Quantum Field Theory

Part III Michaelmas 2019

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Contents

Notation

In the following chapters, we will make extensive use of Fourier theory. To avoid having to drag around factors of $(2\pi)^d$ all the time, which make the equations longer and obtuser than they need to be, we adopt the following notational convention.

Notation (barred differential)**:** We understand the *barred differential* $đq$ to be the following normalisation of the ordinary differential measure

$$đq := \frac{\mathrm{d}q}{2\pi}. \tag{1}$$

In particular, given a d-dimensional vector $\mathbf{q} = (q_1, \dots, q_d)$—most often a Fourier mode—we will frequently use the d-dimensional integration measure in Fourier space:

$$đ^d q = \frac{\mathrm{d}^d q}{(2\pi)^d} = \prod_{i=1}^{d} \frac{\mathrm{d}q_i}{2\pi}. \tag{2}$$

Notation (barred delta)**:** We will denote the *barred Dirac delta* $\bar{\delta}(q)$ to include a normalisation of 2π in the following way

$$\bar{\delta}(q) := 2\pi\delta(q). \tag{3}$$

In particular, given a d-dimensional vector $\mathbf{q} = (q_1, \dots, q_d)$, we will frequently use the d-dimensional distribution

$$\bar{\delta}^d(\mathbf{q}) = (2\pi)^d \delta(\mathbf{q}) = \prod_{i=1}^{d} 2\pi\delta(q_i). \tag{4}$$

1 Introduction

Reference text book: Introduction to Quantum Field Theory - Peskin and Schröder

QFT is a quantum theory with an infinite number of degrees of freedom at each point in spacetime $x^\mu = (t, \mathbf{x})$. They are constructed from classical field theories (eg. EM field) incorporating special relativity. QFT states are multi-particle states. The theory becomes relevant at relativistic energies, where we have a lot of exchange between energy $\leftrightarrow$ mass. This takes the form of particles being destroyed / created in interactions. Particle creation and annihilation occurs in Quantum field theory (QFT) but not in QM, differentiating the two theories. Unlike in QM, particle number is not fixed. The interactions themselves arise from the mathematical structure of the theory. Various principles determine this structure:

- locality
- symmetry
- renormalisation group flow

Definition 1: A *Free QFT* is the limit which has particles but no interactions. Weakly interacting theories are then built from these by using perturbation theory. Strong coupling phenomena are an active area of research, hard to solve and relevant for nature, often approached by a discretisation of spacetime (lattice QCD). Free QFT is a relativistic theory with infinitely many *quantised harmonic oscillators*. Non-relativistic QFTs also exist, describing quasi-particles like phonons.

- Based on D. Tong's notes (videos at PIRSA useful as well)

Units in QFT

As a mathematician, you would use $\pi = i = -1 = 2$ in these lectures. However, since we care about having correct numbers for our experimental measurements, we use special units in which

$\hbar = c = 1$.

$$[c] = LT^{-1}, \qquad [\hbar] = L^2MT^{-1} \Rightarrow L = T = M^{-1}. \tag{1.1}$$

Length and time are measured in inverse mass—or equivalently ($E = m\cancel{c^2}$) inverse energy units. To convert back to metres or seconds, insert relevant powers of c and $\hbar$. E.g. $\lambda = \frac{\hbar}{mc}$, so when we measure $m_e \sim 1 \times 10^6$ eV, we have $\lambda_e = 2 \times 10^{-12}$ m. Similarly $m_p \sim 1$ GeV $\to 1 \times 10^9$ eV .

In QFT, if a quantity x has mass dimension $(\text{mass})^d$, we write $[x] = d$. E.g. $[G_N] = -2$ since

$$G_N = \frac{\cancel{\hbar c}}{M_p^2} = \frac{1}{M_p^2} \tag{1.2}$$

where $M_p \sim 1 \times 10^{19}$ GeV corresponds to the *Planck scale*, where quantum gravity effects became important ($\lambda_p \sim 10^{-35}$m).

Angular momentum, like $\hbar$, is dimensionless in these units: e^- total spin: $\hbar/2 = 1/2$.

Relativistic Schrödinger equation does not work—something always goes wrong (causality violation / energy unbounded from below). In QFT, these faults are fixed by allowing for the creation and annihilation of particles.

2 Classical Field Theory

A field is defined at each point of space and time $(t, \mathbf{x})$. Classical particle mechanics yields a finite number of generalised coordinates $q_a(t)$, where $a = x, y, ...$ is a label.

In field theory, we have a field $\phi_a(\mathbf{x}, t)$, where we can consider $\mathbf{x}$ as a label as well. So a and $\mathbf{x}$ are labels: we have an infinite number of degrees of freedom: one for each $\mathbf{x}$. Position is relegated from a dynamical variable to a mere label.

Example (Electromagnetism)**:**

$$E_i(\mathbf{x}, t), B_i(\mathbf{x}, t) \qquad \{i, j, k\} \in \{1, 2, 3\} \text{ label spatial position} \tag{2.1}$$

These 6 fields are derived from 4 fields $A_\mu(\mathbf{x}, t) = (\phi, \mathbf{A})$, where $\mu \in \{0, 1, 2, 3\}$. The relationship between the electric and magnetic fields and A is

$$E_i = -\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x_i}, \qquad B_i = \frac{1}{2}\varepsilon_{ijk}\frac{\partial A_k}{\partial x_j}, \tag{2.2}$$

where the Einstein summation convention is used.

2.1 Dynamics of Fields

The dynamics of the field is governed by a Lagrangian L. This is a function of the fields $\phi_a(\mathbf{x}, t)$, their time derivatives $\dot{\phi}_a(\mathbf{x}, t)$ and spatial derivatives $\boldsymbol{\nabla}\phi_a(\mathbf{x}, t)$. We can write it as an integral over the Lagrangian density $\mathcal{L}$:

$$L = \int \mathrm{d}^3x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a). \tag{2.3}$$

Confusingly, $\mathcal{L}$ is also often just called the *Lagrangian* in QFT. We will mostly be concerned with Lagrangian densities.

We define the action as

$$S = \int_{t_0}^{t_1} L\, \mathrm{d}t = \sum_a \int \mathrm{d}^4x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a). \tag{2.4}$$

Since $[\mathcal{L}] = 4$ and $[\mathrm{d}^4x] = -4$, the action is dimensionless $[S] = 0$. The *Dynamical Principle* of classical field theory is that fields evolve in such a way that the action S is stationary with respect to those variations of the fields that don't affect the initial and final values.

$$\delta S = \sum_a \int \mathrm{d}^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta(\partial_\mu\phi_a) \right\} \tag{2.5}$$

$$= \sum_a \int \mathrm{d}^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a \right) - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a \right\} \tag{2.6}$$

The total derivative vanishes for any term that decays at spatial ∞ and has $\delta\phi(\mathbf{x}, t_0) = 0 = \delta\phi(\mathbf{x}, t_1)$. So the dynamical principle is

$$\delta S = 0 \;\Rightarrow\; \boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0 \right)} \qquad \text{Euler-Lagrange equations.} \tag{2.7}$$

Example (Klein-Gordon Field ϕ)**:**

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 \tag{2.8}$$

where we use the mostly minus signature $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{2.9}$$

> **Remark:** $\mathcal{L} = T - V$ where $T = \frac{1}{2}\dot{\phi}^2$ is the kinetic and $V = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$ is the potential energy density.

The relevant derivatives of $\mathcal{L}$ appearing the the Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu\phi, \tag{2.10}$$

So substituting $\mathcal{L}$ into (2.7) yield

$$\boxed{\partial_\mu\partial^\mu\phi + m^2\phi = 0} \qquad \text{Klein-Gordon eqn} \tag{2.11}$$

The K-G eqn admits a wave-like solution $\phi = e^{-ip\cdot x}$.

> **Remark:** $p \cdot x = p^0x^0 - \mathbf{p}\cdot\mathbf{x}^T = Et - \mathbf{p}\cdot\mathbf{x}$

Substitute $\phi \to$ KG:

$$(-p^2 + m^2)\phi = 0. \tag{2.12}$$

Since $p^2 = E^2 - |\mathbf{p}|^2 = m^2$, this reproduces the relativistic energy dispersion relation for a particle of mass m and three-momentum $\mathbf{p}$.

Example (Maxwell's Equations)**:** The Lagrangian which allows us to recover the charge-free Maxwell equations is

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A_\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2 \tag{2.13}$$

Noting that $\mathcal{L}$ only depends on derivatives $\partial_\mu A_\nu$ of the scalar field, we only need to compute

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \eta^{\mu\nu}\partial_\rho A^\rho. \tag{2.14}$$

The Euler-Lagrange equations then give the Maxwell equations in four-vector notation:

$$\begin{aligned} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} \right) = 0 &= -\partial_\mu \partial^\mu A^\nu + \partial^\nu(\partial_\rho A^\rho) && (2.15) \\ &= -\partial_\mu(\partial^\mu A^\nu - \partial_\nu A^\mu) && (2.16) \\ &= -\partial_\mu F^{\mu\nu} && (2.17) \end{aligned}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$ is the Maxwell field strength tensor.

In the previous examples, we only considered local Lagrangians, which means that the Lagrangian does not involve any products of fields like $\phi(\mathbf{x}, t)\phi(\mathbf{y}, t)$ where the positions differ $\mathbf{x} \neq \mathbf{y}$.

2.2 Lorentz Invariance

Consider the Lorentz transformation (LT) Λ of some scalar field $\phi(x) \equiv \phi(x^\mu)$. Under Λ, the field changes as $\phi \to \phi'$ where $\phi'(x) = \phi(x')$ and $x'^\mu = (\Lambda^{-1})^\mu{}_\nu x^\nu$.

The defining equation for Lorentz transformations is

$$\Lambda^\mu{}_\sigma \eta^{\sigma\rho} \Lambda_\rho{}^\nu = \eta^{\mu\nu}. \tag{2.18}$$

Examples of a Lorentz transformation include rotation around the x-axis or Lorentz boosts, whose matrix representation is respectively

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\theta & \sin\theta \\ & & -\sin\theta & \cos\theta \end{pmatrix}, \qquad \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{2.19}$$

where $\gamma = (1-v^2)^{-1/2}$, and the empty entries are understood to be zero. The Lorentz transformations form a Lie group under matrix multiplication. This means that applying two Lorentz transformations gives another, and any Lorentz transformation has an inverse. The Lorentz transformations allow a representation on the fields. For a scalar field, this is $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$. This is an *active transformation*, which genuinely rotates the field. This is why the inverse Lorentz transformation Λ^{-1} is needed to used the new coordinates to the old. A *passive transformation* is one where we just relabel the coordinates. Under such a transformation, a scalar field changes as $\phi(x) \to \phi'(\Lambda x)$. *Lorentz invariant* theories are such that the action S, and the dynamics that are described by it, are unchanged under Lorentz transformations. In particular, if $\phi(x)$ satisfies the equations of motion of a Lorentz invariant theory, then so does the transformed field $\phi(\Lambda^{-1}x)$.

Example (Klein-Gordon)**:** Taking the K-G Lagrangian density, we have the action

$$S = \int_{\mathbb{R}^4} \mathrm{d}^4x \, \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi) \tag{2.20}$$

where U is some polynomial. Since we know how ϕ behaves under LT, we can deduce that U behaves under LTs in the following way:

$$U'(x) = U(\phi'(x)) = U(\phi(x')) = U(x'). \tag{2.21}$$

The first term in the action transforms as

$$(\partial_\mu\phi)' = \frac{\partial}{\partial x^\mu}\phi(x') = \frac{\partial x'^\sigma}{\partial x^\mu}\frac{\partial}{\partial x'^\sigma}\phi(x') \tag{2.22}$$

$$= (\Lambda^{-1})^\sigma{}_\mu \partial'_\sigma\phi(x'), \tag{2.23}$$

where we denoted $\partial'_\sigma = \frac{\partial}{\partial x'^\sigma}$. The kinetic term in the Lagrangian is thus

$$\mathcal{L}'_{\text{kin}} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)'(\partial_\nu\phi)' \tag{2.24}$$

$$= \frac{1}{2}\underbrace{\eta^{\mu\nu}(\Lambda^{-1})^\sigma{}_\mu(\Lambda^{-1})^\rho{}_\nu}_{\eta^{\sigma\rho}}\partial'_\sigma\phi(x')\partial'_\rho\phi(x') \tag{2.25}$$

$$= \frac{1}{2}\eta^{\sigma\rho}\partial'_\sigma\phi(x)\partial'_\rho\phi(x'). \tag{2.26}$$

The Lagrangian density transforms like the scalar field ϕ; this transformation law means that $\mathcal{L}$ is itself a scalar field. The action changes under LT as

$$S' = \int \mathrm{d}^4x \, \mathcal{L}(x') = \int \mathrm{d}^4x \, \mathcal{L}(\Lambda^{-1}x). \tag{2.27}$$

We change variables to $y = \Lambda^{-1}x$. Since LTs are in the special orthogonal group, we have unit determinant $\det\Lambda = 1$ which means that the Jacobian is unity and $\int \mathrm{d}^4x = \int \mathrm{d}^4x$. As a result, the action is invariant:

$$S' = \int \mathrm{d}^4y \, \mathcal{L}(y) = S. \tag{2.28}$$

> **Remark:** Under a LT, a vector field A_μ transforms like $\partial_\mu \phi$, so
>
> $$A'_\mu(x) = (\Lambda^{-1})_\mu{}^\sigma A_\sigma(\Lambda^{-1}x). \tag{2.29}$$
>
> If all indices are summed over, the result is Lorentz invariant.

Example: Do Q1 in Ex. sheet 1.

2.3 Noether's theorem

Already noticeable in classical field theory: symmetries are important.

Theorem 1 (Noether's theorem)**:** Every continuous symmetry of the Lagrangian $\mathcal{L}$ gives rise to a *current* $j^\mu(x)$, which is conserved:

$$\partial_\mu j^\mu = \frac{\partial(j^0)}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0. \tag{2.30}$$

Proof. Consider an infinitesimal variation of a field ϕ:

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha \Delta\phi(x), \tag{2.31}$$

where α is seen as an infinitesimal parameter. This is a symmetry if S is unchanged, i.e. $\mathcal{L}$ should be invariant up to a total 4-divergence (which integrates to a surface term and does not affect the E-L equations). In other words, the Lagrangian density changes as

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \partial_\mu X^\mu(x) = 0. \tag{2.32}$$

In fact, often we have $X^\mu = 0$. Recall that $\mathcal{L}$ only depends on ϕ and $\partial_\mu \phi$, so

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \frac{\partial \mathcal{L}}{\partial \phi}\Delta\phi + \alpha \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\Delta\phi) \tag{2.33}$$

$$= \mathcal{L}(x) + \alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right) + \alpha \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)}_{\text{E-L: }=0} \Delta\phi. \tag{2.34}$$

Combining (2.32) and (2.34), we see that the current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - X^\mu \tag{2.35}$$

is conserved: $\partial_\mu j^\mu = 0$. □

Remark: Each conserved current has associated with it a conserved charge:

$$Q = \int_{\mathbb{R}^3} \mathrm{d}^3x \, j^0. \tag{2.36}$$

Proof. The current changes in time as

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} \mathrm{d}^3x \, \partial_0 j^0 \tag{2.37}$$

$$= -\int_{\mathbb{R}^3} \mathrm{d}^3x \, \boldsymbol{\nabla} \cdot \mathbf{j} \tag{2.38}$$

which vanishes by the divergence theorem. □

Starting from the Lagrangian density $\mathcal{L}$ and the symmetry transformation, we can work out what the conserved current is.

Example (Scalar Field)**:**

$$\psi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)) \tag{2.39}$$

We could also use two real fields to describe this theory. However, when using complex fields, we can consider ψ and ψ^* as independent variables. This is because we really have two independent degrees of freedom ϕ_1 and ϕ_2. Using these variables, the Lagrangian is

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - V(|\psi|^2). \tag{2.40}$$

Remark: The potential could be for example:

$$V(|\psi|^2) = m^2 \psi^* \psi + \frac{\lambda}{2}(\psi^* \psi)^2. \tag{2.41}$$

We will see later that the first summand will be a mass term and the second will describe an interaction.

The symmetry is a complex phase rotation:

$$\psi \to e^{i\alpha}\psi \qquad \Longleftrightarrow \qquad \psi^* \to e^{-i\alpha}\psi^*, \tag{2.42}$$

where one equation implies the other by conjugation. Since $\mathcal{L}$ is invariant under this transformation, this is a symmetry with $M^\mu = 0$. The fields change as

$$\Delta\psi = i\alpha\psi \qquad \Delta\psi^* = -i\alpha\psi^*. \tag{2.43}$$

The currents from the two different fields will add:

$$j^\mu = (\psi \partial^\mu \psi - \psi^* \partial^\mu \psi). \tag{2.44}$$

This conserved charge could be the electric charge or some particle number (baryon, lepton, etc.) for example. In QED, we will see something very similar.

Exercise 2.1: Do questions 2 and 3 of Example sheet 1.

Example (Infinitesimal translations)**:** In four vector notation, we write an infinitesimal spacetime translation as

$$x^\mu \to x^\mu - \alpha\varepsilon^\mu. \tag{2.45}$$

By Taylor expansion, we know that the scalar field transforms as

$$\phi(x) \to \phi(x) + \alpha\varepsilon^\mu \partial_\mu \phi(x). \tag{2.46}$$

The derivatives $\partial_\mu\phi$ will also have a similar expansion.

The Lagrangian density transforms as

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\varepsilon^\mu \partial_\mu \mathcal{L}(x) \tag{2.47}$$

$$= \mathcal{L}(x) + \alpha\varepsilon^\nu \partial_\mu \underbrace{(\delta^\mu{}_\nu \mathcal{L})}_{X^\mu}. \tag{2.48}$$

We get one conserved current for each component of ε^ν :

$$(j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu\phi - \delta^\mu{}_\nu \mathcal{L} := T^\mu{}_\nu, \tag{2.49}$$

since $\Delta\phi = \partial_\nu\phi$ for ε^ν and $X^\mu = \delta^\mu{}_\nu\mathcal{L}$. We define this to be the *energy-momentum tensor* $T^\mu{}_\nu$. It has this name since it contains four conserved charges which are associated with

$$\text{Total Energy:} \qquad P^0 := E = \int \mathrm{d}^3x\, T^{00} \tag{2.50}$$

$$\text{Total Momentum:} \qquad P^i = \int \mathrm{d}^3x\, T^{0i}, \quad i = 1, 2, 3 \tag{2.51}$$

Applying (2.49) to the free real valued scalar field theory Lagrangian, $\mathcal{L} = \partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$, we obtain the energy momentum tensor

$$T^{\mu\nu} = \partial^\mu\phi^\nu\phi - \eta^{\mu\nu}\mathcal{L}. \tag{2.52}$$

This corresponds to an energy

$$E = \int \mathrm{d}^3x \left\{\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right\}. \tag{2.53}$$

Notice that in the case of a free scalar field, $T^{\mu\nu}$ is symmetric under interchange of $\mu \leftrightarrow \nu$.

Prop: If $T^{\mu\nu}$ is non-symmetric, we can massage it into a symmetric form.

Proof. Adding a term $\partial_\rho \Gamma^\rho{}_{\mu\nu}$ where $\Gamma^{\rho\mu\nu} = -\Gamma^{\mu\rho\nu} \Rightarrow \partial_\mu\partial_\rho\Gamma^{\rho\mu}{}_\nu = 0$, i.e. it does not change conservation law. We pick it to make the $T^{\mu\nu}$ symmetric. □

Remark: This reparametrisation equivalence can be expressed as a gauge symmetry.

Exercise 2.2: Do question 6 on problem sheet 1.

3 Free Field Theory

3.1 Hamiltonian Formalism

The Hamiltonian formulation also accommodates field theories.

Definition 2 (conjugate momentum)**:** We define the *conjugate momentum*

$$\pi(x) = \frac{\partial \mathcal{L}(x)}{\partial \dot{\phi}(x)}. \tag{3.1}$$

The Hamiltonian density is then

$$\mathcal{H} = \pi(x)\dot{\phi}(x) - \mathcal{L}(x). \tag{3.2}$$

As in classical mechanics, we eliminate $\dot{\phi}$ in favour of π in our equations. Integrating over all space, we would obtain the Hamiltonian.

Example (Scalar field with potential)**:** We add a potential $V(\phi)$ to our free Hamiltonian.

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - V(\phi). \tag{3.3}$$

The conjugate momentum is $\pi = \dot{\phi}$, which means that the Hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + V(\phi). \tag{3.4}$$

The Hamiltonian is

$$H = \int \mathrm{d}^3x\, \mathcal{H}. \tag{3.5}$$

The equations of motion of the scalar field are given by Hamilton's equations:

$$\dot{\phi} = \frac{\delta H}{\delta \pi} \qquad \dot{\pi} = -\frac{\delta H}{\delta \phi}. \tag{3.6}$$

> **Remark:** In this case, the Hamiltonian is the same as the total energy E.

Since the physics remains unchanged when changing from the Lagrangian to the Hamiltonian formulation, the result is Lorentz invariant. However, since we picked out a preferred time, the Hamiltonian formulation is *not* manifestly Lorentz invariant.

3.2 Canonical Quantisation

Recall that in transitioning from classical to quantum mechanics, canonical quantisation tells us to take a set of generalised coordinates q_a and p_a and promote them to operators. We replace Poisson brackets with commutators.

$$[q_a, p^b] = i\delta_a^b \qquad (\hbar = 1). \tag{3.7}$$

Now, in the transition from classical to quantum field theory, we do the analogous thing for the fields $\phi_a(\mathbf{x})$ and $\pi_b(\mathbf{x})$.

Definition 3 (quantum field)**:** A *quantum field* is an operator valued function of space that obeys the following commutation relation: Fields and momenta commute amongst themselves

$$[\phi_a(\mathbf{x}), \phi_b(\mathbf{y})] = 0 \qquad [\pi^a(\mathbf{x}), \pi^b(\mathbf{y})] = 0. \tag{3.8}$$

However, they do not commute between each other

$$[\phi_a(\mathbf{x}), \pi^b(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})\delta_a^b. \tag{3.9}$$

We are in the Schrödinger picture, so the operators $\phi_a(\mathbf{x})$ and $\pi^a(\mathbf{x})$ are *not* functions of t—all t dependence is in states, which evolve according to the Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle. \tag{3.10}$$

As such, we wish to know the spectrum of H, but this is extremely hard due to the infinite number of degrees of freedom (at least 1 for each spatial coordinate $\mathbf{x}$).

For free theories, coordinates evolve independently. These free theories have $\mathcal{L}$ quadratic in the fields ϕ or $\partial_\mu\phi$. As a consequence, this gives linear equations of motion.

Example (Klein-Gordon)**:** We saw that the EL equations for the KG theory of a real scalar field $\phi(\mathbf{x}, t)$ are

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0. \tag{3.11}$$

To see why, we take the Fourier transform

$$\phi(\mathbf{x}, t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t). \tag{3.12}$$

Then we see that the classical KG equation becomes

$$\left[\frac{\partial^2}{\partial t^2} + (\mathbf{p}^2 + m^2)\right]\phi(\mathbf{p}, t) = 0. \tag{3.13}$$

This is the equation for the harmonic oscillator, vibrating at frequency $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. So the solution to the Klein-Gordon equation is a superposition of simple harmonic oscillators (SHOs) each vibrating at a different frequency and amplitude. To quantise $\phi(\mathbf{x}, t)$, we need to quantise this infinite number of SHOs.

> **Remark:** These SHOs are not coupled since this is a free theory. If we include higher powers in the Lagrangian, these become coupled and very difficult to solve.

3.3 Review of SHO in 1d QM

Writing the position as q and momentum as p, the Lagrangian is for the simple harmonic oscillator in 1d quantum mechanics is

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2. \tag{3.14}$$

The Hamiltonian is then easily found to be

$$H = p\dot{q} - L = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2. \tag{3.15}$$

We recall that the canonical quantisation condition is enforced by requiring the position and momentum operators to satisfy the canonical commutation relation:

$$[q, p] = i. \tag{3.16}$$

Definition 4: We define the *ladder* or *creation and annihilation operators* to be

$$a^\dagger = \frac{-ip}{\sqrt{2\omega}} + \sqrt{\frac{\omega}{2}}q \quad \Longleftrightarrow \quad a = \frac{+ip}{\sqrt{2\omega}} + \sqrt{\frac{\omega}{2}}q. \tag{3.17}$$

In terms of these operators, we can write our original operators as

$$q = \frac{a + a^\dagger}{\sqrt{2\omega}}, \qquad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger). \tag{3.18}$$

And the commutation relation becomes

$$[a, a^\dagger] = 1. \tag{3.19}$$

Similarly, we can rewrite the Hamiltonian as

$$H = \frac{1}{2}\omega(aa^\dagger + a^\dagger a) = \omega(a^\dagger a + \frac{1}{2}), \tag{3.20}$$

where we used the commutator in the last equality.

