Symmetries, Fields and Particles

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Contents

A particle with a definite spin defines a vector space with a certain dimension. The angular momentum operators in QM often act on finite dimensional vector spaces.

Example ($\mathbb{C}^2$)**:** The Hilbert space $\mathbb{C}^2$ is spanned by

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.0.1}$$

They correspond to the $2d$ representation $\mathcal{L}(SO(3))$. In place of the angular momentum operators, we have 3 2×2 matrices Σ_i, $i = 1, 2, 3$, which obey the same commutation relations

$$\langle \Sigma_i, \Sigma_i \rangle = i\hbar\Sigma_k. \tag{0.0.2}$$

The Lie algebra is appearing again in a way in which its generators / basis vectors are represented by the Pauli matrices σ_i:

$$\Sigma_i = \frac{1}{2}\hbar\sigma_i. \tag{0.0.3}$$

Definition 1 (Representation)**:** A *representation* of a Lie algebra $\mathcal{L}(G)$ is a map

$$R\colon \mathcal{L}(G) \to \mathrm{Mat}_n(\mathbb{C}) \tag{0.0.4}$$

which preserves the bracket of the Lie algebra.

> **Remark:** In the context of QM systems, we need to know both about the Lie algebra, but also about its representation in terms of finite dimensional matrices. Lie groups are largely determined by their Lie algebra. As seen in the last lecture, Lie algebras itself can be classified. One of the goals of this course will also be the classification of their representations.

In Quantum mechanics, a rotational symmetry manifests itself in the commutation relation

$$\left\langle \hat{\mathcal{H}}, \hat{L}_i \right\rangle = 0 \tag{0.0.5}$$

with the Hamiltonian $\mathcal{H}$. The fact that the angular momentum generators, which move you around in the Hilbert space, commute with the Hamiltonian means that states in any representation of $\mathcal{L}(SO(3))$ have the same energy.

Example: The spin vectors $|\uparrow\rangle$ and $|\downarrow\rangle$ have the same energy in a rotationally invariant system.

0.1 Key Idea

The degeneracies in the spectrum of a quantum system are effectively determined by the representations of the (global) symmetry group. This can be seen as a tool for which we do not yet know the underlying symmetry and the Lagrangian.

Example (Approximate Symmetry of Hadrons)**:** Strongly interacting particles have an observed degeneracy which led Gell-Mann to postulate the approximate symmetry $G = SU(3)$ of 3×3 complex, unitary matrices with unit determinant. This is where group theory really took off in physics.

It was observed that there were sets of particles in the accelerators with approximately the same energy. Their interaction was characterised by conserved charges which could be assigned to integer values.

The particular pattern is illustrated in Fig 0.1.1.

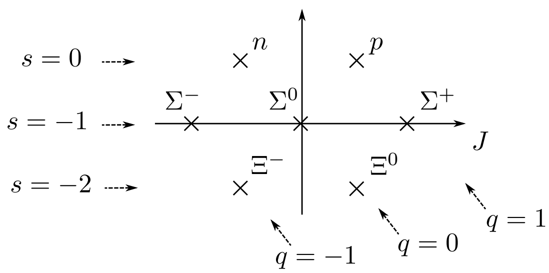


Figure 0.1.1: The "eightfold way" showing the approximate octet formed by the spin-$\frac{1}{2}$ baryons.

This turned out to be precisely explained by the mathematics of Lie groups and Lie algebras, and gives us the main tool of organising structures in particle physics.

Definition 2 (Global symmetries)**:** *Global* symmetries can be understood as operators in some Hilbert space which commute with the Hamiltonian. These include the spacetime symmtries, which fit into the pattern of non-Abelian Lie-groups:

- Rotations $SO(3)$
- Lorentz transformations $SO(3,1)$
- Poincaré group (+ translations)

However, the Poincaré group is not a simple Lie group.

Moreover, we also have the *internal* symmetries

- flavour symmetry (approximate symmetry of strong interactions)
- baryon number
- lepton number

Advancements in particle physics have often hinged on the idea of enlarging the symmetry group. This is where the interaction between mathematics and physics also really took off: There are powerful theorems which prevent the combination of the global and internal symmetry groups, in the context of ordinary Lie algebras. Physicists then started to relax the constraints of Lie algebras, which led to *supersymmetry*.

