Advanced Quantum Field Theory

Part III Lent 2019

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1 Path Integrals in QM

Goal: Schrödinger's equation → path integral

Consider a Hamiltonian in one dimension $\hat{H} = H(\hat{x}, \hat{p})$, where position and momentum operators satisfy the common commutation relations $[\hat{x}, \hat{p}] = i\hbar$. Assume the it takes the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{1.1}$$

Schrödinger's equation then says that the time evolution of a state $|\psi(t)\rangle$ is given by

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle. \tag{1.2}$$

This has a formal solution, giving us the time-evolution operator

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle. \tag{1.3}$$

In the Schrödinger picture, the states are evolving in time whereas operators and their eigenstates are constant in time.

Definition 1 (wavefunction)**:** $\Psi(x,t) := \langle x|\psi(t)\rangle$

The Schrödinger equation then becomes

$$\langle x|\hat{H}|\psi(t)\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi(x,t). \tag{1.4}$$

We will turn this differential equation into an integral equation, where we will sum over particle paths—a path integral. We can introduce an integral by inserting a complete set of states $1 = \int dx_0\,|x_0\rangle\langle x_0|$.

$$\begin{aligned}
\Psi(x,t) &= \langle x|e^{-i\hat{H}t/\hbar}|\psi(0)\rangle & (1.5)\\
&= \int_{-\infty}^{\infty} \mathrm{d}x_0\,\langle x|e^{-i\hat{H}t/\hbar}|x_0\rangle\langle x_0|\psi(0)\rangle & (1.6)\\
&:= \int_{-\infty}^{\infty} \mathrm{d}x_0\,\underbrace{K(x,x_0;t)}_{\text{`kernel'}}\Psi(x_0,0) & (1.7)
\end{aligned}$$

We repeat this insertion for n intermediate times and positions.

Notation: Let $0 := t_0 < t_1 < \cdots < t_n < t_{n+1} := T$

And we also want to factor the exponential into n terms:

$$e^{i\hat{H}T/\hbar} = e^{-\frac{i}{\hbar}\hat{H}(t_{n+1}-t_n)} \cdots e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)}. \tag{1.8}$$

Then

$$K(x, x_0; T) = \int_{-\infty}^{\infty} \left[\prod_{r=1}^{n} \mathrm{d}x_r \left\langle x_{r+1}\right| e^{-\frac{i}{\hbar}\hat{H}(t_{r+1}-t_r)} \left|x_r\right\rangle\right] \left\langle x_1\right| e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)} \left|x_0\right\rangle \tag{1.9}$$

Integrals are over all possible position eigenstates at times $t_r, r = 1, \dots, n$.

Free Theory

Consider the "free" theory, with $V(\hat{x}) = 0$. We will now play a similar but different trick to what we did before. Let us insert a complete set of momentum eigenstates $1 = \int_{-\infty}^{\infty} đp \left|p\right\rangle\left\langle p\right|$. We also note that these momentum eigenstates are plane waves $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$.

Definition 2 (barred differential)**:** We define the normalised differential in Fourier space to be

$$đp := \frac{\mathrm{d}p}{2\pi\hbar}. \tag{1.10}$$

In higher dimensional QFT, this generalises to

$$đ^n p := \frac{\mathrm{d}^n p}{(2\pi\hbar)^n} \tag{1.11}$$

The corresponding kernel is

$$K_0(x, x'; t) = \langle x| \exp\left(\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t\right)\left|x'\right\rangle. \tag{1.12}$$

$$= \int_{-\infty}^{\infty} đp\, e^{-ip^2t/2m\hbar} e^{ip(x-x')/\hbar} \tag{1.13}$$

$$= \sqrt{\frac{m}{2\pi i\hbar t}} e^{\frac{im(x-x')^2}{2\hbar t}}. \tag{1.14}$$

Remark:

$$\lim_{t\to 0}\left\{K_0(x, x'; t)\right\} = \delta(x - x'). \tag{1.15}$$

As expected from $\left\langle x\middle|x'\right\rangle = \delta(x - x')$.

