Statistical Field Theory

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Contents

1 The Ising Model

The Ising model is a simple model for a magnet, which also forms a rich playground for SFT, QFT, etc...

1.1 Basics

We have a lattice of sites in any dimension d. At each lattice site, there is positioned a spin which can either point up or down. The energy of the entire system is

$$E = -B\sum_i S_i - J\sum_{<ij>} S_i S_j. \tag{1.1.1}$$

B is a magnetic field and $< ij >$ denotes nearest neighbour interactions.

If $J > 0$: Spins favour alignment (Ferromagnet)

If $J < 0$: Misaligned (Anti-Ferromagnet)

1.2 Finite Temperature

With finite temperature, this is where the *statistical* bit of the course comes in. We introduced some entropy into the system so the spins can flip. We would expect that even for $J < 0$ this implies that the spins will tend to be misaligned, resulting in no magnetisation, at high temperatures.

In the canonical ensemble:

$$p[S_i] = \frac{e^{-\beta E[S_i]}}{Z} \tag{1.2.1}$$

where $\beta = 1/T$ and the partition function is $Z = \sum_{\{S_i\}} e^{-\beta E[S_i]}$.

Example:

$$F_{\text{thermo}}(T; B) = \langle E\rangle - TS = -T\log Z \tag{1.2.2}$$

Definition 1 (Average Spin)**:**

$$m = \frac{1}{N}\left\langle \sum_i S_i \right\rangle, \qquad m \in [-1, +1] \tag{1.2.3}$$

To calculate the average, we need the probabilities. We write this as

$$m = \frac{1}{N} \sum_{\{S_i\}} \underbrace{\frac{e^{-\beta E[S_i]}}{Z}}_{\text{probability}} \overbrace{\sum_i S_i}^{\text{spin}} = \frac{1}{N\beta} \frac{\partial \log Z}{\partial \beta}. \tag{1.2.4}$$

There are many numerical tools to calculate the partition function. For analytic tools, we work towards *mean field theory* which makes the large number of degrees of freedom tractable.

1.3 Effective Free Energy

Ising model: "Microscopic Degrees of Freedom" (spins)

Partition function: "Macroscopic" (average spin)

The ansatz that we will use is to write the partition function Z as a sum over the magnetisations $m = \frac{1}{N}\sum_i S_i$ of configurations, and then sum over all configuations that give this magnetisation:

$$Z = \sum_m \sum_{\{S_i\}|_m} e^{-\beta E[S_i]} \coloneqq \sum_m e^{-\beta F(m)}. \tag{1.3.1}$$

In the large-N limit:

$$\sum e^{-\beta F(m)} \longrightarrow \frac{N}{2} \int_{-1}^{1} \mathrm{d}m\, e^{-\beta F(m)}. \tag{1.3.2}$$

Saddle Point Approximation: Define $f(m) = \frac{F(m)}{N}$ Ignoring normalisation factors, the partition function becomes

$$Z = \int_{-1}^{+1} \mathrm{d}m\, e^{-\beta N f(m)} \approx e^{-\beta N f(m_{\min})} \implies F_{\text{thermo}} \approx F(m_{\min}) \tag{1.3.3}$$

1.4 Mean Field Theory

1.4.1 Mean Field Approximation

One essentially replaces each spin variable by the mean variable that is the same everywhere:

$$S_i \to \langle S \rangle = m. \tag{1.4.1}$$

The total energy is then

$$E = -B \sum_i m - J \sum_{\langle ij \rangle} m^2 \tag{1.4.2}$$

and the normalised energy

$$\frac{E}{N} = -Bm - \frac{1}{2} Jqm^2. \tag{1.4.3}$$

Here, $q \sim 2d$ is the number of nearest neighbours. For a given magnetisation m, we need to count how many states there are.

