Supersymmetry

Part III Lent 2019

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Lecture notes: www.damtp.cam.ac.uk/user/dbs26

Main book: Hori & Vafa "Mirror Symmetry" (chapters 8-16)

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1 Introduction

1.1 Motivation: What is supersymmetry?

In a theory with bosons and fermions, the Hamiltonian can be written $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$, where $\mathcal{H}_{B(F)}$ has even (odd) number of fermionic excitations.

Such a theory is supersymmetric if there exists an operator Q mapping $\mathcal{H}_B \to \mathcal{H}_F$ and $\mathcal{H}_F \to \mathcal{H}_B$ such that

$$\{Q, Q^\dagger\} = 2H \qquad Q^2 = 0. \tag{1.1}$$

Here, $\{A, B\} = AB + BA$ is the *anti-commutator*, H is the Hamiltonian.

Consequences

i) The Hamiltonian and Q commute

$$[H, Q] = \frac{1}{2}[\{Q, Q^\dagger\}, Q] \tag{1.2}$$

$$= \frac{1}{2}[(QQ^\dagger + Q^\dagger Q)Q - Q(QQ^\dagger + Q^\dagger Q)] \tag{1.3}$$

$$= 0. \tag{1.4}$$

The two inner terms vanish since $Q^2 = 0$ and the two outer terms cancel identically. Therefore, the operator Q is *conserved*, and the transformations it generates will be *symmetries*. We call them *supersymmetries* because they mix bosons and fermions.

ii) All states ψ in our theory have non-negative energy

$$E = \langle \psi | H | \psi \rangle = \frac{1}{2} \langle \psi | \{Q, Q^\dagger\} | \psi \rangle \tag{1.5}$$

$$= \frac{1}{2}\|Q|\psi\rangle\|^2 + \frac{1}{2}\|Q^\dagger \psi\|^2 \geq 0, \tag{1.6}$$

with equality if and only if $Q|\psi\rangle = 0 = Q^\dagger \psi$, meaning that the state is invariant under supersymmetry.

If we have a Lorentz invariant quantum field theory (QFT), then H is part of the momentum vector $P_m = (H, \mathbf{P})$. It is then natural to expect that there is a multiplet of Q's.

Indeed in $d = 4$ we have $\left\{Q_\alpha, Q_{\dot\alpha}^\dagger\right\} = 2\sigma_{\alpha\dot\alpha}^m P_m$, where $\sigma^m = (\mathbb{1}_{2\times2}, \boldsymbol{\sigma})$.

For generic dimension d, this becomes $\left\{Q_A, Q_B^\dagger\right\} = 2\Gamma_{AB}^m P_m$.

Why study supersymmetry?

Traditionally, this question was answered by phenomenology. Supersymmetry was a promising approach to solve ongoing problems in dark matter, the unification of couplings in the standard model, as well as stabilizing the Higgs mass. The involvement of supersymmetry in the last issue was ruled out in experiments at CERN, and the above reasons will not be the motivation that drives us to study supersymmetry in this course.
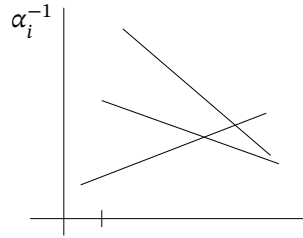

Figure 1.1: In many supersymmetric theories, the running couplings meet at the same point. This was taken to be indicative of a grand unified theory (GUT).

In this course, we will be driven by the following fact: QFT is hard! Usually, we have to study it via perturbation theory, as exemplified in 1.2. This is very different to quantum mechanics, where we
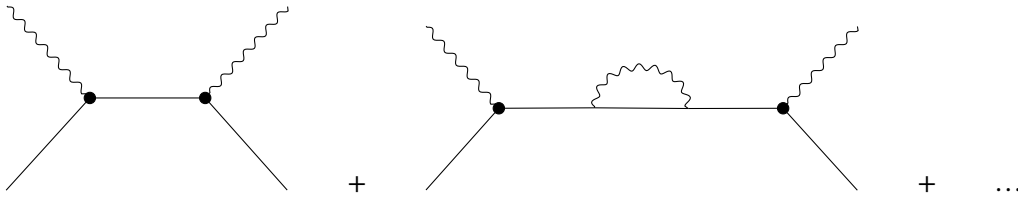

Figure 1.2: In the diagrammatic perturbation series in QFT, particles are almost always propagating freely, except at the interaction vertices.

first practice with exactly solvable systems. The reason for this is twofold. Firstly, they are usually better approximations to reality than a free particle; a spherically symmetric Coulomb potential is a better starting point to describe atoms than a free particle is, even though we still need to consider realistic models like (hyper)fine-structure perturbatively. Secondly, it helps us understand what quantum mechanics actually *is*.

