Advanced Quantum Field Theory

Part III Lent 2019

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- Path integrals
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1 Path Integrals in QM

Goal: Schrödinger's equation → path integral

Consider a Hamiltonian in one dimension $\hat{H} = H(\hat{x}, \hat{p})$, where position and momentum operators satisfy the common commutation relations $[\hat{x}, \hat{p}] = i\hbar$. Assume the it takes the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{1.1}$$

Schrödinger's equation then says that the time evolution of a state $|\psi(t)\rangle$ is given by

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle. \tag{1.2}$$

This has a formal solution, giving us the time-evolution operator

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle. \tag{1.3}$$

In the Schrödinger picture, the states are evolving in time whereas operators and their eigenstates are constant in time.

Definition 1 (wavefunction)**:** $\Psi(x,t) := \langle x|\psi(t)\rangle$

The Schrödinger equation then becomes

$$\langle x|\hat{H}|\psi(t)\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi(x,t). \tag{1.4}$$

We will turn this differential equation into an integral equation, where we will sum over particle paths—a path integral. We can introduce an integral by inserting a complete set of states $1 = \int dx_0\, |x_0\rangle\langle x_0|$.

$$\begin{aligned}
\Psi(x,t) &= \langle x|e^{-i\hat{H}t/\hbar}|\psi(0)\rangle && (1.5)\\
&= \int_{-\infty}^{\infty} \mathrm{d}x_0\, \langle x|e^{-i\hat{H}t/\hbar}|x_0\rangle\langle x_0|\psi(0)\rangle && (1.6)\\
&:= \int_{-\infty}^{\infty} \mathrm{d}x_0\, \underbrace{K(x,x_0;t)}_{\text{‘kernel’}}\Psi(x_0,0) && (1.7)
\end{aligned}$$

We repeat this insertion for n intermediate times and positions.

Notation: Let $0 := t_0 < t_1 < \cdots < t_n < t_{n+1} := T$

And we also want to factor the exponential into n terms:

$$e^{i\hat{H}T/\hbar} = e^{-\frac{i}{\hbar}\hat{H}(t_{n+1}-t_n)} \cdots e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)}. \tag{1.8}$$

Then

$$K(x, x_0; T) = \int_{-\infty}^{\infty} \left[\prod_{r=1}^{n} \mathrm{d}x_r \, \langle x_{r+1} | \, e^{-\frac{i}{\hbar}\hat{H}(t_{r+1}-t_r)} \, |x_r\rangle \right] \langle x_1 | \, e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)} \, |x_0\rangle \tag{1.9}$$

Integrals are over all possible position eigenstates at times $t_r, r = 1, \ldots, n$.

Free Theory

Consider the "free" theory, with $V(\hat{x}) = 0$. We will now play a similar but different trick to what we did before. Let us insert a complete set of momentum eigenstates $1 = \int_{-\infty}^{\infty} đp \, |p\rangle \langle p|$. We also note that these momentum eigenstates are plane waves $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$.

Definition 2 (barred differential)**:** We define the normalised differential in Fourier space to be

$$đp := \frac{\mathrm{d}p}{2\pi\hbar}. \tag{1.10}$$

In higher dimensional QFT, this generalises to

$$đ^n p := \frac{\mathrm{d}^n p}{(2\pi\hbar)^n} \tag{1.11}$$

The corresponding kernel is

$$K_0(x, x'; t) = \langle x| \exp\left(\frac{i}{\hbar} \frac{\hat{p}^2}{2m} t \right) \left|x'\right\rangle . \tag{1.12}$$

$$= \int_{-\infty}^{\infty} đp \, e^{-ip^2 t/2m\hbar} e^{ip(x-x')/\hbar} \tag{1.13}$$

$$= \sqrt{\frac{m}{2\pi i\hbar t}} e^{\frac{im(x-x')^2}{2\hbar t}} . \tag{1.14}$$

Remark:

$$\lim_{t\to 0} \left\{ K_0(x, x'; t) \right\} = \delta(x - x'). \tag{1.15}$$

As expected from $\left\langle x \middle| x' \right\rangle = \delta(x - x')$.

From the Baker-Campbell-Hausdorff formula, we know that

$$e^{\epsilon\hat{A}}e^{\epsilon\hat{B}} = \exp\left(\epsilon\hat{A} + \epsilon\hat{B} + \frac{\epsilon^2}{2}[\hat{A},\hat{B}] + \ldots\right) \neq e^{\epsilon(\hat{A}+\hat{B})} \tag{1.16}$$

$$\text{for small } \epsilon: \qquad e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}}e^{\epsilon\hat{B}}(1+O(\epsilon^2)) \tag{1.17}$$

Letting $\epsilon = 1/n$ and raising the above to the n^{th} power[1] gives

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty}\left\{e^{\hat{A}/n}e^{\hat{B}/n}\right\}^n. \tag{1.18}$$

We will use this to separate kinetic and potential terms.

