Statistical Field Theory

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Contents

1 The Ising Model

The Ising model is a simple model for a magnet, which also forms a rich playground for SFT, QFT, etc...

1.1 Basics

We have a lattice of sites in any dimension d. At each lattice site, there is positioned a spin which can either point up or down. The energy of the entire system is

$$E = -B \sum_i S_i - J \sum_{<ij>} S_i S_j. \tag{1.1}$$

B is a magnetic field and $< ij >$ denotes nearest neighbour interactions.

If $J > 0$: Spins favour alignment (Ferromagnet)

If $J < 0$: Misaligned (Anti-Ferromagnet)

1.2 Finite Temperature

With finite temperature, this is where the *statistical* bit of the course comes in. We introduced some entropy into the system so the spins can flip. We would expect that even for $J < 0$ this implies that the spins will tend to be misaligned, resulting in no magnetisation, at high temperatures.

In the canonical ensemble:

$$p[S_i] = \frac{e^{-\beta E[S_i]}}{Z} \tag{1.2}$$

where $\beta = 1/T$ and the partition function is $Z = \sum_{\{S_i\}} e^{-\beta E[S_i]}$.

Example:

$$F_{\text{thermo}}(T; B) = \langle E \rangle - TS = -T \log Z \tag{1.3}$$

Definition 1 (Average Spin)**:**

$$m = \frac{1}{N} \left\langle \sum_i S_i \right\rangle, \qquad m \in [-1, +1] \tag{1.4}$$

To calculate the average, we need the probabilities. We write this as

$$m = \frac{1}{N} \sum_{\{S_i\}} \underbrace{\frac{e^{-\beta E[S_i]}}{Z}}_{\text{probability}} \overbrace{\sum_i S_i}^{\text{spin}} = \frac{1}{N\beta} \frac{\partial \log Z}{\partial \beta}. \tag{1.5}$$

There are many numerical tools to calculate the partition function. For analytic tools, we work towards *mean field theory* which makes the large number of degrees of freedom tractable.

1.3 Effective Free Energy

Ising model: “Microscopic Degrees of Freedom” (spins)

Partition function: “Macroscopic” (average spin)

The ansatz that we will use is to write the partition function Z as a sum over the magnetisations $m = \frac{1}{N}\sum_i S_i$ of configurations, and then sum over all configuations that give this magnetisation:

$$Z = \sum_m \sum_{\{S_i\}|_m} e^{-\beta E[S_i]} := \sum_m e^{-\beta F(m)}. \tag{1.6}$$

In the large-N limit:

$$\sum e^{-\beta F(m)} \longrightarrow \frac{N}{2} \int_{-1}^{1} \mathrm{d}m\, e^{-\beta F(m)}. \tag{1.7}$$

Saddle Point Approximation: Define $f(m) = \frac{F(m)}{N}$ Ignoring normalisation factors, the partition function becomes

$$Z = \int_{-1}^{+1} \mathrm{d}m\, e^{-\beta N f(m)} \approx e^{-\beta N f(m_{\min})} \implies F_{\text{thermo}} \approx F(m_{\min}) \tag{1.8}$$

1.4 Mean Field Theory

1.4.1 Mean Field Approximation

One essentially replaces each spin variable by the mean variable that is the same everywhere:

$$S_i \to \langle S \rangle = m. \tag{1.9}$$

The total energy is then

$$E = -B \sum_i m - J \sum_{\langle ij \rangle} m^2 \tag{1.10}$$

and the normalised energy

$$\frac{E}{N} = -Bm - \frac{1}{2} Jqm^2. \tag{1.11}$$

Here, $q \sim 2d$ is the number of nearest neighbours. For a given magnetisation m, we need to count how many states there are.

1.4.2 Counting Microstates

$$m = \frac{N_\uparrow - N_\downarrow}{N} = \frac{2N_\uparrow - N}{N} \qquad \Omega = \frac{N!}{N_\uparrow!(N-N_\uparrow)!} \tag{1.12}$$

Use Stirling's formula (can plug this in as an "illuminating" exercise at home)

$$\log\Omega = N\log N - N_\uparrow \log N_\uparrow - (N-N_\uparrow)\log(N-N_\uparrow) \tag{1.13}$$

$$\frac{\log\Omega}{N} \approx \log 2 - \frac{1}{2}(1+m)\log(1+m) - \frac{1}{2}(1-m)\log(1-m). \tag{1.14}$$

Hence,

$$\sum_{\{S_i\}_m} e^{-\beta E[S_i]} \approx \Omega(m) e^{-\beta E(m)} \tag{1.15}$$

$$\approx e^{\frac{N\log\Omega}{N}} e^{-\beta\left(-Bm - \frac{1}{2}Jqm^2\right)} \tag{1.16}$$

This defines

$$f(m) \approx -Bm - \frac{1}{2}Jqm^2 - T\left[\log(2) - \frac{1}{2}(1+m)\log(1+m) - \frac{1}{2}(1-m)\log(1-m)\right]. \tag{1.17}$$

Minimised at

$$\frac{\partial f}{\partial m} = 0 \implies \beta(B + Jqm) = \frac{1}{2}\log\left(\frac{1+m}{1-m}\right) \tag{1.18}$$

$$\implies m = \tanh(\beta B + \beta Jqm), \qquad (B_{\text{eff}} = B + Jqm) \tag{1.19}$$

Plotting left and right hand side, we see that at $T = \infty$ there is only one solution at $m = 0$. At low temperature, we then obtain other solutions.

2 Landau Approach to Phase Transitions

Landau Theory Free energy + Symmetry

Effective Free Energy $F(m) = Nf(m)$, where m is the *order parameter*

$$f(m) = -Bm - Jqm^2 - T[\log(2) - \frac{1}{2}(1+m)\log(1+m) - \frac{1}{2}(1-m)\log(1-m)] \quad (2.1)$$
$$\simeq -T\log(2) - Bm + \frac{1}{2}(T - Jq)m^2 + \frac{1}{12}Tm^4 + \cdots \quad (2.2)$$

Saddle Point Equilibrium at minimum of $f(m)$

2.1 Phase transition in the Ising Model

Without external magentic field $B = 0$, we have

$$f(m) = \frac{1}{2}(T - Jq)m^2 + \frac{1}{12}Tm^5 + \cdots \quad (2.3)$$

As we lower the temperature the potential becomes flat. Below the critical temperature T_C, we obtain two minima at $\pm m_0$, where

$$m_0 = \sqrt{\frac{3(T_c - T)}{T}} \quad (2.4)$$

We call a phase transition an *n-th order phase transition* if it has a discontinuity in the n-th derivative. In this case, we have a second order phase transition.

> **Remark:** The reason for this discontinuity is the large number of degrees of freedom. Since $N \to \infty$, discontinuities show up, but on the level of individual spins, all processes are continuous.

Definition 2 (Heat Capacity)**:** The heat capacity of the system is

$$C = \frac{\partial\langle E\rangle}{\partial T} = \beta^2\frac{\partial^2\log(Z)}{\partial\beta^2}, \qquad E = -\frac{\partial\log(Z)}{\partial\beta} \quad (2.5)$$

where $Z = e^{-\beta Nf(m_{\min})}$.

In the Landau Approach:

$$T > T_C \qquad f(m_{\min}) = 0 \tag{2.6}$$

$$T < T_c \qquad f(m_{\min}) = -\frac{3}{4}\frac{(T_c - T)^2}{T} \tag{2.7}$$

2.2 Spontaneous Symmetry Breaking

At high temperatures, we have

$$f(m) = \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Tm^4. \tag{2.8}$$

This has a $\mathbb{Z}_2$ symmetry under interchange $m \to -m$.

Definition 3 (Spontaneous Symmetry Breaking)**:** The ground state does not respect a symmetry of the action / free energy.

2.3 Non-Zero Magnetic Field

If we have $B \neq 0$, then we have a term that is linear in m:

$$f(m) = -Bm + \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Tm^4 + \cdots \tag{2.9}$$

In the absence of the magnetic field, the two tracks split. However, in this case, the tracks do not meet and there is no discontinuity and no phase transition either. However, we can manufacture a phase transition in the following way.

2.3.1 Manufacturing a Phase Transition

We increase the magnetic field in such a way as to increase the energy of the state in which the system found itself via spontaneous symmetry breaking. The system will remain in that state, even until we increase the magnetic field so much that it is only a local, not the global minimum. After some time spent in this meta-stable state, the magnetisation jumps to the lower energy state. This is a first order phase transition.

At the critical point $T = T_c$, we can solve for the magnetisation as a function of the magnetic field. We find that

$$m \sim \begin{cases} B^{1/3} & B < 0 \\ -|B|^{1/3} & B > 0 \end{cases} \tag{2.10}$$

Definition 4 (Magnetic Susceptibility)**:**

$$\chi = \frac{\partial m}{\partial B}|_T \tag{2.11}$$

Since we have $m \simeq \frac{B}{T - T_c}$, the susceptibility is $\chi \sim |T - T_c|^{-1}$. These are predictions which can be tested in experiments.

2.4 Validity of Mean Field Theory

The mean field theory (MFT) gave an excellent qualitative picture of the phase transitions. But how well does it actually work? There were a number of assumptions: We neglected any spatial variations by assuming that any spin takes the same value as the average. We also neglected any interactions, such as corrections from the quartic term in $f(m)$.

It turns out that whether or not MFT works depends on the number of dimensions:

$d = 1$**:** Total Failure. In one dimension there is never any phase transition. We call this the *lower critical dimension* d_l.

$d = 2, 3$**:** Works-*ish*. The qualitative picture is right, but the details are not correct.

$d \geqslant 4$**:** Works remarkably well. In $d \geq 4$, it gives exactly the right answer and matches experiment. We call this the *upper critical dimension* d_c.

In general theories, not restricted to the Ising model, we call the dimension in which MFT fails completely the *lower critical dimension* d_l and the theory for which it always works the *upper critical dimension* d_c.

> **Remark:** This can be understood in terms of the number of nearest-neighbours. The fluctuations grow in a different way depending on the dimension of the system.

2.5 Critical Exponents

We found that in the absence of an external magnetic field, the magnetisation scaled as $m \sim |T_c - T|^{\beta}$, where $\beta = \frac{1}{2}$. Moreover, the heat capacity scaled as $C \sim C_{\pm}|T - T_c|^{-\alpha}$, where $\alpha = 0$, the susceptibility scaled as $\chi \sim |T - T_c|^{-\gamma}$ where $\gamma = 1$. Finally, the magnetisation also scaled as $m \sim B^{1/\delta}$ where $\delta = 3$.

The actual values are tabulated in 2.1.

	MF	d=2	d=3	d=4
α	0	$O(\log)$	$0.1101\ldots$	
β	$\frac{1}{2}$	$0.3264\ldots$		
γ	1	$\frac{7}{4}$		
δ	3	15	$4.7898\ldots$	

Table 2.1: Critical Exponents in various dimensions.

This leads us to the concept of *universality* where two systems that have different underlying physical processes actually have the same critical exponents.

3 Landau-Ginzburg Free Energy

3.1 Model Building

This will be the first attempt of writing down a *theory* by postulating a free energy. In the case of particle physics, this corresponds to the action. There are certain guiding principles that help construct such a theory. Such 'common sense' principles include

Locality All interactions are *local*. Only nearest neighbours matter. This means the free energy functional is just one integral over one spatial component

$$F[m(x)] = \int \mathrm{d}^d x \, f[m(x)]. \tag{3.1}$$

In general, one could imagine non-local theories where we have terms of the form

$$\int \mathrm{d}^d x \, \mathrm{d}^d y \, f[m(x), m(y)].$$

In the continuum Ising model, we have some almost non-local terms. These are the gradient terms

$$\frac{\partial m(\mathbf{x})}{\partial \mathbf{x}} \simeq \lim_{a \to 0} \frac{m(\mathbf{x} + a\hat{\mathbf{x}}) - m(\mathbf{x})}{a}. \tag{3.2}$$

There are certain pitfalls, such as causality violation, which arise when we can include non-local terms. For more information, see Weinberg Volume 1.

Translational and Rotational Invariance The symmetry of the continuum theory should reflect the symmetries of the lattice. The Ising model has a large-scale rotational symmetry, which will be recovered when going to the continuum limit. Similarly, there is a discrete translational symmetry which will become a continuum symmetry. This means, we have terms like $(\boldsymbol{\nabla} m)^2$, but not $\cancel{\mathbf{n} \cdot \boldsymbol{\nabla} m}$.

Moreover, the Ising model with $B = 0$ has a $\mathbb{Z}_2$ symmetry: $S_i \to -S_i$. Therefore, the continuum limit must have a symmetry under $m(\mathbf{x}) \to -m(\mathbf{x})$.

Analyticity We require the free energy F to have a well-behaved Taylor expansion in the magnetisation field $m(\mathbf{x})$. This allows terms like $m^2(\mathbf{x})$, $(\boldsymbol{\nabla} m(\mathbf{x}))^4$, ..., $m^8(\mathbf{x})$. However, we are not allowed to have non-analytic terms like $\sqrt{m(\mathbf{x})}$, $\log(m(\mathbf{x}))$, or $\frac{1}{m(\mathbf{x})}$.

> **Remark:** Non-analyticities like these usually signal that we overlooked some important ingredient in the theory.

Derivative Expansion If the variation of the gradient is slow over the length scale a of the lattice spacing, $(a\boldsymbol{\nabla})\boldsymbol{\nabla}m(\mathbf{x}) \ll \boldsymbol{\nabla}m(\mathbf{x})$, then higher derivatives can be ignored. As we go to higher number of derivatives, we go to more separated neighbour interactions.

> **Remark:** The Higgs interaction, among many others, starts to generate logarithmic terms when we coarse-grain high-energy processes.

3.2 Zero external field

When $B = 0$, the free energy is

$$F[m(\mathbf{x})] = \int \mathrm{d}^d x \left[\frac{1}{2}\alpha_2(T)m^2 + \frac{1}{4}\alpha_4(T)m^4 + \frac{1}{2}\gamma(T)(\boldsymbol{\nabla}m)^2 + \dots\right], \tag{3.3}$$

where $m = m(\mathbf{x})$, $\alpha_2(T) \simeq T - T_c$, and $\alpha_4(T) \sim \frac{1}{3}T$.

