Quantum Field Theory

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Contents

Notation

In the following chapters, we will make extensive use of Fourier theory. To avoid having to drag around factors of $(2\pi)^d$ all the time, which make the equations longer and obtuser than they need to be, we adopt the following notational convention.

Notation (barred differential)**:** We understand the *barred differential* $đq$ to be the following normalisation of the ordinary differential measure

$$đq\, a = \frac{\mathrm{d}q\, a}{2\pi}. \tag{0.0.1}$$

In particular, given a d-dimensional vector $\mathbf{q} = (q_1, \ldots, q_d)$—most often a Fourier mode—we will frequently use the d-dimensional integration measure in Fourier space:

$$đ^d q\, a = \frac{\mathrm{d}^d q\, a}{(2\pi)^d} = \prod_{i=1}^{d} \frac{\mathrm{d}q_i}{2\pi}. \tag{0.0.2}$$

Notation (barred delta)**:** We will denote the *barred Dirac delta* $\bar{\delta}(q)$ to include a normalisation of 2π in the following way

$$\bar{\delta}(q) = 2\pi\delta(q). \tag{0.0.3}$$

In particular, given a d-dimensional vector $\mathbf{q} = (q_1, \ldots, q_d)$, we will frequently use the d-dimensional distribution

$$\bar{\delta}(\mathbf{q}) = (2\pi)^d \delta(\mathbf{q}) = \prod_{i=1}^{d} 2\pi\delta(q_i). \tag{0.0.4}$$

1 Introduction

Reference text book: Introduction to Quantum Field Theory - Peskin and Schröder

QFT is a quantum theory with an infinite number of degrees of freedom at each point in spacetime $x^\mu = (t, \mathbf{x})$. They are constructed from classical field theories (eg. EM field) incorporating special relativity. QFT states are multi-particle states. The theory becomes relevant at relativistic energies, where we have a lot of exchange between energy $\leftrightarrow$ mass. This takes the form of particles being destroyed / created in interactions. Particle creation and annihilation occurs in Quantum field theory (QFT) but not in QM, differentiating the two theories. Unlike in QM, particle number is not fixed. The interactions themselves arise from the mathematical structure of the theory. Various principles determine this structure:

- locality
- symmetry
- renormalisation group flow

Definition 1: A *Free QFT* is the limit which has particles but no interactions. Weakly interacting theories are then built from these by using perturbation theory. Strong coupling phenomena are an active area of research, hard to solve and relevant for nature, often approached by a discretisation of spacetime (lattice QCD). Free QFT is a relativistic theory with infinitely many *quantised harmonic oscillators*. Non-relativistic QFTs also exist, describing quasi-particles like phonons.

- Based on D. Tong's notes (videos at PIRSA useful as well)

Units in QFT

As a mathematician, you would use $\pi = i = -1 = 2$ in these lectures. However, since we care about having correct numbers for our experimental measurements, we use special units in which $\hbar = c = 1$.

$$[c] = LT^{-1}, \qquad [\hbar] = L^2MT^{-1} \implies L = T = M^{-1}. \tag{1.0.1}$$

Length and time are measured in inverse mass—or equivalently ($E = m\cancel{c^2}$) inverse energy units. To convert back to metres or seconds, insert relevant powers of c and $\hbar$. E.g. $\lambda = \frac{\hbar}{mc}$, so when we measure $m_e \sim 1 \times 10^6\,\text{eV}$, we have $\lambda_e = 2 \times 10^{-12}\,\text{m}$. Similarly $m_p \sim 1\,\text{GeV} \to 1 \times 10^9\,\text{eV}$.

In QFT, if a quantity x has mass dimension $(\text{mass})^d$, we write $[x] = d$. E.g. $[G_N] = -2$ since

$$G_N = \frac{\cancel{\hbar c}}{M_p^2} = \frac{1}{M_p^2} \tag{1.0.2}$$

where $M_p \sim 1 \times 10^{19}\,\text{GeV}$ corresponds to the *Planck scale*, where quantum gravity effects became important ($\lambda_p \sim 10^{-35}\text{m}$).

Angular momentum, like $\hbar$, is dimensionless in these units: e^- total spin: $\hbar/2 = 1/2$.

Relativistic Schrödinger equation does not work—something always goes wrong (causality violation / energy unbounded from below). In QFT, these faults are fixed by allowing for the creation and annihilation of particles.

2 Classical Field Theory

A field is defined at each point of space and time $(t, \mathbf{x})$. Classical particle mechanics yields a finite number of generalised coordinates $q_a(t)$, where $a = x, y, \ldots$ is a label.

In field theory, we have a field $\phi_a(\mathbf{x}, t)$, where we can consider $\mathbf{x}$ as a label as well. So a and $\mathbf{x}$ are labels: we have an infinite number of degrees of freedom: one for each $\mathbf{x}$. Position is relegated from a dynamical variable to a mere label.

Example (Electromagnetism)**:**

$$E_i(\mathbf{x}, t), B_i(\mathbf{x}, t) \qquad \{i, j, k\} \in \{1, 2, 3\} \text{ label spatial position} \tag{2.0.1}$$

These 6 fields are derived from 4 fields $A_\mu(\mathbf{x}, t) = (\phi, \mathbf{A})$, where $\mu \in \{0, 1, 2, 3\}$. The relationship between the electric and magnetic fields and A is

$$E_i = -\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x_i}, \qquad B_i = \frac{1}{2}\varepsilon_{ijk}\frac{\partial A_k}{\partial x_j}, \tag{2.0.2}$$

where the Einstein summation convention is used.

2.1 Dynamics of Fields

The dynamics of the field is governed by a Lagrangian L. This is a function of the fields $\phi_a(\mathbf{x}, t)$, their time derivatives $\dot{\phi}_a(\mathbf{x}, t)$ and spatial derivatives $\boldsymbol{\nabla}\phi_a(\mathbf{x}, t)$. We can write it as an integral over the Lagrangian density $\mathcal{L}$:

$$L = \int \mathrm{d}^3x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a). \tag{2.1.1}$$

Confusingly, $\mathcal{L}$ is also often just called the *Lagrangian* in QFT. We will mostly be concerned with Lagrangian densities.