From the Baker-Campbell-Hausdorff formula, we know that

$$e^{\epsilon\hat{A}}e^{\epsilon\hat{B}} = \exp\left(\epsilon\hat{A} + \epsilon\hat{B} + \frac{\epsilon^2}{2}[\hat{A},\hat{B}] + \ldots\right) \neq e^{\epsilon(\hat{A}+\hat{B})} \tag{1.16}$$

$$\text{for small } \epsilon: \qquad e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}}e^{\epsilon\hat{B}}(1 + O(\epsilon^2)) \tag{1.17}$$

Letting $\epsilon = 1/n$ and raising the above to the n^{th} power[1] gives

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty}\left\{e^{\hat{A}/n}e^{\hat{B}/n}\right\}^n. \tag{1.18}$$

We will use this to separate kinetic and potential terms.

Take $t_{r+1} - t_r = \delta t$ with $\delta t \ll T$ and n large such that $n\delta t = T$.

$$e^{-\frac{i}{\hbar}\hat{H}\delta t} = \exp\left(-\frac{i\hat{p}^2\delta t}{2m\hbar}\right)\exp\left(-\frac{iV(\hat{x})\delta t}{\hbar}\right)[1 + O(\delta t^2)] \tag{1.19}$$

Using the result (1.14),

$$\langle x_{r+1}|\exp\left(-\frac{i\hat{H}}{\hbar}\delta t\right)|x_r\rangle = e^{-iV(x_r)\delta t/\hbar}K_0(x_{r+1}, x_r; \delta t) \tag{1.20}$$

$$= \sqrt{\frac{m}{2\pi i\hbar\delta t}}\exp\left[\frac{im}{2\hbar}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2\delta t - \frac{i}{\hbar}V(x_r)\delta t\right] \tag{1.21}$$

With $T = n\delta t$

$$K(x, x_0; T) = \int\left[\prod_{r=1}^{n}\mathrm{d}x_r\right]\left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{n+1}{2}}\exp\left\{i\sum_{r=0}^{n}\left[\frac{m}{2\hbar}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2 - \frac{1}{\hbar}V(x_r)\right]\delta t\right\} \tag{1.22}$$

In the limit $n \to \infty$, $\delta t \to 0$ with $n\delta t = T$ fixed, the exponent becomes

$$\frac{1}{\hbar}\int_0^T \mathrm{d}t\left[\frac{1}{2}m\dot{x}^2 - V(x)\right] = \int_0^T \mathrm{d}t\, L(x, \dot{x}), \tag{1.23}$$

where L is the classical Lagrangian, the Legendre transformation of the classical Hamiltonian. The classical action is $S = \int \mathrm{d}t\, L(x, \dot{x})$.

The main result therefore is that the path integral for the kernel is

$$\boxed{K(x, x_0; t) := \langle x|\, e^{-i\hat{H}t/\hbar}\, |x_0\rangle = \int \mathcal{D}x\, e^{\frac{i}{\hbar}S}} \tag{1.24}$$

Definition 3 (functional integral)**:**

$$\mathcal{D}x = \lim_{\substack{\delta t\to 0\\ n\delta T \text{ fixed}}}\left\{(\sqrt{\ldots})\prod_{r=1}^{n}(\sqrt{\ldots}dx_r)\right\} \tag{1.25}$$

We do not need to care about normalization factors.

Remark: In the limit $\hbar \to 0$, the interference of amplitudes is dominated by the ones close to the extremal path where δS. This leads to Hamilton's principle of least action.

[1]This step is sometimes called Suzuki–Trotter decomposition.

We may analytically continue this to imaginary time. Let $\tau = it$. In terms of this imaginary time, we have

$$\langle x| \, e^{-\hat{H}\tau/\hbar} \, |x_0\rangle = \int \mathcal{D}x \, e^{-S/\hbar}. \tag{1.26}$$

Mathematically, it makes these integrals much more well-defined, clearly convergent. Here the least-action principle is really evident from the $\hbar = 0$ argument. We also see the connection to statistical physics, interpreting $e^{-S/\hbar}$ as the Boltzmann factor $e^{-\beta H}$.