3.2.1 The Saddle Point Approximation

We want to find minimise the free energy functional $F[m(\mathbf{x})]$. Consider a field configuration $m(\mathbf{x})$ and perturb it by $\delta m(\mathbf{x})$. The free energy then changes by

$$\delta F = \int \mathrm{d}^d x \left[\alpha_2 m\delta m + \alpha_4 m^3 \delta m + \gamma \boldsymbol{\nabla}m \cdot \boldsymbol{\nabla}\delta m\right] \tag{3.4}$$

$$= \int \mathrm{d}^d x \left[\alpha_2 m + \alpha_4 m^3 - \gamma\nabla^2 m\right]\delta m \tag{3.5}$$

The implicit assumption here is that the variation vanishes at the boundaries of the integral. We can take this as a definition of the functional derivative

$$\frac{\delta f}{\delta m} = \alpha_2 m + \alpha_4 m^3 - \gamma\nabla^2 m. \tag{3.6}$$

For an Extremum, $\frac{\delta F}{\delta m}\big|_m = 0 \implies \gamma\nabla^2 m = \alpha_2 m + \alpha_4 m^3$. This is sometimes called the *Euler-Lagrange equation* or the *equation of motion*.

Solutions

The simplest solution is $m = \text{const}$. There are two cases:

$$m = \begin{cases} 0 & \alpha_2 > 0 \\ \pm m_0 = \pm\sqrt{\frac{-\alpha_2}{\alpha_4}} & \alpha_2 < 0 \end{cases} \tag{3.7}$$

This recovers the mean-field approximation, confirming that our original mean-field Ansatz was not so bad after all. Moreover, analysing the other solutions allows us to understand the mean-field solution better. The other solutions are called *domain walls*.

Domain Walls

We can also imagine that different parts of the lattice take on different solutions, so that both $\pm m_0$ coexist. In that case, there will be an asymptotic transition function between the domain walls at spatial infinity. This transition function cannot just be a step function, since the

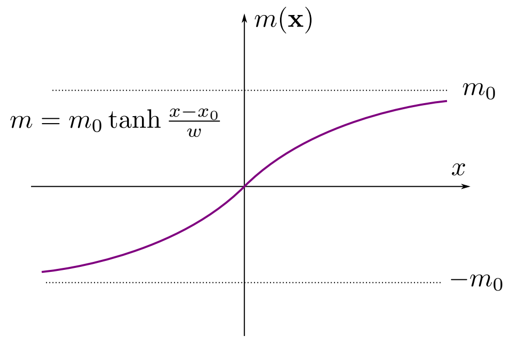


Figure 3.1: The transition function asymptotes to the domain walls as $x = |\mathbf{x}| \to \infty$

derivative would be discontinuous, giving infinite energy. Solving $\gamma \frac{\mathrm{d}^2 m}{\mathrm{d}x^2} = \alpha_2 m + \alpha_4^2$ gives

$$m = m_0 \tanh\left(\frac{x - x_0}{w}\right), \qquad w = \sqrt{\frac{-2\gamma}{\alpha}}. \tag{3.8}$$

The energy associated with this solution can be calculated to give $F_{\mathrm{DW}} - F_{\mathrm{GS}} \sim L^{d-1}\sqrt{\frac{-\gamma\alpha_2^3}{\alpha_4}}$. Here we see something interesting happening; in one dimension, the free energy is independent of the size of the system, whereas in higher dimensions it does. This exponent $d-1$ will tell us about why the mean-field theory fails in one-dimension: the domain-walls completely take over and dominate over the mean-field theory contributions.

This domain wall solution will shed light on the behaviour of mean field theory in lower dimensions.

3.3 Lower Critical Dimension

Let us first consider the case of one domain wall in the system. The probability of having a domain wall at $x = X$ is

$$p(\text{wall at } x = X) = \frac{e^{-\beta F_{\text{wall}}}}{Z}. \tag{3.9}$$

However, the domain wall could form anywhere. Integrating over the probability of having a domain wall anywhere in the system gives

$$p(\text{wall anywhere}) = \frac{e^{-\beta F_{\text{wall}}}}{Z}\frac{L}{w}. \tag{3.10}$$

The probability to have n walls in the system is obtained by considering the product of the probabilities at n different locations:

$$P(n \text{ walls}) = \frac{e^{-n\beta F_{\text{wall}}}}{Z}\frac{1}{w^n}\int_{-L/2}^{L/2}\mathrm{d}x_1\int_{-L/2}^{L/2}\mathrm{d}x_2\cdots\int_{-L/2}^{L/2}\mathrm{d}x_n \tag{3.11}$$

$$= \frac{1}{Zn!}\left(\frac{Le^{-\beta F_{\text{wall}}}}{w}\right)^n. \tag{3.12}$$

We can now consider the sum over these probabilities. The sum over of all even n is giving a hyperbolic cosine. This is the probability of going from left to right and ending up at the same magnetisation m_0 that we started with, since at each domain wall, the magnetisation swaps $m_0 \to -m_0$. Similarly, we can calculate the probability of having an odd number n of domain walls, giving a hyperbolic sine:

$$p(m_0 \to m_0) = \frac{1}{Z}\cosh\left(\frac{Le^{-\beta F_{\text{wall}}}}{w}\right) \tag{3.13}$$

$$p(m_0 \to -m_0) = \frac{1}{Z}\sinh\left(\frac{Le^{-\beta F_{\text{wall}}}}{w}\right). \tag{3.14}$$

Now let us consider what happens in $d = 1$: In one dimension, the free energy of domain wall solutions $F_{DW} \sim L^{d-1}$ is a constant. Therefore, the cosh and sinh terms both diverge as $L \to \infty$. If there are no other divergent processes, then these probabilities will dominate over the partition function Z, and eventually converge to $p(m_0 \to m_0) = p(m_0 \to -m_0) = 50\%$. This is a puzzling result; it appears that as the system size grows, the probability of ending up in a system with domain walls go to unity. In this case, we say that domain walls proliferate. In one dimension, we do not expect the mean-field approximation to work, because spatial variations are important. On the other hand, in the case of $d > 1$, the free energy $F \sim L^{d-1}$ will grow, meaning that the exponential factor $e^{-\beta F} \to 0$ suppresses the otherwise linear growth of probability.

3.4 My First Path Integral

We now define the partition function as a path integral over all possible field configurations. This is an integral over all functions $m(\mathbf{x})$:

$$Z = \int \mathcal{D}m(\mathbf{x})\, e^{-\beta F[m(\mathbf{x})]}. \tag{3.15}$$

We focus on fluctuations around the saddle point. If these fluctuations are too big, we cannot use perturbative methods any more, and the path integral method breaks down, corresponding to strongly coupled field theories.

As is conventional in field theory, we denote the scalar field as $m(\mathbf{x}) \to \phi(\mathbf{x})$. We also drop the dependence on $\mathbf{x}$ in writing $\phi = \phi(\mathbf{x})$. Setting $B = 0$, the free energy is

$$F[\phi(\mathbf{x})] = \int \mathrm{d}^d x \left[\frac{1}{2}\alpha_2(T)\phi^2 + \frac{1}{4}\alpha_4(T)\phi^4 + \frac{1}{2}\gamma(T)(\boldsymbol{\nabla}\phi)^2 + \dots\right]. \tag{3.16}$$

We approximate this action by setting all coefficients $\alpha_{n\geq 4} = 0$. At the moment, this looks like a gross oversimplification; however, renormalisation group theory will show us that these terms are actually irrelevant for our purposes. Now there are two cases: The coefficient to the quadratic term α_2 can either be positive or negative. If we have $\alpha_2 > 0$, this will lead to a *disordered phase*, whereas $\alpha_2 < 0$ will lead to an *ordered* phase. In that case, we can look at the variations of the field about its mean $\langle\phi\rangle$:

$$\widetilde{\phi}(\mathbf{x}) = \phi(\mathbf{x}) - \langle\phi\rangle. \tag{3.17}$$

The action then becomes

$$F[\widetilde{\phi}(\mathbf{x})] = F[\langle\phi\rangle] + \frac{1}{2}\int \mathrm{d}^d x \left[\alpha_2'(T)\widetilde{\phi}^2 + \gamma(T)(\boldsymbol{\nabla}\phi)^2 + \dots\right], \tag{3.18}$$

where $\alpha_2'(T) = -2\alpha_2(T) > 0$.

3.4.1 Fourier Space

To perform this calculation, we have to move to Fourier space. In that case, we express the field $\phi(\mathbf{x})$ as an integral over the modes $\phi_{\mathbf{k}}$ with wavevector $\mathbf{k}$:

$$\phi_{\mathbf{k}} = \int \mathrm{d}^d x\, e^{-i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{x}). \tag{3.19}$$

Since ϕ is a real scalar field, a standard result in Fourier theory is that $\phi^*_{\mathbf{k}} = -\phi_{\mathbf{k}}$.

> **Remark:** For complex fields, this relationship between the Fourier conjugates does not in general hold.

Since we have a smallest lattice scale a in the system, the momentum modes must vanish for some wavenumber scale $|\mathbf{k}| > \Lambda \sim \frac{\pi}{a}$.

Finite Volume In a finite box of side-length L, and volume $V \sim L^d$, the allowed wavevectors are discrete:

$$\mathbf{k} = \frac{2\pi\mathbf{n}}{L}, \qquad \mathbf{n} \in \mathbb{N}^d. \tag{3.20}$$

Following this discretisation, the Fourier expansions of the fields ϕ turn from integrals into sums:

$$\phi(\mathbf{x}) = \frac{1}{V}\sum_{\mathbf{k}} e^{+i\mathbf{k}\cdot\mathbf{x}}\phi_{\mathbf{k}}. \tag{3.21}$$

We sometimes say that the field theory has been 'quantised'.

Infinite Volume If the volume becomes infinite $L \to \infty$, the wavevectors become members of a continuous set $\mathbf{k} \in \mathbb{R}^d$, meaning that the expansion of ϕ into its Fourier modes now looks like

$$\phi(\mathbf{x}) = \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{+i\mathbf{k}\cdot\mathbf{x}}\phi_{\mathbf{k}} \coloneqq \int đ^d k\, e^{+i\mathbf{k}\cdot\mathbf{x}}\phi_{\mathbf{k}}. \tag{3.22}$$

The definition of the normalised measure $đ^d k = (2\pi)^{-d}\,\mathrm{d}^d k$ was chosen to not have to carry around superfluous factors of $(2\pi)^d$ in Fourier space.

Inserting $\phi(\mathbf{x})$ into the free energy, we have

$$F[\phi_{\mathbf{k}}] = \frac{1}{2}\int đ^d\mathbf{k}_1 \int đ^d\mathbf{k}_2 \int \mathrm{d}^d x\,(-\gamma\mathbf{k}_1\cdot\mathbf{k}_2 + \mu^2)\phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}. \tag{3.23}$$

where we chose to write the coefficient to the quadratic term as $\alpha_2 = \mu^2$. This convention is often chosen in field theory, stemming from the analogy to quantum field theory, where the coefficient to the quadratic field ϕ^2 is (twice) the mass of the particle. Performing the integral over $\mathbf{x}$ gives a delta function. The normalisation of that delta function is determined by our Fourier space conventions:

$$\bar{\delta}^d(\mathbf{k}_1+\mathbf{k}_2) \coloneqq (2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_2) = \int \mathrm{d}^d x\, e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}, \tag{3.24}$$

where we again use the horizontal bar notation to denote a Fourier space normalisation that will considerably simplify equations by hiding annoying factors of $(2\pi)^d$. The free energy is then found to be

$$F[\phi_{\mathbf{k}}] = \frac{1}{2}\int đ^d k(\gamma k^2 + \mu^2)\phi_{\mathbf{k}}\phi^*_{\mathbf{k}}, \tag{3.25}$$

A free energy or action of this form is said to be 'diagonalised'. This terminology stems from the analogous case in matrices, where we can write each matrix as a spectral decomposition over its eigenvalues in a certain basis of eigenvectors. In the picture of this analogy, the $|\phi_{\mathbf{k}}|^2$ are the eigenvectors with eigenvalue $(\gamma k^2 + \mu^2)$.

3.5 Path Integral Measure

What exactly do we mean when we write down the path integral measure $\mathcal{D}\phi$? The path integral measure is defined as an integral over all possible field configurations. We define this as

$$\int \mathcal{D}\phi(\mathbf{x}) \coloneqq \prod_{\mathbf{k}} \left[N \int \mathrm{d}\phi_{\mathbf{k}}\, \mathrm{d}\phi^*_{\mathbf{k}} \right]. \tag{3.26}$$

The normalisations N are not important, since they usually cancel out when we compute expectation values and correlation functions. Moreover, we have to keep in mind that ϕ^* is not actually an independent degree of freedom. This actually means that this derivation would technically only be correct for complex fields, and not really well-defined for the scalar fields that we are dealing with. However, we still go through it here to lead up to the result and introduce the concepts. With this definition of the path integral measure, we will aim to compute the partition function

$$Z = \prod_{\mathbf{k}} N \int \mathrm{d}\phi_{\mathbf{k}}\, \mathrm{d}\phi^*_{\mathbf{k}}\, e^{-\frac{\beta}{2}\int \text{đ}^d k (\gamma k^2 + \mu^2)|\phi_{\mathbf{k}}|^2}. \tag{3.27}$$

However, another slightly strange feature of this notation that we will have to make a bit more rigorous is the notion of having an infinite number of integrals over the continuous wavenumber labels $\mathbf{k}$. To make a bit more sense of this, we first deal with a path integral defined over a finite volume V of space, making the set of wavenumbers discrete, and then define the infinite integral as the limit as $V \to \infty$ of that expression.