Gauge Symmetries

The other topic which we will talk about is *Gauge symmetry.* This is not really a symmetry since it does not obey the definition in the first lecture. It is actually a *redundancy* in the mathematical description of the physics. Examples of gauge symmetries are

- phase of the wavefunction: $\psi \to e^{i\delta}\psi$
- electromagnetism: $\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi$, where χ is some arbitrary scalar function.

These transformations, constituting the Gauge group, do not affect any physical quantities.

> **Remark:** In QFT, as far as we know, only Gauge theories can describe in a consistent, renormalisable way the interaction of spin-1 particles.

The Standard Model is a particular type of Gauge theory with $G_{SM} = SO(3) \times SU(2) \times U(1)$.

1 Lie Groups

1.1 Manifold Structure and Coordinates

Definition 3 (Manifold)**:** A manifold $\mathcal{M}$ is a space that locally looks like Euclidean space. For each coordinate patch $\mathcal{P}$, there is a bijective map $\phi_{\mathcal{P}} : \mathcal{P} \leftrightarrow \mathbb{R}^n$. Moreover, the transition functions between the coordinates have to be *smooth*.

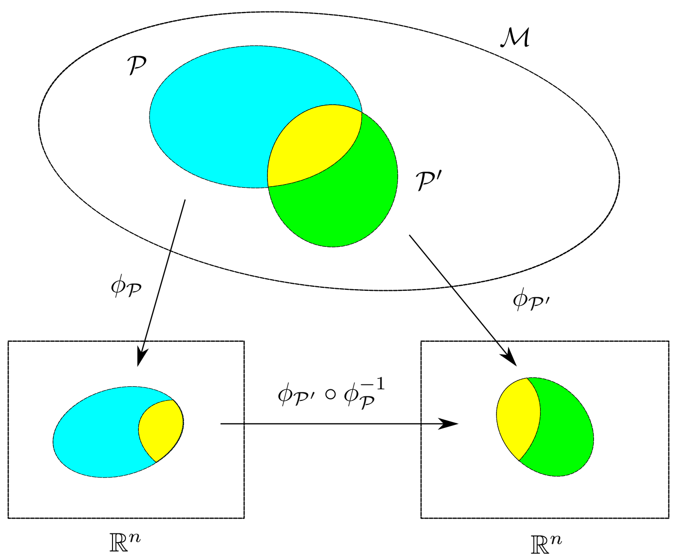


Figure 1.1.1: An illustration of the concept of a manifold.

Definition 4 (Lie groups)**:** A *Lie group* G is a group that is also a manifold. The group operations must define *smooth maps* on the manifold. The *dimension* of the Lie group is the dimension of the manifold.

The definition of the manifold allows us to introduce coordinates $\{\theta^i\}$, $i = 1, \ldots, D = \dim(G)$. In the patch $\mathcal{P}$, the group elements depend continuously on the coordinates $\{\theta^i\}$. WLOG, we can choose coordinates in which identity element lies at the origin: $g(0) = e$.

1.2 Compatibility of Group and Manifold Structures

1.2.1 Multiplication

The group operation of multiplication will define a map on the manifold. Since this map has to be smooth, it will have to be composed of continuous differentiable functions in the coordinate systems.

By closure of the group, and assuming that multiplication gives us another element in the same patch:

$$g(\theta)g(\theta') = g(\varphi) \in G \tag{1.2.1}$$

> **Remark:** In general it is not necessarily the case that the new group element will be in the same coordinate patch $\mathcal{P}$. In that case, we will have to make use of the transition functions.

This defines a map $G \times G \to G$ from a pair of group elements to a third. We can express this map in coordinates as $\varphi^i = \varphi^i(\theta, \theta') : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. In terms of these coordinates, the condition of *compatibility* between a group operation and a manifold structure simply means that these maps φ^i have to be continuous and differentiable.

1.2.2 Inversion

Group inversion also defines a smooth map, this time from G to itself. For all elements $g(\theta) \in G$, where θ describes the position in the coordinate patch $\mathcal{P}$, the group axioms imply that there exists an inverse element $g^{-1}(\theta)$. Assuming that this is still in the same coordinate patch, we can write this as $g^{-1(\theta)} = g(\widetilde{\theta})$. If it is not, we simply apply the relevant transition function.

$$g(\theta)g(\widetilde{\theta}) = g(\widetilde{\theta})g(\theta) = e \tag{1.2.2}$$

In coordinates, $\widetilde{\theta}^i = \widetilde{\theta}^i(\theta)$ have to be continuous and differentiable.