Supersymmetry allows us to get exact results for (some observables) in QFT. This is especially true in $d < 4$, but also in $d = 4$.

These exact results are often closely related to deep maths, such as the Atiyah–Singer theorem (which we will meet in $d = 1$), mirror symmetry and enumerative geometry ($d = 2$), and Donaldson–Seiberg–Witten invariants ($d = 4$).

1.2   Fermions and Super Vector Spaces

Definition 1: A $\mathbb{Z}_2$-graded vector space is of the form $V = V_0 \oplus V_1$.

Definition 2 (parity)**:** We let the *parity* $|v|$ of $v \in V$ be

$$|v| = \begin{cases} 0, & \text{if } v \in V_0 \quad \text{(even / bosonic)} \\ 1, & \text{if } v \in V_1 \quad (odd/fermionic/Grassman) \end{cases} \tag{1.7}$$

Notation: If $\dim_{\mathbb{R}}(V_0) = p$ and $\dim_{\mathbb{R}}(V_1) = q$, then we write $V = \mathbb{R}^{p|q}$.

As usual, the dual V^* of a $\mathbb{Z}_2$-graded vector space (over $\mathbb{R}$) is the space of linear maps $\phi : V \to \mathbb{R}$ with $(V^*)_{0(1)}$ being those linear maps that vanish on $V_{1(0)}$ respectively.

Unsurprisingly, the direct sum of two $\mathbb{Z}_2$-graded vector spaces is

$$V \oplus W = (V \oplus W)_0 \oplus (V \oplus W)_1 \tag{1.8}$$
$$= (V_0 \oplus W_0) \oplus (V_1 \oplus W_1). \tag{1.9}$$

Likewise, we can take the tensor product

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1) \tag{1.10}$$
$$(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0) \tag{1.11}$$

Until now, we are just dealing with usual vector spaces, where we keep track of the fact that some elements have parity 1. To make V a super vector space, we define an unusual exchange operation.

$$\text{usually (bosonic):} \qquad \begin{aligned} s : \quad U \otimes U' &\quad \to \quad U' \otimes U \\ u \otimes u' &\quad \longmapsto \quad u' \otimes u \end{aligned} \tag{1.12}$$

$$\text{super vector space:} \qquad \begin{aligned} s : \quad V \otimes W &\quad \to \quad W \otimes V \\ v \otimes w &\quad \longmapsto \quad (-1)^{|v||w|} w \otimes v. \end{aligned} \tag{1.13}$$

Definition 3 (superalgebra)**:** Closely related is a superalgebra. This is a supervector space A with a multiplication map $\bullet : A \times A \to A$ with $|a \cdot b| = |a| + |b| \pmod 2$.

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Definition 4 (commutative)**:** A is supercommutative (or just commutative) if $ab = (-1)^{|a||b|}ba$.

Example: To treat $\mathbb{R}^{p|q}$ as a superalgebra, we take

$$x^i x^j = x^j x^i, \qquad x^i \psi^a = \psi^a x^i, \qquad \text{but } \psi^a \psi^b = -\psi^b \psi^a, \tag{1.14}$$

where $x^i \in \mathbb{R}^{p|0}$ and $\psi^a \in \mathbb{R}^{0|q}$. In particular, $(\psi^a)^2 = 0$ for any fixed a.

Not all A are (super-)commutative.

Definition 5 (Lie superalgebra)**:** A *Lie superalgebra* is a supervector space $g = g_0 \oplus g_1$ with a bilinear Lie bracket operation $[,] : g \times g \to g$ that

- is 'graded anti-symmetric' $[X, Y] = -(-1)^{|x||y|}[Y, X]$

- obeys $[X, [Y, Z]] + (-1)^{|X|(|Y|+|Z|)}[Y, [Z, X]] + (-1)^{|Y|(|Z|+|X|)}[Z, [X, Y]] = 0$

Definition 6 (polynomials)**:** We can define polynomials on a super vector space $\mathbb{R}^{p|q} = v$ as $O(V) \simeq$ $\text{Sym}^*(V_0^*) \otimes \Lambda^*(V_1^*)$. They are of the form

$$\underbrace{c_{ijk\ldots m}}_{\text{symmetric}} \underbrace{abc\ldots d}_{\text{antisymmetric}} x^i x^j \ldots x^m \psi^a \ldots \psi^d. \tag{1.15}$$

Definition 7 (smooth functions)**:** We define smooth functions on a super vector space to be $C^\infty(V) = C^\infty(V_0) \otimes \Lambda^*(V_1^*)$, so a generic function has an expansion

$$F(x^i, \psi^a) = f(x^i) * \rho_a(x^i)\psi^a + g_{ab}(x^i)\psi^a\psi^b + \cdots + \frac{h(x)}{(\dim V_1)!}\epsilon_{ab\ldots d}\psi^a\psi^b\ldots\psi^d, \tag{1.16}$$

where the coefficients f, ρ_a, g_{ab}, …are smooth functions on V_0. We often call such functions $F(x^i, \psi^a)$ *superfields*, while the smooth functions $f, \rho_a, g_{ab}, \ldots, h$ are the *component fields*. Note that if $F(x, \psi)$ is bosonic, then the component fields with even indices $f, g_{ab} = -g_{ba}, \ldots$ are bosonic whilst the ones with odd indices $\rho_a, \ldots$ are fermionic.