Quantum mechanics is quantum field theory in 0+1 dimensions. We are treating space differently from time: $\hat{x}(t)$ is a field, whereas t is a variable. However, Lorentz invariance forces us to put x and t on the same footing. In QFT, we solve this problem by demoting x from a field to another label. We then talk about fields $\phi(x, t)$ and want to know about the behaviour of these in all of spacetime.

> String theory gives another ansatz to this problem by promoting again, instead of demoting operators to labels.

2 Integrals and their Diagrammatic Expansions

In QFT, we are interested in correlation functions. The following discussion will be very similar to what we have seen in the *Statistical Field Theory* course in Michaelmas term, also no knowledge from that course will be assumed here.

For simplicity, consider a 0 -dimensional field $\varphi \in \mathbb{R}$. As if we are in imaginary time, let

$$Z = \int_{\mathbb{R}} \mathrm{d}\varphi \, e^{-\frac{S(\varphi)}{\hbar}}. \tag{2.1}$$

Assume that the action $S(\varphi)$ is an even polynomial and $S(\varphi) \to \infty$ as $\varphi \to \pm\infty$.

We will be interested in expectation values

$$\langle f \rangle = \frac{1}{Z} \int \mathrm{d}\varphi \, f(\varphi) e^{-S/\hbar}. \tag{2.2}$$

Again, assume f does not grow too fast as $\varphi \to \pm\infty$. Usually, f is polynomial in φ.

2.1 Free Theory

Say we have N scalar fields (in $0+1$ dimensions we should really just say 'variables') φ_a with $a = 1, \ldots, N$, with action

$$S_0(\varphi) = \frac{1}{2} m_{ab} \varphi_a \varphi_b = \frac{1}{2} \varphi^T m \varphi, \tag{2.3}$$

where m is an $N \times N$ symmetric, positive definite ($\det m > 0$) matrix.

We can diagonalise this. There exists some orthogonal P such that $m = P \Lambda P^T$, where Λ is diagonal. Let $\chi = P^T\varphi$. Then the free partition function is

$$Z_0 = \int \mathrm{d}^N\varphi \exp\Big(-\frac{1}{2\hbar}\varphi^T m\varphi\Big) \tag{2.4}$$

$$= \int \mathrm{d}^N\chi \exp\Big(-\frac{1}{2\hbar}\chi^T\Lambda\chi\Big) \tag{2.5}$$

$$= \prod_{c=1}^{N}\int \mathrm{d}\chi_c\, e^{-\frac{\lambda_c}{2\hbar}\chi^2} = \sqrt{\frac{(2\pi\hbar)^N}{\det m}} \tag{2.6}$$

We want to get from the partition function to correlation functions. We can do this by introducing an N-component vector of external sources J to the action

$$S_0(\varphi) \to S_0 + J^T\varphi. \tag{2.7}$$

The partition function is then

$$Z_0(J) = \int \mathrm{d}^N\varphi \exp\Big(-\frac{1}{2\hbar}\varphi^T m\varphi - \frac{1}{\hbar}\Big)J^T\varphi. \tag{2.8}$$

Similar to solving an ordinary Gaussian integral, we complete the square by writing $\widetilde{\varphi} = \varphi + m^{-1}J$. One can then solve this integral to be

$$Z_0(J) = Z_0(0)\exp\Big(\frac{1}{2\hbar}J^T m^{-1}J\Big). \tag{2.9}$$

This is called the *generating function*[1]. Correlation functions are obtained from differentiating with respect to the auxiliary sources J and evaluating the whole expression at $J = 0$:

$$\langle\varphi_a\varphi_b\rangle = \frac{1}{Z_0(0)}\int \mathrm{d}^N\varphi\,\varphi_a\varphi_b \exp\Big(-\frac{1}{2\hbar}\varphi^T m\varphi - \frac{1}{\hbar}J^T\varphi\Big)\Big|_{J=0}. \tag{2.10}$$