We define the action as

$$S = \int_{t_0}^{t_1} L \, \mathrm{d}t = \sum_a \int \mathrm{d}^4x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a). \tag{2.1.2}$$

Since $[\mathcal{L}] = 4$ and $[\mathrm{d}^4x] = -4$, the action is dimensionless $[S] = 0$. The *Dynamical Principle* of classical field theory is that fields evolve in such a way that the action S is stationary with respect to those variations of the fields that don't affect the initial and final values.

$$\delta S = \sum_a \int \mathrm{d}^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a) \right\} \tag{2.1.3}$$

$$= \sum_a \int \mathrm{d}^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right) - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right\} \tag{2.1.4}$$

The total derivative vanishes for any term that decays at spatial ∞ and has $\delta\phi(\mathbf{x}, t_0) = 0 = \delta\phi(\mathbf{x}, t_1)$. So the dynamical principle is

$$\delta S = 0 \implies \boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0 \right)} \qquad \text{Euler-Lagrange equations.} \tag{2.1.5}$$

Example (Klein-Gordon Field ϕ)**:**

$$\mathcal{L} = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \tag{2.1.6}$$

where we use the mostly minus signature $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

$$\mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\boldsymbol{\nabla} \phi)^2 - \frac{1}{2} m^2 \phi^2 \tag{2.1.7}$$

Remark: $\mathcal{L} = T - V$ where $T = \frac{1}{2}\dot{\phi}^2$ is the kinetic and $V = \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2$ is the potential energy density.

The relevant derivatives of $\mathcal{L}$ appearing the the Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi, \tag{2.1.8}$$

So substituting $\mathcal{L}$ into (2.1.5) yield

$$\boxed{\partial_\mu \partial^\mu \phi + m^2 \phi = 0} \qquad \text{Klein-Gordon eqn} \tag{2.1.9}$$

The K-G eqn admits a wave-like solution $\phi = e^{-ip \cdot x}$.

Remark: $p \cdot x = p^0 x^0 - \mathbf{p} \cdot \mathbf{x}^T = Et - \mathbf{p} \cdot \mathbf{x}$

Substitute $\phi \to$ KG:

$$(-p^2 + m^2)\phi = 0. \tag{2.1.10}$$

Since $p^2 = E^2 - |\mathbf{p}|^2 = m^2$, this reproduces the relativistic energy dispersion relation for a particle of mass m and three-momentum $\mathbf{p}$.

Example (Maxwell's Equations)**:** The Lagrangian which allows us to recover the charge-free Maxwell equations is

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A_\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2 \tag{2.1.11}$$

Noting that $\mathcal{L}$ only depends on derivatives $\partial_\mu A_\nu$ of the scalar field, we only need to compute

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \eta^{\mu\nu}\partial_\rho A^\rho. \tag{2.1.12}$$

The Euler-Lagrange equations then give the Maxwell equations in four-vector notation:

$$\begin{aligned}\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)}\right) = 0 &= -\partial_\mu\partial^\mu A^\nu + \partial^\nu(\partial_\rho A^\rho) && (2.1.13)\\ &= -\partial_\mu(\partial^\mu A^\nu - \partial_\nu A^\mu) && (2.1.14)\\ &= -\partial_\mu F^{\mu\nu} && (2.1.15)\end{aligned}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$ is the Maxwell field strength tensor.

In the previous examples, we only considered local Lagrangians, which means that the Lagrangian does not involve any products of fields like $\phi(\mathbf{x}, t)\phi(\mathbf{y}, t)$ where the positions differ $\mathbf{x} \neq \mathbf{y}$.

2.2 Lorentz Invariance

Consider the Lorentz transformation (LT) Λ of some scalar field $\phi(x) \equiv \phi(x^\mu)$. Under Λ, the field changes as $\phi \to \phi'$ where $\phi'(x) = \phi(x')$ and $x'^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu$.

The defining equation for Lorentz transformations is

$$\Lambda^\mu{}_\sigma \eta^{\sigma\rho} \Lambda_\rho{}^\nu = \eta^{\mu\nu}. \tag{2.2.1}$$

Examples of a Lorentz transformation include rotation around the x-axis or Lorentz boosts, whose matrix representation is respectively

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta & \sin\theta \\ & & -\sin\theta & \cos\theta \end{pmatrix}, \qquad \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{2.2.2}$$

where $\gamma = (1 - v^2)^{-1/2}$. The Lorentz transformations form a Lie group under matrix multiplication. This means that applying two Lorentz transformations gives another, and any Lorentz transformation has an inverse. The Lorentz transformations allow a representation on the fields. For a scalar field, this is $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$. This is an *active transformation*, which genuinely rotates the field. This is why the inverse Lorentz transformation Λ^{-1} is needed to used the new coordinates to the old. A *passive transformation* is one where we just relabel the coordinates. Under such a transformation, a scalar field changes as $\phi(x) \to \phi'(\Lambda x)$. *Lorentz invariant* theories are such that the action S, and the dynamics that are described by it, are unchanged under Lorentz transformations. In particular, if $\phi(x)$ satisfies the equations of motion of a Lorentz invariant theory, then so does the transformed field $\phi(\Lambda^{-1}x)$.

Example (Klein-Gordon)**:** Taking the K-G Lagrangian density, we have the action

$$S = \int_{\mathbb{R}^4} \mathrm{d}^4x \, \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi) \tag{2.2.3}$$

where U is some polynomial. Since we know how ϕ behaves under LT, we can deduce that U behaves under LTs in the following way:

$$U'(x) = U(\phi'(x)) = U(\phi(x')) = U(x'). \tag{2.2.4}$$

The first term in the action transforms as

$$\begin{aligned}(\partial_\mu\phi)' &= \frac{\partial}{\partial x^\mu}\phi(x') = \frac{\partial x'^\sigma}{\partial x^\mu}\frac{\partial}{\partial x'^\sigma}\phi(x') &(2.2.5)\\ &= (\Lambda^{-1})^\sigma{}_\mu \partial'_\sigma\phi(x'), &(2.2.6)\end{aligned}$$

where we denoted $\partial'_\sigma = \frac{\partial}{\partial x'^\sigma}$. The kinetic term in the Lagrangian is thus

$$\begin{aligned}\mathcal{L}'_{\text{kin}} &= \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)'(\partial_\nu\phi)' &(2.2.7)\\ &= \frac{1}{2}\underbrace{\eta^{\mu\nu}(\Lambda^{-1})^\sigma{}_\mu(\Lambda^{-1})^\rho{}_\nu}_{\eta^{\sigma\rho}}\partial'_\sigma\phi(x')\partial'_\rho\phi(x') &(2.2.8)\\ &= \frac{1}{2}\eta^{\sigma\rho}\partial'_\sigma\phi(x)\partial'_\rho\phi(x'). &(2.2.9)\end{aligned}$$