Example: The simplest possible example of a Lie group is $G = (\mathbb{R}^D, +)$.

- operation: $\mathbf{x}'' = \mathbf{x} + \mathbf{x}'$ for all $\mathbf{x}$, $\mathbf{x}' \in \mathbb{R}^D$
- inversion: $\mathbf{x}^{-1} = -\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^D$

This is an *Abelian* group.

1.3 Embedded Submanifolds

We can define properties of manifold in an intrinsic way. However, for some of these properties it is significantly easier to define manifolds as subspaces embedded in real space. In general, we can achieve this if we write

$$\mathcal{M} = \{\mathbf{x} \in \mathbb{R}^m \mid \mathcal{F}_\alpha(\mathbf{x}) = 0\} \tag{1.3.1}$$

where $\mathcal{F}_\alpha : \mathbb{R}^m \to \mathbb{R}$, with $\alpha = 1, \cdots, l$ is a smooth map.

Theorem 1 (Embedding Theorem)**:** $\mathcal{M}$ is a manifold of dimension $D = m - l$ if and only if the Jacobian matrix

$$(\mathcal{J})_{\alpha,i} = \frac{\partial \mathcal{F}_\alpha}{\partial x_i} \tag{1.3.2}$$

has maximal rank l everywhere on the manifold $\mathcal{M}$.

Example (S^2)**:** We can realise the two-sphere S^2 as a manifold embedded in three dimensional Euclidean space

$$\mathcal{M} = \left\{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| = R\right\} \tag{1.3.3}$$

The solution space of $x^2 + y^2 + z^2 - R^2 = F$ defines a manifold. In this case, the Jacobian

$$\mathcal{J} = \left(\frac{\partial \mathcal{F}}{\partial x}, \frac{\partial \mathcal{F}}{\partial y}, \frac{\partial \mathcal{F}}{\partial z}\right) = 2(x, y, z) \tag{1.3.4}$$

has rank 1 except at $x = y = z = 0$, but that point is not on the manifold, so that is allowed by the theorem.

Definition 5 (connected)**:** A manifold is said to be *connected* if there is a smooth path between any two paths on the manifold.

Definition 6 (simply connected)**:** A manifold is said to be *simply connected* if all loops are "trivial", in the sense that they can be continuously be contracted to a point.

Example: The spherical surface S^2 is simply connected, while the torus T^2 is not.

Definition 7 (compact)**:** A manifold is said to be a *compact space* if any *closed* and *bounded* subset of $\mathbb{R}^n$ is *compact.*

Example: The sphere is a compact space, while the hyperboloid is not.

1.4 Matrix Lie Groups

Matrix multiplication is *closed* and *associative* and there exists a unit element

$$e = 1_n \in \mathrm{Mat}_n(F). \tag{1.4.1}$$

Here, F is some field. We will mostly be working with $F = \mathbb{R}$ or $\mathbb{C}$. However, the set of matrices $\mathrm{Mat}_n(F)$ is not a group under matrix multiplication since not all matrices are *invertible.*

Definition 8 (general linear group)**:** The general linear group of dimension n is the set of matrices with non-vanishing determinant

$$GL(n,F) = \{M \in \mathrm{Mat}_n(F) \mid \mathrm{det}M \neq 0\} \tag{1.4.2}$$

guaranteeing invertibility.

Definition 9 (special linear group)**:** The special linear group has unit determinant

$$SL(n,F) = \{M \in GL(n,F) \mid \mathrm{det};= 1\} \tag{1.4.3}$$

These are enough to guarantee that these are groups. In particular, the closure property follows from

$$\det(M_1 M_2) = \det(M_1)\det(M_2) \qquad \forall M_1, M_2 \in \mathrm{Mat}_n(F). \tag{1.4.4}$$

This is connected to the embedding theorem in the following way. Taking $SL(n,\mathbb{R})$ we apply the embedding theorem with $m = n^2$. The number of constraints is $l = 1$ due to the determinant being constraint to unity:

$$F_1(M) = \det M - 1 \tag{1.4.5}$$

To apply the embedding theorem we have to calculate the Jacobian. It is useful to recall the definition of a minor:

Definition 10 (minor)**:** Let $M \in \mathrm{Mat}_n(\mathbb{R})$ be an $n \times n$ matrix with real entries. We define the *minor* $\hat{M}^{(ij)}$ of each element of the matrix as the $(n-1)\times(n-1)$ matrix with ith row and jth column deleted.