The commutation relations between these ladder operators and the Hamiltonian are

$$[H, a^\dagger] = \omega a^\dagger, \qquad [H, a] = -\omega a. \tag{3.21}$$

Exercise 3.1: Work these out!

These ensure that $a, a^\dagger$ take us between energy states: If $H|E\rangle = E|E\rangle$, then

$$H(a^\dagger |E\rangle) = (E + \omega)(a^\dagger |E\rangle) \tag{3.22}$$

$$H(a|E\rangle) = (E - \omega)(a|E\rangle). \tag{3.23}$$

We ended up with a ladder of states with energy $\cdots, E-\omega, E, E+\omega, E+2\omega, \cdots$. To ensure a stable system, the energy must be bounded from below by the existence of a *ground-state* $|0\rangle$ that is annihilated by the lowering operator: $a|0\rangle = 0$. Acting on this state with the Hamiltonian, we find the *ground-state energy* $H|0\rangle = \frac{1}{2}\omega|0\rangle$.

Excited states arise from a repeated application of $a^\dagger$:

$$|n\rangle \equiv (a^\dagger)^n |0\rangle, \qquad H|n\rangle = (n + \frac{1}{2})\omega |n\rangle. \tag{3.24}$$

> **Remark:** We have not normalised these states: $\langle n|n\rangle \neq 1$.

> **Remark:** We are, in the absence of gravitational interactions, only interested in *energy differences*, rather than absolute values. As a result, we want to set the ground state energy to zero.

Definition 5 (normal ordering)**:** A *normal ordered* operator $:\mathcal{O}:$, is the same as the operator $\mathcal{O}$, except that we reorder the string of ladder operators so that all the annihilation operators a's are on the right of all the $a^\dagger$s.

> **Remark:** We *do not* use any commutation relations! We simply reorder the string.

Example: For two operators, $:a^\dagger a: = a^\dagger a$, or $:aa: = aa$ and $:a^\dagger a^\dagger: = a^\dagger a^\dagger$, but $:aa^\dagger: = a^\dagger a$. With multiple operators, we have $:a^\dagger a a^\dagger a a: = a^\dagger a^\dagger a a a$.

> **Remark:** Normal ordering is *not* a linear function on operators! To see this, consider
>
> $$a^\dagger a = :aa^\dagger: = :1 + a^\dagger a: \neq :1: + :a^\dagger a: = 1 + a^\dagger a \tag{3.25}$$

The normal ordered Hamiltonian is $:H: = \omega a^\dagger a$. With this definition, the ground-state energy is $H\left|0\right\rangle = 0$.

3.4 The Free Scalar Field

Let us apply the SHO to free fields. We define the Fourier transform of a scalar field as

$$\phi(\mathbf{x}) = \int đ^3 p \, \frac{1}{\sqrt{2\omega_p}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right), \tag{3.26}$$

where $\omega_p^2 = p^2 + m^2$, writing $p = \sqrt{|\mathbf{p}|}^2$. Since we explicitly added a Hermitian conjugate constant $a^\dagger$, this definition of the Fourier integral makes it evident that $\phi = \phi^\dagger$ is a real scalar field. We also expand the conjugate momentum $\pi(\mathbf{x})$ in terms of these Fourier modes

$$\pi(\mathbf{x}) = \int đ^3 p (-i) \sqrt{\frac{\omega_p}{2}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right). \tag{3.27}$$

The process of promoting the constants $a_{\mathbf{p}}$ and $a^\dagger$ to quantum operators—as it turns out, these will be annihilating and creating particles of momentum **p**— acting on a quantum state is called *second quantisation*; we have inserted an infinite number of QHOs into momentum space. As a result, the fields ϕ and π are also quantum operators. For these quantum fields, we want to impose commutation relations:

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = 0, \qquad [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0, \tag{3.28a}$$

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y}). \tag{3.28b}$$

> **Remark:** We are in the Schrödinger picture, where time $t = x^0$ is in the states, not in the fields.

Claim 1: These are equivalent to

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0, \qquad [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0, \tag{3.29a}$$

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = \delta^3(\mathbf{p} - \mathbf{q}). \tag{3.29b}$$

Proof. Let us check that (3.29b) implies (3.28b) .

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = \int đ^3 p \, đ^3 q \, \frac{(-i)}{2} \frac{\omega_q}{\omega_p} \left\{ - \left[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger \right] e^{i\mathbf{p}\cdot\mathbf{x} - i\mathbf{q}\cdot\mathbf{y}} + \left[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}} \right] e^{-i\mathbf{p}\cdot\mathbf{x} + i\mathbf{q}\cdot\mathbf{y}} \right\} \tag{3.30}$$

$$= \int đ^3 p \left(\frac{-i}{2} \right) \left\{ -e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \right\} \tag{3.31}$$

$$= i\delta^3(\mathbf{x} - \mathbf{y}) \tag{3.32}$$

Exercise 3.2: Complete the proof.

□

Now, compute the Hamiltonian H in terms of the ladder operators $a_{\mathbf{p}}$ and $a_{\mathbf{q}}^\dagger$:

$$
\begin{aligned}
H &= \frac{1}{2}\int \mathrm{d}^3x \left[\pi^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2\right] && (3.33)\\
&= \frac{1}{2}\int \mathrm{d}^3x\, đ^3p\, đ^3q \\
&\quad \times\Big\{ -\frac{\sqrt{\omega_p\omega_2}}{2}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right)\left(a_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}} - a_{\mathbf{q}}^\dagger e^{i\mathbf{q}\cdot\mathbf{x}}\right) \qquad \text{from } \pi^2 \\
&\quad - \frac{1}{2}\sqrt{\omega_p\omega_2}\left(ipa_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - ipa_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right)\left(iqa_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}} - iqa_{\mathbf{q}}^\dagger e^{i\mathbf{q}\cdot\mathbf{x}}\right) \qquad \text{from } (\boldsymbol{\nabla}\phi)^2 && (3.34)\\
&\quad - \frac{m^2}{2\sqrt{\omega_p\omega_2}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right)\left(a_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}} + a_{\mathbf{q}}^\dagger e^{-i\mathbf{q}\cdot\mathbf{x}}\right)\Big\} \qquad \text{from } m^2\phi^2 \\
&= \cdots \\
&= \frac{1}{4}\int đ^3p \Big\{ \underbrace{\cancel{\left(-\omega_p + \frac{p^2}{\omega_p} + \frac{m^2}{\omega_p}\right)}}_{\text{since } \omega_p^2 = p^2+m^2} \left(a_{\mathbf{p}}a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger\right) + \left(\omega_p + \frac{p^2}{\omega_p} + \frac{m^2}{\omega_p}\right)(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}})\Big\} && (3.35)
\end{aligned}
$$

$$\therefore H = \frac{1}{2}\int đ^3p\, \omega_p(a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}^\dagger a_{\mathbf{p}}). \tag{3.36}$$

This is the Hamiltonian of an infinite number of uncoupled harmonic oscillators, each at frequency $\omega_p = \sqrt{p^2 + m^2}$.

3.4.1 The Vacuum

Definition 6 (vacuum)**:** The *vacuum* or *ground state* $|0\rangle$ is defined to be the state with the property that it is annihilated by all annihilation operators:

$$a_{\mathbf{p}}\,|0\rangle = 0, \quad \forall p. \tag{3.37}$$

Acting on the vacuum with the Hamiltonian, we expect to find the ground-state energy:

$$
\begin{aligned}
H\,|0\rangle &= \int đ^3p\, \omega_p\left(\cancel{a_{\mathbf{p}}^\dagger a_{\mathbf{p}}} + \frac{1}{2}[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]\right)|0\rangle && (3.38)\\
&= \frac{1}{2}\int đ^3p\, \omega_p \delta^3(0)\,|0\rangle\,. && (3.39)
\end{aligned}
$$

We find an infinite ground-state energy! And worse, this infinity is twofold; we have the obvious factor of $\delta^3(0)$, but also a less obvious infinity that arises because the integral $\int đ^3p\,\omega_p$ does not converge; the energy of each harmonic oscillator diverges $\omega_p = \sqrt{p^2+m^2} \to \infty$ as $p \to \infty$. This latter divergence is called a *high-frequency* or *ultra-violet* divergence. We can resolve this issue by introducing a UV-cutoff Λ as an upper integration bound, which tames the integral.

> **Remark:** Physically, this can be justified by postulating that Λ corresponds to the highest energy (or equivalently, Λ^{-1} the smallest length scale) on which this QFT is valid. On higher energy or smaller length scales, new physics will need to be introduced. In statistical field theory, this smallest length scale is on the order of the lattice spacing. On smaller length scales, the continuum assumptions of the theory break down and new physics—accounting for the lattice effects—will need to be introduced.

Concerning the former infinity, $\delta^3(0)$, we have another resolution up our sleeves: Heuristically, one might claim that in non-gravitational physics, only energy differences are relevant for the dynamics of the system. As such, one might redefine the Hamiltonian to subtract the commutator $[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]$ which led to the divergence. More concretely, this arbitrary choice that we can make in the Hamiltonian points to an ambiguity arising in moving from the classical to the quantum theory:

Classically, the Hamiltonian $H = \frac{1}{2}(\omega q - ip)(\omega q + ip)$ is exactly the same as $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$. However, in QM, the operator order matters, and upon quantisation it gives us $H = \omega a^\dagger a$. The analogous thing happens in QFT with momentum modes, which means that the Hamiltonian that we should really use is

$$H = \int đ^3 p\, \omega_p a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \tag{3.40}$$

Using this Hamiltonian, the ground-state energy of the vacuum has been redefined to $H\,|0\rangle = 0$.

This ambiguity in the definition of QFT operators can be resolved by introducing the concept of normal ordering.

Definition 7 (normal order)**:** A *normal ordered* string of operators $:\ \phi_1(\mathbf{x}_1)\cdots\phi_n(\mathbf{x}_n)\ :$ is defined to be the same string of operators, except that all annihilation operators a are moved to the RHS of all creation operators $a^\dagger$.

Since the annihilation operators annihilate the vacuum, the $\delta^3(0)$ infinity is discarded when using the normally ordered Hamiltonian

$$:H: = \int đ^3 p\, \omega_p a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \tag{3.41}$$

3.4.2 Particles

It is easy to verify the commutation relation with the Hamiltonian

$$[H, a_{\mathbf{p}}^\dagger] = \omega_p a_{\mathbf{p}}^\dagger \qquad \text{and} \qquad [H, a_{\mathbf{p}}] = -\omega_p a_{\mathbf{p}}. \tag{3.42}$$

The operator $a^\dagger_{\mathbf{p}}$ increases the energy by the constant value ω_p.

Now consider the state $\left|\mathbf{p}'\right\rangle = a^\dagger_{\mathbf{p}'}\left|0\right\rangle$. The energy of this state is

$$H\left|p'\right\rangle = \int đ^3p\, \omega_p a^\dagger_{\mathbf{p}} a_{\mathbf{p}} a^\dagger_{\mathbf{p}'}\left|0\right\rangle \tag{3.43}$$

$$= \int đ^3p\, \omega_p a^\dagger_{\mathbf{p}} \left([a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] - a^\dagger_{\mathbf{p}'} a_{\mathbf{p}}\right)\left|0\right\rangle \tag{3.44}$$

$$= \omega_{p'} a^\dagger_{\mathbf{p}'}\left|0\right\rangle = \omega_{p'}\left|\mathbf{p}'\right\rangle. \tag{3.45}$$

Since $\omega_p = \sqrt{p^2 + m^2}$ is the relativistic dispersion relation of a particle of mass m and momentum p, we interpret the state $\left|\mathbf{p}\right\rangle$ as the one-particle momentum eigenstate.

> **Remark:** Mass m comes from the term $\frac{1}{2}m^2\phi^2$ in the Lagrangian; in general, the coefficient of the quadratic field term in $\mathcal{L}$ allows us to identify the mass.

Properties of $\left|\mathbf{p}\right\rangle$

We previously defined the momentum operator $\mathbf{P} = -\int d^3x\, \pi(\mathbf{x})\nabla\phi(\mathbf{x})$.

Exercise 3.3: Show that $\mathbf{P} = \int đ^3p\, \mathbf{p} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$. Then show that $\left|\mathbf{p}\right\rangle$ is actually a momentum eigenstate, i.e. that $P\left|\mathbf{p}\right\rangle = \mathbf{p}\left|\mathbf{p}\right\rangle$.

We could also act with the angular momentum operator J^i to find $J^i\left|\mathbf{p}\right\rangle = 0$. This is a spin-zero state.

Exercise 3.4: Do all of Example sheet 1.

Multi-Particle States $\left|\mathbf{p}_1, \ldots, \mathbf{p}_n\right\rangle$

We define $\left|\mathbf{p}_1, \cdots, \mathbf{p}_n\right\rangle := a_{\mathbf{p}_1} \cdots a_{\mathbf{p}_n}\left|0\right\rangle$. Since the creation operators $a_{\mathbf{p}_i}$ commute with each other, we have $\left|\mathbf{p}, \mathbf{q}\right\rangle = \left|\mathbf{q}, \mathbf{p}\right\rangle$. These multi-particle states are thus symmetric under interchange of particle label, which means that we are dealing with *bosons*.

The full Hilbert space is spanned by the set

$$\{\left|0\right\rangle, \left|\mathbf{p}_1\right\rangle, \left|\mathbf{p}_1, \mathbf{p}_2\right\rangle, \cdots\}. \tag{3.46}$$

A space that is built from these basis states is called a *Fock space*.

We can also introduce a *number operator*

$$N = \int đ^3 p \, a_{\mathbf{p}}^\dagger a_{\mathbf{p}}, \tag{3.47}$$

which counts the number of particles.

Exercise 3.5: Show that $N \, |\mathbf{p}_1, \cdots, \mathbf{p}_n\rangle = n\mathbf{p}_1, \cdots, \mathbf{p}_n$.

Also, the number operator commutes with the Hamiltonian, $[N, H] = 0$, which means that particle number is conserved in the free quantum field theory. This ceased to hold true once we allow interactions between the particles.

Remark: The momentum eigenstates are not localised in space. We can create a localised state via Fourier transform

$$|\mathbf{x}\rangle = \int đ^3 p \, e^{i\mathbf{p}\cdot\mathbf{x}} \, |\mathbf{p}\rangle \,. \tag{3.48}$$

More generally, when we talk about real particles, we describe a wave-packet partially localised in position and partially in momentum space:

$$|\psi\rangle = \int đ^3 p \, e^{i\mathbf{p}\cdot\mathbf{x}} \psi(\mathbf{p}) \, |\mathbf{p}\rangle \tag{3.49}$$

where for example $\psi(\mathbf{p}) \propto e^{-|\mathbf{p}|^2/2m^2}$. Neither $|\mathbf{x}\rangle$ nor $|\psi\rangle$ are eigenstates of H—like in usual QM.

3.5 Relativistic Normalisation

We define the vacuum normalisation to be $\langle 0|0\rangle = 1$. So we have for a general momentum eigenstate,

$$\langle \mathbf{p}|\mathbf{q}\rangle = \langle 0| \, [a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] \, |0\rangle = \deltā^3(\mathbf{p} - \mathbf{q}). \tag{3.50}$$

But is this Lorentz invariant? If we perform a Lorentz transformation, the momenta change as $p^\mu \to \Lambda^\mu{}_\nu p^\nu := p'^\mu$, such that $|\mathbf{p}\rangle \to |\mathbf{p}'\rangle$. We want the states $|\mathbf{p}\rangle$ and $|\mathbf{p}'\rangle$ to be related by a unitary transformation:

$$|\mathbf{p}\rangle \to |\mathbf{p}'\rangle = U(\Lambda) \, |\mathbf{p}\rangle \,. \tag{3.51}$$

This is because the inner product would be invariant under such a transformation:

$$\langle \mathbf{p}|\mathbf{q}\rangle \xrightarrow{\text{LT}} \langle \mathbf{p}| \, U^\dagger(\Lambda) U(\Lambda) \, |\mathbf{q}\rangle = \langle \mathbf{p}|\mathbf{q}\rangle \,. \tag{3.52}$$

Take the identity operator on one-particle states to be

$$1 = \int đ^3 p \, |\mathbf{p}\rangle \langle \mathbf{q}| \,. \tag{3.53}$$

Notice that neither the integral measure $\int đ^3 p$, not the outer product $|\mathbf{p}\rangle \langle \mathbf{q}|$ is Lorentz invariant by itself. However, the identity operator evidently is. Let us analyse this.

Claim 2: The integral $\int \frac{đ^3 p}{2E_p}$ is Lorentz invariant.

Proof. The integral over the full four momentum $\int \mathrm{d}^4 p$ is obviously Lorentz invariant, since Lorentz transformations are elements of SO(1,3), meaning that $\det \Lambda = 1$ and the Jacobian is unity when changing frames. The relativistic dispersion relation for a massive particle $E_p^2 = p_0^2 = |\mathbf{p}|^2 + m^2$ is Lorentz invariant (ie. holds in all frames). The choice of branch for p_0 is also Lorentz invariant.

$$\int \mathrm{d}^4 p \, \delta(p_0^2 - |\mathbf{p}|^2 - m^2)|_{p_0 > 0} \qquad \text{is LI} \tag{3.54}$$

Finally, we use the properties of the delta function

$$\delta(g(x)) = \sum_{x_i \text{ roots of } g} \frac{\delta(x - x_i)}{|g'(x_i)|} \tag{3.55}$$

to show that the above integral is

$$\int \mathrm{d}^4 p\, \delta(p_0^2 - |\mathbf{p}|^2 - m^2)|_{p_0>0} = \int \frac{\mathrm{d}^3 p}{2E_p} \tag{3.56}$$

□

Claim 3: $2E_p \delta^3(\mathbf{p} - \mathbf{q})$ is the Lorentz invariant delta function.

Proof.

$$\int \frac{\mathrm{d}^3 p}{2E_p} 2E_p \delta^3(\mathbf{p} - \mathbf{q}) = 1 \tag{3.57}$$

Since the integral measure is Lorentz invariant by the previous claim, and the right hand side is just a number, the claim follows. □

Hence, we can define relativistically normalised states as

$$|p^\mu\rangle := |p\rangle = \sqrt{2E_p}\, |\mathbf{p}\rangle \sqrt{2E_p} a_{\mathbf{p}}^\dagger |0\rangle . \tag{3.58}$$

Then the inner product is

$$\langle p|q\rangle = (2\pi)^3 2\sqrt{E_p E_q} \delta^3(\mathbf{p} - \mathbf{q}) \tag{3.59}$$

and we re-write the completeness relation on single-particle states as

$$1 = \int \frac{đ^3 p}{2E_p} |p\rangle \langle p| . \tag{3.60}$$

3.6 Free Complex Scalar Field

Let $\psi \in \mathbb{C}$ be a complex scalar field. The associated Lagrangian density is

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - \mu^2 \psi^* \psi, \tag{3.61}$$

where $\mu \in \mathbb{R}$. Applying the Euler-Lagrange equations give

$$\partial_\mu \partial^\mu \psi + \mu^2 \psi = 0, \qquad \partial_\mu \partial^\mu \psi + \mu^2 \psi = 0. \tag{3.62}$$

Again, we can expand this as a Fourier integral. However, this time, unlike in the case of the scalar field ϕ, we want the conjugate fields to be different $\psi \neq \psi^*$. This is achieved by taking

$$\psi = \int \frac{đ^3 p}{\sqrt{2E_p}} \left(b_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + c_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right), \qquad \psi^\dagger = \int \frac{đ^3 p}{\sqrt{2E_p}} \left(b_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} + c_{\mathbf{p}} e^{+i\mathbf{p}\cdot\mathbf{x}} \right). \tag{3.63}$$

The conjugate momenta to these fields are

$$\pi = \int đ^3p(-i)\sqrt{\frac{E_p}{2}}\left(b_{\mathbf{p}}^\dagger e^{i\mathbf{p}\cdot\mathbf{x}} - c_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\right), \qquad \pi^\dagger = \int đ^3p(-i)\sqrt{\frac{E_p}{2}}\left(b_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - c_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right). \tag{3.64}$$

Finally, the commutation relations

$$[\psi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}), \qquad [\psi(\mathbf{x}), \pi^\dagger(\mathbf{y})] = 0, \qquad \text{etc...} \tag{3.65}$$

can then be shown to be equivalent to

$$[b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) = [c_{\mathbf{p}}, c_{\mathbf{q}}^\dagger] \tag{3.66}$$

with all other commutators vanishing. The interpretation of this is that there are two different particles created by $b_{\mathbf{p}}^\dagger$ and $c_p^\dagger$. Both have the same mass m and spin 0. These are interpreted as particle and anti-particle. Retrospectively, we see that for a real scalar field, the particle is its own anti-particle.

The conserved charge associated with a phase rotation invariance of the Lagrangian is

$$\begin{aligned} Q &= i\int \mathrm{d}^3x\,(\dot{\psi}^*\psi - \psi^*\dot{\psi}) && (3.67)\\ &= i\int \mathrm{d}^3x\,(\pi\psi - \psi^\dagger\pi^\dagger). && (3.68)\end{aligned}$$

We then insert the respective Fourier expansions in terms of the ladder operators. After normal ordering, this conserved charge becomes

$$Q = \int đ^3p\left(c_{\mathbf{p}}^\dagger c_{\mathbf{p}} - b_{\mathbf{p}}^\dagger b_{\mathbf{p}}\right) = N_c - N_b. \tag{3.69}$$

Since $[Q, H]$, the number of particles minus the number of anti-particles is conserved.

> **Remark:** We are in the free theory, where N_c and N_b are individually conserved. In the interacting theory, this is not true any more, but the charge Q will still be conserved. This corresponds to the conservation of electric charge, even in interacting theories.

3.7 The Heisenberg Picture

So far, in the Schrödinger picture, the Lorentz invariance has not been overt. In particular, the field operators $\phi(\mathbf{x})$ did only depend on space and not on time, whereas the states evolve in time via

$$i\frac{\mathrm{d}\,|p\rangle}{\mathrm{d}t} = H\,|p\rangle = E_p\,|p\rangle\,. \tag{3.70}$$

Physically, the expectation values $\langle ...|...\rangle$ corresponding to probability amplitudes that we will measure. We can leave these invariant by making the following transformations. Taking the unitary transformation $U(t) = e^{iHt}$, we define operators O_H in the Heisenberg picture as

$$O_H(t) = U(t) O_S U^\dagger(t). \tag{3.71}$$

This implies that the operators evolve in time via

$$\frac{\mathrm{d}O_H}{\mathrm{d}t} = i[H, O_H]. \tag{3.72}$$

Note that at $t = 0$, the Schrödinger and Heisenberg operators coincide. The operators in the Heisenberg picture satisfy equal-time commutation relations:

$$[\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = 0 = [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] \tag{3.73}$$

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}). \tag{3.74}$$

Exercise 3.6: We can check that $\frac{\mathrm{d}\phi}{\mathrm{d}t} = i[H, \phi]$ means that the Heisenberg operator ϕ satisfies the Klein-Gordon equation $\partial_\mu \partial^\mu \phi + m^2 \phi = 0$. This is now a vector equation.

Notation: We denote the full four-vector spacetime dependence as $\phi(\mathbf{x}, t) = \phi(x)$.

We write the Fourier transform of $\phi(x)$ by deriving

$$U(t) a_{\mathbf{p}} U^\dagger(t) = e^{-iE_p t} a_{\mathbf{p}}, \qquad U(t) a_{\mathbf{p}}^\dagger U^\dagger(t) = e^{+iE_p t} a_{\mathbf{p}}^\dagger. \tag{3.75}$$

Then, in the Heisenberg picture, $\phi(x)$ is given by

$$\phi(x) = \int \frac{\text{đ}^3 p}{\sqrt{2E_p}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}} e^{+ip\cdot x} \right). \tag{3.76}$$

Exercise 3.7: Do up to Q4 on Example sheet 2.

3.8 Causality

There is still (potentially) a hint of Lorentz variance because the commutation relations of ϕ and π satisfy equal-time commutation relations; moving to a different frame, they might not be equal-time anymore. What about arbitrarily large space-time separations? The real requirement of causality is that measurements performed at spacelike distances do not influence each other. In the language of quantum mechanics, this means that all spacelike separated operators commute with each other:

$$[O_1(x), O_2(y)] = 0, \qquad \text{whenever } |x-y|^2 < 0 \tag{3.77}$$

Do we have this? Define

$$\begin{aligned}\Delta(x-y) &:= [\phi(x), \phi(y)] && (3.78)\\ &= \int_{\mathbb{R}^6} \frac{đ^3p\, đ^3p'}{\sqrt{4E_pE_{p'}}} \left\{[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}e^{-i(p\cdot x - p'\cdot y)} + [a^\dagger_{\mathbf{p}}, a_{\mathbf{p}'}]e^{+i(p\cdot x-p'\cdot y}\right\} && (3.79)\\ &= \int \frac{đ^3p}{2E_p}\left\{e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right\} && (3.80)\end{aligned}$$

What do we know about this function?