The Lagrangian density transforms like the scalar field ϕ; this transformation law means that $\mathcal{L}$ is itself a scalar field. The action changes under LT as

$$S' = \int \mathrm{d}^4x \, \mathcal{L}(x') = \int \mathrm{d}^4x \, \mathcal{L}(\Lambda^{-1}x). \tag{2.2.10}$$

We change variables to $y = \Lambda^{-1}x$. Since LTs are in the special orthogonal group, we have unit determinant $\det\Lambda = 1$ which means that the Jacobian is unity and $\int \mathrm{d}^4x = \int \mathrm{d}^4x$. As a result, the action is invariant:

$$S' = \int \mathrm{d}^4y \, \mathcal{L}(y) = S. \tag{2.2.11}$$

> **Remark:** Under a LT, a vector field A_μ transforms like $\partial_\mu\phi$, so
>
> $$A'_\mu(x) = (\Lambda^{-1})_\mu{}^\sigma A_\sigma(\Lambda^{-1}x). \tag{2.2.12}$$
>
> If all indices are summed over, the result is Lorentz invariant.

Example: Do Q1 in Ex. sheet 1.

2.3 Noether's theorem

Already noticeable in classical field theory: symmetries are important.

Theorem 1 (Noether's theorem)**:** Every continuous symmetry of $\mathcal{L}$ gives rise to a current $j^\mu(x)$ which is conserved:

$$\partial_\mu j^\mu = \frac{\partial(j^0)}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0. \tag{2.3.1}$$

Proof. Consider an infinitesimal variation of a field ϕ:

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha \Delta\phi(x), \tag{2.3.2}$$

where α is seen as an infinitesimal parameter. This is a symmetry if S is unchanged, i.e. $\mathcal{L}$ should be invariant up to a total 4-divergence (which integrates to a surface term and does not effect the E-L equations). In other words, the Lagrangian density changes as

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \partial_\mu X^\mu(x) = 0. \tag{2.3.3}$$

In fact, often we have $X^\mu = 0$. Recall that $\mathcal{L}$ only depends on ϕ and $\partial_\mu\phi$, so

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \frac{\partial\mathcal{L}}{\partial\phi}\Delta\phi + \alpha \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\Delta\phi) \tag{2.3.4}$$

$$= \mathcal{L}(x) + \alpha\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi \right) + \alpha \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right)}_{\text{E-L: }=0} \Delta\phi. \tag{2.3.5}$$

Combining (2.3.3) and (2.3.5), we see that

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - X^\mu \tag{2.3.6}$$

is conserved: $\partial_\mu j^\mu = 0$. □

Remark: Each conserved current has associated with it a conserved charge:

$$Q = \int_{\mathbb{R}^3} \mathrm{d}^3x\, j^0. \tag{2.3.7}$$

Proof. The current changes in time as

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \mathrm{d}^3x\, \partial_0 j^0 \tag{2.3.8}$$

$$= -\int_{\mathbb{R}^3} \mathrm{d}^3x\, \boldsymbol{\nabla}\cdot\mathbf{j} \tag{2.3.9}$$

which vanishes by the divergence theorem. □

Starting from the Lagrangian density $\mathcal{L}$ and the symmetry transformation, we can work out what the conserved current is.

Example (Scalar Field)**:**

$$\psi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_{28}(x)) \tag{2.3.10}$$

We could also use two real fields to describe this theory. However, when using complex fields, we can consider ψ and ψ^* as independent variables. This is because we really have two independent degrees of freedom ϕ_1 and ϕ_2. Using these variables, the Lagrangian is

$$\mathcal{L} = \partial_\mu \psi * \partial^\mu \psi - V(|\psi|^2). \tag{2.3.11}$$

> **Remark:** The potential could be for example:
>
> $$V(|\psi|^2) = m^2 \psi * \psi + \frac{\lambda}{2}(\psi^* \psi)^2. \tag{2.3.12}$$
>
> We will see later that the first summand will be a mass term and the second will describe an interaction.

The symmetry is a complex phase rotation:

$$\psi \to e^{i\alpha}\psi \implies \psi^* \to e^{-i\alpha}\psi^*. \tag{2.3.13}$$

Since $\mathcal{L}$ is invariant under this transformation, this is a symmetry with $M^\mu = 0$. The fields change as

$$\Delta\psi = i\alpha\psi \qquad \Delta\psi^* = -i\alpha\psi^*. \tag{2.3.14}$$

The currents from the two different fields will add:

$$j^\mu = (\psi \partial^\mu \psi - \psi^* \partial^\mu \psi). \tag{2.3.15}$$

This conserved charge could be the electric charge or some particle number (baryon, lepton, etc.) for example. In QED, we will see something every similar.

Example: Do questions 2 and 3 of Example sheet 1.

Example (Infinitesimal translations)**:** In four vector notation, we write an infinitesimal space-time translation as

$$x^\mu \to x^\mu - \alpha\varepsilon^\mu. \tag{2.3.16}$$

By Taylor expansion, we know that the scalar field transforms as

$$\phi(x) \to \phi(x) + \alpha\varepsilon^\mu\partial_\mu\phi(x). \tag{2.3.17}$$

The derivatives $\partial_\mu\phi$ will also have a similar expansion.