Using this definition, we can differentiate as follows

$$\frac{\partial F}{\partial M_{ij}} = \pm \det\left(\hat{M}^{(ij)}\right). \tag{1.4.6}$$

$\frac{\partial F}{\partial M_{ij}}$ has rank 1, unless all the determinants of the minors vanish. This is equivalent to the determinant of the matrix vanishing

$$\det\left(\hat{M}^{(ij)}\right) = 0 \iff \det(M) = 0 \neq 1 \tag{1.4.7}$$

This shows that $SL(n,\mathbb{R})$ is a smooth manifold of dimension $n^2 - 1$. We could do this with $SL(n,\mathbb{C})$ by splitting the coordinates and then the detminant condition in its real and imaginary parts. For the general linear groups, we define them by removing a condition.

Exercise 1.4.1: Complete the proof that $SL(n,\mathbb{R})$ is a Lie group. For this, you have to convince yourself that matrix multiplication, considered element by element, provides a smooth map.

This gives us four families of matrix Lie groups. We have established that

$$\dim(SL(n,\mathbb{R})) = n^2 - 1. \tag{1.4.8}$$

A similar argument allows us to find that

$$\dim(SL(n,\mathbb{C})) = 2n^2 - 2. \tag{1.4.9}$$

Moreover, for the general linear group we can find that

$$\dim(GL(n,\mathbb{R})) = n^2 \qquad \dim(GL(2,\mathbb{C})) = 2n^2. \tag{1.4.10}$$

Definition 11 (submanifold)**:** A sub manifold is a subspace of a manifold that is also a manifold.

Definition 12 (Lie subgroup)**:** A *Lie subgroup* is a subset of a Lie group which is also a Lie group.

Definition 13 (orthogonal groups)**:** The *orthogonal groups* are defined to be those elements of the general linear groups which satisfy the orthogonality condition

$$O(n) = \left\{ M \in GL(n,\mathbb{R}) \mid MM^T = 1_n \right\}. \tag{1.4.11}$$

Definition 14 (orthogonal transformations)**:** Orthogonal transformations are of the form

$$\mathbf{v} \in \mathbb{R}^n \to \mathbf{v}' = M \cdot \mathbf{v} \in \mathbb{R}^n \tag{1.4.12}$$

where $M \in O(n)$ is an orthogonal matrix.

Orthogonal transformations can be thought of as linear transformations which preserve the length of vectors since

$$|\mathbf{b}'| = \mathbf{v}'^T \cdot \mathbf{v}' = \mathbf{v} \cdot M^T M \cdot \mathbf{v} = \mathbf{v}^T \cdot \mathbf{v} = |\mathbf{v}|^2. \tag{1.4.13}$$

If we have an orthogonal real matrix, we have

$$\det\bigl(MM^T\bigr) = \det(M)^2 = 1, \tag{1.4.14}$$

so its determinant is $\det(M) = \pm 1$.

From this, by continuity, we can tell that $O(n)$ is not a connected manifold. Indeed, $O(n)$ must have 2 *connected components*. Moreover, only one of these can contain the identity element. As such, we can consider the space which only includes the connected components of the identity:

Definition 15 (special orthogonal group)**:**

$$SO(n) = \{M \in O(n) \mid \det(M) = 1\} \tag{1.4.15}$$

How do we distinguish between matrices which have $\det(M) = \pm 1$? Given a frame $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ in $\mathbb{R}^n$ and an orthogonal transformation

$$\mathbf{v}_a \in \mathbb{R}^n \to \mathbf{v}'_a = M \cdot \mathbf{v}_a \in \mathbb{R}^n \tag{1.4.16}$$

with $M \in O(n)$, preserves the *orientation* of a frame, i.e. the sign of the volume element

$$\Omega = \varepsilon^{i_1,\ldots,i_n} v_1^{i_1} v_2^{i_2} \cdots v_n^{i_n}, \tag{1.4.17}$$

where ε is the n-dimensional alternating tensor, only if $M \in SO(n)$. Elements of $SO(n)$ correspond to *rotations*, whereas elements of $O(n)$ with $\det(M) = -1$ correspond to some mixture of *rotation* and *reflection*.

