General Relativity

Part III Michaelmas 2019

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Contents

1 Introducing Differential Geometry

Structure: will do only differential geometry (maths methods) in the first 12 lectures, first 4 weeks. Afterwards the connection to the physics will be made. We will develop a mathematical language that allows us to write valid equations involving vectors, similar to dimensional analysis for physicists. The physics will be introduced with the action principle.

Extra book: "Geometry, Topology, and Physics" by Nakahara.

Office Hours: Friday 4 $\rightarrow$ 5 in B2.13

Extra stuff not examinable. Lectures are what matters.

1.1 Manifolds

An *n-dimensional manifold*, $\mathcal{M}$, is a space that locally looks like $\mathbb{R}^n$. More precisely, we require

1. For each point $p \in \mathcal{M}$, there is a map $\phi : \mathcal{O} \rightarrow U$ where $\mathcal{O} \subset \mathcal{M}$ is an open set with $p \in \mathcal{O}$ and $U \subset \mathbb{R}^n$. We will think of $\phi(p) = (x^1(p), \ldots, x^n(p))$ as coordinates on $\mathcal{O} \subset \mathcal{M}$.

   This map must be a *homeomorphism*:

   (a) injective (or 1-1) $p \neq q \;\Rightarrow\; \phi(p) \neq \phi(q)$

   (b) surjective (or onto) $\phi(\mathcal{O}) = U$ These ensure that ϕ^{-1} exists

   (c) both ϕ and ϕ^{-1} are continuous

2. If $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ are two open sets with

$$\phi_\alpha : \mathcal{O}_\alpha \rightarrow U_\alpha \text{ and } \phi_\beta : \mathcal{O}_\beta \rightarrow U_\beta \tag{1.1}$$

   then the *transition functions* $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \rightarrow \phi_\alpha(\mathcal{O}_\alpha \cap \mathcal{O}_\beta)$ are smooth (ie. infinitely differentiable). The ϕ_α are called *charts*. The idea is that there may be different ways to assign coordinates to a given point $p \in \mathcal{M}$. The collection of all charts is called an *atlas*. We require that these coordinate systems are mutually compatible. This is depicted in Figure 1.1.

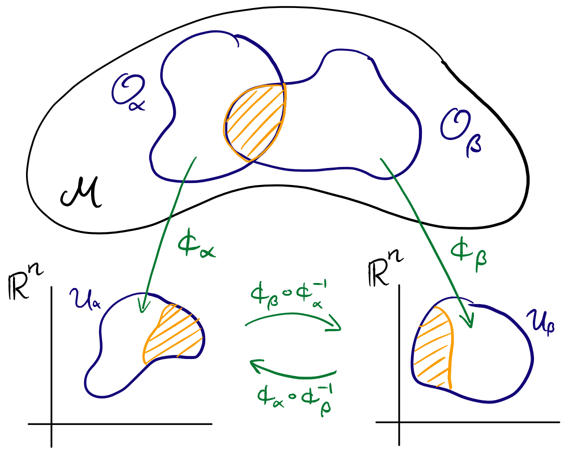

Figure 1.1: An illustration of charts on a manifold

Example ($\mathbb{R}^n$)**:** $\mathbb{R}^n$ or any open subset of $\mathbb{R}^n$ is a manifold. You only need a single chart.

Example (S^1)**:** We can view this as $(\cos\theta, \sin\theta) \in \mathbb{R}^2$ with $\theta \in [0, 2\pi)$. The closed set $[0, 2\pi)$ means that we cannot differentiate at 0. This is not a good chart because it is not an open set. We need at least two charts, depicted in 1.2:

$$\begin{aligned} \phi_1 : \mathcal{O}_1 &\to (0, 2\pi) & \qquad \phi_2 : \mathcal{O}_2 &\to (-\pi, \pi) \\ p &\longmapsto \theta_1 & p &\longmapsto \theta_2 \end{aligned} \tag{1.2}$$

The transition function is

$$\theta_2 = \phi_2(\phi_1^{-1}(\theta_1)) = \begin{cases} \theta_1, & \theta_1 \in (0, \pi) \\ \theta_1 - 2\pi, & \theta_1 \in (\pi, 2\pi) \end{cases} \tag{1.3}$$

> **Remark:** The fact that the coordinates go bad in the case of closed sets, similar to spherical polar coordinates, does not bother us too much for physical applications.

Since we can map $\mathcal{M} \to \mathbb{R}^n$ (at least locally), anything we can do on $\mathbb{R}^n$, we can now also do on $\mathcal{M}$ (e.g. differentiation).

> **Remark:** Note that at the moment, the distance in $\mathbb{R}^n$ cannot be translated back to the manifold. This is because the maps ϕ_α are arbitrary.

Definition 1 (Diffeomorphism)**:** A *diffeomorphism* is a smooth homeomorphism $f : \mathcal{M} \to \mathcal{N}$ between two manifolds. I.e. two manifolds are diffeomorphic if the map $\psi \circ f \circ \phi^{-1} : U \to V$ is smooth for all charts $\phi : \mathcal{M} \to U \subset \mathbb{R}^n$ and $\psi : N \to V \subset \mathbb{R}^n$.

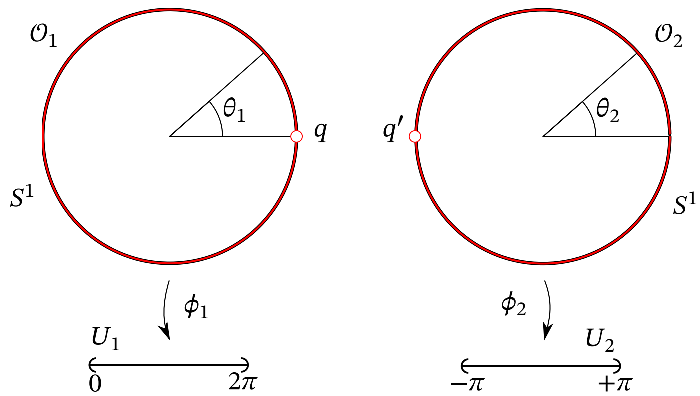

Figure 1.2: The two charts ϕ_1, ϕ_2 form an atlas of the manifold S^1.

Remark: There are interesting properties of S^7 and $\mathbb{R}^4$. You can find two atlasses of these manifolds such that the two atlasses are not diffeomorphic to each other (maybe I didn't understand this correctly). As far as we know, there is yet no application of this in physics.

1.2 Tangent Spaces

So far, we have the idea of a manifold. It is a space that looks locally like $\mathbb{R}^n$, with a set of charts that allow us to map patches to $\mathbb{R}^n$ and differentiate (and later, integrate).

Let $C^\infty(\mathcal{M})$ denote the set of all smooth functions that assign each point in the manifold $\mathcal{M}$ a real number in $\mathbb{R}$. We know how to differentiate on $\mathbb{R}^n$. Smooth functions allow us to differentiate in $\mathbb{R}^n$ before we map back onto the manifold.

Definition 2 (tangent vector)**:** A *tangent vector* X_p is an object that differentiates functions at some point $p \in \mathcal{M}$ on the manifold. Specifically, it is a map $X_p : C^\infty(\mathcal{M}) \to \mathbb{R}$ with certain properties:

1. linearity: $X_p(f+g) = X_p(f) + X_p(g)$ for all $f, g \in C^\infty(\mathcal{M})$
2. constants vanish: $X_p(f) = 0$ when f is constant
3. Leibniz rule: $X_p(fg) = f(p)X_p(g) + X_p(f)g(p)$ for all $f, g \in C^\infty(\mathcal{M})$

Definition 3 (tangent space)**:** The set of all tangent vectors at p is the *tangent space* $T_p(\mathcal{M})$ at p.

One of the early surprises of differential geometry is thinking of vectors as differential operators. In $\mathbb{R}^3$, we used position vectors as displacement. However, this does not generalise to curved spaces. The second idea of a vector in physics is the velocity of a particle. This change in time is analogous to the definition that we use here in differential geometry. By differentiating, the tangent vector tells us how things change when moving in a particular direction.

Claim: In a chart $\phi = (x^1, \cdots, x^n)$, we can write a tangent vector as

$$X_p = X^\mu \partial_\mu|_p, \tag{1.4}$$

with $X^\mu = x_p(x^\mu)$ and $\partial_\mu f = \frac{\partial(f\circ\phi^{-1})}{\partial x^\mu}$ for all functions $f \in C^\infty(\mathcal{M})$. This definition of $\partial_\mu f$ uses the fact that we can differentiate on $\mathbb{R}^n$, which we map to via $f\circ\phi^{-1}$.

Proof. The idea is that given a function f, we use coordinates to push it to $\mathbb{R}^n$, and then we differentiate there. A detailed proof is in the notes. □

> **Remark:** We are going to use the summation convention with indices up/down. The object with coefficients as superscripts are tangent vectors, different from objects with subscript coefficients. Hence, the position of the indices matter even more than in Special Relativity.

Writing $X_p = X^\mu \partial_\mu$ clearly depends on coordinates x^μ. What would happen if we chose different coordinates? If we picked another set of coordinates $\widetilde{x}^\mu$, we could write

$$X_p = X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_p = \widetilde{X}^\mu \left.\frac{\partial}{\partial \widetilde{x}^\mu}\right|_p . \tag{1.5}$$

Notation We write $\partial_\mu := \frac{\partial}{\partial x^\mu}$.

Acting on a function f, (and dropping the implicit action of going to $\mathbb{R}^n$ via ϕ^{-1})

$$X_p(f) = X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_p = X^\mu \left.\frac{\partial \widetilde{x}^\nu}{\partial x^\mu}\right|_{\phi(p)} \left.\frac{\partial f}{\partial \widetilde{x}^\nu}\right|_p . \tag{1.6}$$

We can view this as a change of basis of $T_p(\mathcal{M})$

$$\left.\frac{\partial}{\partial x^\mu}\right|_p = \left.\frac{\partial \widetilde{x}^\nu}{\partial x^\mu}\right|_p \left.\frac{\partial}{\partial \widetilde{x}^\nu}\right|_p . \tag{1.7}$$

Alternatively, we can think of the change of components

$$\widetilde{X}^\nu = X^\mu \left.\frac{\partial \widetilde{x}^\nu}{\partial x^\mu}\right|_{\phi(p)} . \tag{1.8}$$

This is called a *contravariant* transformation; the indices tell us which way things transform.

To see why this is called a tangent vector, we are going to consider a path $\sigma(t) : \mathbb{R} \to \mathcal{M}$ in the manifold such that $\sigma(0) = p$. We can think of the parameter as time for example. Given coordinates, this becomes a path $x^\mu(t)$ in $\mathbb{R}^n$. The tangent in $\mathbb{R}^n$ is $X^\mu = \frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t}|_{t=0}$. We can then use this to define $X_p \in T_p(\mathcal{M})$ as

$$X_p = \left.\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t}\right|_{t=0} \left.\frac{\partial}{\partial x^\mu}\right|_p . \tag{1.9}$$

In this sense, $T_p(\mathcal{M})$ is the space of all tangent vectors at p. By considering all possible paths, one can describe all possible tangent vectors. Physically, it is like the space of velocities of a particle. The way to think about this is as in fig

> **Remark:** Tangent spaces at two different points are two different vector spaces. It is meaningless to try to add vectors of two different tangent spaces . . . for now.

> **Remark:** These manifolds have intrinsic meaning and do not depend on an embedding. Physically, the $3+1$-dimensional spacetime is a manifold that we do not think is embedded in any higher dimensional space.

1.3 Vector Fields

Definition 4 (vector field)**:** A *vector field* X is a smooth assignment of tangent vectors at each point $p \in \mathcal{M}$:

$$X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}) \tag{1.10}$$

with $(Xf)(p) = X_p(f)$. In a given coordinate chart we can write

$$X = X^\mu(x)\partial_\mu. \tag{1.11}$$

Notation: We will denote the space of all vector fields as $\mathfrak{X}(\mathcal{M})$.

> **Remark:** We will make the notion of *smooth* more precise later. Intuitively, we want neighbouring tangent spaces to have similar directions.

Given two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, we can define the *commutator*.

Definition 5 (commutator)**:** The *commutator* $[X, Y] \in \mathfrak{X}(\mathcal{M})$ is a map

$$[X, Y]f = X(Yf) - Y(Xf). \tag{1.12}$$

> **Remark:** Both X, Y are first order differential operator. Thus, each term on the RHS is a second order operator. However, by subtracting these terms from each other, the second order terms cancel by commutation of partial derivatives. Thus, the commutator is a first order differential operator.

Exercise 1.1: In a coordinate basis, you can check

$$[X, Y] = \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\right) \frac{\partial}{\partial x^\nu}. \tag{1.13}$$

> **Remark:** As the course progresses, we will see connections with the *Symmetries, Field and Particles* course and its treatment of Lie algebras though these commutators.

Exercise 1.2: Check that the *Jacobi identity* holds:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{1.14}$$

1.4 Integral Curves

There is a relationship between vector fields and streamlines, also known as flows, on a manifold $\mathcal{M}$.

Definition 6 (flow)**:** A *flow* on $\mathcal{M}$ is a one-parameter family of diffeomorphisms $\sigma_t : \mathcal{M} \to \mathcal{M}$, labelled by $t \in \mathbb{R}$, such that $\sigma_{t=0} = id$ and $\sigma_s \circ \sigma_t = \sigma_{s+t}$.

This defines *flow lines* on $\mathcal{M}$ as illustrated in For each point on the manifold, there is a unique such streamline that flows through it. Each line has coordinates $x^\mu(t)$.

Definition 7 (vector field)**:** We define a vector field X by taking the tangent vector at each point:

$$X^\mu(x(t)) = \frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t}. \tag{1.15}$$

Definition 8 (integral curves)**:** Conversely, given a vector field $X^\mu(x)$, we can integrate (1.15) to find $x^\mu(t)$. These flow lines are called *integral curves*.

Example: On S^2, with (θ, ϕ) polar coordinates, consider $X = \frac{\partial}{\partial \phi}$. From (1.15), $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}\phi}{\mathrm{d}t} = 1$. This means that we have constant $\theta = \theta$.

$$\phi = \phi_0 + t \implies \sigma_t(\theta, \phi) \to (\theta, \phi + t) \tag{1.16}$$

The integral curves are lines of constant lattitude.

1.5 Lie Derivatives

We have learned that vector fields differentiate scalar functions. But how can we differentiate vector fields? Given $X, Y \in \mathfrak{X}(\mathcal{M})$, is there some way to know how Y changes in the direction of X? There is a problem here. For a function on $\mathbb{R}$:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{h\to 0}\left\{\frac{f(t+h)-f(t)}{f(t)}h\right\}. \tag{1.17}$$

Similarly, to differentiate $Y \in \mathfrak{X}(\mathcal{M})$, we need to compare $Y_p \in T_p\mathcal{M}$ with some neighbouring $Y_q \in T_q\mathcal{M}$. The problem is that these are in different vector spaces! As each vector space has different coordinates, simply working with component values gives us a bad definition of a derivative; one which is dependent of coordinate choice. We cannot just add or subtract one vector from another. To differentiate Y, we need to understand how to map vectors in $T_q(\mathcal{M})$ to vectors in $T_p(\mathcal{M})$, so we can compare them.

1.5.1 Push-Forward and Pull-Back

First, we think more generally about maps of the form $\varphi : \mathcal{M} \to \mathcal{N}$, where $\mathcal{N}$ and $\mathcal{M}$ can be manifolds of different dimensions.

Definition 9 (pull-back)**:** Suppose $\varphi : \mathcal{M} \to \mathcal{N}$ is a smooth map between manifolds. Given a smooth function $f : \mathcal{N} \to \mathbb{R}$ on $\mathcal{N}$, the *pull-back* of f by φ is the smooth function $\varphi^* f$ on $\mathcal{M}$ defined by

$$\begin{aligned}(\varphi^* f) &: \mathcal{M} \to \mathbb{R}\\ (\varphi^* f)(p) &\longmapsto f(\varphi(p)).\end{aligned} \tag{1.18}$$

Definition 10 (push-forward)**:** If we have a vector field $Y \in \mathfrak{X}(\mathcal{M})$, then we can construct a new vector field $(\varphi_* Y) \in \mathfrak{X}(\mathcal{N})$ as

$$[(\varphi_* Y)(f)]\,(\varphi(p)) = [Y(\varphi^* f)]\,(p). \tag{1.19}$$

This is called the *push-forward*.

Let x^μ be coordinates on $\mathcal{M}$ and y^μ be coordinates on Y. If $\varphi(x) = y^\alpha(x)$ and $Y : Y^\mu\partial_\mu \in \mathfrak{X}(\mathcal{M})$ is the vector field on $\mathcal{M}$, then we can write the push-forward on $\mathcal{N}$ as

$$(\varphi_* Y)(f) = Y^\mu \frac{\partial f(y(x))}{\partial x^\mu} = Y^\mu \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial f(y)}{\partial y^\alpha}. \tag{1.20}$$

In components:

$$(\varphi_* Y)^\alpha = Y^\mu \frac{\partial y^\alpha}{\partial x^\mu}. \tag{1.21}$$

Some objects are naturally pulled back and some are pushed forward. Note that if we have a diffeomorphism φ^{-1}, we can go both ways.

1.5.2 Differentiation by Lie Derivative

Now we can use this to differentiate. The idea is that given a vector field $X \in \mathfrak{X}(\mathcal{M})$, we can define a flow $\sigma_t : \mathcal{M} \to \mathcal{M}$. We use this to push forward vectors at $T_p\mathcal{M}$ to $T_{\sigma_t(p)}\mathcal{M}$. This is called the *Lie derivative*, denoted $\mathcal{L}_X$. Let us first look at functions:

$$\mathcal{L}_X f = \lim_{t\to 0}\left\{\frac{f(\sigma_t(x)) - f(x)}{t}\right\} \tag{1.22}$$

$$= \frac{\mathrm{d}f(\sigma_t(x))}{\mathrm{d}t}\bigg|_{t=0} = \frac{\partial f}{\partial x^\mu}\,\frac{\partial x^\mu}{\partial t}\bigg|_{t=0} \tag{1.23}$$

$$= X^\mu\,\frac{\partial f}{\partial x^\mu} = X(f) \tag{1.24}$$

As $X^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}$, we have

$$\mathcal{L}_X f = X^\mu(x)\,\frac{\partial f}{\partial x^\mu} = X(f). \tag{1.25}$$

The action of the Lie derivative $\mathcal{L}_X$ on a function is the same as the action of the vector field X. Now we differentiate vector fields: The Lie derivative of a vector field is defined by

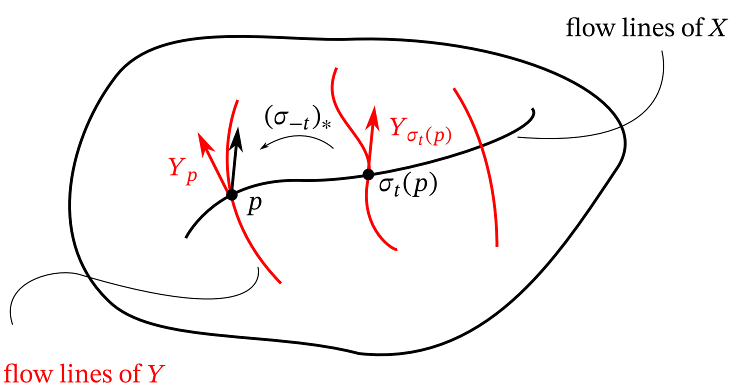

Figure 1.3: The Lie derivative uses the push-forward $(\sigma_{-t})_*$ to be able to compare vectors at a point $p \in \mathcal{M}$ and a bit further along the flow at $\sigma_t(p)$ defined by the vector field X.

$$\mathcal{L}_X Y\big|_p = \lim_{t\to 0}\left\{\frac{((\sigma_{-t})_* Y_p - Y_p}{t}\right\}. \tag{1.26}$$

The minus sign is needed to push with the inverse flow map, $\sigma_{-t} : \sigma_t(p) \longmapsto p$. Let us calculate the action of the Lie derivative on a basis $\{\partial_\mu = \partial/\partial x^\mu\}$:

$$\mathcal{L}_X \partial_\mu = \lim_{t\to 0}\left\{\frac{(\sigma_{-t})_*\partial_\mu - \partial_\mu}{t}\right\}. \tag{1.27}$$

The components of the push-forward are

$$(\varphi_* f)^\alpha = Y^\mu\,\frac{\partial y^\alpha}{\partial x^\mu} \tag{1.28}$$

Now, we use the expression for the push-forward of a tangent vector as given by Equation (1.21), except that we replace the coordinates y^α on $\mathcal{N}$ by the infinitesimal coordinate change induced by the flow σ_t:

$$y^\mu = x^\mu - tX^\mu + \dots \tag{1.29}$$

Therefore, we have for small t:

$$(\sigma_{-t})_* \partial_\mu = (\delta^\nu_\mu - t\frac{\partial X^\nu}{\partial x^\mu} + \dots)\partial_\nu. \tag{1.30}$$

And finally, we see that the Lie derivative acts on a coordinate basis as

$$\mathcal{L}_X \partial_\mu = -\frac{\partial X^\nu}{\partial x^\mu}\partial_\nu. \tag{1.31}$$

To find out how the $\mathcal{L}_X$ acts on a general vector field $Y = Y^\mu \partial_\mu$, we can use Leibniz' formula

$$\mathcal{L}_X Y = \mathcal{L}_X(Y^\mu \partial_\mu) = \mathcal{L}_X(Y^\mu)\partial_\mu + Y^\mu \mathcal{L}_x \partial_\mu \tag{1.32}$$

$$= X^\nu \frac{\partial Y^\mu}{\partial x^\nu}\partial_\mu - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\partial_\nu \tag{1.33}$$

$$= [X, Y]. \tag{1.34}$$

This is a surprising fact! We can think of the commutator as describing how Y changes in the direction of X.