1.4.2 Counting Microstates

$$m = \frac{N_\uparrow - N_\downarrow}{N} = \frac{2N_\uparrow - N}{N} \qquad \Omega = \frac{N!}{N_\uparrow!(N-N_\uparrow)!} \tag{1.4.4}$$

Use Stirling's formula (can plug this in as an "illuminating" exercise at home)

$$\log\Omega = N\log N - N_\uparrow \log N_\uparrow - (N-N_\uparrow)\log(N-N_\uparrow) \tag{1.4.5}$$

$$\frac{\log\Omega}{N} \approx \log 2 - \frac{1}{2}(1+m)\log(1+m) - \frac{1}{2}(1-m)\log(1-m). \tag{1.4.6}$$

Hence,

$$\sum_{\{S_i\}_m} e^{-\beta E[S_i]} \approx \Omega(m) e^{-\beta E(m)} \tag{1.4.7}$$

$$\approx e^{\frac{N\log\Omega}{N}} e^{-\beta\left(-Bm - \frac{1}{2}Jqm^2\right)} \tag{1.4.8}$$

This defines

$$f(m) \approx -Bm - \frac{1}{2}Jqm^2 - T\left[\log(2) - \frac{1}{2}(1+m)\log(1+m) - \frac{1}{2}(1-m)\log(1-m)\right]. \tag{1.4.9}$$

Minimised at

$$\frac{\partial f}{\partial m} = 0 \implies \beta(B+Jqm) = \frac{1}{2}\log\left(\frac{1+m}{1-m}\right) \tag{1.4.10}$$

$$\implies m = \tanh(\beta B + \beta Jqm), \qquad (B_{\text{eff}} = B + Jqm) \tag{1.4.11}$$

Plotting left and right hand side, we see that at $T = \infty$ there is only one solution at $m = 0$. At low temperature, we then obtain other solutions.

2 Landau Approach to Phase Transitions

Landau Theory Free energy + Symmetry

Effective Free Energy $F(m) = Nf(m)$, where m is the *order parameter*

$$f(m) = -Bm - Jqm^2 - T[\log(2) - \frac{1}{2}(1+m)\log(1+m) - \frac{1}{2}(1-m)\log(1-m)] \quad (2.0.1)$$

$$\simeq -T\log(2) - Bm + \frac{1}{2}(T - Jq)m^2 + \frac{1}{12}Tm^4 + \cdots \quad (2.0.2)$$

Saddle Point Equilibrium at minimum of $f(m)$

2.1 Phase transition in the Ising Model

Without external magentic field $B = 0$, we have

$$f(m) = \frac{1}{2}(T - Jq)m^2 + \frac{1}{12}Tm^5 + \cdots \quad (2.1.1)$$

As we lower the temperature the potential becomes flat. Below the critical temperature T_C, we obtain two minima at $\pm m_0$, where

$$m_0 = \sqrt{\frac{3(T_c - T)}{T}} \quad (2.1.2)$$

We call a phase transition an *n-th order phase transition* if it has a discontinuity in the n-th derivative. In this case, we have a second order phase transition.

> **Remark:** The reason for this discontinuity is the large number of degrees of freedom. Since $N \to \infty$, discontinuities show up, but on the level of individual spins, all processes are continuous.

Definition 2 (Heat Capacity)**:** The heat capacity of the system is

$$C = \frac{\partial\langle E\rangle}{\partial T} = \beta^2\frac{\partial^2\log(Z)}{\partial\beta^2}, \qquad E = -\frac{\partial\log(Z)}{\partial\beta} \quad (2.1.3)$$

where $Z = e^{-\beta N f(m_{\min})}$.

In the Landau Approach:

$$T > T_C \qquad f(m_{\min}) = 0 \tag{2.1.4}$$

$$T < T_c \qquad f(m_{\min}) = -\frac{3}{4}\frac{(T_c - T)^2}{T} \tag{2.1.5}$$

2.2 Spontaneous Symmetry Breaking

At high temperatures, we have

$$f(m) = \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Tm^4. \tag{2.2.1}$$

This has a $\mathbb{Z}_2$ symmetry under interchange $m \to -m$.