- Lorentz invariant because $\int đ^3p(2E_p)^{-1}$ is and the integrand is too
- If x, y are spacelike from each other, it vanishes because $x-y$ can be Lorentz transformed to $y-x$ in the first term, giving zero
- It does not vanish for timelike separations

  Example: Using (3.80), we see that for two obviously timelike separated spacetime points, we have

  $$[\phi(\mathbf{x}, 0), \phi(\mathbf{x}, t)] = \int \frac{đ^3p}{2E_p}\left(e^{-iE_pt} - e^{+iE_pt}\right) \neq 0 \tag{3.81}$$

- At equal times:

  $$[\phi(\mathbf{x}, t)\phi(\mathbf{y}, t)] = \int \frac{đ^3p}{2E_p}\left(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x})}\right) = 0 \tag{3.82}$$

  where in the last line we changed variable $\mathbf{p}' = -\mathbf{p}$ in the latter integral. This agrees with equal-time commutation relations.

3.9 Propagators

If we prepare a particle at spacetime point y, what is the probability amplitude associated with finding it at spacetime point x? It is given by the braket

$$\langle 0|\,\phi(x)\phi(y)\,|0\rangle = \int \frac{đ^3p\, đ^3p'}{\sqrt{4E_pE'_p}} \langle 0|\, a_{\mathbf{p}} a^\dagger_{\mathbf{p}'}\, |0\rangle\, e^{-ip\cdot x + ip'\cdot y} \tag{3.83}$$

$$= \int \frac{đ^3p\, đ^3p'}{\sqrt{4E_pE'_p}} \langle 0|\, [a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}]\, |0\rangle\, e^{-ip\cdot x + ip'\cdot y} \tag{3.84}$$

$$= \int \frac{đ^3p}{2E_p} e^{-ip\cdot(x-y)} := D(x-y) \qquad \text{the propagator.} \tag{3.85}$$

For spacelike separations $|x-y|^2 < 0$, one can show that the commutator decays as $D(\mathbf{x}-\mathbf{y}) \sim e^{-m|\mathbf{x}-\mathbf{y}|}$. We say that the quantum field *leaks out of the light-cone*. But we just saw that spacelike measurements commute,

$$\Delta(x-y) = [\phi(x), \phi(y)] = D(y-x) - D(x-y) = 0 \qquad \forall |x-y|^2 < 0. \tag{3.86}$$

We interpret this the following way: there is no Lorentz invariant way to order the events, and a particle can just as easily travel from x to y as vice-versa. In a measurement, this symmetry implies that these amplitudes cancel.

All of this holds for real scalar fields. However, there is also an easy generalisation for complex scalar fields. For a $\mathbb{C}$ scalar, $[\psi(x), \psi^\dagger(y)] = 0$ outside the light-cone. However, for complex scalars, we have particles and anti-particles in our interpretation. We interpret this to say that the amplitude for a particle to go from x to y cancels the one for its anti-particle to go from y to x. This is the more general statement; for the case of real fields, the particle is its own anti-particle.

3.9.1 Feynman Propagator

Definition 8 (Feynman Propagator)**:** The *Feynman propagator* $\Delta_F(x-y)$ is defined as

$$\Delta_F(x-y) = \langle 0|\, T\phi(x)\phi(y)\, |0\rangle = \begin{cases} \langle 0|\, \phi(x)\phi(y)\, |0\rangle, & x^0 > y^0 \\ \langle 0|\, \phi(y)\phi(x)\, |0\rangle, & x^0 < y^0. \end{cases} \tag{3.87}$$

where T is the time-ordering operator.

Claim 4: We can write the Feynman propagator as

$$\boxed{\Delta_F = \int đ^4p\, \frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2}}. \tag{3.88}$$

This is manifestly Lorentz invariant. At the moment this is ill-defined because for each value of $\mathbf{p}$, the integral over p^0 has a pole at $(p^0)^2 = |\mathbf{p}|^2 + m^2$. We need a prescription that tells us how to handle this pole. We define the integration contour to be the one depicted in F2. To use the residue theorem, we write

$$\frac{1}{p^2 - m^2} = \frac{1}{(p^0)^2 - E_p^2} = \frac{1}{((p^0)^2 - E_p)((p^0)^2 + E_p)} \tag{3.89}$$

to see that the residue at $p^0 = \pm E_p$ is $\pm\frac{1}{2E_p}$. For $x^0 > y^0$, we close the contour in the lower-half plane:

$$e^{-ip^0(x^0-y^0)} \to 0 \qquad \text{as} \qquad p^0 \to -i\infty. \tag{3.90}$$

Therefore, (integrating clockwise incurs the minus sign)

$$\Delta_F(x-y) = \int đ^3 p \, \frac{1}{(2\pi)2E_p}(-2\pi i) i e^{-iE_p(x^0-y^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{3.91}$$

$$= \int đ^3 p \, \frac{1}{2E_p} e^{-ip(x-y)} \tag{3.92}$$

This is $D(x-y)$, which we showed was $\langle 0| \phi(x)\phi(y) |0\rangle$.

On the other hand, when $x^0 < x^0$, we close the contour in the upper-half plane to get

$$\Delta_F(x-y) = \int đ^3 p \, \frac{1}{(2\pi)2E_p}(+2\pi i) i e^{+iE_p(x^0-y^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \tag{3.93}$$

$$= \int đ^3 p \, \frac{1}{2E_p} e^{-ip(y-x)} \tag{3.94}$$

$$= D(y-x) = \langle 0| \phi(y)\phi(x) |0\rangle . \tag{3.95}$$

There is a mnemonic that we can use to short-circuit this tedious contour integration. The '$i\epsilon$' prescription moves the poles slightly off the real axis:

$$\boxed{\Delta_F(x-y) = \int đ^4 p \, \frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}} \tag{3.96}$$

which is an equivalent description, once we take $\epsilon \to 0$.

The Feynman propagator Δ_F is the Green's function of the Klein-Gordon operator

$$(\partial_t^2 - \nabla^2 + m^2)\Delta_F(x-y) = \int đ^4 p \, \frac{i(-p^2+m^2)}{p^2 - m^2} = -i\delta^4(x-y). \tag{3.97}$$

3.9.2 Retarded Green's functions

It can be useful to pick other contours. One example is the *retarded* Green's function

$$\Delta_R(x-y) = \begin{cases} [\phi(x), \phi(y)], & x^0 > y^0 \\ 0, & x^0 < y^0 \end{cases} \tag{3.98}$$

which is useful in a context in which we start with an initial field configuration and look at evolution in the presence of a source. For example, $(\partial_\mu \partial^\mu + m^2)\phi(x) = J(x)$.

However, Δ_F is the most useful propagator in QFT.

4 Interacting Fields

For free theories, we have a quadratic Lagrangian. This means that the equations of motion are linear. We can get multi-particle states, but they do not scatter off of each other. We can also exactly quantise the theory.

Interactions are written as higher order terms in the Lagrangian.

Example: For a real scalar field, a general interacting Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n=3}^{\infty}\frac{\lambda_n\phi^n}{n!}. \tag{4.1}$$

4.1 Coupling Constants and Mass Dimension

The factors λ_n are called the *coupling constants*; their values determine the strength of the interactions. We wish those interaction terms to be 'small', so that we can develop a perturbation expansion around the free theory. We might naively require $\lambda_n \ll 1$. However, this equation only makes sense if λ_n is dimensionless. We can work out the dimensionality of λ_n in the following way: Recall that the mass dimension of the action is $[S] = 0$, because it has dimensions of angular momentum and we have set $\hbar = 1$. Since $S = \int \mathrm{d}^4x\,\mathcal{L}$ and $[\mathrm{d}^4x] = -4$, the Lagrangian must have $[\mathcal{L}] = 4$. For the first, kinetic term, we have $[\partial_\mu] = 1$, so that $[\phi] = 1$. For the second term, we therefore have $[m] = 1$ (as expected if we interpret m as the mass of the particle) and for the higher order terms we have $[\lambda_n] = 4 - n$. The interaction terms of coupling constant λ_n can be arranged into three classes according to their high and low energy behaviour:

1. $[\lambda_3] = 1 > 0$: The dimensionless parameter is λ_3/E, where E is the energy-scale of the configuration and $[E] = 1$. Think of this as the scattering energy, say, at the LHC. At high energies, when $E \gg \lambda_3$, the term $\frac{1}{3!}\lambda_3\phi^3$ is a small perturbation to the free Lagrangian. This perturbation is called *relevant* (c.f. renormalisation group), since it is relevant at the low energies that we can measure. These relevant perturbations are renormalisable . In a relativistic setting,

$E > m$; we can therefore make any relevant perturbations small by taking $\lambda_3 \ll m$.

2. $[\lambda_4] = 0$: the term $\frac{1}{4!}\lambda_4\phi^4$ is small if $\lambda_4 \ll 1$. We call this a *marginal* perturbation. These are also renormalisable.

3. $[\lambda_{n>4}] < 0$: The dimensionless parameters $\lambda_n E^{n-4}$ are small at low energies, but large at high energies. We call $\frac{1}{n!}\lambda_n\phi^n$ *irrelevant* perturbations; these are non-renormalisable.

In this course, we will only ever consider *weakly-coupled* QFTs. But the study of *strongly-coupled* QFTs, often using computational techniques, is a lively contemporary research area. We also know about interesting phenomena like the 'confinement' of quarks in QCD, whose understanding is a major research goal.

Example (ϕ^4 theory)**:** In scalar ϕ^4 theory, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{4.2}$$

where $\lambda \ll 1$. We can already take a guess at the effects of the last term. We expect that the extra terms cause $[H, N] \neq 0$, so that particle number will *not* be conserved. Intuitively, interactions will create or destroy particles. Expanding the last term in the Lagrangian, we get some twelve-dimensional integral, including some terms

$$\int \ldots (a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{p}'} a^\dagger_{\mathbf{p}''} a^\dagger_{\mathbf{p}'''} + \ldots) + \int \ldots (a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{p}'} a^\dagger_{\mathbf{p}''} a_{\mathbf{p}'''} + \ldots) + \ldots \tag{4.3}$$

creating and destroying particles.

Example (Scalar Yukawa Theory)**:** Let $\psi \in \mathbb{C}$, $\phi \in \mathbb{R}$. Then

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi - \mu^2\psi^*\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - g\psi^*\psi\phi \tag{4.4}$$

This theory has historical significance, since it can model mesons to some degree. In this case, we now know that ψ and ϕ are not fundamental particles, but composite ones. (c.f. effective field theory) This is a relevant perturbation, so provided that $g \ll m$ and $g \ll \mu$, we can use perturbation theory. This theory has the same phase-rotation symmetry of the complex field; the theory is invariant under

$$\psi \to e^{i\alpha}\psi, \quad \psi^* \to e^{-i\alpha}\psi^*, \quad \phi \to \phi. \tag{4.5}$$

The conserved charge associated with this Noether symmetry is the number of particles ψ minus the number of anti-particles ψ^*. However, there is no conservation for the number of ϕ particles.

4.2 The Interaction Picture

In QM, have the Schrödinger picture, where the operators O_S are independent of time and states evolve according to

$$i\frac{\mathrm{d}\,|\psi\rangle_S}{\mathrm{d}t} = H\,|\psi\rangle_S\,, \tag{4.6}$$

and we also have the Heisenberg picture, which is related to the former by the unitary transformations

$$O_H(t) = e^{iHt} O_S e^{-iHt}, \qquad |\psi\rangle_H = e^{iHt} |\psi\rangle_S . \tag{4.7}$$

The *interaction picture* is a hybrid of the Schrödinger and Heisenberg pictures. We write

$$H = H_0 + H_{\text{int}}. \tag{4.8}$$

where H_0 is the free part of the Hamiltonian.

Example (ϕ^4 theory)**:** The interaction Lagrangian is $\mathcal{L}_{\text{int}} = -\frac{1}{4!}\lambda\phi^4$. The free Hamiltonian is

$$H_0 = \int \mathrm{d}^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right), \tag{4.9}$$

whereas the interacting Hamiltonian is

$$H_{\text{int}} = -\int \mathrm{d}^3x\, \mathcal{L}_I = \int \mathrm{d}^3x\, \frac{\lambda\phi^4}{4!}. \tag{4.10}$$

4.2.1 Relation to Schrödinger Picture

In the interaction picture, the operators are related to the Schrödinger operators as

$$O_I(t) = e^{iH_0t} O_S e^{-iH_0t}, \tag{4.11}$$

so e.g.

$$\phi_I(x) = e^{iH_0t} \phi'(\mathbf{x}) e^{-iH_0t}, \tag{4.12}$$

where $O'(\mathbf{x})$ is the Schrödinger picture field operator.

We can use the same reasoning as in the Heisenberg picture (free theory) to derive that ϕ_I obeys the Klein-Gordon equation $(\partial^2 + m^2)\phi_I(x) = 0$ with solutions

$$\phi_I(x) = \int \mathrm{đ}^3p\, \frac{1}{\sqrt{2E_p}} \left(a_{\mathbf{p}} e^{-ip\cdot x} a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right), \tag{4.13}$$

just like in the free theory. Strictly speaking, we should use indices on the operators $a_{\mathbf{p}}$. We have $\phi(t=0,\mathbf{x}) = \phi_S(\mathbf{x})$. As before, $[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = \delta^3(\mathbf{p}-\mathbf{p}')$ with other brackets vanishing. Moreover, the vacuum $|0\rangle$ is the vacuum of the free theory and satisfies $a_{\mathbf{p}} |0\rangle = 0$.

4.2.2 Relation to Heisenberg Picture

We use our knowledge of the relation between Schrödinger and Heisenberg pictures to find that the interaction picture fields are related to the Heisenberg ones by

$$\phi_H(t,\mathbf{x}) = e^{iHt}\, \underbrace{e^{-iH_0t}\phi_I(t,\mathbf{x})e^{iH_0t}}_{\phi_S(\mathbf{x})}\, e^{-iHt}. \tag{4.14}$$

We will write this as $U(t,t_0)^\dagger \phi_I(x) U(t,t_0)$, were $U(t,t_0) = e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t}$ is the unitary time-evolution operator. It has the properties that $U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$ and $U(t,t) = 1$.

Remark:

$$i\frac{\mathrm{d}U(t,t_0)}{\mathrm{d}t} = i\left[e^{iH_0 t}(iH_0)e^{-iH(t-t_0)}e^{-iH_0 t_0} + e^{iH_0 t}(-iH)e^{-iH(t-t_0)}e^{-iH_0 t_0}\right] \tag{4.15}$$

$$= e^{iH_0 t}(H_{\text{int}})_S e^{iH(t-t_0)} e^{-iH_0 t_0} \tag{4.16}$$

$$= \underbrace{e^{iH_0 t}(H_I)_S e^{-iH_0 t}}_{(H_{\text{int}})_I} \underbrace{e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0}}_{U(t,t_0)}, \tag{4.17}$$

where $(H_{\text{int}})_I$ is the interaction Hamiltonian written in the interaction picture.

If H_I were just a function, we could solve (4.17) by $U = \exp\left[-i\int_{t_0}^{t} H_I(t')\,\mathrm{d}t'\right]$, but this does not work. There are ordering ambiguities since $[H_I(t'), H_I(t'')] \neq 0$ for $t' \neq t''$. From (4.17), we see that $U(t, t_0)$ satisfies

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} \mathrm{d}t'\, H_I(t')U(t',t_0). \tag{4.18}$$

Substituting this back into itself to get an infinite series

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} \mathrm{d}t'\, H_I(t') + (-i)^2\int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t} \mathrm{d}t''\, H_I(t')H_I(t'') + \ldots \tag{4.19}$$

From the ranges of integration, the H_I product is automatically time ordered.

If we integrate over the upper patch in Figure 4.1, time ordering is $T\left\{H_I(t')H_I(t'')\right\} = H_I(t'')H_I(t')$, whereas in the lower patch we have $T\left\{H_I(t')H_I(t'')\right\} = H_I(t')H_I(t'')$. The upper triangle is the initial volume of the integral. But by relabelling, we can show that this is the same as the lower triangle:

$$\int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t} \mathrm{d}t''\, H_I(t')H_I(t'') = \int_{t_0}^{t} \mathrm{d}t'' \int_{t_0}^{t} \mathrm{d}t'\, H_I(t'')H_I(t'). \tag{4.20}$$

So $\frac{1}{2!}$ times the area of the full square is the area of the upper triangle

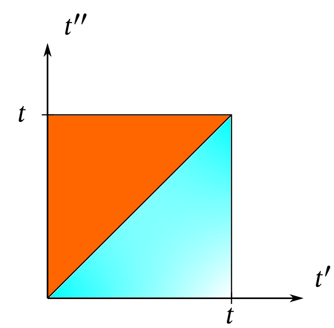


Figure 4.1

$$U(t) = 1 + (-i)\int_{t_0}^{t} \mathrm{d}t'\, H_I(t') + \frac{(-i)^2}{2!}\int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t} \mathrm{d}t''\, T\left\{H_I(t')H_I(t'')\right\} \tag{4.21}$$

Therefore, we obtain *Dyson's formula* by writing

$$U(t,t_0) = T\exp\left\{-i\int_{t_0}^{t} \mathrm{d}t'\, H_I(t')\right\} = T\exp\left\{+i\int_{t_0}^{t} \mathrm{d}^3x\,\mathrm{d}t\, \mathcal{L}_I(t')\right\} \tag{4.22}$$

Example: In scalar Yukawa theory, this is $T\exp\left[-ig\int \mathrm{d}^4x\, \psi^*\psi\phi\right]$.

This is something of a formal result—we expand to finite order in λ_n.

4.3 Scattering Amplitudes

The time evolution used in scattering theory is called the *S-matrix*. Let us define the time evolution operator going from initial to final states as

$$S = \lim_{\substack{t\to\infty,\\ t_0\to-\infty}} \{U(t,t_0)\} \tag{4.23}$$

The initial state $|i\rangle$ and final state $|f\rangle$ are in some sense 'far away' from each other and from the interaction. We assume that $|f\rangle, |i\rangle$ behave like free particles; they are eigenstates of H_0.

The amplitude is

$$\lim_{\substack{t\to\infty \\ t_0\to-\infty}} \{\langle f|\, U(t,t_0)\,|i\rangle\} = \langle f|\, S\,|i\rangle, \tag{4.24}$$

where $\langle f|\, S\,|i\rangle$ is historically called the *S-matrix* or *matrix element*.

Example: Let us go back to scalar Yukawa theory,

$$H_I = g\psi_I^*\psi_I\phi_I. \tag{4.25}$$

Physically, we will model mesons with ϕ and nucleons as ψ. Moreover, we will drop the subscript I from now on for notational ease. Concentrate on creation and annihilation operators in the expansion The operator $\phi \sim a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger$ destroys or creates a meson. For the other two it is slightly more complicated: $\psi \sim b_{\mathbf{p}}, c_{\mathbf{p}}^\dagger$ destroys a nucleon or creates an anti-nucleon, while $\psi^* \sim b_{\mathbf{p}}^\dagger, c_{\mathbf{p}}$ creates a nucleon or destroys an anti-nucleon. The only non-zero commutators are

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = [b_{\mathbf{p}}, b_{\mathbf{p}'}^\dagger] = [c_{\mathbf{p}}, c_{\mathbf{p}'}^\dagger] = \delta^3(\mathbf{p}-\mathbf{p}'). \tag{4.26}$$

The interaction Hamiltonian H_I contains terms like $\int \dots b_{\mathbf{p}}^\dagger c_{\mathbf{p}'}^\dagger a_{\mathbf{p}''} + \dots$, which destroys a meson and produces a nucleon/anti-nucleon pair. This contributes to meson decay: $\phi \to \psi\bar{\psi}$. Figure 4.2b represents this in a diagrammatic form.

> **Remark:** We wrote $\bar{\psi}$ to denote the anti-particle, not the field. In general, particles are labelled like their fields and anti-particles are labelled like the fields with a bar on top.

This interaction arose from the first term in $U(t,t_0)$. At second order in g, we have more complicated terms like $\int \dots (cba^\dagger)(b^\dagger c^\dagger a) + \dots$.

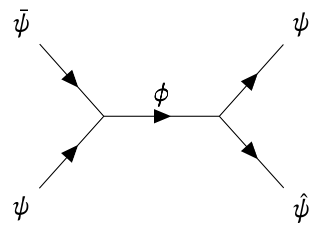
> **Remark:** The momenta $\mathbf{p}, \mathbf{p}', \dots$ are implied.

This term, corresponding to the diagram depicted in Figure 4.2a, contributes to nucleon / anti-nucleon scattering.

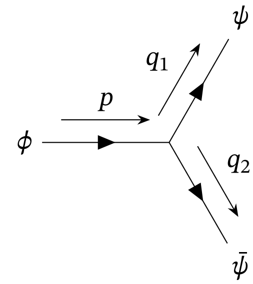
4.3.1 Meson Decay

Let us compute the amplitude for the diagram in Figure 4.2b. We take the initial and final states to be

$$|i\rangle = \sqrt{2E_p}\, a_{\mathbf{p}}^\dagger\,|0\rangle \qquad |f\rangle = \sqrt{4E_{q_1}E_{q_2}}\, b_{\mathbf{q}_1}^\dagger c_{\mathbf{q}_2}^\dagger\,|0\rangle\,. \tag{4.27}$$



(a) Scattering between a nucleon ψ and anti-nucleon $\bar{\psi}$ via the meson ϕ.



(b) A meson ϕ with momentum decays into a nucleon and anti-nucleon pair.

Figure 4.2: Diagrammatic representations of interaction terms.

We need to calculate the matrix element $\langle f|\, S\, |i\rangle$. As previously mentioned, we do this by expanding in the coupling constant g. The zeroth order term vanishes, since $\langle f|i\rangle = \langle 0|\, bca^\dagger\, |0\rangle = 0$. The first order term is

$$-ig\, \langle f|\, T \int \mathrm{d}^4x\, \psi^*(x)\psi(x)\phi(x)\, |i\rangle + O(g^2). \tag{4.28}$$

Note that the time ordering operator T has no effect here, whereas in the higher order terms it will matter. Our general strategy is to expand all field and states, getting rid of all creation and annihilation operators with the commutation relations and the vacuum normalisation $\langle 0|0\rangle = 1$. We will thus reduce this QFT amplitude to a function of 4-momenta.

Let us expand the meson field ϕ and $|i\rangle$ first.

$$\langle f|\, S\, |i\rangle = -ig\, \langle f| \int \mathrm{d}^4x\, \psi^*(x)\psi(x) \int \text{đ}^3k\, \frac{\sqrt{2E_p}}{\sqrt{2E_k}} \left(a_{\mathbf{k}} a^\dagger_{\mathbf{p}} e^{-ik\cdot x} + a^\dagger_{\mathbf{k}} a^\dagger_{\mathbf{p}} e^{ik\cdot x} \right) |0\rangle + O(g^2) \tag{4.29}$$

We can then commute the $a^\dagger_{\mathbf{k}}$ in the second summand directly into the vacuum on the left hand side, using that $[a^\dagger_{\mathbf{k}}, \psi(x)] = 0 = [a^\dagger_{\mathbf{k}}, \psi^*(x)]$. This annihilates the vacuum on the left hand side since

$$a_{\mathbf{k}}\, |0\rangle = 0 \;\Rightarrow\; \langle 0|\, a^\dagger_{\mathbf{k}} = 0. \tag{4.30}$$

In the first term, we can replace $[a_{\mathbf{k}}, a^\dagger_{\mathbf{p}}]\, |0\rangle = \text{đ}^3(\mathbf{p} - \mathbf{k})\, |0\rangle$, so to first order in g,

$$\langle f|\, S\, |i\rangle = -ig\, \langle f| \int \mathrm{d}^4x\, \psi^*(x)\psi(x) e^{-ip\cdot x}\, |0\rangle\,. \tag{4.31}$$

We now expand the ψ fields and the $\langle f|$.

$$\begin{aligned}\langle f|\, S\,|i\rangle &= -ig\,\langle 0|\int \mathrm{d}^4x\,\frac{\mathrm{d}^3k_1\,\mathrm{d}^3k_2}{\sqrt{4E_{k_1}E_{k_2}}}\Big\{\sqrt{4E_{q_1}E_{q_2}}c_{\mathbf{q}_2}b_{\mathbf{q}_1}\left(b^\dagger_{\mathbf{k}_1}e^{ik_1\cdot x}+\cancel{c_{\mathbf{k}_1}}e^{-ik_1\cdot x}\right)\\ &\quad\times\left(\cancel{b_{\mathbf{k}_1}}(\ldots)+c^\dagger_{\mathbf{k}_2}e^{ik_2\cdot x}\right)e^{-ip\cdot x}\,|0\rangle\Big\}\end{aligned} \tag{4.32}$$

$$= -ig\,\langle 0|\int \mathrm{d}^4x\,\frac{\mathrm{d}^3k_2}{\sqrt{2E_{k_2}}}\sqrt{2E_{q_2}}[c_{\mathbf{q}_2},c^\dagger_{\mathbf{k}_2}]e^{i(q_1\cdot x+k_2\cdot x-p\cdot x)}\,|0\rangle \tag{4.33}$$

$$= -ig\,\langle 0|\int \mathrm{d}^4x\,e^{i(q_1+q_2-p)\cdot x}\,|0\rangle \tag{4.34}$$

$$= -ig\delta^4(q_1+q_2-p). \tag{4.35}$$

This four dimensional delta function imposes 4-momentum conservation on initial and final states.

