String Theory

Part III Lent 2020

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Admin Stuff

Books and Lecture Notes

- String Theory Vol. 1 — Polchinski, CUP

  Probably fits this course most closely, although there will be some things that will be in this course that are not in the book and vice-versa.

- Superstrings Vol. 1 — Green et. al, CUP

- A String Theory Primer — Schomerus, CUP

Online Resources

- Lecture notes by the professor are online.

- Probelem sheet 1 by the professor is online.

- David Tong's notes on arXiv:0908.0333.

- 'Why string theory?' Conlon (CRC) for some light reading and history

1 Introduction

1.1 What is String Theory?

We do not actually know. This question still needs fleshing out to be answered well. Most likely, the actual string theory that we are working towards will be very different to the content that we will cover in these lectures.

In some sense, string theory is an attempt at quantising the gravitational field.

Naive quantisation of the Einstein-Hilbert action presents a number of problems.

Conceptual Problems

- The nature of time: Non-relativistic quantum mechanics is based on the Hamiltonian formulation. This is not necessarily a technical problem.

- How to quantise without a pre-existing causal structure?

  One of the first things we learn in QM, when going to QFT, is that it is important to know whether two operators are timelike or spacelike separated. We have a notion that all operators that are spacelike separated commute. However, if we are talking about general relativity, the metric contains information about and determines the causal structure. But this is what we want to quantise. So it is not immediately obvious what the algebra of operators should look like.

- The symmetry of general relativity is diffeomorphism invariance (coordinate reparametrisations).

  This is a gauge symmetry. One thing we will discuss (although not prove) when talking about scattering amplitudes in string theory, the fact is that there are no *local* diffeomorphism-invariant observables. As such, it is not even obvious what the quantum observables should be.

We will not be able to answer these deeper conceptual questions within the framework of string theory.

But perhaps more importantly, there are technical obstacles. You might ask, are there any assumptions that allow us to make progress and return to the difficult conceptual problems later?

Technical obstacles

Let us look at perturbation theory.

In particle physics, we often expand $g_{\mu\nu}$ about some classical solution, e.g. Minkowski spacetime:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x). \tag{1.1}$$

This means that we can use the causal structure of the *background* classical metric $\eta_{\mu\nu}$ to talk about quantisation of the fluctuations $h_{\mu\nu}$. This is what we do in *Field Theory in Cosmology* for inflation. In spirit, this is very close to what we do in string theory.

However, in some sense this split between background and perturbation is artificial and arbitrary. Nonetheless, we can take the Einstein–Hilbert action in general dimensions D

$$S[g] = \frac{1}{k_D} \int \mathrm{d}^D x \sqrt{-g} R(g) \tag{1.2}$$

and expand it out by choosing a gauge. Choosing a gauge wisely, we get an action

$$S[h] \approx \frac{1}{k_D} \int \mathrm{d}^D x \left(h_{\mu\nu} \Box h^{\mu\nu} + \dots \right). \tag{1.3}$$

Since the Ricci scalar involves inverse powers of the metric, this expansion will not terminate. We say that this expression is *non-polynomial* in $h_{\mu\nu}$.
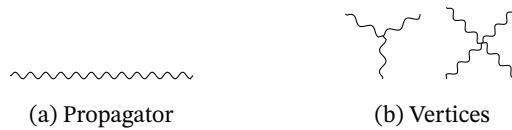

(a) Propagator (b) Vertices

Figure 1.1: Feynman rules

The quadratic term gives us a propagator of Fig. 1.1a and the interaction terms give us vertices illustrated in Fig. 1.1b These are the Feynman rules.

However, loops give divergences, which cannot be dealt with using standard techniques (renormalisation, c.f. *Advanced Quantum Field Theory*).

String theory provides a way to do quantum perturbation theory of the gravitational 'field' (and much more).

String theory gives us a framework to ask meaningful questions concerning quantum gravity (although we are not able to answer all of them at the moment).

There are also other approaches out there, although string theory is most thoroughly studied / understood (although this might be because most resources in this area flow into string theory).

One of the possibilities is trying to make classical gravity consistent with the standard model without needing to quantise it.

There will be a sense in which we will learn quite a lot about QFT from studying string theory.

To a certain extend, the motivation that we will take throughout this course is not whether string theory will give us a theory of the real world, but rather whether it might give us a hint on how to reconcile quantum mechanics and gravity. We have been trying to do this for the better part of a hundred years, so any hint would be appreciated.

1.2   Worldsheets and Embeddings

Popular science books and textbooks alike usually start off a discussion of string theory by starting with the assumption that particles are not point-like, but rather tiny little strings.

The starting point is to consider a worldsheet Σ, a two-dimensional surface swept out by a string. This is analogous to a worldline swept out by a pointlike particle moving through spacetime, as illustrated in Fig. 1.2.
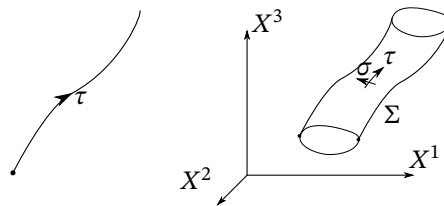
Figure 1.2: Particle worldline and string worldsheet

We put coordinates (σ, τ) on Σ (at least locally) and we define an embedding of Σ in the background spacetime M by the functions $X^\mu(\sigma, \tau)$, where the X^μ are coordinates on M, i.e. $X : \Sigma \to M$.

There are rules (which we shall investigate) for glueing such worldsheets together, in a way that is consistent with the symmetries of the theory.

We shall see that diagrams such as Fig. 1.3 are in one-to-one correspondence with correlation functions in some quantum theory. It is natural to interpret such diagrams as Feynman diagrams in
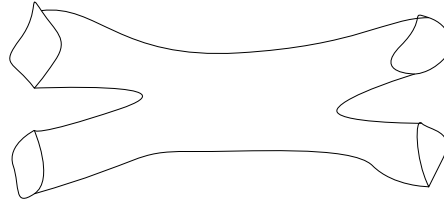
Figure 1.3: Not a particularly beautiful diagram...

a perturbative expansion of some theory about a vacuum.

2 The Classical Particle and String

In this section we will start systematically constructing a theory of quantum strings. As a precursor to this discussion, we will need to talk a bit about non-relativistic quantum mechanics.

In non-relativistic QM, we treat time (t) as a parameter and position ($\hat{x}$) as an operator. Obviously, this kind of distinction should not survive in a relativistic theory. This means there are some choices to be made.

Second quantisation: Both x^i and t are considered parameters. They are not the fundamental objects we quantise. Instead, we quantise quantum fields $\phi(\mathbf{x}, t)$, which are the basic objects of our theory. We require that the fields transform in appropriate ways under Lorentz transformations. This is historically overwhelmingly the most useful way to do it, seeing its success in quantum field theory and the standard model.

**First quantisation:** Here we make the other choice: we elevate t to being an operator. This is the natural framework for describing the relativistic embedding of worldlines (-sheets, -volumes) into spacetime. Here, $X^\mu = (x^i, t)$ is an operator, the fundamental object we quantise, and there will be some other natural parameter entering the theory.

There is such a thing as string field theory, which employs the viewpoint of second quantisation. However, apart from a few exceptions, there is not much that you do not also obtain from first quantisation string theory. However, first quantisation string theory has made significant advances on problems that have not yet be solved with second quantisation.

2.0.1 Worldlines and Particles

We consider the embedding of a worldline $\mathcal{L}$ into spacetime M. The basic field is the embedding $X^\mu : \mathcal{L} \to M$. An action that describes this embedding might be

$$S[X] = -m \int_{x_2}^{x_1} \mathrm{d}s = -m \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \sqrt{-\eta^{\mu\nu}\dot{X}^\mu \dot{X}^\nu}, \tag{2.1}$$

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where τ (a parameter) is the proper time and $X^\mu(\tau_2) = x_2^\mu, X^\mu(\tau_1) = x_1^\mu$ are end points of the world line.
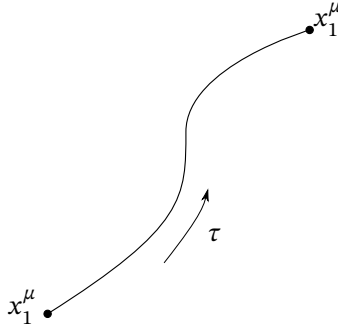
Figure 2.1

The momentum conjugate to $X^\mu(\tau)$ is $P_\mu(\tau) = -m\frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}}$. This satisfies $\dot{X}^2 = \eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu$. This satisfies $P^2 + m^2 = 0$ identically, an *on-shell condition*.

Symmetries

Rigid symmetry: $X^\mu(\tau) \to \Lambda^\mu{}_\nu X^\nu(\tau) - a^\mu$, where $\Lambda^\mu{}_\nu$ is a Lorentz transformation and a^μ is a (constant) displacement.

Reparameterisation invariance: $\tau \to \tau + \xi(\tau)$. The embedding X^μ changes as

$$X^\mu(\tau) \to X^\mu(\tau + \xi) = X^\mu(\tau) + \xi\dot{X}^\mu(\tau) + \dots. \tag{2.2}$$

To first order, $\delta X^\mu(\tau) = \xi\dot{X}^\mu(\tau)$.

There is a rewriting of this action (2.1) that makes life a bit easier. We do this by introducing a new auxiliary field $e(\tau)$, which is a one-form, on the worldline $\mathcal{L}$.

$$S[X, e] = \frac{1}{2}\int_\mathcal{L} \mathrm{d}\tau \left(e^{-1}\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu - em^2\right). \tag{2.3}$$

If you like, you can think of e as something like a one-dimensional metric, which sets a scale for distances on the line.

We will be able to do something analogous for strings, which will be very useful!

The equations of motion for $X^\mu(\tau)$ and $e(\tau)$ are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(e^{-1}\dot{X}^\mu\right) = 0. \tag{2.4}$$

However, the equation of motion for e will not depend on $\dot{e}$, but be purely algebraic:

$$\dot{X}_2 + e^2 m^2 = 0. \tag{2.5}$$

As such, e can be thought of as a Lagrange multiplier, enforcing a constraint. What is the constraint that it enforces?

The momentum conjugate to X^μ is

$$P_\mu = e^{-1}\dot{X}_\mu. \tag{2.6}$$

Combining (2.5) and (2.6), we can eliminate e to find precisely the on-shell condition $P^2 + m^2 = 0$.

▌   The auxiliary field e basically enforces energy-momentum conservation on the worldline.

Exercise 2.1: We can write $e^{-1} = m/|\dot{X}|$. Plug this into the action (2.3) to find that $S[X, e]$, subject to the equation of motion for $e(\tau)$ gives precisely the action (2.1).