Corollary: From the Jacobi identity of commutators, we have for vector fields X, Y, Z:

$$[\mathcal{L}_X, \mathcal{L}_Y]Z = \mathcal{L}_{[X,Y]}Z. \tag{1.35}$$

1.6 Tensors

Dual Spaces

Definition 11 (dual space)**:** Let V be a vector space. The *dual vector space* V^* is the space of all linear maps from V to $\mathbb{R}$.

The easiest way to define the dual space is to work with a given basis $\{e_\mu\}$, $\mu = 1, \dots, n$ of V. We can then introduce a dual basis $\{f^\mu\}$, $\mu = 1, \dots, n$ for V^*, defined by

$$f^\nu(e_\mu) = \delta^\nu_\mu. \tag{1.36}$$

The dual basis elements f^μ are linear maps which pick out their corresponding basis element e_μ and send it to unity. By definition, a general vector $X \in V$ can be decomposed in this basis as $X = X^\mu e_\mu$. Since the maps are linear, the dual basis elements act on a general vector hence as $f^\nu(X) = X^\mu f^\nu(e_\mu) = X^\nu$. For a given basis, the correspondence between the bases provides an isomorphism between V and V^*. In particular, this means that $\dim V = \dim V^*$. However, this map $e_\mu \longmapsto f^\mu$ depends on the basis and is thus a bad isomorphism to work with in a coordinate invariant theory. We can repeat the construction to show that there is a natural isomorphism $(V^*)^* \to V$, which is independent of the choice of basis.

1.6.1 One-Forms

Definition 12 (cotangent space and vector)**:** The *cotangent space* $T^*_p(\mathcal{M})$ at a point $p \in \mathcal{M}$ is the dual space to the tangent space $T_p(\mathcal{M})$ at p. Elements of the cotangent space are called *cotangent vectors* or *covectors*.

Given a basis $\{e_\mu\}$ of the tangent space $T_p(\mathcal{M})$, we can introduce a dual basis $\{f^\mu\}$ for $T^*_p(\mathcal{M})$ and expand any covector as $\omega = \omega_\mu f^\mu$.

Definition 13 (one-form)**:** A *one-form* or *cotangent field* is a smooth assignment of cotangent vectors taken from the cotangent spaces across all points in the manifold. These one-forms are maps from vector fields to real numbers.

Notation: The set of all one-forms on $\mathcal{M}$ is denoted $\Lambda^1(M)$.

Definition 14 (differential)**:** The *differential* df of a smooth function f on $\mathcal{M}$ is a special one-form. It is defined by its action on a vector field X:

$$\begin{aligned} df \in \Lambda^1(\mathcal{M}) :\ \mathfrak{X} &\to \mathbb{R} \\ X &\longmapsto X(f). \end{aligned} \tag{1.37}$$

These differentials can be used to find a basis for the space $\lambda^1(\mathcal{M})$ of all one-forms on $\mathcal{M}$: Let $\{x^\mu\}$ be coordinates on $\mathcal{M}$. This defines a coordinate basis $\{\frac{\partial}{\partial x^\mu} = \partial_\mu\}$ of vector fields. Then, for each value of μ, we take the function f that we used to build the differential to simply be that coordinate x^μ to give a one-form dx^μ. By definition, this acts on the basis of vector fields as

$$dx^\mu(\partial^\nu) = \partial^\nu(x^\mu) = \delta^\mu_\nu. \tag{1.38}$$

Therefore, the set $\{dx^\mu\}$ provides a basis for $\Lambda^1(\mathcal{M})$ that is dual to the coordinate basis $\{\partial_\mu\}$. In general, an arbitrary one-form $\omega \in \Lambda^1(\mathcal{M})$ can hence be expanded as $\omega = \omega_\mu dx^\mu$.

Claim 1: In this basis, the coefficients ω_μ are the partial derivatives; the differential can be expanded as

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu. \tag{1.39}$$

Proof. We can check that this expansion satisfies Equation (1.37) by acting with it on a vector field X:

$$df(X) = \frac{\partial f}{\partial x^\mu} dx^\mu(X^\nu \partial_\nu) = X^\mu \frac{\partial f}{\partial x^\mu} = X(f). \tag{1.40}$$

□

How do one-forms transform under a change of coordinates? Recall that given two charts $\phi = (x^1, \dots, x^n)$ and $\widetilde{\phi} = (\tilde{x}^1, \dots, \tilde{x}^n)$, the partial derivatives, which form the basis of vector fields, are related via the chain rule:

$$\frac{\partial}{\partial \tilde{x}^\mu} = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \frac{\partial}{\partial x^\nu}. \tag{1.41}$$

Since the differentials form the corresponding dual basis, they transform in the inverse manner

$$d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu. \tag{1.42}$$

Claim 2: This transformation law ensures that the new basis $\{d\tilde{x}^\mu\}$ is also dual to the new chart $\widetilde{\phi}$.

Proof. This is an exercise in definitions and symbol manipulation: We need to show that

$$d\tilde{x}^\mu \left(\frac{\partial}{\partial \tilde{x}^\nu} \right) = \delta^\mu_\nu. \tag{1.43}$$

Starting from the left hand side, we first use the two transformation laws

$$d\tilde{x}^\mu \left(\frac{\partial}{\partial \tilde{x}^\nu} \right) = \frac{\partial \tilde{x}^\mu}{\partial x^\rho} dx^\rho \left(\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} \frac{\partial}{\partial x^\sigma} \right) \tag{1.44}$$

Then we can pull out the factor $\partial x^\sigma / \partial \tilde{x}^\nu$ since the one-form dx^ρ only acts on the vector field ∂^σ:

$$\cdots = \frac{\partial \tilde{x}^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial \tilde{x}^\nu} dx^\rho \left(\frac{\partial}{\partial x^\sigma} \right) = \frac{\partial \tilde{x}^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial \tilde{x}^\nu} \delta^\rho_\sigma = \frac{\partial \tilde{x}^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial \tilde{x}^\rho}. \tag{1.45}$$

Lastly, we use the fact that these change of variable matrices are inverses

$$\cdots = \frac{\partial \tilde{x}^\mu}{\partial x^\rho}\frac{\partial x^\rho}{\partial \tilde{x}^\nu} = \delta^\mu_\nu. \tag{1.46}$$

□

Therefore, we can expand a one-form ω in the new basis as well

$$\omega = \omega_\mu dx^\mu = \tilde{\omega}_\mu d\tilde{x}^\mu \quad \text{with} \quad \tilde{\omega}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu}\omega_\nu. \tag{1.47}$$

We say that these transformations of covectors are *covariant* transformations.

1.6.2 Lie Derivative of One-Forms

Example: A $(2,1)$ tensor acts on the one-forms $\omega, \eta \in \Lambda^1(\mathcal{M})$ on the manifold as

$$T(\omega,\eta;X) = T(\omega_\mu f^\mu, \eta_\mu f^\nu; X^\rho e_\rho) = \omega_\mu \eta_\nu X^\rho T^{\mu\nu}{}_\rho, \tag{1.48}$$

where $X \in \mathfrak{X}(\mathcal{M})$. Consider a coordinate transformation $\tilde{e}_\nu = A^\mu{}_\nu e_\mu$ with $A^\mu{}_\nu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}$ and $\tilde{f}^\rho = B^\rho{}_\sigma f^\sigma$ with $B^\rho{}_\sigma = \frac{\partial \tilde{x}^\rho}{\partial x^\sigma}$. Then a $(2,1)$ tensor transforms as

$$\tilde{T}^{\mu\nu}{}_\rho = B^\mu{}_\sigma B^\nu{}_\tau A^\lambda{}_\rho T^{\sigma\tau}{}_\lambda. \tag{1.49}$$

This is often taken as a definition of a tensor. What is really happening is that the tensor is a *linear* map.

Motivation

We care about tensors since we want the physical equations to be independent of the coordinates we are using. To actually do calculations, we cannot use this abstract notation, but we really need to introduce coordinates. However, we need to make sure that the results do not depend on coordinates, but are physically meaningful. This is why tensors are important for us.

1.6.3 Tensor operations

There are a number of operations that we can perform on tensors:

- vector space: can add / subtract or multiply by functions
- tensor product: if S has rank (p,q) and T has rank (r,s), then $S \otimes T$ has rank $(p+r, q+s)$ defined by

$$\begin{aligned} S \otimes T(\omega_1, \dots, \omega_p, \eta_1, \dots, \eta_r; X_1, \dots, X_q, Y_1, \dots Y_s) \\ = S(\omega_1, \dots, \omega_p; X_1, \dots, X_q) T(\eta_1, \dots, \eta_r; Y_1, \dots, Y_s). \end{aligned} \tag{1.50}$$

  In components, this is

$$(S \otimes T)^{\mu_1 \dots \mu_p \nu_1 \dots \nu_r}{}_{\rho_1 \dots \rho_s \sigma_1 \dots \sigma_s} = S^{\eta_1 \dots \eta_p}{}_{\rho_1 \dots \rho_q} T^{\nu_1 \dots \nu_r}{}_{\sigma_1 \dots \sigma_s}. \tag{1.51}$$

- contraction: We can turn an (r,s)-tensor into an $(r-1, s-1)$-tensor. If T is a $(2,1)$ tensor, then

$$S(\omega) = T(\omega, f^\mu; e_\mu) \tag{1.52}$$

  Since we sum over μ, this is basis independent. In components, this is

$$S^\mu = T^{\mu\nu}{}_\nu. \tag{1.53}$$

Remark: This is typically different from $(S')^\mu = T^{\nu\mu}{}_\nu$.

- (anti)-symmetrisation: For example, given a $(0,2)$-tensor T, we can define two new $(0,2)$-tensors

$$S(X,Y) = \frac{1}{2}\left(T(X,Y) + T(Y,X)\right) \qquad A(X,Y) = \frac{1}{2}\left(T(X,Y) - T(Y,X)\right). \tag{1.54}$$

In terms of components, we write

$$S_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) \qquad A_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) \tag{1.55}$$

Remark: This is similar to how we can decompose a matrix into its symmetric and anti-symmetric components.

Notation: We write

$$T_{(\mu\nu)} := \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) \qquad T_{[\mu\nu]} := \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}). \tag{1.56}$$

We can also (anti)-symmetrise over multiple indices.

Example:

$$T^\mu{}_{(\nu\rho\sigma)} = \frac{1}{3!}(T^\mu{}_{\nu\rho\sigma} + 5 \text{ permutations}) \tag{1.57}$$

$$T^\mu{}_{[\nu\rho\sigma]} = \frac{1}{3!}(T^\mu{}_{\nu\rho\sigma} + \text{sign(perm)} \times \text{ permutations}) \tag{1.58}$$

$$\tag{1.59}$$

In general, we divide by $p!$, where p is the number of indices we (anti)-symmetrise over.

- We can define a Lie derivative $\mathcal{L}_X$ on a tensor field.

The next part is ε more interesting than all these preceding definitions. But the good thing is that $\varepsilon > 0$.

1.6.4 Differential Forms

Definition 15 (p-forms)**:** Totally anti-symmetric $(0,p)$-tensors are called *p-forms*. The space of all p-forms is denoted $\Lambda^p(\mathcal{M})$.

Example: 0-forms are simply functions.

In general, if the dimension of the manifold is $\dim M = n$, then p-forms have $\binom{n}{p}$ independent components. n-forms are called *top-forms*.

Definition 16 (wedge product)**:** Given $\omega \in \Lambda^p(\mathcal{M})$ and $\eta \in \Lambda^q(\mathcal{M})$, we can form a $(p+q)$-form by taking the tensor product and anti-symmetrising. This is the *wedge product*

$$(\omega \wedge \eta)_{\mu_1 \dots \mu_p \nu_1 \dots \nu_q} = \frac{(p+q)!}{p!q!} \omega_{[\mu_1 \dots \mu_p} \eta_{\nu_1 \dots \nu_q]} \tag{1.60}$$

Example: For one-forms, we have $(\omega \wedge \eta)_{\mu\nu} = \omega_\mu \eta_\nu - \omega_\nu \eta_\mu$.

Exercise 1.3: Show that in general $\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega$.

Moreover, if you take the wedge product with a form by itself, you get $\omega \wedge \omega =$ for odd forms, but not necessarily for even forms.

Example (A wedge we have seen before)**:** For $\mathcal{M} = \mathbb{R}^3$ and $\omega, \eta \in \Lambda^1(\mathcal{M})$, pick some coordinates x_1, x_2, x_3 and expand the one-form in the coordinate basis as

$$\begin{aligned}(\omega \wedge \eta) &= (\omega_1 dx^1 + \omega_2 dx^2 + \omega_3 dx^3) \wedge (\eta_1 dx^1 + \eta_2 dx^2 + \eta_3 dx^3) &(1.61)\\ &= (\omega_1\eta_2 - \eta_1\omega_2) dx^1 \wedge dx^2 + (\omega_2\eta_3 - \eta_2\omega_3) dx^2 \wedge dx^3 + (\omega_3\eta_1 - \eta_3\omega_1) dx^3 \wedge dx^1 &(1.62)\end{aligned}$$

These are the components of the *cross-product* in $\mathbb{R}^3$! The cross-product is really a wedge product between forms. We thus find out that we really always got back a two-form by taking the cross-product between two one-forms. However, as we will see later, there is a natural correspondence between two-forms and one-forms.

In a coordinate basis, we write

$$\omega = \frac{1}{p!} \omega_{\eta_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}. \tag{1.63}$$

1.7 The Exterior Derivative

This is the second of three different derivatives we will meet, the first having been the Lie derivative. Given a function f, we can construct a 1-form

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu. \tag{1.64}$$

In general, there is a map $d : \Lambda^p(\mathcal{M}) \to \Lambda^{p+1}(\mathcal{M})$. This is called the *exterior derivative*.

Remark: The one that will be coming up is the *covariant derivative*. With the Lie derivative and the covariant one, we get back an object that is the same as the one we feed in. Here, we get an object that is one dimension higher.

There is also a (horrendous) coordinate-free definition, but it is easiest to work in coordinates, where

$$d\omega = \frac{1}{p!}\frac{\partial \omega_{\mu_1 \dots \mu_p}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \tag{1.65}$$

or

$$(d\omega)_{\mu_1 \dots \mu_{p+1}} = (p+1)\partial_{[\mu_1} \omega_{\mu_2 \dots \mu_{p+1}]}. \tag{1.66}$$

Again, we have seen this before in a different guise; we will see this later. Because of anti-symmetry $d(d\omega) = 0$. We write this as $d^2 = 0$.

Remark: Whenever there is a second derivative, even with such a simple equation, there are beautiful things that follow. We will see this later.

It is simple to show the following properties

- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\omega$
- for pull-backs: $d(\varphi^* \omega) = \varphi^* d\omega$
- $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X \omega)$

Definition 17: A p-form is *closed* if $d\omega = 0$ everywhere.

Definition 18: A p-form is *exact* if $\omega = d\eta$ everywhere for some η.

Corollary: The statement that $d^2 = 0$ implies that exact p-forms are automatically closed.

Lemma 1 (Poincaré's Lemma)**:** On $\mathbb{R}^n$, or locally on $\mathcal{M}$,

$$\text{closed} \quad \Rightarrow \quad \text{exact}. \tag{1.67}$$

Example (one-form on $\mathbb{R}^3$)**:** Consider the 1-form $\omega = \omega_\mu(x)dx^\mu$. Then

$$(d\omega)_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu \quad \text{or} \quad d\omega = \frac{1}{2}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu)dx^\mu \wedge dx^\nu. \tag{1.68}$$

On $\mathbb{R}^3$, we have

$$d\omega = (\partial_1\omega_2 - \partial_2\omega_1)dx^1 \wedge dx^2 + (\partial_2\omega_3 - \partial_3\omega_2)dx^2 \wedge dx^3 + (\partial_3\omega_1 - \partial_1\omega_3)dx^3 \wedge dx^1. \tag{1.69}$$

Here we have used the property of the wedge product that $dx^i \wedge dx^i = 0$. These three terms are the components of the curl $\boldsymbol{\nabla} \times \boldsymbol{\omega}$. The correct way to think about the curl is as an exterior derivative as a one-form, which coincidentally has three components as well.

Example (two form on $\mathbb{R}^3$)**:** Consider $B \in \Lambda^2(\mathbb{R}^3)$ with

$$B = B_1(x)dx^2 \wedge dx^3 + B_2(x)dx^3 \wedge dx^1 + B_3(x)dx^1 \wedge dx^2. \tag{1.70}$$

Taking the exterior derivative of this two-form, we get a three form. However, this is a top-form in $\mathbb{R}^3$, so we only have once component.

$$dB = (\partial_1B_1 + \partial_2B_2 + \partial_3B_3)dx^1 \wedge dx^2 \wedge dx^3. \tag{1.71}$$

We have seen this before as well. These are the components of the divergence $\boldsymbol{\nabla} \cdot \mathbf{B}$.

Example: In electromagnetism, the gauge field A^μ should be thought of as a one-form $A \in \Lambda^1(\mathbb{R}^4)$. In components, this is $A = A_\mu dx^\mu$. Taking the exterior derivative we get

$$F = dA = F_{\mu\nu}dx^\mu \wedge dx^\nu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)dx^\mu \wedge dx^\nu. \tag{1.72}$$

Gauge transformations act as $A \to A + d\alpha$, where $\alpha \in \Lambda^0(\mathcal{M}) = C^\infty(\mathcal{M})$. Under this transformation, the field strength is invariant since

$$F = dA \to d(A + d\alpha) = dA. \tag{1.73}$$

Moreover, since $F = dA$ is exact, we have $dF = d^2A = 0$. This is the *Bianchi identity*, which is equivalent to two of the Maxwell equations. We need one more ingredient to write the other two Maxwell equations in terms of differential forms.

1.8 Integration

On a manifold, we integrate forms.

Definition 19 (volume form)**:** A *volume form* v, also called an *orientation*, is a nowhere-vanishing top form. Locally, it can be written as

$$v = v(x)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^\mu, \tag{1.74}$$

with $v(x) \neq 0$ everywhere.

There are a bunch of subtleties here; for some manifold, it is impossible to find a form like this. However, these are not useful in GR; which is why we will brush these subtleties under the carpet. If such a form exists, $\mathcal{M}$ is said to be orientable. The subtleties mean that not all manifolds are orientable, e.g. the Möbius strip, or real projective space $\mathbb{RP}^n$, with n even.