Take $t_{r+1} - t_r = \delta t$ with $\delta t \ll T$ and n large such that $n\delta t = T$.

$$e^{-\frac{i}{\hbar}\hat{H}\delta t} = \exp\left(-\frac{i\hat{p}^2\delta t}{2m\hbar}\right)\exp\left(-\frac{iV(\hat{x})\delta t}{\hbar}\right)[1+O(\delta t^2)] \tag{1.19}$$

Using the result (1.14),

$$\langle x_{r+1}|\exp\left(-\frac{i\hat{H}}{\hbar}\delta t\right)|x_r\rangle = e^{-iV(x_r)\delta t/\hbar}K_0(x_{r+1},x_r;\delta t) \tag{1.20}$$

$$= \sqrt{\frac{m}{2\pi i\hbar\delta t}}\exp\left[\frac{im}{2\hbar}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2\delta t - \frac{i}{\hbar}V(x_r)\delta t\right] \tag{1.21}$$

With $T = n\delta t$

$$K(x,x_0;T) = \int\left[\prod_{r=1}^n \mathrm{d}x_r\right]\left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{n+1}{2}}\exp\left\{i\sum_{r=0}^n\left[\frac{m}{2\hbar}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2 - \frac{1}{\hbar}V(x_r)\right]\delta t\right\} \tag{1.22}$$

In the limit $n\to\infty$, $\delta t\to 0$ with $n\delta t = T$ fixed, the exponent becomes

$$\frac{1}{\hbar}\int_0^T \mathrm{d}t\left[\frac{1}{2}m\dot{x}^2 - V(x)\right] = \int_0^T \mathrm{d}t\, L(x,\dot{x}), \tag{1.23}$$

where L is the classical Lagrangian, the Legendre transformation of the classical Hamiltonian. The classical action is $S = \int \mathrm{d}t\, L(x,\dot{x})$.

The main result therefore is that the path integral for the kernel is

$$\boxed{K(x,x_0;t) := \langle x|\, e^{-i\hat{H}t/\hbar}\,|x_0\rangle = \int \mathcal{D}x\, e^{\frac{i}{\hbar}S}} \tag{1.24}$$

Definition 3 (functional integral)**:**

$$\mathcal{D}x = \lim_{\substack{\delta t\to 0\\ n\delta T \text{ fixed}}}\left\{(\sqrt{\ldots})\prod_{r=1}^n(\sqrt{\ldots}dx_r)\right\} \tag{1.25}$$

We do not need to care about normalization factors.

Remark: In the limit $\hbar\to 0$, the interference of amplitudes is dominated by the ones close to the extremal path where δS. This leads to Hamilton's principle of least action.

[1]This step is sometimes called Suzuki–Trotter decomposition.

We may analytically continue this to imaginary time. Let $\tau = it$. In terms of this imaginary time, we have

$$\langle x| \, e^{-\hat{H}\tau/\hbar} \, |x_0\rangle = \int \mathcal{D}x \, e^{-S/\hbar}. \tag{1.26}$$

Mathematically, it makes these integrals much more well-defined, clearly convergent. Here the least-action principle is really evident from the $\hbar = 0$ argument. We also see the connection to statistical physics, interpreting $e^{-S/\hbar}$ as the Boltzmann factor $e^{-\beta H}$.

Quantum mechanics is quantum field theory in 0+1 dimensions. We are treating space differently from time: $\hat{x}(t)$ is a field, whereas t is a variable. However, Lorentz invariance forces us to put x and t on the same footing. In QFT, we solve this problem by demoting x from a field to another label. We then talk about fields $\phi(x, t)$ and want to know about the behaviour of these in all of spacetime.

String theory gives another ansatz to this problem by promoting again, instead of demoting operators to labels.

2 Integrals and their Diagrammatic Expansions

In QFT, we are interested in correlation functions. The following discussion will be very similar to what we have seen in the *Statistical Field Theory* course in Michaelmas term, also no knowledge from that course will be assumed here.

For simplicity, consider a 0 -dimensional field $\varphi \in \mathbb{R}$. As if we are in imaginary time, let

$$Z = \int_{\mathbb{R}} \mathrm{d}\varphi \, e^{-\frac{S(\varphi)}{\hbar}}. \tag{2.1}$$

Assume that the action $S(\varphi)$ is an even polynomial and $S(\varphi) \to \infty$ as $\varphi \to \pm\infty$.

We will be interested in expectation values

$$\langle f \rangle = \frac{1}{Z} \int \mathrm{d}\varphi \, f(\varphi) e^{-S/\hbar}. \tag{2.2}$$

Again, assume f does not grow too fast as $\varphi \to \pm\infty$. Usually, f is polynomial in φ.

2.1 Free Theory

Say we have N scalar fields (in $0+1$ dimensions we should really just say 'variables') φ_a with $a = 1, \ldots, N$, with action

$$S_0(\varphi) = \frac{1}{2} M_{ab} \varphi_a \varphi_b = \frac{1}{2} \varphi^T M \varphi, \tag{2.3}$$

where M is an $N \times N$ symmetric, positive definite ($\det M > 0$) matrix.

We can diagonalise this. There exists some orthogonal P such that $M = P \Lambda P^T$, where Λ is diagonal. Let $\chi = P^T\varphi$. Then the free partition function is

$$Z_0 = \int \mathrm{d}^N\varphi \exp\left(-\frac{1}{2\hbar}\varphi^T M\varphi\right) \tag{2.4}$$

$$= \int \mathrm{d}^N\chi \exp\left(-\frac{1}{2\hbar}\chi^T \Lambda\chi\right) \tag{2.5}$$

$$= \prod_{c=1}^{N} \int \mathrm{d}\chi_c \, e^{-\frac{\lambda_c}{2\hbar}\chi^2} = \sqrt{\frac{(2\pi\hbar)^N}{\det M}} \tag{2.6}$$

We want to get from the partition function to correlation functions. We can do this by introducing an N-component vector of external sources J to the action

$$S_0(\varphi) \to S_0 + J^T\varphi. \tag{2.7}$$

The partition function is then

$$Z_0(J) = \int \mathrm{d}^N\varphi \exp\left(-\frac{1}{2\hbar}\varphi^T M\varphi - \frac{1}{\hbar}\right)J^T\varphi. \tag{2.8}$$