Definition 3 (Spontaneous Symmetry Breaking)**:** The ground state does not respect a symmetry of the action / free energy.

2.3 Non-Zero Magnetic Field

If we have $B \neq 0$, then we have a term that is linear in m:

$$f(m) = -Bm + \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Tm^4 + \cdots \tag{2.3.1}$$

In the absence of the magnetic field, the two tracks split. However, in this case, the tracks do not meet and there is no discontinuity and no phase transition either. However, we can manufacture a phase transition in the following way.

2.3.1 Manufacturing a Phase Transition

We increase the magnetic field in such a way as to increase the energy of the state in which the system found itself via spontaneous symmetry breaking. The system will remain in that state, even until we increase the magnetic field so much that it is only a local, not the global minimum. After some time spent in this meta-stable state, the magnetisation jumps to the lower energy state. This is a first order phase transition.

At the critical point $T = T_c$, we can solve for the magnetisation as a function of the magnetic field. We find that

$$m \sim \begin{cases} B^{1/3} & B < 0 \\ -|B|^{1/3} & B > 0 \end{cases} \tag{2.3.2}$$

Definition 4 (Magnetic Susceptibility)**:**

$$\chi = \frac{\partial m}{\partial B}|_T \tag{2.3.3}$$

Since we have $m \simeq \frac{B}{T - T_c}$, the susceptibility is $\chi \sim |T - T_c|^{-1}$. These are predictions which can be tested in experiments.

2.4 Validity of Mean Field Theory

The mean field theory (MFT) gave an excellent qualitative picture of the phase transitions. But how well does it actually work? There were a number of assumptions: We neglected any spatial variations by assuming that any spin takes the same value as the average. We also neglected any interactions, such as corrections from the quartic term in $f(m)$.

It turns out that whether or not MFT works depends on the number of dimensions:

$d = 1$**:** Total Failure. In one dimension there is never any phase transition. We call this the *lower critical dimension* d_l.

$d = 2, 3$**:** Works-*ish*. The qualitative picture is right, but the details are not correct.

$d \geqslant 4$**:** Works remarkably well. In $d \geq 4$, it gives exactly the right answer and matches experiment. We call this the *upper critical dimension* d_c.

In general theories, not restricted to the Ising model, we call the dimension in which MFT fails completely the *lower critical dimension* d_l and the theory for which it always works the *upper critical dimension* d_c.

> **Remark:** This can be understood in terms of the number of nearest-neighbours. The fluctuations grow in a different way depending on the dimension of the system.

2.5 Critical Exponents

We found that in the absence of an external magnetic field, the magnetisation scaled as $m \sim |T_c - T|^{\beta}$, where $\beta = \frac{1}{2}$. Moreover, the heat capacity scaled as $C \sim C_{\pm}|T - T_c|^{-\alpha}$, where $\alpha = 0$, the susceptibility scaled as $\chi \sim |T - T_c|^{-\gamma}$ where $\gamma = 1$. Finally, the magnetisation also scaled as $m \sim B^{1/\delta}$ where $\delta = 3$.

The actual values are tabulated in 2.5.1.

	MF	d=2	d=3	d=4
α	0	$O(\log)$	$0.1101\ldots$	
β	$\frac{1}{2}$	$0.3264\ldots$		
γ	1	$\frac{7}{4}$		
δ	3	15	$4.7898\ldots$	

Table 2.5.1: Critical Exponents in various dimensions.

This leads us to the concept of *universality* where two systems that have different underlying physical processes actually have the same critical exponents.

