0$. Previously, we have found that $\dim(R_\Lambda) = \Lambda + 1$ and $\dim(R_{\Lambda'}) = \Lambda' + 1$. Let V_Λ and $V_{\Lambda'}$ be the respective representation spaces. We then form the tensor product representation $R_\Lambda \otimes R_{\Lambda'}$ with the complex representation space $V_\Lambda \otimes V_{\Lambda'}$, by representing each $X \in \mathfrak{su}(2)$ by a matrix $(R_\Lambda \otimes R_{\Lambda'})(X)$ that acts as

$$(R_\Lambda \otimes R_{\Lambda'})(X)(v \otimes v') = (R_\Lambda(X)v) \otimes v' + v \otimes (R_{\Lambda'}(X)v'). \tag{6.67}$$

We know that the dimension is simply the product of the two representation dimensions $\dim(R_\Lambda \otimes R_{\Lambda'}) = (\Lambda + 1)(\Lambda' + 1)$. Moreover, we know that this has to be a fully reducible representation of $\mathfrak{su}(2)$, which means that we can write

$$R_\Lambda \otimes R_{\Lambda'} = \bigoplus_{\Lambda'' \in \mathbb{N}_0} \mathcal{L}^{\Lambda''}_{\Lambda,\Lambda'} R_{\Lambda''} \tag{6.68}$$

where the *multiplicities* $\mathcal{L}^{\Lambda''}_{\Lambda,\Lambda'} \in \mathbb{N}_0$ are sometimes called *Littlewood coefficents*.

> **Remark:** In *Foundations of Quantum Mechanics* and *Group Theory*, we have met the Clebsch-Gordan coefficients. Clebsch-Gordan coefficients are matrix elements that are not always integers! So they are not the same as these multiplicities.

Let us try to calculate these coefficients. V_Λ has a basis $\left\{v^\lambda\right\}$ that are eigenvectors of $R_\Lambda(H)$ with eigenvalues $\lambda \in S_\Lambda = \{-\Lambda, -\Lambda+2, \ldots, +\Lambda\}$. Similarly, we have the same for $V_{\Lambda'}$.

For the Lie algebra of $SU(2)$, we have already describe the full set of irreducible representations R_Λ with $\dim(R) = \Lambda + 1$, where $\Lambda \in \mathbb{N}_0$. Let us consider the tensor product representations $R_\Lambda \otimes R_{\Lambda'}$. Because $\mathfrak{su}(2)$ is simple, it is fully reducible. In other words, we can decompose the product representation into a sum over the irreps $R_\Lambda \otimes R_{\Lambda'} \simeq \bigoplus_{\Lambda'' \in \mathbb{N}_0} \mathcal{L}^{\Lambda''}_{\Lambda,\Lambda'} R_{\Lambda''}$. Our task is now to find the multiplicities $\mathscr{L}$, which describe how often each irrep enters the decomposition. Let us use the Cartan basis. The vector space V_Λ has basis $\{v_\lambda\}$, $\lambda \in S_\Lambda = \{-\Lambda, -\Lambda+2, \ldots, +\Lambda\}$. The eigenvectors $R_\Lambda(H)v_\lambda = \lambda v_\lambda$. Similarly, we have a second vector space $V_{\lambda'}$ with basis $\{v'_{\lambda'}\}$. Construct a basis for $V_\Lambda \otimes V_{\Lambda'}$ as $B = \{v_\lambda \otimes v'_\lambda \mid \lambda \in S_\Lambda, \lambda' \in S_{\Lambda'}\}$. We now want to understand the weights of the new representation. To work out this weight, we act on a general basis vector with the representation of the Cartan element H. By definition, the tensor product representation acts as

$$(R_\Lambda \otimes R_{\Lambda'})(H)(v_\lambda \otimes v_{\lambda'}) = (R_\Lambda(H)v_\lambda) \otimes v'_{\lambda'} + v_\lambda \otimes (R_{\Lambda'}(H)v'_{\lambda'}) \tag{6.69}$$

$$= (\lambda + \lambda')(v_\lambda \otimes v_{\lambda'}). \tag{6.70}$$

The eigenvalues are just the sums of the eigenvalues of the basis elements. Therefore, we can deduce that $R_\Lambda \otimes R_{\Lambda'}$ has weight set $S_{\Lambda,\Lambda'} = \{\lambda + \lambda' \mid \lambda \in S_\Lambda, \lambda' \in S_{\Lambda'}\}$. Sometimes, this sum adds up the same weight in n different cases. We then have to keep track of these n multiplicities. We can then construct the representation, which is the sum of irreducibles, explicitly. The irreps are uniquely determined by identification of their weight set. We must take this weight set and decompose it into unions of the weight sets associated with the irreducibles; we will see that there is a unique way to do this. Consider first the highest weight. This will only come when $\lambda = \Lambda$ and $\lambda' = \Lambda'$. Since there is only one way to obtain the sum $\Lambda + \Lambda'$, the irrep $R_{\Lambda+\Lambda'}$ must appear in the decomposition with multiplicity one! In other words, $\mathcal{L}^{\Lambda+\Lambda'}_{\Lambda,\Lambda'} = 1$. We can decompose the tensor product as

$$R_\Lambda \otimes R_{\Lambda'} = R_{\Lambda+\Lambda'} \oplus \widetilde{R}_{\Lambda,\Lambda'}. \tag{6.71}$$