The Lagrangian density transforms as

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\varepsilon^\mu\partial_\mu\mathcal{L}(x) \tag{2.3.18}$$

$$= \mathcal{L}(x) + \alpha\varepsilon^\nu\partial_\mu\underbrace{(\delta^\mu{}_\nu\mathcal{L})}_{X^\mu}. \tag{2.3.19}$$

We get one conserved current for each component of ε^ν :

$$(j^\mu)_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu{}_\nu\mathcal{L} \tag{2.3.20}$$

since $\Delta\phi = \partial_\nu\phi$ for ε^ν and $X^\mu = \delta^\mu{}_\nu\mathcal{L}$. We define this to be the *energy-momentum tensor* $T^\mu{}_\nu$. It has this name since it contains four conserved charges which are associated with

- Total energy: $E = \int \mathrm{d}^3x\, T^{00}$
- Total momentum: $P^i = \int \mathrm{d}^3x\, T^{0i}$

Applying (2.3.20) to the free real valued scalar field theory

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \tag{2.3.21}$$

we obtain the energy momentum tensor

$$T^{\mu\nu} = \partial^\mu\phi^\nu\phi - \eta^{\mu\nu}\mathcal{L}. \tag{2.3.22}$$

This corresponds to an energy

$$E = \int \mathrm{d}^3x \left\{\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right\}. \tag{2.3.23}$$

Notice that in the case of a free scalar field, $T^{\mu\nu}$ is symmetric under interchange of $\mu \leftrightarrow \nu$.

Prop: If $T^{\mu\nu}$ is non-symmetric, we can massage it into a symmetric form.

Proof. Adding a term $\partial_\rho\Gamma^\rho{}_{\mu\nu}$ where $\Gamma^{\rho\mu\nu} = -\Gamma^{\mu\rho\nu} \implies \partial_\mu\partial_\rho\Gamma^{\rho\mu}{}_\nu = 0$, i.e. it does not change conservation law. We pick it to make the $T^{\mu\nu}$ symmetric. □

> **Remark:** This reparametrisation equivalence can be expressed as a gauge symmetry.

Exercise 2.3.1: Do question 6 on problem sheet 1.

3 Free Field Theory

3.1 Hamiltonian Formalism

The Hamiltonian formulation also accommodates field theories.

Definition 2 (conjugate momentum)**:** We define the *conjugate momentum*

$$\pi(x) = \frac{\partial \mathcal{L}(x)}{\partial \dot{\phi}(x)}. \tag{3.1.1}$$

The Hamiltonian density is then

$$\mathcal{H} = \pi(x)\dot{\phi}(x) - \mathcal{L}(x). \tag{3.1.2}$$

As in classical mechanics, we eliminate $\dot{\phi}$ in favour of π in our equations. Integrating over all space, we would obtain the Hamiltonian.

Example (Scalar field with potential)**:** We add a potential $V(\phi)$ to our free Hamiltonian.

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - V(\phi). \tag{3.1.3}$$

The conjugate momentum is $\pi = \dot{\phi}$, which means that the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + V(\phi). \tag{3.1.4}$$

The Hamiltonian is

$$H = \int \mathrm{d}^3x \, \mathcal{H}. \tag{3.1.5}$$

The equations of motion of the scalar field are given by Hamilton's equations:

$$\dot{\phi} = \frac{\delta H}{\delta \pi} \qquad \dot{\pi} = -\frac{\delta H}{\delta \phi}. \tag{3.1.6}$$

> **Remark:** In this case, the Hamiltonian is the same as the total energy E.

Since the physics remains unchanged when changing from the Lagrangian to the Hamiltonian formulation, the result is Lorentz invariant. However, since we picked out a preferred time, the Hamiltonian formulation is *not* manifestly Lorentz invariant.

3.2 Canonical Quantisation

Recall that in transitioning from classical to quantum mechanics, canonical quantisation tells us to take a set of generalised coordinates q_a and p_a and promote them to operators. We replace Poisson brackets with commutators.

$$[q_a, p^b] = i\delta_a^b \qquad (\hbar = 1). \tag{3.2.1}$$

Now, in the transition from classical to quantum field theory, we do the analogous thing for the fields $\phi_a(\mathbf{x})$ and $\pi_b(\mathbf{x})$.

Definition 3 (quantum field)**:** A *quantum field* is an operator valued function of space that obeys the following commutation relation: Fields and momenta commute amongst themselves

$$[\phi_a(\mathbf{x}), \phi_b(\mathbf{y})] = 0 \qquad [\pi^a(\mathbf{x}), \pi^b(\mathbf{y})] = 0. \tag{3.2.2}$$

However, they do not commute between each other

$$[\phi_a(\mathbf{x}), \pi^b(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y})\delta_a^b. \tag{3.2.3}$$

We are in the Schrödinger picture, so the operators $\phi_a(\mathbf{x})$ and $\pi^a(\mathbf{x})$ are *not* functions of t—all t dependence is in states, which evolve according to the Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle. \tag{3.2.4}$$

As such, we wish to know the spectrum of H, but this is extremely hard due to the infinite number of degrees of freedom (at least 1 for each spatial coordinate $\mathbf{x}$).

For free theories, coordinates evolve independently. These free theories have $\mathcal{L}$ quadratic in the fields ϕ or $\partial_\mu\phi$. As a consequence, this gives linear equations of motion.

Example (Klein-Gordon)**:** We saw that the EL equations for the KG theory of a real scalar field $\phi(\mathbf{x}, t)$ are

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0. \tag{3.2.5}$$

To see why, we take the Fourier transform

$$\phi(\mathbf{x}, t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p}, t). \tag{3.2.6}$$

Then we see that the classical KG equation becomes

$$\left[\frac{\partial^2}{\partial t^2} + (\mathbf{p}^2 + m^2)\right]\phi(\mathbf{p}, t) = 0. \tag{3.2.7}$$

This is the equation for the harmonic oscillator, vibrating at frequency $\omega_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2}$. So the solution to the Klein-Gordon equation is a superposition of simple harmonic oscillators (SHOs) each vibrating at a different frequency and amplitude. To quantise $\phi(\mathbf{x}, t)$, we need to quantise this infinite number of SHOs.

> **Remark:** These SHOs are not coupled since this is a free theory. If we include higher powers in the Lagrangian, these become coupled and very difficult to solve.

