Black Holes

Part III Lent 2019

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Contents

Administrative

- Office hours: Fridays 2pm, B2.09
- Lecture notes: www.damtp.cam.ac.uk/user/hsr1000 and /examples
  - everything in notes is examinable
- Conventions: $G = c = 1$, ignore Λ (negligible for Black holes)
- indices: $\mu, \nu, \ldots$ refer to *specific* basis,

  $a, b, c, \ldots$ 'abstract indices' (Penrose) refer to *any* basis

$$\text{e.g.} \qquad \Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}(g^{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}) \qquad R = g^{ab} R_{ab} \tag{1}$$

- Books listed in lecture notes (Wald etc)

1 Spherical Stars

1.1 Cold stars

Gravitational force, which wants the star to contract, is balanced by pressure of nuclear reactions. If we wait long enough, star will exhaust nuclear fuel and the star will contract. What happens next? Any new source of pressure will have to be non-thermal, since time will cause the star to cool down. There is one such source of pressure coming from the Pauli principle. If you have a gas of fermions, it will resist compression. This is called 'degeneracy pressure'. This is entirely a quantum effect, which is not thermal.

Definition 1: A *white dwarf* is a star in which gravity is balanced by electron degeneracy pressure.

This is a very dense star: a white dwarf with the same mass as our sun, $M = M_{\odot}$ has a radius $R \sim \frac{1}{100} R_{\odot}$.

However, not all stars can end their life this way. The maximum mass of a white dwarf is the Chandrasekhar limit $M_{wd} \leq 1.4 M_{\odot}$.

If matter is sufficiently dense, we have inverse β-decay, which turns the protons in the star into neutrons. We therefore get a second class of star:

Definition 2: A *neutron dwarf* is a star in which gravity is balanced by neutron degeneracy pressure.

These are tiny: taking a neutron star with $M \sim M_{\odot}$, then $R \sim 10$ km. Compare this with the radius of our sun, which is $R_{\odot} \simeq 7 \times 10^5$ km. Because they are so dense, their gravitational force on the surface is very strong. In terms of Newtonian gravity, we have $|\Phi| \sim 0.1$ at the surface. General relativity becomes negligible if $|\Phi| \ll 1$. So here, general relativity is important.

We will show that for any cold star there is a maximal mass around five solar masses. This bound will be independent of our ignorance of the properties of matter at such high densities.

In order to make this problem tractable, we will assume that the star is spherically symmetric and time independent.

1.2 Spherical Symmetry

Definition 3: The *unit round metric* on S^2 is $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$.

Roughly speaking, spherical symmetry is the isometry group of this metric. The isometry group in this case is $SO(3)$.

Definition 4: A spacetime is *spherically symmetric* if its isometry group contains an $SO(3)$ subgroup, whose orbits are 2-spheres.

> Pick a point and act on it with all $SO(3)$ elements. It will then fill out a sphere with unit round metric.

Definition 5: In a spherically symmetric spacetime (M, g) , the *area radius function* is

$$\begin{aligned} r: M &\to \mathbb{R} \\ p &\longmapsto r(p) = \sqrt{\frac{A(p)}{4\pi}} \end{aligned} \tag{1.1}$$

where $A(p)$ is the area of the S^2 orbit through p.

> You can think of r as the radial coordinate. Instead of defining r in terms of distance from the origin (which does not exist on S^2), we define it here via the area.

Remark: The S^2 has induced metric $r(p)^2 d\Omega^2$.

1.3 Time-independence

Definition 6 (stationary)**:** The spacetime (M, g) is *stationary* if there exists a Killing vector field (KVF) k^a, which is everywhere timelike ($g_{ab}k^a k^b < 0$).

> Our spacetime has a time-translation symmetry.

Pick some hypersurface Σ transverse to k^a . We can then pick coordinates x^i , $i = 1, 2, 3$ on Σ.

We assign coordinates (t, x^i) to point parameter distance t along an integral curve k^a through a point on Σ with coordinates x^i . This implies that $k = \partial/\partial t$, implying that the metric in independent of t (since k^a is Killing).

$$ds^2 = g_{00}(x^k)dt^2 + 2g_{0i}(x^k)dt dx^i + g_{ij}(x^k)dx^i dx^j \tag{1.2}$$

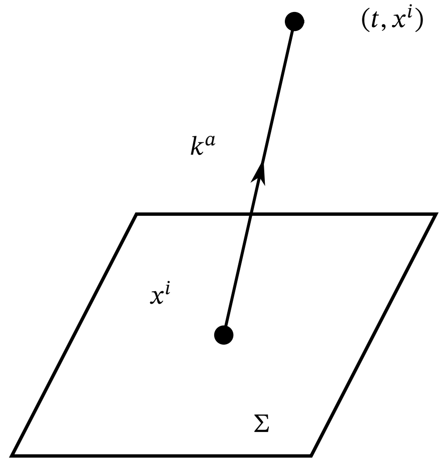

Figure 1.1

with $g_{00} < 0$. Conversely, any metric of this form is stationary.

This is the weakest notion of time-independence we can use. There is also a more refined notion. Before we can introduce that, we need to talk about hypersurface orthogonality.

Claim 1: Let Σ be a hypersurface of constant $f = 0$ on Σ where $f : M \to \mathbb{R}$ a smooth function where $df \neq 0$ on Σ. Then df is normal to Σ.

Proof. Let t^a be a vector that is tangent to Σ. Then

$$df(t) = t(f) = t^\mu \partial_\mu f = 0 \tag{1.3}$$

since f is constant on Σ. □

> Normals to a surface are not unique. For example, we can rescale f to get another normal on Σ. In fact, we can also add something that vanishes on Σ.