Similar to solving an ordinary Gaussian integral, we complete the square by writing $\widetilde{\varphi} = \varphi + M^{-1}J$. One can then solve this integral to be

$$Z_0(J) = Z_0(0)\exp\left(\frac{1}{2\hbar}J^T M^{-1}J\right). \tag{2.9}$$

This is called the *generating function*[1]. Correlation functions are obtained from differentiating with respect to the auxiliary sources J and evaluating the whole expression at $J = 0$:

$$\langle\varphi_a\varphi_b\rangle = \frac{1}{Z_0(0)}\int \mathrm{d}^N\varphi\, \varphi_a\varphi_b \exp\left(-\frac{1}{2\hbar}\varphi^T M\varphi - \frac{1}{\hbar}J^T\varphi\right)\bigg|_{J=0}. \tag{2.10}$$

$$= \frac{1}{Z_0(0)}\int \mathrm{d}^N\varphi \left(-\hbar\frac{\partial}{\partial J_a}\right)\left(-\hbar\frac{\partial}{\partial J_b}\right)\exp(\ldots)\bigg|_{J=0} \tag{2.11}$$

$$= \frac{1}{Z_0(0)}\left(-\hbar\frac{\partial}{\partial J_a}\right)\left(-\hbar\frac{\partial}{\partial J_b}\right)Z_0(J)|_{J=0} \tag{2.12}$$

$$= \hbar\,(M^{-1})_{ab} := \; a \;\text{———}\; b \tag{2.13}$$

Connecting this to the *Quantum Field Theory* course, we identify this as the *free propagator*.

More generally, let $l(\varphi)$ be a *linear* combination of φ_a.

$$l(\varphi) = \sum_{a=1}^{N} l_a\varphi_a, \qquad l_a \in \mathbb{R}. \tag{2.14}$$

Then the steps above are equivalent to swapping $l(\varphi)$ for $l(-\hbar\frac{\partial}{\partial J}) = -\hbar\sum_a l_a\frac{\partial}{\partial J_a}$.

[1] When we go to higher dimensions, where $J = J(x)$ this will be a generating functional $Z[J(x)]$.

The correlation function can again be evaluated explicitly by the introduction of an auxiliary current J:

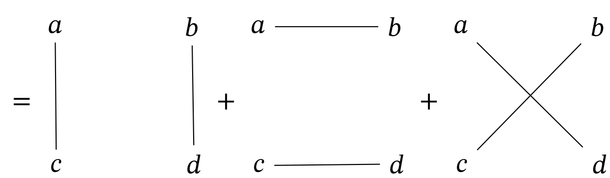
$$\langle l^{(1)}(\varphi)\cdots l^{(p)}(\varphi)\rangle = \frac{1}{Z_0(0)}\int \mathrm{d}^N\varphi \prod_{i=1}^{p} l^{(i)}(\varphi) e^{-\frac{1}{2\hbar}\varphi^T M\varphi - \frac{1}{\hbar}J^T\varphi}\Big|_{J=0} . \tag{2.15}$$

$$= (-\hbar)^p \prod_{i=1}^{p} l^{(i)}(\frac{\partial}{\partial J})\, e^{\frac{1}{2\hbar}J^T M^{-1} J}\Big|_{J=0} \tag{2.16}$$

In other words, if p is odd, the integrand is odd in some φ_a and the integral over $\varphi_a \in (-\infty, +\infty)$ vanishes. For $p = 2k$, the terms which are non-zero as $J \to 0$ have the following form. We need half the derivatives to bring down components of $m^{-1}J$ and half to remove the J -dependence from those terms that earlier derivatives brought down. As such, we get exactly k factors of M^{-1} . This is Wick's theorem.

Example (4-point function)**:** Consider the 4-point correlation function. One can check that the above result gives

$$\langle \varphi_a\varphi_b\varphi_c\varphi_d\rangle = \hbar^2\left[(M^{-1})_{ab}(M^{-1})_{cd} + (M^{-1})_{ac}(M^{-1})_{bd} + (M^{-1})_{ad}(M^{-1})_{bc}\right] \tag{2.17}$$



(2.18)

We end up with three terms, one for each way of grouping the 4 fields into pairs.

In general, for $\langle \varphi_1 \dots \varphi_{2k}\rangle$, the number of terms is the number of distinct ways of pairing the $2k$ fields. This is $(2k+1)!! = (2k)!/(2^k k!)$; the number of permutations of $2k$ fields is $(2k)!$, but we have to divide this by the 2^k permutations within the pairs and the $k!$ ways of rearranging the pairs.

Remark: For complex fields, M is Hermitian but not symmetric anymore. In that case, the order of indices of M^{-1} is important. We keep track of this by drawing the propagator with a directed line

$$\langle \phi_a\phi_b^*\rangle = \hbar\,(M^{-1})_{ab} := a \longrightarrow b \tag{2.19}$$

2.2 Interacting Theory

We want to go beyond the free theory. The way we are going to achieve this is by an expansion about the classical result $\hbar$. The resulting integral will end up not being convergent.

Claim 1: Integrals like

$$\int \mathrm{d}\phi\, f(\phi) e^{-S/\hbar} \tag{2.20}$$

do not have a Taylor expansion about $\hbar = 0$.

Proof (Dyson). If the expansion about $\hbar = 0$ existed for $\hbar > 0$, then in the complex plane, there must be some open neighbourhood of $\hbar$ in which the expansion converges. For $S(\phi)$ has a minimum, the integral is divergent if $\mathrm{Re}(\hbar) < 0$. Therefore, the radius of convergence cannot be greater than zero. □

So the $\hbar$-expansion is at best *asymptotic*.

Definition 4 (asymptotic)**:** A series $\sum_{n=0}^{\infty} c_n \hbar^n$ is asymptotic to a function $I(\hbar)$ if

$$\lim_{\hbar\to 0^+} \frac{1}{\hbar^N} \left| I(\hbar) - \sum_{n=0}^{N} c_n \hbar^n \right| = 0. \tag{2.21}$$

Notation: We write $I(\hbar) \sim \sum_{n=0}^{\infty} c_n \hbar^n$.