4.4 Wick's Theorem

We want to compute various quantities such as

$$\langle f|\, t\,\{H_I(x_1)\dots H_I(x_n)\}\,|i\rangle\,. \tag{4.36}$$

Wick's theorem will simplify these computations. It is handy if we write this in terms of normal ordered products. Take the case of a real scalar field ϕ to illustrate this. Separate the field into creation and annihilation operators

$$\phi(x) = \phi^+(x) + \phi^-(x), \tag{4.37}$$

where the notation is slightly counterintuitive since ϕ^+ are the annihilatin operators and ϕ^- the creation ones:

$$\phi^+(x) := \int \text{đ}^3p\, \frac{1}{\sqrt{2E_p}} a_{\mathbf{p}} e^{-ip\cdot x} \qquad \phi^-(x) := \int \text{đ}^3p\, \frac{1}{\sqrt{2E_p}} a^\dagger_{\mathbf{p}} e^{+ip\cdot x}. \tag{4.38}$$

For $x^0 > y^0$, the time ordered product of two fields is $T\{\phi(x)\phi(y)\} = \phi(x)\phi(y)$. Expanding this out in terms of ϕ^+ and ϕ^-, we get

$$\begin{aligned}
T\{\phi(x)\phi(y)\} &= (\phi^+(x)+\phi^-(x))(\phi^+(y)+\phi^-(y)) && (4.39)\\
&= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^-(y) + [\phi^+(x),\phi^-(y)]. && (4.40)\\
&= \,:\phi(x)\phi(y):\, + \int \frac{\text{đ}^3p'}{\sqrt{2E_{p'}}}\frac{\text{đ}^3p}{\sqrt{2E_p}}[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}]e^{-ip\cdot x + ip'\cdot y} && (4.41)\\
&= \,:\phi(x)\phi(y):\, + \int \frac{\text{đ}^3p}{2E_p} e^{-ip\cdot(x-y)} && (4.42)\\
&= \,:\phi(x)\phi(y):\, + D(x-y). && (4.43)
\end{aligned}$$

For $y^0 > x^0$, we can go through the same argument to give

$$T\{\phi(x)\phi(y)\} = \,:\phi(x)\phi(y):\, + D(y-x). \tag{4.44}$$

Putting these two results together, we find that

$$T\{\phi(x)\phi(y)\} = \,:\phi(x)\phi(y):\, + \Delta_F(x-y). \tag{4.45}$$

Note that the operators $T\{\phi(x)\phi(y)\}$ and $:\phi(x)\phi(y):$ only differ by a function $\Delta_F(x-y)$. This is the simplest example of Wick's theorem.

Notation (Wick contractions)**:** We use a bracket to denote a *contraction* of a pair of fields; the string $\phi(x_1)\overbracket{\phi(x_2)\phi(x_3)\dots\phi}(x_j)\dots\phi(x_n)$ denotes the same product of fields, except that the two contracted fields are replaced by their Feynman propagator $\overbracket{\phi(x_2)\phi}(x_j) = \Delta_F(x_2-x_j)$.

Using this notation, we can write the above result as

$$T\{\phi(x)\phi(y)\} = :\phi(x)\phi(y): + \wick{\c{\phi}(x)\c{\phi}(y)} \tag{4.46}$$

Theorem 2 (Wick's theorem)**:** In general, Wick's theorem states that

$$T\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\} = :\phi(x_1)\dots\phi(x_n): + :\text{all possible contractions}: . \tag{4.47}$$

Example (four fields)**:** For a more concise notation, denote $\phi_i := \phi(x_i)$, for the four fields $i = 1, 2, 3, 4$. The time ordered product of the four fields is then

$$\begin{aligned} T\{\phi_1\phi_2\phi_3\phi_4\} = :\phi_1\phi_2\phi_3\phi_4: + \Big(\wick{\c{\phi}_1\c{\phi}_2} :\phi_3\phi_4: + \wick{\c{\phi}_1\c{\phi}_3} :\phi_2\phi_4: + 4 \text{ similar terms}\Big) \\ + \wick{\c{\phi}_1\c{\phi}_2}\wick{\c{\phi}_3\c{\phi}_4} + \wick{\c{\phi}_1\c{\phi}_3}\wick{\c{\phi}_2\c{\phi}_4} + \wick{\c{\phi}_1\c{\phi}_4}\wick{\c{\phi}_2\c{\phi}_3} \end{aligned} \tag{4.48}$$

This is useful because all normal ordered terms that are sandwiched between vacuum states vanish; only the propagators survive:

$$\begin{aligned} \langle 0| T\{\phi_1\phi_2\phi_3\phi_4\} |0\rangle = \Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) \\ + \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3). \end{aligned} \tag{4.49}$$

Proof of Wick's theorem. Let us proceed by induction. We have already shown that Wick's theorem holds for $n = 2$. Now suppose it is true for $T\{\phi_2 \dots \phi_n\}$ and left-multiply with ϕ_1 with $x_1^0 > x_k^0$, $\forall k \in \{2, \dots, n\}$. Then

$$T\{\phi_1 \dots \phi_n\} = (\phi_1^+ + \phi_1^-)\,[:\phi_2 \dots \phi_n: + :\text{all contractions of } \phi_2 \dots \phi_n:] \tag{4.50}$$

The term ϕ_1^- can stay on the left hand side and go inside the normal ordering. The ϕ_1^+ has to be commuted through to the right hand side of all $\phi_2^- \dots \phi_n^-$, so that the right hand side can be written as a sum over normal-ordered products. Each commutation gives us $D(x_1 - x_k)$, which, for this time ordering, is $\Delta_F(x_1 - x_k) = \wick{\c{\phi}_1\c{\phi}_k}$.

> **Remark:** The textbook 'Relativistic quantum fields' (1965) by Bjřken and Drell contains a clear exposition of Wick's theorem.

□

4.4.1 Consequences of Wick's theorem

- For odd n, the time ordered expectation values just vanish

$$\langle 0| T\phi_1 \dots \phi_n |0\rangle = 0 \tag{4.51}$$

- For even n, we have a sum over all possible Feynman propagators

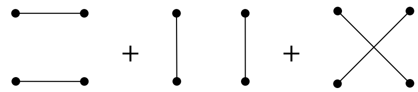
$$\langle 0| T\phi_1 \dots \phi_n |0\rangle = \sum \Delta_F(x_{i_1} - x_{i_2})\Delta_F(x_{i_3} - x_{i_4}) \dots \Delta_F(x_{i_{n-1}} - x_{i_n}), \tag{4.52}$$

where the sum is over all symmetric (even) permutations of $\{i_1, i_2, \dots, i_n\}$.

There is a nice graphical representation of this: We mark $x_1, x_2, \dots, x_n$ as points $\to \bullet$. The sum of equation (4.52) is then obtained by connecting pairs of points in all possible ways and assigning to each line a propagator factor.

Example: For four fields, we draw four dots, representing $x_1, \dots, x_4$, and connect them in all possible ways:

$$\langle 0| T\phi_1\phi_2\phi_3\phi_4 |0\rangle = \tag{4.53}$$



Replacing each line with a propagator, we see that this equation is equivalent to our previous result Equation (4.49).

Example: For a $\mathbb{C}$ scalar ψ, the time ordered product is

$$T\{\psi(x)\psi^*(y)\} = \,:\psi(x)\psi^*(y):\, + \Delta_F(x-y). \tag{4.54}$$

So here we define $\overbracket{\psi(x)\psi^*}(y) = \Delta_F(x-y)$, whilst $\overbracket{\psi(x)\psi}(y) = 0$ and $\overbracket{\psi^*(x)\psi^*}(y) = 0$.

4.5 Nucleon-Nucleon Scattering

Let us apply Wick's theorem to nucleon-nucleon scattering:

$$\psi(p_1)\psi(p_2) \to \psi(p_1')\psi(p_2'). \tag{4.55}$$

Then the initial and final states are

$$|i\rangle = \sqrt{4E_{p_1}E_{p_2}}\, b^\dagger_{\mathbf{p}_1} b^\dagger_{\mathbf{p}_2} |0\rangle := |\mathbf{p}_1, \mathbf{p}_2\rangle \tag{4.56}$$

$$|f\rangle = \sqrt{4E_{p_1'}E_{p_2'}}\, b^\dagger_{\mathbf{p}_1'} b^\dagger_{\mathbf{p}_2'} |0\rangle := \left|\mathbf{p}_1', \mathbf{p}_2'\right\rangle. \tag{4.57}$$

We are not interested in processes where no scattering happens. Therefore, we look at terms in $\langle f| (S-1) |i\rangle$ at $O(g^2)$:

$$\frac{(-ig)^2}{2!} \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2 \left\langle \mathbf{p}_1', \mathbf{p}_2' \right| T\{\psi^*(x_1)\psi(x_1)\phi(x_1)\psi^*(x_2)\psi(x_2)\phi(x_2)\} |\mathbf{p}_1, \mathbf{p}_2\rangle. \tag{4.58}$$

Using Wick's theorem, there exists a term in this of the form

$$:\psi^*(x_1)\psi(x_2)\psi^*(x_2)\psi(x_2):\, \overbracket{\phi(x_1)\phi}(x_2), \tag{4.59}$$

which will contribute: the ψs will annihilate the initial nucleons whereas the ψ^*s will create the final ones. All other terms will vanish once they are commuted through to the vacuum.

$$M = \left\langle \mathbf{p}_1', \mathbf{p}_2' \right| :\psi^+(x_1)\psi(x_1)\psi^*(x_2)\psi(x_2): \left|\mathbf{p}_1, \mathbf{p}_2\right\rangle \tag{4.60}$$

$$\begin{aligned} &= \int \frac{\mathrm{d}^3 q_1 \ldots \mathrm{d}^3 q_4}{\sqrt{2E_{q_1} \ldots 2E_{q_4}}} \sqrt{16 E_{p_1} E_{p_2} E_{p_1'} E_{p_2'}} \langle 0| \, b_{\mathbf{p}_1'} b_{\mathbf{p}_2'} b^\dagger_{\mathbf{q}_1} b^\dagger_{\mathbf{q}_2} b_{\mathbf{q}_3} b_{\mathbf{q}_4} b^\dagger_{\mathbf{p}_1} b^\dagger_{\mathbf{p}_2} \, |0\rangle \\ &\quad \times e^{i(q_1 \cdot x_1 + q_2 \cdot x_2 - q_3 \cdot x_1 - q_4 \cdot x_2)}. \end{aligned} \tag{4.61}$$

Why does the term

$$\psi^*(x_1)\psi(x_1)\psi^*(x_2)\psi(x_2)\phi(x_1)\phi(x_2)\left\langle f\middle|i\right\rangle \tag{4.62}$$

not contribute? The momenta do not change between $\langle f|i\rangle$.

$$M \equiv \left\langle \mathbf{p'}_1, \mathbf{p}'_2\right| :\psi^*(x_1)\psi(x_1)\psi^*(x_2)\psi(x_2): \left|\mathbf{p}_1, \mathbf{p}_2\right\rangle \tag{4.63}$$

$$= \int \frac{\mathrm{d}^3 q_1 \dots \mathrm{d}^3 q_4}{\sqrt{2E_{q_1 \dots 2E_{q_4}}}} \langle 0|\, b_{\mathbf{p}'_1} \dots |0\rangle\, e^{i(q_1\cdot x_1 + q_2\cdot x_2 - q_3\cdot x_1 - q_4\cdot x_2)} \tag{4.64}$$

Using the commutation relations, we can show that

$$\langle 0|\, b_{\mathbf{p}'_1} \dots b^\dagger_{\mathbf{p}_2}\, |0\rangle = [\delta^3(\mathbf{p}'_1 - \mathbf{q}_2)\delta^3(\mathbf{p}'_2 - \mathbf{q}_i) + (p_1 \to p_2)]$$
$$\times [\delta^3(\mathbf{q}_4 - \mathbf{p}_1)\delta^3(\mathbf{q}_3 - \mathbf{p}_2) + (p_1 \leftrightarrow p_2)] \tag{4.65}$$

so

$$M = \left(e^{i(p'_1\cdot x_2 + p'_1\cdot x_1} + e^{i(p'_2\cdot x_2 + p'_1\cdot x_1)}\right) \times \left(e^{-i(p_1\cdot x_2 + p_2\cdot x_1)} + e^{-i(p_1\cdot x_1 + p_2\cdot x_2)}\right). \tag{4.66}$$

And so

$$\langle f|\,(S-1)\,|i\rangle = \frac{(-ig)^2}{2!}\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2 \Big[e^{ix_2\cdot(p'_1 - p_1) + ix_1\cdot(p'_2 - p_2)} \tag{4.67}$$

$$+e^{ix_2\cdot(p'_2 - p_1) + ix_1\cdot(p'_1 - p_2)} + \underbrace{(x_1 \leftrightarrow x_2)}_{\text{cancels the } \frac{1}{2!}}\Bigg] \tag{4.68}$$

$$\times \int \text{đ}^4 k\, \frac{i e^{ik\cdot(x_2 - x_1)}}{k^2 - m^2 + i\varepsilon} \tag{4.69}$$

$$= (-ig)^2 \int \text{đ}^4 k\, \frac{i}{k^2 - m^2 + i\varepsilon}\left[\delta^4(p'_1 - p_1 + k)\delta^4(p'_2 - p_2 - k)\right. \tag{4.70}$$

$$\left. +\delta^4(p'_2 - p_1 + k)\delta^4(p'_1 - p_2 - k)\right] \tag{4.71}$$

$$= (-ig)^2 \left\{\frac{i}{(p_1 - p'_1)^2 - m^2 + i\varepsilon} + \frac{i}{(p'_2 - p_1)^2 - m^2 + i\varepsilon}\right\}\delta^4(p_1 + p_2 - p'_1 - p'_2), \tag{4.72}$$

where we end up with a 4-momentum preserving δ-function.

4.6 Feynman Diagrams

This was a fairly painful process. Doing many of these Wick's theorem calculations, we see that we can make this whole thing more streamlined by introducing Feynman rules.

We draw Feynman diagrams to represent the expansion of the matrix element $\langle f|(S-1)|i\rangle$. We learn to associate functions to each diagram:

- We draw an external line for each particle in $|i\rangle$ and $|f\rangle$, assigning a directional 4-momentum to each.
- For a complex field, we add an arrow to indicate the flow of charge. We choose an ingoing (outgoing) arrow in initial states $|i\rangle$ for particles (anti-particles), and the opposite for final states $|f\rangle$.
- We join the lines together with vertices.

  Example (Scalar Yukawa Theory)**:** Figure 4.3 represents the vertex for scalar Yukawa theory.

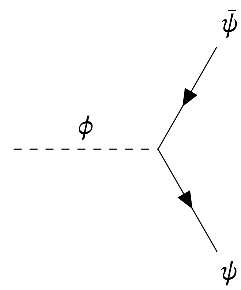
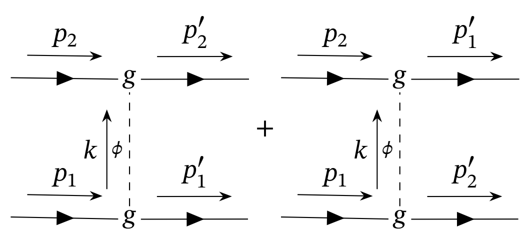


Figure 4.3

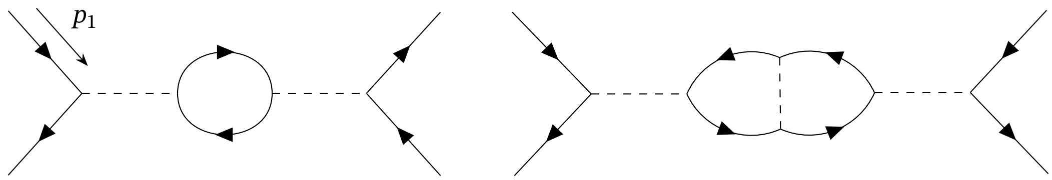
Each such diagram is in one-to one correspondence with terms in $\langle f|(S-1)|i\rangle$.

Example: For meson scattering $\phi+\phi\to\phi+\phi$, we have diagram expansions such as:



(4.73)

At higher orders in H_I, there are more complicated diagrams, such as the one-loop diagram to order $O(g^4)$ in Figure 4.4a or the two-loop diagram to order $O(g^6)$ in Figure 4.4b.



(a) "One-loop" (b) Two-loop

Figure 4.4: Higher order diagrams

> **Remark:** Time flows from left to right in the conventions used in these lectures.

To each diagram, we associate a function using the *Feynman rules*:

1. Associate a 4-momentum to each internal line
2. Assign a factor $(-ig)\delta^4(\sum_i \mathbf{k}_i)$ to each vertex, where $\sum_i \mathbf{k}_i$ is the sum of 4-momenta flowing *into* the vertex.
3. For each internal line with 4-momentum $\mathbf{k}$, write a factor $\int đ^4k\, D(k^2)$, where the propagator is
$$D(k^2) = \begin{cases} \frac{i}{k^2-m^2+i\varepsilon} & \text{for } \phi \\ \frac{i}{k^2-\mu^2+i\varepsilon} & \text{for } \psi \end{cases}. \tag{4.74}$$

Example ($\psi\psi$ scattering revisisted)**:** We use the Feynman diagrams depicted in Equation (4.73). Applying the Feynman rules, these give

$$= (-ig)^2 \int đ^4k\, \frac{i}{k^2-m^2 \pm i\varepsilon}\delta^4(p_1-p_1'-k)\delta^4(p_2+k-p_2') + \delta^4(p_1-p_2'-k)\delta^4(p_2+k-p_1') \tag{4.75}$$

$$= i(-ig)^2 \left\{\frac{1}{(p_1-p_1')^2-m^2+i\varepsilon} + \frac{1}{(p_1-p_2')^2-m^2+i\varepsilon}\right\}\delta^4(p_1+p_2-p_1'-p_2'). \tag{4.76}$$

This is the same as our previous result of Equation (4.72).

> **Remark:** The meson does not necessarily satisfy $k^2 - m^2$—if it does not, it is called *off-shell* or a *virtual* particle. This is a quantum effect. The second diagram enforces *Bose statistics* for the identical final state particles.

Definition 9: We will define the *matrix element* $\mathcal{M}$ by

$$\langle f|\,(S-1)\,|i\rangle = i\mathcal{M} + \delta^4\Big(\sum_{\text{final state particles } i} p_i - \sum_{\text{initial state pts } i} p_i\Big), \tag{4.77}$$

where the factor of i is a convention which was designed to match non-relativistic quantum mechanics. The δ^4-function follows from translation invariance and is common to all S-matrix elements.

We now refine the Feynman rules to compute $i\mathcal{M}$:

1. Draw all possible diagrams with appropriate external lefts and impose 4-momentum conservation at each vertex.
2. Write a factor $(-ig)$ at each vertex.
3. For each internal line, factor in the propagator
4. Integrate over momentum k flowing in any closed loop as $\int đ^4k$.

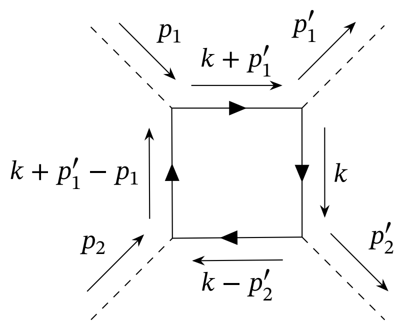


Figure 4.5

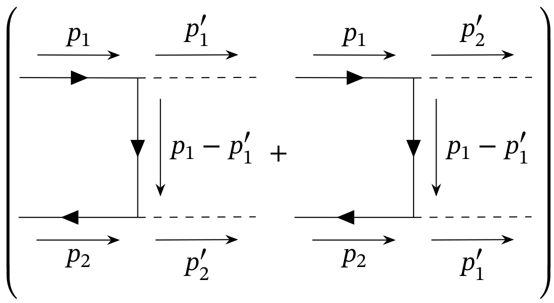
Example ($\phi\phi \to \phi\phi$)**:** The diagram in Figure 4.5 gives

$$i\mathcal{M} = \int \frac{(-ig)^4\, đ^4k\, i^4}{(k^2-\mu^2+i\varepsilon)((k+p_2')^2-\mu^2+i\varepsilon)((k+p_1'-p_1)^2-\mu^2+i\varepsilon)((k+p_1')^2-\mu^2+i\varepsilon)}. \quad (4.78)$$

Amazingly, we can do these integrals! All the one-loop diagrams in four dimensions have been done, however, the two-loop diagrams are still subject to active research.

Example ($\psi\bar{\psi} \to \phi\phi$)**:** The Feynman diagrams that describe this interaction are

$i\mathcal{M} =$

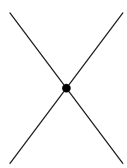


$$= (-ig)^2 \left(\frac{i}{(p_1 - p_1')^2 - \mu^2 + i\epsilon} + \frac{i}{(p_1 - p_2')^2 - \mu^2 + i\epsilon} \right) \quad (4.79)$$

where we evaluated these diagrams by using the Feynman rules.

4.6.1 Feynman Rules for ϕ^4 theory

We now introduce the ϕ^4 theory with the interaction Hamiltonian $\frac{1}{4!}\lambda\phi^4$. This is now drawn as a four-point vertex:



$$\sim -i\lambda \quad (4.80)$$

> **Remark:** Note that there is no factor of 4!. We will see why this cancels in the upcoming discussion of symmetry factors.

Example ($\phi\phi \to \phi\phi$)**:**

$$i\mathcal{M} \sim \frac{-i\lambda}{4!} \left\langle p_1', p_2' \right| : \phi(x)\phi(x)\phi(x)\phi(x) : \left| p_1, p_2 \right\rangle \quad (4.81)$$

(where we neglected $\int \dots$).