3.3 Review of SHO in 1d QM

Writing the position as q and momentum as p, the Lagrangian is for the simple harmonic oscillator in 1d quantum mechanics is

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2. \tag{3.3.1}$$

The Hamiltonian is then easily found to be

$$H = p\dot{q} - L = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2. \tag{3.3.2}$$

We recall that the canonical quantisation condition is enforced by requiring the position and momentum operators to satisfy the canonical commutation relation:

$$[q, p] = i. \tag{3.3.3}$$

We then define operators

$$a^\dagger = \frac{-i}{\sqrt{2\omega}}p + \sqrt{\frac{\omega}{2}}q \qquad a = \frac{+ip}{\sqrt{2\omega}} + \sqrt{\frac{\omega}{2}}q. \tag{3.3.4}$$

In terms of these operators, we can write our original operators as

$$q = \frac{a + a^\dagger}{\sqrt{2\omega}}, \qquad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger). \tag{3.3.5}$$

And the commutation relation becomes

$$[a, a^\dagger] = 1. \tag{3.3.6}$$

Similarly, we can rewrite the Hamiltonian as

$$H = \frac{1}{2}\omega(aa^\dagger + a^\dagger a) = \omega(a^\dagger a + \frac{1}{2}), \tag{3.3.7}$$

where we used the commutator in the last equality.

The commutation relations between these ladder operators and the Hamiltonian are

$$[H, a^\dagger]\omega a^\dagger, \qquad [H, a] = -\omega a. \tag{3.3.8}$$

Exercise 3.3.1: Work these out!

These ensure that $a, a^\dagger$ take us between energy states: If $H|E\rangle = E|E\rangle$, then

$$H(a^\dagger|E\rangle) = (E+\omega)(a^\dagger|E\rangle) \tag{3.3.9}$$

$$H(a|E\rangle) = (E-\omega)(a|E\rangle). \tag{3.3.10}$$

We ended up with a ladder of states with energy $\cdots, E-\omega, E, E+\omega, E+2\omega, \cdots$. To ensure a stable system, the energy must be bounded from below by the existence of a *ground-state* $|0\rangle$ that is annihilated by the lowering operator: $a|0\rangle = 0$. Acting on this state with the Hamiltonian, we find the *ground-state energy* $H|0\rangle = \frac{1}{2}\omega|0\rangle$.

Excited states arise from a repeated application of $a^\dagger$:

$$|n\rangle \equiv (a^\dagger)^n|0\rangle, \qquad H|n\rangle = (n+\frac{1}{2})\omega|n\rangle. \tag{3.3.11}$$

Remark: We have not normalised these states: $\langle n|n\rangle \neq 1$.

Remark: We are, in the absence of gravitational interactions, only interested in *energy differences*, rather than absolute values. As a result, we want to set the ground state energy to zero.

Definition 4 (normal ordering)**:** A *normal ordered* operator $:\mathcal{O}:$, is one in which we put all the a s on the right of all the $a^\dagger$s.

The normal ordered Hamiltonian is $H = \omega a^\dagger a$. With this definition, the ground-state energy is $H|0\rangle = 0$.

3.4 The Free Scalar Field

Let us apply the SHO to free fields. We define the Fourier transform of a scalar field as

$$\phi(\mathbf{x}) = \int đ^3 p \, \frac{1}{\sqrt{2\omega_p}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right), \tag{3.4.1}$$

where $\omega_p^2 = p^2 + m^2$, writing $p = \sqrt{|\mathbf{p}|}^2$. Since we explicitly added a Hermitian conjugate constant $a^\dagger$, this definition of the Fourier integral makes it evident that $\phi = \phi^\dagger$ is a real scalar field. We also expand the conjugate momentum $\pi(\mathbf{x})$ in terms of these Fourier modes

$$\pi(\mathbf{x}) = \int đ^3 p (-i) \sqrt{\frac{\omega_p}{2}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right). \tag{3.4.2}$$

The process of promoting the constants $a_{\mathbf{p}}$ and $a^\dagger$ to quantum operators—as it turns out, these will be annihilating and creating particles of momentum $\mathbf{p}$— acting on a quantum state is called *second quantisation*; we have inserted an infinite number of QHOs into momentum space. As a result, the fields ϕ and π are also quantum operators. For these quantum fields, we want to impose commutation relations:

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = 0, \qquad [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0, \tag{3.4.3a}$$

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y}). \tag{3.4.3b}$$

> **Remark:** We are in the Schrödinger picture, where time $t = x^0$ is in the states, not in the fields.

Claim: These are equivalent to

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0, \qquad [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0, \tag{3.4.4a}$$

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = \eth^3(\mathbf{p} - \mathbf{q}). \tag{3.4.4b}$$

Proof. Let us check that (3.4.4b) implies (3.4.3b) .

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = \int đ^3 p \, đ^3 q \, \frac{(-i)}{2} \frac{\omega_q}{\omega_p} \left\{ - \left[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger\right] e^{i\mathbf{p}\cdot\mathbf{x} - i\mathbf{q}\cdot\mathbf{y}} + \left[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}\right] e^{-i\mathbf{p}\cdot\mathbf{x} + i\mathbf{q}\cdot\mathbf{y}} \right\} \tag{3.4.5}$$

$$= \int đ^3 p \left(\frac{-i}{2} \right) \left\{ -e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \right\} \tag{3.4.6}$$

$$= i\delta^3(\mathbf{x} - \mathbf{y}) \tag{3.4.7}$$

Exercise 3.4.1: Complete the proof.