Claim 2: If n is also normal to Σ, then $n = gdf + fn'$, where $g \neq 0$ on Σ and n' is a smooth 1-form.

Proof. By the rules of the exterior derivative,

$$dn = dg \wedge df + df \wedge n' + f dn' \tag{1.4}$$

Evaluating this on Σ gives

$$dn|_\Sigma = (dg - n') \wedge df \;\Rightarrow\; n \wedge dn|_\Sigma = 0, \tag{1.5}$$

as $n \propto df$ on Σ. □

This is very useful since there is also a converse of this statement:

Theorem 1 (Frobenius)**:** If $n \neq 0$ is a 1-form such that $n \wedge dn \equiv 0$, then $\exists$ functions g, f such that $n = g df$. So n is normal to surfaces of constant f. We say that n is *hypersurface orthogonal*.

Definition 7 (static)**:** A spacetime (M, g) is *static* if there is a hypersurface-orthogonal timelike Killing vector field.

Remark: This is a refinement since static $\Rightarrow$ stationary.

By Frobenius' theorem, we can choose $\Sigma \perp k^a$ when defining (t, x^i) (since k^a is hypersurface-orthogonal). But Σ is $t = 0$, normal to Σ is dt. Therefore, $k_\mu\big|_{t=0} \propto (1,0,0,0)$. In particular, the spatial components in these coordinates are $k_i\big|_{t=0} = 0$, but $k_i = g_{0i}(x^k)$. Therefore, $g_{0i}(x^k) = 0$. In a static spacetime, the off-diagonal elements of the metric are zero.

$$ds^2 = g_{00}(x^i)dt^2 + g_{ij}(x^k)dx^i dx^j \qquad (g_{00} < 0.) \tag{1.6}$$

There is now an additional symmetry present. We have a discrete time-reversal symmetry $(t, x^i) \to (-t, x^i)$.

Roughly speaking, static means 'time-independent and invariant under time-reversal'.

Example: A rotating star can be stationary, but not static.

> Static means non-rotating.

1.4 Static, spherically symmetric spacetimes

Let us talk about a spherical, non-rotating star. More formally, we will assume the isometry group $\mathbb{R} \times SO(3)$.

> $SO(3)$ are the spatial rotations. $\mathbb{R}$ are the time-translations associated to the timelike Killing vector field k^a.

Claim 3: This implies that the spacetime is static (rotation breaks spherical symmetry).

On Σ, choose coordinates $x^i = (r, \theta, \varphi)$, where r is defined via the area-radius. A consequence of the spherical symmetry is that the metric must take the following form on Σ

$$ds^2\big|_\Sigma = e^{2\Psi(r)}dr^2 + r^2 d\Omega^2 \tag{1.7}$$

(this is because $drd\theta$ or $drd\varphi$ break spherical symmetry.)

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Psi(r)}dt^2 + r^2 d\Omega^2. \tag{1.8}$$

The choice of g_{00} is inspired by the Newtonian limit.

At the moment there is no origin. There is no reason to think of r as the distance to the origin. In fact, it is not the distance to the origin.

For a static, spherically symmetric star, we use the metric (1.8). To find Φ and Ψ, we need to solve the Einstein equations. In order to find those, we need to determine what matter the star contains.

We will model the matter inside the star as a perfect fluid with energy-momentum tensor

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}, \tag{1.9}$$

where u_a is the velocity of the fluid, obeying $g_{ab}u^a u^b = -1$. The quantities ρ and p are, respectively, the energy density and the pressure in the fluid's rest frame.

Time-independence implies that $u^a = e^{-\Phi}(\frac{\partial}{\partial t})^a$. The velocity is fixed by the symmetry assumptions, which also imply that $\rho = \rho(r)$ and $p = p(r)$ can only be functions of r. Also $\rho, p = 0$ for $r > R$, where R is the radius of the star.

1.5 Tolman–Oppenheimber–Volkoff Equations

Solving the Einstein equation imposes the fluid equations, so we do not separately need to deal with those. However, in what follows, it will actually be slightly easier to derive one of the following equations by using the fluid equation $\nabla_\mu T^{\mu\nu} = 0$ instead of some components of the Einstein equations.

By symmetry, there are only really 3 equations to solve. To write these down more concisely, we define $m(r)$ by the relation

$$e^{2\Psi(r)} = (1 - \frac{2m(r)}{r})^{-1} \overset{\text{LHS}>0}{\Rightarrow} m(r) < r/2. \tag{1.10}$$

Exercise 1.1 (Sheet 1)**:** Using the $(\mu\nu)$ component of the Einstein equation, we can derive

$$(tt): \qquad \frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho \tag{TOV 1}$$

$$(rr): \qquad \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{m + 4\pi r^3 \rho}{r(r-2m)} \tag{TOV 2}$$

$$\nabla_\mu T^{\mu\nu} = 0 \qquad \frac{\mathrm{d}p}{\mathrm{d}r} = -(p+\rho)\frac{m + 4\pi r^3 p}{r(r-2m)} \tag{TOV 3}$$

where instead of using a third Einstein equation, it is easiest to use $\nabla_\mu T^{\mu\nu} = 0$ to derive (TOV 3).

We have three equations, but four unknowns (m, Φ, ρ, π) . However, luckily we have some extra information coming from *thermodynamics*.

A cold star has $T = 0$ but $T = T(\rho, p)$, so $T = 0$ fixes some relation $p = p(\rho)$. This is known as a “barotopic equation of state”.

We will not need much information about this relation. However, we will assume that $\rho, p > 0$ and $\frac{dp}{d\rho} > 0$.

> Else, we have an unstable fluid: an increase in density $\delta\rho > 0$ would cause a decrease in pressure $\delta p < 0$, which causes more fluid to flow into a given volume, causing in turn an even bigger increase in density.