3 Landau-Ginzburg Free Energy

3.1 Model Building

This will be the first attempt of writing down a *theory* by postulating a free energy. In the case of particle physics, this corresponds to the action. There are certain guiding principles that help construct such a theory. Such 'common sense' principles include

Locality All interactions are *local*. Only nearest neighbours matter. This means the free energy functional is just one integral over one spatial component

$$F[m(x)] = \int \mathrm{d}^d x \, f[m(x)]. \tag{3.1.1}$$

In general, one could imagine non-local theories where we have terms of the form

$$\int \mathrm{d}^d x \, \mathrm{d}^d y \, f[m(x), m(y)].$$

In the continuum Ising model, we have some almost non-local terms. These are the gradient terms

$$\frac{\partial m(\mathbf{x})}{\partial \mathbf{x}} \simeq \lim_{a \to 0} \frac{m(\mathbf{x} + a\hat{\mathbf{x}}) - m(\mathbf{x})}{a}. \tag{3.1.2}$$

There are certain pitfalls, such as causality violation, which arise when we can include non-local terms. For more information, see Weinberg Volume 1.

Translational and Rotational Invariance The symmetry of the continuum theory should reflect the symmetries of the lattice. The Ising model has a large-scale rotational symmetry, which will be recovered when going to the continuum limit. Similarly, there is a discrete translational symmetry which will become a continuum symmetry. This means, we have terms like $(\boldsymbol{\nabla} m)^2$, but not ~~$\mathbf{n} \cdot \boldsymbol{\nabla} m$~~.

Moreover, the Ising model with $B = 0$ has a $\mathbb{Z}_2$ symmetry: $S_i \to -S_i$. Therefore, the continuum limit must have a symmetry under $m(\mathbf{x}) \to -m(\mathbf{x})$.

Analyticity We require the free energy F to have a well-behaved Taylor expansion in the magnetisation field $m(\mathbf{x})$. This allows terms like $m^2(\mathbf{x})$, $(\boldsymbol{\nabla} m(\mathbf{x}))^4$, $\ldots$, $m^8(\mathbf{x})$. However, we are not allowed to have non-analytic terms like $\sqrt{m(\mathbf{x})}$, $\log(m(\mathbf{x}))$, or $\frac{1}{m(\mathbf{x})}$.

> **Remark:** Non-analyticities like these usually signal that we overlooked some important ingredient in the theory.

Derivative Expansion If the variation of the gradient is slow over the length scale a of the lattice spacing, $(a\boldsymbol{\nabla})\boldsymbol{\nabla}m(\mathbf{x}) \ll \boldsymbol{\nabla}m(\mathbf{x})$, then higher derivatives can be ignored. As we go to higher number of derivatives, we go to more separated neighbour interactions.

> **Remark:** The Higgs interaction, among many others, starts to generate logarithmic terms when we coarse-grain high-energy processes.

3.2 Zero external field

When $B = 0$, the free energy is

$$F[m(\mathbf{x})] = \int \mathrm{d}^d x \left[\frac{1}{2}\alpha_2(T)m^2 + \frac{1}{4}\alpha_4(T)m^4 + \frac{1}{2}\gamma(T)(\boldsymbol{\nabla}m)^2 + \ldots\right], \tag{3.2.1}$$

where $m = m(\mathbf{x})$, $\alpha_2(T) \simeq T - T_c$, and $\alpha_4(T) \sim \frac{1}{3}T$.

3.2.1 The Saddle Point Approximation

We want to find minimise the free energy functional $F[m(\mathbf{x})]$. Consider a field configuration $m(\mathbf{x})$ and perturb it by $\delta m(\mathbf{x})$. The free energy then changes by

$$\delta F = \int \mathrm{d}^d x \left[\alpha_2 m\delta m + \alpha_4 m^3\delta m + \gamma\boldsymbol{\nabla}m\cdot\boldsymbol{\nabla}\delta m\right] \tag{3.2.2}$$

$$= \int \mathrm{d}^d x \left[\alpha_2 m + \alpha_4 m^3 - \gamma\nabla^2 m\right]\delta m \tag{3.2.3}$$