The series misses out transcendental terms like $e^{-\frac{1}{\hbar^2}} \sim 0$. However, these can evidently be important since obviously $e^{-\frac{1}{\hbar^2}} \neq 0$ for finite $\hbar$. These are called *non-perturbative contributions*. These become important in particular for non-Abelian gauge theories.

Take the ϕ-fourth action for a real scalar

$$S(\phi) = \underbrace{\frac{1}{2} m^2 \phi^2}_{S_0(\phi)} + \underbrace{\frac{\lambda}{4!} \phi^4}_{S_1(\phi)} \qquad \begin{matrix} m^2 > 0 \\ \lambda > 0. \end{matrix} \tag{2.22}$$

Expand the exponential in the paritition function Z about the minimum of $S(\phi)$, which is $\phi = 0$.

$$Z = \int \mathrm{d}\phi \exp\left[-\frac{1}{\hbar} \left(\frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right) \right] \tag{2.23}$$

$$= \int \mathrm{d}\phi\, e^{-S_0/\hbar} \overbrace{\sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!\hbar} \right)^n \phi^{4n}}^{e^{-S_1/\hbar}}. \tag{2.24}$$

In order to make progress, we truncate the series to be able to swap the order of summation and integration. This misses out transcendental terms. In the end, we end up with a series that is asymptotic to Z:

$$Z \sim \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^{N} \frac{1}{n!} \left(-\frac{\hbar\lambda}{4!m^4}\right)^n 2^{2v} \int_0^\infty \mathrm{d}t\, e^{-t} t^{2n+\frac{1}{2}-1}, \tag{2.25}$$

where $t = \frac{1}{2\hbar} m^2 \phi^2$. We recognise the integral to be the Gamma function

$$\int_0^\infty \mathrm{d}t\, e^{-t} t^{2n+\frac{1}{2}-1} = \Gamma(2n + \frac{1}{2}) = \frac{(4n)!\sqrt{\pi}}{4^{2n}(2n)!}. \tag{2.26}$$

The partition function is

$$Z \sim \frac{\sqrt{2\pi\hbar}}{m} \sum_{n=0}^{N} \left(-\frac{\hbar\lambda}{m^4}\right)^n \underbrace{\frac{1}{(4!)n!}}_{(a)} \underbrace{\frac{(4n)!}{2^{2n}(2n)!}}_{(b)} \tag{2.27}$$

The factor on the right comes in part from (a) the Taylor expansion of the term $S_1(\phi) = (\phi)\frac{\lambda}{4!}\phi^4$ in the exponential and from (b) the number of ways of pairing the $4n$ fields of the n copies of ϕ^4. Stirling's approximation allows us to write $n! \approx e^{n\ln n}$. The factor in the partition function then become

$$\frac{(4n)!}{(4!)^n n! 2^{2n}(2n)!} \approx n!. \tag{2.28}$$

We end up with factorial growth, signalling that the series is not convergent, but asymptotic!

2.2.1 Diagrammatic Method

Let us now introduce a current J to obtain the generating function

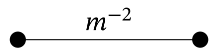
$$Z(J) = \int \mathrm{d}\phi \exp\left[-\frac{1}{\hbar}\left(S_0(\phi) + S_1(\phi) + J\phi\right)\right] \tag{2.29}$$

$$= \exp\left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right] \underbrace{\int \mathrm{d}\phi \exp\left[-\frac{1}{\hbar}(S_0 + J\phi)\right]}_{Z_0(J)} \tag{2.30}$$

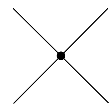
$$\overset{(2.9)}{\propto} \exp\left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right] \exp\left(\frac{1}{2\hbar} J^T M^{-1} J\right), \qquad M = m^2 \tag{2.31}$$

$$\sim \sum_{V=0}^{N} \frac{1}{V!}\left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right]^V \sum_{P=0}^{\infty} \frac{1}{P!}\left(\frac{1}{2\hbar}\frac{J^2}{2\hbar m^2}\right)^P. \tag{2.32}$$

This is called the *double expansion*. Diagrammatically, each of the P propagators, represented by a line as in Fig. 2.1a, give a factor of $M^{-1} = m^{-2}$. We use a large filled circle at the end of a line to represent a source factor J. Each of the V factors $\left(\frac{\partial}{\partial J}\right)^4$, originating from the interaction term $S_1(\phi)$,



(a) Propagator with external sources J at both ends.



$$\sim \quad -\lambda \left(\frac{\partial}{\partial J}\right)^4$$

(b) Vertex

Figure 2.1: Components of the diagrammatic representation of the double series.

are associated with a vertex as in Fig. 2.1b. We use a small dot (or sometimes a small square) to mark a vertex.

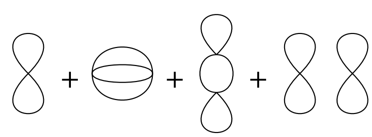
Let us check that we reproduce the result (2.27) for $Z(0)$. For a term to be non-zero when $J = 0$, we need the number of derivations to be equal to the number of source terms coming from the end of propagators.

Notation (external sources)**:** We denote by E the number of external sources, which are left undifferentiated. For P propagtors and V vertices, the number of such sources is

$$E := 2P - 4V. \tag{2.33}$$

For $Z(0)$ we will require $E = 0$, whereas for n-point functions, we will want $E = n$. The first non-trivial terms are $(V, P) = (1, 2), (2, 4), \ldots$.

$Z(0) \propto 1 +$



$+ O(V = 3). \qquad (2.34)$

2.2.2 Symmetry Factors

Definition 5 (symmetry factor)**:** The *symmetry factor* S is the number of ways of redrawing the unlabeled diagram, leaving it unchanged.