4.6.2 Symmetry Factors

There can be other combinatoric factors, also called *symmetry factors*. These are often 2 or 4. Consider $\langle 0| T\{\phi_1 \dots \phi_m S\} |0\rangle$, where $\phi_i := \phi(x_i)$. This is an example of a *correlation function*. The n^{th} term in the perturbation expansion for S gives us

$$\frac{1}{n!} \left(\frac{-i\lambda}{4!} \right)^n \int \mathrm{d}^4 y_1 \dots \mathrm{d}^4 y_n \, \langle 0| T \left\{ \phi_1 \dots \phi_m \phi^4(y_1) \dots \phi^4(y_n) \right\} |0\rangle \quad (4.82)$$

For example, the $n = 1$ and $m = 4$ term is

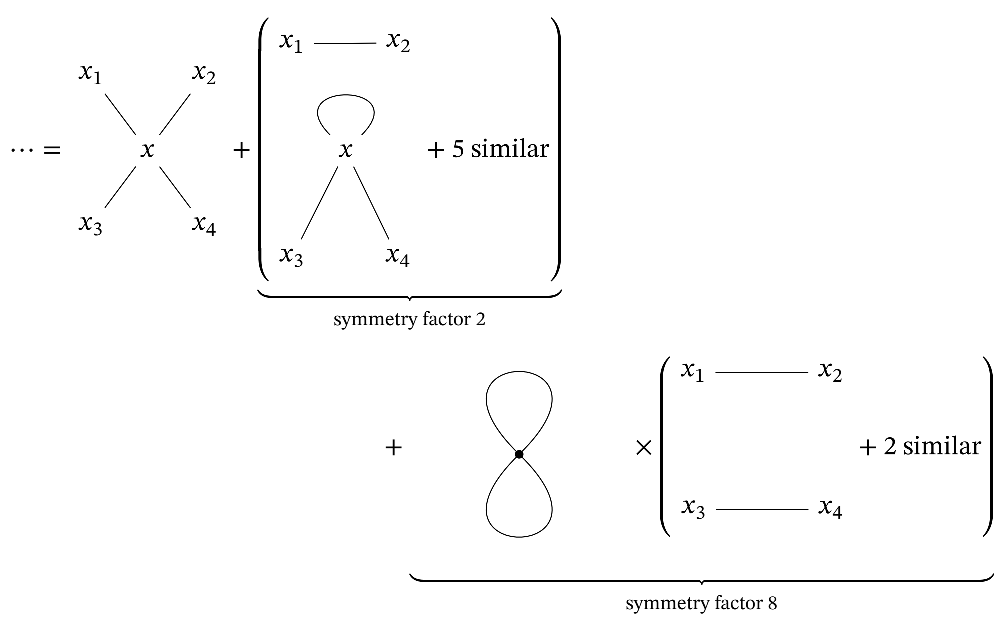
$$\frac{-i\lambda}{4!}\int \mathrm{d}^4x \, \langle 0| \, T\left\{\phi_1\phi_2\phi_3\phi_4\phi_x^4\right\}|0\rangle \tag{4.83}$$

From Wick's theorem, we get many terms:

$$\begin{aligned}
\ldots &= \frac{-i\lambda}{4!}\int \mathrm{d}^4x \, \phi_1\phi_2\phi_3\phi_4\phi_x\phi_x\phi_x\phi_x + (\text{perms of contractions between } \phi_i \text{ and } \phi_x) \\
&+ \frac{(-i\lambda)}{4!}\int \mathrm{d}^4x \, \phi_1\phi_2\phi_3\phi_4\phi_x\phi_x\phi_x\phi_x + (\text{perms where we only contract two } \phi_i\text{'s}) \\
&+ \frac{(-i\lambda)}{4!}\int \mathrm{d}^4x \, \phi_1\phi_2\phi_3\phi_4\phi_x\phi_x\phi_x\phi_x + (\text{perms where we contract all pairs } \phi_x)
\end{aligned} \tag{4.84}$$

$$\begin{aligned}
&= -i\lambda\int \mathrm{d}^4x \, \Delta_F(x_1 - x)\Delta_F(x_2 - x)\Delta_F(x_3 - x)\Delta_F(x_4 - x) \\
&+ \underbrace{\frac{(-i\lambda)}{2}}\int \mathrm{d}^4x \, \Delta_F(x_1 - x_2)\Delta_F(x_3 - x)\Delta_F(x_4 - x)\Delta_F(x - x) + (\text{similar}) \\
&\quad \text{factor 2 since there are 12 ways of pairing } \phi_3, \phi_4 \text{ with } \phi_x \\
&+ \underbrace{\frac{(-i\lambda)}{8}}\int \mathrm{d}^4x \, \Delta_F(x_1 - x_1)\Delta_F(x_3 - x_4)\Delta_F(x - x)\Delta_F(x - x) + (\text{perms}) \\
&\quad \text{factor 8 due to 3 ways of pairing } \phi_x\text{'s.}
\end{aligned} \tag{4.85}$$

Diagrammatically, the result (4.85) can be written



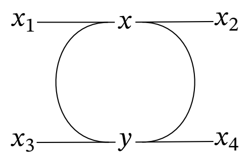
(4.86)

As a second example, the case $n = 2$ and $m = 4$ gives

$$\frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2 \int \langle 0|\, T\left\{\phi_1\phi_2\phi_3\phi_4\phi_x^4\phi_y^4\right\}|0\rangle\, \mathrm{d}^4x\, \mathrm{d}^4y \tag{4.87}$$

One of the terms that contributes is

$$\phi_1\phi_2\phi_3\phi_4\phi_x\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y\phi_y = \tag{4.88}$$



This gives

$$\frac{(-i\lambda)^2}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, \Delta_F(x_1 - x)\Delta_F(x_2 - x)\Delta_F(x_3 - y)\Delta_F(x_4 - y)\Delta_F(x - y)^2 \tag{4.89}$$

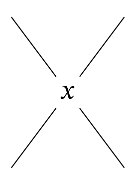
The symmetry factor is

$$\frac{1}{2} = \frac{1}{2!(4!)^2} \times \underbrace{4\times 3}_{x_1,x_2,x} \times \underbrace{12}_{x_3,x_4,y} \times \underbrace{2}_{x,y} \times \underbrace{2}_{x\leftrightarrow y} \tag{4.90}$$

are distinct diagrams, which also contribute.

The Feynman diagram rules are thus:

- $\langle 0|\, T\left\{\phi_1 \ldots \phi_m \exp\left(\frac{-i\lambda}{4!}\int \mathrm{d}^4x\, \phi_x^4\right)\right\}|0\rangle$ is the sum of all diagrams with m external points and any number of vertices connected by propagator lines.
- For each diagram, there exists and integral containing
  - propagator
    $$x \text{ ——— } z \ = \Delta_F(y - z) \tag{4.91}$$
  - vertex



$$= -i\lambda \int \mathrm{d}^4x \tag{4.92}$$

- Divide by symmetry factor

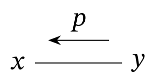
Since

$$\Delta_F(x - y) = \int \frac{đ^4p\, i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}, \tag{4.93}$$

we have

4.6.3 ϕ^4 Feynman Rules with Momenta

- propagator



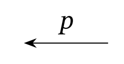
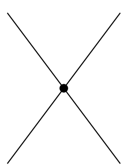
(4.94)

assign $e^{ip\cdot y}$ to y vertex (arrows out) and $e^{-ip\cdot x}$ to x vertex (arrows in) and $\frac{i}{p^2-m^2+i\epsilon}$ to the line, with $\int \text{đ}^4 p$.

The integral at vertex x becomes

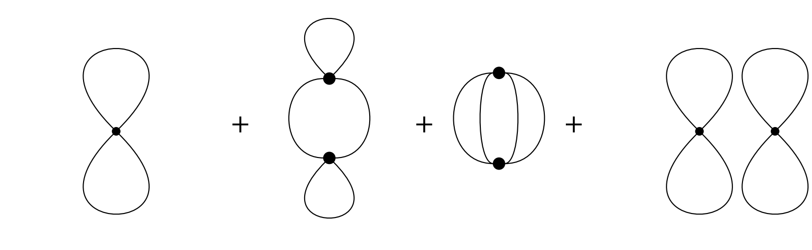
$$\int \mathrm{d}^4x\, e^{-i(p_1+p_3-p_4-p_2)\cdot x} = \not{\delta}^4(p_1+p_4-p_2-p_4) \tag{4.95}$$

Summary

1. propagator:  $= \frac{i}{p^2-m^2+i\epsilon}$
2. vertex:  $= -i\lambda$
3. Impose 4-momentum conservation at each vertex
4. Integrate over all undetermined momenta $\int \text{đ}^4 k$
5. Divide by symmetry factor

4.6.4 Vacuum Bubbles and Connected Diagrams

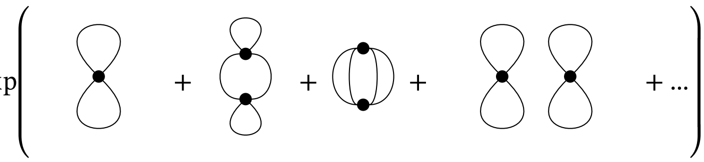
The diagrammatic expansion of $\langle 0| S |0\rangle$ in ϕ^4 theory is

1 +  + ... (4.96)

1st order

These are called *vacuum bubbles*. The combinatorial factors are such that

$$\langle 0|\, S\, |0\rangle = \exp$$



(4.97)

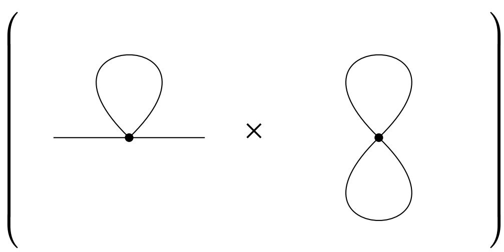
$$= \exp(\text{sum over all distinct vacuum bubble types}) \tag{4.98}$$

Exercise 4.1: This is done in the second example sheet.

Remember that our correlation function $\langle 0|\, T\{\phi_1 \dots \phi_m S\}\, |0\rangle$, also called the *m-point function* is the sum over all the diagrams with m external points. A typical diagram has some vacuum bubbles in it. Remarkably, these vacuum bubbles add to the *same* exponential series as above. In other words

$$\langle 0|\, T\{\phi_1 \dots \phi_m S\}\, |0\rangle = \sum(\text{connected diagrams}) \times \langle 0|\, S\, |0\rangle\,. \tag{4.99}$$

Example: Let us look at the second order term in a two point function. In general, there will be a term

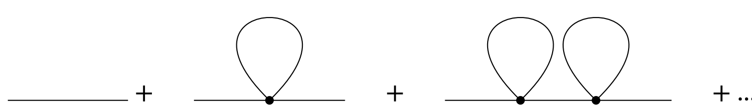


(4.100)

Only the first diagram is connected

Definition 10: A diagrams is said to be *connected* if every part of the diagram is connected to at least 1 external point.

Example: Connected diagrams for the two-point function are



(4.101)

4.7 Green's Functions and the Vacuum

Up until now, we have been describing amplitudes in terms of operators sandwiched between the *free* vacuum states: $\langle 0| \ldots |0\rangle$. What if we have interactions? We will see that we have been doing so far is actually correct, but we should show that explicitly. Let $|\Omega\rangle$ be the vacuum of the *interacting* theory. In analogy to $H_0 |0\rangle = 0$ for the free Hamiltonian, we normalise the Hamiltonian with interactions as $H = H_0 + H_{\text{int}}$ such that $H|\Omega\rangle = 0$ and $\langle\Omega|\Omega\rangle = 1$. The n-point Green function is then defined to be

$$G^{(n)}(x_1, \ldots, x_n) := \langle\Omega| T\{\phi_H(x_1) \ldots \phi_H(x_n)\} |\Omega\rangle . \tag{4.102}$$

We will write $\phi_H(x_i) := \phi_{iH}$ for the fields in the interacting picture.

Claim 5: Removing the vacuum bubbles takes us back to the free vacuum:

$$\langle\Omega| T\{\phi_{1H} \ldots \phi_{MH}\} |\Omega\rangle = \tag{4.103}$$

$$\frac{\langle 0| T\{\phi_{1I} \ldots \phi_{mI} S\} |0\rangle}{\langle 0| S |0\rangle} = \sum(\text{connected diagrams with } m \text{ external points}). \tag{4.104}$$

> **Remark:** This explains why the computations that we have been doing make sense at all.

Proof. Without loss of generality, we may order the coordinates in time as $x_1^0 > x_2^0 > \cdots > x_m^0$. Remember, $U(t, t_0) = e^{iH_0 t} e^{iH(t-t_0)} e^{-iH_0 t_0}$. Expanding out the S operator, the left hand side of the claim is equal to

$$\langle 0| T\{\phi_{1I} \ldots \phi_{mI} S\} |0\rangle = \langle 0| U(\infty, t_1)\phi_{1I} U(t_1, t_2)\phi_{2I} \ldots U(t_{m-1}, t_m)\phi_{mI} U(t_m, -\infty) |0\rangle \tag{4.105}$$

$$\begin{aligned} &= \langle 0| U(\infty, t_1) U(t_1, 0) \underbrace{U(0, t_1)\phi_{1I} U(t_1, 0)}_{\phi_{1H}} U(0, t_2)\phi_{2I} \ldots \\ &\ldots \underbrace{U(0, t_m)\phi_{mI} U(t_m 0)}_{\phi_{mH}} U(0, -\infty) |0\rangle \end{aligned} \tag{4.106}$$

This insertion in the last line allows us to convert back into the Heisenberg picture.

$$\begin{aligned} \ldots &= \underbrace{U(\infty, 0)\phi_{1H} \ldots \phi_{mH}}_{:=\langle\psi|} U(0, -\infty) |0\rangle & (4.107) \\ &= \lim_{t_0 \to -\infty} \{\langle\psi| U(0, t_0) |0\rangle\} & (4.108) \\ &= \lim_{t_0 \to -\infty} \left\{\langle\psi| e^{iHt_0} |0\rangle\right\}, & (4.109) \end{aligned}$$

where we used that $H_0 |0\rangle = 0$. We now insert a complete set of interacting states to get

$$\ldots = \lim_{t_0 \to -\infty} \left\{ \langle \psi | e^{-iHt_0} \left[|\Omega\rangle \langle \Omega| + \sum_{n=1}^{\infty} \left[\prod_{j=1}^{n} \frac{đ^3 p_j}{\sqrt{2E_{p_j}}} \right] |p_1, \ldots, p_n\rangle \langle p_1, \ldots, p_n| \right] |0\rangle \right\} \tag{4.110}$$

$$= \langle\langle|\Omega\rangle + \lim_{t_0 \to -\infty} \left\{ \sum_n \int \left(\prod_j \frac{đ^3 p_j}{\sqrt{2E_{p_j}}} \right) e^{-i \sum_{j=1}^{n} E_{p_j} t_0} \langle \psi | p_1, \ldots, p_n\rangle \langle p_1, \ldots, p_n | 0\rangle \right\} \tag{4.111}$$

The second term vanishes because of the Riemann-Lebesgue lemma:

$$\lim_{\mu \to \infty} \left\{ \int_a^b \mathrm{d}x\, f(x) e^{i\mu x} \right\} = 0 \tag{4.112}$$

Therefore, we find

$$\cdots = \langle 0| U(\infty, 0) \underbrace{\phi_{1H} \ldots \phi_{mH} |\Omega\rangle \langle \Omega | 0\rangle}_{|\psi\rangle} . \tag{4.113}$$

Similar to above we have

$$\langle 0| U(\infty, 0) |\psi\rangle = \lim_{t_0 \to \infty} \left\{ \langle 0| e^{iHt_0} |\psi\rangle \right\} \tag{4.114}$$

$$= \langle \Omega | \phi_{1H} \ldots \phi_{mH} |\Omega\rangle \langle \Omega | 0\rangle \langle 0 | \Omega\rangle . \tag{4.115}$$

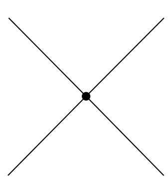
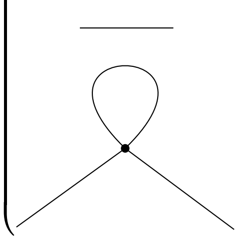
The denominator in the right hand side of the claim is $(m = 0)$

$$\langle 0| S |0\rangle = \langle 0|\Omega\rangle \langle \Omega | 0\rangle , \tag{4.116}$$

which completes the proof. $\square$

> **Remark:** Green's functions are ultimately the simplest way to think about scattering.

Example: Let us consider the previous example, we have

$\langle \Omega | T \{ \phi_{1H} \ldots \phi_{4H} |\Omega\rangle \} =$  $+ \Big($  $+ 5$ similar $\Big) + /$ (4.117)

LSZ Reduction

To describe scattering in the interacting theory, the external states (eg $|p_1, p_2\rangle$) should be those of the interacting theory. This means that when we have a Feynman diagram with an interaction on an external leg, then that loop gets absorbed into the definition of an external interacting state. In other words, we should exclude loops on external legs.

> **Remark:** A more detailed explanation of these *amputated diagrams* is described further in the *Advanced Quantum Field Theory* course in Lent term.

4.8 Scattering

4.8.1 Kinematics

Mandelstram Variables

We would like to describe the kinematics in terms of Lorentz invariant theories. This means we do not have to worry about switching frames. Consider $2 \to 2$ scattering depicted in Figure 4.6. By

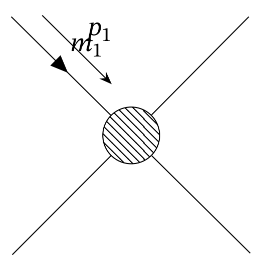
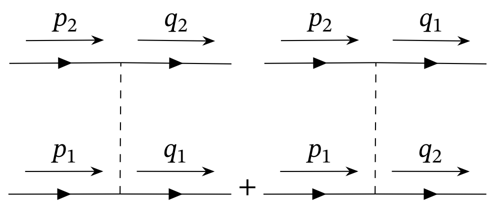


Figure 4.6: Not a Feynman Diagram

momentum conservation, we must have $p_1^\mu + p_2^\mu = q_1^\mu + q_2^\mu$. The Mandelstram variables are then defined as

$$s := (p_1 + p_2)^2 \qquad t := (p_1 - q_1)^2 \qquad u := (p_1 - q_2)^2. \tag{4.118}$$

Exercise 4.2: Let $s + t + u = m_1^2 + m_2^2 + m_1'^2 + m_2'^2$. Nucleon scattering is then given as



$$(4.119)$$

$$i\mathcal{M} = (-ig)^2 \left\{ \frac{1}{(p_1 - q_1)^2 - m^2} + \frac{1}{(p_1 - q_2)^2 - m^2} \right\} \tag{4.120}$$

$$= (-ig)^2 \left\{ \frac{1}{t - m^2} + \frac{1}{u - m^2} \right\} \tag{4.121}$$

$$\tag{4.122}$$

Decay Rates and Cross-Sections

Probability of scattering is expressed in terms of

$$\langle f | (S - 1) | i \rangle = i\mathcal{M}\delta^4(p_i - \sum_i q_i) \tag{4.123}$$

Now $|f\rangle$ and $|i\rangle$ are states of definite 4-momentum. We then get a normalised probability as

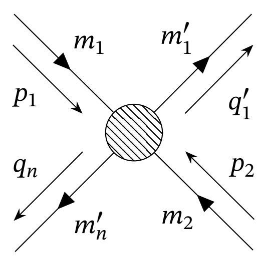


Figure 4.7

$$P = \frac{|\langle f | (S - 1) | i \rangle|^2}{\langle f | f \rangle \langle i | i \rangle}. \tag{4.124}$$

Defining $d\sigma$ as the effective cross-sectional area for scattering into $|f\rangle$ as depicted in FThis can be shown (eg Peskin and Schröder) to be

$$d\sigma = \frac{\delta^4(p_1 + p_2 - \sum_i q_i)|\mathcal{M}|^2}{\mathcal{F}} \tag{4.125}$$

where the *flow factor* $\mathcal{F}$ is

$$\mathcal{F} = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}. \tag{4.126}$$

To find the total $i \to f$ cross-section, we integrate over the final state momentum in the usual Lorentz invariant manner:

$$\sigma = \int \frac{1}{\mathcal{F}} \, \mathrm{d}p_f \, |\mathcal{M}|^2, \tag{4.127}$$

where

$$\int \mathrm{d}p_f = \int \left(\prod_{r=1}^{n} \frac{\text{đ}^3 q_r}{\sqrt{2E_{q_r}}} \right) \bar{\delta}^4 \Big(\sum_i p_i - q_i\Big). \tag{4.128}$$

4.9 2 → 2 Scattering

We define the Mandelstram variable

$$t := (p_1 - q_1)^2 = m_1^2 + m_2'^2 - 2E_{p_1}E_{q_1} + 2\mathbf{p}_1 \cdot \mathbf{q}_1 \tag{4.129}$$

We will take the derivative

$$\frac{\mathrm{d}t}{\mathrm{d}\cos\theta} = 2|\mathbf{p}_1||\mathbf{q}_1|, \tag{4.130}$$

where $\cos\theta$ is the angle between $\mathbf{p}_1$ and $\mathbf{q}_1$. This depends on the frame in which it is defined and is not a Lorentz invariant quantity. Often, we define our expressions in the centre-of-mass frame.

Definition 11: The *centre of mass frame* is the frame in which the total momentum is zero.

We will want to know how the cross section varies with angle.

Let us for a second talk about another Mandelstram variable. In the centre of mass frame, $s = (p_1 + p_2)^2$ is the centre of mass energy—a constant of the scattering. At the LHC, people talk about $\sqrt{s}$ to describe the energy scale of the scattering. This is the quantity that was cited as $13TeV$ in the previous run.

We can write

$$\frac{\mathrm{d}^3 q_2}{2E_{q_2}} = \mathrm{d}^4 q_2\, \delta(q_2^2 - m_2'^2)\theta(q_2^0), \tag{4.131}$$

where we defined the Heaviside-θ-function as

$$\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}. \tag{4.132}$$

In spherical polar coordinates, we can write

$$\frac{\mathrm{d}^3 q_1}{2E_{q_1}} = \frac{|\mathbf{q}_1|^2\, \mathrm{d}|q_2|\, \mathrm{d}(\cos\theta)\, \mathrm{d}\phi}{2E_{q_1}} = \frac{1}{4|\mathbf{p}_1|}\, \mathrm{d}E_{q_1}\, \mathrm{d}\phi\, \mathrm{d}t\,. \tag{4.133}$$

We then get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{8\pi\mathcal{F}|\mathbf{p}_1|}\int \mathrm{d}E_{q_1}\, |\mathcal{M}|^2 \delta(s - m_2' + m_1'^2 - 2q_1 \cdot (p_1 + p_2)). \tag{4.134}$$

We now boost with a Lorentz transformation to the centre of mass frame. Writing the components of the four momentum as

$$p_1^\mu = (\sqrt{|\mathbf{p}_1|^2 + m_1^2}, \mathbf{p}_1), \tag{4.135}$$

we know that, since the momenta must add up to zero, we must have

$$p_2^\mu = (\sqrt{|\mathbf{p}_1|^2 + m_1^2}, -\mathbf{p}_1), \tag{4.136}$$

Since s is obtained from the sum of these momenta, we find that

$$s = \left(\sqrt{|\mathbf{p}_1|^2 + m_1^2} + \sqrt{|\mathbf{p}_1|^2 + m_2^2}\right)^2 \tag{4.137}$$

Expanding this out, one can show that

$$|\mathbf{p}_1| = \frac{\lambda^{1/2}(s_1 m_1^2 m_2^2)}{2\sqrt{s}}, \tag{4.138}$$

where $\lambda(x, y, z) := x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ is the *Källen function*. Moreover, one can check that the flux is $\mathcal{F} = 2\sqrt{\lambda(s, m_1^2, m_2^2)}$.

Substituting all of this back into (4.134), we get

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{|\mathcal{M}|^2}{6\pi\lambda(s, m_1^2, m_2^2)}}. \tag{4.139}$$

For each different collision, we therefore know the probability of scattering.

4.10 Decay Rates

A decay is an interaction in which we start from one particle and end up with multiple particles.

Definition 12: The particle decay rate for a one-particle initial state $|i\rangle$ to an n-particle final state $|f\rangle$ is called the *partial width*.

$$\Gamma_f = \frac{1}{2E_{p_i}} \int \mathrm{d}p_f \, |\mathcal{M}|^2 \tag{4.140}$$

Note that $2E_{p_i}$ is not Lorentz invariant! This is expected since time-dilation tells us that the average decay rate depends on the frame. The convention is to quote Γ_f in the rest frame of the initial particle, where the energy $E_{p_i} = m_i$ is the mass of the initial state.

Definition 13: The *total width* Γ is the sum over all partial widths:

$$\Gamma = \sum_f \Gamma_f. \tag{4.141}$$

Definition 14: The *branching ratio* for $|i\rangle \to |f\rangle$, is

$$\mathrm{BR}(i \to f) = \frac{\Gamma_f}{\Gamma}. \tag{4.142}$$

Putting this into units, meaning that we reintroduce the relevant powers of $\hbar$ and c, we notice that the average lifetime τ is

$$\tau = 6.6 \times 10^{-25} \left(\frac{1 \text{ GeV}}{\Gamma}\right) \text{ seconds} \tag{4.143}$$

Exercise 4.3: Show that the mass of the decay results has to be lower than or equal to the mass of the initial particle.

Exercise 4.4: Complete the second example sheet.

5 The Dirac Equation

Consider a Lorentz transformation $x^\mu \xrightarrow{LT} x'^\mu = \Lambda^\mu{}_\nu x^\nu$. Under this transformation, scalars transform as $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$. Most particles (but not ϕ) have some intrinsic angular momentum—*spin*.

Example: Consider the spin-1 vector $A^\mu(x)$. Under a Lorentz transformation, this transforms as $A_\mu(X) \to A'_\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$.

In general, we have $\phi^a(x) \to D^a{}_b(\Lambda)\phi^b(\Lambda^{-1}x)$, where the $D^a{}_b(\Lambda)$ form a *representation* of the Lorentz group. This means that

- $D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2)$
- $D(\Lambda^{-1}) = D^{-1}(\Lambda)$
- and $D(I) = \mathbb{1}$.

To find representations, we look at the Lorentz algebra, considering infinitesimal transformations. Write $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon\omega^\mu{}_\nu + O(\epsilon^2)$, where ϵ is an infinitesimal parameter. The defining relation of the Lorentz transformations Λ is

$$\Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho \eta^{\sigma\rho} = \eta^{\mu\nu}. \tag{5.1}$$

Therefore, we have to first order

$$(\delta^\mu_\sigma + \epsilon\omega^\mu{}_\sigma)(\delta^\nu_\rho + \epsilon\omega^\nu{}_\rho)\eta^{\sigma\rho} = \eta^{\mu\nu} + O(\epsilon^2) \tag{5.2}$$

$$\Rightarrow\ \omega^{\mu\nu} + \omega^{\nu\mu} = 0. \tag{5.3}$$

Therefore, $\omega^{\mu\nu}$ is *anti-symmetric*. It has $4 \times 3/x$ components—three rotations and three boosts. We can introduce a basis of six 4×4 matrices for ω:

$$(M^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}. \tag{5.4}$$

Notice that this is anti-symmetric in $\mu \leftrightarrow \nu$ and $\rho \leftrightarrow \sigma$. If we lower the ν with the Minkowski metric, we get the following

$$(M^{\rho\sigma})^{\mu}{}_{\nu} = \eta^{\rho\mu}\delta^{\sigma}_{\nu} - \eta^{\sigma\mu}\delta^{\rho}_{\nu}. \tag{5.5}$$

Let us think of this in the following way: The first outer index μ labels the rows while ν labels the columns of each matrix. The inner indices $\rho\sigma$ tell us which matrix we are considering.