There are two reasons why we should consider this new action instead. Firstly, in the $m \to 0$ limit, it will be easy to show that it describes null worldlines. Secondly, since there is no square root in this theory, it will be easier to quantise. This is at the cost of having to introduce a new non-dynamical field e.

The action $S[X, e]$ has the following symmetries:

- Poincaré invariance (where e is invariant)

- Reparameterisation invariance:

$$\delta X^\mu = \xi \dot{X}^\mu \qquad \delta e = \frac{\mathrm{d}}{\mathrm{d}\tau}(\xi e) \tag{2.7}$$

  provided these variations vanish on the endpoints.

▌   Note that this transformation law under diffeomorphisms shows that e needs to transform like a one-form.

▌   Locally, we can think of e as a pure gauge.

Remark: We could generalise $\eta_{\mu\nu} \to g_{\mu\nu}(X(\tau))$. The model becomes highly non-linear. We will look at this later in the context of strings.

2.1   Classical Strings: The Nambu–Goto Action

Our starting point will be the *Nambu-Goto Action*. We use units where $\hbar = c = 1$ throughout. The
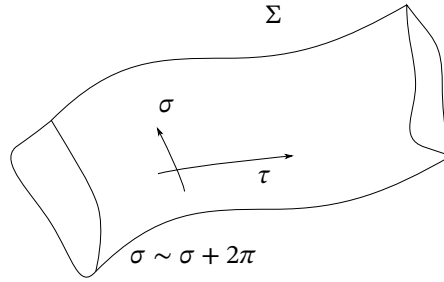
Figure 2.2

fundamental degree of freedom is

$$X : \sigma \to M, \tag{2.8}$$

where M is called the *target space*.

Definition 1 (Nambu–Goto Action)**:** The Nambu–Goto action is

$$S[X] = -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}\tau\,\mathrm{d}\sigma\,\sqrt{-\det\!\left(\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\right)}, \tag{2.9}$$

where α' is a constant with dimensions of (spacetime) area and $\sigma^a = (\tau, \sigma)$ and $\partial_a = \frac{\partial}{\partial\sigma^a}$.

Remark: One often speaks of the *string length* $l_2 = 2\pi\sqrt{\alpha'}$.

Definition 2 (string tension)**:** We introduce the *string tension* $T = \frac{1}{2\pi\alpha'}$.

> This turns out to be a good starting point; quantising this puts us on the right track. However, it is horrendously difficult to actually perform this quantisation. Instead, we will consider another action.

Definition 3 (Polyakov Action)**:** The Polyakov action is

$$S[X, h] = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}\,h^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu. \tag{2.10}$$

The new feature as compared to the Nambu–Goto action is the new field h_{ab}, which is a metric on Σ. It is non-dynamical; there are no Einstein-Hilbert terms and similarly to e previously it can be thought of as enforcing a constraint. We will find that h_{ab} is extremely important.

> This form is quite suggestive. All this is, when quantised, is a two-dimensional massless Klein–Gordon field in three dimensions. It is difficult to find an easier theory than this.

Chapter 2. The Classical Particle and String

Let us look at the equations of motion for the Polyakov action.

Exercise 2.2: If we vary the worldsheet metric $h_{ab} \to h_{ab} + \delta h_{ab}$, then the action changes as

$$\delta S = -\frac{1}{2\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h} T_{ab} \delta h^{ab}, \tag{2.11}$$

with *stress tensor* $T_{ab} = \frac{1}{\alpha'}\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}h_{ab}h^{cd}\partial_c X^\mu \partial_d X_\mu$.

> **Remark:** Indices are raised and lowered with the Minkowski metric.

So the h_{ab} equation of motion is

$$\boxed{T_{ab} = 0} \tag{2.12}$$

Remark: • The trace vanishes $h^{ab}T^{ab} = 0$, because $h_{ab}h^{ab} = 2$.

- Symmetric $T_{ab} = T_{ba}$ because $h_{ab} = h_{ba}$.

- The X^μ equation of motion is

$$\frac{1}{\sqrt{-h}}\partial_a\left(\sqrt{-h}h^{ab}\partial_b X^\mu\right) = 0. \tag{2.13}$$

If the $h_{ab} = \text{diag}(-1, 1)$, then this is the wave equation.

2.2 Classical Equivalence of Polyakov and Nambu–Goto

It is useful to define $G_{ab} = \eta_\mu \partial_a X^\mu \partial_b X^\nu$. The Nambu–Goto action is then

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma \sqrt{-\det(G_{ab})}. \tag{2.14}$$

By imposing the metric's equation of motion, the vanishing of T_{ab} tells us that

$$G_{ab} - \frac{1}{2}h_{ab}\underbrace{G_{cd}h^{cd}}_{G:=\text{tr}(G_{ab})} = 0. \tag{2.15}$$

The determinant is

$$\det(G_{ab}) = \frac{1}{4}G^2\det(h_{ab}) = \frac{1}{4}G^2 h. \tag{2.16}$$

So $\sqrt{-h}G = 2\sqrt{-\det G_{ab}}$ and

$$\frac{1}{2}\sqrt{-h}h^{cd}\partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \sqrt{-\det(G_{ab})}. \tag{2.17}$$

Remark: This is because h_{ab} is not appearing dynamically in the action (no derivatives), so its equation of motion is just a constraint.

Chapter 2. The Classical Particle and String

2.3 The Polyakov Action

$$S[X, h] = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h}\, h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu.$$

Can we generalise this action?

- We could replace $\eta_{\mu\nu}$ with a general $g_{\mu\nu}(x)$ (more later).

- What about a 2-D Einstein–Hilbert term, to make h_{ab} dynamical?

$$\frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{-h}\, R(h) = \chi. \tag{2.18}$$

  In 2-D, this is a topological invariant: the Euler characteristic χ of the worldsheet.

- What about a cosmological constant on Σ?

$$\Lambda \int_\Sigma d^2\sigma \sqrt{-h} \tag{2.19}$$

  The equations of motion for h_{ab} will be of the form

$$T_{ab} \propto \Lambda h_{ab}. \tag{2.20}$$

  However, we know that T_{ab} is traceless. We therefore have $h^{ab} T_{ab} \propto 2\Lambda$, so we need $\Lambda = 0$.

- We could include background fields in the spacetime.

  For example, there could be a 2-form field $B(X) = \frac{1}{2} B_{\mu\nu} dX^\mu \wedge dX^\nu$. We could include the term

$$-\frac{1}{2\pi\alpha'} \int_\Sigma B = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h}\, \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}. \tag{2.21}$$

  Or we might have a scalar field $\phi(X)$. The sort of term we might have is something like

$$\frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{-h}\, \phi(X) R(h). \tag{2.22}$$

2.3.1 Symmetries

- Poincaré invariance: $X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu$, where $\Lambda^\mu{}_\nu$ and a^μ are independent of (σ, τ). (These are rigid / global symmetries, as opposed to gauge / local symmetries). The metric $h^{ab} \to h^{ab}$ does not change.

- Diffeomorphism invariance.

  Infinitesimally: $\sigma^a \rightarrow \sigma^a + \xi^a$. Then

$$\delta X^\mu = \xi^a \partial_a X^\mu \tag{2.23}$$

$$\delta h_{ab} = \xi^c \partial_c h_{ab} + (\partial_a \xi^c) h_{bc} + (\partial_b \xi^c) h_{ac}. \tag{2.24}$$

- Weyl invariance: $X^\mu \rightarrow X^\mu$ does not change. But the worldsheet is rescaled by some position dependent factor $h_{ab} \rightarrow e^{2\Lambda(\sigma,\tau)} h_{ab}$.

  Infinitesimally:

$$\delta X^\mu = 0, \qquad \delta h_{ab} = 2\Lambda h_{ab}. \tag{2.25}$$

2.3.2 Classical Solutions

The two-dimensional metric h_{ab} has three degrees of freedom. We can use the diffeomorphism invariance to fix two of the degrees of freedom in h_{ab} (at least locally) and write it as

$$h_{ab} = e^{2\Phi} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.26}$$

Moreover, Weyl invariance means that the factor $e^{2\phi}$ drops out, eliminating the final degree of freedom.

The action then becomes

$$S[X] = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \left(-\dot{X}^2 + (X')^2 \right), \qquad \dot{X}^\mu := \frac{\partial X^\mu}{\partial \tau}, \quad (X')^\mu := \frac{\partial X^\mu}{\partial \sigma}. \tag{2.27}$$

The equation of motion is

$$\Box X^\mu = 0 \qquad \Box = -\partial_\tau^2 + \partial_\sigma^2. \tag{2.28}$$

Solutions are split between left- and right-movers.

$$X^\mu(\sigma, \tau) = X_R^\mu(\tau + \sigma) + X_L^\mu(\tau + \sigma). \tag{2.29}$$

Introduce the Fourier modes α_n^μ and $\overline{\alpha}_n^\mu$ (which are not complex conjugate to each other, but it will be useful to think of them as if they were). The solutions then become

$$X_R^\mu(\tau - \sigma) = \frac{1}{2} x^\mu + \frac{\alpha'}{2} p^\mu (\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in(\tau - \sigma)} \tag{2.30}$$

$$X_R^\mu(\tau + \sigma) = \frac{1}{2} x^\mu + \frac{\alpha'}{2} p^\mu (\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\overline{\alpha}_n^\mu}{n} e^{-in(\tau + \sigma)}. \tag{2.31}$$

The x^μ and p^μ are the centre of mass position and momentum of the string in spacetime.

Notation: We also introduce the notation

$$\alpha_0^\mu = \overline{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu \tag{2.32}$$

X^μ is real, so we also require that $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$.

2.4　Classical Hamiltonian Dynamics of the String

We continue to work in conformal gauge, where $h_{ab} = e^{\phi} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Definition 4 (canonical momentum)**:** We define the *canonical momentum field*, conjugate to X^{μ} as the functional derivative

$$P_{\mu}(\sigma, \tau) = \frac{\delta S(x)}{\delta X^{\mu}(\sigma, \tau)} = \frac{1}{2\pi\alpha'} \dot{X}_{\mu}. \tag{2.33}$$

Definition 5: Given the Lagrangian density $\mathscr{L}$, the Hamiltonian density is

$$\mathscr{H} = P_{\mu}\dot{X}^{\mu} - \mathscr{L} = \frac{1}{4\pi\alpha'}(\dot{X}^2 + X'^2). \tag{2.34}$$

Definition 6 (Poisson brackets)**:** We introduce Poisson brackets $\{\cdot, \cdot\}_{\text{PB}}$. For a particle theory, where our coordinates $x^{\mu}(\tau)$ and momenta $p_{\mu}(\tau)$ are our fundamental variables, it is useful to define

$$\{f, g\}_{\text{PB}} = \frac{\partial f}{\partial x^{\mu}} \frac{\partial g}{\partial p_{\mu}} - \frac{\partial f}{\partial p_{\mu}} \frac{\partial g}{\partial x^{\mu}}. \tag{2.35}$$

Example 2.4.1: We have for example, $\{x^{\mu}, p_{\nu}\} = \delta^{\mu}_{\nu}$.