Given a volume form, we can integrate any function $f : \mathcal{M} \longmapsto \mathbb{R}$ over $\mathcal{M}$. To do this, we map it to $\mathbb{R}$ via a chart, and then integrate over $\mathbb{R}^n$ as we would usually do. In a chart $\mathcal{O} \in \mathcal{M}$, we have

$$\int_{\mathcal{O}} f v = \int_U dx^1 \dots dx^\mu f(x) v(x). \tag{1.75}$$

We then sum over patches to integrate over $\mathcal{M}$.

> **Remark:** In the language of integration, the volume form is the measure. This tells us how to weight functions on the manifold. This is because we have no notion of distance between points on the manifold without it.

Definition 20 (submanifold)**:** A manifold Σ of dimension $k < n$ is a *submanifold* of $\mathcal{M}$ if there exists a bijection $\phi : \Sigma \to \mathcal{M}$, which embeds Σ into $\mathcal{M}$, such that $\phi_* : T_p(\Sigma) \to T_p(\mathcal{M})$ is also a bijection.

These definitions make sure that everything is nice and smooth. We can then integrate any $\omega \in \Lambda^k(\mathcal{M})$ over the submanifold Σ by

$$\int_{\phi(\Sigma)} \omega = \int_\Sigma \phi^* \omega, \tag{1.76}$$

where ω^* is the pull-back.

Example: Let σ be a map $\sigma : C \subset \mathbb{R} \to \mathcal{M}$. This defines a non-intersecting curve in $\mathcal{M}$, parametrised by τ. Then, if $A \in \Lambda^1(\mathcal{M})$, in coordinates x^μ, the integral is

$$\int_{\sigma(C)} A = \int_C \sigma^* A = \int \mathrm{d}\tau\, A_\mu(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}. \tag{1.77}$$

The action in Minkowski space is obtained by pulling back the one-form in Minkowski space $\mathbb{M}^4$ to the real line, and then integrating over this line.

Theorem 2 (Stokes' Theorem)**:** Let $\mathcal{M}$ be a manifold with boundary ∂M, as illustrated in 1.4. If $\omega \in \Lambda^{n-1}(\mathcal{M})$

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega. \tag{1.78}$$

Example: Let $\mathcal{M}$ be the one-dimensional interval I with $x \in [a, b]$. The zero-form $\omega(x)$ is a function, and $d\omega = \frac{\mathrm{d}\omega}{\mathrm{d}x} dx$ is a one-form. Stokes' theorem says that

$$\int_{\mathcal{M}} d\omega = \int_a^b \frac{\mathrm{d}\omega}{\mathrm{d}x}\,\mathrm{d}x \quad \text{and} \quad \int_{\partial\mathcal{M}} \omega = \omega(b) - \omega(a) \tag{1.79}$$

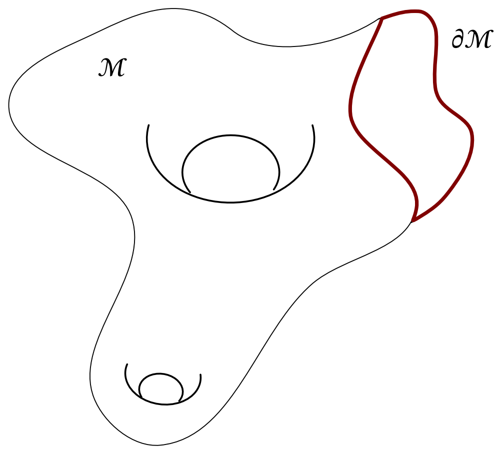

Figure 1.4: Manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$.

where the minus sign is related to the subtleties at the boundary, which we have previously swept under the carpet. This is the *fundamental theorem of calculus*.

Example: Let $\mathcal{M} \subset \mathbb{R}^2$ and $\omega \in \Lambda^1(\mathcal{M})$. Then

$$\int_{\mathcal{M}} d\omega = \int_{\mathcal{M}} \left(\frac{\partial\omega_2}{\partial x^1} - \frac{\partial\omega_1}{\partial x^2}\right) dx^1 \wedge dx^2 \tag{1.80}$$

$$\text{and} \quad \int_{\partial\mathcal{M}} \omega = \int_{\partial\mathcal{M}} \omega_1 dx^1 + \omega_2 dx^2. \tag{1.81}$$

The equality between left and right hand sides is *Green's theorem* in the plane.

Example: Let $\mathcal{M} \subset \mathbb{R}^3$ and $\omega \in \Lambda^2(\mathcal{M})$. Then

$$\int_{\mathcal{M}} d\omega = \int dx^1 dx^2 dx^3 \left(\partial_1\omega_1 + \partial_2\omega_2 + \partial_3\omega_3\right) \tag{1.82}$$

$$\text{and} \quad \int_{\partial\mathcal{M}} \omega = \int_{\partial\mathcal{M}} \omega_1 dx^2 dx^3 + \omega_2 dx^3 dx^1 + \omega_3 dx^1 dx^2. \tag{1.83}$$

The equality between these two sides is *Gauss' divergence theorem*.

2 Introducing Riemannian Geometry

Riemannian geometry is differential geometry with a metric. You can tell that the metric is important, because it literally changes the name of the whole subject. As we will see, we should really be differentiating between Riemannian and Lorentzian manifolds, but one should be aware that people often just refer to a manifold with metric as a Riemannian manifold.

2.1 The Metric

The metric defines an inner product on the space of tangent vectors. It has a few properties, which guarantee that the metric is a good inner product.

Definition 21 (metric)**:** A *metric* g is a $(0,2)$-tensor that is

- symmetric: $g(X,Y) = g(Y,X)$
- non-degenerate: if $g(X,Y)_p = 0$ for all $Y_p \in T_p(\mathcal{M})$, then $X_p = 0$

> **Remark:** The non-degeneracy ensures that the metric is invertible.

In a particular coordinate basis, the metric gets two indices $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$, with $g_{\mu\nu} = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$.

Notation: We often write this as a *line element* $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

Definition 22 (signature)**:** If we diagonalise $g_{\mu\nu}$, it has positive and negative elements (none are zero, since it is non-degenerate). The number of negative elements is called the *signature* of the metric.

> **Remark:** There is a theorem (Sylvester's law of inertia) in matrix algebra that says that the signature is unchanged under a change of basis.

There are two metrics that are going to be of interest to us.

2.1.1 Riemannian Metrics

Definition 23: A *Riemannian manifold* is a manifold with metric with signature $(+ + \cdots +)$.

Example: Euclidean space is $\mathbb{R}^n$ with $g = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n$.

A metric gives us a way to measure

- the length of a vector $X \in T_p(\mathcal{M})$
$$|X| = \sqrt{g(X, X)}, \tag{2.1}$$
- the angle between vectors
$$g(X, Y) = |X||Y| \cos\theta, \tag{2.2}$$
- the distance between two points p and q along a curve $\sigma : [a, b] \to \mathcal{M}$, with end points $\sigma(a) = p$ and $\sigma(b) = q$
$$\text{distance} = \int_a^b \mathrm{d}t \sqrt{g(X, X)|_{\sigma(t)}}, \tag{2.3}$$
where X is tangent to the curve. If the curve has coordinates $x^\mu(t)$, this is
$$\text{distance} = \int_a^b \mathrm{d}t \sqrt{g_{\mu\nu}(t) \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \frac{\mathrm{d}x^\nu}{\mathrm{d}t}} \tag{2.4}$$

2.1.2 Lorentzian Metrics

Definition 24: A *Lorentzian manifold* is a manifold equipped with a metric of signature $(- + + \cdots +)$.

Example: *Minkowski space* is $\mathbb{R}^n$ with metric
$$\eta = -dx \otimes dx^0 + dx^1 \otimes dx^1 + \cdots + dx^{n-1} \otimes dx^{n-1} \tag{2.5}$$
with components $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$.

Because of this minus sign in the metric, 'lengths' and 'distances'—in the sense of inner products of vectors, can be negative. We classify vectors $X \in T_p(\mathcal{M})$ as

$$g(X,X)\begin{cases} < 0 & \text{timelike} \\ = 0 & \text{null} \\ > 0 & \text{spacelike} \end{cases}. \tag{2.6}$$

At each point $p \in \mathcal{M}$, we can draw null tangent vectors called *lightcones*.

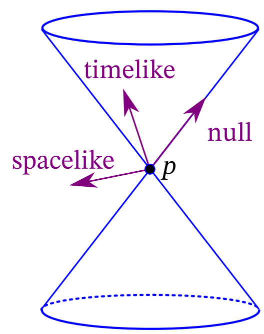

Figure 2.1: The causal structure around a point p in a Lorentzian manifold.

Definition 25: A curve is called *timelike* if its tangent vector is everywhere timelike.

Definition 26 (proper time)**:** In this case, we can measure the distance between two points

$$\tau = \int_a^b \mathrm{d}t \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}}. \tag{2.7}$$

This is the *proper time* between a and b.

> **Remark:** The parametrisation t is arbitrary.

2.1.3 The Joys of a Metric

The metric gives a natural (basis independent) isomorphism at every point

$$\begin{aligned} g : T_p(\mathcal{M}) &\to T_p^*(\mathcal{M}) \\ X &\longmapsto g(X,\bullet). \end{aligned} \tag{2.8}$$

Given a vector field $X \in \mathfrak{X}(\mathcal{M})$, we can construct $g(X,\bullet) \in \Lambda^1(\mathcal{M})$. If $X = X^\mu \partial_\mu$, then the corresponding 1-form is

$$g_{\mu\nu}X^\mu dx^\nu := X_\nu dx^\nu. \tag{2.9}$$

Because g is non-degenerate, there is an inverse. In components, $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$. This defines a rank $(2,0)$ tensor $\hat{g} = g^{\mu\nu}\partial_\mu \otimes \partial_\nu$. We use this to raise indices

$$X^\mu := g^{\mu\nu}X_\nu. \tag{2.10}$$

> **Remark:** As physicists, we often say that we can use the metric to *raise* and *lower* indices. This is what we really mean with that; the covariant and contravariant vectors really live in different mathematical spaces, and the *lowering* of an index is the statement that the metric provides a natural isomorphism. This isomorphism allowed us to jump between vectors and 1-forms without even worrying about the fact that they are different objects. In spaces other than Euclidean space, their difference becomes important.

The metric also gives a natural volume form.

- On a Riemannian manifold,

$$v = \sqrt{\det g_{\mu\nu}}dx^1 \wedge \cdots \wedge dx^n. \tag{2.11}$$

  We usually write $g = \det g_{\mu\nu}$.

- On a Lorentzian manifold,

$$v = \sqrt{-g}dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{n-1}. \tag{2.12}$$

Claim 3: This is independent of coordinates.

Proof. In new coordinates, $dx^\mu = A^\mu{}_\nu d\tilde{x}^\nu$ with $A^\mu{}_\nu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}$,

$$dx^1 \wedge \cdots \wedge dx^n = A^1{}_{\mu_1} \dots A^1{}_{\mu_n} \underbrace{d\tilde{x}^{\mu_1} \wedge \cdots \wedge d\tilde{x}^{\mu_n}}_{\text{rearrange to } d\tilde{x}^1\wedge\cdots\wedge d\tilde{x}^n} \tag{2.13}$$

$$= \sum_{\text{perm } \pi} \text{sign}(\pi) A^1{}_{\pi(1)} \dots A^n{}_{\pi(n)} d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n \tag{2.14}$$

$$= \det(A) d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n. \tag{2.15}$$

The determinant $\det A > 0$ if the coordinate change preserves orientation. Meanwhile,

$$g_{\mu\nu} = \frac{\partial \tilde{x}^\rho}{\partial x^\mu}\frac{\partial \tilde{x}^\sigma}{\partial x^\nu}\tilde{g}_{\rho\sigma} \tag{2.16}$$

$$= (A^{-1})^\rho{}_\mu (A^{-1})^\sigma{}_\nu \tilde{g}_{\rho\sigma} \tag{2.17}$$

$$\Rightarrow \det g_{\mu\nu} = (\det A^{-1})^2 \det \tilde{g}_{\rho\sigma} = \frac{\det \tilde{g}_{\rho\sigma}}{(\det A)^2} \tag{2.18}$$

$$\Rightarrow v = \sqrt{|\tilde{g}|} d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n. \tag{2.19}$$

□

In components, $v = \frac{1}{n!} v_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$, with

$$v_{\mu_1 \dots \mu_n} = \sqrt{|g|} \varepsilon_{\mu_1 \dots \mu_n}, \tag{2.20}$$

where ε is the totally anti-symmetric tensor. We can then integrate any function as

$$\int_{\mathcal{M}} f v = \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|} f(x). \tag{2.21}$$

So the metric provides the measure in the form of $\sqrt{|g|}$.

There are two more things we can do with a metric. The metric provides a map from p-forms $\omega \in \Lambda^p(\mathcal{M})$ to $n-p$-forms $(\star\omega) \in \Lambda^{n-p}(\mathcal{M})$, where $n = \dim \mathcal{M}$:

$$(\star\omega)_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!}\sqrt{|g|}\varepsilon_{\mu_1 \dots \mu_{n-p}\nu_1 \dots - n_p}\omega^{\nu_1 \dots \nu_p}. \tag{2.22}$$

This map is the *Hodge dual*.

Exercise 2.1: You can check that $\star(\star\omega) = \pm(-1)^{p(n-p)}\omega$, where the sign depends on the signature of the metric; $+$ for Riemannian and $-$ for Lorentzian metrics.

> **Remark:** For $\mathbb{R}^3$, this is what is really going on when we were able to conflate 1-forms and 2-forms.

We can then define an inner product on forms.

Definition 27 (inner product on forms)**:** Given $\omega, \eta \in \Lambda^p(\mathcal{M})$, let

$$\langle \eta, \omega \rangle = \int_{\mathcal{M}} \underbrace{\eta \wedge \star\omega}_{\text{this is a top-form}} . \tag{2.23}$$

Claim 4: For $\omega \in \Lambda^p(\mathcal{M})$ and $\alpha \in \Lambda^{p-1}(\mathcal{M})$, then there exists an operation $d^\dagger : \Lambda^p(\mathcal{M}) \to \Lambda^{p-1}(\mathcal{M})$ such that

$$\langle d\alpha, \omega \rangle = \langle \alpha, d^\dagger \omega \rangle. \tag{2.24}$$

This operation is given by $d^\dagger = \pm(-1)^{np+n-1} \star d\star$, where the sign again depends on the signature.

> **Remark:** This is integration by parts. Compare this to the Hermitian conjugate of an operator in quantum mechanical inner products on a Hilbert space. Most of the time, we do not play with forms in QM. However, there is a close relationship between forms in differential geometry and fermions in QFT; this relationship will be fleshed out in the Lent term's *Supersymmetry* course.

Proof. On a closed manifold, Stokes' theorem implies that

$$0 = \int_{\mathcal{M}} d(\alpha \wedge \star\omega) = \int_{\mathcal{M}} \left[d\alpha \wedge (\star\omega) + (-1)^{p-1}\alpha \wedge d(\star w)\right] \tag{2.25}$$

$$= \langle d\alpha, \omega + (-1)^{p-1}\mathrm{sign}\langle \alpha, \star d \star \omega \rangle\rangle \tag{2.26}$$

Fixing the sign gives the result. □

2.2 Connections and Curvature

We have met two derivatives so far. We will now meet the final, and ultimately most useful one.

Definition 28: A *connection* is a map $\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ We write this as $\nabla(X, Y) = \nabla_X Y$ to make it look more like differentiation. Here, ∇_X is called the *covariant derivative* and it satisfies the following properties:

- linearity on the first argument: $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$ for all functions g, g
- some linearity on the second argument: $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$
- but functions in the second argument obey the Leibniz rule: $\nabla_X(fY) = f\nabla_X Y + (\nabla_X f)Y$, where $\nabla_X f := X(f)$.

Remark: This last term in the Leibniz rule is the reason why this is not a linear map, and thus not a tensor.

Remark: We often refer to the abstract object and also its components both simply as the *connection*.

On a basis $\{e_\mu\}$ of vector fields, not necessarily a coordinate basis, we write

$$\nabla_{e_\rho} e_\nu = \Gamma_{\rho\nu}{}^{\mu} e_\mu. \tag{2.27}$$

This defines Γ, but it is simply the most general form it can be. We also use the notation $\nabla_{e_\mu} = \nabla_\mu$ to make the connection look like a partial derivative. Then for a general vector

$$\nabla_X Y = \nabla_X(Y^\mu e_\mu) = X(Y^\mu)e_\mu + Y^\mu \nabla_X e_\mu. \tag{2.28}$$

$$= X^\nu e_\nu(Y^\mu)e_\mu + Y^\mu X^\nu \nabla_\nu e_\mu = X^\nu(e_\nu(Y^\mu) + \Gamma^\mu{}_{\nu\rho} Y^\rho)e_\mu. \tag{2.29}$$

Because the components X^ν sits out front, we can write $\nabla_X Y = X^\nu \nabla_n Y$ with

$$\nabla_n Y = (e_\nu(Y^\mu) + \Gamma^\mu{}_{\nu\rho} Y^\rho)e_\mu, \tag{2.30}$$

or with a slightly slippery notation $(\nabla_\nu Y)^\mu := \nabla_\nu Y^\mu = e_\nu(Y^\mu) + \Gamma^\mu{}_{\nu\rho} Y^\rho$.

Remark: For $\mathcal{L}_X$ depends on X and ∂X, so we cannot write something like "$\mathcal{L}_X = X^\mu \mathcal{L}_\mu$". This is ultimately why the covariant derivative is more useful. The flip side is that the Lie derivative comes for free, while the numbers Γ are extra structure that needs to be specified on the manifold.

If we take a coordinate basis $\{e_\mu\} = \{\partial_\mu\}$, then

$$\nabla_\nu Y^\mu = \partial_\nu Y^\mu + \Gamma^\mu{}_{\nu\rho} Y^\rho. \tag{2.31}$$

Remark: This should be familiar from previous courses in GR or fluid dynamics.

Notation: We often write the covariant derivative as $\nabla_\nu Y^\mu := Y^\mu{}_{;\nu}$ with a semicolon and the partial derivative with a comma as $\partial_\nu Y^\mu := Y^\mu{}_{,\nu}$.

Remark: The name *connection* hints that we can use this to connect tangent spaces at different points in the manifold.

Claim 5: The connection is *not* a tensor.

Proof. Consider a change of basis $\tilde{e}_\nu = A^\mu{}_\nu e_\mu$ with $A^\mu{}_\nu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}$ for a coordinate bases. Then

$$\nabla_{\tilde{e}_\rho} \tilde{e}_\nu = \tilde{\Gamma}^\mu{}_{\rho\nu} \tilde{e}_\mu \tag{2.32}$$

$$= \nabla_{(A^\sigma{}_\rho e_\sigma)} (A^\lambda{}_\nu e_\lambda) \tag{2.33}$$

$$= A^\sigma{}_\rho \nabla_\sigma (A^\lambda{}_\nu e_\lambda) \tag{2.34}$$

Using the Leibniz rule we have

$$\ldots = A^\sigma{}_\rho (A^\lambda{}_\nu \Gamma^\tau{}_{\sigma\lambda} e_\tau + e_\lambda \partial_\sigma A^\lambda{}_\nu) \tag{2.35}$$

$$= A^\sigma{}_\rho (A^\lambda{}_\nu \Gamma^\tau{}_{\sigma\lambda} + \partial_\sigma A^\tau{}_\nu) e_\tau, \qquad e_\tau = (A^{-1})^\mu{}_\tau \tilde{e}_\mu \tag{2.36}$$

$$\Rightarrow \tilde{\Gamma}^\mu{}_{\rho\nu} = (A^{-1})^\mu{}_\tau A^\sigma{}_\rho A^\lambda{}_\nu \Gamma^\tau{}_{\sigma\lambda} + \underbrace{(A^{-1})^\mu{}_\tau A^\sigma{}_\rho \partial_\sigma A^\tau{}_\nu}_{\text{this is why it is not a tensor}}. \tag{2.37}$$

□

Remark: There are other objects, where things transform as something → something + ∂other. The Maxwell tensor and the Yang-Mills gauge potential are examples of these. Mathematicians actually call both of these objects *connections*, precisely because of this property. The connections will become more obvious in the Lent term's *Applications of Differential Geometry to Physics* course.