$$= \frac{1}{Z_0(0)}\int \mathrm{d}^N\varphi\Big(-\hbar\frac{\partial}{\partial J_a}\Big)\Big(-\hbar\frac{\partial}{\partial J_b}\Big)\exp(\ldots)\Big|_{J=0} \tag{2.11}$$

$$= \frac{1}{Z_0(0)}\Big(-\hbar\frac{\partial}{\partial J_a}\Big)\Big(-\hbar\frac{\partial}{\partial J_b}\Big)Z_0(J)|_{J=0} \tag{2.12}$$

$$= \hbar\,(m^{-1})_{ab} := a \;\text{———}\; b \tag{2.13}$$

Connecting this to the *Quantum Field Theory* course, we identify this as the *free propagator*.

More generally, let $l(\varphi)$ be a *linear* combination of φ_a.

$$l(\varphi) = \sum_{a=1}^{N} l_a\varphi_a, \qquad l_a \in \mathbb{R}. \tag{2.14}$$

Then the steps above are equivalent to swapping $l(\varphi)$ for $l(-\hbar\frac{\partial}{\partial J}) = -\hbar\sum_a l_a\frac{\partial}{\partial J_a}$.

[1]When we go to higher dimensions, where $J = J(x)$ this will be a generating functional $Z[J(x)]$.

The correlation function can again be evaluated explicitly by the introduction of an auxiliary current J:

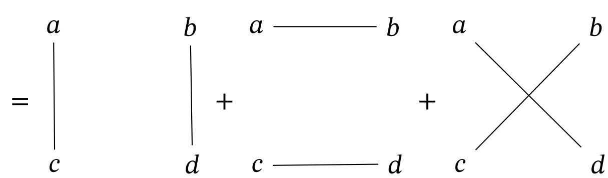
$$\left\langle l^{(1)}(\varphi)\cdots l^{(p)}(\varphi)\right\rangle = \frac{1}{Z_0(0)}\int \mathrm{d}^N\varphi \prod_{i=1}^{p} l^{(i)}(\varphi)e^{-\frac{1}{2\hbar}\varphi^T m\varphi-\frac{1}{\hbar}J^T\varphi}\bigg|_{J=0} . \tag{2.15}$$

$$= (-\hbar)^p \prod_{i=1}^{p} l^{(i)}(\frac{\partial}{\partial J})\, e^{\frac{1}{2\hbar}J^T m^{-1}J}\bigg|_{J=0} \tag{2.16}$$

In other words, if p is odd, the integrand is odd in some φ_a and the integral over $\varphi_a \in (-\infty,+\infty)$ vanishes. For $p = 2k$, the terms which are non-zero as $J \to 0$ have the following form. We need half the derivatives to bring down components of $m^{-1}J$ and half to remove the J -dependence from those terms that earlier derivatives brought down. As such, we get exactly k factors of m^{-1} . This is Wick's theorem.

Example (4-point function)**:** Consider the 4-point correlation function. One can check that the above result gives

$$\langle\varphi_a\varphi_b\varphi_c\varphi_d\rangle = \hbar^2\left[(m^{-1})_{ab}(m^{-1})_{cd} + (m^{-1})_{ac}(m^{-1})_{bd} + (m^{-1})_{ad}(m^{-1})_{bc}\right] \tag{2.17}$$



(2.18)

We end up with three terms, one for each way of grouping the 4 fields into pairs.

In general, for $\langle\varphi_1 \ldots \varphi_{2k}\rangle$, the number of terms is the number of distinct ways of pairing the $2k$ fields. This is $(2k+1)!! = (2k)!/(2^k k!)$; the number of permutations of $2k$ fields is $(2k)!$, but we have to divide this by the 2^k permutations within the pairs and the $k!$ ways of rearranging the pairs.