Exercise 1.4.2: Use the embedding theorem to check that $O(n)$ is a manifold and that its dimension is

$$\dim(O(n)) = \dim SO(n) = \frac{1}{2}n(n-1). \tag{1.4.18}$$

Remember to show that the Jacobian matrix has maximal rank.

Recall the two defining properties of the orthogonal group $O(n)$ with respect to the matrix eigenvalue equation

$$M\mathbf{v}_\lambda = \lambda \mathbf{v}_\lambda, \qquad M \in O(n): \tag{1.4.19}$$

1. Complex conjugate: If λ is an eigenvalue, then λ^* is an eigenvalue as well.
2. Normalisation: $|\lambda|^2 = 1$.

Proof. 1. Complex conjugating both sides of (1.4.19) gives

$$M\mathbf{v}_\lambda^* = \lambda^* \mathbf{v}_\lambda^* \tag{1.4.20}$$

2. First, note that we have $(M\mathbf{v}^*)^T \cdot M\mathbf{v} = \mathbf{v}^\dagger M^T M \mathbf{v} = \mathbf{v}^\dagger \cdot \mathbf{v}$. Then, if $\mathbf{v} = \mathbf{v}_\lambda$:

$$(M\mathbf{v}_\lambda^*)^T \cdot M\mathbf{v}_\lambda = |\lambda|^2 \mathbf{v}_\lambda^\dagger \mathbf{v}_\lambda = \mathbf{v}_\lambda^\dagger \mathbf{v}_\lambda \implies |\lambda|^2 = 1. \tag{1.4.21}$$

□

Example (G = SO(2))**:** Let M be a matrix in $SO(2)$. Then M has eigenvalues $\lambda = e^{i\theta}, e^{-i\theta}$ for small $\theta \in \mathbb{R}$, with the identification $\theta \sim \theta + 2\pi$. In a matrix representation, we write

$$M = M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{1.4.22}$$

Although this is a real matrix, its eigenvalues are complex. Provided that we made the identification $\theta \sim \theta + 2\pi$, the matrix is uniquely specified by θ. Therefore, the manifold of this Lie group is $M(SO(2)) \cong S^1$. Moreover, since the matrices are commutative, $M(\theta_1)M(\theta_2) = M(\theta_2)M(\theta_1) = M(\theta_1 + \theta_2)$, this is an Abelian Lie group.

> **Remark:** This is in fact the simplest compact Lie group.

Example (G = SO(3))**:** We consider now matrices M in the three-dimensional special orthogonal group $SO(3)$. The eigenvalues are $\lambda = e^{+i\theta}, e^{-i\theta}, +1$, where we again have made the identification $\theta \sim \theta + 2\pi$. To parametrise a rotation matrix in three dimensions, consider the normalised eigenvector corresponding to the $\lambda = +1$ eigenvalue:

$$\hat{\mathbf{n}} \in \mathbb{R}^3, \qquad M\hat{\mathbf{n}} = \hat{\mathbf{n}}, \qquad \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1. \tag{1.4.23}$$

The direction of $\hat{\mathbf{n}}$ parametrises the axis, and θ parametrises the angle of rotation.

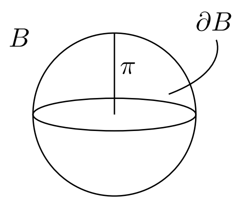
Exercise 1.4.3: One can write a general group element of SO(3) as

$$M(\hat{\mathbf{n}}, \theta)_{ij} = \cos\theta \delta_{ij} + (1 - \cos\theta) n_i n_j - \sin\theta \varepsilon_{ijk} n_k. \tag{1.4.24}$$

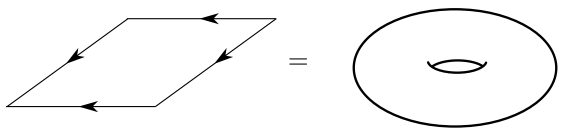
We want to specify the elements uniquely. Above, one needs to be careful about the uniqueness, due to two issues:

1. Identification: $M(\hat{\mathbf{n}}, 2\pi - \theta) = M(-\hat{\mathbf{n}}, \theta)$
2. If $\theta = 0$, then for all directions $\hat{\mathbf{n}}$, we have $M(\hat{\mathbf{n}}, 0) = I_3$,

To be precise, we need to identify these rotations. To get a better parametrisation, define the parameter $\boldsymbol{\omega} = \theta\hat{\mathbf{n}}$. Consider the ball $B_3 \subset \mathbb{R}^3 = \left\{\boldsymbol{\omega} \in \mathbb{R}^3 \mid |\boldsymbol{\omega}| \leqslant \pi\right\}$. The group manifold associated with SO(3) is obtained by taking B_3 and identifying antipodal points on the boundary.



(a) The ball B_3 with radius π.



(b) Identifying opposing edges of a sheet of paper gives a space topologically equivalent to a torus.

Figure 1.4.1: The group manifold associated with SO(3) is obtained by identifying antipodal points on the boundary ∂B of the ball B_3.

> **Remark:** In general, freely acting quotients give a manifold. Here, the group we quotiented out is the group of inversion $\mathbb{R}_2$.

The resulting manifold is connected, but not simply connected. This is because loops that come out and back via the identification cannot be contracted to a point; antipodal points are always antipodal. This is illustrated in Figure 1.4.2. As such, we have

$$\pi_1(SO(3)) \neq \{0\}, \qquad \pi_1(SO(3)) \simeq \mathbb{Z}_2 = \{+1, -1\}. \tag{1.4.25}$$

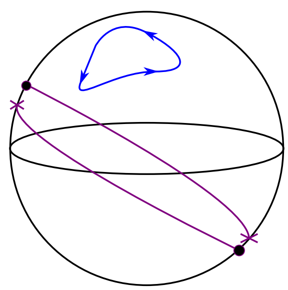


Figure 1.4.2: Loops passing through the identification of antipodal points cannot be contracted to a point. Note that the purple loop is constructible. This can be seen by "rotating" one half of the loop until it matches up with the other.

1.4.1 Non-compact subgroups of $GL(n,\mathbb{R})$

Orthogonal matrices obey $MM^T = I_n$. We can also read this as $MI_nM^T = I_n$; orthogonal transformations preserve the Euclidean metric $g = \mathrm{diag}\underbrace{(1,\cdots,1)}_{n \text{ times}}$ on $\mathbb{R}^n$. Similarly, we can generalise to say that $O(p,q)$ transformations preserve the metric of signature p,q :

$$O(p,q) = \left\{M \in GL(n,\mathbb{R}) \mid M^T\eta M = \eta, \text{where } \eta = \mathrm{diag}(\underbrace{-1,\ldots,-1}_{p \text{ times}},\underbrace{+1,\ldots,+1}_{q \text{ times}})\right\} \tag{1.4.26}$$

SO(2) $\sim \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, with $\theta \sim \theta + 2\pi$

SO(1, 1) $\sim \begin{pmatrix} \cosh\varphi & -\sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix}$, with $\varphi \in \mathbb{R}$, $M\big(SO(1,1)\big) \simeq \mathbb{R}$.

1.4.2 Subgroups of $GL(n,\mathbb{C})$

Claim: Let $U \in U(n) = \left\{U \in GL(n,\mathbb{C}) \mid U^\dagger U = I_n\right\}$ be a matrix in the unitary group. Under such a unitary transformation, the length of a vector $\mathbf{v}$ is unchanged.

Proof. The vector $\mathbf{v}$ transforms as $\mathbf{v} \in \mathbb{C}^n \to \mathbf{v}' = U\mathbf{v} \in \mathbb{C}^n$. Using the property $UU^\dagger = 1_n$ of unitary matrices, we have

$$|\mathbf{v}'|^2 = \mathbf{v}'^\dagger \cdot \mathbf{v}' = (\mathbf{v}^\dagger U^\dagger) \cdot (U\mathbf{v}) = \mathbf{v}^\dagger \cdot \mathbf{v} = |\mathbf{v}|^2 \tag{1.4.27}$$

□

Claim: Let $U \in U(n)$ be an element of the group of unitary $n \times n$ matrices. Then $\det U = e^{i\delta}$, where $\delta \in \mathbb{R}$.