□

Now, compute the Hamiltonian H in terms of the ladder operators $a_{\mathbf{p}}$ and $a_{\mathbf{q}}^\dagger$:

$$H = \frac{1}{2}\int \mathrm{d}^3x \left[\pi^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right] \tag{3.4.8}$$

$$\begin{aligned} &= \frac{1}{2}\int \mathrm{d}^3x\, đ^3p\, đ^3q \\ &\quad \times \Big\{ -\frac{\sqrt{\omega_p\omega_2}}{2}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right)\left(a_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}} - a_{\mathbf{q}}^\dagger e^{i\mathbf{q}\cdot\mathbf{x}}\right) \qquad \text{from } \pi^2 \\ &\quad - \frac{1}{2}\sqrt{\omega_p\omega_2}\left(ipa_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - ipa_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right)\left(iqa_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}} - iqa_{\mathbf{q}}^\dagger e^{i\mathbf{q}\cdot\mathbf{x}}\right) \qquad \text{from } (\boldsymbol{\nabla}\phi)^2 \\ &\quad - \frac{m^2}{2\sqrt{\omega_p\omega_2}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right)\left(a_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}} + a_{\mathbf{q}}^\dagger e^{-i\mathbf{q}\cdot\mathbf{x}}\right)\Big\} \qquad \text{from } m^2\phi^2 \end{aligned} \tag{3.4.9}$$

$$= \cdots$$

$$= \frac{1}{4}\int đ^3p\,\Big\{\underbrace{\left(-\omega_p + \frac{p^2}{\omega_p} + \frac{m^2}{\omega_p}\right)}_{\omega_p^2 = p^2+m^2 \implies =0}\left(a_{\mathbf{p}}a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger\right) + \left(\omega_p + \frac{p^2}{\omega_p} + \frac{m^2}{\omega_p}\right)\left(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right)\Big\} \tag{3.4.10}$$

$$\therefore H = \frac{1}{2}\int đ^3p\,\omega_p(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}). \tag{3.4.11}$$

This is the Hamiltonian of an infinite number of uncoupled harmonic oscillators, each at frequency $\omega_p = \sqrt{p^2+m^2}$.

3.4.1 The Vacuum

Definition 5 (vacuum)**:** The *vacuum* or *ground state* $|0\rangle$ is defined to be the state with the property that it is annihilated by all annihilation operators:

$$a_{\mathbf{p}}\,|0\rangle = 0, \quad \forall p. \tag{3.4.12}$$

Acting on the vacuum with the Hamiltonian, we expect to find the ground-state energy:

$$H\,|0\rangle = \int đ^3p\,\omega_p\left(\cancel{a_{\mathbf{p}}^\dagger a_{\mathbf{p}}} + \frac{1}{2}[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]\right)|0\rangle \tag{3.4.13}$$

$$= \frac{1}{2}\int đ^3p\,\omega_p \delta\!\!\bar{}^3(0)\,|0\rangle\,. \tag{3.4.14}$$

We find an infinite ground-state energy! And worse, this infinity is twofold; we have the obvious factor of $\delta\!\!\bar{}^3(0)$, but also a less obvious infinity that arises because the integral $\int đ^3p\,\omega_p$ does not converge; the energy of each harmonic oscillator diverges $\omega_p = \sqrt{p^2+m^2} \to \infty$ as $p \to \infty$. This latter divergence is called a *high-frequency* or *ultra-violet* divergence. We can resolve this issue by introducing a UV-cutoff Λ as an upper integration bound, which tames the integral.

Remark: Physically, this can be justified by postulating that Λ corresponds to the highest energy (or equivalently, Λ^{-1} the smallest length scale) on which this QFT is valid. On higher energy or smaller length scales, new physics will need to be introduced. In statistical field theory, this smallest length scale is on the order of the lattice spacing. On smaller length scales, the continuum assumptions of the theory break down and new physics—accounting for the lattice effects—will need to be introduced.

Concerning the former infinity, $\eth^3(0)$, we have another resolution up our sleeves: Heuristically, one might claim that in non-gravitational physics, only energy differences are relevant for the dynamics of the system. As such, one might redefine the Hamiltonian to subtract the commutator $[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]$ which led to the divergence. More concretely, this arbitrary choice that we can make in the Hamiltonian points to an ambiguity arising in moving from the classical to the quantum theory:

Classically, the Hamiltonian $H = \frac{1}{2}(\omega q - ip)(\omega q + ip)$ is exactly the same as $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$. However, in QM, the operator order matters, and upon quantisation it gives us $H = \omega a^\dagger a$. The analogous thing happens in QFT with momentum modes, which means that the Hamiltonian that we should really use is

$$H = \int đ^3 p \, \omega_p a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \tag{3.4.15}$$

Using this Hamiltonian, the ground-state energy of the vacuum has been redefined to $H\left|0\right\rangle = 0$.

This ambiguity in the definition of QFT operators can be resolved by introducing the concept of normal ordering.

Definition 6 (normal order)**:** A *normal ordered* string of operators $:\phi_1(\mathbf{x}_1)\cdots\phi_n(\mathbf{x}_n):$ is defined to be the same string of operators, except that all annihilation operators a are moved to the RHS of all creation operators $a^\dagger$.

Since the annihilation operators annihilate the vacuum, the $\eth^3(0)$ infinity is discarded when using the normally ordered Hamiltonian

$$: H := \int đ^3 p \, \omega_p a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \tag{3.4.16}$$

3.4.2 Particles

It is easy to verify the commutation relation with the Hamiltonian

$$[H, a_{\mathbf{p}}^\dagger] = \omega_p a_{\mathbf{p}}^\dagger \qquad \text{and} \qquad [H, a_{\mathbf{p}}] = -\omega_p a_{\mathbf{p}}. \tag{3.4.17}$$

The operator $a_{\mathbf{p}}^\dagger$ increases the energy by the constant value ω_p.

Now consider the state $|\mathbf{p}'\rangle = a^\dagger_{\mathbf{p}'}|0\rangle$. The energy of this state is

$$H|p'\rangle = \int đ^3p\,\omega_p a^\dagger_{\mathbf{p}} a_{\mathbf{p}} a^\dagger_{\mathbf{p}'}|0\rangle \tag{3.4.18}$$

$$= \int đ^3p\,\omega_p a^\dagger_{\mathbf{p}}\left([a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] - a^\dagger_{\mathbf{p}'} a_{\mathbf{p}}\right)|0\rangle \tag{3.4.19}$$

$$= \omega_{p'} a^\dagger_{\mathbf{p}'}|0\rangle = \omega_{p'}|\mathbf{p}'\rangle . \tag{3.4.20}$$

Since $\omega_p = \sqrt{p^2+m^2}$ is the relativistic dispersion relation of a particle of mass m and momentum p, we interpret the state $|\mathbf{p}\rangle$ as the one-particle momentum eigenstate.

