General Relativity: Example Sheet 2

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1. The metric of Minkowski spacetime in the coordinates of an inertial frame is

$$ds^2 = -dt^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

(a) Show that if we replace (x, y, z) with spherical polar coordinates (r, θ, ϕ) defined by

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = z/r, \quad \tan\phi = y/x,$$

then the metric takes the form

$$ds^2 = -dt^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2).$$

(b) Find the components of the metric in "rotating coordinates" defined by

$$t' = t, \qquad x' = \sqrt{x^2+y^2}\cos(\phi - \omega t), \qquad y' = \sqrt{x^2+y^2}\sin(\phi - \omega t), \qquad z' = z$$

where $\tan\phi = y/x$.

2. Consider a change of basis $\tilde{e}_\mu = (A^{-1})^\nu{}_\mu e_\nu$. Show that the components of a connection in the new basis are related to its components in the old basis by

$$\tilde{\Gamma}^\rho_{\mu\nu} = A^\rho{}_\lambda (A^{-1})^\sigma{}_\mu \left[(A^{-1})^\tau{}_\nu \Gamma^\lambda_{\sigma\tau} + e_\sigma((A^{-1})^\lambda{}_\nu)\right]$$

Show further that the difference of two connections, $(\Gamma_1)^\rho_{\mu\nu} - (\Gamma_2)^\rho_{\mu\nu}$, transforms as a tensor.

3. Let ∇ be a connection that is not torsion-free. Let $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ where X and Y are vector fields. Show that this defines a $(1, 2)$ tensor field T. This is called the *torsion tensor*. Show that, for any function f,

$$2\nabla_{[\mu}\nabla_{\nu]} f = -T^\rho{}_{\mu\nu}\nabla_\rho f$$

4. Let ∇ be a torsion-free connection. Derive the analogue of the Ricci identity for a 1-form ω,

$$\nabla_{[\mu}\nabla_{\nu]}\omega_\rho = -R^\sigma{}_{\rho\mu\nu}\,\omega_\sigma$$

5. The Riemann tensor constructed from the Levi-Civita connection obeys the Bianchi identity $R^\mu{}_{\nu[\rho\sigma;\lambda]} = 0$. Use this fact to derive the contracted Bianchi identity $G^\mu{}_{\nu;\mu} = 0$ where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor.

6*. The *Reissner-Nordstrom* solution of the Einstein-Maxwell equations has metric

$$ds^2 = -f(r)^2\, dt^2 + f(r)^{-2} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

with

$$f(r)^2 = 1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}$$

and a Maxwell field strength $F = dA$, with

$$A = -\frac{Q}{r} dt - P\cos\theta\, d\phi$$

where M, P, Q are constants. M can be interpreted as the total mass of this spacetime. Assume that (t, r, θ, ϕ) is a right handed coordinate chart. Show that

$$\frac{1}{4\pi}\int_{\mathbf{S}^2_\infty} \star F = Q \quad \text{and} \quad \frac{1}{4\pi}\int_{\mathbf{S}^2_\infty} F = P \tag{1}$$

where $\mathbf{S}^2_\infty$ is a sphere at $r = \infty$ on a surface of constant t. What is the physical interpretation of Q and P?

7. A vector field Y is parallely propagated (with respect to the Levi-Civita connection) along an affinely parameterized geodesic with tangent vector X in a Riemannian manifold. Show that the magnitudes of the vectors X, Y and the angle between them are constant along the geodesic.

On the unit sphere a unit vector Y is initially tangent to the line $\phi = 0$ at a point on the equator. It is then moved by parallel propagation first along the equator to the point $\phi = \phi_0$, from there along the line $\phi = \phi_0$ to the North pole, and then back along the line $\phi = 0$ to its original position. By how much has it changed, and why?

8. In Q7 of examples sheet 1, we showed that

$$\begin{aligned}(\mathcal{L}_X\omega)_\mu &= X^\nu\partial_\nu\omega_\mu + \omega_\nu\partial_\mu X^\nu \\ (\mathcal{L}_X g)_{\mu\nu} &= X^\rho\partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu X^\rho + g_{\rho\nu}\partial_\mu X^\rho\end{aligned}$$

Use normal coordinates to argue that one can replace partial derivatives with covariant derivatives to obtain the basis-independent results

$$\begin{aligned}(\mathcal{L}_X\omega)_\mu &= X^\nu\nabla_\nu\omega_\mu + \omega_\nu\nabla_\mu X^\nu \\ (\mathcal{L}_X g)_{\mu\nu} &= \nabla_\mu X_\nu + \nabla_\nu X_\mu\end{aligned}$$

with ∇ is the Levi-Civita connection.

9. How many independent components does the Riemann tensor (of the Levi-Civita connection) have in two, three and four dimensions? Show that in two dimensions

$$R_{\mu\nu\rho\sigma} = \tfrac{1}{2}R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}).$$

Discuss the implications for general relativity in two spacetime dimensions.

10. In a d-dimensional spacetime, define a tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \alpha(R_{\mu\rho}g_{\nu\sigma} + R_{\nu\sigma}g_{\mu\rho} - R_{\mu\sigma}g_{\nu\rho} - R_{\nu\rho}g_{\mu\sigma}) + \beta R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$$

where α and β are constants. Show that $C_{\mu\nu\rho\sigma}$ has the same symmetries as $R_{\mu\nu\rho\sigma}$.

What values of α and β give $C^{\mu}{}_{\nu\mu\sigma} = 0$? Determine them. With this extra condition $C_{\mu\nu\rho\sigma}$ is called the *Weyl tensor*. Show that it vanishes if $d = 2, 3$.

Setting $d = 4$, how many independent components do $R_{\mu\nu}$ and $C_{\mu\nu\rho\sigma}$ have? Show that in vacuum

$$\nabla^{\mu}C_{\mu\nu\rho\sigma} = 0.$$

What does the Weyl tensor represent physically?

11. Use the Bianchi identity to derive the *Penrose equation* for a vacuum spacetime

$$\nabla^{\lambda}\nabla_{\lambda}R_{\mu\nu\rho\sigma} = 2R^{\kappa}{}_{\mu\lambda\sigma}R^{\lambda}{}_{\rho\kappa\nu} - 2R^{\kappa}{}_{\nu\lambda\sigma}R^{\lambda}{}_{\rho\tau\mu} - R^{\kappa}{}_{\lambda\sigma\rho}R^{\lambda}{}_{\kappa\mu\nu}$$

12*. Consider metrics of the form

$$ds^2 = -f(r)^2\,dt^2 + f(r)^{-2}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$$

Use the action for a test particle to write down the geodesic equations in this metric, and hence extract the Christoffel symbols in coordinates (t, r, θ, ϕ).

Use a basis of vierbeins to determine the curvature 2-form, and hence the components of the Riemann tensor in coordinates (t, r, θ, ϕ).