1.6 Outside a Star: Schwarzschild Solution

Outside the star, at $r > R$, we have no matter and therefore $\rho = p = 0$.

From (TOV 1), we then find that $m(r) = M$ is a constant. One can then integrate (TOV 2) to find that $\Phi(r) = \frac{1}{2}\ln\left(1 - \frac{2M}{r}\right) + \Phi_0$, where Φ_0 is some constant of integration.

However, Φ_0 is not physical: as $r \to \infty$, $\Phi(r) \to \Phi_0$, so $g_{tt} \to e^{-2\Phi_0}$ as $r \to \infty$. This means that we can eliminate Φ_0 by absorbing it into the time coordinate via the coordinate transform $A' = e^{\Phi_0}t$. Without loss of generality, we may therefore set $\Phi_0 = 0$. The resulting metric is the *Schwarzschild solution*

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{1.11}$$

We interpret M to be the mass of the star.

At $r = 2M$, the “Schwarzschild radius”, the metric components $g_{\mu\nu}$ (in a coordinate basis) are singular. Since in our derivation every step was sound, this singularity must be inside the star, where the metric is not valid. The star must have

$$R > 2M. \tag{1.12}$$

Remark: To get from GR to Newtonian physics, we take the limit of $c \to \infty$. The inequality

$$R > 2M \quad \xleftrightarrow[\text{units}]{\text{reinstate}} \quad \frac{GM}{c^2R} < \frac{1}{2} \tag{1.13}$$

then becomes trivial, meaning that there is no Newtonian analogue of this new GR effect.

> **Remark:** This is not true for black holes: they violate the assumption of static spacetime.

This inequality is certainly true for the sun, which has a Schwarzschild radius of $2M_\odot \approx 3km$ and a radius of $R_\odot \approx 7 \times 10^5$ km.

1.7 Interior Solution

From (TOV 1), we have that

$$m(r) = 4\pi \int_0^r \rho(r')r'^2 \, \mathrm{d}r' + m_*, \tag{1.14}$$

where m_* is some integration constant.

Let Σ_t denote a surface of constant time t. The metric induced on such a surface is

$$ds^2\big|_{\Sigma_t} = e^{2\Psi(r)}dr^2 + r^2 d\Omega^2. \tag{1.15}$$

We want the metric to be smooth at $r = 0$. This implies that the spacetime is locally flat at $r = 0$. For small r , this spacetime will look like Euclidean space $\mathbb{R}^3$.

As such, a point on S^2 of small radius r must be a distance r from the origin $r = 0$ (since this is true in $\mathbb{E}^3$). For small r we have

$$\therefore \quad r \approx \int_0^r e^{\Phi(r')} \, \mathrm{d}r' \approx e^{\Phi(0)} r \implies \Phi(0) = 0 \tag{1.16}$$

Else, there is some kind of singularity at the origin; the origin would not be smooth.

This means that $m(0) = 0 \implies m_* = 0$ in Eq. (1.14).

This was outside R. Continuity tells us that

$$m(r) = M = 4\pi \int_0^R \rho(r)r^2 \, \mathrm{d}r \,. \tag{1.17}$$

The fact that this is the same as in Newtonian physics is a coincidence; this is not in general true for general relativity.

More specifically, in general relativity, the total energy is obtained by integrating the energy density $\rho(r)$ over the appropriate volume form. On Σ_t, this is

$$e^{\Psi(r)} \underbrace{r^2 \sin\theta dr \wedge d\theta \wedge d\varphi}_{\text{usual volume form on } \mathbb{E}^3} . \tag{1.18}$$

The energy of matter on Σ_t is then

$$E = 4\pi \int_0^R e^{\Psi(r)} \rho(r) r^2 \, \mathrm{d}r \,. \tag{1.19}$$

Since $m > 0$, we find that $e^{\Psi} > 1$. Therefore $E > M$: the energy of the matter in the star is larger than the total energy of the star. This means that there is some gravitational binding energy $E - M$.

Finally, reduces to (TOV 3) $\frac{\mathrm{d}p}{\mathrm{d}r} < 0$. Together with the previously mentioned assumption that $\frac{\mathrm{d}p}{\mathrm{d}\rho} > 0$, this implies that $\frac{\mathrm{d}\rho}{\mathrm{d}r} < 0$.

Exercise 1.2 (Sheet 1)**:** One can then show that

$$\frac{m(r)}{r} < \frac{2}{9}\left[1 - 6\pi r^2 p(r) + (1 + 6\pi r^2 p(r))^{1/2}\right] \tag{1.20}$$

At $r = R$, the surface of the star, the pressure vanishes $p = 0$. This then reduces to the "Buchdahl inequality"

$$R > \frac{9}{4}M, \tag{1.21}$$

which is an improvement on Eq. (1.12).

Now (TOV 1) and (TOV 3) are coupled ordinary differential equations for m and ρ (via $p = p(\rho)$). These can be solved numerically given initial conditions. Eq. (1.14) automatically implies $m(0) = 0$, so we only need to specify $\rho(0) = \rho_c$.

In particular, (TOV 3) implies that the pressure p decreases as we move out towards higher r. We define the radius R by $p(R) = 0$ giving us $R = R(\rho_c)$. Similarly, once we have done this Eq. (1.17) fixes $M = M(\rho_c)$. Finally, we fix Φ by solving (TOV 2) in $r < R$ with initial condition

$$\Phi(R) = \frac{1}{2}\ln\left(1 - \frac{2M}{R}\right). \tag{1.22}$$

As such, for a given equation of state, cold stars form a one-parameter family labelled uniquely by the energy density ρ_c at the center of the star.

1.8 Maximum Mass of Cold Star

The maximum mass M_{max} depends on the equation of state.