The implicit assumption here is that the variation vanishes at the boundaries of the integral. We can take this as a definition of the functional derivative

$$\frac{\delta f}{\delta m} = \alpha_2 m + \alpha_4 m^3 - \gamma\nabla^2 m. \tag{3.2.4}$$

For an Extremum, $\frac{\delta F}{\delta m}\big|_m = 0 \implies \gamma\nabla^2 m = \alpha_2 m + \alpha_4 m^3$. This is sometimes called the *Euler-Lagrange equation* or the *equation of motion*.

Solutions

The simplest solution is $m = \text{const}$. There are two cases:

$$m = \begin{cases} 0 & \alpha_2 > 0 \\ \pm m_0 = \pm\sqrt{\frac{-\alpha_2}{\alpha_4}} & \alpha_2 < 0 \end{cases} \tag{3.2.5}$$

This recovers the mean-field approximation, confirming that our original mean-field Ansatz was not so bad after all. Moreover, analysing the other solutions allows us to understand the mean-field solution better. The other solutions are called *domain walls*.

Domain Walls

We can also imagine that different parts of the lattice take on different solutions, so that both $\pm m_0$ coexist. In that case, there will be an asymptotic transition function between the domain walls at spatial infinity. This transition function cannot just be a step function, since the

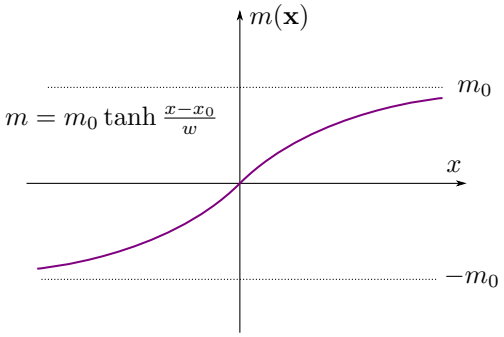

Figure 3.2.1: The transition function asymptotes to the domain walls as $x = |\mathbf{x}| \to \infty$

derivative would be discontinuous, giving infinite energy. Solving $\gamma \frac{\mathrm{d}^2 m}{\mathrm{d}x^2} = \alpha_2 m + \alpha_4^2$ gives

$$m = m_0 \tanh\left(\frac{x - x_0}{w}\right), \qquad w = \sqrt{\frac{-2\gamma}{\alpha}}. \tag{3.2.6}$$

The energy associated with this solution can be calculated to give $F_{\mathrm{DW}} - F_{\mathrm{GS}} \sim L^{d-1}\sqrt{\frac{-\gamma\alpha_2^3}{\alpha_4}}$. Here we see something interesting happening; in one dimension, the free energy is independent of the size of the system, whereas in higher dimensions it does. This exponent $d-1$ will tell us about why the mean-field theory fails in one-dimension: the domain-walls completely take over and dominate over the mean-field theory contributions.

This domain wall solution will shed light on the behaviour of mean field theory in lower dimensions.

3.3 Lower Critical Dimension

Let us first consider the case of one domain wall in the system. The probability of having a domain wall at $x = X$ is

$$p(\text{wall at } x = X) = \frac{e^{-\beta F_{\text{wall}}}}{Z}. \tag{3.3.1}$$

However, the domain wall could form anywhere. Integrating over the probability of having a domain wall anywhere in the system gives

$$p(\text{wall anywhere}) = \frac{e^{-\beta F_{\text{wall}}}}{Z}\frac{L}{w}. \tag{3.3.2}$$

The probability to have n walls in the system is obtained by considering the product of the probabilities at n different locations:

$$P(n \text{ walls}) = \frac{e^{-n\beta F_{\text{wall}}}}{Z}\frac{1}{w^n}\int_{-L/2}^{L/2}\mathrm{d}x_1\int_{-L/2}^{L/2}\mathrm{d}x_2\cdots\int_{-L/2}^{L/2}\mathrm{d}x_n \tag{3.3.3}$$

$$= \frac{1}{Zn!}\left(\frac{Le^{-\beta F_{\text{wall}}}}{w}\right)^n. \tag{3.3.4}$$