Definition 6 (pre-diagram)**:** A *pre-diagram* for a (V, P) term in the double expansion is a collection of V vertices and P propagators, where the ends of the vertex lines are labeled by numbers and the ends of the propagators labelled by letters.

We count the number of times each diagram appears in the double expansion by using such pre-diagrams.

Example ($V = 1$)**:** Consider the first diagram with only a single vertex and two loops attached to it. There are $A = 4!$ ways of matching the sources a, a', b, b' to the derivatives at 1, 2, 3, and 4. This is

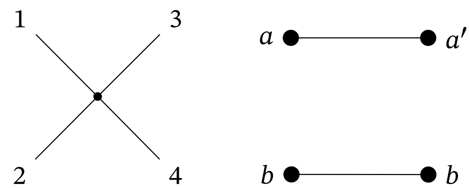


Figure 2.2: Pre-diagram of the $V = 1$ diagram with $P = 2$ loops.

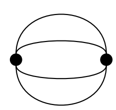
cancelled by a 4! in the denominator $F = (V!)(4!)^V(P!)2^P = 4! \cdot 2 \cdot 2^2$ of Eq. (2.27).

So the $(V,P) = (1,2)$ diagram comes with a prefactor of $\frac{A}{F} = \frac{1}{8}$ (times $-\hbar\lambda m^{-4}$).

In general, S is given by the relation $\frac{A}{F} = \frac{1}{S}$, where A is the number of ways of assigning the sources to the derivatives and F the number of non-equivalent permutations of all vertices, each vertex's legs, all propagators, and their ends. However, the symmetry of each particular graph is important: If the diagram has a particular symmetry, then some permutations in F may be identical and have been double-counted. For the above diagram, consider the pairing $(1a, 2a', 3b, 4b')$. Swapping $a \leftrightarrow a'$ and $1 \leftrightarrow 2$ gives exactly the same graph, so it should not be counted twice.

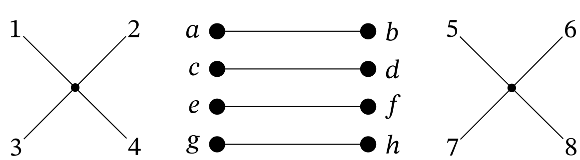
An alternative way to determine S is to consider the actions, which leave invariant the unlabelled diagram. These are called the automorphisms of the graph. For $(1,2)$, we can swap the direction of upper and lower loops (2^2) and also swap upper and lower loops (2). Therefore, we obtain $S = 2 \cdot 2^2 = 8$.

Example (basketball)**:** Let us look at a slightly more complicated example. The *basketball* diagram has the symmetry factor



$$S = 4! \cdot 2 = 48 \tag{2.35}$$

The pre-diagram associated to this is



(2.36)

We can simply calculate $F = 2 \cdot (4!)^2 \cdot 4! \cdot 2^4 = 4^3 \cdot 2^{14}$ and from the pre-diagram we determine the ways to pair currents and derivatives

$$A = \underbrace{(2\cdot 4)}_{a}\ \underbrace{(4)}_{b}\ \underbrace{(2\cdot 3)}_{c}\ \underbrace{(3)}_{d}\ \underbrace{(2\cdot 2)}_{e}\ \underbrace{(2)}_{f}\ \underbrace{(2\cdot 1)}_{g}\ \underbrace{(1)}_{h} = 3^2 \cdot 2^{10}. \tag{2.37}$$

. There are probably multiple ways to obtain this factor, but the reasoning here was as follows: For the letter a, we have a choice (factor 2) whether to connect to the left or the right vertex. In each case, we have 4 numbers to connect to. Since the basketball shape has no loops, this means that b has no choice in which vertex to use; it always has to be the one that we did not choose for a. For b we only have a choice of 4 numbers to connect to. For c, we again have a choice of two vertices, but only three remaining numbers (since the others are filled by a or b). We proceed in the same way for the remaining letters. Thus $A/F = 1/48$.

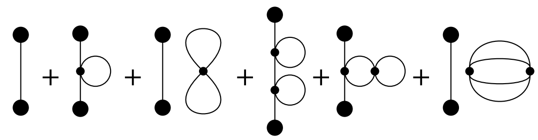
For the other diagrams, we have

$$\frac{Z(0)}{Z_0(0)} = 1 - \frac{\lambda\hbar}{8m^4} + \frac{\hbar^2\lambda^2}{m^8}\left(\frac{1}{48} + \frac{1}{16} + \frac{1}{128}\right) \tag{2.38}$$

2.2.3 Diagrams with External Sources

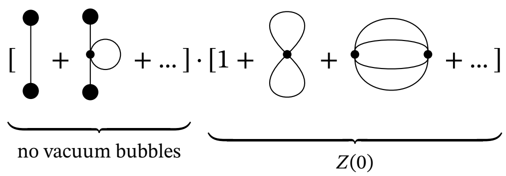
As we have previously mentioned, diagrams with $E = n$ external sources need to be considered for the n-point correlation functions. Let us focus on those diagrams that have $E = 2$ external currents.

$$Z(J) \supset [\quad \cdots \quad + \ldots] \tag{2.39}$$



Factor out vacuum bubble diagrams

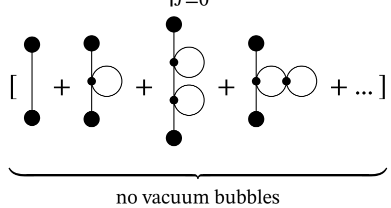
$$[\ldots] = [\quad \cdots \quad + \ldots] \tag{2.40}$$



The 2-point expectation value is then given by

$$\langle \phi^2 \rangle = \frac{(-\hbar)^2}{Z(0)} \left. \frac{\partial^2 Z(J)}{\partial J^2} \right|_{J=0} \tag{2.41}$$

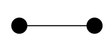
$$= [\quad \cdots \quad + \ldots] \tag{2.42}$$

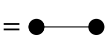


Symmetry factors

From $Z(J)$ (2.32), the $E = 2$, $V = 0$ ($P = 1$) term is

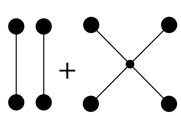
$$\frac{1}{2\hbar} \frac{J^2}{m^2} \tag{2.43}$$



We have $F = 2$ and $A = 1$, so $\frac{A}{F} = \frac{1}{2} = \frac{1}{S}$. The expectation value is $\langle \phi^2 \rangle = \frac{\hbar}{m^2} =$  as expected!