Remark: This is reminiscent of a *polyform* $F \in \Omega^*(V)$

$$F(x^i, dx^i) = f(x) + \rho_i(x)dx^i + g_{ij}(x)dx^i \wedge dx^j + \cdots + h(x)dx^i \wedge \cdots \wedge dx^n. \tag{1.17}$$

There is a fundamental difference: the polyform indices run over i, while the superfield indices a need not be the same as i. However, if they have the same indexing set, then these are really similar.

1.3   Differentiation and Integration of Fermions

Definition 8 (derivation)**:** A *derivation* of a (super-)algebra A is a linear map $D : A \to A$ obeying

$$D(ab) = (Da)b + (-1)^{|a||D|}a(Db) \qquad \text{(graded Leibniz rule)}. \tag{1.18}$$

Example: On $\mathbb{R}^{p|q}$, we have even derivatives $\frac{\partial}{\partial x^i}$ and odd derivatives $\frac{\partial}{\partial \psi^a}$, which act in the way you would expect on single fields:

$$\frac{\partial}{\partial x^i}(x^j) = \delta_i^j \qquad \frac{\partial}{\partial x^i}\psi^a = 0 \qquad \frac{\partial x^j}{\partial \psi^a} = 0 \qquad \frac{\partial}{\partial \psi^a}(\psi^b) = \delta_a^b. \tag{1.19}$$

However,

$$\frac{\partial}{\partial \psi^a}(\psi^b\psi^c) = \delta_a^b\psi^c - \psi^b\delta_a^c. \tag{1.20}$$

More generally, a (smooth) vector field on $\mathbb{R}^{p|q}$ is

$$X(x, \psi) = X_0^i(x, \psi)\frac{\partial}{\partial x^i} + X_1^a(x, \psi)\frac{\partial}{\partial \psi^a}, \tag{1.21}$$

where $X_0^i, x_1^a \in C^\infty(\mathbb{R}^{p|q})$.

For integration, since $f(\psi) = \rho + a\psi$, we only need to define $\int 1 \, \mathrm{d}\psi$ and $\int \psi \, \mathrm{d}\psi$. We require our measure $\mathrm{d}\psi$ to be translation invariant[1]: if $\psi' = \psi + \eta$ for some fixed fermionic $\eta \in \mathbb{R}^{0|1}$, then we want

$$\int \psi' \, \mathrm{d}\psi' = \int (\psi + \eta) \, \mathrm{d}\psi = \int \psi \, \mathrm{d}\psi + \eta \int \mathrm{d}\psi \implies \boxed{\int 1 \, \mathrm{d}\psi = 0}. \tag{1.22}$$

We then normalise our measure by defining

$$\boxed{\int \psi \, \mathrm{d}\psi = 1} \tag{1.23}$$

These rules are called *Berezin integration*.

> **Remark:** Differentiation and integration is really the same thing. Not unlike complex variables.

Remark: These imply that

$$\int \frac{\partial}{\partial \psi} (F(\psi, \dots)) \, \mathrm{d}\psi = 0. \tag{1.24}$$

In other words, when we perform integration by part for fermions, we never have to worry about boundary terms as long as we are careful about minus signs.

Suppose that we instead have a general case of n fermionic variables ψ^a. Then by iterated application of the previous rules, we define

$$\int \psi^1 \psi^2 \dots \psi^n \, \mathrm{d}^n\psi = 1 \tag{1.25}$$

if they all appear, and zero otherwise. If they all appear, but not in the correct order, then we get extra minus signs

$$\int \underbrace{\psi^a \psi^b \dots \psi^c}_{n \text{ fermions}} \mathrm{d}^n\psi = e^{ab\dots c}. \tag{1.26}$$

> **Remark:** Note in particular that if any index appears twice, the square on the left-hand side vanishes, just like the Levi-Civita symbol on the right.