The problem is reduced to finding the remainder $\widetilde{R}_{\Lambda,\Lambda'}$, which will have weight set $\widetilde{S}_{\Lambda,\Lambda'}$ where $S_{\Lambda,\Lambda'} = S_{\Lambda+\Lambda'} \cup \widetilde{S}_{\Lambda,\Lambda'}$. We remove the weight set

$$S_{\Lambda+\Lambda'} = \{-\Lambda - \Lambda', \ldots, +\Lambda + \Lambda'\}, \tag{6.72}$$

and find the highest weight of the remainder $\widetilde{R}_{\Lambda,\Lambda'}$ and keep repeating until the remainder is empty.

Example $(\Lambda = \Lambda' = 1)$**:** In this case, the weight set of both representations consists of two elements

$$S_1 = \{-1, +1\}. \tag{6.73}$$

This is the representation a spin-$\frac{1}{2}$ particle carries. Then the weight set of the tensor product is

$$S_{1,1} = \{-1,+1\} + \{-1,+1\} = \{-2, 0, 0, +2\}. \tag{6.74}$$

$$= \{-2, 0, +2\} \cup \{0\}, \tag{6.75}$$

where $\{-2, 0, +2\}$ is the weight set of the highest weight representation R_2. We find that

$$R_1 \otimes R_1 = R_2 \oplus R_0. \tag{6.76}$$

In quantum mechanics, this means that two spin-$\frac{1}{2}$ particles form states of spin-1 (triplet) and spin-0 (singlet). Note that the dimensions match as well.

Exercise 6.2 (Sheet 2, Qu 8)**:** Consider two irreps of $SU(2)$: $\Lambda' = M$ and $\Lambda = N$. Then show, by applying the above algorithm, that $R_N \otimes R_M = R_{|M-N|} \oplus R_{|M-N|+2} \oplus \cdots \oplus R_{N+M}$. For $SU(2)$ this is essentially the whole story.

7 The Killing Form

We can specify an inner product on $\mathbb{R}^3$, by mapping $\mathbf{u}, \mathbf{v} \to \mathbf{u} \cdot \mathbf{v}$. Similarly, we might specify this scalar product by giving a metric such as $u_i, v_j \to \delta^{ij} u_i v_j$.

Definition 52 (Inner Product)**:** Given a vector space V over $F = \mathbb{R}$ or $\mathbb{C}$, an *inner product* is a bilinear, symmetric map $i : V \times V \to F$.

Definition 53: We say that i is *non-degenerate* if $\forall v \in V, v \neq 0$ there is a $w \in V$ such that the inner product $i(v, w) \neq 0$.

> **Remark:** This amounts to saying that there is no vector which is orthogonal to itself and all other vectors in the vector space.

Is there a 'natural' inner product that we can write down on a Lie algebra $\mathfrak{g}$? Yes! This is the *Killing form*.

Definition 54: The *Killing form* κ of a Lie algebra $\mathfrak{g}$ over a field F is the map

$$\begin{array}{rccc} \kappa\colon & \mathfrak{g} \times \mathfrak{g} & \to & F \\ & (X, Y) & \mapsto & \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \end{array} \tag{7.1}$$

Let us find out what this means explicitly in components. The action of the inner *ad*-map composition is

$$\begin{array}{rcl} (ad_X \circ ad_Y)\colon \mathfrak{g} & \to & \mathfrak{g} \\ Z & \mapsto & [X, [Y, Z]]. \end{array} \tag{7.2}$$

Choose a basis $\{T^a\}$, where $a = 1, \ldots, D = \dim(\mathfrak{g})$, for the Lie algebra $\mathfrak{g}$. We then know that by definition of the structure constants $f^{ab}{}_c$, we have $[T^a, T^b] = f^{ab}{}_c T^c$. Expanding the elements of the Lie algebra in terms of this basis, we see that

$$[X,[Y,Z]] = X_a Y_b Z_c [T^a,[T^b,T^c]] \tag{7.3}$$

$$= X_a Y_b Z_c f^{ad}{}_e f^{bc}{}_d T^e \tag{7.4}$$

$$= M(X,Y)^a{}_e Z_c T^e \tag{7.5}$$

with $M(X,Y)^c{}_e = X_a Y_b f^{ad}{}_e f^{bc_d}$. This is the explicit form of the inner ad-map composition. To find the explicit map defined by the Killing form, we take the trace of this:

$$\kappa(X,Y) = \mathrm{Tr}_D[M(X,Y)] = \kappa^{ab} X_a Y_b, \tag{7.6}$$

where, concretely, the Killing form is given by

$$\boxed{\kappa^{ab} = f^{ad}_c f^{bc}_d} \tag{7.7}$$

This is manifestly symmetric.