> **Remark:** Mass m comes from the term $\frac{1}{2}m^2\phi^2$ in the Lagrangian; in general, the coefficient of the quadratic field term in $\mathcal{L}$ allows us to identify the mass.

Properties of $|\mathbf{p}\rangle$

We previously defined the momentum operator $\mathbf{P} = -\int \mathrm{d}^3x\,\pi(\mathbf{x})\boldsymbol{\nabla}\phi(\mathbf{x})$.

Exercise 3.4.2: Show that $\mathbf{P} = \int đ^3p\,\mathbf{p}a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$. Then show that $|\mathbf{p}\rangle$ is actually a momentum eigenstate, i.e. that $P|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$.

We could also act with the angular momentum operator J^i to find $J^i|\mathbf{p}\rangle = 0$. This is a spin-zero state.

Exercise 3.4.3: Do all of Example sheet 1.

Multi-Particle States $|\mathbf{p}_1,\ldots,\mathbf{p}_n\rangle$

We define $|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle :- a_{\mathbf{p}_1}\cdots a_{\mathbf{p}_n}|0\rangle$. Since the creation operators $a_{\mathbf{p}_i}$ commute with each other, we have $|\mathbf{p},\mathbf{q}\rangle = |\mathbf{q},\mathbf{p}\rangle$. These multi-particle states are thus symmetric under interchange of particle label, which means that we are dealing with *bosons*.

The full Hilbert space is spanned by the set

$$\{|0\rangle, |\mathbf{p}_1\rangle, |\mathbf{p}_1,\mathbf{p}_2\rangle, \cdots\} . \tag{3.4.21}$$

A space that is built from these basis states is called a *Fock space*.

We can also introduce a *number operator*

$$N = \int đ^3p\, a^\dagger_{\mathbf{p}}a_{\mathbf{p}}, \tag{3.4.22}$$

which counts the number of particles.

Exercise 3.4.4: Show that $N|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle = n\mathbf{p}_1,\cdots,\mathbf{p}_n$.

Also, the number operator commutes with the Hamiltonian, $[N,H]=0$, which means that particle number is conserved in the free quantum field theory. This ceased to hold true once we allow interactions between the particles.

Remark: The momentum eigenstates are not localised in space. We can create a localised state via Fourier transform

$$|\mathbf{x}\rangle = \int đ^3p \, e^{i\mathbf{p}\cdot\mathbf{x}} \, |\mathbf{p}\rangle \, . \tag{3.4.23}$$

More generally, when we talk about real particles, we describe a wave-packet partially localised in position and partially in momentum space:

$$|\psi\rangle = \int đ^3p \, e^{i\mathbf{p}\cdot\mathbf{x}} \psi(\mathbf{p}) \, |\mathbf{p}\rangle \tag{3.4.24}$$

where for example $\psi(\mathbf{p}) \propto e^{-|\mathbf{p}|^2/2m^2}$. Neither $|\mathbf{x}\rangle$ nor $|\psi\rangle$ are eigenstates of H—like in usual QM.

3.5 Relativistic Normalisation

We define the vacuum normalisation to be $\langle 0|0\rangle = 1$. So we have for a general momentum eigenstate,

$$\langle \mathbf{p}|\mathbf{q}\rangle = \langle 0| \, [a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] \, |0\rangle = \delta\!\!\!{}^-{}^3(\mathbf{p}-\mathbf{q}). \tag{3.5.1}$$

But is this Lorentz invariant? If we perform a Lorentz transformation, the momenta change as $p^\mu \to \Lambda^\mu{}_\nu p^\nu :- p'^\mu$, such that $|\mathbf{p}\rangle \to |\mathbf{p}'\rangle$. We want the states $|\mathbf{p}\rangle$ and $|\mathbf{p}'\rangle$ to be related by a unitary transformation:

$$|\mathbf{p}\rangle \to |\mathbf{p}'\rangle = U(\Lambda) \, |\mathbf{p}\rangle \, . \tag{3.5.2}$$

This is because the inner product would be invariant under such a transformation:

$$\langle \mathbf{p}|\mathbf{q}\rangle \xrightarrow{\text{LT}} \langle \mathbf{p}| \, U^\dagger(\Lambda) U(\Lambda) \, |\mathbf{q}\rangle = \langle \mathbf{p}|\mathbf{q}\rangle \, . \tag{3.5.3}$$

Take the identity operator on one-particle states to be

$$1 = \int đ^3p \, |\mathbf{p}\rangle \, \langle \mathbf{q}| \, . \tag{3.5.4}$$

Notice that neither the integral measure $\int đ^3p$, not the outer product $|\mathbf{p}\rangle \, \langle \mathbf{q}|$ is Lorentz invariant by itself. However, the identity operator evidently is. Let us analyse this.

Claim: $\int \frac{\mathrm{đ}^3p}{2E_p}$ is Lorentz invariant.

Proof. The integral over the full four momentum $\int \mathrm{d}^4p$ is obviously Lorentz invariant, since Lorentz transformations are elements of SO(1,3), meaning that $\det \Lambda = 1$ and the Jacobian is unity when changing frames. The relativistic dispersion relation for a massive particle $E_p^2 = p_0^2 = |\mathbf{p}|^2 + m^2$ is Lorentz invariant (ie. holds in all frames). The choice of branch for p_0 is also Lorentz invariant.

$$\int \mathrm{d}^4p \, \delta(p_0^2 - |\mathbf{p}|^2 - m^2)|_{p_0>0} \qquad \text{is LI} \tag{3.5.5}$$

Finally, we use the properties of the delta function

$$\delta(g(x)) = \sum_{x_i \text{ roots of } g} \frac{\delta(x - x_i)}{|g'(x_i)|} \tag{3.5.6}$$

to show that the above integral is

$$\int \mathrm{d}^4 p\, \delta(p_0^2 - |\mathbf{p}|^2 - m^2)|_{p_0>0} = \int \frac{\mathrm{d}^3 p}{2E_p} \tag{3.5.7}$$

□

Claim: $2E_p \delta^3(\mathbf{p} - \mathbf{q})$ is the Lorentz invariant delta function.