Finite Volume When we choose to work in a finite volume V, the integral over continuous wavevectors in the argument of the exponential function turns into a sum over discrete wavevectors. The partition function for a finite volume is therefore

$$Z = \prod_{\mathbf{k}} \left[N \int \mathrm{d}\phi_{\mathbf{k}}\, \mathrm{d}\phi^*_{\mathbf{k}} \right] e^{-\frac{\beta}{2V}\sum_{\mathbf{k}}(\gamma k^2 + \mu^2)|\phi_{\mathbf{k}}|^2} \tag{3.28}$$

$$= \prod_{\mathbf{k}} N \left[\int \mathrm{d}\phi_{\mathbf{k}}\, \mathrm{d}\phi^*_{\mathbf{k}}\, e^{-\frac{\beta}{2V}(\gamma k^2 + \mu^2)|\phi_{\mathbf{k}}|^2} \right]. \tag{3.29}$$

Now, we can actually solve these integrals exactly! To see this, consider the well-known Gaussian integral over the real numbers $x \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-\frac{x^2}{2a}} = \sqrt{2\pi a}. \tag{3.30}$$

Each of the integrals over a particular $\phi_{\mathbf{k}}$ or $\phi^*_{\mathbf{k}}$ can then be solved as a Gaussian integral. The total partition function is a product over these integrals, two for each wavevector $\mathbf{k}$ (one for the field ϕ and one for the conjugate field ϕ^*). This procedure yields the following partition function:

$$Z = \prod_{\mathbf{k}} N \sqrt{\frac{2\pi T_v}{\gamma k^2 + \mu^2}}. \tag{3.31}$$

As we will see, the Gaussian theory is one of the few path integrals that we can solve exactly due to this analogy with the Gaussian integral. All higher order terms, corresponding to 'interactions' between the fields, have to be dealt with perturbatively.

Let us look in a bit more detail at the path integral above. We are in fact dealing not with a complex variable, which has two degrees of freedom, but with a scalar field, where $\phi^*_{\mathbf{k}} = \phi_{-\mathbf{k}}$. This constraint complicates the integral. If we just had worked with a complex variable, then the path integral can be solved. The real space result which was quoted above is the square root of this, which is explained by the additional constraint.

In the free, non-interacting theory, the path integral is completely solvable. Although they look very frightening, there are actually ways to tame the infinite integrals in the path integral and make sense of them.

3.6 Path integral in thermodynamics

The thermodynamic free energy is related to the partition function as $Z = e^{-\phi F_{\text{therm}}}$. Taking logarithms and dividing by the system's volume V, we have

$$\begin{aligned}\frac{F_{\text{therm}}}{V} &= -\frac{T}{V}\log(Z) && (3.32)\\ &= -\frac{T}{2V}\sum_{\mathbf{k}}\log\left(\frac{2\pi TVN^2}{\gamma k^2+\mu^2}\right) && (3.33)\\ &\to -\frac{T}{2}\int đ^4k\log\left(\frac{2\pi TVN^2}{\gamma k^2+\mu^2}\right) && (3.34)\end{aligned}$$

where we took the continuum limit $V \to \infty$ in the last line.

We can use this to calculate various physical quantities.

3.6.1 Heat Capacity

The normalised heat capacity $c = \frac{C}{V}$ is given by

$$\begin{aligned}c &= \frac{\beta^2}{V}\frac{\partial^2}{\partial B^2}\log(Z) && (3.35)\\ &= \frac{1}{2}(T^2\frac{\partial^2}{\partial T^2} + 2T\frac{\partial}{\partial T})\int đ^4k\log\left(\frac{2\pi TVN^2}{\gamma k^2+\mu^2}\right) && (3.36)\end{aligned}$$

where $\mu^2 = T - T_c$. We have an infinite number of degrees of freedom; an infinite number of spins—one for each lattice site. This means that the integral

$$c = \frac{1}{2}\int_0^\Lambda đ^dk\left[1 - \frac{2T}{\gamma k^2+\mu^2} + \frac{T^2}{(\gamma k^2+\mu^2)^2}\right] \tag{3.37}$$

actually diverges. We can calculate this integral by using $\mathrm{d}^dk = k^{d-1}\,\mathrm{d}k\,\mathrm{d}\Omega_d$. Let us look at the middle term. This has dimension dependent behaviour

$$\int_0^\Lambda \mathrm{d}k\,\frac{k^{d-1}}{\gamma k^2+\mu^2} = \begin{cases}\Lambda^{d-2} & d>2\\ \log\Lambda & d=2\\ \frac{1}{\mu} & d=1.\end{cases} \tag{3.38}$$

Similarly, the last term scales with dimension, although in a different way

$$\int_0^\Lambda \mathrm{d}k\, \frac{k^{d-1}}{(\gamma k^2+\mu^2)^2} = \begin{cases} \Lambda^{d-4} & d>4 \\ \log\Lambda & d=4 \\ \mu^{d-4} & d<4. \end{cases} \tag{3.39}$$

This means that something interesting is happening when $d=4$. When $d<4$, we have $c \sim |T-T_c|^{-\alpha}$, where $\alpha = 2-\frac{d}{2}$. Note that this result also ignored interactions. However, as opposed to mean-field theory, we have at least included fluctuations. By doing this, we have learned that the critical exponent α has changed. This tells us that fluctuations matter. Moreover, we also learned that when $d>4$, the heat capacity is dominated by the last term, which is dominated by Λ, which is independent of temperature. This is why the critical exponents do not differ from each other once we cross the upper critical dimension $d=4$.

3.7 Correlation Functions

The big step was that the order parameter depends on space, and is allowed to fluctuate about the mean-field value. We can therefore now ask questions about the system which depend on position. In mean field theory, we had no such local information:

$$\langle\phi(x)\rangle = \begin{cases} 0 & T>T_c \\ \pm m_0 & T<T_c. \end{cases} \tag{3.40}$$

The information about these fluctuations are encoded in *correlation functions*, sometimes also referred to as $Green's functions$. The next simplest correlation function beyond the simple mean above, is the *two-point* correlation function

$$\langle\phi(x)\phi(y)\rangle \tag{3.41}$$

This is known from QFT, where it is the *propagator* that tells us how waves propagate from one place to another. In our case, it is related to the value of the local magnetisation. Since we want to know about the fluctuations about the mean, we can work with the *connected correlation function*, which is defined by subtracting off the mean

$$\langle\phi(x)\phi(y)\rangle_c = \langle\phi(x)\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle. \tag{3.42}$$

In statistics, this is known as the *second cumulant*.

3.7.1 Computing Correlation Functions

This section will be very useful for any studies in field theory. Consider the free energy for a position dependent magnetic field

$$F[\phi(x)] = \int \mathrm{d}^d x \left[\frac{\gamma}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{\mu^2}{2}\phi^2 + B(x)\phi\right]. \tag{3.43}$$

The partition function then depends on the configuration of the magnetic field:

$$Z[B(x)] = \int \mathcal{D}\phi \, e^{-\beta F[\phi,B]}. \tag{3.44}$$

The partition function gives the probability for each individual spin configuration. The mean of some random variable O was then given by

$$\langle O \rangle = \sum_i o_i p_i \tag{3.45}$$

where o_i are the values of the random variable, and p_i their associated probabilities. We can get objects like this from the partition function in a slick fashion by performing a functional differentiation:

$$\frac{1}{\beta}\frac{\delta \log Z}{\delta B(\mathbf{x})} = \frac{1}{\beta Z}\frac{\delta Z}{\delta B(\mathbf{x})} = -\frac{1}{Z}\int \mathcal{D}\phi(x)\,\phi(x)e^{-\phi F[\phi(x),B(x)]} = -\langle \phi(\mathbf{x})\rangle|_B. \tag{3.46}$$

We say that the external magnetic field B, which linearly multiplies the ϕ in the free energy, is called a *source*. Similarly, if we wanted to calculate the correlation function for O, we can add a source term to the free energy which allows us to recover $\langle O \rangle$ by methods of functional differentiation of Z.

We can take this further. There are more interesting quantities than the average value $\langle\phi(x)\rangle$; we can obtain the *connected two-point function* by taking two functional derivatives

$$\frac{1}{\beta^2}\frac{\delta^2 \log Z}{\delta B(x)\delta B(y)} = \langle\phi(x)\phi(y)\rangle_B - \langle\phi(x)\rangle\langle\phi(y)\rangle_B. \tag{3.47}$$

This automatically calculates the fluctuations about the average, which is quite neat. Setting $B = 0$ for $T > T_c$, we get

$$\frac{1}{\beta^2}\left.\frac{\delta^2 \log(Z)}{\delta B(x)\delta B(y)}\right|_{B=0} = \langle\phi(x)\phi(y)\rangle_{B=0}. \tag{3.48}$$

This two-point function is a very important quantity that pops up in various places all around theoretical physics.

3.8 The Gaussian Path Integral

Moving to Fourier space, we have the action

$$F[\phi_{\mathbf{k}}] = \int đ^d k \left[\frac{1}{2}(\gamma k^2 + \mu^2)\phi_{\mathbf{k}}\phi_{-\mathbf{k}} + B_{-\mathbf{k}}\phi_{\mathbf{k}}\right]. \tag{3.49}$$

We can define a shifted momentum mode

$$\hat{\phi}_{\mathbf{k}} = \phi_{\mathbf{k}} + \frac{B_{\mathbf{k}}}{\gamma k^2 + \mu^2}. \tag{3.50}$$

When we do this, the integral over momentum space simplifies greatly:

$$F[\hat{\phi}_{\mathbf{k}}] = \int đ^d k \left[\frac{1}{2}(\gamma k^2 + \mu^2)\left|\hat{\phi}_{\mathbf{k}}\right|^2 + \frac{1}{2}\frac{\left|B_{\mathbf{k}}\right|^2}{(\gamma k^2 + \mu^2)}\right]. \tag{3.51}$$

The path integral measure, written out as a product of integrals, is

$$Z = \left[\prod_{\mathbf{k}}\int \mathrm{d}\phi_{\mathbf{k}}\,\mathrm{d}\phi_{-\mathbf{k}}\right] e^{-\beta F[\hat{\phi}_{\mathbf{k}}]}. \tag{3.52}$$

And therefore, we have the full partition function with external magnetic field as

$$Z[B_{\mathbf{k}}] = e^{-\beta F_{\text{thermo}}}\exp\left(\frac{\beta}{2}\int đ^d k\,\frac{\left|B_{\mathbf{k}}\right|^2}{(\gamma k^2 + \mu^2)}\right). \tag{3.53}$$

We have separated out the path integral over ϕ into the front. One might think that this drops all the information about the fields ϕ. However, this information is still in the right side of the integral, since the form of the factor $(\gamma k^2 + \mu^2)$ was determined by ϕ. Taking the inverse Fourier transform, we have

$$Z[B(\mathbf{x})] = e^{-\beta F_{\text{thermo}}} e^{\frac{\beta}{2}\int \mathrm{d}^d x \mathrm{d}^d y B(\mathbf{x}) G(\mathbf{x}-\mathbf{y}) B(\mathbf{x})}, \tag{3.54}$$

where we have

$$G(\mathbf{x}-\mathbf{y}) = \int đ^d k\,\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{(\gamma k^2 + \mu^2)}. \tag{3.55}$$

Therefore, we have

$$\langle \phi(\mathbf{x})\phi(\mathbf{y})\rangle|_{B=0} = \frac{1}{\beta} G(\mathbf{x}-\mathbf{y}) \tag{3.56}$$

This is our version of the Feynman propagator. There is no special direction picked out, so $G(\mathbf{x}-\mathbf{y}) = G(r)$, where $r^2 = \sum_i x_i^2$.

$$G(r) = \frac{1}{\gamma}\int đ^d k \, \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{k^2 + \frac{1}{\varepsilon^2}}, \qquad \varepsilon^2 = \frac{\gamma}{\mu^2}. \tag{3.57}$$

The only length scale in the theory is the *correlation length* ε. We can perform this integral explicitly. However, we can also write

$$\frac{1}{k^2 + \frac{1}{\varepsilon^2}} = \int_0^\infty \mathrm{d}t \, e^{-t(k^2 + \frac{1}{\varepsilon^2})} \tag{3.58}$$

> **Remark:** c.f. Laplace transform and Schwinger parameter.

Then

$$\begin{aligned} G(r) &= \frac{1}{\gamma}\int đ^d k \int_0^\infty \mathrm{d}t \, e^{-t(k^2+\varepsilon^{-2}) - i\mathbf{k}\cdot\mathbf{x}} && (3.59)\\ &= \frac{1}{\gamma}\int đ^d k \int_0^\infty \mathrm{d}t \, e^{-t(k+\frac{ix}{2t})^2 - \frac{r^2}{4t} - \frac{t}{\varepsilon^2}} && (3.60)\\ &= \frac{1}{(4\pi)^{d/2}\gamma}\int_0^\infty \mathrm{d}t \, t^{-d/2} e^{-\frac{r^2}{4t} - \frac{t}{\varepsilon^2}} && (3.61)\\ &\sim \int_0^\infty \mathrm{d}t \, e^{-S(t)}, && (3.62) \end{aligned}$$

where we can use the saddle point approximation

$$\begin{aligned} S(t) &= \frac{r^2}{4t} + \frac{t}{\varepsilon^2} + \frac{d}{2}\log(t) && (3.63)\\ &\approx S(t_*) + \frac{S''(t_*)t^2}{2} + \dots, \qquad t_* = \frac{\varepsilon^2}{2}\left(-\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{r^2}{\varepsilon}}\right). && (3.64) \end{aligned}$$

There are two interesting limits.

1. $r \gg \varepsilon$: We find that t_* scales as $t_* \sim \frac{1}{2} r\varepsilon$. This means that the Green's function scales as

$$G(r) \sim \varepsilon^{-(d-3)/2} \frac{e^{-r/\varepsilon}}{r^{(d-1)/2}}. \tag{3.65}$$

   The correlations are dying off exponentially fast; on large distance scales, it looks like a completely uncorrelated theory. There are no modes propagating easily on large distances.

2. $r \ll \varepsilon$: We have $t_* \sim \frac{r^2}{4d}$, and therefore $G(r) \sim r^{2-d}$. On small distances, we have the same polynomial scaling as one would expect from dimensional analysis. There correlations and modes that propagate freely.

In particle physics, this ε is related to the mass of the particle. Heavy particles give the Yukawa potential and they do not propagate. Light or massless particles, like with gravity or electromagnetism, have polynomial behaviour as in the second type. For higher order interactions, like ϕ^4, we would not be able to solve this explicitly.

