Advanced Quantum Field Theory

Part III L 2019

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1 Path Integrals in QM

Goal: Schrödinger's equation → path integral

Consider a Hamiltonian in one dimension $\hat{H} = H(\hat{x}, \hat{p})$, where position and momentum operators satisfy the common commutation relations $[\hat{x}, \hat{p}] = i\hbar$. Assume the it takes the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{1.1}$$

Schrödinger's equation then says that the time evolution of a state $|\psi(t)\rangle$ is given by

$$i\hbar\frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle . \tag{1.2}$$

This has a formal solution, giving us the time-evolution operator

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle . \tag{1.3}$$

In the Schrödinger picture, the states are evolving in time whereas operators and their eigenstates are constant in time.

Definition 1 (wavefunction)**:** $\Psi(x,t) := \langle x|\psi(t)\rangle$

The Schrödinger equation then becomes

$$\langle x| \hat{H} |\psi(t)\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi(x,t). \tag{1.4}$$

We will turn this differential equation into an integral equation, where we will sum over particle paths—a path integral. We can introduce an integral by inserting a complete set of states $1 = \int dx_0 |x_0\rangle\langle x_0|$.

$$\begin{aligned}
\Psi(x,t) &= \langle x| e^{-i\hat{H}t/\hbar} |\psi(0)\rangle &(1.5)\\
&= \int_{-\infty}^{\infty} \mathrm{d}x_0 \langle x| e^{-i\hat{H}t/\hbar} |x_0\rangle \langle x_0|\psi(0)\rangle &(1.6)\\
&:= \int_{-\infty}^{\infty} \mathrm{d}x_0 \underbrace{K(x,x_0;t)}_{\text{'kernel'}} \Psi(x_0,0) &(1.7)
\end{aligned}$$

We repeat this insertion for n intermediate times and positions.

Notation: Let $0 := t_0 < t_1 < \cdots < t_n < t_{n+1} := T$

And we also want to factor the exponential into n terms:

$$e^{i\hat{H}T/\hbar} = e^{-\frac{i}{\hbar}\hat{H}(t_{n+1}-t_n)} \cdots e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)}. \tag{1.8}$$

Then

$$K(x, x_0; T) = \int_{-\infty}^{\infty} \left[\prod_{r=1}^{n} \mathrm{d}x_r \, \langle x_{r+1}| \, e^{-\frac{i}{\hbar}\hat{H}(t_{r+1}-t_r)} \, |x_r\rangle\right] \langle x_1| \, e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)} \, |x_0\rangle \tag{1.9}$$

Integrals are over all possible position eigenstates at times $t_r, r = 1, \ldots, n$.

Free Theory

Consider the "free" theory, with $V(\hat{x}) = 0$. We will now play a similar but different trick to what we did before. Let us insert a complete set of momentum eigenstates $1 = \int_{-\infty}^{\infty} \text{đ}p \, |p\rangle \langle p|$. We also note that these momentum eigenstates are plane waves $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$.

Definition 2:

$$\text{đ}p := \frac{\mathrm{d}p}{2\pi\hbar}. \tag{1.10}$$

The corresponding kernel is

$$K_0(x, x'; t) = \langle x| \exp\left(\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t\right)|x'\rangle. \tag{1.11}$$

$$= \int_{-\infty}^{\infty} \text{đ}p \, e^{-ip^2t/2m\hbar} e^{ip(x-x')/\hbar} \tag{1.12}$$

$$= \sqrt{\frac{m}{2\pi i\hbar t}} e^{\frac{im(x-x')^2}{2\hbar t}}. \tag{1.13}$$

Remark:

$$\lim_{t\to 0}\{K_0(x, x'; t)\} = \delta(x - x'). \tag{1.14}$$

As expected from $\langle x|x'\rangle = \delta(x - x')$.

From the Baker-Campbell-Hausdorff formula, we know that

$$e^{\epsilon\hat{A}}e^{\epsilon\hat{B}} = \exp\left(\epsilon\hat{A} + \epsilon\hat{B} + \frac{\epsilon^2}{2}[\hat{A}, \hat{B}] + \ldots\right) \neq e^{\epsilon(\hat{A}+\hat{B})} \tag{1.15}$$

$$\text{for small } \epsilon: \qquad e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}}e^{\epsilon\hat{B}}(1 + O(\epsilon^2)) \tag{1.16}$$

Letting $\epsilon = 1/n$ and raising the above to the n^{th} power[1] gives

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty} \left\{ e^{\hat{A}/n} e^{\hat{B}/n} \right\}^n . \tag{1.17}$$

We will use this to separate kinetic and potential terms.

Take $t_{r+1} - t_r = \delta t$ with $\delta t \ll T$ and n large such that $n\delta t = T$.

$$e^{-\frac{i}{\hbar}\hat{H}\delta t} = \exp\left(-\frac{i\hat{p}^2 \delta t}{2m\hbar} \right) \exp\left(-\frac{iV(\hat{x})\delta t}{\hbar} \right) [1 + O(\delta t^2)] \tag{1.18}$$

Using the result (1.13),

$$\langle x_{r+1} | \exp\left(-\frac{i\hat{H}}{\hbar} \delta t \right) | x_r \rangle = e^{-iV(x_r)\delta t/\hbar} K_0(x_{r+1}, x_r; \delta t) \tag{1.19}$$

$$= \sqrt{\frac{m}{2\pi i \hbar \delta t}} \exp\left[\frac{im}{2\hbar} \left(\frac{x_{r+1} - x_r}{\delta t} \right)^2 \delta t - \frac{i}{\hbar} V(x_r) \delta t \right] \tag{1.20}$$

With $T = n\delta t$

$$K(x, x_0; T) = \int \left[\prod_{r=1}^{n} \mathrm{d}x_r \right] \left(\frac{m}{2\pi i \hbar \delta t} \right)^{\frac{n+1}{2}} \exp\left\{ i \sum_{r=0}^{n} \left[\frac{m}{2\hbar} \left(\frac{x_{r+1} - x_r}{\delta t} \right)^2 - \frac{1}{\hbar} V(x_r) \right] \delta t \right\} \tag{1.21}$$

In the limit $n \to \infty$, $\delta t \to 0$ with $n\delta t = T$ fixed, the exponent becomes

$$\frac{1}{\hbar} \int_0^T \mathrm{d}t \left[\frac{1}{2} m \dot{x}^2 - V(x) \right] = \int_0^T \mathrm{d}t \, L(x, \dot{x}), \tag{1.22}$$

where L is the classical Lagrangian, the Legendre transformation of the classical Hamiltonian. The classical action is $S = \int \mathrm{d}t \, L(x, \dot{x})$.

The main result therefore is that the path integral for the kernel is

$$\boxed{K(x, x_0; t) := \langle x | \, e^{-i\hat{H}t/\hbar} \, | x_0 \rangle = \int \mathcal{D}x \, e^{\frac{i}{\hbar} S}} \tag{1.23}$$

Definition 3 (functional integral)**:**

$$\mathcal{D}x = \lim_{\delta t \to 0 n\delta T \text{ fixed}} \left\{ (\sqrt{\ldots}) \prod_{r=1}^{n} (\sqrt{\ldots} dx_r) \right\} \tag{1.24}$$

We do not need to care about normalization factors.

[1] This step is sometimes called Susuki–Trotter (?) decomposition.