Now what does 'natural' mean?

Claim 15: The Killing form κ remains invariant under the adjoint action of the Lie algebra $\mathfrak{g}$, meaning that $\forall X,Y,Z \in \mathfrak{g}$

$$\boxed{\kappa([Z,X],Y) + \kappa(X,[Z,Y]) = 0}. \tag{7.8}$$

Proof. By the defining property of the adjoint representation, we have

$$ad_{[Z,X]} = ad_Z \circ ad_X - ad_X \circ ad_Z. \tag{7.9}$$

Therefore, the first term of (7.8) is

$$\kappa([Z,X],Y) = \mathrm{Tr}\big[ad_{[Z,X]} \circ ad_Y\big] \tag{7.10}$$

$$= \mathrm{Tr}[ad_Z \circ ad_X \circ ad_Y] - \mathrm{Tr}[ad_X \circ ad_Z \circ ad_Y] \tag{7.11}$$

Moreover, the second term of (7.8) is

$$\kappa(X,[Z,Y]) = \mathrm{Tr}[ad_X \circ ad_Z \circ ad_Y] - \mathrm{Tr}[ad_X \circ ad_Y \circ ad_Z]. \tag{7.12}$$

By cyclic invariance of the trace $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$, we find that these two terms exactly cancel. $\square$

In fact, for simple Lie algebras, the Killing form κ is the unique invariant inner product, up to an overall scalar multiple.

Definition 55: A Lie algebra is *semi-simple* if it has no Abelian ideals.

Exercise 7.1 (Sheet 2 Qu 9b)**:** Show that a finite-dimensional semi-simple Lie algebra can be written as the direct sum of a finite number of simple Lie algebras.

We will now try to find out under what conditions the Killing form is non-degenerate.

Theorem 4 (by Cartan)**:** The Killing form κ is non-degenerate if and only if the associated Lie algebra $\mathfrak{g}$ is semi-simple.

Proof of forward direction only. Assume that κ is non-degenerate. Suppose for contradiction that $\mathfrak{g}$ is not semi-simple. This means that $\mathfrak{g}$ has an Abelian ideal $\mathfrak{j}$. Denote $\dim(\mathfrak{g}) = D$ and $\dim(\mathfrak{j}) = d$. Choose a basis

$$B = \{T^a\} = \left\{T^i \mid i = 1, \dots, d\right\} \cup \{T^\alpha \mid \alpha = 1, \dots, D - d\}\,, \tag{7.13}$$

where $\left\{T^i\right\}$ span $\mathfrak{j}$. As $\mathfrak{j}$ is Abelian, we must have $[T^i, T^j] = 0$, $\forall i, j$. Moreover, as $\mathfrak{j}$ is an ideal, $[T^\alpha, T^j] = f^{\alpha j}{}_\kappa T^k \in \mathfrak{j}$ and therefore $f^{ij}_a = 0$ and $f^{\alpha j}_\beta = 0$. For $X = X_a T^a \in \mathfrak{g}$ and $Y = Y_j T^j \in \mathfrak{j}$, we have $\kappa[X, Y] = \kappa^{ai} X_a Y_i$ with

$$\kappa^{ai} = f^{ad}_c f^{ic}_d = f^{aj}_\alpha f^{i\alpha}_j = 0. \tag{7.14}$$

Therefore, we have $K[X, Y] = 0$ for all $X \in \mathfrak{j}$ and all $X \in \mathfrak{g}$. In other words, κ is degenerate, which contradicts the assumption. Hence, $\mathfrak{g}$ is semi-simple. ☐

7.1 Complexification

Given a *real* Lie algebra $\mathfrak{g}$, we can find a basis $\{T^a\}$, $a = 1, \dots, \dim \mathfrak{g}$, with real structure constants

$$[T^a, T^b] = f^{ab}_c T^c, \qquad f^{ab}_c \in \mathbb{R}. \tag{7.15}$$

Definition 56: Given a real Lie algebra $\mathfrak{g} = \text{Span}_\mathbb{R}\{T^a\}$, we define the *complexification* $g_\mathbb{C} = \text{Span}_\mathbb{C}\{T^a\}$.