3.9 Green's Functions

Consider a multi-dimensional Gaussian. This is not a path integral, but rather a large but finite number of integrals. We can then diagonalise this, separating this integral into n Gaussian integrals. This will evaluate to the square root of the product of the eigenvalues, which gives the determinant

$$\int_{-\infty}^{\infty} \mathrm{d}^n y\, e^{-\frac{1}{2}\mathbf{y}^T\cdot G^{-1}\cdot\mathbf{y}} = \sqrt{\det(2\pi G)}. \tag{3.66}$$

Similarly, we can deal with an extra linear term $\mathbf{B}^T \cdot \mathbf{y}$ by changing variables and completing the square. This gives

$$\int_{-\infty}^{\infty} \mathrm{d}^n y\, e^{-\frac{1}{2}\mathbf{y}^T\cdot G^{-1}\cdot\mathbf{y}+\mathbf{B}^T\cdot\mathbf{y}} = \sqrt{\det(2\pi G)}e^{\frac{1}{2}\mathbf{B}^T\cdot G\cdot\mathbf{B}}. \tag{3.67}$$

Let us apply this to path integrals now. Consider again the action

$$\begin{aligned} F[\phi(x)] &= \int \mathrm{d}^d x \left[\frac{1}{2}\phi(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}\mu^2\phi^2 + B\phi\right] && (3.68)\\ &= \iint \mathrm{d}^d x\, \mathrm{d}^d y\, \frac{1}{2}\phi(x)G^{-1}(x,y)\phi(y) + \int \mathrm{d}^d x\, B(x)G(x) && (3.69)\end{aligned}$$

Then we have the inverse $G^{-1}(x,y) = \delta^d(x-y)(-\gamma\nabla_y^2+\mu^2)$ in the sense that it acts on $G(x,y)$ to give, in analogy to matrix multiplication,

$$\int \mathrm{d}^4 z\, G^{-1}(x,z)G(z,y) = \delta^d(x,y). \tag{3.70}$$

The correlation function is then

$$(-\phi\nabla_x^2+\mu^2)\langle\phi(x)\phi(0)\rangle = \frac{1}{\beta}\delta^d(x). \tag{3.71}$$

This is telling us that the fluctuations solve the Euler-Lagrange equations.

Remark: Green's functions allow us to solve differential equations: Consider a differential operator $O(x)$ acting on a function $F(x)$ to give

$$O(x)F(x) = H(x). \tag{3.72}$$

If we can solve the equation

$$O(x)G(x,y) = \delta^d(x-y), \tag{3.73}$$

then we can solve equation (3.72) by integrating

$$F(x) = \int \mathrm{d}^d y\, H(y)G(x-y) \tag{3.74}$$

so that as expected we have

$$O(x)F(x) = \int \mathrm{d}^d y\, H(y)\delta^d(x-y) = H(x). \tag{3.75}$$

3.9.1 Connection to Susceptibility

In the mean field theory, we defined the magnetic susceptibility to be $\chi = \partial m/\partial B$. In the continuum limit $m \to \phi(x)$, where we go from mean-field theory to a theory with local fluctuations, we can define a local version of the magnetic susceptibility as

$$\chi(x,y) \coloneqq \frac{\delta\phi(x)}{\delta B(y)} = \beta\langle\phi(x)\phi(y)\rangle \sim \int \mathcal{D}\phi\, \frac{\delta\phi}{\delta B} e^{\beta F[\phi]} \tag{3.76}$$

We can also define a global susceptibility by integrating over all space

$$\chi = \int \mathrm{d}^d x\, \chi(x,0) = \beta \int \mathrm{d}^d x\, \langle\phi(x)\phi(0)\rangle. \tag{3.77}$$

We can now start to ask a few more physical questions. Recall that we found the functional form of the correlator to have a particular scaling behaviour:

$$\langle\phi(x)\phi(y)\rangle \sim \begin{cases} \frac{1}{r^{d-2}} & r \ll \varepsilon \\ \frac{e^{-r/2}}{r^{(d-1)/2}} & r \gg \varepsilon. \end{cases} \implies \mu^2 \sim |T - T_c|, \text{ and } \xi \sim \frac{1}{|T-T_c|^{1/2}}. \tag{3.78}$$

This means that correlations are related across the entire system, across all distance scales, around the critical temperature T_c. Physically, we intuitively understand that at high temperatures $T \gg T_c$, there is so much energy in the system, so many fluctuations, that different parts of the system are practically uncorrelated. As we lower the temperature, the fluctuations decrease in magnitude and the size of patches with aligned spins grows, meaning that the correlation length starts to grow. As $T \to T_c$, we find patches of all shapes and sizes at all scales, as the correlation length goes to infinity.

3.10 Critical Exponents (Again...)

The correlation length goes like $\xi \sim |T - T_c|^{-\nu}$, where $\nu = \frac{1}{2}$ and $\langle\phi(x)\phi(y)\rangle \sim r^{-(d-2+\eta)}$, where $\eta = 0$.

	MFT	$d=2$	$d=3$
η	0	$\frac{1}{4}$	$0.0363\ldots$
ν	$\frac{1}{2}$	1	0.6300

Table 3.1: caption

3.10.1 Upper Critical Dimension

Recall that from mean-field theory, the one point correlation is $\langle\phi(x)\rangle = \pm m_0$. The path integral told us that there are also fluctuations. A sensible question to ask is: How big are the fluctuations relative to m_0? This tells us whether the mean-field theory is good or bad.

$$R = \frac{\int_0^\xi \mathrm{d}^d x \, \langle \phi(x)\phi(0) \rangle}{\int_0^\xi \mathrm{d}^d x \, m_0^2} \sim \frac{1}{m_0^2 \xi^d} \int_0^\xi \mathrm{d}r \, \frac{r^{d-1}}{r^{d-2}} \sim \frac{\xi^{2-d}}{m_0^2} \tag{3.79}$$

where we used that in polar coordinates, $\mathrm{d}^d x = r^{d-1} \, \mathrm{d}r \, \mathrm{d}\Omega_d$. Therefore $m_0 \sim |T - T_c|^{1/2}$ and $\xi = |T - T_c|^{-1/2}$ implies that $R = |T - T_c|^{(d-4)/2}$.

3.11 Translating to QFT

There is a very strong and non-accidental connection between the tools we are using in this course and the tools used in quantum field theory. In statistical field theory, we have a path integral $Z = \int \mathcal{D}\phi \, e^{-\beta \int \mathrm{d}^d x F[\phi]}$. Here, d is entirely spatial. There is no imaginary term.

In Quantum field theory, we have

$$Z = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar} \int \mathrm{d}^d x \mathcal{L}[\phi]}. \tag{3.80}$$

Here d is spacetime and we have a factor of $\frac{i}{\hbar}$.

To translate from one to the other, we can use a *Wick rotation*, which is a change of coordinates such that $\tau = it$. This means that our operators become

$$\partial_t^2 - \nabla^2 \to -(\partial_\tau^2 + \nabla^2). \tag{3.81}$$

The quantum field theory path integral then becomes a Euclidean path integral such as the one in statistical field theory:

$$Z = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar} S[\phi]} \to \int \mathcal{D}\phi \, e^{-\frac{1}{\hbar} S_E[\phi]}. \tag{3.82}$$

High energy physics	Statistical field theory
Quantum fluctuations	Thermal fluctuations
QFT in $d-1$ dimensions	SFT in d dimensions
$\hbar$	β

Table 3.2: Analogy between QFT and SFT.

There are places where this analogy breaks down. For example, there are situations where there are important topological properties of QFT. In this case, the analogy becomes limited; it is difficult to phrase topological questions in the language of statistical mechanics.

4 The Renormalisation Group

The conceptional tools that we learn in the context of RG could be some of most important theoretical ideas of the last century. There are aspects of RG that we knew before, but the framework, developed by Kadanoff, Fischer, Wilson, and others, gave a unifying picture of these ideas. We will mostly be working in momentum space; these guys mostly worked in real space, using the language of operator product expansions.

> **Remark:** Wilson was a genius, but famously did not publish very often; he had published something like two papers when Pauli suggested for him to be tenured at Cornell. Nowadays, another Wilson would not exist; he likely would not even get a postdoc...

4.1 Basic Introduction

Consider a very general free energy

$$F[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}\mu^2\phi^2 + g\phi^4 + \cdots + g_{126}\phi^{126} + \cdots + \tilde{g}_2(\nabla^2\phi)^2 + \ldots\right], \tag{4.1}$$

where $\mu^2 \sim T - T_c$. The form of F is only restricted by symmetry and analyticity. We refer to the g_i as *Wilson coefficients*. And we will be consistent in referring to the free energy by the more general term *action*. The partition function is then

$$Z = \int \mathcal{D}\phi \, e^{-F[\phi]}. \tag{4.2}$$

However, to fully describe the validity of the theory, we also need to specify a cutoff $\Lambda \sim \frac{\pi}{a}$, such that the high momentum fields vanish $\phi_{\mathbf{k}} = 0$ if $|\mathbf{k}| > \Lambda$. The renormalisation group (RG) answers the question of what happens to the coupling constants when this cutoff is changed. We usually refer to change of the Wilson coefficients as RG *flow*.

4.2 Flowing to the IR

The IR (infrared) is a historical term that refers to long distances. Similarly, going to the UV (ultraviolet) refers to smaller distances—often used in synonym for high energy theories that we do not yet know about.

Imagine we only care about predictions on distance scales $L \gg a$, where a is the original coarse-graining scale. We can then define a theory with a new, slightly lower cutoff $\Lambda' = \Lambda/\varepsilon$, where $\varepsilon > 1$. This means that all Fourier modes with $1/L < |\mathbf{k}| < \Lambda$ have to be accounted for. With this new cutoff, something must have happened to the Fourier modes which were between Λ' and Λ that are now gone. We say that these modes have been *integrated out*. As we will see, this will be a well-defined concept.

We can expand the Fourier modes as

$$\phi_{\mathbf{k}} = \phi^-_{\mathbf{k}} + \phi^+_{\mathbf{k}}, \tag{4.3}$$

where

$$\phi^-_{\mathbf{k}} = \begin{cases} \phi_{\mathbf{k}} & |\mathbf{k}| < \Lambda' \\ 0 & |\mathbf{k}| > \Lambda' \end{cases}, \qquad \phi^+_{\mathbf{k}} = \begin{cases} \phi_{\mathbf{k}} & \Lambda' < |\mathbf{k}| < \Lambda \\ 0 & |\mathbf{k}| > \Lambda \end{cases}. \tag{4.4}$$

We refer to ϕ^- as the IR modes and ϕ^+ as the UV modes. The action then splits into the form

$$F[\phi] = F_0[\phi^-] + F_0[\phi^+] + F_I[\phi^-, \phi^+], \tag{4.5}$$

where F_0 is the Gaussian part and F_I is the non-Gaussian part, which is often called the *interaction part*, of the action.

4.3 Partition Function

Let us see what happens to the partition function Z on such a split of Fourier modes:

$$\begin{aligned} Z &= \int \left[\prod_{|\mathbf{k}|<\Lambda} \mathrm{d}^d\phi_{\mathbf{k}} \right] e^{-F[\phi_{\mathbf{k}}]} && (4.6) \\ &= \int \left[\prod_{|\mathbf{k}|<\Lambda'} \mathrm{d}^d\phi^-_{\mathbf{k}} \right] e^{-F_0[\phi^-_{\mathbf{k}}]} \int \left[\prod_{\Lambda'<|\mathbf{k}|<\Lambda} \mathrm{d}^d\phi^+_{\mathbf{k}} \right] e^{-F_0[\phi^+_{\mathbf{k}}} e^{-F_I[\phi^+_{\mathbf{k}},\phi^+_{\mathbf{k}}]} && (4.7) \\ &= \int \left[\prod_{|\mathbf{k}|<\Lambda'} \mathrm{d}^d\phi^-_{\mathbf{k}} \right] e^{-F'[\phi^-_{\mathbf{k}}]}. && (4.8) \end{aligned}$$

This defines the *Wilsonian effective action* F'. In general, this will be of the same form as our original action in Equation (4.1), except that the coupling constants are shifted

$$F'[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}\gamma'(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}\mu'^2\phi^2 + g'\phi^4 + \dots \right]. \tag{4.9}$$

However, we cannot directly compare F' and F to see how the theory is changed. This is because of the different cutoffs in the Fourier integrals.

4.4 Momentum Rescaling

To compare the theories, we need to restore the cutoff from Λ' to Λ of the original theory. We therefore rescale $\mathbf{k} \to \mathbf{k}' = \zeta\mathbf{k}$, which corresponds to $\mathbf{x} \to \mathbf{x}' = \mathbf{x}/\zeta$, where $\zeta = \Lambda/\Lambda'$. Finally, we can also also do a field rescaling $\phi \to \phi'$ to absorb γ' into the definition of the field: $\gamma'(\boldsymbol{\nabla}\phi)^2 \to (\boldsymbol{\nabla}\phi')^2$. Note that these rescalings do not change any physical predictions; they only let us compare the two theories with each other. The new action, with the rescaled modes and fields, is

$$F_\zeta[\phi'] = \int \mathrm{d}^d x \left[\frac{1}{2}(\boldsymbol{\nabla}\phi')^2 + \frac{1}{2}\mu^2(\zeta)\phi'^2 + g(\zeta)\phi'^4 + \dots\right]. \tag{4.10}$$

If $\varepsilon \approx 1$—if we only integrate out an infinitesimal region in Fourier space close to the cutoff—then the coupling constants change continuously. We can apply the RG transformations repeatedly to obtain an RG flow in the space of values of coupling constants.

This space of coupling constant configurations might have *basins of attractions*—regions from which you always flow to a certain fixed points, no matter which configuration of $\{g_i\}$ you started with. At these fixed points, the theory does not change anymore under a rescaling. This scale invariance is mathematically expressed in *conformal symmetry*. These conformal field theories are extremely interesting to study! The flow of the coupling constants with scale is described by the *beta functions* $\mathrm{d}g/\mathrm{d}\zeta \propto \sum \beta_i$. Note that the renormalisation group is not really a group, since there does not exist a unique inverse to each transformation—it is a semi-group.