We can also use the connection to differentiate other tensors. We simply ask that it obeys the Leibniz rule.

Example: We can use the covariant derivative to differentiate a function. Let us think of a one-form $\omega \in \Lambda^1(\mathcal{M})$ acting on a vector field $Y \in \mathfrak{X}(\mathcal{M})$ as that function. Using the Leibniz rule,

$$\nabla_X(\omega(Y)) = X(\omega(Y)) = (\nabla_X\omega)(Y) + \omega(\nabla_X Y) \tag{2.38}$$

$$\Rightarrow \ (\nabla_X\omega)(Y) = (\omega(Y)) - \omega\nabla_X(Y). \tag{2.39}$$

In terms of coordinates, we have

$$X^\mu(\nabla_\mu\omega_\nu)Y^\nu = X^\mu\partial_\mu(\omega_\nu X^\nu) - \omega_\nu X^\mu(\partial_\mu Y^\nu + \Gamma^\nu{}_{\mu\rho}Y^\rho \tag{2.40}$$

$$= X^\mu(\partial_\mu\omega_\rho - \Gamma^\nu{}_{\mu\rho}\omega_\nu)Y^\rho \tag{2.41}$$

$$\Rightarrow \ \nabla^\mu\omega_\rho = \partial_\mu\omega_\rho - \Gamma^\nu{}_{\mu\rho}\omega_\nu. \tag{2.42}$$

For a general (p,q)-tensor, we have one term for each index:

$$\nabla_\rho T^{\mu_1\dots\mu_p}{}_{\nu_1\dots\nu_q} = \partial_\rho T^{\mu_1\dots\mu_p}{}_{\nu_1\dots\nu_q} + \Gamma^{\mu_1}{}_{\rho\sigma}T^{\sigma\mu_2\dots\mu_p}{}_{\nu_1\dots\nu_q} + \cdots - \Gamma^{\sigma}_{\rho\nu_1}T^{\mu_1\dots\mu_p}{}_{\sigma\nu_2\dots\nu_q} - \dots. \tag{2.43}$$

Claim 6: Given a connection, we can construct two tensors. Let $\omega \in \Lambda^1(\mathcal{M})$ and $X, Y \in \mathfrak{X}(\mathcal{M})$, then

Torsion is a rank $(1,2)$-tensor (field)

$$T(\omega; X, Y) := \omega(\nabla_X Y - \nabla_Y X - [X,Y]) \tag{2.44}$$

We can also think of T as a map

$$\begin{aligned} T:\ \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) &\to \mathfrak{X}(\mathcal{M}) \\ (X,Y) &\longmapsto T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \end{aligned} \tag{2.45}$$

Curvature is a rank $(1,3)$-tensor

$$R(\omega; X, Y, Z) = \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \tag{2.46}$$

This is the *Riemann tensor*. We can also think of it as a map

$$\begin{aligned} R:\ \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) &\to \text{differential operator on } \mathfrak{X}(\mathcal{M}) \\ (X,Y) &\longmapsto R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \end{aligned} \tag{2.47}$$

Proof. To show that these are tensors, we need to check linearity in all arguments.

Example:

$$T(\omega; fX, Y) = \omega(\nabla_{fX} Y - \nabla_Y (fX) - [fX, Y]) \tag{2.48}$$

$$= \omega\left[f\nabla_X Y - f\nabla_Y X - Y(f)\cdot X - (f[X,Y] - Y(f)X)\right] \tag{2.49}$$

$$= f\omega(\nabla_X Y - \nabla_Y X - [X,Y]) = fT(\omega; X, Y) \tag{2.50}$$

Similar calculations for X, Y, Z for torsion and curvature show that these are all linear. □

In a coordinate basis $\{e_\mu = \partial_\mu\}$ and $\{f^\mu = dx^\mu\}$, we have

$$T^\rho{}_{\mu\nu} = T(f^\rho; e_\mu, e_\nu) \tag{2.51}$$

$$= f^\rho(\nabla_\mu e_\nu - \nabla_\nu e_\mu - \underbrace{[e_\mu, e_\nu]}_{\text{vanishes in coord. basis}}) \tag{2.52}$$

$$= \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}. \tag{2.53}$$

Remark: Therefore, we learn that although Γ is not a tensor, the anti-symmetrisation of it is!

Remark: In this lecture, since we never raise or lower indices on the Christoffel symbols, we do not care about the position of its indices.

A connection with $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu}$ has $T^{\rho}{}_{\mu\nu} = 0$ and is said to be *torsion free*. We also have

$$R^{\sigma}{}_{\rho\mu\nu} = R(f^{\sigma}; e_{\mu}, e_{\nu}, e_{\rho}) \tag{2.54}$$

Remark: This is a slightly odd ordering on the right hand side.

$$\begin{aligned} \dots &= f^{\sigma}\left(\nabla_{\mu}\nabla_{\nu}e_{\rho} - \nabla_{\nu}\nabla_{\mu}e_{\rho} - \nabla_{[e_{\mu},e_{\nu}]}e_{\rho}\right) && (2.55)\\ &= f^{\sigma}(\nabla_{\mu}(\Gamma^{\lambda}_{\nu\rho}e_{\lambda}) - \nabla_{\nu}(\Gamma^{\lambda}_{\mu\rho}e_{\lambda}) && (2.56)\\ &= \partial_{\mu}\Gamma^{\sigma}_{\nu}\rho - \partial_{\nu}\Gamma^{\sigma}_{\mu\rho} + \Gamma^{\lambda}_{\nu\rho}\Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\lambda}_{\mu\rho}\Gamma^{\sigma}_{\nu\lambda}. && (2.57) \end{aligned}$$

Remark: We mentioned that Γ transforms similarly to the gauge potential. We have a similar thing in Yang-Mills theory. This Riemann tensor is to GR and curvature R what the field strength F is to Maxwell theory.

Clearly, we have anti-symmetry in the last indices:

$$R^{\sigma}{}_{\rho\mu\nu} = -R^{\sigma}{}_{\rho\nu\mu}. \tag{2.58}$$

There are a few more subtle symmetry properties of R, which we will prove in the upcoming sections.

So far, the connection is completely independent of the metric. However, it turns out that, given a metric, we can define a natural connection from it.

2.2.1 Levi-Civita Connection

Theorem 3 (Fundamental Theorem of Riemannian Geometry)**:** There exists a unique, torsion-free connection with the property $\nabla_X g = 0$ for all vector fields $X \in \mathfrak{X}(\mathcal{M})$.

Proof. Suppose that the connection exists. We then follow a series of manipulation which gives us the desired result. Consider the object $X(g(Y,Z))$. Since $g(Y,Z)$ is a function, we have

$$X(g(Y,Z)) = \nabla_X[g(Y,Z)] \tag{2.59}$$

By the Leibniz rule, we have

$$\begin{aligned} \ldots &= \nabla_X g(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z) && (2.60)\\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) && (2.61)\end{aligned}$$

However, the torsion vanishes by assumption. Therefore we have

$$\nabla_X Y - \nabla_Y X = [X,Y]. \tag{2.62}$$

We use this equation in (2.61) to find that

$$X(g(Y,Z)) = g(\nabla_Y X, Z) + g(\nabla_X Z, Y) + g([X,Y],Z). \tag{2.63}$$

Now we repeat these exact same calculations twice over, cyclically permuting X, Y, and Z. These give

$$\begin{aligned} Y(g(Z,X)) &= g(\nabla_Z Y, X) + g(\nabla_Y X, Z) + g([Z,Y],X), && (2.64)\\ Z(g(X,Y)) &= g(\nabla_X Z, Y) + g(\nabla_Z Y, X) + g([Z,X],Y). && (2.65)\end{aligned}$$

Then, taking (2.63) + (2.64) − (2.65), we have

$$\begin{aligned} g(\nabla_Y X, Z) = \frac{1}{2}\big[&X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))\\ &- g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)\big]. \end{aligned} \tag{2.66}$$

In a coordinate basis $\{e_\mu = \partial_\mu\}$, this becomes

$$\begin{aligned} g(\nabla_\nu e_\mu, e_\rho) = \Gamma^\lambda_{\nu\mu} g_{\lambda\rho} &= \frac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) && (2.67)\\ = \Gamma^\lambda_{\mu\nu} &= \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) && (2.68)\end{aligned}$$

This is the *Levi-Civita connection* and the explicit representation $\Gamma^\lambda_{\mu\nu}$ are the *Christoffel symbols*.

Exercise 2.2: We showed that this is the unique form of the connection if it exists. We now still have to show that it is actually a connection, by considering how it transforms.

□

We basically only introduced torsion to show that given a metric, a torsion-free connection is this unique one. From now on, we will talk about this object when we talk about a connection.

Theorem 4 (Divergence theorem)**:** Consider a Riemannian manifold $\mathcal{M}$ with metric g and boundary $\partial\mathcal{M}$. Let n^μ be an outward-pointing unit vector orthogonal to (any tangent vectors in) the boundary. Then, for any X^μ,

$$\int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|} \nabla_\mu X^\mu = \int_{\partial\mathcal{M}} \mathrm{d}^{n-1} x \sqrt{|\gamma|} n_\mu X^\mu, \tag{2.69}$$

where γ_{ij} is the pull-back of g onto $\partial\mathcal{M}$. On a Lorentzian manifold, this also holds with $\sqrt{g} \to \sqrt{-g}$ provided that $\partial\mathcal{M}$ is purely timelike or purely spacelike.

> **Remark:** Note that this follows on from Stokes' theorem. However, we did not prove Stokes' theorem, and since we will use the divergence theorem over and over again, we will prove this explicitly.

Lemma 5:

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{\sqrt{g}}\partial_\nu\sqrt{g} \tag{2.70}$$

Proof of Lemma 5. Introducing the notation that $\hat{g}$ is a matrix, we have

$$\begin{aligned}
\Gamma^{\mu}_{\mu\nu} &= \frac{1}{2}g^{\mu\rho}\partial_\nu g_{\mu\rho} = \frac{1}{2}\operatorname{tr}\left(\hat{g}^{-1}\partial_\nu\hat{g}\right) &(2.71)\\
&= \frac{1}{2}\operatorname{tr}(\partial_\nu\log\hat{g}) &(2.72)\\
&= \frac{1}{2}\partial_\nu\log\det\hat{g}, \qquad \operatorname{tr}\log A = \log\det A &(2.73)\\
&= \frac{1}{2}\frac{1}{\det\hat{g}}\partial_\nu\det\hat{g} &(2.74)\\
&= \frac{1}{\sqrt{\det\hat{g}}}\partial_\nu\sqrt{\det\hat{g}} &(2.75)\\
&= \frac{1}{\sqrt{g}}\partial_\nu\sqrt{g} &(2.76)
\end{aligned}$$

□

Proof of Theorem 4. Using Lemma 5, we have that

$$\sqrt{g}\nabla_\mu X^\mu = \sqrt{g}(\partial_\mu X^\mu + \Gamma^\mu{}_{\mu\nu}X^\nu) \tag{2.77}$$

$$= \sqrt{g}(\partial_\mu X^\mu + X^\nu \frac{1}{\sqrt{g}}\partial_\nu\sqrt{g}) \tag{2.78}$$

$$= \partial_\mu(\sqrt{g}X^\mu). \tag{2.79}$$

Let us now restrict our discussion to a particular metric

$$g_{\mu\nu} = \begin{pmatrix} \gamma_{ij} & 0 \\ 0 & N^2 \end{pmatrix} \tag{2.80}$$

where the boundary $\partial\mathcal{M}$ is a surface with $x^n = \text{const.}$. We then use standard integration by parts over this boundary to give

$$\int_{\mathcal{M}} \mathrm{d}^n x\,\sqrt{g}\nabla_\mu X^\mu = \int_{\mathcal{M}} \mathrm{d}^n x\,\partial_\mu(\sqrt{g}X^\mu) \tag{2.81}$$

$$= \int_{\partial\mathcal{M}} \mathrm{d}^{n-1} x\,\sqrt{\gamma N^2}X^n. \tag{2.82}$$

The unit normal vector is chosen $n^\mu = (0, 0, \dots, 0, 1/N)$, so that when measured with respect to the metric, we indeed satisfy the definition for a unit vector: $g_{\mu\nu}n^\mu n^\nu = 1$. This means that when we lower with the metric we have $n_\mu = (0, 0, \dots, N)$ and

$$\int_{\mathcal{M}} \mathrm{d}^n x\,\partial_\mu(\sqrt{g}X^\mu) = \int_{\partial\mathcal{M}} \mathrm{d}^{n-1} x\,\sqrt{\gamma}n_\mu X^\mu. \tag{2.83}$$

The final expression is covariant, and so holds in all coordinate systems. □

2.3 Parallel Transport

The general story is that differentiation requires us to compare the vectors at two 'neighbouring' points. But these points have different tangent spaces, and we cannot simply add vectors in these different tangent spaces. The connection allows us to define a map between vector spaces that live at different points. This map is called *parallel transport*.

Definition 29 (parallel transport)**:** Take a vector field X and an integral curve C with the property

$$X^\mu|_C = \frac{\mathrm{d}x^\mu(\tau)}{\mathrm{d}\tau}. \tag{2.84}$$

A tensor T is said to be *parallel transported* along C if

$$\nabla_X T = 0. \tag{2.85}$$

Example: A vector field $Y \in \mathfrak{X}(\mathcal{M})$ is parallel transported along C if it obeys $\nabla_X Y = 0$. In components, this is

$$X^\nu(\partial_\nu Y^\mu + \Gamma^\mu_{\nu\rho} Y^\rho) = 0. \tag{2.86}$$

In particular, if we restrict to the curve C, consider $Y^\mu(x(\tau))$, which, from obeys (2.86)

$$\frac{\mathrm{d}Y^\mu}{\mathrm{d}\tau} + X^\nu \Gamma^\mu_{\nu\rho} Y^\rho = 0. \tag{2.87}$$

Given an initial condition $Y(x(\tau = 0)) \in T_p(\mathcal{M})$, where $p = x(\tau = 0)$, equation (2.87) determines a unique vector at each point along C. This is simply because it is a first order differential equation.

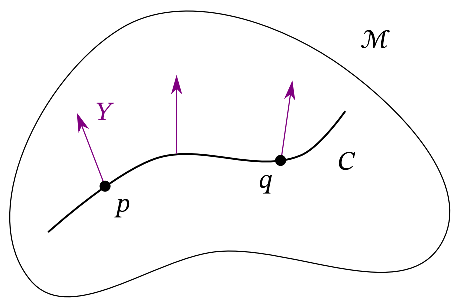

Figure 2.2: Parallel transport

> **Remark:** We can define parallel transport with only one curve. However, we need a vector field to cover the whole manifold. The parallel transport property depends on the path along which we parallel transport. Different paths will in general give different results.

2.4 Geodesics

Definition 30: A *geodesic* is a curve tangent to $X \in \mathfrak{X}(\mathcal{M})$ that is parallel transported along itself: it obeys $\nabla_X X = 0$.

From (2.87), the coordinates $x^\mu(\tau)$ along the curve obey the (affinely parametrised) geodesic equation:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\rho\nu} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0. \tag{2.88}$$

The geodesic equation also arises as the equation of motion from the action

$$S = \int \mathrm{d}\tau\, g_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}. \tag{2.89}$$

> **Remark:** If you are ever given a metric and you need to compute the Christoffel symbols, it gets quite difficult quickly. Do not compute the long way round The quick way round is to plug it into the action (2.89) and then work out the equations of motion. Comparing this with the geodesic equation (2.88) gives the Christoffel symbols $\Gamma^{\mu}_{\rho\nu}$. There is a small subtlety with symmetrisation.

2.5 Normal coordinates

Claim 7 (Riemann Normal Coordinates)**:** We can always find *normal coordinates* such that at a point $p \in \mathcal{M}$,

$$g_{\mu\nu}(p) = \delta_{\mu\nu} \quad \text{and} \quad g_{\mu\nu,\rho}(p) = 0. \tag{2.90}$$

Motivation. To see that this is possible, start with coordinates $\widetilde{x}^{\mu}$ and change to x^{μ}, such that $\widetilde{x}^{\mu}(p) = x^{\mu}(p) = 0$. Then

$$\widetilde{g}_{\rho\sigma}\frac{\partial\widetilde{x}^{\rho}}{\partial x^{\mu}}\frac{\partial\widetilde{x}^{\sigma}}{\partial x^{\nu}} = g_{\mu\nu} \tag{2.91}$$

Taylor expanding we get

$$\widetilde{x}^{\rho}\ \frac{\partial\widetilde{x}^{\rho}}{\partial x^{\mu}}\bigg|_{x=0} x^{\mu} + \frac{1}{2}\ \frac{\partial^2\widetilde{x}^{\rho}}{\partial x^{\mu}\partial x^{\nu}}\bigg|_{x=0} x^{\mu}x^{\nu} + \dots \tag{2.92}$$

At leading order, we want

$$\widetilde{g}_{\rho\sigma}\ \frac{\partial\widetilde{x}^{\rho}}{\partial x^{\mu}}\bigg|_{x=0}\ \frac{\partial\widetilde{x}^{\sigma}}{\partial x^{\nu}}\bigg|_{x=0} = \delta_{\mu\nu}. \tag{2.93}$$

We have $\frac{1}{2}n(n+1)$ conditions, but n^2 coefficients in $\partial\widetilde{x}^{\rho}/\partial x^{\mu}$. We can do it with $\frac{1}{2}n(n-1) = \dim SO(n)$ left over. At second order, we want to set $g_{\mu\nu,\rho} = 0$. This gives $\frac{1}{2}n(n+1)\times n$ conditions. Meanwhile, $\partial^2\widetilde{x}^{\rho}\big/\partial x^{\mu}\partial x^{\nu}$ has $n \times \frac{1}{2}n(n+1)$ components. In terms of degrees of freedom, this suggests that we can do it. To actually prove it we will have to do some more work. However, the usefulness of this counting comes in when we consider the second derivative of the metric: If we try to go further, we have $\frac{1}{2}n(n+1)\times\frac{1}{2}n(n+1)$ components in $g_{\mu\nu,\rho\sigma}$. And the third derivative of the coordinate transformation $\partial^3\widetilde{x}^{\rho}\big/\partial x^{\mu}\partial x^{\nu}\partial x^{\lambda}$ has $n\times\frac{1}{3!}n(n+1)(n+2)$ components. We find that we do not have enough degrees of freedom to do this; we are short by $\frac{1}{12}n^2(n^2-1)$. This is precisely the number of independent coordinates of the Riemann tensor $R^{\sigma}{}_{\rho\mu\nu}$.

Let us now actually give the coordinates that satisfy the claim. We can construct normal coordinates using geodesics and the *exponential map* (c.f. *Symmetries, Fields, and Particles*).

$$\mathrm{Exp}:\ T_p(\mathcal{M}) \ \to\ \mathcal{M} \tag{2.94}$$

We 'follow along' the geodesic such that $d\widetilde{x}^{\mu}/d\tau\,|_{\tau=0} = \widetilde{X}^{\mu}$ for distance $\tau = 1$. Now, pick an or-

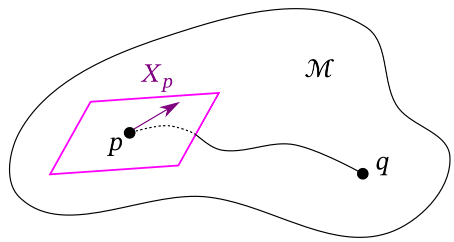

Figure 2.3

thonormal basis $\{e_\mu\}$ of $T_p(\mathcal{M})$ such that $X_p = X^\mu e_\mu$. Normal coordinates are $x^\mu(q) = X^\mu$. The completion of the proof is in the printed notes. $\square$

> **Remark:** This is a mathematical version of the equivalence principle.

Corollary: In normal coordinates, $\Gamma^\mu_{\nu\rho}(p) = 0$.