Proof. $U^\dagger U = 1_n \implies |\det U|^2 = 1 \implies \det U = e^{i\delta}$, $\delta \in \mathbb{R}$. □

> **Remark:** Since $\delta \in \mathbb{R}$ is able to vary continuously, $U(n)$ is connected, whereas $O(n)$ was not.

1.4.3 Special Unitary Groups

$$SU(n) = \{U \in U(n) \mid \det U = 1\} \tag{1.4.28}$$

The groups $U(n)$ and $SU(n)$ are indeed Lie groups $\subset GL(n,\mathbb{C})$.

Claim: Their dimensions are given by

$$\dim(U(n)) = 2n^2 - n^2 = n^2, \qquad \dim(SU(n)) = n^2 - 1. \tag{1.4.29}$$

Proof. To see this, take any matrix $M \in \mathrm{Mat}(n, \mathbb{C}) \leftrightarrow \mathbb{R}^{2n^2}$ (real and complex components) and apply the embedding theorem. The constraint for $U(n)$ is $\mathcal{F} = UU^\dagger - I = 0$. This gives quadratic constraints in the matrix elements. Since $H = UU^\dagger$ is Hermitian, these are actually n^2 constraints instead of $2n^2$. For $SU(n)$, we require $\mathcal{F} = \det U - 1 = 0$. Since $\det U = e^{i\varphi}$, this is only one additional constraint. $\square$

In this course, we are interested in classifying Lie groups and Lie algebras. We have isomorphisms and homeomorphisms, maps which preserve group and manifold structure respectively.

Definition 16 (isomorphism)**:** Two Lie groups G and G' are *isomorphic* ($G \simeq G'$) if there exists a one-to-one smooth map $J : G \to G'$ such that for all $g_1, g_2 \in G$, we have $J(g_1 g_2) = J(g_1)J(g_2)$.

Let us look at some low-dimensional examples of unitary groups:

Example ($G = U(1)$)**:** Let $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. A general element $z = e^{i\theta}$ of $G = U(1)$, parametrised by $\theta \in \mathbb{R}$ with identification $\theta \sim \theta + 2\pi$, corresponds to a unique element

$$M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.4.30}$$

of $G' = SO(2)$ via the map

$$\begin{aligned} J\colon U(1) &\to SO(2) \\ z(\theta) = e^{i\theta} &\mapsto M(\theta). \end{aligned} \tag{1.4.31}$$

The map J is one-to-one and

$$J(z(\theta_1)z(\theta_2)) = M(\theta_1 + \theta_2) = M(\theta_1)M(\theta_2) = J(z(\theta_1))J(z(\theta_2)). \tag{1.4.32}$$

This implies that $U(1) \simeq SO(2)$.

Example ($G = SU(2)$)**:** This is a three-dimensional group. The matrix parametrised as $U = a_0 I_2 + i\mathbf{a} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and $a_0 \in \mathbb{R}$, $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$, is an element of $SU(2)$ provided that

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1. \tag{1.4.33}$$

This implies that $M(SU(2)) \simeq S^3 \subset \mathbb{R}^4$. Since

$$\pi_1(SU(2)) \simeq \{1\}\,, \qquad \pi_1(SO(3)) = \mathbb{R}_2, \tag{1.4.34}$$

this means that $SU(2) \not\simeq SO(3)$.

2 Lie Algebras

Definition 17 (Lie algebra)**:** A *Lie algebra* $\mathfrak{g}$ is a vector space over a field $F = \mathbb{R}, \mathbb{C}$ with a bracket,

$$[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{2.0.1}$$

with the following properties for all $X, Y, Z \in \mathfrak{g}$:

1. Anti-symmetry: $[X, Y] = -[Y, X]$.
2. (Bi-)linearity: $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$, for all coefficients $\alpha, \beta \in F$.
3. Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Let V be a vector space that has a product $* : V \times V \to V$, which is associative, meaning that for all $X, Y, Z \in V$,

$$(X * Y) * Z = X * (Y * Z). \tag{2.0.2}$$

Moreover, the product is distributive over the field

$$Z * (\alpha X + \beta Y) = (\alpha Z * X + \beta Z * Y). \tag{2.0.3}$$

Then, we obtain a Lie algebra from the following definition of a Lie bracket

$$[X, Y] = X * Y - Y * X. \tag{2.0.4}$$

One example we have in mind is the case where V is the vector space of matrices and $*$ is matrix multiplication.