Remark: For complex fields, m is Hermitian but not symmetric anymore. In that case, the order of indices of m^{-1} is important. We keep track of this by drawing the propagator with a directed line

$$\langle\phi_a\phi_b^*\rangle = \hbar\,(m^{-1})_{ab} := a \longrightarrow b \tag{2.19}$$

2.2 Interacting Theory

We want to go beyond the free theory. The way we are going to achieve this is by an expansion about the classical result $\hbar$. The resulting integral will end up not being convergent.

Claim 1: Integrals like

$$\int \mathrm{d}\phi\, f(\phi) e^{-S/\hbar} \tag{2.20}$$

do not have a Taylor expansion about $\hbar = 0$.

Dyson. If the expansion about $\hbar = 0$ existed for $\hbar > 0$, then in the complex plane, there must be some open neighbourhood of $\hbar$ in which the expansion converges. For $S(\phi)$ has a minimum, the integral is divergent if $\mathrm{Re}(\hbar) < 0$. Therefore, the radius of convergence cannot be greater than zero. □

So the $\hbar$-expansion is at best *asymptotic*.

Definition 4 (asymptotic)**:** A series $\sum_{n=0}^{\infty} c_n \hbar^n$ is asymptotic to $I(\hbar)$ if

$$\lim_{\hbar \to 0^+} \frac{1}{\hbar^N} \left| I(\hbar) - \sum_{n=0}^{N} c_n \hbar^n \right| = 0. \tag{2.21}$$

Notation: We write $I(\hbar) \sim \sum_{n=0}^{\infty} c_n \hbar^n$.

The series misses out transcendental terms like $e^{-\frac{1}{\hbar^2}} \sim 0$. However, these can evidently be important since obviously $e^{-\frac{1}{\hbar^2}} \neq 0$ for finite $\hbar$. These are called *non-perturbative contributions*. These become important in particular for non-Abelian gauge theories.

Take the ϕ-fourth action

$$S(\phi) = \underbrace{\frac{1}{2} m^2 \phi^2}_{S_0(\phi)} + \underbrace{\frac{\lambda}{4!} \phi^4}_{S_1(\phi)} \qquad \begin{matrix} m^2 > 0 \\ \lambda > 0. \end{matrix} \tag{2.22}$$

Expand about the minimum of $S(\phi)$, which is $\phi = 0$.

$$Z = \int \mathrm{d}\phi\, e^{-S/\hbar} \tag{2.23}$$

$$= \int \mathrm{d}\phi\, e^{-S_0/\hbar} \overbrace{\sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!\hbar} \right)^n \phi^{4n}}^{e^{-S_1/\hbar}} . \tag{2.24}$$

In order to make progress, we truncate the series and swap summation and integration. This misses out transcendental terms. In the end, we end up with a series that is asymptotic to Z:

$$Z \sim \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^{N} \frac{1}{n!} \left(-\frac{\hbar\lambda}{4!m^4}\right)^n 2^{2v} \int_0^\infty \mathrm{d}t\, e^{-t} t^{2n+\frac{1}{2}-1}, \tag{2.25}$$

where $t = \frac{1}{2\hbar} m^2 \phi^2$. We recognise the Gamma function

$$\int_0^\infty \mathrm{d}t\, e^{-t} t^{2n+\frac{1}{2}-1} = \Gamma(2n + \frac{1}{2}) = \frac{(4n)!\sqrt{\pi}}{4^{2n}(2n)!}. \tag{2.26}$$

The partition function is

$$Z \sim \frac{\sqrt{2\pi\hbar}}{m} \sum_{n=0}^{N} \left(-\frac{\hbar\lambda}{m^4}\right)^n \frac{(4n)!}{(4!)^n n! 2^{2n} (2n)!} \tag{2.27}$$

The factor on the right comes in part from the Taylor expansion of $e^{-S_1/\hbar}$ and from the number of ways of pairing the $4n$ fields of the n copies of ϕ^4. Stirling's approximation allows us to write $n! \approx e^{n \ln n}$. The factor in the partition function then become

$$\frac{(4n)!}{(4!)^n n! 2^{2n} (2n)!} \approx n!. \tag{2.28}$$

We end up with factorial growth!