We can now consider the sum over these probabilities. The sum over of all even n is giving a hyperbolic cosine. This is the probability of going from left to right and ending up at the same magnetisation m_0 that we started with, since at each domain wall, the magnetisation swaps $m_0 \to -m_0$. Similarly, we can calculate the probability of having an odd number n of domain walls, giving a hyperbolic sine:

$$p(m_0 \to m_0) = \frac{1}{Z}\cosh\left(\frac{Le^{-\beta F_{\text{wall}}}}{w}\right) \tag{3.3.5}$$

$$p(m_0 \to -m_0) = \frac{1}{Z}\sinh\left(\frac{Le^{-\beta F_{\text{wall}}}}{w}\right). \tag{3.3.6}$$

Now let us consider what happens in $d = 1$: In one dimension, the free energy of domain wall solutions $F_{DW} \sim L^{d-1}$ is a constant. Therefore, the cosh and sinh terms both diverge as $L \to \infty$. If there are no other divergent processes, then these probabilities will dominate over the partition function Z, and eventually converge to $p(m_0 \to m_0) = p(m_0 \to -m_0) = 50\%$. This is a puzzling result; it appears that as the system size grows, the probability of ending up in a system with domain walls go to unity. In this case, we say that domain walls proliferate. In one dimension, we do not expect the mean-field approximation to work, because spatial variations are important. On the other hand, in the case of $d > 1$, the free energy $F \sim L^{d-1}$ will grow, meaning that the exponential factor $e^{-\beta F} \to 0$ suppresses the otherwise linear growth of probability.

3.4 My First Path Integral

We now define the partition function as a path integral over all possible field configurations. This is an integral over all functions $m(\mathbf{x})$:

$$Z = \int \mathcal{D}m(\mathbf{x})\, e^{-\beta F[m(\mathbf{x})]}. \tag{3.4.1}$$

We focus on fluctuations around the saddle point. If these fluctuations are too big, we cannot use perturbative methods any more, and the path integral method breaks down, corresponding to strongly coupled field theories.

As is conventional in field theory, we denote the scalar field as $m(\mathbf{x}) \to \phi(\mathbf{x})$. We also drop the dependence on $\mathbf{x}$ in writing $\phi = \phi(\mathbf{x})$. Setting $B = 0$, the free energy is

$$F[\phi(\mathbf{x})] = \int \mathrm{d}^d x \left[\frac{1}{2}\alpha_2(T)\phi^2 + \frac{1}{4}\alpha_4(T)\phi^4 + \frac{1}{2}\gamma(T)(\boldsymbol{\nabla}\phi)^2 + \ldots\right]. \tag{3.4.2}$$

We approximate this action by setting all coefficients $\alpha_{n\geq 4} = 0$. At the moment, this looks like a gross oversimplification; however, renormalisation group theory will show us that these terms are actually irrelevant for our purposes. Now there are two cases: The coefficient to the quadratic term α_2 can either be positive or negative. If we have $\alpha_2 > 0$, this will lead to a *disordered phase*, whereas $\alpha_2 < 0$ will lead to an *ordered* phase. In that case, we can look at the variations of the field about its mean $\langle\phi\rangle$:

$$\widetilde{\phi}(\mathbf{x}) = \phi(\mathbf{x}) - \langle\phi\rangle. \tag{3.4.3}$$

The action then becomes

$$F[\widetilde{\phi}(\mathbf{x})] = F[\langle\phi\rangle] + \frac{1}{2}\int \mathrm{d}^d x \left[\alpha_2'(T)\widetilde{\phi}^2 + \gamma(T)(\boldsymbol{\nabla}\phi)^2 + \ldots\right], \tag{3.4.4}$$

where $\alpha_2'(T) = -2\alpha_2(T) > 0$.