> In particular, choosing the density of state for the degenerate electron gas gives the Chandrasekhar limit!

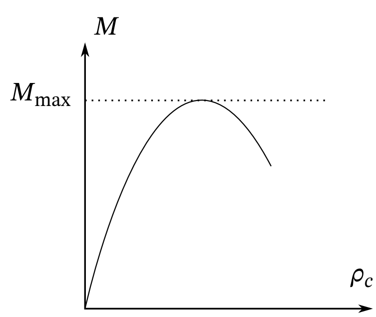

Figure 1.2

Experimentally, we can only know the equation of state up to nuclear density ρ_0 .

Claim 4: The maximum mass is always $M_{\max} \lesssim 5M_\odot$ whatever happens for $\rho > \rho_0$.

Proof. We know that ρ decreases with r. Let us now define two regions:

core region where $\rho > \rho_0$ ($r < r_0$)

envelope region where $\rho < \rho_0$ ($r_0 < r < R$)

We then define the 'core mass' to be $M_0 := m(r_0)$. Then Eq. (1.14) gives

$$M_0 > \frac{4}{3}\pi r_0^3 \rho_0. \tag{1.23}$$

The core mass has a higher density than nuclear density ρ_0.

On the other hand, eq. (1.20) for $r = r_0$ gives that $\frac{m_0}{r_0} < \frac{2}{9}\left[1 - 6\pi r_0^2 p_0 + (1 + 6\pi r_0^2 p_0)^{\frac{1}{2}}\right]$, but we know the quantity $p_0 = p(r_0)$ from the equation of state. Now the right-hand side of this is a decreasing function of p_0 , so to simplify, we can evaluate this at $p_0 = 0$. This then gives the *Bookdahl bound*

$$m_0 < \frac{4}{9} r_0, \tag{1.24}$$

which is satisfied by the core alone. We can of course get a sharper inequality by not restricting to $p_0 = 0$, but this is not needed here. Now the intersection of (1.23) and (1.24), as illustrated in

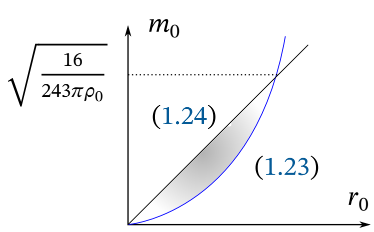

Figure 1.3

Fig. 1.3, turns out to be

$$m_0 < \sqrt{\frac{16}{243\pi\rho_0}} \simeq 5M_\odot, \tag{1.25}$$

where in evaluating this last expression we used the nuclear density ρ_0.

For any (m_0, r_0) in the allowed region, we solve (TOV 1) and (TOV 3) in the envelope region with $\rho = \rho_0$, $m = m_0$ at $r = r_0$. This fixes M in terms of (m_0, r_0). Numerically, we find that the total mass M is maximised when the core mass m_0 is maximised. At this maximum, the envelope is small, so $M_{\max} \lesssim 5M_\odot$. If we possess extra information, we can lower this bound further. However, this result as it is holds independent of the densities at $\rho > \rho_0$. □

2 The Schwarzschild Black Hole

In contrast to cold stars, which cannot have masses more than a few times $M_\odot$, hot stars will undergo complete gravitational collapse to form a *black hole*. The simplest black hole solution is described by the Schwarzschild metric, which we will assume to be valid everywhere in this chapter.

2.1 Birkhoff's Theorem

In *Schwarzschild coordinates* (t, r, θ, ϕ) , the Schwarzschild metric is the one-parameter family

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.1}$$

where the parameter $M > 0$ is interpreted as a mass. This is a solution to the vacuum Einstein equations for $0 < r < r_S = 2M$, the *Schwarzschild radius*. This is spherically symmetric, but it turns out that staticity is not required.

Theorem 2 (Birkhoff)**:** Any spherically symmetric solution of the vacuum Einstein equations is isometric to the Schwarzschild solution.

Proof. See Hawking and Ellis. □

The theorem assumes only spherical symmetry, but the Schwarzschild solution has an additional isometry: $\partial/\partial t$ is a hypersurface-orthogonal Killing vector field, which is timelike for $r > 2M$, so the corresponding Schwarzschild solution is static.

Birkhoff's theorem implies that the spacetime outside any spherical body is the time-independent (exterior) Schwarzschild spacetime, even if the body itself is time-dependent. In particular, the Schwarzschild solution is a good description of the spacetime outside a spherical star during its gravitational collapse collapse.

2.2 Gravitational Redshift

Let A and B be two observers in Schwarzschild spacetime at fixed (r, θ, ϕ) with $r_B > r_A$. Now A sends two photons to B, separated by a coordinate time Δt as measured by A. Since $\partial/\partial t$ is an isometry, the two photons follow the same paths, separated by a time translation of Δt.

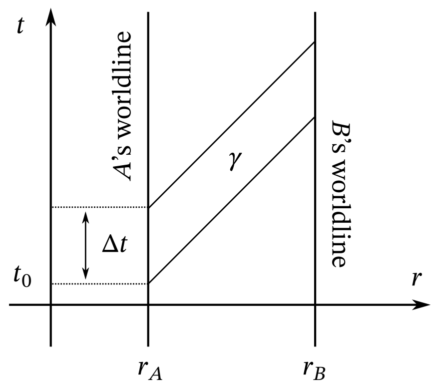

Figure 2.1

Claim 5: A measures the *proper* time between the photons to be $\Delta\tau_A = \sqrt{1 - \frac{2M}{r_A}}\Delta t$.