2.6 Path Dependence and Curvature

How does parallel transport depend on the path taken? Let $X, Y \in \mathfrak{X}(\mathcal{M})$ be vector fields such that $[X, Y]$. Consider two curves generated by X and Y. Now consider a vector Z_p living in the tangent

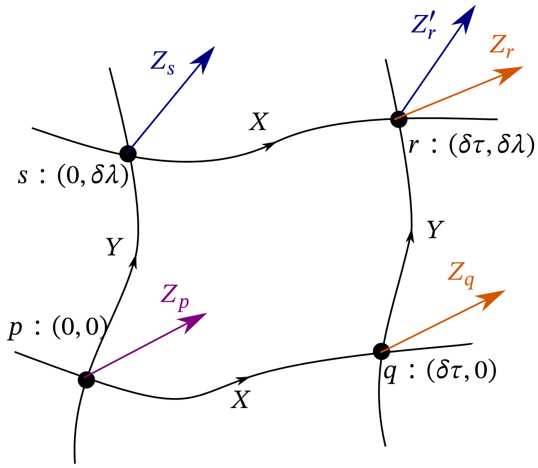

Figure 2.4: Path dependence of parallel transport

space $T_p(\mathcal{M})$ at point p. As illustrated in Figure 2.4, the result of parallel transporting this vector from p to another point r on the manifold will depend on the path taken. The expression for their difference is most clearly expressed in the case of infinitesimal translations. Moreover, life will be much easier if we pick normal coordinates at p with $x^\mu = (\tau, \lambda, 0, 0, \dots)$. Along $p \to q$, the parallel transport equation is a first order differential equation:

$$\frac{\mathrm{d}Z^\mu}{\mathrm{d}\tau} + X^\nu \Gamma^\mu_{\rho\nu} Z^\rho = 0. \tag{2.95}$$

Let us first see which vector Z_r we end up with when taking the path

$$p \xrightarrow{\delta\tau} q \xrightarrow{\delta\lambda} r. \tag{2.96}$$

The first step is to Taylor expand Z_q about the point $p : (\tau = 0, \lambda = 0, \dots)$. Let

$$Z^\mu_q = Z^\mu_p + \left.\frac{\mathrm{d}Z^\mu}{\mathrm{d}\tau}\right|_{\tau=0} d\tau + \frac{1}{2} \left.\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\tau^2}\right|_{\tau=0} d\tau^2 + \dots \tag{2.97}$$

$$\left.\frac{\mathrm{d}Z^\mu}{\mathrm{d}\tau}\right|_p = -(X^\nu Z^\rho \Gamma^\mu_{\rho\nu})_p = 0. \tag{2.98}$$

The last expression vanishes since $\Gamma^\mu_{\rho\nu}(p) = 0$ in normal coordinates. Moreover, we have

$$\left.\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\tau^2}\right|_{\tau=0} = -\left(X^\nu Z^\rho \frac{\mathrm{d}\Gamma^\mu_{\rho\nu}}{\mathrm{d}\tau} + \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} Z^\rho \Gamma^\mu_{\rho\nu} + X^\nu \frac{\mathrm{d}Z^\rho}{\mathrm{d}\tau} \Gamma^\mu_{\rho\nu} \right)_p . \tag{2.99}$$

By the previous reasoning, the latter two summands vanish. Using $\frac{\mathrm{d}}{\mathrm{d}\tau} = X^\sigma \partial_\sigma$, we obtain

$$\left.\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\tau^2}\right|_{\tau=0} = -\left(X^\nu X^\sigma Z^\rho \Gamma^\mu_{\rho\nu,\sigma}\right)_p . \tag{2.100}$$

Hence, we have

$$Z^\mu_q = Z^\mu_p - \frac{1}{2}(X^\nu X^\sigma Z^\rho \Gamma^\mu_{\rho\nu,\sigma})_p d\tau^2 + \ldots \tag{2.101}$$

Next, we will go from $q \to r$:

$$Z^\mu_r = Z^\mu_q + \left.\frac{\mathrm{d}Z^\mu}{\mathrm{d}\lambda}\right|_q d\lambda + \frac{1}{2}\left.\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\lambda^2}\right|_q d\lambda^2 + \ldots \tag{2.102}$$

For the first summand, we use the earlier result, while for the second term we use parallel transport

$$\begin{aligned}
\left.\frac{\mathrm{d}Z^\mu}{\mathrm{d}\lambda}\right|_q &= -\left(Y^\nu Z^\rho \Gamma^\mu_{\rho\nu}\right)_q & (2.103)\\
&= -\left(Y^\nu Z^\rho \frac{\mathrm{d}\Gamma^\mu_{\rho\nu}}{\mathrm{d}\tau}\right)_p d\tau + \ldots & (2.104)\\
&= -\left(Y^\nu Z^\rho X^\sigma \Gamma^\mu_{\rho\nu,\sigma}\right) d\tau + \ldots & (2.105)
\end{aligned}$$

Similarly, the second derivative is

$$\left.\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\lambda^2}\right|_q = -\left(Y^\nu Y^\sigma Z^\rho \Gamma^\mu_{\rho\nu,\sigma}\right)_q + \ldots . \tag{2.106}$$

The higher order terms ($\ldots$) include $\Gamma^\mu_{\rho\nu}(q) \sim \frac{\mathrm{d}}{\mathrm{d}t}\Gamma^\mu_{\rho\nu} d\tau$. Putting all of this together, the components of the vector Z at r is

$$Z^\mu_r = Z^\mu_p - \frac{1}{2}(\Gamma^\mu_{\rho\nu,\sigma})_p \left[X^\nu X^\sigma Z^\rho d\tau^2 + 2Y^\nu Z^\rho X^\sigma d\tau d\lambda + Y^\nu Y^\sigma Z^\rho d\lambda^2\right]_p + \ldots . \tag{2.107}$$

If we go the other way, we have

$$(Z')^\mu_r = Z^\mu_p - \frac{1}{2}(\Gamma^\mu_{\rho\nu,\sigma})_p \left[X^\nu X^\sigma Z^\rho d\tau^2 + 2X^\nu Z^\rho Y^\sigma d\lambda d\tau + Y^\nu Y^\sigma Z^\rho d\lambda^2\right] + \ldots \tag{2.108}$$

The difference between the two is

$$\begin{aligned}
\Delta Z^\mu_r &= Z^\mu_r - (Z')^\mu_r & (2.109)\\
&= \left(\Gamma^\mu_{\rho\nu,\sigma} - \Gamma^\mu_{\rho\sigma,\nu}\right)_p (Y^\nu Z^\rho X^\sigma)_p \, d\lambda d\tau + \ldots & (2.110)\\
&= (R^\mu{}_{\rho\nu\sigma} Y^\nu X^\rho X^\sigma)_p d\lambda d\tau + \ldots & (2.111)
\end{aligned}$$

We picked very special coordinates. However, the final expression is a tensor equation, which has to hold in all coordinate systems, even though we used special coordinates to derive it.

Remark: If a tensor evaluates to zero in one coordinate system, it does so in all coordinate systems. This is not true of the Christoffel symbols.

Remark: There is a whole area of geometry, called *holonomy*, which deals with the possible rotations of the vector that can be obtained when going along certain paths along a manifold. Calabi-Yau manifolds are related to *special holonmies*.

2.6.1 Geodesic Deviation

Consider the one-parameter family of geodesics $x^\mu(\tau; s)$; for a fixed s, $x^\mu(\tau; s)$ is a geodesic with affine parameter τ. Figure 2.5 depicts these geodesics, which we can think of being integral curves

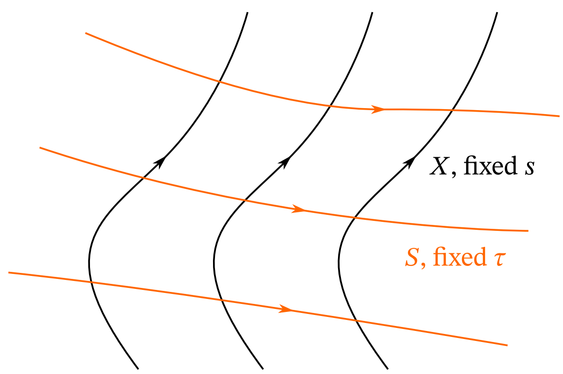

Figure 2.5

generated by a vector field $X^\mu = \partial x^\mu/\partial\tau|_s$. Moreover, the curves of fixed τ can be thought of as being generated by a vector fields $S^\mu = \partial x^\mu/\partial s|_\tau$ We will not prove it, but we can always pick τ, s such that $[S, X] = 0$.

Claim 8:

$$\nabla_X\nabla_X S = R(X,S)X \tag{2.112}$$

$$\text{or} X^\nu\nabla_\nu(X^\rho\nabla_\rho S^\mu) = R^\mu{}_{\nu\rho\sigma}X^\nu X^\rho S^\sigma \tag{2.113}$$

$$\text{or } \frac{\mathrm{d}^2 S^\mu}{\mathrm{d}\tau^2} = R^\mu{}_{\nu\rho\sigma}X^\nu X^\rho S^\sigma \tag{2.114}$$

Proof. We are dealing with the Levi-Civita connection, which is torsion free. For a torsion-free connection, we have

$$[X,S] = 0 \;\Rightarrow\; \nabla_X S = \nabla_S X. \tag{2.115}$$

We use this in the left hand side of (2.113) to find

$$\nabla_X \nabla_X S = \nabla_X \nabla_S X = \nabla_S \nabla_X X + R(X, S)X. \tag{2.116}$$

Now $\nabla_S \nabla_X X = 0$ as X is a geodesic. □

> **Remark:** In the physics-part of this lecture course, we will see this again when we talk about gravitational waves.

2.7 More on the Riemann Tensor

Recall that the definition of the Riemann tensor in terms of the Christoffel symbols is

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\rho\nu} - \partial_\nu \Gamma^{\sigma}_{\rho\mu} + \Gamma^{\lambda}_{\nu\rho}\Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\lambda}_{\mu\rho} - \Gamma^{\lambda}_{\mu\rho}\Gamma^{\sigma}_{\nu\lambda}. \tag{2.117}$$

This satisfies certain symmetry properties which are more evident once we lower the first index.

Claim 9: $R_{\sigma\rho\mu\nu} = g_{\sigma\lambda} R^{\lambda}{}_{\rho\mu\nu}$ obeys

- $R_{\sigma\rho\mu\nu} = -R_{\sigma\rho\nu\mu}$
- $R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\mu\nu}$
- $R_{\sigma\rho\mu\nu} = R_{\mu\nu\sigma\rho}$
- $R_{\sigma[\rho\mu\nu]]} = 0$, the *first Bianchi identity*

Proof. Use normal coordinates

$$R_{\sigma\rho\mu\nu} = g_{\sigma\lambda}(\partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho}) \tag{2.118}$$

$$= \frac{1}{2}(\partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\mu \partial_\sigma g_{\nu\rho} - \partial_\nu \partial_\rho g_{\mu\sigma} + \partial_\nu \partial_\sigma g_{\mu\rho}) \tag{2.119}$$

Now stare at this. □

Claim 10 (The Second Bianchi Identity)**:**

$$\nabla_{[\lambda} R_{\sigma\rho]\mu\nu} = 0 \quad \text{or equivalently} \quad R_{\sigma\rho[\mu\nu;\lambda]} = 0 \tag{2.120}$$

Proof. The Riemann tensor is, schematically, $R \sim \partial\Gamma + \Gamma\Gamma$ and so its derivative is $\partial R \sim \partial^2\Gamma + 2\Gamma\partial\Gamma$. In normal coordinates at a point, we have $\Gamma = 0$ (but $\partial\Gamma \neq 0$ in general), so the last summand vanishes and the partial derivative of the Riemann tensor is

$$R_{\sigma\rho\mu\nu,\lambda} = \frac{1}{2}(g_{\nu\sigma,\rho\mu\lambda} - g_{\nu\rho,\sigma\mu\lambda} - g_{\mu\sigma,\rho\nu\lambda} + g_{\mu\rho,\sigma\nu\lambda}) \tag{2.121}$$

The partial derivatives $\partial_\mu \partial_\lambda$ in the first term commute: $g_{\nu\sigma,\rho\mu\lambda} = g_{\nu\sigma,\rho\mu\lambda}$. Therefore, anti-symmetrising over μ and λ makes this term vanish. In the same way, the other three terms vanish when we anti-symmetriese over μ, ν, and λ. Thus $R_{\sigma\rho[\mu\nu,\lambda]} = 0$. In normal coordinates, this is the same as $R_{\sigma\rho[\mu\nu;\lambda]} = 0$, which is a tensor equation and therefore holds in all coordinate systems. □

> **Remark:** The Bianchi identity here implies a similar identity on the Ricci tensor, which we will see soon. There is a very elegant proof, which we will discuss, of that identity from the action principle of GR.

We can easily construct new tensors from the Riemann tensor. The Riemann tensor has four indices; to build a new one, we just raise index and contract it with one of the lower ones. However, due to the symmetry properties, we have to contract one of the first pair indices to one index from the second pair.

Definition 31: The *Ricci tensor* $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$. This obeys $R_{\mu\nu} = R_{\nu\mu}$.

Definition 32: The *Ricci scalar* $R = g^{\mu\nu} R_{\mu\nu}$.

We are loosing information every time we contract indices like this. However, in some sense the information that is left over is the most essential part of the information. We can also apply the Bianchi identity to get

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R. \tag{2.122}$$

Definition 33: The *Einstein tensor* $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$.

The whole point of building this tensor is that, using the Riemann tensor's Bianchi identity, these two tensors cancel to give a Bianchi identity for the Einstein tensor:

$$\nabla^{\mu} G_{\mu\nu} = 0. \tag{2.123}$$

Proving the Bianchi identity for the Riemann tensor is a pain, but in Section 3.2 we will show that the Bianchi identity of the Einstein tensor follows from the diffeomorphism invariance of the Einstein-Hilbert action.

2.8 Connection 1-forms

These objects are, as we will see, closely related to two-forms. Connection 1-forms are a technology designed to make it easier to calculate the Riemann tensor for a specific metric. It is not a slick trick like the one that we saw to find the Levi-Civita connection components.

Given a coordinate basis $\{e_{\mu}\} = \{\partial_{\mu}\}$. The idea is that we will take a different basis of tangent vectors so that things start to look simpler. Given the coordinate basis, we can always introduce a different basis $\{\hat{e}_a = e_a{}^{\mu} \partial_{\mu}\}$. There is a particularly clever basis to choose: On a Riemannian / Lorentzian manifold, we pick a basis such that when we evaluate the metric on this new metric, we get

$$g(\hat{e}_a, \hat{e}_b) = g_{\mu\nu} e_a{}^{\mu} e_b{}^{\nu} = \begin{cases} \delta_{ab} & \text{Riemannian} \\ \eta_{ab} & \text{Lorentzian} \end{cases} \tag{2.124}$$

Note that this is not the metric in a set of coordinates (such as in Riemann normal coordinates). This is rather a diagonalised version of the metric. The components $e_a{}^{\mu}$ are called *vielbeins*. In general, having an n-component basis is called an n-bein (where we use the German word for n, e.g. $n = 2$: zweibeins). We raise and lower greek indices with $g_{\mu\nu}$ and latin indices using δ_{ab}. The basis of one-forms $\left\{\hat{\theta}^a\right\}$ obey

$$\hat{\theta}^a(\hat{e}_b) = \delta^a_b. \tag{2.125}$$

The right hand side is a Krönicker delta for both Lorentzian and Riemannian metrics. They are $\hat{\theta}^a = e^a{}_\mu dx^\mu$ with $e^a{}_\mu e_b{}^\mu = \delta^a_b$ and $e^a{}_\mu e_a{}^\nu = \delta^\nu_\mu$. The metric is

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = \delta_{ab}\hat{\theta}^a \otimes \hat{\theta}^b \;\Rightarrow\; g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \delta_{ab}. \tag{2.126}$$

Example: Consider the metric

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2(d\theta^2 + \sin^2(\theta) d\phi^2) \tag{2.127}$$

This is one of the most important class of metric in General relativity, since for a particular choice of f we obtain the *Schwarzschild solution*. In our new notation, this is $ds^2 = \eta_{ab}\hat{\theta}^a \otimes \hat{\theta}^b$ with non-coordinate 1-forms

$$\hat{\theta}^0 = f dt \quad \hat{\theta}^1 = f^{-1} dr \quad \hat{\theta}^2 = r d\theta \quad \hat{\theta}^3 = r \sin\theta d\phi. \tag{2.128}$$

Remark: Note that this is a slightly annoying convention; $\hat{\theta}$ has nothing to do with θ or $d\theta$.

In the basis $\{\hat{e}_a\}$, the components of the connection are by definition

$$\nabla_{\hat{e}_c} \hat{e}_b := \Gamma^a_{cb} \hat{e}_a. \tag{2.129}$$

Remark: Another slightly annoying convention: the type of indices (Roman or Greek) determine which object Γ we are looking at. $\Gamma^\mu{}_{\rho\nu} \neq \Gamma^b{}_{cb}$.

Definition 34: Given this, we define the *connection 1-form* or *spin connection* $\omega^a{}_b = \Gamma^a_{cb}\hat{\theta}^c$.

Remark: The reason that this is called the spin-connection is that we have to use this object when we want to couple to Dirac spinors.

Claim 11 (First Cartan Structure Equation)**:** Defining $\omega_{ab} = \delta_{ac}\omega^c{}_b$, we have

$$d\hat{\theta}^a + \omega^a{}_b \wedge \hat{\theta}^b = 0 \tag{2.130}$$

Proof. No. □

Claim 12: For the Levi-Civita connection, the lowered spin connection is anti-symmetric:

$$\omega_{ab} = -\omega_{ba}. \tag{2.131}$$

Definition 35: In the vielbein basis, ${R^a}_{bcd} = R(\hat{\theta}^a; \hat{e}_c, \hat{e}_d, \hat{e}_b)$ (again, the indices tell us what kind of object this is), with ${R^a}_{bcd} = -{R^a}_{bdc}$. We define the *curvature 2-form*

$$\mathcal{R}^a{}_b = \frac{1}{2} {R^a}_{bcd} \hat{\theta}^c \wedge \hat{\theta}^d. \tag{2.132}$$

> **Remark:** This has the full information of the Riemann tensor. Note that when we write this in components $\hat{\theta} = {e^a}_\mu dx^\mu$, we find that the components of the curvature 2-form are $(\mathcal{R}^a{}_b)_{\mu\nu}$.

Claim 13 (Second Cartan Structure Equation)**:**

$$\mathcal{R}^a{}_b = d{\omega^a}_b + {\omega^a}_c \wedge {\omega^c}_b \tag{2.133}$$

Proof. No. □

If we are given a metric, the quickest way to compute the Riemann tensor is to calculate the curvature one-forms $\hat{\theta}$. Then we can easily take the exterior derivative, which allows us to use the first Cartan structure equation. Plugging this into the second structure equation, we can find the components of the Riemann tensor.

Back to the Example

Compute $d\hat{\theta}^a$:

$$\hat{\theta}^0 = f' dr \wedge dt \qquad d\hat{\theta}^1 = f' dr \wedge dr = 0 \tag{2.134}$$

$$d\hat{\theta}^2 = dr \wedge d\theta \qquad d\hat{\theta}^3 = \sin\theta dr \wedge d\phi - r\cos\theta d\theta \wedge d\phi. \tag{2.135}$$

Use $d\hat{\theta}^0 = -{\omega^0}_b \wedge \hat{\theta}^b$ to convince yourself that ${\omega^0}_1 = f' f dt = f'\hat{\theta}^0$ and ${\omega^0}_1 = -\omega_{01} = +\omega_{10} = {\omega^1}_0$, where we used the Minkowski metric in the first equality.