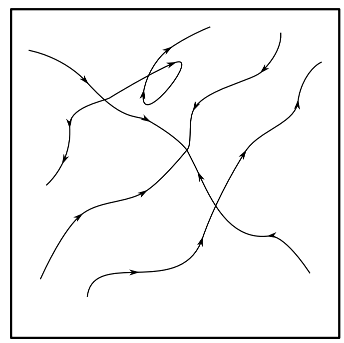
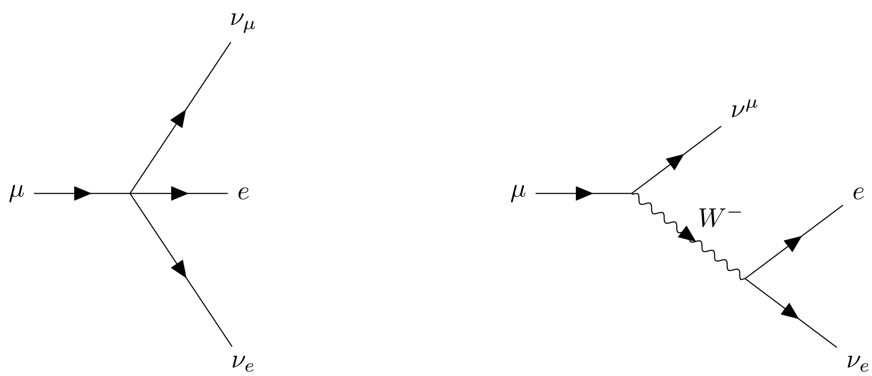


Figure 4.1: RG flow in parameter space

Example: Beta decay was historically understood (before we discovered the W and Z bosons) as a four field interaction as depicted in Figure 4.2a. However this neglected the UV interactions. Nowadays, we understand this interaction to be mediated by a W-boson as depicted in Figure 4.2b.



(a) IR theory: the four fields interact at a point. (b) UV theory: the interaction is mediated by a W-boson.

Figure 4.2: Meson decay

Let us recap. The renormalisation tells us how the theory changes on coarse graining to a different scale. The renormalisation steps are

1. 'Integrate out' high momentum modes, $1/\zeta < |\mathbf{k}| < \Lambda$
2. Rescale momenta: $\mathbf{k} \to \mathbf{k}' = \zeta\mathbf{k}$
3. Rescale fields such that

$$F_\zeta[\phi'] = \int \mathrm{d}^d x \left[\frac{1}{2}(\boldsymbol{\nabla}\phi')^2 + \frac{1}{2}\mu^2(\zeta)\phi'^2 + g(\zeta)\phi'^4 + \dots\right] \tag{4.11}$$

Previously, we included all possible interaction terms in F. Therefore, the new action will be of the same form, except with different coupling constants. As we have briefly discussed, under the RG, the coupling constants can flow in different ways. We will discuss this further in upcoming sections.

4.5 Returning to Universality

We discussed the idea of universality when we introduced the path integral and looked at how fluctuations can change the model. For example, the phase transition of a ferromagnet can be described by the same physical theory as the phase transition in liquid to gas, or even in early universe cosmology.

Let us think about where we will land if we coarse-grain to larger and larger distance scales. There are two possibilities:

- couplings flow to infinity
- couplings converge towards a fixed point
  - theory becomes scale-invariant
  - the correlation length becomes either $\varepsilon = 0$ or $\varepsilon = \infty$

Example: Performing the RG to the theory of QCD, we flow towards infinity as we go towards the IR. In the perturbative framework, we cannot make calculations anymore.

Example: In QED, below $500 KeV$ we go below the point where we integrate out electrons. We end up with a scale-invariant theory with free photons that do not interaction.

> **Remark:** Q: can we have loops in RG parameter space?
> A: Schwimmer and Commagautzki (?) proved in 2010 (?) that RG cannot flow back on itself. This is related to the A and C theorems in conformal field theory.

4.5.1 Ising Model Critical Points

Let us begin at $T > T_c$ and increase the temperature $T \to \infty$. Physically, we know that at these high temperatures the correlation length will vanish $\varepsilon \to 0$. However, if we start at $T < T_c$ and let $T \to 0$. All the spins will line up, there are no fluctuations and correlation is maximal. Therefore, there is zero correlation length $\varepsilon \to 0$ also in this case. We see that the trivial fixed points have $\varepsilon \to 0$; the most interesting fixed points will be those with $\varepsilon \to \infty$; this will happen for the Ising model at $T = T_c$. These fixed points, where we have fluctuations on all length-scales, are known as *critical points*. In general, we expect to find these at second order phase transitions.

4.5.2 Relevant, Irrelevant, Marginal

We can deform our theory by changing one of the values of the couplings. For *irrelevant* operators, the deformation does not change the flow behaviour; we end up at the same point in RG parameter space. The perturbed theory is in the same universality class as the original one. On the other hand, *relevant* deformations put us into a different universality class. In practice, there are usually not that many relevant operations. The last kind of deformation we can perform is *marginal* deformation.

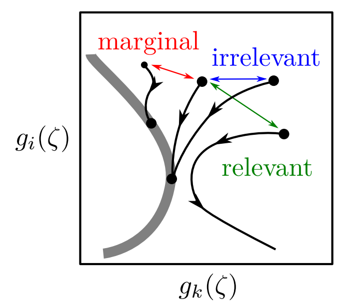


Figure 4.3

4.5.3 Scaling

Let us think about how particular physical quantities change under rescaling. In a scale-invariant system, a correlation function $\langle\phi(\mathbf{x})\phi(0)\rangle$ can only depend on $r = |\mathbf{x}|$. We have

$$\langle\phi(\mathbf{x})\phi(0)\rangle \sim \frac{1}{r^{d-2+\eta}}. \tag{4.12}$$

Let us use the RG to calculate η. In units of $[\text{length}]^{-1}$, we have

$$[x] = -1, \qquad [\frac{\partial}{\partial x}] = +1, \qquad F[\phi] = 0, \qquad [\phi(\mathbf{x})] = \frac{d-2}{2}, \qquad \Longrightarrow \eta = 0. \tag{4.13}$$

A hand-waving explanation is that under the RG procedure, $\mathbf{x} \to \mathbf{x}/\zeta$, so any other quantity will be rescaled as $\phi(\mathbf{x}) \to \zeta^{\Delta\phi}\phi(\mathbf{x})$, where

$$\Delta_\phi = \frac{d-2+\eta}{2}, \tag{4.14}$$

where $[\phi(\mathbf{x})] = (d-2)/2$ is called the *engineering dimension* and η the *anomalous dimension*. There is no reason to think that the exponent of ζ in the RG rescaling should match the engineering dimension. If we remember that we can have factors of the cutoff a, this anomalous scaling can be matched

$$\langle\phi(\mathbf{x})\rangle \sim \frac{a^n}{r^{d-2+\eta}}. \tag{4.15}$$

4.5.4 Return to Critical Exponents

We defined the correlation length $\varepsilon \sim t^{-\nu}$. By definition, ε is a length scale with scaling dimension $\Delta_\varepsilon = -1$. This implies that $t \to \zeta^{\Delta t} t$ with $\Delta t = 1/\nu$.

Consider the free energy at $B = 0$. This will be a function of the reduced temperature:

$$F_{\text{thermo}}(t) = \int \mathrm{d}^d x\, f(t) \implies f(t) \sim t^{d\nu}. \tag{4.16}$$

Recall that the total heat capacity was defined as $c = \partial^2 f \big/ \partial t^2 \sim t^{d\nu - 2} \sim t^{-\alpha}$, with $\alpha = 2 - d\nu$. Moreover, the magnetisation is $\phi \sim t^\beta$. This implies that $\Delta\phi = \beta\Delta t$ and so $\beta = \nu\Delta\phi$. Therefore $\beta = (f - 2 + \eta)/2\nu$. Let us now add a magnetic field $\int \mathrm{d}^d x\, B\phi$. This gives $\Delta_B = (d + 2 - \eta)/2\nu$. Finally, the susceptibility scales as: $\chi = \frac{\partial\phi}{\partial B}\Big|_T \sim t^{-\gamma} \implies \gamma = \nu(2 - \eta)$.

Heat Capacity	$c = \frac{\partial^2 f}{\partial t^2} \sim t^{-\alpha}$	$\alpha = 2 - d\nu$
Magnetisation	$\phi \sim t^{-\beta}$	$\beta = \frac{d-2+\mu}{2}$
Susceptibility	$\chi \sim t^{-\gamma}$	$\gamma = \nu(2-\eta)$
External Field	$\phi \sim B^{1/\delta}$	$\delta = \frac{d+2-\eta}{d-2+\eta}$

Table 4.1: Critical Exponents

From 4.1, we can see that we have found relations between exponents that writes the four parameters in terms of only ν and η.

	α	β	γ	δ	η	ν
MF	$(4-d)/2$	$1/2$	1	3	0	$1/2$
$d=2$	0	$1/8$	$7/4$	15	$1/4$	1
$d=3$	0.11	0.33	1.24	4.79	0.04	0.63

Table 4.2

This shows that the scaling arguments that we employed are extremely powerful. We were able to recover physical relationships between things like magnetisation and heat capacity by simply considering the scaling of certain quantities.

4.5.5 Relevant, Irrelevant, Marginal?

We saw that the free energy is an integral over many terms. Amongst these, we have terms of the following form

$$F[\phi] \sim \int \mathrm{d}^d x \, \{\cdots + g_O O(x) + \dots\} \tag{4.17}$$

Suppose that $O(\mathbf{x}) \to O'(\mathbf{x}') = \xi^{\Delta O} O(\mathbf{x})$. If we have an operator that scales like this, we know by dimensional analysis that the scaling dimension of the associated coupling constant is $\Delta g_O = d - \Delta O$. Now we have various possible cases. We say that the operator is *relevant* if $\Delta O < d$, *irrelevant* if $\Delta 0 > d$, and *marginal* if $\Delta O = d$.

4.5.6 The Gaussian Fixed Point

Consider a free action

$$F_0[\phi] = \int_{\mathbb{R}^d} \mathrm{d}^d x \left[\frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}\mu_0^2\phi^2\right] = \int_0^\Lambda \text{đ}^d k \, \frac{1}{2}(|\mathbf{k}|^2 + \mu_0^2)\phi_{\mathbf{k}}\phi_{-\mathbf{k}}. \tag{4.18}$$

Following the RG steps, we split the field into UV and IR fields, giving

$$F_0[\phi] = F_0[\phi^-] + F_0[\phi^+]. \tag{4.19}$$

The Wilsonian effective action is then obtained by considering

$$e^{-F'[\phi']} = \int \mathcal{D}\phi^+ \, e^{-F_0[\phi^+]} e^{-F_0[\phi^-]} = N e^{-F_0[\phi^-]}. \tag{4.20}$$

We then need to rescale $\mathbf{k} \to \mathbf{k}' = \zeta \mathbf{k}$ and $\phi_{\mathbf{k}} \to \phi'_{\mathbf{k}} = \zeta^{-\omega}\phi^-_{\mathbf{k}}$. Then, the effective action is

$$F_0[\phi'] = \int_0^\Lambda \text{đ}^d k \, \frac{1}{2\zeta^d}\left(\frac{k^2}{\zeta^2} + \mu_0^2\right)\zeta^{2\omega}\phi'_{\mathbf{k}}\phi'_{-\mathbf{k}}. \tag{4.21}$$

To make this agree with the previous action, we rescale $\mu^2(\zeta) = \zeta^2\mu_0^2$. Therefore, the correlation length $\epsilon \sim 1/\mu^2$ also gets rescaled to $\epsilon \to \epsilon/\zeta$. We have fixed points whenever $\frac{\mathrm{d}\mu^2}{\mathrm{d}\zeta} = 0$. There are only two possibilities for this. Either, $\mu^2 = \infty$. Since $\mu^2 \sim (T - T_c)$, this corresponds to a state of infinite temperature. The other possibility is $\mu^2 = 0$. This is the interesting one because the correlation length becomes infinite. We call this the *Gaussian fixed point*. Even within this very simple calculation, the RG analysis allows us to tell that the interesting physics lies at the point where the correlation length becomes infinite.

Around the Fixed Point

Let us now look at some of the other couplings. We can consider a general action with higher order interactions

$$F[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}\mu_0^2\phi^2 + \sum_{n=4}^{\infty} g_{0,n}\phi^n\right] \tag{4.22}$$

After splitting the fields, we then get an extra interaction term $F_0[\phi^-, \phi^+]$, which mixes the low and high wavenumbers, as compared to (4.19) when splitting the action

$$F_0[\phi] = F_0[\phi^-] + F_0[\phi^+] + F_0[\phi^-, \phi^+]. \tag{4.23}$$

However, let us first consider a simple spatial rescaling. When rescaling $\mathbf{x} = \mathbf{x}/\zeta$, and $\phi'(\mathbf{x}) \to \zeta^{\Delta\phi}\phi(\mathbf{x})$, we get

$$F'[\phi'] = \int \mathrm{d}^d x \, \zeta^d \left[\frac{1}{2}\zeta^{-2-2\Delta\phi}(\boldsymbol{\nabla}'\phi')^2 + \frac{1}{2}\mu_0^2\zeta^{-2\Delta\phi}\phi'^2 + \sum g_{0,n}\zeta^{-n\Delta\phi}\phi'^n\right]. \tag{4.24}$$

We see that $g_n(\zeta) = \zeta^{d-n\Delta\phi}g_{0,n}$. In particular, considering the ϕ^4 coupling is irrelevant for $d > 4$, marginal for $d = 4$, and relevant for $d < 4$. This model shows us why $d = 4$ is so special. As we are going toward the IR, this tells us that there is an enormous difference between $d < 4$ and $d > 4$. We can tune our coupling constants in such a way that we are exactly at the Gaussian fixed point. In general, we will have to tune each relevant parameter in the theory to hit this fixed point. This can be seen in figure , explaining the difference of the Ising model and liquid-gas phase transitions.

4.5.7 Symmetry Breaking

We have already seen that turning on a magnetic field B in the Ising model corresponds to a $\mathbb{Z}_2$-breaking. There is a very important thing to know: Typically, the RG evolution will not break symmetries. We started off with some action which respected some symmetry. This symmetry is independent of momentum. This means that when breaking the fields into UV and

IR parts, the new action (4.23) will also respect the symmetry. The statement of symmetry is an RG independent statement. However, we can also have *spontaneous symmetry breaking*, where the symmetry is still present, but invisible in the IR. If we turn on a magnetic field, we have $F \sim \alpha\phi$.