Proof. Proper time from a point a to b is given by $\tau = \int_a^b \sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}d\lambda$. We want to measure the proper time elapsing along the worldline of A in between sending the two photons. The relevant points are $a = (t_0, r_A, \theta, \phi)$ and $b = (t_0 + \Delta t, r_A, \theta, \phi)$. Parametrising this path with $\lambda = t$, we have

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0} = \delta_0^\mu \quad \Rightarrow \quad \Delta\tau_A = \int_{t_0}^{t_0+\Delta t} \sqrt{g_{00}}\,\mathrm{d}t = \sqrt{1 - \frac{2M}{r_A}}\Delta t. \tag{2.2}$$

□

By the same argument, the proper time along B 's worldline is $\Delta\tau_B = \sqrt{1 - \frac{2M}{r_B}}\Delta t$. The difference here is due to the difference in metric: the curvature of the Schwarzschild spacetime at r_B is different from the curvature at r_A. Eliminating Δt gives

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \sqrt{\frac{1 - 2M/r_B}{1 - 2M/r_A}} > 1. \tag{2.3}$$

Note that this diverges as $r_A \to 2M$. We can apply this argument to two successive wavecrests of light waves propagaing from A to B to relate the period $\Delta\tau_A$ of waves emitted by A to the period $\Delta\tau_B$ of the waves received by B. For light $\Delta\tau = \lambda$ (with $c = 1$), where λ is the wavelength of the light.

Hence $\lambda_B > \lambda_A$: the light becomes redshifted as it climbs out of the gravitational field. If $r_B \gg 2M$, the redshift z is given by

$$1 + z := \frac{\lambda_B}{\lambda_A} \approx \sqrt{\frac{1}{1 - 2M/r_A}}. \tag{2.4}$$

Refering back to the Buchdahl inequality (1.21), the maximum possible redshift of light emitted from the surface $r_A = R > 9M/4$ of a spherical star is therefore $z = 2$.

2.3 Geodesics of the Schwarzschild Solution

Let $x^\mu(\tau)$ be an affinely parametrised geodesic with tangent vector $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$. Since $k = \partial/\partial t$ and $m = \partial/\partial\phi$ are Killing vector fields, we have the conserved quantities

$$E = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} \qquad \text{and} \qquad h = r^2 \sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}\tau}. \tag{2.5}$$

We can interpret these quantities by evaluating the expressions at large r, where the metric is almost flat, and comparing these with analogous results from special relativity. For a timelike geodesic, chossing τ to be proper time gives E and h the interpretations of energy and angular momentum, both per unit rest mass, respectively. For a null geodesic, we can rescale the affine parameter and E and h do not have clear physical interpretations. However, the ratio h/E is invariant under this rescaling. For a null geodesic which propagates to large r, $b = |h/E|$ is the *impact parameter*.

Claim 6: We can always choose coordinates θ and ϕ so that the geodesic is confined to the equatorial plane.

Proof. Using the geodesic Lagrangian $L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, the Euler–Lagrange equation for $\theta(\tau)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\dot{\theta}\right) - r^2 \sin\theta\cos\theta\dot{\phi}^2 = 0 \tag{2.6}$$

$$r^2\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right) - h^2\frac{\cos\theta}{\sin^3\theta} = 0. \tag{2.7}$$

We can choose coordinates (θ, ϕ) on S^2 so that the geodesic initally lies in the equatorial plane $\theta(0) = \frac{\pi}{2}$ and moves tangentially to it with $\left.\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right|_{\tau=0} = 0$. For any function $r(\tau)$, Eq.(2.7) is then a second order ODE for θ with two initial conditions. One solution to this is $\theta(\tau) = \pi/2$ and uniqueness results for ODEs guarantee that this is in fact the unique solution. □

Claim 7: The radial motion of the geodesic is determined by the same equation as a Newtonian particle of unit mass and energy $E^2/2$ moving in a $1d$ potential

$$V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\sigma + \frac{h^2}{r^2}\right), \qquad \sigma = \begin{cases} 1, & \text{timelike} \\ 0, & \text{null} \\ -1, & \text{spacelike} \end{cases}. \tag{2.8}$$

Proof. We will use the relation $g_{\mu\nu}u^\mu u^\nu = -\sigma$.

$$-\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 = -\sigma \tag{2.9}$$

$$-E^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)\frac{h^2}{r^2} = -\left(1 - \frac{2M}{r}\right)\sigma \tag{2.10}$$

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V(r) = \frac{1}{2}E^2. \tag{2.11}$$

□

2.4 Eddington–Finkelstein Coordinates

In this section we will have a closer look at radial null geodesics.

Definition 8 (radial)**:** A geodesic is *radial* if θ and ϕ are constant along it.

We evidently have $h = 0$, but for null geodesics we can also rescale the affine parameter τ so that $E = 1$. The geodesic equations are

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2M}{r}\right)^{-1} \qquad \frac{\mathrm{d}r}{\mathrm{d}\tau} = \begin{cases} +1, & \text{outgoing} \\ -1, & \text{ingoing.} \end{cases} \tag{2.12}$$

An ingoing geodesic at some $r > 2M$ will reach $r = 2M$ in finite affine parameter. Dividing gives

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left(1 - \frac{2M}{r}\right)^{-1}. \tag{2.13}$$

This has a simple pole at $r = 2M$, so t diverges logarithmically as $r \to 2M$.

2.4.1 Coordinate Singularity

Definition 9 (Regge–Wheeler)**:** To investigate what is happening at $r = 2M$, we define the *Regge–Wheeler radial coordinate* r_* by

$$dr_* = \frac{dr}{\left(1 - \frac{2M}{r}\right)}. \tag{2.14}$$

Making a choice of integration, we get $r_* = r + 2M \ln\left|\frac{r}{2M} - 1\right|$. Note that $r_* \sim r$ for large r and $r_* \to -\infty$ as $r \to 2M$. This is illustrated in Fig. 2.2. Along a radial null geodesic we have $\frac{\mathrm{d}t}{\mathrm{d}r_*} = \pm 1$, so $t \mp r_*$ is constant.