The Hamiltonian $\mathscr{H}$ is the generator of time translations

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \frac{\partial f}{\partial \tau} + \{f, \mathscr{H}\}. \tag{2.36}$$

Our field theoretic generalisation requires

$$\{X^{\mu}(\sigma, \tau), P_{\mu}(\sigma', \tau)\} = \delta^{\mu}_{\nu} \delta(\sigma - \sigma'). \tag{2.37}$$

Recall that the $X^{\mu}(\sigma, \tau)$ can be written in terms of Fourier modes α_n^{μ} and $\overline{\alpha}_n^{\mu}$, where $\sigma \sim \sigma + 2\pi$ is periodic, which is the reason why n takes on discrete values.

Claim 1: The Poisson bracket relationship between X^{μ} and P_{μ} requires

$$\{\alpha_m^{\mu}, \alpha_n^{\nu}\}_{\text{PB}} = -im\eta^{\mu\nu}\delta_{m+n,0} \tag{2.38a}$$

$$\{\overline{\alpha}_m^{\mu}, \overline{\alpha}_n^{\nu}\}_{\text{PB}} = -im\eta^{\mu\nu}\delta_{m+n,0} \tag{2.38b}$$

$$\{\alpha_m^{\mu}, \overline{\alpha}_n^{\nu}\}_{\text{PB}} = 0 \tag{2.38c}$$

Proof. Without loss of generality, let us check this at $\tau = 0$:

$$X^\mu(\sigma) = x^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{+in\sigma} + \overline{\alpha}_n^\mu e^{-in\sigma} \right), \tag{2.39}$$

$$P^\mu(\sigma) = \frac{p^\mu}{2\pi} + \frac{1}{2\pi} \frac{1}{\sqrt{2\alpha'}} \sum_{n \neq 0} \left(\alpha_n^\mu e^{+in\sigma} + \overline{\alpha}_n^\mu e^{-in\sigma} \right). \tag{2.40}$$

$$\tag{2.41}$$

The Poisson bracket is

$$\left\{ X^\mu(\sigma), P^\nu(\sigma') \right\}_{\text{PB}} = \frac{1}{2\pi} \left\{ x^\mu, p^\nu \right\}_{\text{PB}} + \frac{i}{4\pi} \sum_{m,n \neq 0} \frac{1}{m} \left(\left\{ \alpha_m^\mu, \alpha_n^\nu \right\}_{\text{PB}} e^{i(m\sigma + n\sigma')} + \left\{ \overline{\alpha}_m^\mu, \overline{\alpha}_n^\nu \right\}_{\text{PB}} e^{-i(m\sigma + n\sigma')} \right) \tag{2.42}$$

$$= \frac{\eta^{\mu\nu}}{2\pi} + \frac{\eta^{\mu\nu}}{2\pi} \sum_{n \neq 0} e^{in(\sigma - \sigma')} = \frac{\eta^{\mu\nu}}{2\pi} \sum_{n} e^{in(\sigma - \sigma')}, \tag{2.43}$$

but $\frac{1}{2\pi} \sum_{n} e^{in(\sigma - \sigma')}$ is just the periodic version of the Dirac δ-function, so

$$\left\{ X^\mu(\sigma), P^\nu(\sigma') \right\}_{\text{PB}} = \eta^{\mu\nu} \delta(\sigma - \sigma'). \tag{2.44}$$

$$\square$$

2.5 The Stress Tensor and Witt Algebra

Let us introduce (worldsheet) light-cone coordinates $\sigma^\pm = \tau \pm \sigma$. In these coordinates, the worldsheet metric looks like $e^\phi \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ and $\partial_\pm = \frac{\partial}{\partial\sigma^\pm}$. The action and equations of motion become

$$S = -\frac{1}{2\pi\alpha'} \int d\sigma^+ \, d\sigma^- \, \partial_+ X \cdot \partial_- X, \qquad \partial_+ \partial_- X^\mu = 0. \tag{2.45}$$

The stress tensor T_{ab} is

$$T_{++} = -\frac{1}{\alpha'} \partial_+ X \cdot \partial_+ X, \qquad T_{--} = -\frac{1}{\alpha'} \partial_- X \cdot \partial_- X, \qquad \underbrace{T_{+-} = T_{-+} = 0}_{\text{effectively trace of } T_{ab}}. \tag{2.46}$$

The constraint is $T_{\pm\pm} = 0$. It is useful to introduce the Fourier modes of $T_{\pm\pm}$. We define (at $\tau = 0$) the charges

$$L_n = -\frac{1}{2\pi} \int_0^{2\pi} d\sigma \, T_{--}(\sigma) e^{-in\sigma} \qquad \overline{L}_n = -\frac{1}{2\pi} \int_0^{2\pi} d\sigma \, T_{++}(\sigma) e^{+in\sigma}. \tag{2.47}$$

Recall that

$$\partial_- X^\mu(\sigma^-) = \sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n e^{-in\sigma^-}, \tag{2.48}$$

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where $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$. We find

$$L_n = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \, \partial_- X^\mu(\sigma) \partial_- X_\mu(\sigma) \tag{2.49}$$

$$= \frac{1}{4\pi} \sum_{m,p} \alpha_m \cdot \alpha_p \int_0^{2\pi} d\sigma \, e^{-i(m+p-n)\sigma} \tag{2.50}$$

$$= \frac{1}{4\pi} \sum_{m,p} \alpha_m \cdot \alpha_p 2\pi \delta_{p,n-m}, \tag{2.51}$$

and similarly for $\bar{L}_n$. We have

$$\boxed{L_n = \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m, \qquad \bar{L}_n = \frac{1}{2} \sum_m \bar{\alpha}_{n-m} \cdot \bar{\alpha}_m,} \tag{2.52}$$

The constraint can be written as $L_n = 0 = \bar{L}_n$. Using the algebra (2.38) for the α_n^μ ($\bar{\alpha}_n^\mu$), we can compute the algebra for the L_n ($\bar{L}_n$) to be

$$\{L_m, L_n\}_{\text{PB}} = -i(m-n)L_{m+n}, \tag{2.53a}$$

$$\{\bar{L}_m, \bar{L}_n\}_{\text{PB}} = -i(m-n)\bar{L}_{m+n}, \tag{2.53b}$$

$$\{L_m, \bar{L}_n\}_{\text{PB}} = 0. \tag{2.53c}$$

This is called the *Witt algebra*. We will see that if we set $L_n = 0 = \bar{L}_n$ at a given τ, then the evolution of the system preserves $L_n = 0 = \bar{L}_n$.

2.6　A First Look at the Quantum Theory

If we choose the metric to be Minkowski up to Weyl transformation, we get a two-dimensional Klein–Gordon equation of motion. However, there is also the constraint $T_{ab} = 0$, which distinguishes the two-dimensional massless Klein–Gordon field from bosonic string theory. There are two approaches we can take. Either we impose the constraint first and then quantise, or we do it the other way around. *Lightcone quantisation* goes the first route, which has historically been very successful. However, in this course, we will not do this!

Instead, we will quantise the unconstrained theory and then impose the constraint $T_{ab} = 0$ as a physical condition on the Hilbert space of states.

2.6.1　Canonical Quantisation

We quantise by replacing our Poisson brackets with commutators:

$$\left\{ \ , \ \right\}_{\text{PB}} \to -i[\ , \ ]. \tag{2.54}$$

Giving equal time commutators:

$$[X^\mu(\sigma), X^\nu(\sigma')] = 0, \qquad [P_\mu(\sigma), P_\nu(\sigma')] = 0, \qquad [P_\mu(\sigma), X^\nu(\sigma')] = -i\delta_\mu{}^\nu \delta(\sigma - \sigma'). \tag{2.55}$$

Using the mode expansions for X^μ and P_μ , e.g.

$$X^\mu(\sigma, \tau) = x^\mu + p^\mu \alpha' \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{-in(\tau - \sigma)} + \overline{\alpha}_n{}^\mu e^{-in(\tau + \sigma)} \right). \tag{2.56}$$

The commutation relations for the modes are

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}, \qquad [\alpha_m^\mu, \overline{\alpha}_n^\nu] = 0, \qquad [\overline{\alpha}_m^\mu, \overline{\alpha}_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}. \tag{2.57}$$

For each direction μ, these are an infinite number of ladder operator relations ($(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu$).

Definition 7 (vacuum)**:** We introduce a *vacuum state* $|0\rangle$ on the worldsheet Σ such that

$$\alpha_n^\mu |0\rangle = 0 \qquad n \geq 0. \tag{2.58}$$

We recall the Fourier modes of T_{ab} are L_n and $\overline{L}_n$, where

$$L_m = \frac{1}{2}\Sigma_n \alpha_{m-n} \cdot \alpha_n. \tag{2.59}$$

These are called the *Virasoro operators*.

> The contraction is given by the spacetime Minkowski metric. Later we will briefly talk about how things change with a curved metric.

This expression Eq. (2.59) is ambiguous for $m = 0$ because α_n and α_{-n} do not commute. Going from the classical to the quantum theory, operator ordering starts to matter. We shall take

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0} \alpha_{-n} \cdot \alpha_n. \tag{2.60}$$

> The ambiguity is still there as we will see later. We just shifted the ambiguity into the states.

Instead of thinking about x^μ, p_μ and T_{ab}, it will be more useful to think about the mode operators $\alpha_m^\mu, \overline{\alpha}_m^\mu$ and L_m and $\overline{L}_m$.

2.6.2 Physical State Conditions

We shall be imposing the conditions $T_{ab} = 0$ on the Hilbert space of the theory.

Notation: Let us define $N = \sum_{n>0} \alpha_{-n} \cdot \alpha_n$ and $\overline{N} = \sum_{n>0} \overline{\alpha}_{-n} \cdot \overline{\alpha}_n$.

> We think of the α_n with positive (negative) n to be annihilation (creation) operators. In that case N is like a number operator. However, they are not quite like the ladder operators for the quantum harmonic operator, and N is not quite a number operator; it is more like a weighted number operator.

Then $L_0 = \frac{\alpha'}{4}p^2 + N$, and $\overline{L}_0 = \frac{\alpha'}{4}p^2 + \overline{N}$, where $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$.

It would be too strong a condition for the operator to vanish $T_{ab} = 0$ identically as an operator expression; there would not be much of a theory left if we did that.