Proof.

$$\int \frac{\mathrm{d}^3 p}{2E_p} 2E_p \delta^3(\mathbf{p} - \mathbf{q}) = 1 \tag{3.5.8}$$

Since the integral measure is Lorentz invariant by the previous claim, and the right hand side is just a number, the claim follows. □

Hence, we can define relativistically normalised states as

$$|p^\mu\rangle :- |p\rangle = \sqrt{2E_p}\, |\mathbf{p}\rangle\, \sqrt{2E_p} a^\dagger_{\mathbf{p}}\, |0\rangle\,. \tag{3.5.9}$$

Then the inner product is

$$\langle p|q\rangle = (2\pi)^3 2\sqrt{E_p E_q}\delta^3(\mathbf{p} - \mathbf{q}) \tag{3.5.10}$$

and we re-write the completeness relation on single-particle states as

$$1 = \int \frac{đ^3 p}{2E_p}\, |p\rangle\, \langle p|\,. \tag{3.5.11}$$

3.6 Free Complex Scalar Field

Let $\psi \in \mathbb{C}$ be a complex scalar field. The associated Lagrangian density is

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - \mu^2 \psi^* \psi, \tag{3.6.1}$$

where $\mu \in \mathbb{R}$. Applying the Euler-Lagrange equations give

$$\partial_\mu \partial^\mu \psi + \mu^2 \psi = 0, \qquad \partial_\mu \partial^\mu \psi + \mu^2 \psi = 0. \tag{3.6.2}$$

Again, we can expand this as a Fourier integral. However, this time, unlike in the case of the scalar field ϕ, we want the conjugate fields to be different $\psi \neq \psi^*$. This is achieved by taking

$$\psi = \int \frac{đ^3 p}{\sqrt{2E_p}} \left(b_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + c^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\right), \qquad \psi^\dagger = \int \frac{đ^3 p}{\sqrt{2E_p}} \left(b^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} + c_{\mathbf{p}} e^{+i\mathbf{p}\cdot\mathbf{x}}\right). \tag{3.6.3}$$

The conjugate momenta to these fields are

$$\pi = \int đ^3p(-i)\sqrt{\frac{E_p}{2}}\left(b^\dagger_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - c_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right), \qquad \pi^\dagger = \int đ^3p(-i)\sqrt{\frac{E_p}{2}}\left(b_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - c^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right). \tag{3.6.4}$$

Finally, the commutation relations

$$[\psi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}), \qquad [\psi(\mathbf{x}), \pi^\dagger(\mathbf{y})] = 0, \qquad \text{etc.}\ldots \tag{3.6.5}$$

can then be shown to be equivalent to

$$[b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) = [c_{\mathbf{p}}, c^\dagger_{\mathbf{q}}] \tag{3.6.6}$$

with all other commutators vanishing. The interpretation of this is that there are two different particles created by $b^\dagger_{\mathbf{p}}$ and $c^\dagger_p$. Both have the same mass m and spin 0. These are interpreted as particle and anti-particle. Retrospectively, we see that for a real scalar field, the particle is its own anti-particle.

The conserved charge associated with a phase rotation invariance of the Lagrangian is

$$\begin{aligned} Q &= i\int \mathrm{d}^3x\,(\dot{\psi}^*\psi - \psi^*\dot{\psi}) &(3.6.7)\\ &= i\int \mathrm{d}^3x\,(\pi\psi - \psi^\dagger\pi^\dagger). &(3.6.8)\end{aligned}$$

We then insert the respective Fourier expansions in terms of the ladder operators. After normal ordering, this conserved charge becomes

$$Q = \int đ^3p\left(c^\dagger_{\mathbf{p}}c_{\mathbf{p}} - b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\right) = N_c - N_b. \tag{3.6.9}$$

Since $[Q, H]$, the number of particles minus the number of anti-particles is conserved.

> **Remark:** We are in the free theory, where N_c and N_b are individually conserved. In the interacting theory, this is not true any more, but the charge Q will still be conserved. This corresponds to the conservation of electric charge, even in interacting theories.

3.7 The Heisenberg Picture

So far, in the Schrödinger picture, the Lorentz invariance has not been overt. In particular, the field operators $\phi(\mathbf{x})$ did only depend on space and not on time, whereas the states evolve in time via

$$i\frac{\mathrm{d}\,|p\rangle}{\mathrm{d}t} = H\,|p\rangle = E_p\,|p\rangle\,. \tag{3.7.1}$$

Physically, the expectation values $\langle\ldots|\ldots\rangle$ corresponding to probability amplitudes that we will measure. We can leave these invariant by making the following transformations. Taking the unitary transformation $U(t) = e^{iHt}$, we define operators O_H in the Heisenberg picture as

$$O_H(t) = U(t)O_SU^\dagger(t). \tag{3.7.2}$$

This implies that the operators evolve in time via

$$\frac{\mathrm{d}O_H}{\mathrm{d}t} = i[H, O_H]. \tag{3.7.3}$$

Note that at $t = 0$, the Schrödinger and Heisenberg operators coincide. The operators in the Heisenberg picture satisfy equal-time commutation relations:

$$[\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = 0 = [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] \tag{3.7.4}$$

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}). \tag{3.7.5}$$

Exercise 3.7.1: We can check that $\frac{\mathrm{d}\phi}{\mathrm{d}t} = i[H, \phi]$ means that the Heisenberg operator ϕ satisfies the Klein-Gordon equation $\partial_\mu \partial^\mu \phi + m^2\phi = 0$. This is now a vector equation.

Notation: We denote the full four-vector spacetime dependence as $\phi(\mathbf{x}, t) = \phi(x)$.

We write the Fourier transform of $\phi(x)$ by deriving

$$U(t) a_{\mathbf{p}} U^\dagger(t) = e^{-iE_p t} a_{\mathbf{p}}, \qquad U(t) a_{\mathbf{p}}^\dagger U^\dagger(t) = e^{+iE_p t} a_{\mathbf{p}}^\dagger. \tag{3.7.6}$$

Then, in the Heisenberg picture, $\phi(x)$ is given by

$$\phi(x) = \int \frac{\mathrm{đ}^3 p}{\sqrt{2E_p}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}} e^{+ip\cdot x} \right). \tag{3.7.7}$$