Check $d\hat{\theta}^1 = -{\omega^1}_b \wedge \hat{\theta}^b = -{\omega^1}_0 \wedge \hat{\theta}^0 + \cdots = -f'\hat{\theta}^0 \wedge \hat{\theta}^0 + \ldots$, where the $\hat{\theta}^0 \wedge \hat{\theta}^0$ vanishes. ✓

Proceed in the same way to find that the only non-vanishing ones are

$$\omega^0{}_1 = \omega^1{}_0 = f'\hat{\theta}^0 \tag{2.136}$$

$$\omega^2{}_1 = -\omega^1{}_2 = \frac{f}{r}\hat{\theta}^2 \tag{2.137}$$

$$\omega^3{}_1 = -\omega^1{}_3 = \frac{f}{r}\hat{\theta}^3 \tag{2.138}$$

$$\omega^3{}_2 = -\omega^2{}_3 = \frac{\cot\theta}{r}\hat{\theta}^3 \tag{2.139}$$

Now the curvature tensor is

$$\mathcal{R}^0{}_1 = d\omega^0{}_1 + \overbrace{\omega^0{}_c \wedge \omega^c{}_1}^{\text{In this case, these vanish}} \tag{2.140}$$
$$= f' d\hat{\theta}^0 + f'' dr \wedge \hat{\theta}_0 \tag{2.141}$$
$$= ((f')^2 + f'' f) dr \wedge dt \tag{2.142}$$
$$= -((f')^2 + f'' f)\hat{\theta}^0 \wedge \hat{\theta}^1 \tag{2.143}$$
$$\Rightarrow R_{0101} = f f'' + (f')^2 \tag{2.144}$$

We convert back to Greek indices using the vielbeins:

$$R_{\mu\nu\rho\sigma} = e^a{}_\mu e^b{}_\nu e^c{}_\rho e^d{}_\sigma R_{abcd} \tag{2.145}$$

In our case, we find $R_{trtr} = f f'' + (f')^2$.

3 The Einstein Equations

It is time to do some physics! All of the mathematics that we developed will pay off quickly in that we will quickly understand why we have to write down the Einstein equations.

We should think about gravity in much the same way as we think about other forces. Spacetime is a manifold $\mathcal{M}$, equipped with a Lorentzian metric g. This metric is a field, which tells us how gravity is to behave. This is similar to the fields in electromagnetism or other forces. The right way to describe a fundamental force in physics is with an action.

3.1 The Einstein-Hilbert Action

The dynamics is governed by the *Einstein-Hilbert action*,

$$S = \int \mathrm{d}^4x \sqrt{-g} R \tag{3.1}$$

Remark: This is where we see the payoff: There is not very much that we can write down. From differential geometry, we know that we need to cook up a top-form. Thankfully, given the metric, there is a natural volume form $\sqrt{-g}$ (minus sign for Lorentzian metric).

We then integrate over a scalar function. The simplest function—a constant function—gives us the volume of the manifold, but it is not quite the right thing to give dynamics. The next simples thing that we can write down is the Ricci scalar R. Note that $R \sim \partial\Gamma + \Gamma\Gamma$ and $\Gamma \sim g^{-1}\partial g$. Therefore, the Ricci scalar R has two derivatives. This is what we have come to expect from an action for a bosonic scalar field, or the Maxwell action.

There are other choices, like the square of the Riemann tensor, or others. However, these would be four-derivative terms. It turns out that with four derivatives we have exactly three options and at six derivatives we have even more. We will only consider two-derivatives, like in the other action principles we use in theoretical physics.

Remark: This has to be the Levi-Civita connection, since we just have the metric. Other than g and $\mathcal{M}$, there is nothing else to play with in this theory.

To derive the equations of motion, we vary the field, which in this case is simply the metric,

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \tag{3.2}$$

in the usual way. We can also consider how the inverse of the metric changes

$$g_{\rho\mu} = \delta^{\nu}_{\rho} \implies \delta g^{\rho\mu} + g_{\rho\mu}\delta g^{\mu\nu} = 0 \tag{3.3}$$

$$\implies \delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}. \tag{3.4}$$

It turns out that it is actually marginally simpler to look at how the inverse changes. We have

$$\delta S = \int \mathrm{d}^4x \left[\delta(\sqrt{-g})g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\left(\delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right)\right]. \tag{3.5}$$

Claim 14: $\delta\sqrt{-g} = \frac{-1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$.

Proof. We use $\log\det A = \operatorname{tr}\log A$

$$\implies \frac{1}{\det A}\delta(\det A) = \operatorname{tr}\left(A^{-1}\delta A\right). \tag{3.6}$$

Remark: We vary the log of a matrix as

$$B = \log A \implies A = e^B \tag{3.7}$$

$$\implies \operatorname{Tr}(\delta A) = \operatorname{Tr}\left(e^B\delta B\right) = \operatorname{Tr}(A\delta B) \tag{3.8}$$

$$\implies \operatorname{Tr}(\delta(\log A)) = \operatorname{Tr}\left(A^{-1}\delta A\right) \tag{3.9}$$

$$\implies \delta\sqrt{-g} = \frac{1}{2}\frac{1}{\sqrt{-g}}(-g)g^{\mu\nu}\delta g_{\mu\nu} \tag{3.10}$$

$$= \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \tag{3.11}$$

□

Claim 15:

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\nabla_\mu\delta g_{\sigma\nu} + \nabla_\nu\delta g_{\sigma\mu} - \nabla_\sigma\delta g_{\mu\nu}) \tag{3.12}$$

Proof. First, note that $\Gamma^{\mu}_{\rho\nu}$ is *not* a tensor, but the variation $\delta\Gamma^{\mu}_{\rho\nu}$ is the difference between two connections and *is* a tensor. In normal coordinates, at some point,

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu \delta g_{\sigma\nu} + \partial_\nu \delta g_{\sigma\mu} - \partial_\sigma \delta g_{\mu\nu}) \tag{3.13}$$

The $\delta g^{\rho\sigma}$ multiplies ∂g, which is $\partial g = 0$ in normal coordinates. Moreover, in normal coordinates we have $\Gamma = 0$ and therefore $\partial \to \nabla$.

$$\cdots = \frac{1}{2}g^{\rho\sigma}(\nabla_\mu \delta g_{\sigma\nu} + \nabla_\nu \delta g_{\sigma\mu} - \nabla_\sigma \delta g_{\mu\nu}). \tag{3.14}$$

This is a tensor equation, and is therefore true in all coordinates. □

Claim 16:

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^{\rho}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\rho}_{\mu\rho} \tag{3.15}$$

Proof.

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\nu\rho} - \partial_\nu \Gamma^{\sigma}_{\mu\rho} \tag{3.16}$$

$$\Rightarrow \delta R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \delta\Gamma^{\sigma}_{\nu\rho} - \partial_\nu \delta\Gamma^{\sigma}_{\mu\nu} \tag{3.17}$$

$$= \nabla_\mu \delta\Gamma^{\sigma}_{\nu\rho} - \nabla_\nu \delta\Gamma^{\sigma}_{\mu\nu} \tag{3.18}$$

as $\Gamma 00$ in normal coordinates. □

Therefore, the Einstein-Hilbert action becomes

$$\delta S = \int \mathrm{d}^4 x \sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}R_{\gamma\mu\nu}\right]\delta g^{\mu\nu} + \nabla_\mu X^\mu, \tag{3.19}$$

with $X^\mu = g^{\rho\nu}\delta\Gamma^{\mu}_{\rho\nu} - g^{\mu\nu}\delta\Gamma^{\rho}_{\nu\rho}$. This last term is a total derivative and we will just drop it.

> **Remark:** Unlike in other field theories, you do not have to look too far to find situations where these boundary terms become important. You can introduce the Gibbons-Hawking term to deal with these boundary terms. These might well become important in next term's *Black Holes* course.

Requiring $\delta S = 0$ for all variations $\delta g^{\mu\nu}$, we obtain the *Einstein equations*:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{3.20}$$

In fact, we can simplify this by contracting on the right with $g^{\mu\nu}$: we get $R = 0$ which gives us

$$R_{\mu\nu} = 0. \tag{3.21}$$

These are the *Einstein vacuum equations*.

> **Remark:** Note that this does not say that spacetime has no curvature! It is the more subtle type of curvature, which the Riemann tensor has and the Ricci scalar has not, that is relevant for spacetime.

Dimensional Analysis

The action should have exactly the same dimension as $\hbar$, which is $(energy) \times (time) = ML^2T^{-1}$. Meanwhile $[\mathrm{d}^4x] = L^4$ (or L^3T) and $[\sqrt{-g}] = 0$ (which is sort-of-true) and $[R] = L^{-2}$.

> **Remark:** We insist that in $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, the dimensions are hiding in the dx^μ, and $g_{\mu\nu}$ is dimensionless.

Therefore, for dimensional consistency, we have

$$S = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}R. \tag{3.22}$$

At the moment, this does not change the equations of motion. However, once we introduce matter, this will have an effect.

Definition 36: The *Planck mass* is $M_{\mathrm{pl}}^2 = \frac{\hbar c}{8\pi G}$ and $M_{\mathrm{pl}} \sim 10^{18}$ GeV.

We work in units with $c = 1$ and $\hbar = 1$. In this case, the action can be written as

$$S = \frac{1}{2}M_{\mathrm{pl}}^2\int \mathrm{d}^4x\,\sqrt{-g}R. \tag{3.23}$$

Depending on the physics that we are interested in, we use these different forms of the action. We will also keep G in all the formulae; other relativists tend to set it to one, but we will try to keep gravity on an equal footing with the other forces in the universe.

The Cosmological Constant

We could add a further term to the action

$$S = \frac{1}{2}M_{\mathrm{pl}}^2\int \mathrm{d}^4x\,\sqrt{-g}\,(R - 2\Lambda) \tag{3.24}$$

We can think of this constant Λ as a potential term for gravity. Doing the same variations as usual, we obtain

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\Lambda g_{\mu\nu} \;\Rightarrow\; R = 4\Lambda \;\Rightarrow\; R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{3.25}$$

We will study gravity by itself first. Only later we will try to understand how gravity couples to other fields, like masses.

> **Remark:** Einstein called Λ to be his biggest mistake, since he picked it to balance out the expansion of the universe in the solutions he found, when compared against the redshift observed by Slyther (?) and Hubble.

3.2 Diffeomorphisms Revisited

In part, we revisit this section to count the number of degrees of freedom of these equations. The metric has $\frac{1}{2} \times 4 \times 5 = 10$ components. But two metrics related by a change of coordinates $x^\mu \to \widetilde{x}x^\mu(x)$ are physically equivalent. Therefore, the metric contains $10-4 = 6$ degrees of freedom. You might worry that the 10 equations of (3.25) over-determine the metric; we will see that this is not true. The change of coordinates can be viewed as a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$.

> **Remark:** This is a bit like the active vs passive transformation issue.

Such diffeomorphisms (diffeos) are the "gauge symmetry" of GR.

We will look at diffeomorphism of the action. However, the action does not depend on coordinates we use; the change of coordinates is like a gauge theory of the action. Similarly to Noether's theorem, where symmetries give rise to conserved quantities, this gauge symmetry will give rise to something. We will try to find out what that something is.

Claim 17: Acting with an infinitesimal diffeomorphism along a vector field $X \in \mathfrak{X}(\mathcal{M})$, the metric changes as the Lie derivative

$$\delta g_{\mu\nu} = (\mathcal{L}_X g)_{\mu\nu}. \tag{3.26}$$

Proof. Consider a diffeomorphism that take points x^μ to nearby points $\widetilde{x}^\mu$ as

$$x^\mu \to \widetilde{x}^\mu(x) = x^\mu + \delta x^\mu. \tag{3.27}$$

We also know that a vector field on a manifold induces a flow; we will use this to take us to this nearby point. As such, we can think of the associated coordinate change as being generated by a vector field $X^\mu = -\delta x^\mu$. The metric transforms as

$$g_{\mu\nu}(x) \to \widetilde{g}_{\mu\nu}(\widetilde{x}) = \frac{\partial x^\rho}{\partial \widetilde{x}^\mu}\frac{\partial x^\sigma}{\partial \widetilde{x}^\nu} g_{\rho\sigma}(x). \tag{3.28}$$

The Jacobian associated with the change of coordinate $\tilde{x}^\mu = x^\mu - X^\mu(x)$ can be inverted:

$$\frac{\partial \tilde{x}^\mu}{\partial x^\rho} = \delta^\mu_\rho - \partial_\rho X^\mu \quad \Rightarrow \quad \frac{\partial x^\rho}{\partial \tilde{x}^\mu} = \delta^\rho_\mu + \partial_\mu X^\rho + \dots, \tag{3.29}$$

where the higher order terms can be neglected for infinitesimal X^μ. We then have

$$\tilde{g}_{\mu\nu}(\tilde{x}) = (\delta^\rho_\mu + \partial_\mu X^\rho)(\delta^\sigma_\nu + \partial_\nu X^\sigma) g_{\rho\sigma}(x) \tag{3.30}$$

$$= g_{\mu\nu}(x) + g_{\mu\rho}(x)\partial_\nu X^\rho + g^{\nu\rho}(x)\partial_\mu X^\rho \tag{3.31}$$

Meanwhile, we may Taylor expand $\tilde{g}_{\mu\nu}(\tilde{x})$ on the left-hand side around x to give

$$\widetilde{g}_{\mu\nu}(\tilde{x}) = \tilde{g}_{\mu\nu}(x + \delta x) = \widetilde{g}_{\mu\nu}(x) - X^\lambda \partial_\lambda g_{\mu\nu}(x). \tag{3.32}$$

Thus, we find the change $\delta g_{\mu\nu}(x)$ of the metric at the point x to be

$$\delta g_{\mu\nu} := \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = X^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\rho}\partial_\nu X^\rho + g_{\nu\rho}\partial_\mu X^\rho \tag{3.33}$$

$$= (\mathcal{L}_X g)_{\mu\nu}. \tag{3.34}$$

□

Alternatively, we can write

$$\delta g_{\mu\nu} = \partial_\mu X_\nu + \partial_\nu X_\mu + X^\rho \underbrace{(\partial_\rho g_{\mu\nu} - \partial_\mu g_{\rho\nu} - \partial_\nu g_{\mu\rho})}_{2g^{\rho\sigma}\Gamma^\sigma_{\mu\nu}} \tag{3.35}$$

$$= \nabla_\mu X_\nu + \nabla_\nu X_\mu. \tag{3.36}$$

Now look at the action

$$\delta S = \int \mathrm{d}^4x\,\sqrt{-g}G^{\mu\nu}\delta g_{\mu\nu}. \tag{3.37}$$

The symmetries of the action are those for which the change in the action is zero. If we restrict to these changes of coordinates, it must be that for all choices of X_ν, we have

$$\cdots = 2\int \mathrm{d}^4x\,\sqrt{-g}G^{\mu\nu}\nabla_\mu X_\nu = 0 \tag{3.38}$$

because changing coordinates is a gauge symmetry. Integrating by parts, and using the divergence theorem,

$$... = -2\int \mathrm{d}^4x\,\sqrt{-g}(\nabla_\mu G^{\mu\nu})X^\nu \tag{3.39}$$

$$\Rightarrow \;\nabla^\mu G_{\mu\nu} = 0. \tag{3.40}$$

But we know that this is true for any metric. This is the Bianchi identity. This is essentially another derivation of the Bianchi identity using the path integral, using the diffeomorphism invariance of the action.

Remark: The metric only actually has 6 pieces of information in it. However, the Einstein equations $G_{\mu\nu} = 0$ are 10 equations. You might think that this over-determines the metric, but the Bianchi identity saves us. In other words: $\nabla_\mu G^{\mu\nu} = 0$ are 4 conditions, which mean that the Einstein equations $G_{\mu\nu} = 0$ are really only 6 equations, which is the right number to determine the metric $g_{\mu\nu}$.

3.3 Some Simple Solutions

There are three cases that we should consider, depending on the cosmological constant: zero, positive and negative. We will spend a lot of time understanding the differences between these solutions. Even the solutions that look completely trivial—such as the one we will write down shortly—have interesting subtleties about them.

3.3.1 $\Lambda = 0$: Minkowski Spacetime

Need to solve $R_{\mu\nu} = 0$. The solution $g_{\mu\nu}$ is not allowed!

Remark: Usually, the simplest solution that solves a field theory is always $\phi = 0$. On geometrical grounds, it is obvious that the metric cannot vanish, since we have to have an inverse. Nonetheless, this restriction, that none of the eigenvalues cross zero, is a very weird constraint to put on a dynamical field. What this might be telling us is that the metric is not a fundamental field of nature; the metric might emerge. For example, the other case where a field like this pops up is in fluid mechanics. When we write down the Navier-Stokes equations, we do not want the energy-density to cross zero, since the approximations that lead to the fluid equations break down otherwise. There are lots of hints that gravity is like fluid mechanics.

Note that there are infinitely many solutions of these. We are nowhere close to understanding the properties of the general solution.

The simplest solution is *Minkowski spacetime*.

$$ds^2 = -dt^2 + d\mathbf{x}^2 \tag{3.41}$$

3.3.2 $\Lambda > 0$: De Sitter Spacetime

There are many solutions, many of them very hard to solve since these are coupled second order differential equations. We will look for solutions to $R_{\mu\nu} = \Lambda g_{\mu\nu}$ of the form

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2(d\theta^2 + \sin^2(\theta) d\phi^2). \tag{3.42}$$

We can compute $R_{\mu\nu}$ (using for example the curvature two-forms) to find

$$R_{tt} = -f^4 R_{rr} = f^3(f'' + \frac{2f'}{r} + \frac{(f')^2}{f}) \tag{3.43}$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta} = (1 - f^2 - 2ff'r)\sin^2\theta. \tag{3.44}$$

With $R_{\mu\nu} = \Lambda g_{\mu\nu}$, this becomes

$$\mu,\nu = tt, rr \;\Rightarrow\; f'' + \frac{2f'}{r} + \frac{(f')^2}{f} = -\frac{\Lambda}{f} \tag{3.45}$$

$$\mu,\nu = \theta\theta, \phi\phi \;\Rightarrow\; 1 - 2ff'r - f^2 = \Lambda r^2 \tag{3.46}$$

Both equations are solved by

$$f(r) = \sqrt{1 - \frac{r^2}{R^2}}, \qquad \text{with } R^2 = \frac{3}{\Lambda}. \tag{3.47}$$

This is *de Sitter spacetime* dS (or the *static patch* of dS).

Remark: $r \in [0, R]$ but the metric appears to be singular at $r = R$.

Let us now build up some intuition of what it is like to live in a universe determined by metric (3.42). We can look at geodesics. With σ denoting proper time, and $\dot{(\ldots)} = \frac{\mathrm{d}(\ldots)}{\mathrm{d}\sigma}$, we have the action:

$$S = \int \mathrm{d}\sigma \left[-f(r)^2 (\dot{t})^2 + f(r)^{-2} (\dot{r})^2 + r^2 (\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2) \right]. \tag{3.48}$$

Given an action it is really tempting to find the equations of motion. However, in this case this is the wrong thing to do; the right thing to do is to consider the symmetries. There are two conserved quantities:

$$\text{angular momentum} \qquad l = \frac{1}{2}\frac{\partial L}{\partial \dot{\phi}} = r^2 \sin^2\theta \dot{\phi} \tag{3.49}$$

$$\text{energy} \qquad E = -\frac{1}{2}\frac{\partial L}{\partial \dot{t}} = f(r)^2 \dot{t}. \tag{3.50}$$

For the action without the square root, we should also enforce a constraint that tells us whether things are timelike, spacelike, or null. For massive particles, we require that the trajectory is timelike. With θ being proper time, this means that the Lagrangian itself should be -1:

$$-f^2 \dot{t}^2 + f^{-2}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2) = -1. \tag{3.51}$$

For spacelike or timelike trajectories, we would have 1 or 0 on the right hand side respectively. Let us look for geodesics with $\theta = \pi/2$ and $\dot{\theta} = 0$. This is similar to the Kepler problem, where we say that angular momentum is conserved, so all the motion has to be in the plane normal to the angular momentum vector. Having done this, we do not look at the equations of motion, but instead see that the constraint becomes

$$\Rightarrow \dot{r}^2 + V_{\text{eff}}(r) = E^2, \tag{3.52}$$

with

$$V_{\text{eff}}(r) = \left(1 + \frac{l^2}{r^2}\right)\left(1 - \frac{r^2}{R^2}\right). \tag{3.53}$$

Here our intuition of relativistic Newtonian mechanics starts to kick in. This is the usual angular momentum barrier, illustrated in Figure 3.1. Notice that the potential is unbounded from below, but it does so after the point at which we cannot trust the metric; it is singular at $r = R$ and so we should think of the particle to only be moving at $r < R$. This is a bit like an inside-out black hole! We get pushed outwards. For $l = 0$, we have

$$r(\sigma) = R\sqrt{E^2 - 1}\sinh\left(\frac{\sigma}{R}\right). \tag{3.54}$$

We can look for weird things that happen at the singularity $r = R$. However, nothing weird happens. In fact, this hits $r = R$ in some finite proper time σ. Meanwhile, we can ask what happens to coordinate time. Coordinate time solves

$$\frac{\mathrm{d}t}{\mathrm{d}\sigma} = E\left(1 - \frac{r^2}{R^2}\right)^{-1}. \tag{3.55}$$

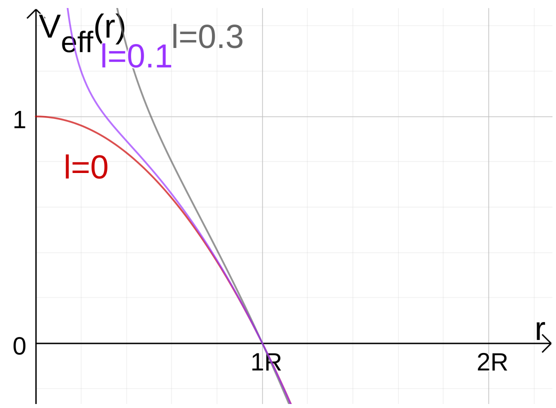

Figure 3.1

Claim 18: Solutions to this have $t \to \infty$ as $r \to R$.