$\langle \phi^{2n} \rangle$ proceeds similarly, but note that there *are* disconnected diagrams

$$\langle \phi^4 \rangle = \quad \cdots \quad + COPYFROMNOTES \tag{2.44}$$



2.3 Effective Actions

Definition 7 (Wilson effective action)**:** We define $W(\phi)$ such that $Z(J) = e^{-W(J)/\hbar}$.

Claim 2: $W(0)$ is the sum of all connected vacuum diagrams and $W(J)$ is the sum of all connected diagrams.

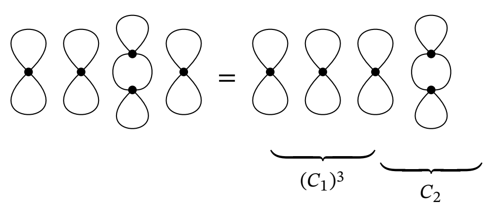
Proof. We denote the set of connected diagrams as $\{C_I\}$, which are taken to contain their respective symmetry factors. Any diagram D is a product of connected diagrams:

$$D = \frac{1}{S_D} \prod_I (C_I)^{n_I}, \tag{2.45}$$

where n_I is the number of times C_I appears in D, and S_D is the symmetry factor associated with rearranging diagrams C_I's, given by

$$S_D = \prod_I n_I!. \tag{2.46}$$

Example: Consider the diagram $D \propto (C_1)^3 \cdot C_2$:



(2.47)

The disconnected parts commute. We have $n_1 = 3$ parts of type C_1 and $n_2 = 1$ of C_2. The symmetry factor associated with the permutations of disconnected pieces is $S_D = 3! \cdot 1! = 6$.

Let $\{n_I\}$ be the set of integers specifying D, then

$$\begin{aligned} \frac{Z}{Z_D} = \sum_{\{n_I\}} D = \sum_{\{n_I\}} \prod_I \frac{1}{n_I!}(c_I)^{n_I} && (2.48)\\ = \prod_I \sum_{n_I} \frac{1}{n_I!}(c_I)^{n_I} && (2.49)\\ = \exp\left(\sum_I c_I\right) && (2.50)\\ = \exp(\text{sum of unique connected diagrams}) && (2.51)\\ := e^{-(W-W_0)/\hbar}, && (2.52) \end{aligned}$$

where $W = W_0 - \hbar \sum_I c_I$. □

Claim 3: $W(J)$ is the generating function for *connected correlation functions*.

Example ($\langle\phi^2\rangle$)**:** Taking logarithms, we have

$$-\frac{1}{\hbar}W(J) = \ln(Z(J)). \tag{2.53}$$

Differentiating with respect to J twice and evaluating at $J = 0$ gives

$$-\frac{1}{\hbar}\frac{\partial^2}{\partial J^2}W\bigg|_{J=0} = \frac{1}{Z(0)}\frac{\partial^2 Z}{\partial J^2}\bigg|_{J=0} - \frac{1}{(Z(0))^2}\left(\frac{\partial Z}{\partial J}\right)^2\bigg|_{J=0} \tag{2.54}$$

$$= -\frac{1}{\hbar^2}\left[\langle\phi^2\rangle - \cancel{\langle\phi\rangle^2}\right] = \frac{1}{\hbar^2}\langle\phi^2\rangle_{\text{connected}}. \tag{2.55}$$

There are also theories where $\langle\phi\rangle \neq 0$.

Example ($\langle\phi^4\rangle$)**:** Less trivially,

$$-\frac{1}{\hbar}\frac{\partial^4 W}{\partial J^4}\bigg|_{J=0} = \frac{1}{Z(0)}\frac{\partial^4 Z}{\partial J^4}\bigg|_{J=0} - \left(\frac{1}{Z(0)}\frac{\partial^2 Z}{\partial J^2}\right)^2\bigg|_{J=0} \tag{2.56}$$

$$= \langle\phi^4\rangle - \langle\phi^2\rangle^2 = \langle\phi^4\rangle_{\text{connected}} \tag{2.57}$$

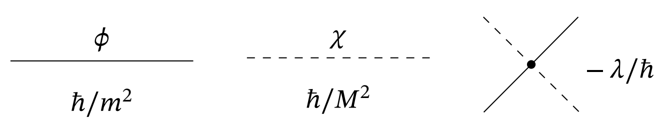
Remark: In statistics and statistical field theory, where ϕ is taken to be a random variable, the n-point correlation function $\langle\phi^n\rangle$ is often called the n^{th} *moment* of ϕ. The connected correlation functions $\langle\phi^n\rangle_c$ are then called *cumulants*. In this context $W(J)$ is the *cumulant generating function*[1], which is the natural logarithm of the *moment generating function* Z.