Definition 8 (physical states)**:** Instead, we require that $T_{ab}\ket{\phi} = 0$ for $\ket{\phi}$ to be a *physical state*.

This means that $L_n\ket{\phi} = 0$ and $\bra{\phi} L_{-n} = 0$ for $n > 0$.

We shall also require

$$L_0\ket{\phi} = a\ket{\phi} \quad \text{and} \quad \overline{L}_0\ket{\phi} = a\ket{\phi}, \tag{2.61}$$

where the constant a reflects the ambiguity in defining the ordering of operators constituting L_0, Eq. (2.60), in the quantum theory.

For now we shall take $a = 1$.

▌ We will find that we can interpret the canonical quantised states in a natural way when $a = 1$.

Definition 9: Let us define $L^{\pm} = L_0 \pm \overline{L}_0$.

Then our conditions for a state $|\phi\rangle$ to be physical translate to

$$(L_0^+ - 2)|\phi\rangle = 0, \qquad L_0^-|\phi\rangle = 0, \qquad L_n|\phi\rangle = 0 = \overline{L}_n|\phi\rangle \quad \text{for } n > 0. \qquad (2.62)$$

2.7 The Spectrum

2.7.1 The Tachyon

Definition 10: We can construct a (spacetime) momentum eigenstate as

$$|k\rangle = e^{ik \cdot x}|0\rangle \qquad (2.63)$$

In terms of a position basis in the target space, the momentum operator is $p_\mu = -i\frac{\partial}{\partial x^\mu}$, so $p_\mu|k\rangle = k_\mu|k\rangle$.

From the point of view of a massless Klein–Gordon theory, we would just stop here. However, we have got to check the physicality conditions, which we imposed. Straightforwardly, we get from (2.62) that

$$L_n|k\rangle = 0 = \overline{L}_n|k\rangle. \qquad (2.64)$$

For the other two, we need to work a bit harder. We can write $L_0^- = N - \overline{N}$. This is sometimes called *level matching*.

▌ This is the only constraint coupling the left- and right-moving sectors together.

Here, $N = \overline{N} = 0$. Finally, we only need to check that $(L_0^+ - 2)|k\rangle \overset{!}{=} 0$.

$$(L_0^+ - 2)|k\rangle = \left(\frac{\alpha'}{2}p^2 + N + \overline{N} - 2\right)|k\rangle = \left(\frac{\alpha'}{2}k^2 - 2\right)|k\rangle = 0. \qquad (2.65)$$

This gives us a condition on $k^2 : k^2 - \frac{4}{\alpha'} = 0$. If we compare this with the energy-momentum condition $k^2 + M^2 = 0$, we see that the spacetime interpretation of this is that the state has negative mass squared:

$$\boxed{M^2 = -\frac{4}{\alpha'}} \qquad (2.66)$$

The state $|k\rangle$ has a spacetime interpretation as a tachyon.

Chapter 2. The Classical Particle and String

We found that $|k\rangle = e^{ik \cdot x} |0\rangle$ has a tachyon in the spectrum, so we are off to an inauspicious start. This problem can be cured by adding supersymmetry, but we will not do that in this course.

2.7.2 Massless States

Consider states of the form

$$|\epsilon\rangle = \epsilon_{\mu\nu} \alpha^{\mu}_{-1} \overline{\alpha}^{\nu}_{-1} |k\rangle, \tag{2.67}$$

where $\epsilon_{\mu\nu}$ is a constant tensor.

Physical State Conditions

Clearly $N = \overline{N} = 1$, so $L_0^- |\phi\rangle = 0$. We can look at the energy-momentum condition $(L_0^+ - 2) |\epsilon\rangle = 0$. Since $N = \overline{N} = 1$, a calculation like (2.65) implies that $\frac{1}{2}\alpha' k^2 = 0$, so we require that $k^2 = 0$ is null.

Consider next $L_1 |\epsilon\rangle = 0$:

$$0 = \frac{1}{2}\left(\sum_n \alpha_{1-n} \cdot \alpha_n\right)\epsilon_{\mu\nu}\alpha^{\mu}_{-1}\overline{\alpha}^{\nu}_{-1}|k\rangle = \epsilon_{\mu\nu}\overline{\alpha}^{\nu}_{-1}\alpha_0 \cdot \alpha_1 \alpha^{\mu}_{-1}|k\rangle. \tag{2.68}$$

We require then

$$0 = \epsilon_{\mu\nu}k_\rho \alpha^\rho_1 \alpha^\mu_{-1}|k\rangle = \epsilon_{\mu\nu}k_\rho\left([\alpha^\rho_1, \alpha^\mu_{-1}] + \alpha^\mu_{-1}\alpha^\rho_1\right)|k\rangle \tag{2.69}$$

$$= \epsilon_{\mu\nu}\eta^{\mu\rho}k_\rho|k\rangle = \epsilon_{\rho\nu}k^\rho|k\rangle = 0. \tag{2.70}$$

The condition $L_1 |\epsilon\rangle = 0$ requires us to impose $\epsilon_{\mu\nu}k^\mu = 0$. Interpreting k^μ as the centre of mass momentum, this means that there are no longitudinal polarisations. Similarly, $\overline{L_1} |\epsilon\rangle = 0$ requires that $\epsilon_{\mu\nu}k^\nu = 0$.

For all higher n, the L's will commute with anything and annihilate the $|k\rangle$ in the definition (2.67) of $|\epsilon\rangle$; there is nothing else to check.

In summary, the physical state conditions on $|\epsilon\rangle$ are:

$$k^2 = 0, \qquad \epsilon_{\mu\nu}k^\mu = 0, \qquad \epsilon_{\mu\nu}k^\nu = 0. \tag{2.71}$$

We can decompose $\epsilon_{\mu\nu}$ into traceless symmetric ($h_{\mu\nu}$), traceless antisymmetric ($b_{\mu\nu}$), and trace part (ϕ).

We call these parts by the following names:

$$\text{Dilaton}: \qquad |\phi\rangle = \phi\alpha^\mu_{-1}\overline{\alpha}_{-1\mu}|k\rangle \tag{2.72}$$

$$\text{Graviton}: \qquad |h\rangle = h_{\mu\nu}\alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}|k\rangle \qquad\qquad (h_{\mu\nu} = h_{\nu\mu}) \tag{2.73}$$

$$\text{B-field}: \qquad |b\rangle = b_{\mu\nu}\alpha^\mu_{-1}\overline{\alpha}_{-1}{}^\nu|k\rangle \qquad\qquad (b_{\mu\nu} = -b_{\nu\mu}) \tag{2.74}$$

2.7.3 Massive States

We could then look at the states with $N = \overline{N} = 2$.

$$A_{\mu\nu}\alpha^{\mu}_{-2}\overline{\alpha}_{-2}{}^{\nu}\,|k\rangle + A_{\mu\nu\lambda}\alpha^{\mu}_{-2}\overline{\alpha}_{-1}{}^{\nu}\overline{\alpha}_{-1}{}^{\lambda}\,|k\rangle + \widetilde{A}_{\mu\nu\lambda}\overline{\alpha}_{-2}{}^{\mu}\alpha^{\nu}_{-1}\alpha^{\lambda}_{-1}\,|k\rangle + A_{\mu\nu\lambda\rho}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\overline{\alpha}_{-1}{}^{\lambda}\overline{\alpha}_{-1}{}^{\rho}\,|k\rangle . \quad (2.75)$$

The mass of such states is $m^2 = \frac{4}{\lambda'}$.

2.8 The Big(-ish) Picture

We started with our Polyakov action

$$S = -\frac{1}{4\pi\alpha'}\int_{\Sigma}\mathrm{d}^2\sigma\,\eta_{\mu\nu}\partial_a X^{\mu}\partial^a X^{\nu}.$$

We could deform this theory by adding a small (plane wave) deformation to the spacetime metric:

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}e^{ik\cdot X}. \quad (2.76)$$

The action changes by

$$\Delta S = -\frac{1}{4\pi\alpha'}\int_{\Sigma}\mathrm{d}^2\sigma\,h_{\mu\nu}\partial_a X^{\mu}\partial^a X^{\nu}e^{ik\cdot X}. \quad (2.77)$$

The idea is that any deformation of the spacetime metric is in some sense associated with an operator

$$\mathcal{O} = h_{\mu\nu}\partial_a X^{\mu}\partial^a X^{\nu}e^{ik\cdot X}. \quad (2.78)$$

What makes string theory (or any 2-dimensional CFT) special is that there is a one-to-one correspondence between such operators $\mathcal{O}$, which correspond to a physical[1] deformation of the theory, and states in the Hilbert space. This *state-operator correspondence* will look something like

$$''\lim_{\tau\to-\infty}\mathcal{O}\,|0\rangle = |h\rangle'' \quad (2.79)$$

We will make this more precise as we go on.

> The physical Hilbert space contains the information on how to deform the spacetime metric. This is in part why string theory allows us to talk about quantum gravity, where worldline field theory does not.

Comment

What if we choose to start with a general metric?

$$S = -\frac{1}{4\pi\alpha'}\int_{\Sigma}\mathrm{d}^2\sigma\,g_{\mu\nu}(X)\partial_a X^{\mu}\partial^a X^{\nu}. \quad (2.80)$$

---

[1] We will make this notion of 'physical' or 'reasonable' more precise in the language of CFT.

There are two things to note about this:

1. This is highly *nonlinear*. We might expand this metric as a power series in X in Riemann normal coordinates. We get additional interaction terms in addition to the free kinetic term.

2. If we want to quantise this, we want all the classical symmetries to go through. Requiring the condition of Weyl-invariance ($h_{ab} \to e^{\omega(\sigma,\tau)} h_{ab}$) in the quantum theory constrains what $g_{\mu\nu}(X)$ we can have. One finds that $g_{\mu\nu}(X)$ has to satisfy

$$R_{\mu\nu}(g) + O(\alpha') = 0. \tag{2.81}$$

To leading order in α', the metric must satisfy the vacuum Einstein equations. The higher order terms are *stringy corrections* to general relativity. More generally, if we have other background fields, we find that Weyl-invariance requires the full Einstein equations to be satisfied to leading order in α'.