Definition 10 (Eddington–Finkelstein)**:** The *Eddington–Finkelstein coordinates* (v, r, θ, ϕ) are obtained by defining a new coordinate $v = t + r_*$, which is constant along ingoing radial null geodesics.

We eliminate t by $t = v - r_*(r)$ and hence

$$dt = dv - \frac{dr}{\left(1 - \frac{2M}{r}\right)}. \tag{2.15}$$

In these coordinates, the metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2. \tag{2.16}$$

As a matrix, we have

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 1 & & \\ 1 & & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \\ & , & & \end{pmatrix} \tag{2.17}$$

with all empty entries being zero.

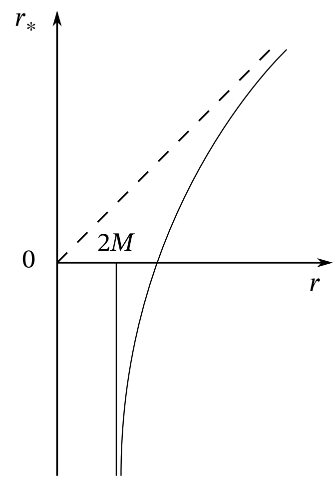

Figure 2.2: Regge–Wheeler radial coordinates.

Unlike Schwarzschild coordinates, the metric components in Eddington–Finkelstein coordinates are smooth for all $r > 0$, including $r = 2M$. The determinant $\det(g_{\mu\nu}) = -r^4 \sin^2\theta$ means that the metric is non-degenerate for all $r > 0$.[1] This means that the signature is Lorentzian for $r > 0$, since a change of signature would require an eigenvalue passing through zero.

Definition 11 (real analytic)**:** A *real analytic function* can be expanded as a convergent power series about any point.

The metric components are real analytic functions of the above coordinates. If a real analytic metric satisfies the Einstein equations in some open set, then it will satisfy them everywhere. Without encountering any problems, the Schwarzschild spacetime can therefore be *extended* though the surface $r = 2M$ to a new region with $r < 2M$. The metric (2.16) is a solution to the vacuum Einstein equations for all $r > 0$.

Remark: The new region $0 < r < 2M$ is spherically symmetric. This is consistent with Birkhoff's theorem since we can just transform back to coordinates (t, r, θ, ϕ) to obtain the Schwarzschild metric in Schwarzschild coordinates, but now with $r < 2M$.

[1]Except at $\theta = 0, \pi$, because the coordinates (θ, ϕ) are not defined at the poles.

2.4.2 Curvature Singularity

Ingoing radial null geodesics in Eddington–Finkelstein coordinates obey $\frac{\mathrm{d}r}{\mathrm{d}\tau} = -1$ and will reach $r = 0$ in finite affine parameter. Since the metric is Ricci flat, the simplest non-trivial scalar constructed from the metric is

$$R_{abcd}R^{abcd} \propto \frac{M^2}{r^6}. \tag{2.18}$$

This diverges as $r \to 0$. Since it is a scalar, it diverges in all coordinate charts. Therefore, there exists no chart for which the metric can be smoothly extended through $r = 0$, which is an example of a *curvature singularity*, where tidal forces become infinite and general relativity ceases to hold. Strictly speaking, $r = 0$ is not part of the spacetime manifold because the metric is not defined there.

Recall that $k = \partial/\partial t$ is a Killing vector field of the Schwarzschild solution for $r > 2M$. In ingoing Eddington–Finkelstein coordinates x^μ , this is

$$k = \frac{\partial}{\partial t} = \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial v}, \tag{2.19}$$

since the Eddington–Finkelstein coordinates are independent of t except for $v = t + r_*(r)$. This equation can be used to extend the definition of k to $r \leq 2M$. Since $k^2 = g_{vv}$, k is null at $r = 2M$ and spacelike for $0 < r < 2M$. Hence the extended Schwarzschild solution is static only in the $r > 2M$ region.

2.5 Finkelstein Diagram

Definition 12: *Outgoing* radial null geodesics in $r \gg M$ have $t - r_* =$ constant. This means that

$$v = 2r + 4M \ln \left| \frac{r}{2M} - 1 \right| + \text{const.} \tag{2.20}$$

Exercise 2.1: Consider radial null geodesics in ingoing Eddington–Finkelstein coordinates. Show that these are either (i) ingoing $v =$ constant or (ii) 'outgoing'—either (2.20) or $r \equiv 2M$.

Plot:

In $r < 2M$, *both* families have decreasing r and reach $r = 0$ in finite τ.

2.6 Gravitational Collapse

Eventually the star collapses through the Schwarzschild surface, forming a black hole. It continues to collapse until it hits the curvature singularity in finite time, which marks its end. In fact, that time is very short.

Exercise 2.2 (Sheet 1)**:** Proper time along any timelike curve in the region $r \leq 2M$ cannot exceed πM.

Taking the Sun $M = M_\odot$, then you get to live 10^{-5}s before you are destroyed in a singularity.

Assume it is not you falling into the black hole, just your friend who is going to be destroyed. You will see that the light is gradually more reshifted and time slows down. An observer at $r > 2M$ never sees the star cross $r = 2M$. They see an ever-increasing redshift causing the star to fade away.

2.7 Black Hole Region

Definition 13 (causal)**:** A vector is *causal* if it is timelike or null (and therefore not the zero-vector). A curve is causal if its tangent vector is not causal.

Definition 14 (time-orientability)**:** A spacetime (M, g) is *time-orientable* if there exists a *time-orientation*: a causal vector field T^a.