Example: We generate a boost in the x^1 direction with

$$(M^{01})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \tag{5.6}$$

Example: We generate rotations in the $x^1 - x^2$ plane with

$$(M^{12})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix} \tag{5.7}$$

We can expand ω in terms of this basis

$$\omega^{\mu}{}_{\nu} := \frac{1}{2}(\Omega_{\rho\sigma}M^{\rho\sigma})^{\mu}{}_{\nu}. \tag{5.8}$$

$M^{\rho\sigma}$ are called the *generators* of the Lorentz group and $\Omega^{\rho\sigma}$ are six independent parameters, which are anti-symmetric in $\rho \leftrightarrow \sigma$. The factor of $1/2$ is conventional.

5.0.1 The Algebra of the Lorentz group

...is defined by the algebra of the generators:

$$[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau}M^{\rho\nu} - \eta^{\rho\tau}M^{\sigma\nu} + \eta^{\rho\nu}M^{\sigma\tau} - \eta^{\sigma\nu}M^{\rho\tau}. \tag{5.9}$$

Note that this is anti-symmetric under $(\rho \leftrightarrow \sigma)$, $(\tau \leftrightarrow \nu)$ and also $(\rho\sigma \leftrightarrow \tau\nu)$. We write it concisely as

$$[M^{\rho\sigma}, M^{\tau\nu}] = [\eta^{\sigma\tau}M^{\rho\nu} - (\rho \leftrightarrow \sigma)] - (\tau \leftrightarrow \nu) \tag{5.10}$$

The finite Lorentz transformation Λ, a 4×4 matrix, is connected to the identity $\mathbb{1}$ as

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right). \tag{5.11}$$

5.1 the Spinor Representation

We found that the Lorentz algebra is

$$[M^{\rho\sigma}, M^{\tau\nu}] = [\eta^{\sigma\tau}M^{\rho\nu} - \gamma^{\rho\tau}M^{\sigma\nu} + \eta^{\rho\nu}M^{\sigma\tau} - \gamma^{\sigma\nu}M^{\rho\tau}]. \tag{5.12}$$

We will look for other matrices other than M that satisfy this algebra.

Let us start with the *Clifford algebra*.

Definition 15: The anti-commutator of two matrices $X, Y \in GL(n)$ is

$$\{X, Y\} = XY + YX. \tag{5.13}$$

Definition 16: The *Clifford algebra* is defined, in any number of dimensions, by the relation

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}}, \tag{5.14}$$

where γ^μ are a set of 4 matrices, $\mu \in \{0, 1, 2, 3\}$. We have $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\mu \neq \nu$ and $(\gamma^0)^2 = \mathbb{1}$ and $(\gamma^i)^2 = -\mathbb{1}$, for $i \in \{1, 2, 3\}$.

The simplest representation of the Clifford algebra is in terms of 4×4 matrices, for example the *chiral representation*:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{5.15}$$

where σ^i are the *Pauli matrices*:

$$\sigma^i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.16}$$

These satisfy $\{\sigma^i, \sigma^j\} = 2\delta^{ij}\mathbb{1}_2$ and $[\sigma^j, \sigma^k] = 2i\epsilon^{jkl}\sigma^l$, for $i, j, k, l \in \{1, 2, 3\}$. Note that these are not the only matrices which work as a representation. In fact, any similarity transformation $U\gamma^\mu U^{-1}$, with constant invertible matrix U, will give a valid representation.

Definition 17:

$$S^{\rho\sigma} := \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \begin{cases} 0, & \rho = \sigma \\ \frac{1}{2}\gamma^\rho\gamma^\sigma, & \rho \neq \sigma, \end{cases} \tag{5.17}$$

$$= \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma}\mathbb{1} \tag{5.18}$$

Claim 6:

$$[S^{\mu\nu}, \gamma^\rho] = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu} \tag{5.19}$$

Claim 7: S provides a representation of the Lorentz group, meaning that

$$[S^{\rho\sigma}, S^{\tau\nu}] = \eta^{\sigma\tau} S^{\rho\nu} - \eta^{\rho\tau} S^{\sigma\nu} + \eta^{\rho\nu} S^{\sigma\tau} - \eta^{\sigma\nu} S^{\rho\tau}. \tag{5.20}$$

Proof. We use the previous claim and (5.18). □

Definition 18: The *Dirac spinor* $\psi_\alpha(x)$, $\alpha \in \{1,2,3,4\}$ is defined by

$$\psi_\alpha(x) \xrightarrow{L.T.} S[\Lambda]^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x), \tag{5.21}$$

where $\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$ and $S[\Lambda] = \exp\left(\frac{1}{2}\Omega^{\rho\sigma}S^{\rho\sigma}\right)$.

Claim 8: The spinor representation is *not* equivalent to the usual vector representation.

Proof. We can see this by looking at a specific LT, such as *rotations*:

$$S^{ij} = \frac{1}{4}\left[\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\right] = -\frac{i\epsilon^{ijk}}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \tag{5.22}$$

where we made use of the algebra of σ^i. Interpreting ϕ physically as the rotation angle, we then write $\Omega_{ij} := -\epsilon_{ijk}\phi^k$, (eg. $\Omega_{12} = -\phi^3$). We then have

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) = \begin{pmatrix} e^{i\phi\cdot\sigma/2} & 0 \\ 0 & e^{-i\phi\cdot\sigma/2} \end{pmatrix}, \tag{5.23}$$

Remark: check sign

where we get an extra factor of two due to the two ϵ dotted together. Consider a rotation of 2π around the x^3-axis: $\phi = 0, 0, 2\pi$ gives

$$S[\Lambda] = \begin{pmatrix} e^{i\pi\sigma^3} & 0 \\ 0 & e^{-i\pi\sigma^3} \end{pmatrix} = -\mathbb{1}. \tag{5.24}$$

Therefore, a rotation of 2π takes $\psi_\alpha(x) \to -\psi_\alpha(x)$. This is different to a vector, which goes as

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\mu\nu}M^{\mu\nu}\right) = \exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2\pi & 0 \\ 0 & -2\pi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbb{1}_4, \tag{5.25}$$

as expected.

Let us now consider what happens to *boosts of spinors* Using (5.18), we get

$$S^{01} = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{5.26}$$

Writing the boost parameter as $\Omega_{0i} := \chi_i(= -\Omega_{i0})$, we have

$$S[\Lambda] = \begin{pmatrix} e^{-\chi\cdot\sigma/2} & 0 \\ 0 & e^{\chi\cdot\sigma/2} \end{pmatrix} \tag{5.27}$$

For rotations, $S[\Lambda]$ is unitary, since $S[\Lambda]^\dagger S[\Lambda] = \mathbb{1}$, but for boosts it is not! □

Claim 9: There are no finite-dimensional unitary representations of the Lorentz boosts.

Proof. The $S[\Lambda] = \exp\left[\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right]$ are only unitary if $S^{\sigma\rho}$ are anti-Hermitian, i.e. $(S^{\rho\sigma})^\dagger = -S^{\rho\sigma}$. $(S^{\rho\sigma})^\dagger = -\frac{1}{4}[(\gamma^\rho)^\dagger, (\gamma^\sigma)^\dagger]$ can be anti-Hermitian only if *all* γ^μ are Hermitian, or if all γ^μ are anti-Hermitian. This cannot be arranged: $(\gamma^0)^2 = \mathbb{1}$ implies that γ^0 has real eigenvalues. Therefore it cannot be anti-Hermitian, since these have imaginary eigenvalues. Similarly, $(\gamma^i)^2 = -\mathbb{1}$ implies that γ^i cannot be Hermitian. In general, there is no way to pick γ^μ such that $S^{\mu\nu}$ are anti-Hermitian. □

5.2 Constructing a Lorentz Invariant Action of ψ

$$\psi^\dagger(x) = (\psi^*)^T(x) \tag{5.28}$$

Is $\psi^\dagger(x)\psi(x)$ a Lorentz scalar? Under a LT, we have

$$\psi^\dagger(x)\psi(x) \xrightarrow{LT} \psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger S[\Lambda]\psi(\Lambda^{-1}x). \tag{5.29}$$

This is not a Lorentz scalar; this is where the non-unitarity of S matters!

> **Remark:** In the chiral representation, $(\gamma^0)^\dagger = (\gamma^0)$, and $\gamma^i = -(\gamma^i)^\dagger$. Using the Clifford algebra, and the fact that $(\gamma^0)^2 = \mathbb{1}$, we can encapsulate both of these facts as
>
> $$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0. \tag{5.30}$$

Using (5.30), we have

$$(S^{\mu\nu})^\dagger = -\frac{1}{4}[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger] = -\gamma^0 S^{\mu\nu}\gamma^0, \tag{5.31}$$

which we can use to show, by repeatedly using $(\gamma^0)^2 = \mathbb{1}$, that

$$S[\Lambda]^\dagger = \exp\left(\frac{1}{2}\Omega_{\mu\nu}(S^{\mu\nu})^\dagger\right) = \gamma^0 S[\Lambda]^{-1}\gamma^0. \tag{5.32}$$

The fact that these γ^0's pop up points to the existence of another adjoint. With this in mind, we define the following.

Definition 19: The *Dirac adjoint* of ψ is $\bar{\psi}(x) := \psi^\dagger(x)\psi^0$.

Claim 10: $\bar{\psi}\psi$ is a scalar.

Proof. Under Lorentz transformations, this goes to

$$\begin{aligned}\bar{\psi}(x)\psi(x) = \psi^\dagger(x)\gamma^0\psi(x) \to &\psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger\gamma^0 S[\Lambda]\psi(\Lambda^{-1}x) &(5.33)\\ &= \psi^\dagger(\Lambda^{-1}x)\gamma^0\psi(\Lambda^{-1}x) &(5.34)\\ &= \bar{\psi}(\Lambda^{-1}x)\psi(\Lambda^{-1}x). &(5.35)\end{aligned}$$

□

Claim 11: $\bar{\psi}(x)\gamma^\mu\psi(x)$ is a Lorentz vector.

Proof. Let us suppress the argument. Under Lorentz transformations, we have

$$\bar{\psi}\gamma^\mu\psi \to \bar{\psi}S[\Lambda]^{-1}\gamma^\mu S[\Lambda]\psi. \tag{5.36}$$

If $\bar{\psi}\gamma^\mu\psi$ is to be a Lorentz transformation, we must have

$$S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu. \tag{5.37}$$

Recall that $\Lambda^\mu{}_\nu = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$ and $S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)$. Infinitesimally (taking the Ω terms to be small and throwing out higher order terms in Ω), we get

$$(M^{\rho\sigma})^\mu{}_\nu\gamma^\nu = -[S^{\rho\sigma}, \gamma^\mu]. \tag{5.38}$$

$$(\eta^{\rho\mu}\delta^\sigma_\nu - \gamma^{\sigma\mu}\delta^\rho_\nu)\gamma^\nu) = \eta^{\rho\mu}\gamma^\sigma - \gamma^\rho\eta^{\sigma\mu}. \tag{5.39}$$

But by claim 4.1 (TODO: correct numbering), the right hand side is $-[S^{\rho\sigma}, \gamma^\mu]$ □

We find that

$$\boxed{S = \int \mathrm{d}^4x\, \underbrace{\bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)}_{\mathcal{L}_D}} \tag{5.40}$$

is Lorentz invariant.

5.3 The Dirac Equation

Vary $\psi, \bar{\psi}$ independently to get the Euler Lagrange equations:

$$\bar{\psi}: \qquad (i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{5.41}$$

> **Remark:** First order ODE, unlike K-G.

Definition 20 (slash notation)**:** We write $A_\mu \gamma^\mu := \not{A}$.

In slash notation, we have $(i\not{\partial} - m)\psi = 0$.

Each component of ψ solves the Klein-Gordon equations:

$$(i\not{\partial} + m)(i\not{\partial} - m)\psi = 0 \tag{5.42}$$

$$\Rightarrow \; -(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi = 0 \tag{5.43}$$

$$-(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu + m^2)\psi = 0 \tag{5.44}$$

$$-(\partial^2 + m^2)\psi_\alpha = 0. \tag{5.45}$$

5.4 Chiral Spinors

Since $S[\Lambda]$ is block-diagonal, it is *reducible*. In other words, we may decompose it into two irreps, $\psi = \begin{pmatrix} u_L \\ u_R \end{pmatrix}$, where u_L, u_R are two $\mathbb{C}$-component objects: Weyl or Dirac spinors. They transform identically under rotations, but oppositely under boosts:

$$u_{L/R} \to e^{i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}/2} u_{L/R} \qquad u_{L/R} \to e^{\mp\chi\cdot\boldsymbol{\sigma}/2} u_{L/R} \tag{5.46}$$

$$(SU(2), SU(2)): \quad u_L \sim \left(\frac{1}{2}, 0\right),\ u_R \sim \left(0, \frac{1}{2}\right), \quad \psi \text{ is in } \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right). \tag{5.47}$$

5.5 The Weyl Equation

$$\mathcal{L}_D = \bar{\psi}(i\partial\!\!\!/ - m)\psi = iu_L^\dagger \sigma^\mu \partial_\mu u_L + iu_R^\dagger \overline{\sigma}^\mu \partial_\mu u_R - m(u_L^\dagger u_R - u_R^\dagger u - L), \tag{5.48}$$

where $\sigma^\mu := (\mathbb{1}_2, \boldsymbol{\sigma})$, $\overline{\sigma}^\mu := (\mathbb{1}_2, -\boldsymbol{\sigma})$ A massive fermion requires both u_L, u_R. A massless particle $m = 0$ requires only u_L or u_R.

$$m = 0: \qquad \left.\begin{aligned} i\sigma^\mu \partial_\mu u_L &= 0 \\ i\overline{\sigma}^\mu \partial_\mu u_R &= 0 \end{aligned}\right\} \text{Weyl's equations} \tag{5.49}$$

In classical particle mechanics, the number of degrees of freedom N is half the dimension of phase space (If it is a field theory, this is really the number of degrees of freedom at each spacetime point). For a scalar field ϕ, we have $\pi_\phi = \dot{\phi}$, so $N = \frac{1}{2} \times 2 = 1$. For a spinor ψ_α, we have $\pi_\psi = i\psi^\dagger$ (not $\dot{\psi}$), so the number of degrees of freedom is $N = \frac{1}{2} \times 8 = 4$ (spin-$\uparrow$, spin-$\downarrow$ particle $\times$ spin-$\uparrow$, spin-$\downarrow$ anti-particle).

5.5.1 γ^5-Matrix

For other bases, $S[\Lambda]$ is not necessarily block-diagonal. $\gamma^\mu \to U\gamma^\mu U^{-1}$, $\psi \to U\psi$. Let $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$ to define Weyl spinors in a basis-independent way.

$$\{\gamma^\mu, \gamma^5\} = 0, \qquad (\gamma^5)^2 = \mathbb{1}_4. \tag{5.50}$$

Define projection operators $P_L := \frac{1}{2}(\mathbb{1}) - \gamma^5$, and $P_R := \frac{1}{2}(\mathbb{1} + \gamma^5)$. These satisfy $P_{L/R}^2 = P_{L/R}$ and $P_L P_R = 0 = P_R P_L$. We then call $\psi_L := P_L\psi$ *left-handed* and $\psi_R := P_R\psi$ *right-handed*. Under

Lorentz transformation, this behaves like a *pseudo scalar*, which means that

$$\overline{\psi}(x)\gamma^5\psi(x) \xrightarrow{LT} \overline{\psi}(\Lambda^{-1}x)S[\Lambda]^{-1}\gamma^{-5}S[\Lambda]\psi \tag{5.51}$$

$$= \overline{\psi}(\Lambda^{-1}x)\gamma^5\psi(\Lambda^{-1}x). \tag{5.52}$$

Check: $[S_{\mu\nu},\gamma^5] = 0$, $\overline{\psi}\gamma^5\gamma^\mu\psi$. We call this an *axial vector*.

5.5.2 Parity

ψ_L and ψ_R are related by *parity* transformations. This is a symmetry similar to others we have already met:

$$\text{Lorentz group:} \qquad x^\mu \to \Lambda^\mu{}_\nu x^\nu \text{ s.t } \Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma \eta^{\nu\sigma} = \eta^{\mu\rho} \tag{5.53}$$

$$\text{Time reversal } T\text{:} \qquad x^0 \to -x^0, \quad x^i \to x^i \tag{5.54}$$

$$\text{Parity } P\text{:} \qquad x^0 \to x^0, \quad x^i \to -x^i. \tag{5.55}$$

Note that the last two, time reversal and parity, are disconnected from identity! Under P, rotations do not change sign, but boosts do. Using (5.46), this means that P exchanges $u_L \leftrightarrow u_R$. Recall that we defined $\psi_{L/R} := \frac{1}{2}(\mathbb{1} \mp \gamma^5)$. Hence, P exchanges left-handed and right-handed spinors:

$$P: \quad \psi_{L/R}(\mathbf{x},t) \to \psi_{R/L}(-\mathbf{x},t), \qquad \text{i.e. } P: \quad \psi(\mathbf{x},t) \to \gamma^0\psi(-\mathbf{x},t). \tag{5.56}$$

Terms that we might want to add to $\mathcal{L}$ include:

$$\overline{\psi}\psi(\mathbf{x},t) \xrightarrow{P} \overline{\psi}\psi(-\mathbf{x},t) \qquad \text{scalar} \tag{5.57}$$

$$\overline{\psi}\gamma^\mu\psi(\mathbf{x},t) \xrightarrow{P} \begin{cases} \mu = 0: & \overline{\psi}\gamma^0\psi(-\mathbf{x},t) \\ \mu = i: & \overline{\psi}\gamma^0\gamma^i\gamma^0\psi(-\mathbf{x},t) = -\overline{\psi}\gamma^i\psi(-\mathbf{x},t) \end{cases} \qquad \text{vector} \tag{5.58}$$

$$\overline{\psi}\gamma^5\psi(\mathbf{x},t) \xrightarrow{P} \overline{\psi}\gamma^0\gamma^5\gamma^0\psi(-\mathbf{x},t) = -\overline{\psi}\gamma^5\psi \qquad \text{pseudo scalar} \tag{5.59}$$

$$\overline{\psi}\gamma^5\gamma^\mu\psi \xrightarrow{P} \overline{\psi}\gamma^0\gamma^5\gamma^\mu\gamma^0\psi = \begin{cases} \mu = 0: & -\overline{\psi}\gamma^5\gamma^0\psi \\ \mu = i: & \overline{\psi}\gamma^5\gamma^i\psi \end{cases} \qquad \text{axial vector} \tag{5.60}$$

The spinor bilinears are summarised in 5.1, showing that the total number of bilinears is $1 + 4 + (4 \times 3/2) + 4 + 1 = 16$, and therefore the maximal number for a 4-component object.

We can now add these extra terms to $\mathcal{L}$ that involve γ^5. These terms can break parity invariance of our theories! One example of this is $\mathcal{L} = gW_\mu\overline{\psi}\gamma^\mu\frac{1-\gamma^5}{2}\psi$, which describes a W-boson vector field coupling only to left-handed ψ's. However, they do not always break parity; considet the term $\phi\overline{\psi}\gamma^5\psi$, which does not break partiy if ψ is a pseudo scalar.

Definition 21: If $\mathcal{L}$ treats ψ_L and ψ_R equally, it is called a *vector-like* theory. If they appear differently, it is called a *chiral theory*.

Spinor Bilinear	Transforms as...	#
$\overline{\psi}\psi$	scalar	1
$\overline{\psi}\gamma^\mu\psi$	vector	4
$\overline{\psi}S^{\mu\nu}\psi$	tensor	6
$\overline{\psi}\gamma^5\psi$	pseudo scalar	1
$\overline{\psi}\gamma^5\gamma^\mu\psi$	axial vector	4

Table 5.1: Spinor bilinears

5.6 Symmetries & currents of Spinors

Spacetime translations $x^\mu \to x^\mu - \epsilon^\mu$

The spinor transforms as $\delta\psi = \epsilon^\mu\partial^\mu$. Now the Lagrangian $\mathcal{L}_D$ (5.40) depends on ∂_μ but not on $\partial_\mu\overline{\psi}$. Using (2.49), we calculate the energy-momentum tensor to be

$$T^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathcal{L}. \tag{5.61}$$

The fact that the current is conserved implies that the equations of motion are obeyed, so we can impose them on $T^{\mu\nu}$. For scalar fields, the equations of motion are second order in derivatives, which does not teach us anything about the first order energy-momentum tensor. However, for spinor fields, the Lagrangian is $\mathcal{L}_D = \overline{\psi}(i\partial\!\!\!/ - m)\psi = 0$, which vanishes due to the equations of motion. We therefore have $T^{\mu\nu} = i\overline{\psi}\gamma^\mu\partial^\nu\psi$.

Lorentz Transformations $x \to \Lambda x$

The Dirac spinor transforms as $\psi^\alpha \to S[\Lambda]^\alpha{}_\beta \psi^\beta(x^\mu - \omega^\mu{}_\nu x^\nu)$, where $\omega^\mu{}_\nu = \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu$ and $(M^{\sigma\rho})^{\mu\nu}$ are the generators of the Lorentz algebra, given by (5.5). Substituting these into the above, we have $\omega^{\mu\nu} = \Omega^{\mu\nu}$, which means that under infinitesimal Lorentz transformation we have

$$\delta\psi^\alpha = -\omega^\mu{}_\nu x^\nu \partial_\mu \psi^\alpha + \frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta \psi^\beta \tag{5.62}$$

$$= -\omega^{\mu\nu}\left[x_\nu \partial_\mu \psi_\alpha - \frac{1}{2}(S_{\mu\nu})^\alpha{}_\beta \psi^\beta\right] \tag{5.63}$$

$$\delta\overline{\psi}_\alpha = -\omega^{\mu\nu}\left[x_\nu \partial_\mu \overline{\psi}_\alpha + \frac{1}{2}\overline{\psi}_\beta (S_{\mu\nu})^\beta{}_\alpha\right]. \tag{5.64}$$

Again, the spinor equations of motion set $\mathcal{L} = 0$ and the conserved current can be calculated to be

$$(\mathcal{J}^\mu)^{\rho\sigma} = \underbrace{x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}}_{\text{looks like } \mathbb{R} \text{ scalar}} - i\overline{\psi}\gamma^\mu S^{\rho\sigma}\psi. \tag{5.65}$$

After quantisation, the last term will give rise to the internal angular momentum of spin-$\frac{1}{2}$ single particle states.

$$(\mathcal{J}^0)^{ij} = -i\overline{\psi}\gamma^0 S^{ij}\psi = \frac{1}{2}\varepsilon^{ijk}\psi^\dagger \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}\psi \tag{5.66}$$

Internal Vector Symmetry The Dirac Lagrangian is symmetric under rotation of the spinor phase

$$\psi \to e^{i\alpha}\psi \quad \Rightarrow \quad \delta\psi = i\alpha\psi \tag{5.67}$$

By Noether's theorem, this implies the existence of the conserved current $j^\mu_V = \overline{\psi}\eta^\mu\psi$. Here, the index V for *vector* emphasises the fact that ψ_L and ψ_R transform the same way under this symmetry. The associated conserved charge is

$$Q = \int d^3x\, \overline{\psi}\gamma^0\psi = \int d^3x\, \psi^\dagger\psi. \tag{5.68}$$

As we will see soon, this will be interpreted as electric charge.

Axial Symmetry When $m = 0$, the Dirac Lagrangian is invariant under the axial symmetry transformation

$$\psi_\alpha \to (e^{i\alpha\gamma^5})_\alpha{}^\beta \psi_\beta, \tag{5.69}$$

which rotates left-handed and right-handed spinors in opposite directions. We have a conserved axial vector current $j^\mu_A = \overline{\psi}\gamma^\mu\gamma^5\psi$.

> **Remark:** Once we couple this theory to gauge fields, we will see that this is the first example of an *anomaly*—a symmetry that is present in the classical theory, but ceases to hold once we perform quantisation.

5.6.1 Plane Wave Solutions

We want to find solutions to the Dirac equation $(i\partial\!\!\!/ - m)\psi = 0$. Consider the plane wave ansatz $\psi = u_{\mathbf{p}} e^{-ip\cdot x}$, where $u_{\mathbf{p}}$ is a constant spinor depending on the three-momentum $\mathbf{p}$. Substituting this into the Dirac equation (using the chiral representation of γ^μ), we have

$$(\gamma^\mu p_\mu - m)u_{\mathbf{p}} = \begin{pmatrix} -m & p^\mu \sigma^\mu \\ p_\mu \overline{\sigma}^\mu & -m \end{pmatrix} u_{\mathbf{p}} = 0. \tag{5.70}$$

We have again used the definition that $\sigma^\mu = (1, \sigma^i)$ and $\overline{\sigma}^\mu = (1, -\sigma^i)$.