2.9 Spurious States and Gauge-invariance

We took $a = 1$ in the conditions (2.61), which are $(L_0 - a)|\phi\rangle = (\bar{L}_0 - a)|\phi\rangle = 0$. Why? Consider the state

$$|\chi\rangle = \sqrt{\frac{2}{\alpha'}} \left(\lambda_\lambda \alpha^\mu_{-1} \bar{L}_{-1} + \tilde{\lambda}_\mu \bar{\alpha}_{-1}{}^\mu L_{-1} \right) |k\rangle \tag{2.82}$$

Clearly $|\chi\rangle$ is orthogonal to all physical states: If $|\phi\rangle \in \mathcal{H}$, then $\langle\phi|\chi\rangle = 0$ because $L_1|\phi\rangle = \bar{L}_1|\phi\rangle = 0$. What conditions do λ_μ, $\tilde{\lambda}_\mu$, and k have to satisfy for $|\chi\rangle$ to be physical? It is useful to write $|\chi\rangle$ as

$$|\chi\rangle = \left(\lambda_\mu k_\nu + \tilde{\lambda}_\nu k_\mu \right) \alpha^\mu_{-1} \bar{\alpha}^\nu_{-1} |k\rangle. \tag{2.83}$$

Keeping a arbitrary, we find

$$(L_0^+ - 2a)|\chi\rangle = 0 \quad \Rightarrow \quad k^2 = \frac{4(a-1)}{\alpha'}. \tag{2.84}$$

We see that the physical state conditions are, unless $n = 1$, trivially satisfied since $N = \bar{N} = 1$. For these special cases with $n = 1$, we need to do a small calculation, which gives

$$L_1|\chi\rangle = 0 \qquad\qquad \text{if} \quad (\lambda \cdot k)k_\mu + \tilde{\lambda}_\mu k^2 = 0 \tag{2.85}$$

$$\bar{L}_1|\chi\rangle = 0 \qquad\qquad \text{if} \quad (\tilde{\lambda} \cdot k)k_\mu + \lambda_\mu k^2 = 0 \tag{2.86}$$

Is there any situation in which we can make sense of this? If $a = 1$, then $k^2 = 0$, and $|\chi\rangle$ is physical if in addition $\lambda \cdot k = 0 = \bar{\lambda} \cdot k$. Since $\langle\chi|\chi\rangle = \lambda^2 k^2 + 2(\lambda \cdot k)(\bar{\lambda} \cdot k) + \bar{\lambda}2k^2$, this guarantees that $\langle\chi|\chi\rangle = 0$.

They are orthogonal to every physical state, so in a correlation function they do not affect anything, even though they are physical.

Consider $\lambda_\mu = \widetilde{\lambda}_\mu$ and let us add $|\chi\rangle$ to the graviton state $|h\rangle = \epsilon_{\mu\nu}\alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}$, so

$$\left|h'\right\rangle = |h\rangle + |\chi\rangle. \tag{2.87}$$

Since $|\chi\rangle$ decouples entirely, its inner product with all physical states vanishing, this should give identical physics. Then

$$\left|h'\right\rangle = (\epsilon_{\mu\nu} + \lambda_\mu k_\nu + \lambda_\nu k_\mu)\alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}\,|k\rangle \tag{2.88}$$

$$= \epsilon'_{\mu\nu}\alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}\,|k\rangle. \tag{2.89}$$

All this does it that it changes the graviton to another graviton with a different polarisation. We recognise the transformation

$$\epsilon_{\mu\nu} \to \epsilon'_{\mu\nu} = \epsilon_{\mu\nu} + \lambda_\mu k_\nu + \lambda_\nu k_\mu \tag{2.90}$$

as a (linearised) diffeomorphism, the symmetry of general relativity. These states are interpreted as being gauge exact when we look at BRST quantisation later. This is a hint suggesting that $a = 1$ might be a sensible physical choice.

3 Path Integral Quantisation

This will be our favourite way to quantise the string, since the path integral formalism gives an easy way to computing correlation functions in terms of integrals over spaces of fields with given boundary conditions.

Example 3.0.1 ($d = 1$): In non-relativistic quantum mechanics, we may be interested in calculating the transition amplitude for a particle to go from $x_i(t_i)$ to $x_f(t_f)$. The corresponding amplitude is given by a weighted sum of all possible ways of going from the start point to the end point

$$\langle x_i, t_i : x_f, t_f \rangle \propto \int_i^f \mathcal{D}X \, e^{iS[X]/\hbar}. \tag{3.1}$$

For most of the calculations we want to do we can treat this just as a generating function for correlation functions and we do not need to worry too much about the path integral measure $\mathcal{D}X$.
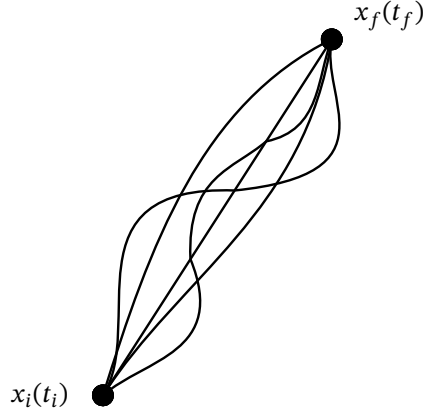

Figure 3.1: Path Integral

In string theory we take a very similar starting point. We take a weighted sum of all worldsheets with initial and final conditions specified. This is illustrated as shown in Fig.

We want to make sense of ($\hbar = 1$)

$$\langle \Psi_i | \Psi_f \rangle = \frac{1}{|\text{Diff} \times \text{Weyl}|} \int \mathcal{D}X \, \mathcal{D}h \, e^{iS[X,h]}, \tag{3.2}$$

where $S[X, h]$ is the Polyakov action (2.10) in flat spacetime. We want to separate the integral over h_{ab} into those h_{ab} related to each other by the action of diffeomorphisms and Weyl transformations, Diff $\times$ Weyl, and those that are 'physically distinct'. The sort of thing we want to do is to split the measure $\mathcal{D}h$ into

$$\mathcal{D}h = \mathcal{D}v \times \mathcal{D}h_{\text{phys}} \times \mathcal{J}, \tag{3.3}$$

where $\mathcal{D}v$ is the volume element of the infinite dimensional group Diff $\times$ Weyl, and $\mathcal{J}$ is some Jacobian coming from the separation, which we can see as a change of variable. Recall the general transformation of h_{ab}:

$$\delta h_{ab} = \delta_v h_{ab} + \delta_w h_{ab} + \delta_t h_{ab}, \tag{3.4}$$

where

- $\delta_v h_{ab} = \nabla_a v_b + \nabla_b v$, are diffeomorphisms (with parameter v), which take the form of a Lie derivative,

- $\delta_w h_{ab} = 2\omega h_{ab}$ are Weyl rescalings of h_{ab},

- $\delta_t h_{ab}$ are physically distinct changes in h_{ab}.
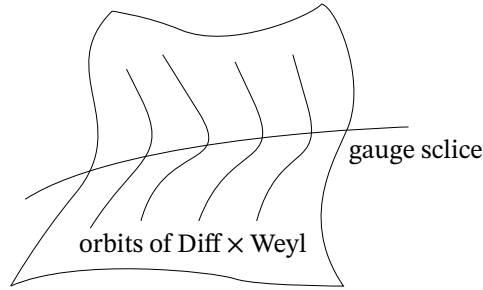
Figure 3.2: The 'space of h_{ab}'. Although we have not proved it, the gauge slice is chosen to cut through all orbits of Diff $\times$ Weyl exactly once.

We notice that $\delta_n h_{ab}$ has non-zero trace. It is useful to absorb this trace part into $\delta_w h_{ab}$. We introduce a traceless version of $\delta_v h_{ab}$:

$$\boxed{(Pv)_{ab} = \nabla_a v_b + \nabla_b v_a - h_{ab}(\nabla_c v^c)}. \tag{3.5}$$

And the Weyl transformation becomes

$$\delta_{\overline{\omega}} = 2\overline{\omega} h_{ab} \qquad \overline{\omega} = \omega + \frac{1}{2} \nabla \cdot v. \tag{3.6}$$

Chapter 3. Path Integral Quantisation

Definition 11: We call the space of physically distinct metrics h_{ab} the *moduli space*.

And $\delta_t h_{ab}$ is a variation of the metric in the moduli space.

> So far, we are only talking about diffeomorphisms connected to the identitiv since we are working infinitesimally. We can extend this to 'large diffeomorphisms'.

We will work on worldsheets in which (at least infinitesimally) all metrics can be related to each other by diffeomorphisms and Weyl transformations.

3.1 A Crash Course on Riemann Surfaces

Potentially, we will work on all sorts of worldsheets with all sorts of topologies.

Definition 12 (Riemann surface)**:** A *Riemann surface* is a 2 (real) dimensional Riemannian manifold, where we consider metrics related by a Weyl transformation $h_{ab} \to e^{2\omega} h_{ab}$ to be equivalent.

Schematically, we might say that the space of Riemann surfaces is the space of Riemannian manifolds modulo Weyl transformations.

3.1.1 Worldsheet Genus and Punctures

These things will distinguish different Riemann surfaces.

Definition 13 (genus)**:** For Σ without boundary (closed strings)[1] the topology of Σ is encapsulated in the Euler characteristic

$$\chi = \frac{1}{4\pi} \int_{\Sigma} \mathrm{d}^2\sigma \sqrt{-h} R(h) = 2 - 2g, \tag{3.7}$$

where $g = 0, 1, \dots$ is the *genus* of Σ.

Considering Fig., this starts to look a bit like a loop expansion, where g will be related to the loop order in perturbation theory.

Definition 14 (punctures)**:** *Punctures* on Σ are marked points on Σ, i.e. points at a particular position.

Example 3.1.1: The Riemann surface illustrated in Fig has $g = 1$ and 3 marked points.

---

[1] For a cylinder, we might need to worry about the infinite past and future? We will discuss this later, but it turns out that it is not quite like a boundary. We will discuss this in terms of punctures.

Consider now the final piece $\delta_t h_{ab}$ of the transformation (3.4) of the metric. We can write these as

$$t^I \mu_{abI}, \qquad \mu_{abI} = \frac{\partial h_{ab}}{\partial m^I},$$

(3.8)

where m^I are coordinates of the moduli space of Riemann surfaces M_g and $t^I \approx \delta m^I$ are tangent vectors to M_g.

3.1.2   Moduli Space of (Unpunctured) Riemann Surfaces

The moduli space M_g of a genus g Riemann surface Σ_g is schematically

$$M_g = \frac{\{\text{metrics}\}}{\{\text{Diff} \times \text{Weyl}\}}.$$

(3.9)

The (real) dimension of M_g is as follows

$$\dim(M_g) := s = \begin{cases} 0, & \text{if } g = 0 \\ 2, & \text{if } g = 1 \\ 6g - 6, & \text{if } g \geq 2. \end{cases}$$

(3.10)

Example 3.1.2 ($g = 0$): All metrics on $g = 0$ surfaces[1] may be brought to the form $e^{2\omega} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ by diffeomorphisms. In other words, they are trivial up to Weyl transformations.