> **Remark:** Compare this to the Lie algebra of differential operators/vectors in differential geometry.

Definition 18 (dimension)**:** The dimension of a Lie algebra $\mathfrak{g}$ is the dimension of its underlying vector space.

Choose a basis B for $\mathfrak{g}$

$$B = \{T^a, a = 1, \ldots, n = \dim(\mathfrak{g})\}. \tag{2.0.5}$$

Then any $X \in \mathfrak{g}$ can be written as

$$X = X_a T^a :- \sum_{a=1}^{n} X_a T^a \tag{2.0.6}$$

where $X_a \in F$. Bracket of elements $X, Y \in \mathfrak{g}$ can then be written as

$$[X, Y] = X_a Y_b [T^a, T^b]. \tag{2.0.7}$$

Therefore, knowing the brackets of basis elements allows us to construct the full Lie algebra:

$$[T^a, T^b] = f^{ab}{}_c T^c. \tag{2.0.8}$$

The *structure constants* $f^{ab}{}_c$ therefore define the Lie algebra, and two Lie algebras are isomorphic if they have the same structure constants. Note however, that structure constants are basis dependent. We will want to find a way to classify Lie algebras that is independent of our choice of basis.

2.1 Structure Constants

Let $f^{ab}{}_c \in F$, $a, b, c = 1, \ldots, \dim(\mathfrak{g})$ be structure constants of a Lie algebra $\mathfrak{g}$. The axioms of Lie algebras then imply

1. $\implies f^{ab}{}_c = -f^{ba}{}_c$
2. $\implies f^{ab}{}_c f^{cd}{}_e + f^{da}{}_c f^{cb}{}_e + f^{bd}{}_c f^{ca}{}_e = 0$

Definition 19 (isomorphism)**:** Two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ are said to be *isomorphic*, $\mathfrak{g} \simeq \mathfrak{g}'$ if there exists a linear, one-to-one map $f : \mathfrak{g} \to \mathfrak{g}'$ such that

$$[f(X), f(Y)] = f([X, Y]), \qquad \forall X, Y \in \mathfrak{g}. \tag{2.1.1}$$

Definition 20 (subalgebra)**:** A *subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ which is also a Lie algebra.

Definition 21 (ideal)**:** An *ideal* of $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with

$$[X, Y] \in \mathfrak{h}, \qquad \forall X \in \mathfrak{g}, Y \in \mathfrak{h} \tag{2.1.2}$$

The notion of ideal roughly corresponds to the concept of a normal subgroup.

Example (trivia algebras)**:** Every Lie algebra $\mathfrak{g}$ has two 'trivial' ideals:

$$\mathfrak{h} = \{0\} \qquad \text{and} \qquad \mathfrak{h} = \mathfrak{g}. \tag{2.1.3}$$

Example (derived algebra)**:** The *derived algebra*

$$i = [\mathfrak{g}, \mathfrak{g}] :-= \mathrm{Span}_F \{[X, Y] \mid X, Y \in \mathfrak{g}\} \tag{2.1.4}$$

is an ideal of $\mathfrak{g}$.

Example (centre)**:** The *centre* of $\mathfrak{g}$

$$\xi(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X,Y] = 0 \quad \forall Y \in \mathfrak{g}\}\,. \tag{2.1.5}$$

Definition 22 (abelian)**:** An Abelian Lie algebra is such that all brackets vanish:

$$[X,Y] = 0 \qquad \forall X, Y \in \mathfrak{g} \tag{2.1.6}$$

For an Abelian Lie algebra, the Lie algebra is equal to its own centre $\mathfrak{g} = \xi(\mathfrak{g})$ and the ideal is trivial $i(g) = \{0\}$.

Definition 23 (simple)**:** A Lie algebra $\mathfrak{g}$ is said to be *simple* if it is non-Abelian and it has no non-trivial ideals.

This implies that for simple Lie algebras, $\xi(\mathfrak{g}) = \{0\}$ and $i(\mathfrak{g}) = \mathfrak{g}$.

The main theorem that we will work up to is the *Cartan classification*, which will allow us to classify all finite-dimensional, simple, complex Lie algebras $\mathfrak{g}$.