3.4.1 Fourier Space

To perform this calculation, we have to move to Fourier space. In that case, we express the field $\phi(\mathbf{x})$ as an integral over the modes $\phi_{\mathbf{k}}$ with wavevector $\mathbf{k}$:

$$\phi_{\mathbf{k}} = \int \mathrm{d}^d x\, e^{-i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{x}). \tag{3.4.5}$$

Since ϕ is a real scalar field, a standard result in Fourier theory is that $\phi_{\mathbf{k}}^* = -\phi_{\mathbf{k}}$.

> **Remark:** For complex fields, this relationship between the Fourier conjugates does not in general hold.

Since we have a smallest lattice scale a in the system, the momentum modes must vanish for some wavenumber scale $|\mathbf{k}| > \Lambda \sim \frac{\pi}{a}$.

Finite Volume In a finite box of side-length L, and volume $V \sim L^d$, the allowed wavevectors are discrete:

$$\mathbf{k} = \frac{2\pi \mathbf{n}}{L}, \qquad \mathbf{n} \in \mathbb{N}^d. \tag{3.4.6}$$

Following this discretisation, the Fourier expansions of the fields ϕ turn from integrals into sums:

$$\phi(\mathbf{x}) = \frac{1}{V} \sum_{\mathbf{k}} e^{+i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}}. \tag{3.4.7}$$

We sometimes say that the field theory has been 'quantised'.

Infinite Volume If the volume becomes infinite $L \to \infty$, the wavevectors become members of a continuous set $\mathbf{k} \in \mathbb{R}^d$, meaning that the expansion of ϕ into its Fourier modes now looks like

$$\phi(\mathbf{x}) = \int_{\mathbb{R}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} e^{+i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}} \coloneqq \int \mathrm{đ}^d k \, e^{+i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}}. \tag{3.4.8}$$

The definition of the normalised measure $\mathrm{đ}^d k = (2\pi)^{-d} \, \mathrm{d}^d k$ was chosen to not have to carry around superfluous factors of $(2\pi)^d$ in Fourier space.

Inserting $\phi(\mathbf{x})$ into the free energy, we have

$$F[\phi_{\mathbf{k}}] = \frac{1}{2} \int \mathrm{đ}^d \mathbf{k}_1 \int \mathrm{đ}^d \mathbf{k}_2 \int \mathrm{d}^d x \, (-\gamma \mathbf{k}_1 \cdot \mathbf{k}_2 + \mu^2) \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} e^{i(\mathbf{k}_1 + \mathbf{k}_2)\cdot\mathbf{x}}. \tag{3.4.9}$$

where we chose to write the coefficient to the quadratic term as $\alpha_2 = \mu^2$. This convention is often chosen in field theory, stemming from the analogy to quantum field theory, where the coefficient to the quadratic field ϕ^2 is (twice) the mass of the particle. Performing the integral over $\mathbf{x}$ gives a delta function. The normalisation of that delta function is determined by our Fourier space conventions:

$$\bar{\delta}^d(\mathbf{k}_1 + \mathbf{k}_2) \coloneqq (2\pi)^d \delta^d(\mathbf{k}_1 + \mathbf{k}_2) = \int \mathrm{d}^d x \, e^{i(\mathbf{k}_1 + \mathbf{k}_2)\cdot\mathbf{x}}, \tag{3.4.10}$$

where we again use the horizontal bar notation to denote a Fourier space normalisation that will considerably simplify equations by hiding annoying factors of $(2\pi)^d$. The free energy is then found to be

$$F[\phi_{\mathbf{k}}] = \frac{1}{2} \int \mathrm{đ}^d k (\gamma k^2 + \mu^2) \phi_{\mathbf{k}} \phi^*_{\mathbf{k}}, \tag{3.4.11}$$

A free energy or action of this form is said to be 'diagonalised'. This terminology stems from the analogous case in matrices, where we can write each matrix as a spectral decomposition over its eigenvalues in a certain basis of eigenvectors. In the picture of this analogy, the $|\phi_{\mathbf{k}}|^2$ are the eigenvectors with eigenvalue $(\gamma k^2 + \mu^2)$.