2.2.1 Diagrammatic Method

Let us introduce a current J

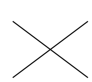
$$Z(J) = \int \mathrm{d}\phi \exp\left\{-\frac{1}{\hbar}\left(S_0(\phi) + S_1(\phi) + J\phi\right)\right\} \tag{2.29}$$

$$= \exp\left[-\frac{1}{\hbar} S_1(-\hbar\frac{\partial}{\partial J})\right] \underbrace{\int \mathrm{d}\phi \exp\left\{-\frac{1}{\hbar}(S_0 + J\phi)\right\}}_{Z_0(J)} \tag{2.30}$$

$$\propto \exp\left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right] \exp\left(\frac{1}{2\hbar} J^T M^{-1} J\right), \qquad M = m^2 \tag{2.31}$$

$$\sim \sum_{n=0}^{N} \frac{1}{n!} \left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right]^n \sum_{p=0} \frac{1}{p!} \left(\frac{1}{2\hbar} J m^{-2} J\right)^p. \tag{2.32}$$

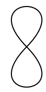
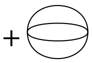
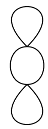
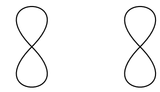
This is called the *double expansion*. Diagrammatically, we use the propagator and vertex

$$J \overset{m^{-2}}{\text{———}} J$$  $$-\lambda\left(\frac{\partial}{\partial J}\right)^4. \tag{2.33}$$

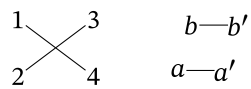
Let us check $Z(0)$. For a term to be non-zero when $J = 0$, we need the number of derivations to be equal to the number of propagators. Denoting by E the number of external sources, left undifferentiate

$$E := 2P - 4n = 0. \tag{2.34}$$

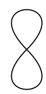
The first non-trivial terms are $(n, p) = (1, 2), (2, 4), \ldots$.

$Z(0) \propto 1 +$  $+$  $+$  $+$  $+ O(n = 3)$. (2.35)

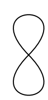
Count the number of times each diagram appears by using a *pre-diagram*



(2.36)

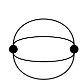
The numerator $A = 4!$ ways of matching derivatives to sources. Denominator of (**??**) is $F = (n!)(4!)^n(p!)2^p = 4! \cdot 2 \cdot 2^2$. So  comes with a prefactor of $\frac{A}{F} = \frac{1}{8}$. More generally, F accounts for permutations of

- all vertices $n!$
- each vertex's legs $4!$
- all propagators $p!$
- both end of each propagator 2

The symmetry of each particular graph is important. For the above diagram, take pairing $(1a, 2a', 3b, 4b')$. Swapping $a \leftrightarrow a'$ and $1 \leftrightarrow^2$ gives exactly the same graph. We define the *symmetry factor* S to be $\frac{A}{F} = \frac{1}{S}$. S is the number of ways of redrawing the unlabelled graph, leaving it unchanged. These are called the automorphisms of the graph. For  , we can swap the direction of upper and lower loops (2^2) and also swap upper and lower loops (2). Therefore, we obtain $S = 2 \cdot 2^3 = 8$.

Let us look at a slightly more complicated example.

Example (basketball)**:**

 $S = 4! \cdot 2 = 48$ (2.37)

The pre-diagram associated to this is

(2.38)

We obtain $F = 2(4!)^2 4! 2^4 = 4^3 \cdot 2^{14}$ and $A = 8 \cdot 6 \cdot 4 \cdot 2 \cdot 4! = 3^2 \cdot 2^{10}$. Thus $A/F = 1/48$.

For the other diagrams, we have

$$\frac{Z(0)}{Z_0(0)} = 1 - \frac{\lambda\hbar}{8m^4} + \frac{\hbar^2\lambda^2}{m^8}\left(\frac{1}{48} + \frac{1}{16} + \frac{1}{128}\right) \tag{2.39}$$