Together with the bracket (7.15), $g_\mathbb{C}$ is a *complex Lie algebra.*

Example: Consider the Lie algebra of $SU(2)$:

$$\mathscr{L}(SU(2)) = \mathfrak{su}(2) = \text{span}_\mathbb{R}\left\{T^a = -\frac{i\sigma_a}{2} \mid a = 1, 2, 3\right\} \tag{7.16}$$

$$= \{2 \times 2 \text{ traceless anti-Hermitian matrices}\}\,. \tag{7.17}$$

Its complexification is

$$\mathscr{L}_{\mathbb{C}}(SU(2)) = \mathfrak{su}_{\mathbb{C}}(2) = \mathrm{Span}_{\mathbb{C}}\left\{T^a = -\frac{i\sigma_a}{2} \mid a = 1,2,3\right\} \tag{7.18}$$

$$= \{2 \times 2 \text{ traceless complex matrices}\}\,. \tag{7.19}$$

The Cartan-Weyl basis for $\mathfrak{su}_{\mathbb{C}}(2)$ is given by $H = 2iT^3$ and $E_\pm = iT^1 \pm T^2$ with brackets

$$[H, E_\pm] = \pm 2E_\pm \qquad [E_+, E_-] = H. \tag{7.20}$$

Appendix

Starting from a representation R of $\mathfrak{su}_{\mathbb{C}}(2)$ with

$$[R(H), R(E_\pm)] = \pm 2R(E_\pm) \tag{7.21a}$$

$$[R(E_+), R(E_-)] = R(H). \tag{7.21b}$$

Pass back to original basis via

$$R(T^1) = \frac{1}{2i}\big(R(E_+) + R(E_-)\big) \tag{7.22a}$$

$$R(T^2) = \frac{1}{2}\big(R(E_+)) - R(E_-)\big) \tag{7.22b}$$

$$R(T^3) = \frac{1}{2i}R(H). \tag{7.22c}$$

For all $X \in \mathfrak{su}(2)$, we can expand $X = X_a T^a$ for some $X_a \in \mathbb{R}$. Set $R(X) = X_a R(T^a)$ to get a representation of $\mathfrak{su}(2)$.

7.2 Cartan Classification

Can classify *all* finite-dimensional simple complex $\mathfrak{g}$ (Cartan 1894).

Definition 57: We say that $X \in \mathfrak{g}$ is *ad-diagonalisable* (AD), if $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is diagonalisable.

Definition 58: A *Cartan subalgebra* (CSA) $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal Abelian subalgebra,

1. $H \in \mathfrak{h} \implies H$ is (AD)
2. $H, H' \in \mathfrak{h} \to [H, H'] = 0$
3. if $X \in \mathfrak{g}$ and $[X, H] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$

In fact, all possible CSAs of $\mathfrak{g}$ are isomorphic and have the same dimension, $r = \dim \mathfrak{h} \in \mathbb{N}$, which is the *rank* of $\mathfrak{g}$.

Example: $\mathfrak{g} = \mathfrak{su}_{\mathbb{C}}(2) = \text{span}_{\mathbb{C}} \{H, E_\pm\} \implies \text{rank}(\mathfrak{g}) = 1$, H is (AD): $[H, E_\pm] = \pm 2E_\pm$ and $[H, H] = 0$, but $E_\pm$ are *not*.

$\mathfrak{h} = \text{span}_{\mathbb{C}} \{H\}$ is choice of CSA. Choose basis $\left\{H^i \mid i = 1, \dots, r\right\}$, $[H^i, H^j] = 0$.

Example: $\mathfrak{su}_{\mathbb{C}}(N) = \{\text{traceless complex } n \times n\text{-matrices}\}$ (why not anti-hermitian?). Choose $(H^i)_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta i} - \delta_{\alpha i+1}\delta_{\beta i+1}$. Then $\text{rank}[\mathfrak{su}_{\mathbb{C}}(N)] = N - 1$, $[H^i, H^j] = 0$ for all $i, j = 1, \dots, r$.

$\implies$ $(ad_{H^i} \circ ad_{H^j} - ad_{H^j} \circ ad_{H^i}) = 0$ (def. property of adjoint rep)

$\implies$ r linear maps $ad_{H^i} : \mathfrak{g} \to \mathfrak{g}$, with $i = d, \dots, r$ are *simultaneously diagonalisable*

$\implies$ $\mathfrak{g}$ is spanned by simultaneous eigenvectors ad_{H^i}

$$\boxed{ad_{H^i}(E^\alpha) = [H^i, E^\alpha] = \alpha^i E^\alpha, \quad i = d, \dots, r} \tag{7.23}$$