Claim 12: The solution is $u_{\mathbf{p}} = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\overline{\sigma}}\xi \end{pmatrix}$ for any two-component spinor ξ, which is normalised such that $\xi^\dagger \xi = 1$.

Proof. Let us write $u_{\mathbf{p}} = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$, and sub into (5.70):

$$(p\cdot\sigma)u_2 = mu_2 \tag{5.71a}$$

$$(p\cdot\overline{\sigma})u_1 = mu_2. \tag{5.71b}$$

Either of these implies the other, since

$$(p\cdot\sigma)(p\cdot\overline{\sigma}) = p_0^2 - p_i p_j \sigma^i \sigma^j \tag{5.72}$$

$$= p_0^2 - p_i p_j \underbrace{\frac{1}{2}\{\sigma^i, \sigma^j\}}_{\delta^{ij}} \tag{5.73}$$

$$= p_\mu p^\mu = m^2. \tag{5.74}$$

Now we try the ansatz $u_1 = (p\cdot\sigma)\xi'$ for a two-component spinor ξ'. Sub into (5.71) to give $u_2 = \frac{1}{m}(p\cdot\overline{\sigma})(p\cdot\sigma)\chi' = m\xi'$. So any vector of form $u_{\mathbf{p}} = A\begin{pmatrix} (p\cdot\sigma)\xi' \\ m\xi' \end{pmatrix}$, where A is constant. To make this look more symmetric, choose $A = \frac{1}{m}$ and $\xi' := \sqrt{p\cdot\overline{\sigma}}$, with ξ constant. Then $u_{\mathbf{p}} = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\overline{\sigma}}\xi \end{pmatrix}$. $\square$

Example (p = 0): Have $u_{\mathbf{p}=0} = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$ for any ξ. Under spatial rotations, $\xi \to e^{i\boldsymbol{\sigma}\cdot\boldsymbol{\phi}/2}\xi$. After quantisation, ξ describes the *spin* of the spinor, e.g. $\xi = \begin{pmatrix}1\\0\end{pmatrix}$ is spin-↑ along x^3-axis: to see this, consider the particle boosted along the x^3-direction.

$$u_{\mathbf{p}} = \begin{pmatrix}\sqrt{E-p^3}\begin{pmatrix}1\\0\end{pmatrix}\\ \sqrt{E+p^3}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \xrightarrow[E\to p^3]{m\to 0} \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \tag{5.75}$$

For $\xi = \begin{pmatrix}0\\1\end{pmatrix}$,

$$u_{\mathbf{p}} = \begin{pmatrix}\sqrt{E+p^3}\begin{pmatrix}0\\1\end{pmatrix}\\ \sqrt{E-p^3}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} \xrightarrow{m\to 0} \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\sqrt{2E}. \tag{5.76}$$

Definition 22 (helicity)**:** The *helicity* operator h projects the angular momentum along the direction of motion.

$$h = \hat{\mathbf{p}}\cdot\mathbf{s} = \frac{1}{2}\hat{p}_k\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}. \tag{5.77}$$

A massless spin-↑ operator has $h = +\frac{1}{2}$, whereas a massless spin-↓ has $h = -\frac{1}{2}$.

5.6.2 Negative Frequency Solutions

$$\psi = v_{\mathbf{p}}e^{+ip\cdot x} \tag{5.78}$$

Dirac equation also has a negative frequency solution

$$v_{\mathbf{p}} = \begin{pmatrix}\sqrt{p\cdot\sigma}\eta\\ -\sqrt{p\cdot\bar{\sigma}}\eta\end{pmatrix}, \tag{5.79}$$

where $\eta^\dagger\eta = 1$ and η is a constant two-spinor.

Exercise 5.1: Go through the same derivation as above, but with the negative frequency solution.

5.6.3 Quantising the Dirac Field

In the Schrödinger picture, where the fields are not functions of time, we have

$$\psi(\mathbf{x}) = \sum_{s=1}^{2} \int \frac{\text{đ}^3 p}{\sqrt{2E_p}} \left[b_{\mathbf{p}}^s u_{\mathbf{p}}^s e^{+i\mathbf{p}\cdot\mathbf{x}} + (c_{\mathbf{p}}^s)^\dagger v_{\mathbf{p}}^s e^{-i\mathbf{p}\cdot\mathbf{x}} \right] \tag{5.80}$$

$$\psi^\dagger(\mathbf{x}) = \sum_{s=1}^{2} \int \frac{\text{đ}^3 p}{\sqrt{2E_p}} \left[(b_{\mathbf{p}}^s)^\dagger (u_{\mathbf{p}}^s)^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} + c_{\mathbf{p}}^s (v_{\mathbf{p}}^s)^\dagger e^{+i\mathbf{p}\cdot\mathbf{x}} \right] \tag{5.81}$$

$$\tag{5.82}$$

In the Heisenberg picture, specify these at an initial time and use the Dirac equation to determine the full evolution.

At this stage, we would usually specify commutation relations. However, as we will illustrate a bit later, something pathological would go wrong if we did that. It turns out that in spinor quantisation, we require *anti*-commutation relations, $\{A, B\} = AB + BA$, such as those we have already seen in the Clifford algebra. We have

$$\{\psi_\alpha(\mathbf{x}), \psi_\beta(\mathbf{y})\} = 0 = \left\{\psi_\alpha^\dagger(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\right\} \tag{5.83a}$$

$$\left\{\psi_\alpha(\mathbf{x}), \psi_\beta^\dagger(\mathbf{y})\right\} = \delta_{\alpha\beta}\delta^3(\mathbf{x} - \mathbf{y}). \tag{5.83b}$$

Claim 13: These are equivalent to

$$\left\{b_{\mathbf{p}}^r, (b_{\mathbf{q}}^s)^\dagger\right\} = \delta^3(\mathbf{p} - \mathbf{q})\delta^{rs} = \left\{c_{\mathbf{p}}^r, (c_{\mathbf{q}}^s)^\dagger\right\} \tag{5.84}$$

and all other anti-commutators vanish.

Hamiltonian

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} \tag{5.85}$$

$$= \cancel{i\psi^\dagger\dot{\psi}} - \cancel{i\overline{\psi}\gamma^0\partial_0\psi} - i\overline{\psi}\gamma^i\partial_i\psi + m\overline{\psi}\psi \tag{5.86}$$

$$= \overline{\psi}(-i\gamma^i\partial_i + m)\psi \tag{5.87}$$

Plug in $\psi, \overline{\psi}$ from above, use anti-commutation relations and some results on the inner product of spinors, namely:

$$(u_{\mathbf{p}}^r)^\dagger u_{\mathbf{p}}^s = (v_{\mathbf{p}}^r)^\dagger v_{\mathbf{p}}^s = 2p_0\delta^{rs} \tag{5.88a}$$

$$(u_{\mathbf{p}}^r)^\dagger v_{\mathbf{p}}^s = 0 = (v_{\mathbf{p}}^r)^\dagger u_{\mathbf{p}}^s \tag{5.88b}$$

to obtain

$$H = \int đ^3 p \, E_p \sum_{s=1}^{2} \left((b_{\mathbf{p}}^s)^\dagger b_{\mathbf{p}}^s + (c_{\mathbf{p}}^s)^\dagger c_{\mathbf{p}}^s \right). \tag{5.89}$$

This is where the difference between commutation and anti-commutation relations comes in; had we specified commutation relations instead of the anti-commutation relations (5.83), then we would have a minus sign in this Hamiltonian: $H = \ldots b_{\mathbf{p}}^s - (c_{\mathbf{p}}^s)^\dagger \ldots$. This would then mean that we can lower the energy indefinitely by creating anti-particles, thus making the theory unstable. This is a first glimpse of how the spin-statistics theorem follows from QFT: scalars and vectors obey commutation relations, whereas spinors obey anti-commutation relations.

Dirac Hole Interpretation

$$i\frac{\partial\psi}{\partial t} = (-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + m\beta)\psi \qquad \boldsymbol{\alpha} = -\gamma^0\boldsymbol{\gamma} \quad \beta = \gamma^0. \tag{5.90}$$

This was interpreted as a 1-particle Hamiltonian $\hat{H}$. This is a very different view to ours; we see ψ as a field that we need to quantise. If we interpret this as a 1-particle Hamiltonian, we get positive and negative energy solutions. And of course, the negative energy solutions create instabilities in the system. Dirac's solution was to postulate all negative energy states to be filled. The only thing we could then measure is charge difference. This lead to the prediction of the positron, although we now realise that the idea of the Dirac sea is flawed. In field theory, we understand that the Dirac equation applies to fields, and the particles, which arise from quantisation of this field, correspond to electrons and positrons.

5.7 Fermi-Dirac Statistics

We had operators b and c with spin indices s. These annihilate the vacuum

$$b^i_{\mathbf{p}}\,|0\rangle = 0 = c^s_{\mathbf{p}}\,|0\rangle\,. \tag{5.91}$$

These obey anti-commutation relations. However, from these, one can derive that the Hamiltonian H has the usual commutation relations

$$[H,(b^r_{\mathbf{p}})^\dagger] = E_p(b^r_{\mathbf{p}})^\dagger \qquad [H,b^r_{\mathbf{p}}] = -E_p b^r_{\mathbf{p}}, \tag{5.92}$$

Notation: We define $|\mathbf{p}_1, r_1\rangle := (b^{r^1}_{\mathbf{p}_1})\,|0\rangle$.

The anti-commutation relations give us, e.g.

$$|\mathbf{p}_1, r_1; \mathbf{p}_2, r_2\rangle = -\,|\mathbf{p}_2, r_2; \mathbf{p}_1, r_1\rangle\,. \tag{5.93}$$

Heisenberg Picture

The field is now a function of spacetime $\psi(x)$ satisfying $\frac{\partial\psi}{\partial t} = i[H,\psi]$. It is solved by

$$\psi_\alpha(x) = \sum_{s=1}^{2}\int \frac{đ^3 p}{\sqrt{2E_p}}\left(b^s_{\mathbf{p}}(u^s_{\mathbf{p}})\alpha e^{-ip\cdot x} + (c^s_{\mathbf{p}})^\dagger(v^s_{\mathbf{p}})_\alpha e^{+ip\cdot x}\right), \tag{5.94}$$

with α denoting the spinor index. There is also an analogous expression for $\psi^\dagger_\alpha$. For real scalars, we wrote down a propagator. In analogy with $\Delta(x-y) = [\phi(x),\phi(y)]$, we define for spinors the anti-commutator

$$iS_{\alpha\beta}(x-y) = \left\{\psi_\alpha(x), \overline{\psi}_\beta(y)\right\}. \tag{5.95}$$

S is a 4×4 matrix; however, we will often drop the indices α, β. Substitute in the field expansion in terms of creation and annihilation operators

$$\begin{aligned} iS(x-y) &= \sum_{r,s} \int \frac{đ^3 p}{\sqrt{2E_p}} \frac{đ^3 q}{\sqrt{2E_q}} \left[\left\{b^s_{\mathbf{p}}, (b^r_{\mathbf{q}})^\dagger\right\} e^{-i(p\cdot x - q\cdot y)} u^s_{\mathbf{p}} \overline{u}^r_{\mathbf{q}} + \left\{(c^s_{\mathbf{p}})^\dagger, c^r_{\mathbf{q}}\right\} v^s_{\mathbf{p}} \overline{v}^r_{\mathbf{q}} e^{+i(p\cdot x - q\cdot y)}\right] && (5.96) \\ &= \int \frac{đ^3 p}{2E_p} \left[\underbrace{\sum_s (u^s_{\mathbf{p}})_\alpha (\overline{u}^s_{\mathbf{p}})_\beta}_{(\not{p}+m)} e^{-ip\cdot(x-y)} + \underbrace{\sum_{s=1} (v^s_{\mathbf{p}})_\alpha (\overline{v}^s_{\mathbf{p}})_\beta}_{(\not{p}-m)} e^{+ip\cdot(x-y)} \right] && (5.97) \\ &= (i\not{\partial}_x + m) D(x-y) - (i\not{\partial}_x + m) D(y-x) && (5.98) \end{aligned}$$

The subscript x denotes what ∂_μ is with respect to.

> **Remark:** Recall, $D(x-y) = \int \frac{đ^3 p}{2E_p} e^{-ip\cdot(x-y)}$

$$\cdots = (i\not{\partial}_x + m\mathbb{1})[D(x-y) - D(y-x)] \tag{5.99}$$

Comments

- For $(x-y)^2 < 0$, $D(x-y) - D(y-x) = 0$. We now have $\left\{\psi_\alpha(x), \overline{\psi}_\beta(y)\right\} = 0$ for all $(x-y)^2 < 0$. So what about causality? Our observables are *bilinear* in fermions. $\therefore$ they do commute at spacelike separations—the theory is causal.

- Away from singularities,

$$i(i\not{\partial}_x - m) S(x-y) = 0 \tag{5.100}$$

Proof. By substituting in for iS from (5.99), we can write the left hand side as

$$\begin{aligned} \text{LHS} &= (i\not{\partial}_x - m)(i\not{\partial}_x + m)[D(x-y) - D(y-x)] && (5.101) \\ &= -(\partial_x^2 + m^2)[D(x-y) - D(y-x)] && (5.102) \\ &= 0 \quad \text{using } p^\mu p_\mu = m^2. && (5.103) \end{aligned}$$

$\square$

5.7.1 The Feynman Propagator

A similar calculation gives us

$$\langle 0|\, \psi_\alpha(x)\overline{\psi}_\beta(y)\, |0\rangle = \int \frac{\text{đ}^d p}{2E_p}(\not{p}+m)_{\alpha\beta}e^{-ip\cdot(x-y)} \tag{5.104}$$

$$\langle 0|\, \overline{\psi}_\beta(y)\psi_\alpha(x)\, |0\rangle = \int \frac{\text{đ}^d p}{2E_p}(\not{p}-m)_{\alpha\beta}e^{+ip\cdot(x-y)}. \tag{5.105}$$

We want to define our Feynman propagator $(S_F)_{\alpha\beta}(x-y)$ with the time ordering as

$$(S_F)_{\alpha\beta}(x-y) = \langle 0|\, T\psi_\alpha(x)\overline{\psi}_\beta(y)\, |0\rangle = \begin{cases} \langle 0|\, \psi_\alpha(x)\overline{\psi}_\beta(y)\, |0\rangle & x^0 > y^0 \\ -\langle 0|\, \overline{\psi}_\beta(y)\psi_\alpha(x)\, |0\rangle & x^0 < y^0 \end{cases} \tag{5.106}$$

The minus sign is required for Lorentz invariance: when we have spacelike separation $(x-y)^2 < 0$, there is no Lorentz invariant way to order the times; different frames give different orderings of x^0, y^0. In this case the anti-commutator vanishes $\left\{\psi(x)\overline{\psi}(y)\right\} = 0$ and so T as defined is Lorentz invariant.

- The same is true for *strings* of fermionic operators in T: they *anti*-commute.
- We also have the same behaviour for normal-ordered products:

$$:\psi_1\psi_2: = -:\psi_2\psi_1: \tag{5.107}$$

Then, with all these definitions, we have Wick's theorem for fermions

$$T(\psi(x)\overline{\psi}(y)) = :\psi(x)\overline{\psi}(y): + \contraction{}{\psi}{(x)}{\overline{\psi}}\psi(x)\overline{\psi}(y). \tag{5.108}$$

Instead of 3-momentum, have a 4-momentum expression

$$S_F(x-y) = i\int \mathrm{d}^4x\, \frac{e^{-ip\cdot(c-y)}}{p^2-m^2+i\epsilon}(\not{p}+m). \tag{5.109}$$

Exercise 5.2: Check that this satisfies

$$(i\not{\partial}_x - m)S_F(x-y) = i\delta^4(x-y), \tag{5.110}$$

i.e. S_F is a Green's function of the Dirac operator.

Let us now rejig Yukawa theory; nucleons are really fermions, and we have not accounted for that yet.

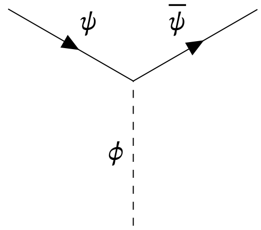


Figure 5.1: Yukawa 3-point interaction.

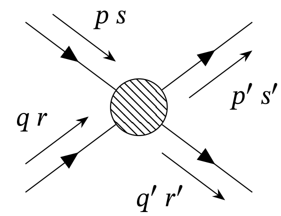


Figure 5.2: Nucleon-nucleon scattering.

5.8 Fermionic Yukawa Theory

We have fermions $\psi, \overline{\psi}$—representing nucleons and anti-nucleons—and a real scalar field ϕ—the mesons.

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi}_{\text{scalar kinetic}} + \overbrace{\frac{1}{2}\mu^2\phi^2}^{scalarmass} + \underbrace{\overline{\psi}(i\partial\!\!\!/ - m)\psi}_{\text{nucleon kinetic and mass}} - \overbrace{\lambda\phi\overline{\psi}\psi}^{\text{Yukawa interaction}} \tag{5.111}$$

The 3-point interaction is depicted in Fig. 5.1.

Dimensional analysis gives $[\mathcal{L}] = 4$, $[\phi] = 1$, $[\psi] = \frac{3}{2} \stackrel{!}{=} \left[\overline{\psi}\right]$. This means that $[\lambda] = 0$; no problems with renormalisability.

Example ($\psi\psi \to \psi\psi$): In nucleon-nucleon scattering, as illustrated in Fig. 5.2, we label our initial and final states

$$|i\rangle = \sqrt{4E_p E_q}(b^s_{\mathbf{p}})^\dagger (b^r_{\mathbf{q}})^\dagger |0\rangle := |\mathbf{p}, s; \mathbf{q}, r\rangle \tag{5.112}$$

$$|f\rangle = \sqrt{4E_{\mathbf{p}'} E_{\mathbf{q}'}}(b^{s'}_{\mathbf{p}'})^\dagger (b^{r'}_{\mathbf{q}'})^\dagger |0\rangle := \left|\mathbf{p}', s'; \mathbf{q}', r'\right\rangle \tag{5.113}$$

Note that there is no minus when taking the adjoint $(b^s_{\mathbf{p}} b^r_{\mathbf{q}})^\dagger = (b^r_{\mathbf{q}})^\dagger (b^s_{\mathbf{p}})^\dagger$. We have

$$\langle f| = \sqrt{4E_{\mathbf{p}'} E_{\mathbf{q}'}}\,\langle 0|\, b^{r'}_{\mathbf{q}'} b^{s'}_{\mathbf{p}'}. \tag{5.114}$$

Look for $O(\lambda^2)$ terms in the matrix element $\langle f|\,(S-1)\,|i\rangle$ with all fields in the interaction picture, as usual.

Can use Dyson's formula (4.22), second order piece of expansion is

$$\langle f|\,\frac{(-i\lambda)^2}{2!}\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,T\left[\overline{\psi}(x_1)\psi(x_1)\phi(x_1)\overline{\psi}(x_2)\psi(x_2)\phi(x_2)\right]|i\rangle\,. \tag{5.115}$$

Using Wick's theorem, we find that the contribution to scattering comes from the contraction

$$\langle f|\,:\overline{\psi}(x_1)\psi(x_1)\overline{\psi}(x_2)\psi(x_2):\,\phi(x_1)\phi(x_2)\,|i\rangle\,. \tag{5.116}$$

If the ϕ were not contracted, they would annihilate the vacuum. Then the ψ's annihilate the $|i\rangle$ and the $\overline{\psi}$'s create $\langle f|$.

We focus on how the fermionic fields act on $|i\rangle$. Let us expand out the ϕ fields first, ignoring the $c^\dagger$ terms, since they give no contribution at order λ^2:

$$:\overline{\psi}_1\psi_1\overline{\psi}_2\psi_2:(b^s_{\mathbf{p}})^\dagger(b^r_{\mathbf{q}})^\dagger|0\rangle = -\int \frac{đ^3k_1\, đ^3k_2}{\sqrt{4E_{\mathbf{k}_1}E_{\mathbf{k}_2}}}[\overline{\psi}_1\cdot u^m_{\mathbf{k}_1}][\overline{\psi}_2\cdot u^n_{\mathbf{k}_2}]e^{-i(k_1\cdot x_1+k_2\cdot x_2)}b^m_{\mathbf{k}_1}b^n_{\mathbf{k}_2}(b^s_{\mathbf{p}})^\dagger(b^r_{\mathbf{q}})^\dagger|0\rangle, \tag{5.117}$$

where we wrote $\phi(x_i)=\phi_i$ and $u(\mathbf{k}_i)=u_{\mathbf{k}_i}$ for brevity and used square brackets to indicate in which manner the spinor indices are contracted. The minus sign that sits out front came from commuting ϕ_1 past $\overline{\psi}_2$. Anti-commuting the b's past the $b^\dagger$'s, we have

$$b_{\mathbf{k}_1}b^n_{\mathbf{k}_2}b^{s\,\dagger}_{\mathbf{p}}b^{r\,\dagger}_{\mathbf{q}}|0\rangle = (b^m_{\mathbf{k}_1}\{b^n_{\mathbf{k}_2},b^{s\,\dagger}_{\mathbf{p}}\}b^{r\,\dagger}_{\mathbf{q}} - b^m_{\mathbf{k}_1}b^{s\,\dagger}_{\mathbf{p}}\cdot b^n_{\mathbf{k}_2}b^{r\,\dagger}_{\mathbf{q}})|0\rangle \tag{5.118}$$

$$= \left[\delta^3(\mathbf{k}_2-\mathbf{p})\delta^{ns}\left\{b^m_{\mathbf{k}_1},b^{r\,\dagger}_{\mathbf{q}}\right\} - \left\{b^m_{\mathbf{k}_1},b^{s\,\dagger}_{\mathbf{p}}\right\}\left\{b^n_{\mathbf{k}_2},b^{n\,\dagger}_{\mathbf{q}}\right\}\right]. \tag{5.119}$$

where in the second lne, we introduce $\{\dots\}$ because $b^s_{\mathbf{p}}|0\rangle=0$.

$$\cdots = [\delta^3(\mathbf{k}_2-\mathbf{p})\delta^3(\mathbf{k}_1-\mathbf{q})\delta^{ns}\delta^{mr} - \delta^3(\mathbf{k}_1-\mathbf{p})\delta^3(\mathbf{k}_2-\mathbf{q})\delta^{ms}\delta^{nr}]\,|0\rangle. \tag{5.120}$$

Pluggin this back into Eq. (5.117), we have

$$:\overline{\psi}_1\psi_1\overline{\psi}_2\psi_2:(b^s_{\mathbf{p}})^\dagger(b^r_{\mathbf{q}})^\dagger|0\rangle = \frac{-1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\Big([\overline{\psi}_1\cdot u^r_{\mathbf{q}}][\overline{\psi}_2\cdot u^s_{\mathbf{p}}]e^{-i(p\cdot x_2+q\cdot x_1)} - [\overline{\psi}_1\cdot u^s_{\mathbf{p}}][\overline{\psi}_2\cdot u^r_{\mathbf{q}}]e^{-i(p\cdot x_1+q\cdot x_2)}\Big)|0\rangle. \tag{5.121}$$

Note the relative minus sign that appears between these two terms. Now let us left multiply with $\langle f|$. Consider

$$\langle 0|\,b^{r'}_{\mathbf{q}'}b^{s'}_{\mathbf{p}'}b^{m\,\dagger}_{\mathbf{k}_1}b^{n\ \dagger}_{\mathbf{k}_2} = \langle 0|\,b^{r'}_{\mathbf{q}'}(\left\{b^{r'}_{\mathbf{p}'},b^{m\,\dagger}_{\mathbf{k}_1}\right\} - b^{m\,\dagger}_{\mathbf{k}_1}b^{s'}_{\mathbf{p}'})b^{n\ \dagger}_{\mathbf{k}_2} \tag{5.122}$$

$$= \langle 0|\,[\delta^{s'm}\delta^{r'n}\delta^3(\mathbf{p}'-\mathbf{k})\delta^3(\mathbf{q}'-\mathbf{k}_2) - (m\leftrightarrow n,\mathbf{k}_1\leftrightarrow\mathbf{k}_2)], \tag{5.123}$$

where we proceeded in a similar manner to (5.118). Using this to left multiply (5.121) by $\langle f|$, we get

$$\langle 0|\,b^{r'}_{\mathbf{q}'}b^{s'}_{\mathbf{p}'}[\overline{\psi}_1\cdot u^r_{\mathbf{q}}][\overline{\psi}_2\cdot u^s_{\mathbf{p}}]\,|0\rangle = \frac{1}{2\sqrt{E_{\mathbf{p}'}E_{\mathbf{q}'}}}\Big\{e^{+i(\mathbf{p}'\cdot x_1+\mathbf{q}'\cdot x_2)}[\overline{u}^{s'}_{\mathbf{p}'}\cdot u^r_{\mathbf{q}}][\overline{u}_{\mathbf{q}'}{}^{r'}\cdot u^s_{\mathbf{p}}] - e^{+i(\mathbf{p}'\cdot x_2+\mathbf{q}'\cdot x_1)}[\overline{u}_{\mathbf{q}'}{}^{r'}\cdot u^r_{\mathbf{q}}][\overline{u}_{\mathbf{p}'}{}^{s'}\cdot u^s_{\mathbf{p}}]\Big\}. \tag{5.124}$$