Example 3.1.3 ($g = 1$ (torus)): We can construct a torus by taking the complex plane $\mathbb{C}$ and imposing identifications such as

$$z \sim z + \lambda_1 m + \lambda_2 n,$$

(3.11)

where λ_1, λ_2 are lattice vectors and $m, n \in \mathbb{Z}$. One can show that the ratio

$$\tau = \frac{\lambda_1}{\lambda_2} \in \mathbb{C}$$

(3.12)

is invariant under Diff $\times$ Weyl .

Remark: This τ is not worldsheet time! It is called the *complex structure*.

In other words, using diffeomorphisms and Weyl transformations, we can always bring the metric to the form

$$ds_2 = \left| dz + \tau d\overline{z} \right|^2.$$

(3.13)

---

[1] Everything we are saying pertains to two dimensional manifolds.

There are two degrees of freedom, which are encoded in this one complex degree of freedom. Without loss of generality, we can choose $\text{Im}\,\tau > 0$. We could write the identification $z \sim z + n_a \lambda^a$, where $n_a = (m, n)$ and $\lambda^a = \begin{pmatrix} \lambda_1 \\ l_2 \end{pmatrix}$. If we act with an $SL(2)$ transformation

$$n_a \to U_a{}^b n_b \tag{3.14}$$

$$\lambda^a \to (U^{-1})^a{}_b \lambda^b \tag{3.15}$$

this statement is preserved. Moreover, if $U \in SL(2; \mathbb{Z})$, which has integer entries, then (n, m) remain integer. So the moduli space M_1 at $g = 1$ can be identified with the upper half plane modulo $SL(2; \mathbb{Z})$.
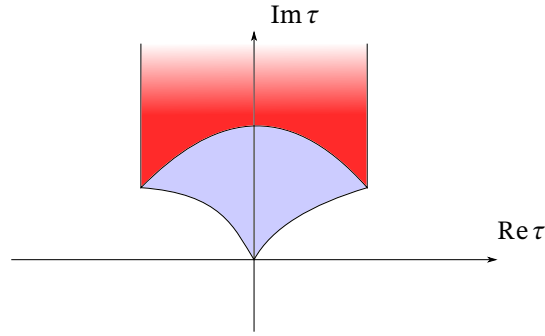
Figure 3.3

Example 3.1.4 ($g \geq 2$)**:** It gets hard but there are analogues of the $SL(2; \mathbb{Z})$ (modular group) in all cases.

3.1.3 Conformal Killing Vectors

Definition 15 (Conformal Killing group)**:** There is an overlap Diff∩Weyl called the *conformal Killing group* (CKG).

Is it possible to undo the effect of a diffeomorphism by a Weyl transformation? In other words, we want to find a pair (v^a, ω) such that

$$\delta h_{ab} = \nabla_a v_b - \nabla_b v_a + 2\omega h_{ab} = 0 \tag{3.16}$$

We take the trace to find that if

$$\omega = -\frac{1}{2} \nabla_a v^a, \tag{3.17}$$

then the diffeomorphism and Weyl transformation cancel out. We defined Pv in Eq. (3.5).

Definition 16 (Conformal Killing vectors)**:** Vectors that satisfy $(Pv)_{ab} = 0$ are what we shall call *conformal Killing vectors* (CKVs).

The CKG is generated by the CKVs. For a given genus, the (real) dimension of the conformal Killing group is

$$\dim_{\mathbb{R}}(\mathrm{CKG}) := \kappa = \begin{cases} 6, & \text{if } g = 0 \\ 2, & \text{if } g = 1 \\ 0, & \text{if } g \geq 2 \end{cases} \tag{3.18}$$

In particular, notice that these are finite-dimensional. We can figure out that for $g = 0$, the CKG is $SL(2, \mathbb{C})$ and on the torus $g = 1$ it is $U(1) \times U(1)$. Moreover, for $g \geq 2$, the CKG is empty; there is no overlap for higher g's.

Example 3.1.5 ($g = 2$)**:** The conformal Killing group is $SL(2; \mathbb{C})$[1]. This is called the *Möbius group*. If $z \in \mathbb{C}$, then this transformation acts as

$$z \to \frac{az + b}{cz + d}, \qquad ad - bc = 1, \qquad a, b, c, d \in \mathbb{Z}. \tag{3.19}$$

We can specify a particular element of the conformal Killing group at genus $g = 0$ by describing how three distinct points transform under the given map.

3.1.4   Moduli Space of Punctured Riemann Surfaces $M_{g,n}$

Suppose we have a Riemann surface of genus g with n punctures (or marked points). What is its moduli space? Naively, we might expect

$$M_{g,n} = M_g \otimes \Sigma^n. \tag{3.20}$$

But we also have to worry about the CKG. If we have a genus $g = 0$ surface with n punctures, we can use the conformal Killing group to fix three of the locations of the punctures (i.e. 'three degrees of freedom'), leaving only $n - 3$ free punctures. Similarly, we can fix the CKG on the torus ($g = 1$) by fixing the location of one puncture.

❚   We will see very explicitly how this works when we look at the path integral.

So we should really take the moduli space (as relating to our path integral) to be

$$M_{g,n} = M_g \otimes \Sigma^{n - \frac{\kappa}{2}}, \tag{3.21}$$

since each puncture has two bits (σ, τ) of information. Therefore, the dimension of the moduli space is

$$\boxed{\dim(M_{g,n}) = 6g - 6 + 2n} \tag{3.22}$$

---

[1]Strictly speaking this should be the projective group $PSL(2; \mathbb{C})$. However, this distinction will not matter much for our purposes.

3.2 Fadeev–Popov Determinant

Our goal will be to find some effective way to make sense of the path integral. When integrating over the worldsheet metrics, we want to count each physical metric only once, and avoid overcounting from to diffeomorphism and Weyl equivalence.

In other words, we heuristically want to mod out the equivalences of physical states

$$Z[0] = \frac{1}{|\text{Diff} \times \text{Weyl}|} \int \mathcal{D}X \, \mathcal{D}h \, e^{iS[h,X]} \tag{3.23}$$

Definition 17 (Fadeev–Popov determinant)**:** To that end, we will introduce the *Fadeev–Popov* (FP) determinant Δ_{FP}, defined by

$$1 = \Delta_{\text{FP}}[h] \int_{M_g} \mathrm{d}^3 t \int_{\text{Diff} \times \text{Weyl}} \mathcal{D}\omega \, \mathcal{D}v \, \delta[h - \hat{h}] \prod_{\substack{i=1 \\ a=1,2}}^{k} \delta(v^a(\hat{\sigma}_i)) \tag{3.24}$$

where M_g is the integral over all Riemann surfaces of genus g and the delta-functional

$$\delta[h - \hat{h}] = \prod_{\substack{a,b \\ \sigma,\tau}} \delta(h_{ab}(\sigma,\tau) - \hat{h}_{ab}(\sigma,\tau)), \tag{3.25}$$

fixes the metric to be a specifically chosen metric, where

$$h_{ab} - \hat{h}_{ab} = \delta h_{ab} = (Pv)_{ab} + 2\overline{\omega} h_{ab} + t^I \mu_{I\,ab}, \qquad \overline{\omega} = \omega + \dots \tag{3.26}$$

We need to fix the CKG (Diff $\cap$ Weyl).

> We will use the physical path integral methods, but in this particular case one can do the functional analysis rigorously, say with heat kernel methods.

3.3 Gauge-Fixing the Path Integral

Let us put (3.24) into (3.23):

$$Z[0] = \frac{1}{\text{Diff} \times \text{Weyl}} \int_{\text{Diff} \times \text{Weyl}} \mathcal{D}\omega \, \mathcal{D}v \prod_{a,i} \delta(v^a(\sigma_i)) \int_{M_g} \mathrm{d}^3 t \int \mathcal{D}X \, \mathcal{D}h \, \delta[h - \hat{h}] \Delta_{\text{FP}} e^{iS[h,X]} \tag{3.27}$$

$$= \frac{1}{\text{Diff} \times \text{Weyl}} \int \mathcal{D}X \int_{M_g} \mathrm{d}^3 t \left(\int_{\text{Diff} \times \text{Weyl}} \mathcal{D}\omega \, \mathcal{D}v \prod_{a,i} \delta(v^a(\hat{\sigma}_i)) \right) e^{iS[\hat{h},X]} \Delta_{\text{FP}}[\hat{h}]. \tag{3.28}$$

In the parentheses, we have

$$\int_{\text{Diff} \times \text{Weyl}} \mathcal{D}\omega \, \mathcal{D}v \prod_{a,i} \delta(v^a(\hat{\sigma}_i)) = \frac{|\text{Diff} \times \text{Weyl}|}{|\text{CKG}|}. \tag{3.29}$$

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> If this makes you uncomfortable, it should. We are trying to divide out infinite dimensional equivalence classes. There are also ways to make this rigorous using the theory of Riemann surfaces. However, they lack the physical intuition and would take much longer to derive.

We find a more respectable starting point

$$Z[0] = \frac{1}{|CKG|} \int_{M_g} \mathrm{d}^3 t \int \mathcal{D}X \, e^{iS[X,\hat{h}]} \Delta_{\mathrm{FP}}[\hat{h}]. \tag{3.30}$$

This looks much more reasonable. The conformal Killing group is finite-dimensional. For this to be a good starting point for the quantisation of the theory, we have to think a bit more about $\Delta_{\mathrm{FP}}[\hat{h}]$.

3.4   A Field Theory Representation for Δ_{FP}

Inverting (3.24), we find that the inverse of the Fadeev–Popov determinant is

$$\Delta_{\mathrm{FP}}^{-1}[\hat{h}] = \int_{M_g} \mathrm{d}^3 t \int_{\mathrm{Diff} \times \mathrm{Weyl}} \mathcal{D}\overline{\omega} \, \mathcal{D}v \, \delta[\delta h] \prod_{a,i} \delta(v^a(\hat{\sigma}_i)). \tag{3.31}$$

Let us find an integral expression for the δ-function(al)s by introducing auxiliary fields $\beta_{ab}(\sigma, \tau)$ and $2k$ elements ξ_a^i:

$$\Delta_{\mathrm{FP}}^{-1}[\hat{h}] = \int_{M_g} \mathrm{d}^3 t \int \mathcal{D}v \, \mathcal{D}\overline{\omega} \, \mathcal{D}\beta \, \mathrm{d}^{2k}\xi \, \exp\left(i(\beta \mid Pv + 2\overline{\omega}h + t^I \mu_I) + i \sum_{i=1}^{k} \xi_a^i \delta(v^a(\hat{\sigma}_i)) \right), \tag{3.32}$$

where the inner product is defined as

$$(\beta \mid Pv + 2\overline{\omega}h + t^I \mu_I) := \int_{\xi} \mathrm{d}^2\sigma \sqrt{-h} \beta^{ab}((Pv)_{ab} + 2\overline{\omega}h_{ab} + t^I \mu_{I\,ab}). \tag{3.33}$$

We can do the $\overline{\omega}$ integral, which constrains $\beta_{ab}h^{ab} = 0$.