4.6 RG with interactions

Recall the action for ϕ^4 theory:

$$F[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}\mu_0^2\phi^2 + g_0\phi^4\right]. \tag{4.25}$$

Splitting into $\phi_{\mathbf{k}} = \phi_{\mathbf{k}}^- + \phi_{\mathbf{k}}^+$ gives

$$F[\phi] = F_0[\phi^-] + F_0[\phi^+] + F_I[\phi^-, \phi^+]. \tag{4.26}$$

So now the Wilsonian effective action is obtained via the integral

$$e^{-F'[\phi^-]} = e^{-F_0[\phi_{\mathbf{k}}^-]} \int \mathcal{D}\phi_{\mathbf{k}}^+ \, e^{-F_0[\phi_{\mathbf{k}}^+]} e^{-F_I[\phi_{\mathbf{k}}^-, \phi_{\mathbf{k}}^+]} \tag{4.27}$$

$$:= e^{-F_0[\phi_{\mathbf{k}}^-]} \left\langle e^{-F_i[\phi_{\mathbf{k}}^-, \phi_{\mathbf{k}}^+]} \right\rangle]_+. \tag{4.28}$$

Taking logarithms, we can expand

$$\log \left\langle e^{-F_I[\phi_{\mathbf{k}}^-, \phi_{\mathbf{k}}^+]} \right\rangle] \simeq -\langle F_I \rangle_+ + \frac{1}{2}[\langle F^2 \rangle_+ - \langle F_I \rangle_+^2] + \dots \tag{4.29}$$

First Order Expansion

To first order in g_0, we have

$$F_I[\phi_{\mathbf{k}}^-, \phi_{\mathbf{k}}^+] = g_0 \int \left[\prod_{i=1}^4 \text{đ}^d k_i\right] (\text{terms with } \phi) \times \tilde{\delta}^d(\sum \mathbf{k}_i) \tag{4.30}$$

When splitting the fields $\phi_{\mathbf{k}}$ into UV and IR modes, we get various possible products of four fields. The possibilities are

1. All UV ($\phi_{\mathbf{k}_1}^- \phi_{\mathbf{k}_2}^- \phi_{\mathbf{k}_3}^- \phi_{\mathbf{k}_4}^-$): This is trivial since the integral does not affect any of the UV fields.
2. One IR ($4\phi_{\mathbf{k}_1}^- \phi_{\mathbf{k}_2}^- \phi_{\mathbf{k}_3}^- \phi_{\mathbf{k}_4}^+$): Odd number of IR fields vanish when integrated over the Gaussian ensemble.
3. Two IR ($6\phi_{\mathbf{k}_1}^- \phi_{\mathbf{k}_2}^- \phi_{\mathbf{k}_3}^+ \phi_{\mathbf{k}_4}^+$): This is the interesting term, which is discussed below.
4. Three IR: Vanishes as above.
5. All IR ($\phi_{\mathbf{k}_1}^+ \phi_{\mathbf{k}_2}^+ \phi_{\mathbf{k}_3}^+ \phi_{\mathbf{k}_4}^+$): The integral gets rid of all the fields; this contributes only a constant to the action, which does not change any of the connected correlation functions, and therefore does not alter any of the physics.

We thus only have to calculate the contribution of the third term. That is

$$6g_0 \int \left[\prod_{i=1}^4 \text{đ}^d k_i\right] \phi_{\mathbf{k}_1}^- \phi_{\mathbf{k}_2}^- \langle \phi_{\mathbf{k}_3}^+ \phi_{\mathbf{k}_4}^+ \rangle_+ \tilde{\delta}^d(\sum \mathbf{k}_i) \tag{4.31}$$

Since $\langle \dots \rangle_+$ is an integral over the Gaussian ensemble only, we can now use our previous result for the free propagator G_0 of a Gaussian path integral:

$$\langle \phi^+_{\mathbf{k}_3} \phi^+_{\mathbf{k}_4} \rangle_+ = \delta\!\!\bar{}^d(\mathbf{k}_3 + \mathbf{k}_4) G_0(\mathbf{k}_3), \qquad G_0(k) = \frac{1}{k^2 + \mu_0^2}, \tag{4.32}$$

where we write as always $k^2 := |\mathbf{k}|^2$. Inserting this and using the $\delta\!\!\bar{}$-function, and relabelling the integral over the ϕ^+ fields as $\mathbf{k} \to \mathbf{q}$, we have

$$\dots = 6 g_0 \int_0^{\Lambda/\zeta} \mathrm{đ}^d k \, \phi^-_{\mathbf{k}} \phi^-_{-\mathbf{k}} \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{đ}^d q}{q^2 + \mu_0^2}. \tag{4.33}$$

We see that this is term is quadratic in ϕ^- and therefore changes the value of the mass term μ_0 to

$$\mu_0^2 \to \mu'^2 = \mu_0^2 + 12 g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{đ}^d q}{q^2 + \mu_0^2} \tag{4.34}$$

Finally, we conclude the RG transformations by rescaling $\mathbf{k}' = \zeta \mathbf{k}$ and $\phi'_{\mathbf{k}} = \zeta^{-\omega} \phi^+_{\mathbf{k}/\zeta}$, $\omega = (d+2)/2$. Restoring the form of the action, this means that we have to absorb an overall scaling of ζ^2 to the mass term. In particular, we end up with a renormalised mass coupling constant

$$\mu^2(\zeta) = \zeta^2 \left[\mu_0^2 + 12 g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{\mathrm{đ}^d q}{q^2 + \mu_0^2} \right]. \tag{4.35}$$

Remark: This is related to the *hierarchy problem* in particle physics: for the Higgs mass, we do not know why the μ_0 term is so small. In particle physics, we do not know what the macroscopic modes are. It is then surprising that $\mu^2(\zeta)$ is small, although we expect the γ_0 terms to contribute; there are theories like *Supersymmetry*, where these interaction contributions precisely cancel.

We find that even if at some length scale $\mu_0 = 0$, then in the presence of the interaction term $g_0 \phi^4$, this is not RG stable; it will become non-zero in the large-scale theory.

Second Order Expansion

The second term $\frac{1}{2}[\langle F_I^2 \rangle_+ - \langle F_I \rangle_+^2]$ contributes 256 terms! However, thankfully most of these will cancel. Let us consider a non-vanishing term: Inside $\frac{1}{2}\langle F_I^2 \rangle_+$ there are terms of the form

$$\frac{1}{2} \frac{4!}{2!} g_0^2 \int_0^{\Lambda/\zeta} \left[\prod_{i=1}^4 \mathrm{đ}^d k_i \, \phi^-_{\mathbf{k}_i} \right] \int_{\Lambda/\zeta}^{\Lambda} \left\langle \left[\prod_{j=1}^4 \mathrm{đ}^d q_j \, \phi^+_{\mathbf{q}_j} \right] \right\rangle_+ \delta\!\!\bar{}^d(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{q}_1 + \mathbf{q}_2) \delta\!\!\bar{}^d(\mathbf{k}_3 + \mathbf{k}_4 + \mathbf{q}_3 + \mathbf{q}_4) \tag{4.36}$$

To deal with the term $\langle \phi^+_{\mathbf{q}_1} \phi^+_{\mathbf{q}_2} \phi^+_{\mathbf{q}_3} \phi^+_{\mathbf{q}_4} \rangle$, we will introduce Wick's theorem (as we have seen in *Quantum Field Theory*).

4.6.1 Wick's Theorem

To prove Wick's theorem, we will first introduce a Lemma.

Lemma 1: Let ϕ be a vector of n variables and let G be an $n \times x$ matrix. We then define $\langle \dots \rangle_G$ to mean

$$\langle f(\phi) \rangle_G \coloneqq \frac{1}{N} \int_{-\infty}^{\infty} \mathrm{d}^n\phi \, f(\phi) e^{-\frac{1}{2}\phi^T G^{-1} \phi}, \tag{4.37}$$

where $N = \sqrt{\det(2\pi G)}$. Under this definition, we have

$$\langle e^{B_a \phi_a} \rangle = e^{\frac{1}{2} B_a \langle \phi_a \phi_b \rangle B_b}, \tag{4.38}$$

where we use the Einstein summation convention.

Proof. We will complete the square in the exponential

$$\langle e^{B_a \phi_a} \rangle = \frac{1}{N} \int_{-\infty}^{\infty} \mathrm{d}^n\phi^T \, e^{-\frac{1}{2}\phi G^{-1} \phi + B\phi} \tag{4.39}$$

$$= \frac{1}{N} \int_{-\infty}^{\infty} \mathrm{d}^n\phi \, e^{-\frac{1}{2}(\phi - GB)^T G^{-1} (\phi - GB)} e^{\frac{1}{2} B^T G B} \tag{4.40}$$

$$= e^{\frac{1}{2} B^T G B} \tag{4.41}$$

$$= e^{\frac{1}{2} B_a \langle \phi_a \phi_b \rangle B_b}, \tag{4.42}$$

where we used that $\langle \phi_a \phi_b \rangle = G_{ab}$. □

Theorem 2 (Wick's Theorem)**:** Any n-point correlation function can be decomposed into the sum of all possible two-point correlation functions that can be formed with the string of fields.

> **Remark:** These two-point functions correspond to the contractions used in the *Quantum Field Theory* course.

Proof. Let us Taylor expand the result from the previous lemma:

$$\langle e^{B_a \phi_a} \rangle = 1 + B_a \langle \phi_a \rangle + \frac{1}{2} B_a B_b \langle \phi_a \phi_b \rangle + \frac{1}{3!} B_a B_b B_c \langle \phi_a \phi_b \phi_c \rangle + \dots \tag{4.43}$$

$$e^{B_a \langle \phi_a \phi_b \rangle B_b} = 1 + \frac{1}{2} B_a B_b \langle \phi_a \phi_b \rangle + \frac{1}{8} B_a B_b B_c B_d \langle \phi_a \phi_b \rangle \langle \phi_c \phi_d \rangle + \text{symmetrise} \dots \tag{4.44}$$

$$\implies \langle \phi_a \phi_b \phi_c \phi_d \rangle = \langle \phi_a \phi_b \rangle \langle \phi_c \phi_d \rangle + \langle \phi_a \phi_c \rangle \langle \phi_b \phi_d \rangle + \text{symmetrise} \dots \tag{4.45}$$

□

Equipped with Wick's theorem, we are now able to deal with the term $\langle\phi^+_{\mathbf{q}_1}\phi^+_{\mathbf{q}_2}\phi^+_{\mathbf{q}_3}\phi^+_{\mathbf{q}_4}\rangle$ in (4.36). Let us insert the result (4.45). It turns out that the term $\langle\phi^+_{\mathbf{q}_1}\phi^+_{\mathbf{q}_2}\rangle\langle\phi^+_{\mathbf{q}_3}\phi^+_{\mathbf{q}_4}\rangle$ is cancelled by the corresponding term in $\langle F_I\rangle^2$. This corresponds to the fact that we only need to calculate connected cumulants in the logarithmic expansion $\log\langle e^{-F_I}\rangle$. The next term we consider is $\langle\phi^+_{\mathbf{q}_1}\phi^+_{\mathbf{q}_3}\rangle\langle\phi^+_{\mathbf{q}_2}\phi^+_{\mathbf{q}_4}\rangle$. The second order contribution to the effective action associated with this term is

$$\int\left[\prod_{j=1}^{4}\mathrm{d}^d q_i\right]\langle\phi^+_{\mathbf{q}_1}\phi^+_{\mathbf{q}_3}\rangle\langle\phi^+_{\mathbf{q}_2}\phi^+_{\mathbf{q}_4}\rangle_+\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{q}_1+\mathbf{q}_2)\delta^d(\mathbf{k}_3+\mathbf{k}_4+\mathbf{q}_3+\mathbf{q}_4) \tag{4.46}$$

$$=\int\mathrm{d}^d q_1\,\mathrm{d}^d q_2\,G_0(q_1)G_0(q_2)\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{q}_1+\mathbf{q}_2)\delta^d(\mathbf{k}_3+\mathbf{k}_4-\mathbf{q}_1-\mathbf{q}_2) \tag{4.47}$$

$$=\int\mathrm{d}^d q\,G_0(q)G_0(|\mathbf{k}_q+\mathbf{k}_2+\mathbf{q}|)\delta^d(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4) \tag{4.48}$$

Where we used out previous result for the propagator:

$$\langle\phi^+_{\mathbf{q}_1}\phi^+_{\mathbf{q}_3}\rangle_+=\delta^d(\mathbf{k}_1+\mathbf{k}_3)G_0(\mathbf{k}_1),\qquad G_0(\mathbf{k}_1)=\frac{1}{\mathbf{k}_1^2+\mu_0^2}. \tag{4.49}$$

As a result, we find that

$$\frac{1}{2}\langle F_I^2\rangle_+\sim\left(\frac{4!}{2!}\right)^2 g_0^2\int_0^{\Lambda/\zeta}\left[\prod_{i=1}^{4}\text{đ}^d k_i\,\phi^-_{\mathbf{k}_i}\right]f(|\mathbf{k}_1+\mathbf{k}_2|)\delta^d(\sum_i\mathbf{k}_i), \tag{4.50}$$

where the function f is

$$f(\mathbf{k})=\int_{\Lambda/\zeta}^{\Lambda}\text{đ}^d q\left[\frac{1}{q^2+\mu_0^2}\times\frac{1}{(\mathbf{k}+\mathbf{q})^2+\mu_0^2}\right]. \tag{4.51}$$

We can Taylor expand this

$$f(\mathbf{k})\approx\int_{\Lambda/\zeta}^{\Lambda}\frac{\text{đ}^d q}{(q^2+\mu_0^2)^2}\left[1+O(|\mathbf{k}|^2)\right]. \tag{4.52}$$

Finally, we rescale

$$g_0\to g_0'=g_0-36g_0^2\int_{\Lambda/\zeta}^{\Lambda}\frac{\mathrm{d}^d q}{(q^2+\mu_0^2)^2}. \tag{4.53}$$

Remark: 'Do not dismiss this calculation, as it underpins your very own existence.'

4.6.2 Feynman Diagrams

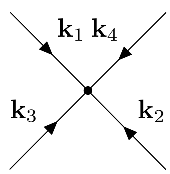
In Feynman diagram language, things like the cancellation of disconnected cumulants are translated very naturally in the fact that only connected diagrams are allowed. When we expand $\log\langle e^{-F_I[\phi^-_{\mathbf{k}},\phi^+_{\mathbf{k}}]}\rangle$, we get terms of the form $g_0^p(\phi^-)^n(\phi^+)^l$. Each of these terms gives an integral. We can keep track of the various different terms arising from this expansion by using Feynman diagrams. Let us recap the Rules (similar to *Quantum Field Theory*):

- Each $\phi^-_{\mathbf{k}}$ is an external solid line.
- Each $\phi^+_{\mathbf{k}}$ is an internal dotted line.
- Dotted lines are always connected at both ends.
- Each vertex joins 4 lines with g_0.
- Each line has momentum $\mathbf{k}$, conserved at vertices.
- Symmetry factors

> **Remark:** A common misconception is that Feynman diagrams are just a useful visualisation for the physics of scattering of particles. However, Feynman diagrams in this context are much more than that; we should think of each diagram as an equation; computing terms in the perturbative expansion of the effective action with the above rules.