Proof. To see this, suppose $r(\sigma_*) = R$ and expand $\sigma = \sigma_* - \epsilon$. Find that

$$\frac{\mathrm{d}t}{\mathrm{d}\epsilon} \approx -\frac{-\alpha}{\epsilon} \tag{3.56}$$

for some constant α. This implies that $t \sim -\alpha \log(\epsilon/R) \to \infty$ as $\epsilon \to 0$. □

This means that it takes a particle a finite proper time σ, but infinite coordinate time t, to move outwards from the origin and reach the singular point $r = R$.

Remark: Again, this looks very similar to an inside-out black hole. Ultimately, we will find out that this is where we live; our universe is like an inside-out black hole...

Definition 37: de Sitter Spacetime is

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right) dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{3.57}$$

where $d\Omega_2^2$ is the metric on S^2, i.e. $d\theta^2 + \sin^2\theta d\phi^2$.

Claim 19: In fact, de Sitter spacetime can be embedded in five-dimensional Minkowski space, which is $\mathbb{M}^5 = (\mathbb{R}^{1,4}, g)$ with metric

$$ds^2 = -(dX^0)^2 + \sum_{i=1}^{4}(dX^i)^2 \tag{3.58}$$

as the surface $-(X^0)^2 + \sum_{i=1}^4 (X^i)^2 = R^2$. The metric gets pulled back onto this surface.

Proof. Let $r^2 = \sum_{i=1}^3 (X^i)^2$ and $X^0 = \sqrt{R^2 - r^2}\sinh(t/R)$, $X^4 = \sqrt{R^2 - r^2}\cosh(t/R)$. In particular, this means that $(X^4)^2 - (X^0)^2 = R^2 - r^2$.

Exercise 3.1: You can check that if you compute $dX^0 = f dR + g dt$ and $dX^4 = \dots$, and plug these into the Minkowski spacetime metric (3.58), you recover the de Sitter metric.

□

These coordinates are not particularly symmetric, meaning that X^4 has been singled out from $X^{1,2,3}$, and, moreover, cover only $X^4 \geq 0$. A better choice of coordinates is

$$X^0 = R\sinh\left(\frac{\tau}{R}\right) \quad \text{and} \quad X^i = R\cosh\left(\frac{\tau}{R}\right) y^i, \tag{3.59}$$

where y^i parametrise a 3-sphere, i.e. $\sum_{i=1}^4 (y^i)^2 = 1$. These are called *global coordinates* on dS.

Exercise 3.2: Another small calculation you can do is to look at the variation again and plug it into (3.58) to give

$$ds^2 = -d\tau^2 + R^2\cosh^2\left(\frac{\tau}{R}\right) d\Omega_3^2, \tag{3.60}$$

where $d\Omega_3^2$ is the metric on S^3. This is called the *FRW* form of the metric.

We know that this metric must also solve the Einstein equation with the same cosmological constant (as it must!), since there is a coordinate transformation that takes us from de Sitter (dS) spacetime to this; they describe the same spacetime. This form of the metric has a natural cosmological interpretation in terms of an initially contracting and later expanding universe. For the past three billion years or so, this has been a fairly good approximation to our universe.

> **Remark:** Note that the first metric is the one that de Sitter found very soon after Einstein's theory was published. He found that immoving two observers move away from each other, and also calculated the redshift that emerged when those two observers sent light rays between each other. In particular, the first observations of Hubble and Slyther of the redshift were called the de Sitter effect. However, Hubble never believed in the expanding universe, which everyone credited him with, because the first metric seems to be independent of time. The second metric is more closely related to our notion of time, in which the universe actually (first contracts and then) expands with increasing τ.

3.3.3 $\Lambda < 0$: Anti-de Sitter Spacetime

Let us look at some more solutions to the Einstein equations. Specifically, let us consider a negative cosmological constant $\Lambda < 0$. Again, we look for solutions of the kind

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2 d\Omega_2^2. \tag{3.61}$$

Exactly the same calculation as before gives us $f(r) = \sqrt{1 + r^2/R^2}$ with $R^2 = -3/\Lambda$. This is *anti-de Sitter* (AdS) spacetime. The plus sign rather than minus sign means that there is no coordinate singularity. Repeating the geodesic calculations, we find that this time, massive geodesics obey $\dot{r}^2 + V_{\text{eff}}(r) = E^2$ (designed to look like Newtonian mechanics) with $V_{\text{eff}}(r) = (1 + l^2/r^2)(1 + r^2/R^2)$. l is to be interpreted as angular momentum. This is illustrated in 3.2. In particular, massive particles

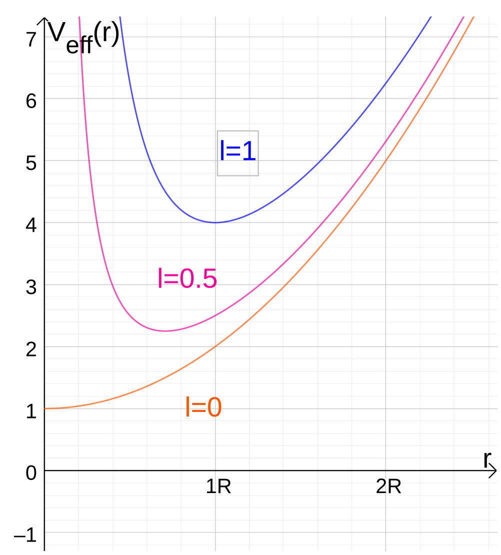

Figure 3.2

are confined to the centre of AdS. With a finite amount of energy, there is only so far we can get before we fall back towards low r. Massless particles follow null geodesics

$$-f^2\dot{t}r^2 + f^{-2}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0. \tag{3.62}$$

The right hand side is now zero. At $\theta = \pi/2$, $\dot{\theta} = 0$, we have

$$\dot{r}^2 + V_{\text{null}}(r) = E^2 \qquad V_{\text{null}} = \frac{l^2}{2r^2}\left(1 + \frac{r^2}{R^2}\right). \tag{3.63}$$

This qualitatively changes the behaviour of the potential, as illustrated in Figure 3.3. There is noth-

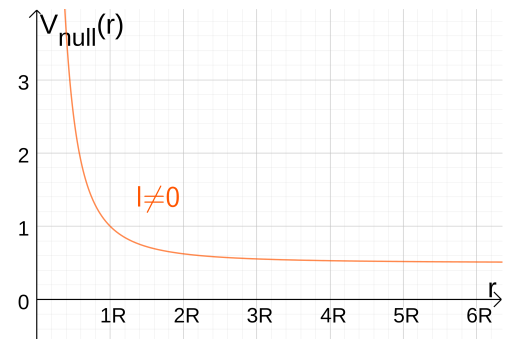

Figure 3.3

ing that stops the light rays to go as far as they like to $r \to \infty$.

Introduce new coordinates $r = R\sinh\rho$. The AdS metric becomes

$$ds^2 = -\cosh^2\rho dt^2 + R^2 d\rho^2 + R^2\sinh^2\rho(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.64}$$

The sinh is chosen to simplify the radial coordinate. The null geodesic equation (3.63) is

$$R\dot{\rho} = \pm\frac{E}{\cosh\rho} \qquad \Rightarrow \qquad R\sinh\rho = E(\sigma - \sigma_0), \tag{3.65}$$

where σ is an affine parameter. Massless particles hit $\rho \to \infty$ as $\sigma \to \infty$. However, going back to $E = \cosh^2\rho\dot{t}$, the equation which defines the energy, we have $R\tan(t/R) = E(\sigma - \sigma_0)$. So $t \to \frac{\pi R}{2}$ as $\sigma \to \infty$. Massless particles reach infinity of AdS in finite coordinate time.

Claim 20: Like dS, AdS can be viewed as a hyperboloid in $\mathbb{R}^{2,3}$

$$-(X^0)^2 - (X^4)^2 + \sum_{i=1}^{3}(X^i)^2 = R^2. \tag{3.66}$$

Proof. To see this, let

$$X^0 = R\cosh\rho\sin\sin\left(\frac{t}{R}\right) \tag{3.67a}$$

$$X^4 = R\cosh\rho\cos\left(\frac{t}{R}\right) \tag{3.67b}$$

$$X^i = Ry^i\sinh\rho, \tag{3.67c}$$

where $\sum(y^i)^2 = 1$ defines a two-sphere and we recover the AdS metric in the form (3.64). □

There is one last set of coordinates, which AdS is most often written in in the physics literature:

$$X^i = \frac{\widetilde{r}}{R}x^i, \qquad i = 0, 1, 2 \tag{3.68a}$$

$$X^4 - X^3 = \widetilde{r} \tag{3.68b}$$

$$X^4 + X^3 = \frac{R^2}{\widetilde{r}} + \frac{\widetilde{r}}{R^2}\eta_{ij}x^i x^j \tag{3.68c}$$

Computing the metric from this parametrisation gives

$$ds^2 = R^2\frac{d\widetilde{r}^2}{\widetilde{r}^2} + \frac{\widetilde{r}^2}{R^2} + \frac{\widetilde{r}^2}{R^2}\eta_{ij}dx^i dx^j. \tag{3.69}$$

This is called the *Poincaré patch*. It does not cover the whole of AdS. Unlike dS spacetime, AdS is not connected to our universe (as far as we can tell); however, it is an important spacetime in which we understand the behaviour of quantum gravity, which makes it worth studying!

3.4 Symmetries

There are three basic spacetimes in GR. We have intuition of why Minkowski is important; the other two have so far just been pulled out of thin air. The thing that makes these spaces special are the symmetries. They are the *maximally symmetric* spacetimes in general relativity.

Let us first define what a symmetry is. Any intuition you have about this is probably correct. Consider a sphere; it has the symmetry $SO(3)$. A rugby ball on the other hand, with the same topology, has only the symmetry $SO(2)$. (In general, this would be $O(3)$, but in these lectures we will only consider groups which are continuously connected to the identity.)

Consider a one-parameter family of diffeomorphisms $\sigma_t : \mathcal{M} \to \mathcal{M}$. Recall that this is associated to a vector field

$$K^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}, \tag{3.70}$$

which is tangent to the flow lines. We will define a symmetry by flowing from one point to a closeby one; if the point looks the same as the point we came from, this is a symmetry. More specifically, this flow is called an *isometry* if the metric looks the same at each point along the flow. From Claim 17, we know that this translates to

$$\mathcal{L}_K g = 0 \qquad \Longleftrightarrow \qquad \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0. \tag{3.71}$$

This is the *Killing equation*. Any K obeying this is called a *Killing vector*. These describe the symmetries of the metric.

> **Remark:** How is this related to the Killing form of *Symmetries*?

> **Remark:** $\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X = \mathcal{L}_{[X,Y]}$. This means that there is a Lie algebra structure emerging from continuous symmetries of metrics.

Example (Minkowski Space)**:** The Killing equation (3.71) becomes $\partial_\mu K_\nu + \partial_\nu K_\mu = 0$. The solutions to this are

$$K_\mu = c_\mu + \omega_{\mu\nu} x^\nu, \tag{3.72}$$

where c_μ are translations and $\omega_{\mu\nu} = -\omega_{\nu\mu}$ correspond to rotations or boosts. These are the components of the Killing vector. Similarly, we can define the Killing vectors themselves as

$$P_\mu = \frac{\partial}{\partial x^\mu} \quad \text{and} \quad M_{\mu\nu} = \eta_{\mu\rho} x^\rho \frac{\partial}{\partial x^\nu} - \eta_{\nu\rho} x^\rho \frac{\partial}{\partial x^\mu}. \tag{3.73}$$

Once we have two Killing vectors, we can compute their commutator as

$$\begin{gathered} [P_\mu, P_\nu] = 0 \qquad [M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\sigma} \\ [M_{\mu\nu}, P_\sigma] = -\eta_{\mu\sigma} P_\nu + \eta_{\sigma\nu} P_\mu \end{gathered} \tag{3.74}$$

These are the commutation relations of the Poincaré group.

Example: The isometries of dS and AdS are inherited from the $5d$ embedding.

dS has isometry group $SO(1,4)$

AdS has isometry group $SO(2,3)$

Note: both groups have dim(10), just like the Poincaré group of $\mathbb{M}^4$. Each of these three spaces is just as symmetric as the other two.

In $5d$, the Killing vectors are

$$M_{AB} = \eta_{AC} X^C \frac{\partial}{\partial X^B} - \eta_{BC} X^C \frac{\partial}{\partial X^A}, \tag{3.75}$$

where $\eta = (-+++)$ in dS and $\eta = (--++)$ in AdS. And X^A with $A = 0, 1, 2, 3, 4$. The flows induced by M_{AB} map the embedding hyperboloid to itself. This implies that these are isometries of (A)dS.

Claim 21: If the metric $g_{\mu\nu}(x)$ does not depend on some coordinate y, then $K = \frac{\partial}{\partial y}$ is a Killing vector.

Proof. This is easy to see when we look at what the Lie derivative does:

$$(\mathcal{L}_{\frac{\partial}{\partial y}} g)_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial y} = 0. \tag{3.76}$$

□

Example (de Sitter in static patch coordinates)**:** For the static patch, we expect $\frac{\partial}{\partial t}$ to be a Killing vector. We had $X^0 = \sqrt{R^2 - r^2} \sinh\left(\frac{t}{R}\right)$ and $X^4 = \sqrt{R^2 - r^2} \cosh\left(\frac{t}{R}\right)$. Look at $\frac{\partial}{\partial t} = \frac{\partial X^a}{\partial t} \frac{\partial}{\partial X^A} = \frac{1}{R}(X^4 \frac{\partial}{\partial X^0} + X^0 \frac{\partial}{\partial X^4})$.

Comment: Timelike Killing vectors, such that $g_{\mu\nu} K^\mu K^\nu < 0$, are used to define energy. Both $\mathbb{M}^4$ and AdS have such objects. And dS has such an object in the static patch, but *not* globally! Consider the Killing vector $K = X^4 \frac{\partial}{\partial X^0} + X^0 \frac{\partial}{\partial X^4}$. The first term increases the timelike direction X^0 when $X^4 > 0$ and decreases X^0 when $X^4 < 0$. The Killing vector is positive and timelike only in the static patch. Elsewhere, it is spacelike. Energy is a subtle concept in dS!

> **Remark:** Timelike geodesics do not take us out of the static patch; the problems with energy only arise when we think more globally than what a single particle is doing.

3.4.1 Conserved Quantities

Claim 22: Consider a particle moving on a geodesic $x^\mu(\tau)$ in a spacetime with some Killing vector K^μ. Then $Q = K_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ is conserved.

Proof. To see this,

$$\frac{\mathrm{d}Q}{\mathrm{d}\tau} = \partial_\nu K_\mu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + K_\mu \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} \tag{3.77}$$

$$= \partial_\nu K_\mu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} K_\mu \Gamma^\mu_{\rho\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} \tag{3.78}$$

$$= \nabla_\nu K_\mu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \overset{(3.71)}{=} 0. \tag{3.79}$$

We can also see this from the action

$$S = \int \mathrm{d}\tau\, \mathbf{g}_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu. \tag{3.80}$$

Consider $\delta x^\mu(\tau) = K^\mu(x)$. We have

$$\delta S = \int \mathrm{d}\tau \left[\partial_\rho g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} K^\rho + 2g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right]. \tag{3.81}$$

Using

$$g_{\mu\nu}\frac{\mathrm{d}K^\nu}{\mathrm{d}\tau} = \frac{\mathrm{d}K_\mu}{\mathrm{d}\tau} - \frac{\mathrm{d}g_{\mu\nu}}{\mathrm{d}\tau}K^\rho \tag{3.82}$$

$$= (\partial_\nu K_\mu - \partial_\nu g_{\mu\nu} K^\rho)\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \tag{3.83}$$

in the second argument, we have

$$\to \delta S = \int \mathrm{d}\tau\, 2\nabla_\mu K_\nu \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}, \tag{3.84}$$

where the Γ 's in ∇ come from the ∂g 's.

Exercise 3.3: Check this!

This means that we have

$$\delta S = 0 \quad \Longleftrightarrow \quad \nabla_{(\mu)}K^{\nu)} = 0, \tag{3.85}$$

which is the Killing equation. □

3.5 Asymptotics of Spacetime

Given a spacetime $\mathcal{M}$, with metric $g_{\mu\nu}(x)$, we consider the *conformal transformation*

$$\widetilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x), \tag{3.86}$$

where Ω is smooth, non-zero. $g_{\mu\nu}$ and $\widetilde{g}_{\mu\nu}$ describe different spacetimes, but they have the same causal structure since

$$g_{\mu\nu} X^\mu X^\nu = 0 \quad \Longleftrightarrow \quad \widetilde{g}_{\mu\nu} X^\mu X^\nu = 0. \tag{3.87}$$

So null / spaceile / timelike in $g_{\mu\nu}$ maps to null / spacelike / timelike in $\widetilde{g}_{\mu\nu}$.

3.6 Penrose Diagrams

The idea is to use conformal transformations to bring infinity a little closer.

> **Note:** We will try to draw a finite picture of the whole manifold $\mathcal{M}$ by stretching an squeezing our coordinates. But we still want the picture to accurately depict the causal structure of the spacetime $(\mathcal{M}, g)$.

3.6.1 Minkowski Space

Start with $\mathbb{R}^{1,1}$ and metric $ds^2 = -dt^2 + dx^2$. Introduce lightcone coordinates

$$u = t - x \qquad v = t + x \tag{3.88}$$

in which the metric is $ds^2 = -du dv$ with $u, v \in (-\infty, \infty)$. We now map this to a finite range

$$u = \tan \widetilde{u} \quad \text{and} \quad v = \tan \widetilde{v}, \tag{3.89}$$

with $\widetilde{u}, \widetilde{v} \in (-\frac{\pi}{2}, +\frac{\pi}{2})$.

$$ds^2 = -\frac{1}{\cos^2 \widetilde{u} \cos^2 \widetilde{v}} d\widetilde{u} d\widetilde{v} \tag{3.90}$$

Consider the new metric

$$d\widetilde{s}^2 = \cos^2 \widetilde{u} \cos^2 \widetilde{v} ds^2 = -d\widetilde{u} d\widetilde{v}. \tag{3.91}$$

> **Note:** The reason we use lightcone coordinates is the following: if we just proceed with $t = \tan \tilde{t}$ and $x = \tan \tilde{x}$, then we end up with $ds^2 = -\frac{1}{\cos^2 \tilde{t}} d\tilde{t} + \frac{1}{\cos^2 \tilde{x}} d\tilde{x}$. And trying to pull out the time prefactor, we get an ugly factor $\frac{cos^2 \tilde{t}}{cos^2 \tilde{x}}$ in front of $d\tilde{x}$.

There is a bit of a technicality there. Strictly speaking the points at the edge of the spacetime were not included in the first place, whereas $\tilde{u}, \tilde{v} \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$. Adding the points $\pm\frac{\pi}{2}$, which used to be $\pm\infty$, is called a *conformal compactification*.