Given such a vector field, every point in spacetime either lives in the future or past lightcone with respect to it. However, this is not always possible since there can be obstructions to this. If the spacetime is time-orientable, there are only two inequivalent choices.

Definition 15: A causal vector is future-directed (past-directed) if it lies in the same (opposite) light cone as T^a.

For the Schwarzschild solution with $r > 2M$, the obvious choice is the Killing vector field $k = \partial/\partial t$ as a time-orientation. In Eddington–Finkelstein coordinates, $k = \partial/\partial v$ works for $r > 2M$ but becomes spacelike for $r < 2M$. However, the component $g_{rr} = 0$, the vector fields $\pm\partial/\partial r$ are null and therefore causal. Which one do we choose?

Claim 8: Choosing $-\partial/\partial r$ gives an equivalent time-orientation as k for $r > 2M$.

Proof. Let us take the inner product

$$k \cdot (-\partial/\partial r) = -g_{vr} = -1. \tag{2.21}$$

If the product of two timelike vector fields is negative, they are in the same lightcone. Therefore, $-\partial/\partial r$ is in the same cone as k for $r > 2M$ and defines the time orientation for $r > 0$ (tangent to ingoing radial null geodesic). □

Claim 9: Let $x^\mu(\lambda)$ be a future-directed causal curve such that initially $r(\lambda_0) \leq 2M$. Then $r(\lambda) \leq 2M$ for all $\lambda > \lambda^0$.

Proof. The tangent vector $V^\mu = \frac{\partial x^\mu}{\partial \lambda}$ is future-directed causal. Therefore, since $-\partial/\partial r$ is also future-directed causal, their inner product is non-positive. Evaluating this gives

$$0 \geq \left(-\frac{\partial}{\partial r}\right) \cdot V = -g_{r\mu}V^\mu = -V^v = -\frac{\mathrm{d}v}{\mathrm{d}\lambda} \tag{2.22}$$

$$\therefore \frac{\mathrm{d}v}{\mathrm{d}\lambda} \geq 0 \tag{2.23}$$

$$\Rightarrow \; -2\frac{\mathrm{d}v}{\mathrm{d}\lambda}\frac{\mathrm{d}r}{\mathrm{d}\lambda} = \underbrace{-V^2}_{\geq 0} + \underbrace{\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}\lambda}\right)^2}_{\geq 0 \text{ in } r \leq 2M} + \underbrace{r^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda}\right)^2}_{r \geq 0} \geq 0 \text{ in } r \leq 2M \tag{2.24}$$

$$\Rightarrow \; \frac{\mathrm{d}v}{\mathrm{d}\lambda}\frac{\mathrm{d}r}{\mathrm{d}\lambda} \leq 0 \text{ in } r \leq 2M. \tag{2.25}$$

Assume for contradiction that $\frac{\mathrm{d}r}{\mathrm{d}\lambda} > 0$ at each point in $r \leq 2M$. Therefore $\frac{\mathrm{d}v}{\mathrm{d}\lambda} \leq 0$. But (2.23) then means that $\frac{\mathrm{d}v}{\mathrm{d}\lambda} = 0$. Then (2.24) implies that $-V^2 = 0$ or $\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda}\right)^2 = 0$. Thus, the only non-zero component of V^μ is $V^r = \frac{\mathrm{d}r}{\mathrm{d}\lambda} > 0$. This means that V^μ is a positive multiple of $\partial/\partial r$, meaning that V^μ is past-directed. This is a contradiction.

Therefore, $\frac{\mathrm{d}r}{\mathrm{d}\lambda} \leq 0$ in $r \leq 2M$. With the initial condition $r(\lambda_0) \leq 2M$, we have that $r(\lambda) \leq 2M$ $\forall \lambda \geq \lambda_0$. □

Definition 16 (black hole)**:** A *black hole* is a region of spacetime from which no signal can reach infinity[1].

We have shown that for $r \leq 2M$ of the Schwarzschild ingoing Eddington–Finkelstein coordinates is a black hole.

Definition 17 (event horizon)**:** The boundary $r = 2M$ is called the *event horizon*.

2.8 Detecting Black Holes

There are two key properties of black holes

- There is no upper bound on the mass of a black hole (unlike for a cold star).
- Black holes are very small.

In practice, we infer the existence of black holes by looking at their gravitational effect on nearby orbiting stars. This is what makes us confident that there is a $4 \times 10^6 M_\odot$ supermassive black hole at the centre of our galaxy. It is still unknown how supermassive black holes (with $M \geq 10^6 M_\odot$) can form in the first place.

[1] We will define infinity more rigorously later in the course, but at the moment the intuitive notion is satisfactory.

Definition 18: A *solar mass black hole* has a mass $M \lesssim 100M_\odot$. These are formed by the gravitational collapse of a star.

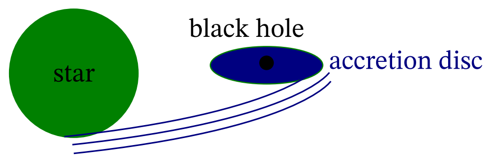

Figure 2.3

Approximate disc particles as following circular orbits.

As the energy decreases (say by friction), the radius slowly decreases. A particle from the disc reaches the ISCO. If it has energy $E = \sqrt{8/9}$ it falls into the hole. A fraction of $1 - \sqrt{8/9} \approx 6\%$ of the rest mass is lost to friction. This colossal amount of energy is converted to electromagnetic radiation.

2.9 White Holes

Consider the $r > 2M$ Schwarzschild solution.

Definition 19 (outgoing EF coords)**:** Now define $u := t - r_*$, which is constant along *outgoing* radial null geodesics. Then (u, r, θ, ϕ) define *outgoing Eddington–Finkelstein coordinates*.