At lowest order, we have the following diagrams:

'Tree'

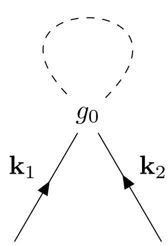


$$\implies g_0 \int \mathrm{d}^4x\,(\phi^-)^4 \tag{4.54}$$

'Loop'

$$\sim g_0(\phi^+)^4 \tag{4.55}$$

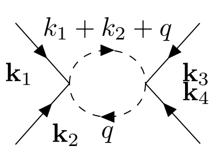
Interesting



$$= 6g_0 \int_{\Lambda/\zeta}^{\Lambda} \frac{đ^d q}{q^2 + m_0^2} \int \mathrm{d}^d x\,(\phi^-)^2 \tag{4.56}$$

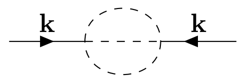
> **Remark:** Symmetry factor: how many ways are there to connect two lines to four dots?

At second order, we have the following diagrams



$$\implies 36g_0^2 \int_0^{\Lambda/\zeta} \left[\prod_{i=1}^{4} đ^d k_i\,\phi^-_{\mathbf{k}_i}\right] f(\mathbf{k}_1 + \mathbf{k}_2)\bar{\delta}^d(\sum_i \mathbf{k}_i) \tag{4.57}$$

Also we have disconnected diagrams, which are cancelled by $\langle F_I \rangle^2$. We also have some extra diagrams at second order, which correspond to diagrams that we have not considered before



$$\implies g_0^2 \int đ^d k \, \frac{1}{2} A(k,\Lambda) \phi^-_{\mathbf{k}} \phi^-_{-\mathbf{k}}, \tag{4.58}$$

where $A(k,\Lambda)$ can be expanded as

$$A(k,\Lambda) \approx A(0) + \frac{1}{2} k^2 A''(0) + \dots \tag{4.59}$$

The first term $A(0)$ is a new correction to μ^2 while $A''(0)$ corrects $(\boldsymbol{\nabla}\phi)^2$. The fact that this type of correction only pops up at two loops is very special to ϕ^4 theory.

4.6.3 β-Functions

Let us write the new cutoff as $\Lambda' = \Lambda/\zeta = \Lambda e^{-s}$. We then define the beta functions $\beta_n(g_n)$ as

$$\beta_n(g_n) \coloneqq \frac{\mathrm{d}g_n}{\mathrm{d}s} \begin{cases} > 0 & g_n \text{ in the IR} \\ < 0 & \text{otherwise} \end{cases} \tag{4.60}$$

$$\frac{\mathrm{d}g_n}{\mathrm{d}s} = (d - \frac{1}{2} nd + n) g_n, \qquad g_n(s) = \exp\left\{ \frac{d - nd/2 + n}{s} \right\} g_{0,n} \tag{4.61}$$

With our RG transformations, we saw that the couplings flow with the RG parameter ζ. This flow can be described by the β-function differential equations.

Let us derive the β-functions for the quartic interaction. To leading order, we found that the interactions give us the following flow

$$\mu^2(\zeta) = \zeta^2(\mu_0^2 + a g_0) \qquad g(\zeta) = \zeta^{4-d}(g_0 - b g_0^2), \tag{4.62}$$

$$a = 12\int_{\Lambda/\zeta}^{\Lambda} \frac{\text{đ}^d q}{q^2+\mu_0^2} \qquad b = 36\int_{\Lambda/\zeta}^{\Lambda} \frac{\text{đ}^d q}{(q^2+\mu_0^2)^2} \tag{4.63}$$

which corrects the mass term and the quartic interactions.

Let us write $\zeta = e^s$, where s is infinitesimal. We can then solve the integrals in a and b as

$$\implies \frac{\mathrm{d}}{\mathrm{d}d}\int_{\Lambda e^{-s}}^{\Lambda} \mathrm{d}q\, f(q) = \Lambda f(\Lambda) \tag{4.64}$$

which becomes exact for $s \to 0$.

Applying this to (4.62), we find the following β-functions:

$$\frac{\mathrm{d}\mu^2}{\mathrm{d}s} = 2\mu^2 + \frac{3g}{2\pi^2}\frac{\Lambda^4}{\Lambda^2+\mu^2} \qquad \frac{\mathrm{d}g}{\mathrm{d}s} = -\frac{9}{2\pi^2}\frac{\Lambda^4}{(\Lambda^2+\mu^2)^2}g^2 \tag{4.65}$$

where we used that $\mathrm{d}^d q = \mathrm{d}\Omega_{d-1}\, q^{d-1}\,\mathrm{d}q$ with $d = 4$. Together with (4.64), this leads to the factor of Λ^4 on top.

> **Remark:** I believe there might be some slight-of-hand happening to switch $\mu_0 \to \mu$ here.

4.7 Wilson-Fischer Fixed Point

4.7.1 ϵ-Expansion

Since we now have the β-functions, it seems like the dimension d does not necessarily need to be an integer anymore—unlike previously where we had integrals over $\mathrm{d}^d q$.

We derived the β-functions (4.65) in $d = 4$. Let us now consider $d = 4 - \epsilon$. The β-functions then turn out to be

$$\frac{\mathrm{d}\mu^2}{\mathrm{d}s} = 2\mu^2 + \frac{3}{2\pi^2}\frac{\Lambda^4}{\Lambda^2+\mu^2}\widetilde{g} \qquad \frac{\mathrm{d}\widetilde{g}}{\mathrm{d}s} = \epsilon\widetilde{g} - \frac{9}{2\pi^2}\frac{\Lambda^4}{(\Lambda^2+\mu^2)^2}\widetilde{g}^2 + \dots \tag{4.66}$$

where we used an 'ϵ-expansion': $\Lambda_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \simeq 2\pi^2 + O(\epsilon)$.

We can solve for the fixed point, where the β-functions vanish. Denoting the fixed-point solutions with a star, we have

$$\mu_*^2 = -\frac{3}{4\pi^2}\frac{\Lambda^4}{\Lambda^2+\mu_*^2}\widetilde{g}_* \qquad \widetilde{g}_* = \frac{2\pi^2}{9}\frac{(\Lambda^2+\mu_*^2)^2}{\Lambda^4}\epsilon, \tag{4.67}$$

$$= -\frac{1}{6}\Lambda^2\epsilon \qquad \simeq \frac{2\pi^2}{9}\epsilon. \tag{4.68}$$

where in the last line, we solved these two equations simultaneously.

This is called the *Wilson-Fischer Fixed Point.* Expanding the coupling constants in perturbations around this fixed point,

$$\mu^2 = \mu_*^2 + \delta\mu^2 \qquad \widetilde{g} = \widetilde{g}_* + \delta\widetilde{g}, \tag{4.69}$$

we can write the flow of the perturbations as

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\delta\mu^2 \\ \delta\widetilde{g}\end{pmatrix} = \begin{pmatrix}2-\epsilon/3 & \frac{3}{2\pi^2}\Lambda^2(1+\frac{\epsilon}{6}) \\ 0 & -\epsilon\end{pmatrix}\begin{pmatrix}\delta\mu^2 \\ \delta\widetilde{g}\end{pmatrix}. \tag{4.70}$$

We have $\Delta_t = 2 - \epsilon/3$, and $\Delta_g = -\epsilon$.

4.7.2 Critical Exponents Again

$$t \to \zeta^{\Delta_t} t = e^{S\Delta_t} t \tag{4.71}$$

The correlation length ξ scales as

$$\xi \sim t^{-\nu}, \qquad \nu = \frac{1}{\Delta_t} = \frac{1}{2} + \frac{\epsilon}{12} \tag{4.72}$$

Old scaling relations:

$$c \sim t^{-\alpha} \qquad \alpha = \frac{\epsilon}{6} \tag{4.73}$$

$$\Delta_\phi \sim \frac{d-2}{2} = 1 - \frac{\epsilon}{2} \tag{4.74}$$

This gives

$$\beta = \frac{1}{2} - \frac{\epsilon}{6} \qquad \gamma = 1 + \frac{\epsilon}{6} \qquad \delta = 3 + \epsilon. \tag{4.75}$$

There is no physical system, which we can study with $d = 4-\epsilon$ dimensions. However, we can look at system which are close, for example $d = 3$ with $\epsilon = 1$. Since the expansion parameter is $\epsilon = 1$, all hell should break loose since we do not have convergence of our asymptotic expansions. However, we get surprisingly close results, listed in 4.3.

Table 4.3: Comparing the results of the ϵ-expansion with the numerical results.

	α	β	γ	δ	η	ν
MF	0	$\frac{1}{2}$	1	3	0	$\frac{1}{2}$
$\epsilon = 1$	0.17	0.33	1.17	4	0	0.58
$d = 3$	0.1101	0.3264	1.2371	4.7898	0.0363	0.6300

$\mathbf{d = 2}$

In $d = 2$, the scaling of ϕ is $\Delta_\phi = (d-2)/2 = 0$. We therefore have an infinite number of relevant terms in the effective action

$$F[\phi] = \int \mathrm{d}^2 x \left[\frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + g_{2(n+1)}\phi^{2(n+1)} + \dots \right] \tag{4.76}$$

In a sense, there is no price to be paid to add another ϕ^2 to your free action. This means that we have an infinite set of fixed points, since the couplings are not scaling—at least to leading order—under RG flow. This is a very bizarre situation; previously we only needed to care about the relevant interaction. Here we need a different tool to study this theory. This takes us towards *conformal symmetry.*

4.8 Towards Conformal Symmetry

Let us think a bit more about scale invariance: In a scale invariant theory, we may rescale $\mathbf{x} \to \lambda\mathbf{x}$ without changing the physics. We know by definition, this symmetry exists at a fixed point. However, it turns out that at a fixed point, this symmetry is actually enhanced to the full conformal symmetry: $\mathbf{x} \to \widetilde{\mathbf{x}}(\mathbf{x})$, where

$$\frac{\partial \widetilde{x}_i}{\partial x_k}\frac{\partial \widetilde{x}_j}{\partial x_l}\delta_{ij} = \phi(\mathbf{x})\delta_{kl}. \tag{4.77}$$

The class of conformal transformations, which satisfy this condition, are of the form

$$\widetilde{x}^i = \frac{x^i - (\mathbf{x}\cdot\mathbf{x})a^i}{1 - 2(\mathbf{x}\cdot\mathbf{a}) + (\mathbf{a}\cdot\mathbf{a})(\mathbf{x}\cdot\mathbf{x})}. \tag{4.78}$$

It is highly non-trivial that conformal symmetry should hold at the RG fixed point. $d = 2$ conformal field theory is a very rich subject to study.

4.9 Continuous Symmetries

This will be our first step beyond the Ising model. Like with music and musical notes and chords, there is an infinite collection of theories—determined by their free energy—we can study. However, just like in music, only a small subset of these are worth considering.

Phases of matter are characterised by symmetries. There are external (spacetime) and internal (field, like U(1)) symmetries. The symmetry of the ground state (H) may be different from the symmetry of the free energy (G). For example, in the Ising model, the ground state at high temperature $T > T_c$ respects the $\mathbb{Z}_2$ symmetry of the theory. However, once we go to low temperatures $T < T_c$, the spins pick out a direction and the ground state has no symmetry. We say the symmetry of the theory has been *spontaneously broken*. Of course, if we had a magnetic field, then the theory had no $\mathbb{Z}_2$ symmetry to begin with.

In general, we build our theories by making a hypothesis of G, and seeing whether their predictions match up with nature. Many systems are characterised by their symmetrised. One example is given by christals. However, some of the most interesting symmetries we come across are internal symmetries.

> **Remark:** Sometimes, these things mix up. Think of lowering the temperature in a crystal. We spontaneously break spacetime symmetries at around the same time as we break internal symmetries.

4.10 O(N) Models

We have N scalar fields $\phi(\mathbf{a}) = (\phi_1(\mathbf{x}), \dots, \phi_N(\mathbf{x}))$. Of the theory respects $O(n)$ symmetry, the theory will be invariant under

$$\phi_a(\mathbf{x}) \to {R_a}^b \phi_b(\mathbf{x}) \qquad R^T R = \mathbb{1}. \tag{4.79}$$

We may now write down the most general action

$$F[\phi(\mathbf{x})] = \int \mathrm{d}^d x \left[\frac{\gamma}{2} \sum_i (\boldsymbol{\nabla}\phi_i) \cdot (\boldsymbol{\nabla}\phi_i) + \frac{\mu^2}{2} \phi \cdot \phi + g(\phi \cdot \phi)^2 + \dots \right], \tag{4.80}$$

Example (O(2): XY-Model)**:** A particularly interesting model of this is the *XY-Model*, which is simply $O(2)$. We package our fields $\phi = (\phi_1, \phi_2)$ into a complex field $\psi(\mathbf{x}) = \phi_1(\mathbf{x}) + i\phi_2(\mathbf{x})$, using the isomorphism between $U(1) \sim SO(2)$. The free energy is then

$$F[\psi(x)] = \int \mathrm{d}^d x \left[\boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\psi + \frac{\mu^2}{2} |\psi|^2 + g|\psi|^4 + \dots \right] \tag{4.81}$$

As we will see in the next sections, this model describes Bose-Einstein condensates and superfluids.

Example (O(3): Heisenberg Model)**:**

These models are kind of 'Ising-Plus'. However, these symmetries also give us some more interesting phenomena.

4.10.1 Goldstone Bosons

Theorem 3: For every spontaneously broken continuous symmetry, you get an exactly massless scalar.

These (Nambu-)Goldstone bosons show up in all areas of physics. Why does this happen? For discrete symmetries we have a finite number of vacua.

Example ($\mathbb{Z}_2$)**:** In the Ising model, the spins can either all point up or all down, $\langle\phi\rangle = \pm m_0$. Therefore, there are two different vacua in the low temperature theory.