Interactions

Consider an action for two distinguishable fields

$$S(\phi,\chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2\chi^2. \tag{2.58}$$

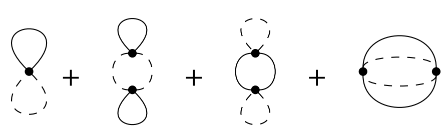
There is no factorial behind the factor 4. The Feynman rules are



(2.59)

The sum of connected vacuum diagrams is

$$-\frac{W}{\hbar} \sim$$

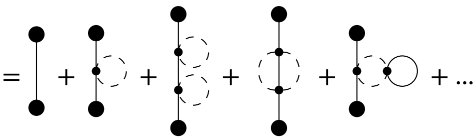


(2.60)

$$= -\frac{\hbar\lambda}{4m^2M^2} + \frac{\hbar^2\lambda^2}{m^4M^4}\left[\frac{1}{16} + \frac{1}{16} + \frac{1}{8}\right]. \tag{2.61}$$

As in (2.42), the two-point correlator $\langle \phi^2 \rangle$ is given by the sum of Feynman diagrams with two external source insertions, represented by a large dot:

$$\langle \phi^2 \rangle = \tag{2.62}$$



$$= \frac{\hbar}{m^2} - \frac{\hbar^2 \lambda}{2m^4 M^2} + \frac{\hbar^3 \lambda^2}{m^6 M^4}\left[\frac{1}{4} + \frac{1}{2} + \frac{1}{4}\right] + \ldots \tag{2.63}$$

In the following section we will arrive at these results in a different way: Say we do not care about the field χ, maybe its mass is $M \gg m$ and it is never produced in our experiments. In such a case, we would want to 'integrate out' χ by defining $W(\phi)$ as

$$e^{-W(\phi)/\hbar} = \int \mathrm{d}\chi \, e^{-S(\phi,\chi)/\hbar}. \tag{2.64}$$

From this viewpoint, we treat $\phi^2\chi^2$ as a source term $J = \phi^2$ for χ^2. Once found, we can then use $W(\phi)$ to calculate expectation values $\langle f(\phi) \rangle$. It is in this sense that $W(\phi)$ indeed plays the role of an effective action for the field ϕ—one in which all the quantum corrections due to χ are taken into account.

2.3.1 Effective Field Theory: Integrating out χ

From the action (2.58), we obtained Feynman diagrams for both fields and calculated expressions for W and $\langle \phi^2 \rangle$. Let us now show that we can get the same thing by first removing χ from the theory and then calculating expectation values in the effective theory. In other words, we want to calculate correlation functions only involving ϕ fields as

$$\langle f(\phi) \rangle = \frac{1}{Z} \int \mathrm{d}\phi\, \mathrm{d}\chi\, f(\phi) e^{-S(\phi,\chi)/\hbar} = \frac{1}{Z}\, \mathrm{d}\phi\, f(\phi) e^{-W(\phi)/\hbar}. \tag{2.65}$$

In this simple example,

$$e^{-W(\phi)/\hbar} = \int \mathrm{d}\chi\, e^{-S(\phi,\chi)/\hbar} = e^{-m^2\phi^2/2\hbar} \sqrt{\frac{2\pi\hbar}{M^2 + \frac{1}{2}\lambda\phi^2}}. \tag{2.66}$$

Taking logarithms, we obtain

$$W(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\hbar}{2} \ln\!\left(1 + \frac{\lambda}{2M^2}\phi^2\right) + \frac{\hbar}{2} \ln\!\left(\frac{M^2}{2\pi\hbar}\right). \tag{2.67}$$

The final term is constant and in non-gravitational physics does not effect any expectation values.

> In gravitational theories, this term is taken to be the origin of the cosmological constant.

We expand the logarithm

$$W(\phi) = \left(\frac{m^2}{2} + \frac{\hbar\lambda}{4M^2}\right)\phi^2 - \frac{\hbar\lambda^2}{16M^4}\phi^4 + \frac{\hbar\lambda^3}{48M^6}\phi^6 + \dots \tag{2.68}$$

From a theory which did not have any self-interaction, integrating out the χ -field gave us infinitely many self-interaction terms for all the even powers. We can think of the first term as an effective mass and write the other couplings as

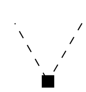
$$W(\phi) := \frac{m_{\text{eff}}^2}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_6}{6!}\phi^6 + \cdots + \frac{\lambda_{2k}}{(2k)!}\phi^{2k} + \dots, \tag{2.69}$$

where

$$m_{\text{eff}}^2 = m^2 + \frac{\hbar\lambda}{2M^2} \qquad \lambda_{2k} = (-1)^{k+1}\hbar \frac{(2k)!}{2^{k+1}k} \frac{\lambda^k}{M^{2k}}. \tag{2.70}$$

In higher dimensions, we usually need to calculate $W(\phi)$ perturbatively. From $S(\phi,\chi)$ and the path integral over χ, we have the Feynman rules

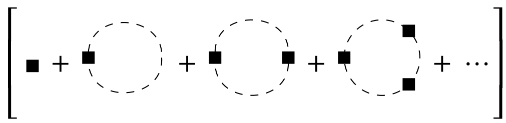
$\hbar/M^2$ and



$$\sim -\frac{\lambda\phi^2}{2\hbar}. \tag{2.71}$$

The propagator is the same as in the previous Feynman rules (2.59), but the factor associated with the vertex now accounts for the fact that we are treating the interaction term $\frac{\lambda}{4}\phi^2\chi^2$ as a source term for χ^2. The effective action is then given by the sum of connected diagrams

$$W(\phi) \sim -\hbar \tag{2.72}$$

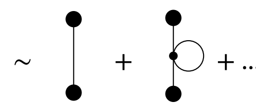


$$= S(\phi) + \frac{1}{2}\frac{\hbar\lambda}{2M}\phi^2 - \frac{1}{4}\frac{\hbar\lambda^2}{4M^4}\phi^4 + \frac{1}{3!}\frac{\hbar\lambda^3}{8M^6}\phi^6 + \cdots, \tag{2.73}$$

where in the first term $S(\phi) = S(\phi, 0)$ is the part of the action that is unaffected by the integral over χ.