> **Remark:** There is a theorem by Penrose that shows that this is essentially the unique way of doing conformal complexification.

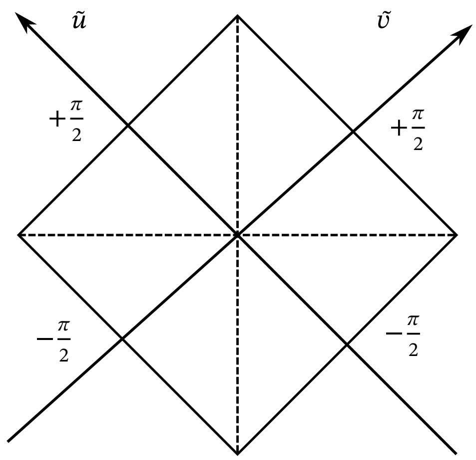

Figure 3.4

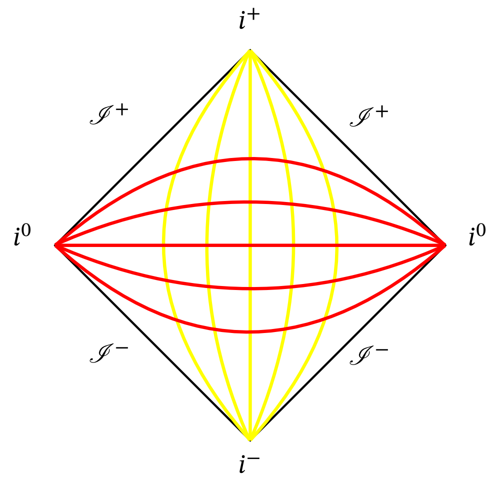

Figure 3.5

Figure 3.6: Penrose Diagrams

Now we draw the spacetime with light rays at 45° and time vertical. These diagrams, as depicted in Figure 3.4 are called *Penrose diagrams*. Note that we cannot trust distances in these diagrams; things that look close might be very far apart in Minkowski space. However, we can trust the causal structure. We can draw various geodesics on this diagram. In particular, we can draw timelike geodesics (constant x) and spacelike geodesics (constant t) as in Figure 3.5.

Note: All timelike geodesics start at $i^- : [-\frac{\pi}{2}, -\frac{\pi}{2}]$ and end at $i^+ : [+\frac{\pi}{2}, +\frac{\pi}{2}]$. These are called *past / future timelike infinity*.

- All spacelike geodesics start / end at two points $i^0 : [-\frac{\pi}{2}, +\frac{\pi}{2}]$ or $[+\frac{\pi}{2}, -\frac{\pi}{2}]$. These are *spacelike infinity*.
- All null curves start at $\mathscr{I}^-$ 'scri-minus' and end at $\mathscr{I}^+$ 'scri-plus'. These are *past / future null infinity*.

Note: If these names do not make sense, consider that we write $\mathscr{I}$ as \mathscr{I} in LaTeX.

Remark: In some sense, most of infinity is given by the diagonal lines $\mathscr{I}$.

The Penrose diagram immediately tells us basic things about the spacetime. For example, and two points in the spacetime have a common future and a common past. For any two points, draw 45° lines, as illustrated in Figure 3.7.

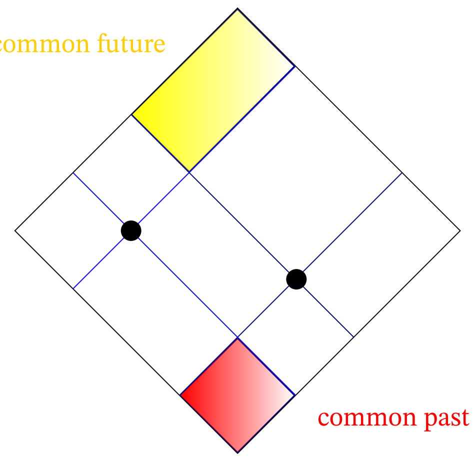

Figure 3.7: Any two points, even spacelike, share a common future and common past—spacetime regions in which their lightcones overlap.

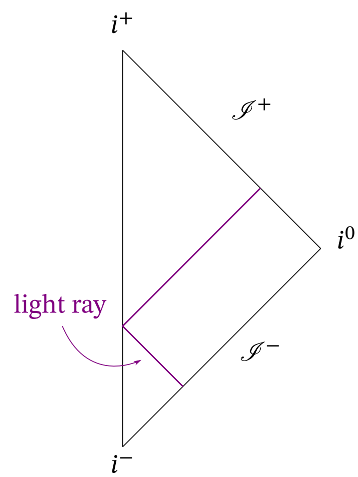

Figure 3.8: Penrose diagram for $\mathbb{R}^{1,3}$ in polar coordinates. Since $r \geq 0$, we only draw the half where $\tilde{v} \geq \tilde{u}$.

Since we can trust the causal structure of Penrose diagrams, this illustrates clearly that any two points, even if they are spacelike separated, they have a common future and common past since their lightcones overlap. If they follow timelike geodesics, they will eventually end up in their common future.

For $\mathbb{R}^{1,3}$, we do something similar. The metric is best written in polar coordinates

$$ds^2 = -dt^2 + dt^2 + r^2 d\Omega_2^2, \tag{3.92}$$

where $d\Omega_2^2$ is the metric on the two-sphere S^2. Let $u = t - r = \tan\widetilde{u}$ and $v = t + r = \tan\widetilde{v}$. Then

$$\begin{aligned} ds^2 &= -du dv + \frac{1}{4}(u-v)^2 d\Omega_2^2 &(3.93)\\ &= \frac{1}{4\cos^2\widetilde{u}\cos^2\widetilde{v}}\left(-4d\widetilde{u}d\widetilde{v} + \sin^2(\widetilde{u}-\widetilde{v})d\Omega_2^2\right). &(3.94)\end{aligned}$$

> **Note:** Recall the identity $\sin(\tilde{v})\cos(\tilde{u}) = \frac{1}{2}(\sin(\tilde{v}+\tilde{u}) + \sin(\tilde{v}-\tilde{u}))$ to show that
> $$\tan(\tilde{v}) - \tan(\tilde{u}) = \frac{\sin(\tilde{v}-\tilde{u})}{\cos(\tilde{v})\cos(\tilde{u})}. \tag{3.95}$$

There is one other subtlety, which stems from the coordinates. Unlike in the $2d$ case, the radial coordinate r has to be non-negative $r \geq 0$. This means that $v \geq u$ and therefore

$$-\frac{\pi}{2} \leq \tilde{u} \leq \tilde{v} \leq \frac{\pi}{2}. \tag{3.96}$$

We drop the S^2 and draw the Penrose diagram, which is now only the half for which $\tilde{v} \geq \tilde{u}$. This is shown in 3.8. The 'boundary' on the left is *not* a boundary of spacetime! It is $\tilde{u} = \tilde{v}$, so $r = 0$, and the two-sphere S^2 shrinks to zero size. In particular, notice the behaviour of a light ray.

3.6.2 de Sitter

We are going to work in global coordinates, since they cover the whole of spacetime:

$$ds^2 = -d\tau^2 + R^2\cosh^2\left(\frac{\tau}{R}\right)d\Omega_3^2. \tag{3.97}$$

We now introduce what cosmologists call *conformal time* $\eta \in (-\frac{\pi}{2}, +\frac{\pi}{2})$.

$$\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = \frac{1}{R\cosh(\tau/R)} \quad \Rightarrow \quad \cos\eta = \frac{1}{\cosh(\tau/R)}. \tag{3.98}$$

The purpose of conformal time is to pull out the factor of $R^2\cosh^2$ to the front of the metric. Plugging this into (3.97), we have

$$ds^2 = \frac{R^2}{\cos^2\eta}(-d\eta^2 + d\Omega_3^2). \tag{3.99}$$

Here, the metric on the three-sphere is $d\Omega_3^2 = d\chi^2 + \sin^2\chi d\Omega_2^2$, where $\chi \in [0,\pi]$. This means that de Sitter is conformal to

$$ds^2 = -d\eta^2 + d\chi^2 + \sin^2\chi d\Omega_2^2. \tag{3.100}$$

The difference is that these are not lightcone coordinates. This means that the Penrose diagram for dS is just a square, as in Figure 3.9. The vertical lines are not boundaries of the spacetime. They correspond to the north pole of S^3 and south pole of S^3 respectively. The horizontal lines do correspond the boundaries. We see that the boundary of dS is spacelike. There is no spatial infinty in dS; the spatial regions are compact manifolds without asymptotic region. Rotating by 45° has changed the physics dramatically. None of the above statements about $\mathbb{M}$ are true. In particular, no matter how long you wait, you cannot see the whole space; nor can you influence the whole space.

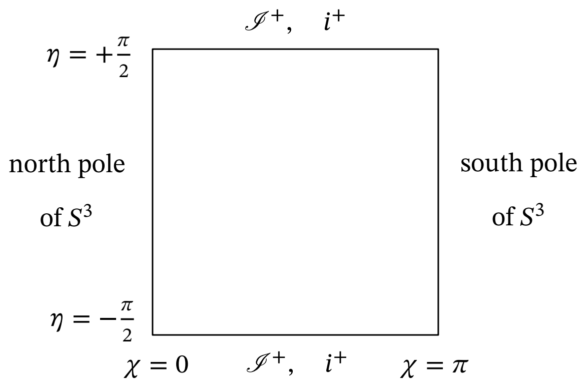

Figure 3.9

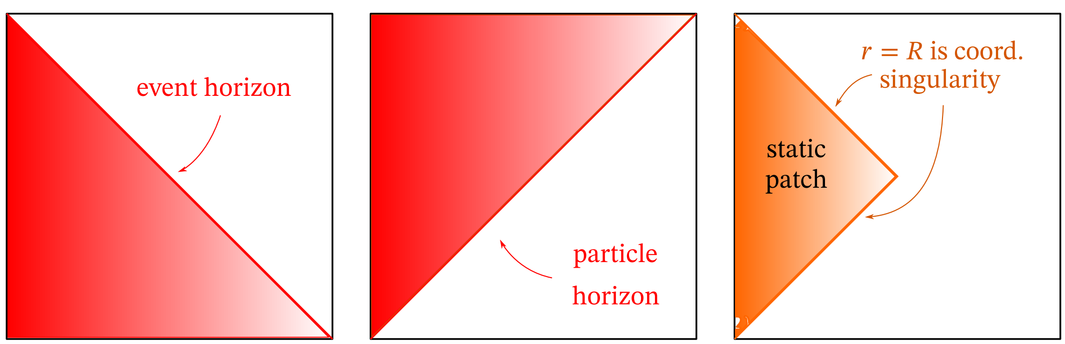

Figure 3.10 Figure 3.11 Figure 3.12

Suppose we have an observer sitting at the north pole, as illustrated in Fig. 3.10. There will be regions of `dS`, which she will never see; the boundary between what she can and cannot see is called the *event horizon*. Unlike in the case of black holes, every observer has a different event horizon. Similarly, if she sits on the south pole, we have the case of Figure 3.11. This in cosmology is called the *particle horizon*.

Exercise 3.4: Check that the static patch coordinates map to the intersection of the event and particle horizons, as depicted in Fig. 3.12.

In some sense, the static patch are the natural coordinates, since they are the region, which an observer can see and influence. However, they come with a coordinate singularity at $r = R$.

3.6.3 Anti-de Sitter Space

The metric in AdS is

$$ds^2 = -\cosh^2\rho\, dt^2 + R^2 d\rho^2 + R^2 \sinh^2\rho\, d\Omega_2^2, \tag{3.101}$$

where $\rho \in [0, \infty)$ is the radial coordinate. Compared to dS, the space and time components are swapped.

Let us introduce a new radial coordinate as $\frac{d\psi}{d\rho} = \frac{1}{\cosh\rho}$, giving $\cos\psi = \frac{1}{\cosh\rho}$. The metric becomes

$$ds^2 = \frac{1}{\cos^2\psi}\left(-d\tilde{t}^2 + d\psi^2 + \sin^2\psi d\Omega_2^2\right), \tag{3.102}$$

with $\tilde{t} = \frac{t}{R}$. AdS is conformal to a new metric

$$d\tilde{s}^2 = -d\tilde{t}^2 + d\psi^2 + \sin^2\psi d\Omega_2^2, \tag{3.103}$$

where $\tilde{t} \in (-\infty, +\infty)$ and $\psi \in [0, \frac{\pi}{2}]$. Again, we sketch the Penrose diagram in the usual way by ignoring the two-sphere.

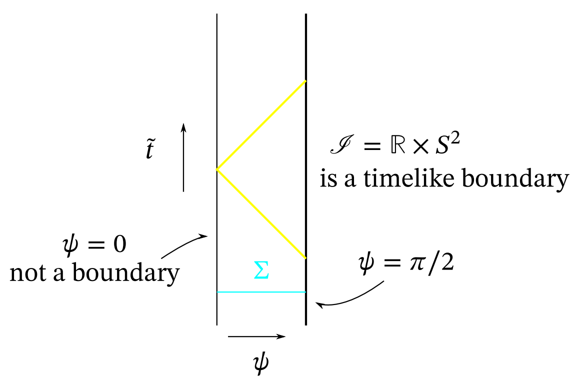

Figure 3.13

Note that a light ray as depicted in Fig. 3.13 hits the boundary in finite coordinate time t.

Remark: There is also something strange in AdS: We cannot specify initial conditions on some spacelike surface Σ and watch it evolve; at least not without knowledge of the boundary.

However, there are natural boundary conditions, such as saying that there will not be a random light ray coming out of the boundary at some point. In that case, we can evolve things in time!

Remark: It is only in AdS that we have this problem. For the cases of $\mathbb{M}$ and dS, see 3.14.

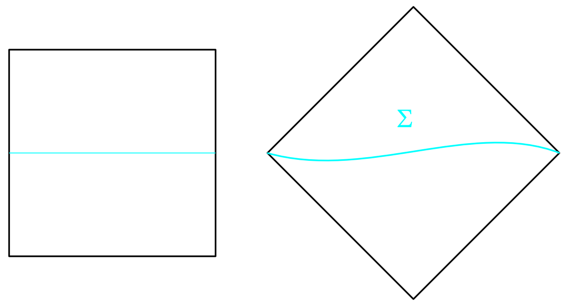

Figure 3.14: Spacelike surfaces of initial conditions in $\mathtt{dS}$ and $\mathbb{M}$.

3.7 Coupling Matter

In Minkowski space $\mathbb{M}$, the action for a scalar field is

$$S_{\text{scalar}} = \int \mathrm{d}^4x \left(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right). \tag{3.104}$$

In curved spacetime, we generalise this to

$$S_{\text{Scalar}} = \int \mathrm{d}^4x \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right) \tag{3.105}$$

Remark: $\nabla_\mu\phi = \partial_\mu\phi$ for a scalar. However, it is good practice to switch to covariant derivatives so that we can integrate by parts with the divergence theorem 4.

This is not quite the unique choice. We can actually add some more terms, which vanish when we go to flat space. In particular,

$$\int \mathrm{d}^4x \sqrt{-g}\frac{1}{2}\xi R\phi^2, \tag{3.106}$$

where ξ is just a dimensionless constant. Varying ϕ gives equations of motion

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{\partial V}{\partial\phi} - \xi R\phi = 0. \tag{3.107}$$

3.7.1 Maxwell Theory

There are other obvious fields that we can generalise to curved space. Using the language of differential forms, where $F = dA$ is a two-form, the Maxwell action is

$$\begin{aligned} S_{\text{Maxwell}} &= -\frac{1}{4}\int F\wedge\star F && (3.108)\\ &= -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}, && (3.109)\end{aligned}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. The last equality is true because the Christoffel symbols cancel due to the anti-symmetrisation. This is why we could differentiate forms before we had a connection; the difference cancels out.

The equation of motion, exactly the same story as in flat space, except with covariant derivatives, turn out to be

$$\nabla_\mu F^{\mu\nu} = 0. \tag{3.110}$$

This is the generalisation of the usual field theories to curved space. You can either view this as the field changing and the metric changing. The action for gravity and matter is then the sum of the Einstein-Hilbert action (3.24) with the matter action S_m:

$$S = \frac{1}{2}M_{pl}^2 \int \mathrm{d}^4x\,\sqrt{-g}(R - 2\Lambda) + S_m. \tag{3.111}$$

Definition 38: The *energy-momentum tensor*

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{3.112}$$

> **Note:** Suppose we had the Lagrangian density $\mathcal{L}(x)$ differentiated by $g^{\mu\nu}(x)$. This gives a delta function, which is killed by the action S_m, which is why we have to vary S_m and not $\mathcal{L}$.

Remark: There is a very slick argument: the usual $T^{\mu\nu}$ you get in QFT is a Noether current. The slick way to calculate the current (in notes) is to pretend that the parameter depends on spacetime.

In other words, take some symmetry for which $\delta S = 0$. Now take some symmetry which depends on space. The change of the action has to be of the form

$$\delta S = \int \mathrm{d}^4x\, J^\mu \partial_\mu \epsilon(x), \tag{3.113}$$

where $\epsilon(x)$ is the symmetry parameter. Integrate by parts to get

$$\cdots = -\int \mathrm{d}^4x\, \partial_\mu J^\mu \epsilon(x). \tag{3.114}$$

This means that $\partial_\mu J^\mu$ must vanish for constant ϵ. The details are in the printed notes.

Varying the metric gives the *Einstein equation*

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{3.115}$$

Recall that if we vary the metric by a diffeomorphism, the metric changes as

$$\delta g_{\mu\nu} = (\mathcal{L}_X g)_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu. \tag{3.116}$$

The change in the matter action is

$$\delta S_m = -2 \int \mathrm{d}^4x \sqrt{-g} T_{\mu\nu} \nabla^\mu X^\nu. \tag{3.117}$$

Requiring $\delta S_m = 0$ by diffeomorphism invariance, it must be that $\nabla_\mu T^{\mu\nu} = 0$. In other words, the energy-momentum tensor is *covariantly conserved*.

Remark: In flat space, this becomes genuine conservation. In curved space, this conservation law is slightly more subtle.

Example: Let $S = \int \mathrm{d}^4x \sqrt{-g} \left(\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right)$. We want to vary this with respect to $g_{\mu\nu}$. There are two ones, one in the $\sqrt{-g}$ and one hiding in the contraction $\nabla_\mu\phi\nabla^\mu\phi$. We obtain

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu - g_{\mu\nu}(\frac{1}{2}\nabla_\rho\phi\nabla^\rho\phi + V(\phi)). \tag{3.118}$$

Example: For Maxwell theory, $S = -\frac{1}{4}\int \mathrm{d}^4x \sqrt{-g} F_{\mu\nu}F^{\mu\nu}$. This gives

$$T_{\mu\nu} = g^{\rho\sigma} F_{\mu\rho}F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}F^{\rho\sigma}. \tag{3.119}$$

3.7.2 Fluids

If you really care about situations in which matter is backreacting on matter, the right description is in terms of a fluid, where lots of particles interact with each other. Often in GR, we model matter as a perfect fluid. These have a velocity field $u_\mu(\mathbf{x}, t)$, with $u_\mu u^\mu = -1$, which tells us that the particles are travelling on timelike geodesics. Then the stress-energy tensor is in its most general form:

$$T_{\mu\nu} = (\text{something})u_\mu u_\nu + (\text{something else})g_{\mu\nu}. \tag{3.120}$$

It turns out that we have

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \tag{3.121}$$

where $P(\mathbf{x}, t)$ is the *pressure* and $\rho(\mathbf{x}, t)$ the *energy density*. Usually, there is some relation between these two, called the *equation of state* $P = P(\rho)$.

Remark: This is the description of a fluid with microscopic interactions and pressure. In particular, there is no gravity yet. We could pick flat space for example. If you want to find out how they interact using gravity, you have to solve the Einstein equations.

For a fluid at rest, $u^\mu = (1, 0, 0, 0)$, and a flat metric $g_{\mu\nu} = \eta_{\mu\nu}$, then

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix}. \tag{3.122}$$

All stress tensors have to obey the conservation law.

Exercise 3.5: Apply the conservation law $\nabla_\mu T^{\mu\nu} = 0$ to this stress tensor. You get two equations out: mass-energy conservation and the Euler equations of fluid dynamics.