The $[\overline{\psi}_1\cdot u^s_{\mathbf{p}}][\overline{\psi}_2\cdot u^r_{\mathbf{p}}]$ term in (5.121) doubles up with this, cancelling the factor of $\frac{1}{2}$ in front of (5.115). Meanwhile, the $1/\sqrt{E}$ terms cancel the relativistic state normalisation. Putting these together, we find that $\langle f|\,(S-1)\,|i\rangle$ is given by

$$(-i\lambda)^2 \int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\text{đ}^4k\,\overbrace{\frac{ie^{ik\cdot(x_1-x_2)}}{k_2-\mu_2+i\epsilon}}^{\phi_1\phi_2}\Big([\overline{u}^{s'}_{\mathbf{p}'}\cdot u^s_{\mathbf{p}}][\overline{u}_{\mathbf{q}'}{}^{r'}\cdot u^r_{\mathbf{q}}]e^{+i[x_1\cdot(q'-q)+\mathbf{x}_2\cdot(p'-p)]} \\ -[\overline{u}_{\mathbf{p}'}{}^{s'}\cdot u^r_{\mathbf{q}}][\overline{u}_{\mathbf{q}'}{}^{r'}\cdot u^s_{\mathbf{p}}]e^{+i[x_1\cdot(p'-q)ix_2\cdot(q'-p)]}\Big), \quad (5.125)$$

where we have put the ϕ propagator back in. Performing the integrals over $\mathbf{x}_1$ and $\mathbf{x}_2$, this becomes

$$\int \frac{\text{đ}^4k\,i(-i\lambda)^2}{k^2-\mu^2+i\epsilon}\Big([\overline{u}_{\mathbf{p}'}{}^{s'}\cdot u^s_{\mathbf{p}}][\overline{u}_{\mathbf{q}'}{}^{r'}\cdot u^r_{\mathbf{q}}]\delta^4(q'-q'+k)\delta^4(p'-p-k) \\ -[\overline{u}_{\mathbf{p}'}{}^{s'}\cdot u^r_{\mathbf{q}}][\overline{u}_{\mathbf{q}'}{}^{r'}\cdot u^s_{\mathbf{p}}]\delta^4(p'-q+k)\delta^4(q'-p-k)\Big) \quad (5.126)$$

Finally, writing the S-matrix element in terms of the amplitude in the usual way, $\langle f|\,(S-1)\,|i\rangle = i\mathcal{M}\delta^4(p+q-p'-q')$, we have

$$\boxed{\mathcal{M} = (-i\lambda^2)\left\{\frac{[\overline{u}^{r'}_{\mathbf{q}'}\cdot u^r_{\mathbf{q}}][u^{s'}_{\mathbf{p}'}\cdot u^s_{\mathbf{p}}]}{(p'-p)^2-\mu^2+i\epsilon} - \frac{[u^{s'}_{\mathbf{p}'}\cdot u^r_{\mathbf{q}}][\overline{u}^{r'}_{\mathbf{q}'}\cdot u^s_{\mathbf{p}}]}{(q'-p)^2-\mu^2+i\epsilon}\right\}}, \quad (5.127)$$

which is our final answer for the amplitude.

$\psi\psi \to \psi\psi$	ϕ
b, c	a
$b, c = 0$	$[a, a^\dagger] = \ldots$

Table 5.2

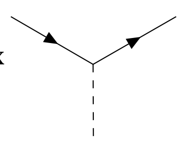
In addition to the commutation relations of 5.2, we can also work out the commutation relations

$$[a, b] = 0 = [a, c]. \tag{5.128}$$

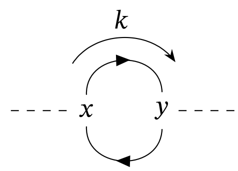
5.9 Momentum Space Feynman Rules for Fermionic Amplitudes

Diagrams:

$$- \tag{5.129}$$

- No *clash* allowed:
- Dirac fermions preserve fermion number.
- Incoming fermion $(u^s_{\mathbf{p}})_\alpha$
- Outgoing fermion $(\overline{u}^s_{\mathbf{p}})$
- Incoming anti-fermion $\overline{v^s_{\mathbf{p}}}$
- Outgoing anti-fermion $v^s_{\mathbf{p}}$
- 4-momentum conservation at each vertex
- vertex  $\sim (-i\lambda)$
- propagator $\alpha \xrightarrow{p} \beta \sim \frac{i(\not{p}+m)_{\beta\alpha}}{p^2-m^2+i\epsilon}$ Indices are contracted in the *oppostite direction* to the arrows.
- To be sure of relative minuses, follow the ordering of operators

- In a closed fermionic loop, e.g.



$$\sim\ :\overline{\psi}_\alpha(x)\psi_\alpha(x)\overline{\psi}_\beta(y)\psi_\beta(y): \tag{5.130}$$

  Contraction is defined on $\psi\overline{\psi}$

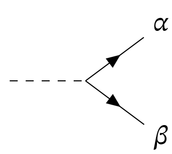
$$= -\ :\psi_\beta(y)\overline{\psi}_\alpha(x)\psi_\alpha(x)\overline{\psi}_\beta(y):\ . \tag{5.131}$$

  Hence there is an additional minus sign for a closed fermionic loop, as well as the usual $\int \mathrm{d}^4k$ around the loop momenta.

What if $\mathcal{L}_{\text{int}} = -\lambda\phi\overline{\psi}_\alpha(\gamma^5)_{\alpha\beta}\psi_\beta$?

> **Note:** This only preserves P if ϕ is a pseudo-scalar, i.e. $P\phi(\mathbf{x},t) = -\phi(-\mathbf{x},t)$.

The Feynman rule for the interaction is



$$\sim (-i\lambda)(\gamma^5)_{\alpha\beta}. \tag{5.132}$$

How do we deal with spin and the $|\mathcal{M}|^2$ in the cross-section calculation?

In most experiments, the initial spin states $|i\rangle$ are random. In that case we average over them (e.g. for $\psi\psi \to \psi\psi$, it would be $\frac{1}{4}\sum_{r,s=1}^{2}$). Typically, the final state particles $|f\rangle$ are not observed. We sum over them.

> **Note:** There are particular experiments with polarised beams, where we have some biases. Peskin & Schroeder [1] deals with this.

Any properties that are not observed can / do happen, so we have to sum over all unobserved possibilities in the final state.

Remark: $\mathcal{M} = B - A$ in $\psi\psi \to \psi\psi$, where B & A are the different terms. They can be written as dot products of spinors; the indices α, β are summed over.

We write appropriate spin sums / averages with a bar:

$$\overline{|\mathcal{M}|^2} = \overline{|A|^2} + \overline{|B|^2} - \overline{A^\dagger B} - \overline{B^\dagger A}. \tag{5.133}$$

$$A = \frac{\lambda^2 (\overline{u}^s_{p'})_\alpha (u^r_r)_\alpha (\overline{u}^{r'}_{q'})_\beta (u^s_p)_\beta}{(u - \mu_2 + i\epsilon)} \tag{5.134}$$

$$:= \lambda^2 \frac{[\overline{u}^{s'}_{p'} \cdot u^r_q][\overline{u}^{r'}_{q'} \cdot u^s_p]}{(u - \mu^2)}, \tag{5.135}$$

where we leave the $i\epsilon$ as "understood" from now on.

Remark: Since $(\gamma^0)^\dagger = \gamma^0$, $[\overline{u}^{s'}_p \cdot u^r_q]^\dagger = [\overline{u}^r_q \cdot u^{s'}_{p'}]$

$$\overline{|A|^2} = \frac{|\lambda|^4}{4} \frac{\sum_{rss'r'} (\overline{u}^{s'}_{p'})_\alpha \overbrace{(u^r_q)_\alpha (\overline{u}^r_q)_\beta}^{\text{spin sum over } r \text{ gives } (\not{q}+m)_\alpha b} (u^{s'}_{p'})^\beta (\overline{u}^{r'}_{q'})_\gamma \overbrace{(u^s_p)_\gamma (\overline{u}^s_p)_\delta}^{(\not{p}+m)_{\gamma\delta}} (u^r_{q'})_\delta}{(u - \mu^2)^2} \tag{5.136}$$

$$= \frac{|\lambda|^4}{4} \frac{(\not{p}' + m)_{\alpha\beta} (\not{q} + m)_{\beta\alpha} (\not{q}' + m)_{\gamma\delta} (\not{p} + m)_{\delta\gamma}}{(u - \mu^2)^2} \tag{5.137}$$

$$= \frac{|\lambda|^4}{4} \frac{\mathrm{Tr}\big[(\not{p}' + m)(\not{q} + m)\big]\, \mathrm{Tr}\big[(\not{q}' + m)(\not{p} + m)\big]}{(u - \mu^2)^2}. \tag{5.138}$$

This is why knowing how to deal with traces (PS3) is important. They come out in the spin sums.

Often we are in the high energy limit, where we can neglect particle masses, $\mu, m \to 0$.

Example: At the LHC, $\sqrt{s} = 13\text{TeV} >> m_p = 1\text{GeV}$.

In that case, then

$$\overline{|A|^2} = \frac{|\lambda|^4}{4u^2} \mathrm{Tr}(\not{p}'\not{q})\, \mathrm{Tr}(\not{q}'\not{p}). \tag{5.139}$$

Exercise 5.3: Check that a similar calculation gives

$$\overline{|B|^2} = \frac{|\lambda|^4}{4t^2} \mathrm{Tr}(\not{q}'\not{q})\, \mathrm{Tr}(\not{p}'\not{p}). \tag{5.140}$$

We also want to know $-\overline{A^\dagger B} - \overline{B^\dagger A} = -2\,\mathrm{Re}\,\overline{(A^\dagger B)}$.

$$\overline{A^\dagger B} = \frac{|\lambda|^4}{4ut} \sum_{rr'ss'} (\overline{u}^r_q)_\beta (u^{s'}_{p'})_\beta (\overline{u}^s_{\mathbf{p}})_\alpha (u^{r'}_{\mathbf{q}'})_\alpha (\overline{u}^{r'}_{\mathbf{q}'})_\gamma (u^r_{\mathbf{q}})_\gamma (\overline{u}^{s'}_{\mathbf{p}'})_\delta (u^s_{\mathbf{p}})_\delta \tag{5.141}$$

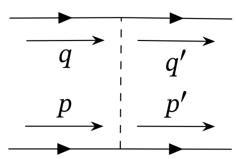
$$= \frac{|\lambda|^4}{4ut} \mathrm{Tr}\big(\not{p}\not{p}'\not{q}\not{q}'\big) \qquad \text{in limit } \mu, m \to 0. \tag{5.142}$$

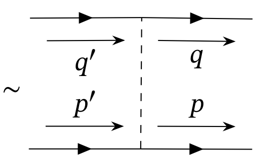
5.9.1 Feynman Rules for Spin Summed $|\mathcal{M}|^2$ diagrams

- Complex conjugation switches $|i\rangle \leftrightarrow |f\rangle$ in diagram
- Fermion lines are joined with identical momenta on LHS and RHS
- After a spin sum, a closed fermion line in the $|\mathcal{M}|^2$ diagram is given by a trace over γ matrices, with appropriate γ^5's etc in vertices at the correct position in the trace.

Trace follows fermion arrows *backwards*.

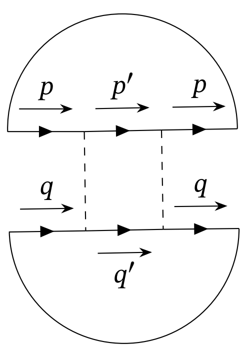
Example: Omitting the s labels, we have

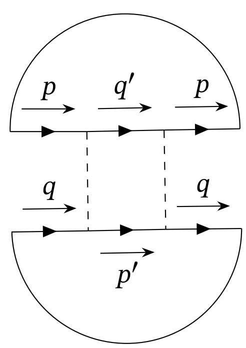
$B \sim$ 

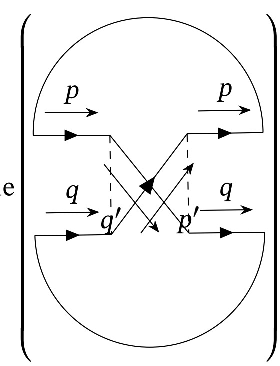
$\Rightarrow$ $B_{\dagger} \sim$ 

(5.143)

Then the equation $\overline{|\mathcal{M}|^2} = \overline{|B|^2} + \overline{|A|^2} - 2\,\mathrm{Re}(\overline{AB^\dagger})$ can be expressed diagrammatically as

$\overline{|\mathcal{M}|^2} =$ 

$+$ 

$-\,2\,\mathrm{Re}$ 

(5.144)

$$= \frac{|\lambda|^4}{4}\left\{\frac{\mathrm{Tr}(\not{q}\not{q}')\,\mathrm{Tr}(\not{p}\not{p}')}{t^2} + \frac{\mathrm{Tr}(\not{q}'\not{p})\,\mathrm{Tr}(\not{p}'\not{q})}{u^2} - \frac{2\,\mathrm{Re}\,\mathrm{Tr}(\not{p}\not{p}'\not{q}\not{q}')}{ut}\right\} \tag{5.145}$$

Remark: The minus sign that we started with ended up in the $2\,\mathrm{Re}$ term. It has an effect on the differential cross-section—a truly quantum interference effect.

5.9.2 $\psi\psi \to \psi\psi$ Scattering

$$\overline{|M|^2} 0 \frac{|\lambda|^4}{4}\left\{\frac{\mathrm{Tr}(\not{q}\not{q}')\,\mathrm{Tr}(\not{p}\not{p}')}{t^2} + \frac{\mathrm{Tr}(\not{q}'\not{p})\,\mathrm{Tr}(\not{p}'\not{q})}{u^2} - \frac{2\,\mathrm{Re}\,\mathrm{Tr}(\not{p}\not{p}'\not{q}\not{q}')}{ut}\right\} \tag{5.146}$$

Use trace techniques (see PS3) to show that

$$\cdots = \frac{|\lambda|^4}{4}\left\{\frac{4q\cdot q' 4p\cdot p'}{t^2} + \frac{4q'\cdot q 4p'\cdot q}{u^2} - \frac{8}{ut}(p\cdot q' p'\cdot q + p\cdot p' q\cdot q' - p\cdot q p'\cdot q')\right\}. \tag{5.147}$$

It is customary to express this in terms of Mandelstram variables s, t, and u.

- By definition, $s = (p+q)^2$ and by 4-momentum conservation, we have $(p+q)^2 = (p'+q')^2$. Therefore, $p\cdot q = p'\cdot q' = \frac{s}{2}$ in the massless limit, where $p\cdot p = 0$ etc.
- $t = (p-p')^2 = (q-q')^2 \quad \Rightarrow \quad p\cdot p' = q\cdot q' = -\frac{t}{2}$
- $u = (p-q')^2 = (q-p')^2 \quad \Rightarrow \quad p\cdot q' = q\cdot p' = -\frac{u}{2}$

Substituting these identities back into (5.147), we get

$$\overline{|M|^2} = |\lambda|^4\left[1 + 1 - \frac{1}{2ut}(u^2 + t^2 - s^2)\right]. \tag{5.148}$$

Recall that $s + t + u = \sum m_i^2 = 0$ here, so $u = -s - t$ and

$$\boxed{\overline{|M|^2} = 3|\lambda|^4}. \tag{5.149}$$

Cross Section

Let us now find the cross-section. In the centre of mass frame,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\overline{|M|^2}}{16\pi\lambda(s, \underbrace{0, 0}_{\text{masses}})} \qquad \text{and} \qquad \left.\frac{\mathrm{d}t}{\mathrm{d}\cos\theta}\right|_{CoM} = 2|\mathbf{p}||\mathbf{p}'|. \tag{5.150}$$

And writing the solid angle Ω: $d\Omega = \mathrm{d}\cos\theta\,\mathrm{d}\phi$, we have

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{CoM} = \frac{\overline{|M|^2}}{64\pi^2 s}. \tag{5.151}$$

We have *identical* particles in the final state, and as illustrated in Figure 5.3. Since we cannot tell which particle is which, we have to integrate the final state angles over the *hemi*-sphere. This gives us

$$\boxed{\sigma = \frac{3|\lambda|^4}{32\pi s}} \tag{5.152}$$

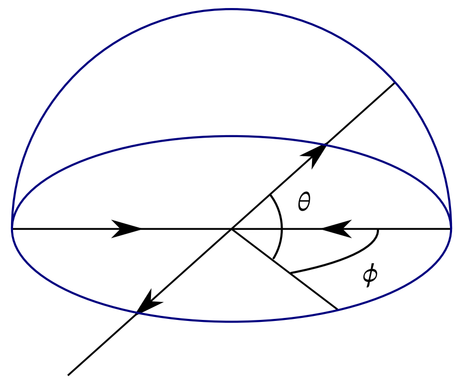


Figure 5.3: Scattering of identical particles. We cannot tell which particle is which.

The scattering cross-section σ is a measurable physical quantity and therefore needs to be non-negative, even with negative quantum interference effects. Since $[\lambda] = 0$, we have $[\sigma] = [s]^{-1} = -2$, which is exactly what we expect: the scattering cross-section is an area.

6 Quantum Electrodynamics

Quantum Electrodynamics (`QED`) is a quantum field theory, in which the photon is described by a 4-component vector field A_μ . The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{6.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the *field strength-tensor*. Since we use $\eta = \mathrm{diag}(+---)$, we have to be very consistent with signs of three-vectors.

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \dot{\mathbf{A}} \tag{6.2}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{6.3}$$

where $\boldsymbol{\nabla} = (\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}) = \partial_i$ and $A^\mu = (\phi, \mathbf{A})$, where $\mathbf{A} := (A^1, A^2, A^3)$.

The electric field is $\mathbf{E} = (F_{01}, F_{02}, F_{03}) = (-F^{01}, -F^{02}, -F^{03})$. Equation (6.2) comes from $F_{0i} = \partial_0 A_i - \partial_i A_0$. The magnetic field is $\mathbf{B} = (B_1, B_2, B_3)$. Looking at Eq. (6.3), we have for example

$$B_3 = \partial_1 A^2 - \partial_2 A^1 = -\partial_1 A_2 + \partial_2 A_1 = -F_{12} \quad \text{etc} \tag{6.4}$$

Doing this for all components, we find that

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \tag{6.5}$$

where the first index denotes the row and the second the column.

6.0.1 Maxwell's Equations

$F_{\mu\nu}$ satisfies the *Bianchi identity*:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = \partial_{[\lambda} F_{\mu\nu]} = 0 \tag{6.6}$$

From this, $\lambda = 3$, $\mu = 1$, and $\nu = 3$ gives $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, and $\lambda = 0$, $\mu = i$, $\nu = j$, $i \neq j$ gives $\dot{\mathbf{B}} = -\boldsymbol{\nabla}\times\mathbf{E}$.

Furthermore, the Euler-Lagrange equations are

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0 = \partial_\mu F^{\mu\nu}, \tag{6.7}$$

which give the other two vacuum Maxwell equations:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 0 \qquad \dot{\mathbf{E}} = \boldsymbol{\nabla}\times\mathbf{B}. \tag{6.8}$$

6.0.2 Comments on Degrees of Freedom

The massless vector field A_μ starts with 4 real degrees of freedom $\mu = 0, \dots, 3$. But we know that a photon γ only has two polarisation states! We note two things to help explain this

i A_0 is *not* dynamical $\leftrightarrow$ no kinetic term in $\mathcal{L}$. In fact, given $A_i(\mathbf{x}, t_0)$ and $\dot{A}_i(\mathbf{x}, t_0)$, th en A_0 is fully determined since $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ and therefore $\nabla^2 A_0 + \boldsymbol{\nabla}\cdot\dot{\mathbf{A}} = 0$. This has solution

$$A_0(\mathbf{x}, t_0) = \int \mathrm{d}^3x' \, \frac{\boldsymbol{\nabla}\cdot\dot{\mathbf{A}}(\mathbf{x}', t_0)}{4\pi|\mathbf{x}-\mathbf{x}'|} \tag{6.9}$$

Exercise 6.1: Check this. *Hint:* show that $\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\delta^3(\mathbf{x}-\mathbf{x}')$.

ii There is a large symmetry group. Consider the *gauge transformation*

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda(x), \tag{6.10}$$

where $\lambda(x)$ is some function such that $\lim_{|\mathbf{x}|\to\infty}\{\lambda(\mathbf{x})\} \to 0$. Under this transformation,

$$F_{\mu\nu} \to \partial_\mu(A_\nu + \partial_\nu\lambda) - \partial_\nu(A_\mu + \partial_\mu\lambda) = F_{\mu\nu}. \tag{6.11}$$

Since $F_{\mu\nu}$ is invariant, the Lagrangian $\mathcal{L}$ is too. This is a new kind of symmetry that we have not met before—a *gauge symmetry*, viewed as a redundancy in our description.

6.0.3 Gauge Fixing

Our EL equations give $\partial_\rho F^{\rho\nu} = 0$. Acting on the left with $\eta_{\mu\nu}$ gives

$$\eta_{\mu\nu}\partial_\rho F^{\rho\nu} = 0 \quad \Rightarrow \quad [\eta_{\mu\nu}\partial_\rho\partial^\rho - \partial_\mu\partial_\nu]A^\nu = 0. \tag{6.12}$$

The operator $O_{\mu\nu} := \eta_{\mu\nu}\partial_\rho\partial^\rho - \partial_\mu\partial_\nu$ is *not* invertible, since it annihilates any function of the form $\partial^\nu\lambda(x)$. This means that there is no way to *uniquely* determined the evolution of A_μ given A_i and $\dot{A}_i$ at t_0 .

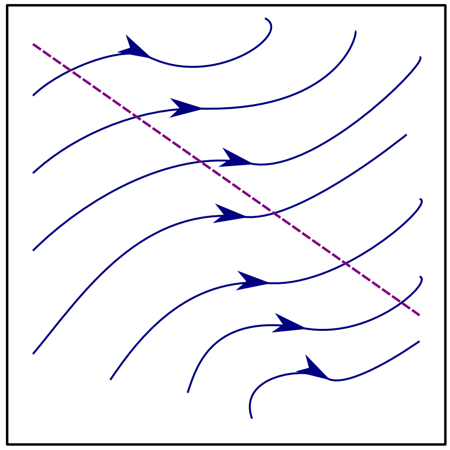


Figure 6.1: A_μ configuration space. The cut chooses one representative of each gauge orbit (c.f. fibres and sections in the *Differential Geometry* lectures).

The reason is that we cannot distinguish between A_μ and $A_\mu + \partial_\mu \lambda$, which are identified with the same physical state.

Definition 23: A gauge orbit is the set of physically equivalent states, which are related by gauge transformation.

As depicted in Fig. 6.1, the gauge orbits fill out the configuration space of A_μ. Cutting a section of this configuration space, we can choose one representative of each gauge orbit. This process is called *fixing a gauge*. It does not matter which representation we choose from each, since they are all physically equivalent. There are many possible gauges, some of which make certain calculations easier.

Example: The *Lorentz gauge* is given by $\partial_\mu A^\mu = 0$.

Claim 14: We can always pick a representative λ such that this is the case.

Proof. Start with A'_μ such that $\partial_\mu (A')^\mu = f(x)$. Then choose $A_\mu = (A')_\mu + \partial_\mu \lambda(x)$, where $\partial_\mu \partial^\mu (\lambda(x)) = -f(x)$. We can always solve the latter equation. $\square$

This condition does not pick a *unique* representation from the orbit: we can make a further transformation with $\partial_\mu \partial^\mu \lambda(x) = 0$, which has non-trivial solutions. The advantage of the Lorentz gauge is that the gauge condition is Lorentz invariant.

Remark: We will pick this gauge in our `QED` calculations.

Example: The *Coulomb gauge* or *radiation gauge* is $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$.

We can perform an argument similar to the one above. From Eq. (6.9), $A_0 = 0$. The advantage of this gauge lies in the ease of seeing the physical degrees of freedom: the three components of $\mathbf{A}$ satisfy a physical constraint, leaving behind two physical degrees of freedom—the polarisation states.

Remark: See David Tong's notes or Peskin & Schröder for more.

Bibliography

[1] Michael E. Peskin and Dan V. Schroeder, 'An Introduction to Quantum Field Theory' (1995)