> We have $\Delta_{\mathrm{FP}^{-1}}$ but we want Δ_{FP}. We introduce Grassmann variables. These are variables θ_i that anticommute $\theta_1\theta_2 = -\theta_2\theta_1$. They have lots of interesting properties, such as $\theta_i^2 = 0$. This means we can expand a function $f(x,\theta) = f(x) + f'(x)\theta$. Moreover, the delta function of a Grassmann variable is simply $\delta(\theta) = \theta$ and integral and derivative are equal $\int \mathrm{d}\theta \, f(\theta) = \frac{\partial f}{\partial \theta}$.

Let us consider some finite-dimensional Gaussian integral over a complex vector space V:

$$\frac{1}{\det M} = \int_V \mathrm{d}z \, \mathrm{d}\overline{z} \, e^{-(\overline{z}, Mz)}, \tag{3.34}$$

where $(\cdot, \cdot)$ denotes some inner product on V. Writing the same type of integral with Grassmann variables,

$$\det M = \int \mathrm{d}\theta \, \mathrm{d}\overline{\theta} \, e^{-(\overline{\theta}, M\theta)}. \tag{3.35}$$

(See Ryder's QFT book if curious). We can get from an integral expression for a determinant to the reciprocal by replacing the c-numbers with Grassmann numbers.

Similarly, we replace all the fields we integrate over in $\Delta_{\mathrm{FP}}^{-1}$ with Grassmann-valued fields:

$$v^a(\hat{\sigma}_i) \to c^a(\sigma, \tau), \qquad \beta_{ab}(\sigma, \tau) \to b_{ab}(\sigma, \tau), \qquad t^I \to \zeta^I, \qquad \xi_a^i \to \eta^i{}_a. \qquad (3.36)$$

Just like β_{ab}, we take b_{ab} to be traceless also. With these Grassmann fields in place, we take $\Delta_{\mathrm{FP}}[\hat{h}]$ to be given by

$$\Delta_{\mathrm{FP}}[\hat{h}] = \int \mathrm{d}^3\zeta \, \mathcal{D}c \, \mathcal{D}b \, \mathrm{d}^{2k}\eta \, \exp\left[i(b \mid Pc + \zeta^I\mu_I + i\sum_{i=1}^{\kappa} \eta^i{}_a c^a(\hat{\sigma}_i))\right], \qquad (3.37)$$

where $\kappa = |\mathrm{CKG}|$. We can do these finite-dimensional integrals. The integral over ζ^I gives a δ-functional.

$$\prod_{I=1}^{s} \delta(b \mid \mu_I) = \prod_{I=1}^{s} (b \mid \mu_I), \qquad (3.38)$$

where $s = \dim M_g$.

The $\eta^i{}_a$ integral gives

$$\prod_{\substack{a=i,? \\ i=1,\dots,\kappa}} \delta(c^a(\hat{\sigma}_i)) = \prod_{i,a} c^a(\hat{\sigma}_i). \qquad (3.39)$$

The expression for the Fadeev–Popov determinant becomes rather simple

$$\boxed{\Delta_{\mathrm{FP}}[\hat{h}] = \int \mathcal{D}c \, \mathcal{D}b \, e^{iS[b,c]} \prod_{I=1}^{s} (b \mid \mu_I) \prod_{i,a} c^a(\hat{\sigma}_i)} \qquad (3.40)$$

where the action for the b, c (*ghost*) fields is

$$S[b,c] = (b \mid Pc) = \int_\Sigma \mathrm{d}^2\sigma \sqrt{-\hat{h}} b^{ab}(Pc)_{ab} = 2\int_\Sigma \mathrm{d}^2\sigma \sqrt{-\hat{h}} b^{ab}(\nabla_a c_b), \qquad (3.41)$$

where we used that b_{ab} is symmetric and traceless.

We shall refer to the (Grassmann) fields b_{ab} and c^a as Fadeev–Popov *ghosts*.

3.5 Calculating Observables

Our final form for $Z[0]$ for Riemann surfaces of genus g is

$$Z_g[0] = \frac{1}{|\text{CKG}|} \int_{M_g} d^3t \int \mathcal{D}X \, \mathcal{D}b \, \mathcal{D}c \, e^{iS[X,b,c]} \prod_I (b \mid \mu_I) \prod_{i,a} c^a(\hat{\sigma}_i). \tag{3.42}$$

In general, we want to sum over all Riemann surfaces

$$Z[0] = \sum_{g=0}^{\infty} e^{\lambda \chi} Z_g[0], \tag{3.43}$$

where λ is a constant and $\chi = lg - l$ is the Euler characteristic of the worldsheet. It will turn out that the constant weighting λ will be determined by the theory rather than being a free parameter.

We can compute correlation functions of observables by including them in our path integral

$$\langle \phi_1 \dots \phi_n \rangle = \mathcal{N} \int \mathcal{D}\phi \, \phi_1 \dots \phi_n e^{iS[\phi]} \tag{3.44}$$

What might our observables look like? They will certainly need to be invariant under diffeomorphisms and Weyl transformations. We can build diffeomorphism-invariant observables by taking an operator $\mathcal{O}(\sigma, \tau)$ on a genus g worldsheet Σ_g and integrating it over Σ_g:

$$\mathcal{O} = \int_{\Sigma_g} d^2\sigma \, \mathcal{O}(\sigma, \tau). \tag{3.45}$$

We will need to choose the $\mathcal{O}(\sigma, \tau)$ carefully in order for this to be Weyl invariant as well. We will talk about this when we come to CFTs.

So a correlation function of such observables would be:

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \sum_{g=0}^{\infty} \frac{e^{\lambda \chi}}{|\text{CKG}|} \int \prod_{i=0}^{n} d^2\sigma_i \int_{M_g} d^3t \int \mathcal{D}X \, \mathcal{D}b \, \mathcal{D}c \, e^{iS[X,b,c]} \prod_I (b \mid \mu_I) \prod_{i,a} c^a(\hat{\sigma}_i) \mathcal{O}_1(\sigma_1) \dots \mathcal{O}_n(\sigma_n),$$

$$\tag{3.46}$$

using the shorthand $\sigma_i = (\sigma_i, \tau_i)$.

Remark: We have the instruction to divide by the conformal Killing group, and to integrate over the moduli space of Riemann surfaces M_g as well as all locations σ_i. We know we can fix the CKG on a sphere by keeping three locations fixed, and one on a torus. Instead of integrating over all points of the Riemann surface, we can thus write the instruction of dividing by the CKG as restricting what we integrate over:

$$\frac{1}{\text{CKG}} \int_{M_g} d^3t \int \prod_{i=1}^{n} d^2\sigma_i = \int_{M_g} d^3t \int_{i=1}^{n-\kappa} d^2\sigma_i = \int_{M_{g,n}} . \tag{3.47}$$

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So when computing the correlation functions, we end up integrating only over $n - \kappa$ locations

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \sum_{g=0}^{\infty} e^{\lambda \chi} \int_{M_{g,n}} \int \mathcal{D}X \, \mathcal{D}b \, \mathcal{D}c \, e^{iS[X,b,c]} \int_I (b \mid \mu_I) \prod_{i,a} c^a(\hat{\sigma}_i) \mathcal{O}_1 \dots \mathcal{O}_n \qquad (3.48)$$

This might seem like the final result depends on some locations, but as we will see the ghosts will take care of this.

3.6   Conformal Field Theory

This is a wonderful topic. There could be a whole Part III course in CFT alone. It is important in String Theory but also in Statistical and Condensed Matter Physics. Moreover, when trying to really understand QFT with mathematical rigour, your best bet is to start with low-dimensional CFT.

3.6.1   Conformal Invariance in General Dimension

Let us consider transformations that preserve angles. In particular, transformations $x^\mu \to x'^\mu(x)$ such that the metric on our space transforms as

$$\Lambda(x) \eta_{\mu\nu} = \eta_{\rho\sigma} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu}, \qquad (3.49)$$

preserving the metric up to a local scale $\Lambda(x)$.

Infinitesimally, we write a transformation as $x^\mu \to x^\mu(x) = x^\mu + \epsilon v^\mu(x) + \dots$, where $\epsilon \ll 1$. The first order in ϵ, the metric transforms as

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + \epsilon(\partial_\mu v_\nu + \partial_\nu v_\mu) = \Lambda(x) \eta_{\mu\nu}. \qquad (3.50)$$

Let $\Lambda(x) = e^{\epsilon \omega(x)} \approx 1 + \epsilon \omega(x) + \dots$. Thus, vector fields v^μ must satisfy

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \omega(x) \eta_{\mu\nu}. \qquad (3.51)$$

Taking the trace of both sides,

$$\omega(x) = \frac{2}{d} \partial_\mu v^\mu, \qquad (3.52)$$

where d is the dimension few the spacetime. Therefore, vector fields that generate such conformal transformations satisfy

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{d} \partial_\lambda v^\lambda \eta_{\mu\nu}. \qquad (3.53)$$

3.6.2   In Two Dimensions

Something special happens in $d = 2$ dimensions. Let us take the coordinates in two dimensions to be σ and τ. We take η to be the Euclidean metric, $\text{diag}(1, 1)$. With this metric, Eq. (3.53) boils down to two independent equations

$$\boxed{\frac{\partial v_\tau}{\partial \tau} = \frac{\partial v_\sigma}{\partial \sigma}, \qquad \frac{\partial v_\sigma}{\partial \tau} = -\frac{\partial v_\tau}{\partial \sigma}} \tag{3.54}$$

where $v = \begin{pmatrix} v_\sigma \\ v_\tau \end{pmatrix}$. These are the Cauchy–Riemann equations!

Remark: We can now bring to bear the full power of complex analysis to understand quantum gravity.

If we introduce complex coordinates

$$w = \tau + i\sigma, \qquad \overline{w} = \tau - i\sigma, \qquad (\omega \in \mathbb{C}), \tag{3.55}$$

then the condition for the vector v to generate a conformal transformation is that v is holomorphic, meaning that $\bar{\partial} v = 0$.

If a theory is conformally invariant, then there is a conserved charge associated with every holomorphic v; there is an infinite number of conserved charges!

Remark: Given our current understanding, this is special in $d = 2$. This allows us to do quantum field theory with an easy that we just do not see in higher dimensions.

A particularly useful set of coordinates for our worldsheet is the exponential of w:

$$z = e^{\tau + i\sigma}, \qquad \overline{z} = e^{\tau - i\sigma}. \tag{3.56}$$

Under this map, the cylindrical worldsheet is mapped to the complex plane (if we choose a Euclidean metric, which we will do from now on).