However, for continuous symmetries, the magnitude of the direction is fixed, whereas the direction of the field is not fixed. $\langle|\phi|\rangle = M_0 = \sqrt{-\mu^2/(4g)}$. This means that we have a sphere of ground states.

The ground state is at the bottom of the Mexican hat potential depicted in F1. If we generalise this to the $O(n)$ model, the vacua are the $(n-1)$-sphere S^{n-1}. This means it costs us no energy to move around the bottom of the potential. There is no Boltzmann suppression in producing these; they are massless. Other names for these Goldstone bosons are *spin waves*, *soft modes*, or *gapless modes*. At low temperatures, you are bound to produce these modes since they cost no energy; they dominate the low energy behaviour.

We can count the number of Goldstone bosons we expect to have, by considering the number of generators. The number of Goldstone bosons is the number of broken symmetries $\dim(G) - \dim(H)$. In each case we get a vacuum manifold, which can be written as the quotient group

$$S^{N-1} = \frac{O(N)}{O(N-1)}. \tag{4.82}$$

The vacuum expectation value of ϕ in $O(N)$ is $\langle\phi\rangle = (M_0, 0, \ldots, 0)$, where the zero vector remaining is the $O(N-1)$ symmetry. This means that the number of Goldstone bosons for $O(N)$ is $N-1$.

Example (XY-Model)**:** Parametrise $\psi(\mathbf{x}) = (M_0 + \widetilde{M}(\mathbf{x}))e^{i\theta(\mathbf{x})}$. The free energy takes the form

$$F[M,\theta] = \int \mathrm{d}^d x \left\{ \frac{\gamma}{2}(\boldsymbol{\nabla}\widetilde{M})^2 + \left|\mu^2\right|\widetilde{M}^2 + g\widetilde{M}^4 + \cdots + \frac{\gamma}{2}M_0^2(\boldsymbol{\nabla}\theta)^2 + \gamma M_0\widetilde{M}(\boldsymbol{\nabla}\theta)^2 + \ldots \right\} \tag{4.83}$$

The symmetry is being remembered in the vacuum, meaning that it costs no energy to rotate; the theory has a *shift symmetry* since $\boldsymbol{\nabla}\theta \to \boldsymbol{\nabla}(\theta + a) = \boldsymbol{\nabla}\theta$.

We saw that approaching the critical point, the correlation length went to infinity, effectively giving us a massless mode, which has long-range correlations. Here, we do not have to tune anything; below the critical temperature, we get Goldstone bosons with infinite correlation length without having to tune the temperature, the magnetic field etc, to get to the critical point. Indeed, they show some very non-trivial behaviour.

4.10.2 Critical Exponents

If we sit at the critical temperature, we expect to have a theory with critical exponents as we had before. Although the $O(n)$ models look similar to the Ising model, they are actually in a different universality class.

In $d = 3$, the exponents are particularly interesting, as tabulated in 4.4. This tiny difference

	η	ν
MF	0	1/2
Ising	0.0363	0.6300
$N = 2$	0.0385	0.6718
$N = 3$	0.0386	0.702

Table 4.4: Critical exponents

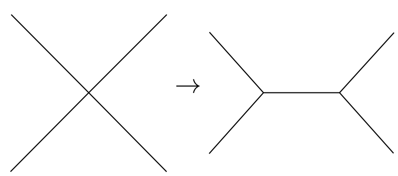
in critical exponent sets apart superfluids from magnets; this is known as the *lambda transition*. In particular, the heat capacity $C \sim |T - T_c|^{-\alpha}$, $\alpha = 2 - 3\nu$ is given by $\alpha_{O(2)} = -0.16$, which gives a divergence at $T = T_c$ for superfluids.

4.10.3 RG in the O(N) Model

Naive dimensional analysis gives us

$$[\phi] = \frac{d-2}{2}, \qquad [\mu_0^2] = 2, \qquad [g_0] = 4 - d. \tag{4.84}$$

However, we now have $F_I[\phi] \sim \int \mathrm{d}^d x \, g\phi^4$, where $\phi^4 \sim \phi \cdot \phi\phi \cdot \phi$, which we realise in Feynman diagrams by splitting the vertex



(4.85)

To order $O(g_0)$, we can calculate the contributions from the following Feynman diagrams Performing the epsilon expansion, we get

$$\frac{\mathrm{d}\mu^2}{\mathrm{d}s} = 2\mu^2 + \frac{N+2}{2\pi^2}\frac{\Lambda^4}{(\Lambda^2+\mu^2)}\widetilde{g}^2 \qquad \frac{\mathrm{d}\widetilde{g}}{\mathrm{d}s} = \epsilon\widetilde{g} - \frac{N+8}{2\pi^2}\frac{\Lambda^4}{\Lambda^2+\mu^2}\widetilde{g}^2, \tag{4.86}$$

where $\widetilde{g} = \Lambda^{-\epsilon} g$. Flowing towards the Wilson-Fischer fixed point, we have

$$\mu_*^2 = -\frac{1}{2}\frac{N+2}{N+8}\Lambda^2\epsilon, \qquad \widetilde{g}_* = \frac{2\pi^2}{N+8}\epsilon \tag{4.87}$$

4.10.4 Epsilon Expansion in O(N) Models

As before, as we go away from $d = 4$, the Wilson-Fischer fixed point opens up.

$$\mu_*^2 = -\frac{1}{2}\frac{N+2}{N+8}\Lambda^2\epsilon \qquad \widetilde{g}_*\frac{2\pi^2}{N+8}\epsilon. \tag{4.88}$$

From here we can go ahead and calculate all the critical exponents of the $O(N)$ Model.

$$\alpha = \frac{4-N}{2(N+8)}\epsilon, \quad \beta = \frac{1}{2} - \frac{3}{2(N+8)}\epsilon, \quad \gamma = 1 + \frac{N+2}{2(N+8)}\epsilon, \quad \delta = 3+\epsilon. \tag{4.89}$$

4.10.5 Goldstone Bosons in $d = 2$

We saw before that under spontaneous continuous symmetry breaking, we get Goldstone bosons. The arguments that led to Goldstone bosons (difference of generators of symmetries) did not seem to care about the number of dimensions. However, in $d = 2$ something very special happens. Recall again the discrete $\mathbb{Z}_2$ symmetry. We saw that in the lower critical dimension $d = 1$ for the Ising model, the domain walls dominate the physics and destroy the ordered phase. This is because it cost us some energy to change from the vacuum at $-m^0$ to $+m^0$ or vice-versa. The fluctuations that take us from one vacuum to the other destroy the ordered phase. For continuous symmetries, it costs us no energy. We will discover, with a slightly hand-waving argument again, that the domain wall fluctuations in the vacuum will destroy the ordered phase in $d = 2$. Afterwards, we will perform the full-blown RG calculation to see what is really going on.

XY-Model

Consider a system, which starts out with the expectation value of its Goldstone modes at $\langle\theta(\mathbf{x})\rangle = 0$. Now let us calculate fluctuations around the vacuum with the two-point correlation

$$\langle[\theta(\mathbf{x}) - \theta(0)]^2\rangle = 2\langle\theta^2(\mathbf{x})\rangle - 2\langle\theta(\mathbf{x})\theta(0)\rangle \tag{4.90}$$

The second term is

$$\langle\theta(\mathbf{x})\theta(0)\rangle = \frac{1}{\theta M_0^2}\int_0^\Lambda đ^d k\, \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{k^2} \sim \begin{cases} \Lambda^{d-2} - r^{2-d} & d > 2 \\ \log(\Lambda r) & d = 2 \\ r - \Lambda^{-1} & d = 1 \end{cases} \tag{4.91}$$

This is a manifestation of the following theorem:

Theorem 4 (Mermin-Wagner Theorem)**:** A continuous symmetry cannot be spontaneously broken in $d = 2$.

Goldstone's theorem tells us that there should be massless modes when symmetries are spontaneously broken. However, when we study the behaviour of massless Goldstone modes in $d = 2$, we see that the fluctuations grow to arbitrary large scales, similar to the domain walls in the Ising model.

Sigma Models

Historically, particle theorists wanted to understand pions, sigma mesons, and other particles, in terms of the theory of Goldstone bosons. Let $\phi(\mathbf{x}) = (\phi^1(\mathbf{x}), \ldots, \phi_N(\mathbf{x}))$. Ordered phase $\langle|\phi|\rangle \neq 0 \implies$ Vacuum Manifold $\sim S^{N-1}$. We get ungapped Goldstone modes and gapped (massive) longitudinal modes. The mass is the second derivative of a potential. Have $\phi\phi = M_0^2$. Rescale $\phi \to \mathbf{n} \cdot \mathbf{n} = 1$. The potential is $V(\phi) = \lambda/2(\phi \cdot \phi - M_0^2)$. Free energy:

$$Fn = \int \mathrm{d}^d x \, \frac{1}{2e^2} (\boldsymbol{\nabla}\mathbf{n}) \cdot (\boldsymbol{\nabla}\mathbf{n}), \qquad e^2 = \frac{1}{\gamma M_0^2}. \tag{4.92}$$

Path Integral

$$Z = \int \mathcal{D}\mathbf{n} \, \delta(\mathbf{n}(\mathbf{x})^2 - 1) \exp\left(-\frac{1}{2e^2} \int \mathrm{d}^d x \, (\boldsymbol{\nabla}\mathbf{n}) \cdot (\boldsymbol{\nabla}\mathbf{n})\right), \qquad [e^2] = 2 - d. \tag{4.93}$$

We can think of this as a coupling. Rewriting $\mathbf{n}(\mathbf{x}) = (\boldsymbol{\pi}(\mathbf{x}), \sigma(\mathbf{x}))$, where π is an $N-1$ dimensional vector, while σ is just one field. This tells us that $\sigma^2(\mathbf{x}) = 1 - \boldsymbol{\pi}(\mathbf{x}) \cdot \boldsymbol{\pi}(\mathbf{x})$. The free energy then takes the following form:

$$\begin{aligned} F[\boldsymbol{\pi}(\mathbf{x})] &= \int \mathrm{d}^d x \, \frac{1}{2e^2} \left\{ (\boldsymbol{\nabla}\boldsymbol{\pi})^2 + (\boldsymbol{\nabla}\sigma)^2 \right\} &\quad (4.94)\\ &= \int \mathrm{d}^d x \, \frac{1}{2e^2} \left[(\boldsymbol{\nabla}\boldsymbol{\pi})^2 + \frac{(\boldsymbol{\pi} \cdot \boldsymbol{\nabla}\boldsymbol{\pi})^2}{1 - \boldsymbol{\pi} \cdot \boldsymbol{\pi}} \right]. &\quad (4.95) \end{aligned}$$

We see that the Goldstone bosons actually have interactions.

Background Field Method

We want to see what happens to the Goldstone bosons as we go to the IR with the renormalisation group. Polyakov developed the background field method, which uses the geometry. We first split the field into IR and UV modes as usual. The long wavelengths also satisfy the constraint $\widetilde{n} \cdot \widetilde{n} = 1$. To introduce the short wavelength modes, we introduce *frame fields*.

Definition 5: The *frame fields* are basis of $N-1$ unit vectors $e^a_\alpha(\mathbf{x})$, with $a = 1, \ldots, N$ and $\alpha = 1, \ldots, N-1$ that are orthogonal to $\widetilde{n}^a$, meaning that $e^a_\alpha e^a_\beta = \delta_{\alpha\beta}$ and $\widetilde{n}^a e^a_\alpha = 0$ for all α.

> **Remark:** There is a rotational ambiguity in the definition of this basis. We just pick any one for this calculation.

Introduce short wavelength modes $\chi_\alpha(x)$ as

$$n^a(\mathbf{x}) = \widetilde{n}^a(\mathbf{x})(1 - \chi^2(x))^{1/2} + \sum_{\alpha=1}^{N-1} \chi_a(x) e^a_\alpha(x) \implies n^2 = 1. \tag{4.96}$$

The interaction term of the action is

$$F_I[\widetilde{n}^a, \chi_\alpha] = \frac{1}{2e^2} \int \mathrm{d}^d x \left[-\chi^2 (\boldsymbol{\nabla} \widetilde{n})^2 + \chi_\alpha \chi_\beta \boldsymbol{\nabla} e^a_\alpha \boldsymbol{\nabla} e^a_\beta + \cancel{2 \boldsymbol{\nabla} \widetilde{n}^a \boldsymbol{\nabla} (\chi_\alpha a^a_\alpha)} \right] \tag{4.97}$$

$$\langle F_I \rangle = \frac{1}{2e^2} \int \mathrm{d}^d x \left(-\delta_{\alpha\beta} (\boldsymbol{\nabla} \widetilde{n}^a)^2 + \boldsymbol{\nabla} e^a_\alpha \boldsymbol{\nabla} e^a_\beta \right) \langle \chi_\alpha \chi_\beta \rangle \tag{4.98}$$

$$\langle \chi_\alpha(\mathbf{x}) \chi_\beta(\mathbf{x}) \rangle = e^2 \delta_{\alpha\beta} I_d, \qquad I_d = \frac{\Omega_{d-1}}{(2\pi)^d} \Lambda^{d-2} \times \begin{cases} \zeta - 1 & d = 1 \\ \log(\zeta) & d = 2 \\ 1 - \zeta^{2-d} & d \geq 3 \end{cases} \tag{4.99}$$

Using $\widetilde{n}^a \widetilde{n}^b + e^a_\alpha e^b_\alpha = \delta^{ab}$,

$$\Rightarrow \boldsymbol{\nabla} e^a_\alpha \boldsymbol{\nabla} e^a_\alpha = \boldsymbol{\nabla} \widetilde{n}^a \boldsymbol{\nabla} \widetilde{n}^a + \dots \tag{4.100}$$

Finally, $\langle F_I \rangle = (2 - N) I_d \int \mathrm{d}^d x \, \frac{1}{2} (\boldsymbol{\nabla} \widetilde{n})^2$. This gives a correction

$$\frac{1}{(e')^2} = \frac{1}{e_0^2} + (2 - N) I_d \tag{4.101}$$

After the whole RG calculation, we find

$$\Longrightarrow \frac{1}{e^2(\zeta)} = \zeta^{d-2} \left[\frac{1}{e_0^2} + (2 - N) I_d \right]. \tag{4.102}$$