The metric in these coordinates is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2 d\Omega^2. \tag{2.26}$$

Just as in the ingoing case, we can extend this though $r = 2M$ to $r \leq 2M$ until the curvature singularity at $r = 0$.

Claim 10: This is not the same as the previous $r \leq 2M$ region!

Proof. Consider for example the outgoing radial null geodesics $u =$ const. and $\frac{\mathrm{d}r}{\mathrm{d}\tau} = +1$. In this $r < 2M$ region, r is increasing, so it cannot be the same region as before. □

Exercise 2.3: Repeat the calculation as for the ingoing case to show $k = \frac{\partial}{\partial u}$ in ingoing EF coordinates and $\frac{\partial}{\partial r}$ is the time-orientation equivalent to k in $r > 2M$.

The fundamental confusion of calculus: $\frac{\partial}{\partial r}$ in the ingoing coordinates is not the same as in the outgoing coordinates, since we are holding different coordinates fixed.

Definition 20 (white hole)**:** A *white hole* is a region that cannot receive a signal from ∞.

The $r \le 2M$ region is a *white hole*!

A white hole is essentially a time-reverse of a black hole. If we substitute $u = -v$, we recover the metric from the ingoing coordinates. Therefore, $u = -v$ is an isometry mapping the white hole to the black hole, which reverses the time orientation.

White holes are unphysical[1], since there is no mechanism for forming them; you would have to start with the singularity at $r = 0$ and get the white hole emerging from it. Black holes are stable; small perturbations will decay. Since white holes are time-reversals of black holes, they are unstable objects.

2.10 Kruskal Extension

Definition 21: For $r > 2M$, take the *Kruskal–Szekeres* coordinates (U, V, θ, ϕ) with $0 > U = -e^{-u/4M}$ and $0 < V = +e^{v/4M}$.

We then have

$$UV = -e^{r_*/2M} = -e^{r/2M}\left(\frac{r}{2M} - 1\right). \tag{2.27}$$

The right hand side is monotonic. Therefore, if we know U and V , we can determine $r = r(U, V)$ uniquely. Similarly,

$$\frac{V}{U} = -e^{t/2M} \tag{2.28}$$

fixes $t(U, V)$.

Exercise 2.4: Show that in these coordinates the metric is

$$ds^2 = -\frac{32M^3}{r(U,V)} e^{-r(U,V)/2M} dUdV + r(U,V)^2 d\Omega^2. \tag{2.29}$$

We can smoothly extend this metric to a larger range of U and V , since it remains smooth and invertible. We can now use (2.27) to define $r(U, V)$ for $U \ge 0$ or $V \le 0$. The metric can then be analytically extended with $\det g_{\mu\nu} \ne 0$ to new regions, where either $U > 0$ or $V < 0$.

[1]As discussed in *General Relativity*, our universe (emerging from the big bang singularity) looks a bit like the inside of a white hole in 5 dimensions.

What does the Schwarzschild radius $r = 2M$ correspond to? We have $UV = 0$, which corresponds either to $U = 0$ or $V = 0$. In fact, this is not one surface but two!

What about the curvature singularity at $r = 0$? Equation (2.27) gives $UV = 1$, a hyperbola. Radial

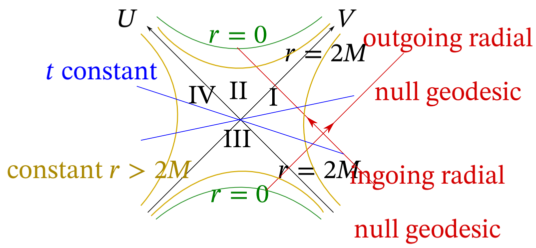

Figure 2.4: Kruskal diagram

null geodesics correspond to constant U or V.

We have four regions

I: $r > 2M$ Schwarzschild

II: Black hole region

III: White hole region

IV: new region with $r > 2M$ isometric to I via $(U, V) \to (-U, -V)$

Ingoing EF cover I and II, while outgoing EF cover I and III.

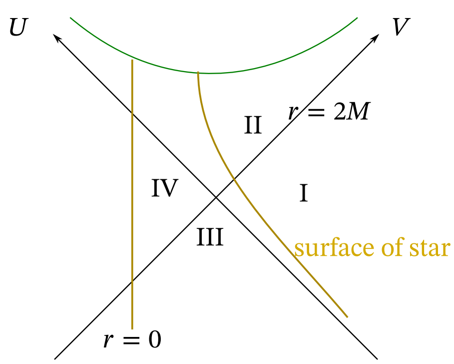

Figure 2.5: A star collapsing to form a black hole. The interior of the star covers up regions IV and III.

Exercise 2.5: Show

$$k = \frac{1}{4M}\left(V\frac{\partial}{\partial V} - U\frac{\partial}{\partial U}\right) \qquad k^2 = -\left(1 - \frac{2M}{r}\right), \tag{2.30}$$

timelike in I, IV, spacelike in II, III, and null at $U = 0$ or $V = 0$.

$\{U = 0\}$ and $\{V = 0\}$ are fixed by k. $k = 0$ on 'bifurcation 2-sphere' $U = V = 0$ (also fixed by k).

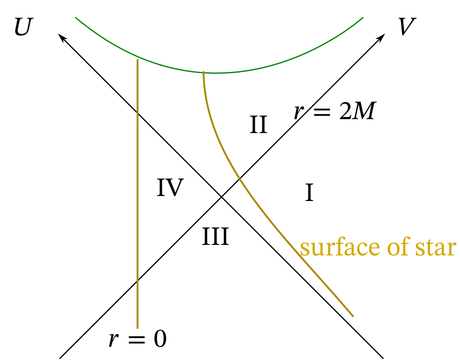

Figure 2.6: Orbits (integral curves) of k.