We can now use $W(\phi)$ to calculate the expectation value

$$\langle\phi^2\rangle = \frac{1}{Z}\int \mathrm{d}\phi\, \phi^2 e^{-W(\phi)/\hbar} \sim \tag{2.74}$$



$$= \frac{\hbar}{m_{\text{eff}}^2} - \frac{\lambda_4\hbar^2}{2m_{\text{eff}}^6} + \ldots \tag{2.75}$$

The five diagrams of (2.62) have been reduced to just two diagrams with the effective action $W(\phi)$.

2.3.2 Quantum Effective Action Γ

Definition 8: We define the average field $\Phi = \langle\phi\rangle_J$ in the presence of an external source J as

$$\Phi := \frac{\partial W}{\partial J} = -\frac{\hbar}{Z(J)}\frac{\partial}{\partial J}\int \mathrm{d}\phi\, e^{-(S+J\phi)/\hbar} := \langle\phi\rangle_J, \tag{2.76}$$

where we have assumed that $S(\phi)$ as before is even in ϕ and has a minimum at $\phi = 0$.

Definition 9 (quantum effective action)**:** We have a Legendre transformation from the Wilsonian effective action $W(J)$ to the *quantum effective action* $\Gamma(\Phi)$:

$$\Gamma(\Phi) = W(J) - \Phi J. \tag{2.77}$$

Claim 4: As usual for Legendre transformations, we have

$$\boxed{\frac{\partial\Gamma}{\partial\Phi} = -J} \tag{2.78}$$

So $J \to 0$ corresponds to an extremum (in practice a minimum) of the effective action $\Gamma(\Phi)$.

Proof. Using the product rule, the chain rule, and the definition (2.76) of Φ,

$$\frac{\partial\Gamma}{\partial\Phi} = \frac{\partial W}{\partial\Phi} - J - \Phi\frac{\partial J}{\partial\Phi} = \frac{\partial W}{\cancel{\partial J}}\frac{\cancel{\partial J}}{\partial\Phi} - J - \Phi\cancel{\frac{\partial J}{\partial\Phi}} = -J \tag{2.79}$$

□

In higher dimensions, one performs a derivative expansion

$$\Gamma(\Phi) = \int \mathrm{d}^d x \left[-V(\Phi) - \frac{1}{2}\partial^\mu \Phi \partial_\mu \Phi + \ldots \right] \tag{2.80}$$

where the first term in this expansion defines the effective potential $V(\Phi)$. The effective potential might shift the minimum of the action when including quantum effects. These quantum corrections can lead to spontaneous symmetry breaking.

2.3.3 Analogy with Statistical Mechanics

The W is like the Helmholtz free energy F. In the presence of some external magnetic field h in some spin system, it is defined via

$$e^{-\beta F(h)} = \int \mathcal{D}s \, e^{-\beta H}. \tag{2.81}$$

We can define the magnetisation to be $M = -\frac{\partial F}{\partial h}$. One can then switch to the Gibbs free energy, analogous to Γ, by defining

$$G(M) = F(h) + Mh. \tag{2.82}$$

As $h \to 0$, the magnetisation of the system is the minimum of G.

2.3.4 Perturbative Calculation of $\Gamma(\Phi)$

We want to treat Φ as ϕ and write down a new Wilsonian effective action

$$e^{-W_\Gamma(J)/g} = \int \mathrm{d}\Phi \, e^{-(\Gamma(\Phi)+J\Phi)/g}, \tag{2.83}$$

where g is a new, fictitious Planck constant and J a source. This is in analogy to before, replacing the action S with the quantum effective action Γ. As before, $W_\Gamma(J)$ is the sum of connected vacuum diagrams and can be written as a power series in g

$$W_\Gamma(J) = \sum_{l=0}^{\infty} g^l W_\Gamma^{(l)}(J), \tag{2.84}$$

where l counts the loops. In particular, $W_\Gamma^{(0)}$ is composed of all the *tree* diagrams with Φ as legs.

In the limit of $g \to 0$, $W_\Gamma(J) \to W_\Gamma^{(0)}(J)$. Also as $g \to 0$, the integral in Φ is dominated by the minimum of the exponent, which is Φ such that

$$\frac{\partial \Gamma}{\partial \Phi} = -J \qquad \text{(steepest descent)} \tag{2.85}$$

Therefore, by analogy to the earlier definition with with action $S(\phi) + J\phi$, we have

$$W_\Gamma^{(0)}(J) = \Gamma(\Phi) + J\Phi = W(J). \tag{2.86}$$

The moral of the story is that the sum of connected diagrams in a theory with action $S(\phi) + J\phi$ (i.e. $W(J)$) can be constructed from a sum of tree diagrams with action $\Gamma(\Phi) + J\Phi$.

Definition 10 (bridge)**:** An edge in a connected graph is a *bridge* if removing it would leave the graph disconnected.

An example is shown in Fig. 2.3.

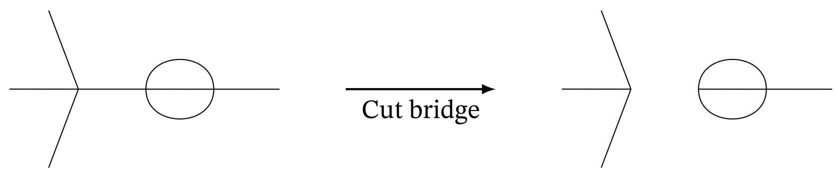


Figure 2.3: Cutting a bridge disconnects a graph.

Definition 11 (1PI)**:** A connected graph is called *one-particle irreducible* (1PI) if it has no bridges.