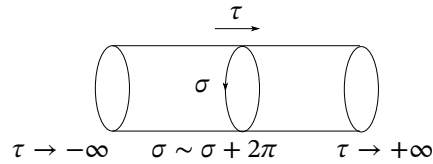$$\tau \to -\infty \qquad \sigma \sim \sigma + 2\pi \qquad \tau \to +\infty$$

Figure 3.4

Cylinder becomes radial evolution on the complex plane.
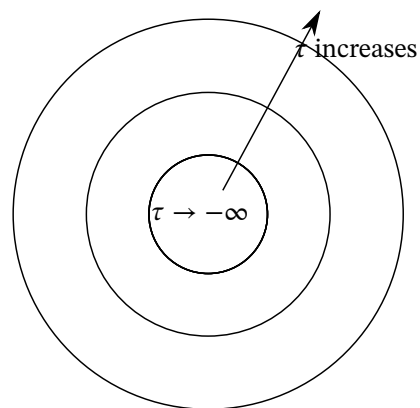
Figure 3.5

3.6.3 The Witt Algebra

We have seen that conformal transformations are generated by (anti-)holomorphic vectors $v(z)$ and $\overline{v}(z)$, i.e.

$$z \rightarrow z + v(z) + \ldots \qquad \overline{z} \rightarrow \overline{z} + \overline{v}(\overline{z}) + \ldots \tag{3.57}$$

We can write $v(z)$ as a power series expansion

$$v(z) = \sum_n v_n z^{n+1}. \tag{3.58}$$

We might allow for the possibility to have some isolated poles in this sum, but we will not worry too much about that for now. For now the exponent z^{n+1} is just a convention. Infinitesimally, we have the transformation

$$z \rightarrow z + \sum_n v_n z^{n+1}, \tag{3.59}$$

which we can think of as being generated by the operators

$$l_n = -z^{n+1} \frac{\partial}{\partial z} \qquad \text{i.e.} \quad z \rightarrow z - \sum_n v_n l_n z. \tag{3.60}$$

The operators l_n satisfy the Witt algebra

$$\boxed{[l_n, l_m] = (n - m)l_{n+m}} \tag{3.61}$$

The first time we saw this, the l_n were the Fourier modes of the stress tensor. Here, another way to look at these is as the generators of conformal transformations. This hints at a connection between the stress tensor and conformal transformations.

3.6.4 Conformal Fields

Some definitions:

Definition 18 (chiral field)**:** A *chiral field* is a field $\phi(z)$ that depends on z alone.

Definition 19 (conformal dimension)**:** The *conformal dimension* is a number $\Delta = h + \overline{h}$ that tells us how a field transforms under scaling.

Suppose we have a field $\phi(z, \overline{z})$ and rescale $(z, \overline{z}) \rightarrow (\lambda z, \overline{\lambda}\overline{z}) = (z', \overline{z}')$, where λ and $\overline{\lambda}$ need not necessarily be complex conjugates of each other. Then

$$\phi(z, \overline{z}) \rightarrow \phi'(z', \overline{z}') = \lambda^h \overline{\lambda}^{\overline{h}} \phi(\lambda z, \overline{\lambda}\overline{z}). \tag{3.62}$$

Remark: A chiral field has $\overline{h} = 0$.

Definition 20 (primary field)**:** Under the transformation $z \to f(z)$, a *primary field* transforms as

$$\phi(z, \overline{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}} \phi(f(z), \overline{f}(\overline{z})), \tag{3.63}$$

where f and $\overline{f}$ need not be complex conjugates of each other. What we are really saying is that primary fields are tensors under conformal transformations (not just under diffeomorphisms).

Similarly to how we focused on just the left-moving fields before, we will look only at chiral primary fields. (The correspondence between left (right) moving modes and (anti) holomorphic fields is given by the definition of our coordinates (3.56).)

For an infinitesimal transformation $f(z) = z + v(z) + \dots$

$$\left(\frac{\partial f}{\partial z}\right)^h \approx (1 + \partial v)^h \approx 1 + h \partial v \tag{3.64}$$

$$\phi(f(z)) = \phi(z + v + \dots) = \phi(z) + v \partial \phi(z) + \dots \tag{3.65}$$

and similarly for $\overline{z} \to \overline{z} + \overline{v}(\overline{z}) + \dots$. We find

$$\boxed{\delta_{v, \overline{v}} \phi(z, \overline{z}) = \left(h \partial v + \overline{h \partial v} + v \partial + \overline{v \partial}\right) \phi(z, \overline{z})} \tag{3.66}$$

3.6.5 Conformal Transformations and the Stress Tensor

We start with out favourite Polyakov action in flat space,

$$S[X] = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \, \partial_a X^\mu \partial^a X^\nu \eta_{\mu\nu}. \tag{3.67}$$

Under a conformal transformation, the embedding field transforms as

$$\delta_v X^\mu = v^a \partial_a X^\mu. \tag{3.68}$$

We would like to find the Noether current associated with this. Under such a transformation, the action changes by

$$\delta_v S[X] = \frac{1}{2\pi} \int_\Sigma \mathrm{d}^2\sigma \, (\partial^a v^b) T_{ab}. \tag{3.69}$$

We can integrate by parts to obtain

$$\delta_v S[X] = -\frac{1}{2\pi} \int_\Sigma \mathrm{d}^2\sigma \, v^a (\partial^b T_{ab}) = 0. \tag{3.70}$$

For this to be a symmetry under all v^a, there has to be a conserved current

$$\boxed{\partial^a T_{ab} = 0 = \partial^b T_{ab}} \tag{3.71}$$

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We could define conserved charges $Q_\pm$, which in lightcone coordinates ($\sigma^\pm = \tau \pm \sigma$) look like

$$Q_\pm = \frac{1}{2\pi} \oint \mathrm{d}\sigma \, v^\pm(\sigma) T_{\pm\pm}(\sigma), \tag{3.72}$$

where the normalisation is a convention. Infinitesimal conformal transformations are given by the Poisson bracket of of some field with $Q_\pm$. In particular,

$$\delta_v X^\mu = \{Q_+ + Q_-, X^\mu\}_{\mathrm{PB}}, \tag{3.73}$$

where $\{\ ,\ \}_{\mathrm{PB}}$ is the classical Poisson bracket.

This classical statement gives us some hint that the stress tensor, thoguht of as a constraint, generates these conformal transformations and explains why the Witt algebra is appearing in these two apparently different circumstances. Put another way, the constraints coming from the vanishing of the stress tensor put physical conditions on the spacetime. We would like to recast this intuition into understanding how conformal symmetry gives rise to various properties of spacetime physics. We will soon look at a quantum version of (3.73).

3.7 Complex Coordinates

In this section, we use $z = e^{\tau+i\sigma}, \overline{z} = e^{\tau-i\sigma}$ and take h_{ab} to be Euclidean on Σ. In these coordinates, the stress tensor has only two non-trivial coordinates

$$T := T_{zz} = -\frac{1}{\alpha'}\partial X^\mu \partial X^\nu \eta_{\mu\nu}, \qquad \overline{T} := T_{\overline{z}\overline{z}} = -\frac{1}{\alpha'}\overline{\partial} X^\mu \overline{\partial} X^\nu \eta_{\mu\nu}. \tag{3.74}$$

The off-diagonal entries $T_{z\overline{z}} = 0$ vanish as they are the trace of the stress tensor. Moreover,

$$\mathrm{d}t \, \mathrm{d}\sigma = -\frac{\mathrm{d}z \, \mathrm{d}\overline{z}}{2i|z|^2}. \tag{3.75}$$

The path integral becomes

$$\int \mathcal{D}X \, e^{iS[X]} \to \int \mathcal{D}X \, e^{-S[X]}, \tag{3.76}$$

where the action is

$$S[X] = -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}^2 z \, \partial X^\mu \overline{\partial} X^\nu \eta_{\mu\nu}. \tag{3.77}$$

The equation of motion for the X^μ is now $\partial\overline{\partial} X^\mu = 0$. This means that we have the split

$$X^\mu(z, \overline{z}) = X_L^\mu(z) + X_R^\mu(\overline{z}). \tag{3.78}$$

Finally, our conservation equation (3.71) becomes

$$\overline{\partial} T(z) = 0 \qquad \partial \overline{T}(\overline{z}) = 0. \tag{3.79}$$

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3.7.1 Ward Identities and Conformal Transformations

Given an infinitesimal conformal transformation, how do we compute it in terms of the operators in our theory. Ultimately, the fields that we are interested in are the embedding fields X^μ, but also the ghosts b and c. We will describe a general primary field as $\phi(z, \overline{z})$. Later we shall choose ϕ to be either X^μ or one of the b, c ghost fields, which came from the quantum field theoretic description of the Fadeev–Popov determinant.

Let us consider the change in a correlation function $\langle \phi_1 \dots \phi_n \rangle = \int \mathcal{D}\phi \, e^{-S[\phi]} \phi_1 \dots \phi_n$ resulting from an infinitesimal transformation $\phi \to \phi' = \phi + \delta\phi$.

$$\langle \phi'(z_1) \dots \phi'(z_n) \rangle = \int \mathcal{D}\phi' \, e^{-S[\phi']} \phi'(z_1) \dots \phi'(z_n), \tag{3.80}$$

where we write $\phi'(z_i)$ for $\phi'(z_i, \overline{z}_i)$. We shall assume that the measure $\mathcal{D}\phi' = \mathcal{D}\phi$ does not change (meaning that there are no anomalies in the theory). However, in general the action might change. To leading order, we have

$$\langle \phi'(z_1) \dots \phi'(z_n) \rangle = \int \mathcal{D}\phi \, e^{-S[\phi] - \delta S[\phi]} (\phi(z_1) + \delta\phi(z_1)) \dots (\phi(z_n) + \delta\phi(z_n)) \tag{3.81}$$

$$= \langle \phi(z_1) \dots \phi(z_n) \rangle - \int \mathcal{D}\phi \, e^{-S[\phi]} \delta S[\phi] \phi(z_1) \dots \phi(z_n) \tag{3.82}$$

$$+ \int \mathcal{D}\phi \, e^{-S[\phi]} \sum_{k=1}^{n} \phi(z_1) \dots \phi(z_{k-1}) \delta\phi(z_k) \phi_{z_{k+1}} \dots \phi(z_n). \tag{3.83}$$

For our theory to be invariant under the change $\phi \to \phi'$ we require $\langle \phi'1 \dots \phi'_n \rangle = \langle \phi_1 \dots \phi_n \rangle$. Thus

$$\langle \delta S[\phi] \phi(z_1) \dots \phi(z_n) \rangle = \sum_{k=1}^{n} \langle \phi(z_1) \dots \delta\phi(z_k) \dots \phi(z_n) \rangle. \tag{3.84}$$