Suppose $\chi^a = N^a{}_b \psi^b$ for some $N \in GL(n, \mathbb{R})$. Then by linearity

$$\int \chi^{a_1} \dots \chi^{a_n} \, \mathrm{d}^n\psi = N^{a_1}{}_{b_1} \dots N^{a_n}{}_{b_n} \int \psi^{b_1} \dots \psi^{b_n} \, \mathrm{d}^n\psi \tag{1.27}$$

$$= N^{a_1}{}_{b_1} \dots N^{a_n}{}_{b_n} \epsilon^{b_1 b_2 \dots b_n} \tag{1.28}$$

$$= \det(N) \epsilon^{a_1 \dots a_n} \tag{1.29}$$

$$= \det(N) \int \chi^{a_1} \chi^{a_2} \dots \chi^{a_n} \, \mathrm{d}^n\chi. \tag{1.30}$$

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[1] In particular, this will be necessary to derive Ward identities in QFT.

We conclude that if $\chi^a = N^a{}_b \psi^b$, then $\mathrm{d}^n\chi = \frac{1}{\det(N)}\,\mathrm{d}^n\psi$.

> **Remark:** This is not the same as if you were doing bosonic integration, where you do not have the inverse of the determinant.

Example: If $\chi = a\psi$, then $\mathrm{d}\psi = d(a\psi) = \frac{1}{a}\,\mathrm{d}\psi$.

1.4   QFT in Zero Dimensions

1.4.1   Bosonic Theory

In $d = 0$, our whole universe is just a single point $M = \{\mathrm{pt}\}$. So a bosonic field is just a map $x : M \to \mathbb{R} \simeq \{\mathrm{pt}\} \to \mathbb{R}$, which is nothing else than a real variable. With n such real fields, the space of all field configurations is $\mathcal{C} = \mathbb{R}^n$. The path-integral measure $[\mathcal{D}X]$ is just the usual Lebesgue measure $\mathrm{d}^n x$. Then the partition function becomes $Z = \int_{\mathbb{R}^n} e^{-S(x)/\hbar}\,\mathrm{d}^n x$, where $S : \mathbb{R}^n \to \mathbb{R}$ is the action.

> Compare this with today's lecture on *Advanced Quantum Field Theory*.

There cannot be any kinetic terms since the universe is just a point and there cannot be anything we differentiate with respect to. However, we can have a mass term and maybe some sort of interaction such as

$$S(x^i) = \frac{m_2}{2}\delta_{ij}x^i x^j + \frac{\lambda^{ijkl}}{4}x^i x^j x^k x^l. \tag{1.31}$$

Now in $d = 0$ this is a finite-dimensional integral. But nonetheless, it is a difficult integral! Expanding the action to quadratic order around its stationary point, we can find that in the limit $\hbar \to 0^+$, the integral is asymptotic to

$$\int_{\mathbb{R}^n} e^{-S(x)/\hbar}\,\mathrm{d}^n x \sim (2\pi\hbar)^{n/2}\frac{e^{-S(x_*)}}{(\det \partial_i \partial_j S)^{1/2}|_{x=x_*}}\left(1 + \hbar A_1 + \hbar^2 A_2 + \dots\right), \qquad \text{(steepest descent)} \tag{1.32}$$

where x_* is a minimum of $S(x)$.

This is complicated! And approximate (zero radius of convergence)!

Remark: In the whole of the QFT course we basically just computed different numerators of this. In AQFT we will go on to loop diagrams and compute the denominator as well as the first order expansion. If you end up doing a PhD in the wrong area, you might compute higher and higher terms. But what is even the point? This series doesn't even converge!

1.4.2   Fermionic Theory

Let us now consider a purely fermionic theory. We need at least two fermions. Take $S = A\psi^1\psi^2$.

$$Z = \int e^{-S(\psi)/\hbar}\, \mathrm{d}^2\psi = \int \left(1 - \frac{A}{\hbar}\psi^1\psi^2\right)\mathrm{d}^2\psi = -\frac{A}{\hbar}. \tag{1.33}$$

More generally, for $2m$ fermions ψ^a and antisymmetric matrix A_{ab} ,

$$Z = \int e^{-\frac{A_{ab}}{2\hbar}\psi^a\psi^b}\, \mathrm{d}^{2m}\psi = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\hbar)^k k!}\int (A_{ab}\psi^a\psi^b)^k\, \mathrm{d}^{2m}\psi \tag{1.34}$$

$$= \frac{(-1)^m}{(2\hbar)^m m!}\epsilon^{a_1 b_1 \ldots a_m b_m} A_{a_1 b_1} \ldots A_{a_m b_m} \tag{1.35}$$

$$= \left(\frac{-1}{\hbar}\right)^m \mathrm{Pfaff}(A). \tag{1.36}$$

Exercise 1.1: For anatisymmetric A, show that $\mathrm{Pfaff}(A) = \sqrt{\det A}$.

Remark: Again, this is the inverse to what you expect from the bosonic counterpart.