3.5 Path Integral Measure

What exactly do we mean when we write down the path integral measure $\mathcal{D}\phi$? The path integral measure is defined as an integral over all possible field configurations. We define this as

$$\int \mathcal{D}\phi(\mathbf{x}) \coloneqq \prod_{\mathbf{k}} \left[N \int \mathrm{d}\phi_{\mathbf{k}} \, \mathrm{d}\phi_{\mathbf{k}}^* \right]. \tag{3.5.1}$$

The normalisations N are not important, since they usually cancel out when we compute expectation values and correlation functions. Moreover, we have to keep in mind that ϕ^* is not actually an independent degree of freedom. This actually means that this derivation would technically only be correct for complex fields, and not really well-defined for the scalar fields that we are dealing with. However, we still go through it here to lead up to the result and introduce the concepts. With this definition of the path integral measure, we will aim to compute the partition function

$$Z = \prod_{\mathbf{k}} N \int \mathrm{d}\phi_{\mathbf{k}} \, \mathrm{d}\phi_{\mathbf{k}}^* \, e^{-\frac{\beta}{2} \int \mathrm{d}^d k (\gamma k^2 + \mu^2) |\phi_{\mathbf{k}}|^2}. \tag{3.5.2}$$

However, another slightly strange feature of this notation that we will have to make a bit more rigorous is the notion of having an infinite number of integrals over the continuous wavenumber labels $\mathbf{k}$. To make a bit more sense of this, we first deal with a path integral defined over a finite volume V of space, making the set of wavenumbers discrete, and then define the infinite integral as the limit as $V \to \infty$ of that expression.

Finite Volume When we choose to work in a finite volume V, the integral over continuous wavevectors in the argument of the exponential function turns into a sum over discrete wavevectors. The partition function for a finite volume is therefore

$$Z = \prod_{\mathbf{k}} \left[N \int \mathrm{d}\phi_{\mathbf{k}} \, \mathrm{d}\phi_{\mathbf{k}}^* \right] e^{-\frac{\beta}{2V} \sum_{\mathbf{k}} (\gamma k^2 + \mu^2) |\phi_{\mathbf{k}}|^2} \tag{3.5.3}$$

$$= \prod_{\mathbf{k}} N \left[\int \mathrm{d}\phi_{\mathbf{k}} \, \mathrm{d}\phi_{\mathbf{k}}^* \, e^{-\frac{\beta}{2V} (\gamma k^2 + \mu^2) |\phi_{\mathbf{k}}|^2} \right]. \tag{3.5.4}$$

Now, we can actually solve these integrals exactly! To see this, consider the well-known Gaussian integral over the real numbers $x \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{x^2}{2a}} = \sqrt{2\pi a}. \tag{3.5.5}$$

Each of the integrals over a particular $\phi_{\mathbf{k}}$ or $\phi_{\mathbf{k}}^*$ can then be solved as a Gaussian integral. The total partition function is a product over these integrals, two for each wavevector $\mathbf{k}$ (one for the field ϕ and one for the conjugate field ϕ^*). This procedure yields the following partition function:

$$Z = \prod_{\mathbf{k}} N \sqrt{\frac{2\pi T_v}{\gamma k^2 + \mu^2}}. \tag{3.5.6}$$

As we will see, the Gaussian theory is one of the few path integrals that we can solve exactly due to this analogy with the Gaussian integral. All higher order terms, corresponding to 'interactions' between the fields, have to be dealt with perturbatively.