General Relativity: Example Sheet 3

David Tong, November 2019

1*. Obtain the form of the general timelike geodesic in a 2d spacetime with metric

$$ds^2 = \frac{1}{t^2}(-\mathrm{d}t^2 + \mathrm{d}x^2)$$

Hint: You should use the symmetries of the Lagrangian. You will probably find the following integrals useful:

$$\int \frac{dt}{t\sqrt{1+p^2t^2}} = \frac{1}{2}\ln\left(\frac{\sqrt{1+p^2t^2}-1}{\sqrt{1+p^2t^2}+1}\right) \quad \text{and} \quad \int \frac{d\tau}{\sinh^2\tau} = -\coth\tau,$$

2. The *Brans-Dicke* theory of gravity has an extra scalar field ϕ which acts like a dynamical Newton constant. The action is given

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big[R\phi - \frac{\omega}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big] + S_M$$

where ω is a constant and S_M is the action for matter fields. Derive the resulting Einstein equation and the equation of motion for ϕ.

3. *M-theory* is a quantum theory of gravity in $d = 11$ spacetime dimensions. It arises from the strong coupling limit of string theory. At low-energies, it is described by $d = 11$ supergravity whose bosonic fields are the metric and a 4-form $G = dC$ where C is a 3-form potential. The action governing these fields is

$$S = \frac{1}{2}M_{\text{pl}}^9\left[\int d^{11}x\,\sqrt{-g}\left(R - \frac{1}{48}G_{\mu\nu\rho\sigma}G^{\mu\nu\rho\sigma}\right) - \frac{1}{6}\int C\wedge G\wedge G\right]$$

i) Show that, up to surface terms, this action is gauge invariant under $C \to C + d\Lambda$ where Λ is a 2-form.

ii) Vary the metric to determine the Einstein equation for this theory.

iii) Vary C to obtain the equation of motion for the 4-form,

$$d\star G = \frac{1}{2}G\wedge G$$

4. i) Let X and Y be two vector fields. Show that

$$\mathcal{L}_X(\mathcal{L}_Y Q) - \mathcal{L}_Y(\mathcal{L}_X Q) = \mathcal{L}_{[X,Y]} Q,$$

when Q is either a function or a vector field. Use the Leibniz property of the Lie derivative to show that this also holds when Q is a one-form.

ii) Demonstrate that if a Riemannian or Lorentzian manifold has two "independent" isometries then it has a third, and define what is meant by independent here.

iii) Consider the unit sphere with metric

$$ds^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2.$$

Show that

$$X = \frac{\partial}{\partial\phi} \qquad \text{and} \qquad Y = \sin\phi\,\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\,\frac{\partial}{\partial\phi}$$

are Killing vectors. Find a third, and show that they obey the Lie algebra of $so(3)$.

5. Let K^μ be a Killing vector field and $T_{\mu\nu}$ the energy momentum tensor. Let $J^\mu = T^\mu{}_\nu K^\nu$. Show that J^μ is a conserved current, meaning $\nabla_\mu J^\mu = 0$.

6. Show that a Killing vector field K^μ satisfies the equation

$$\nabla_\mu\nabla_\nu K^\rho = R^\rho{}_{\nu\mu\sigma}K^\sigma$$

[*Hint:* use the identity $R^\rho{}_{[\mu\nu\sigma]} = 0$.]

Deduce that in Minkowski spacetime the components of Killing covectors are linear functions of the coordinates.

7. Consider Minkowski spacetime in an inertial frame, so the metric is $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. Let K^μ be a Killing vector field. Write down Killing's equation in the inertial frame coordinates.

Using the result of Q6, show that the general solution can be written in terms of a constant antisymmetric matrix $a_{\mu\nu}$ and a constant covector b_μ.

Identify the isometries corresponding to Killing fields with

- $a_{\mu\nu} = 0$
- $a_{0i} = 0$, $b_\mu = 0$,
- $a_{ij} = 0$, $b_\mu = 0$

where $i,j = 1,2,3$. Identify the conserved quantities along a timelike geodesic corresponding to each of these three cases.

8*. The *Einstein Static Universe* has topology $\mathbf{R} \times \mathbf{S}^3$ and metric

$$ds^2 = -dt^2 + d\chi^2 + \sin^2\chi \, d\Omega_2^2$$

where $t \in (-\infty, +\infty)$ and $\chi \in [0, \pi)$ and $d\Omega_2^2$ is the round metric on $\mathbf{S}^2$. If we suppress the $\mathbf{S}^2$, this space can be pictured as an infinite cylinder. Show that Minkowski, de Sitter and anti-de Sitter spacetimes are all conformally equivalent to submanifolds of the Einstein static universe. Draw these submanifolds on a cylinder.

9. The Lagrangian for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}$$

where $F = dA$. Show that this Lagrangian reproduces the Maxwell equations when A_μ is varied and reproduces the energy-momentum tensor when $g_{\mu\nu}$ is varied.

10. i) A scalar field obeying the Klein-Gordon equation $\nabla^\mu \nabla_\mu \phi - m^2 \phi = 0$ has energy-momentum tensor

$$T_{\mu\nu} = \nabla_\mu \Phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \left(\nabla^\rho \phi \nabla_\rho \phi + m^2 \phi^2 \right)$$

Show that $T_{\mu\nu}$ is covariantly conserved.

ii) The energy-momentum for a Maxwell field $F_{\mu\nu}$ is

$$T_{\mu\nu} = g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$$

Show that $T_{\mu\nu}$ is covariantly conserved when the Maxwell equations are obeyed.

iii) The energy-momentum tensor of a perfect fluid, with energy density ρ, pressure P and 4-velocity u^μ with $u^\mu u_\mu = -1$ is

$$T^{\mu\nu} = (\rho + P) u^\mu u^\nu + P g^{\mu\nu}$$

Show that conservation of the energy-momentum tensor implies

$$u^\mu \nabla_\mu \rho + (\rho + p) \nabla_\mu u^\mu = 0 \quad \text{and} \quad (\rho + p) u^\nu \nabla_\nu u_\mu = -(g_{\mu\nu} + u_\mu u_\nu) \nabla^\nu p$$

11. A test particle of rest mass m has a (timelike) worldline $x^\mu(\lambda)$, $0 \le \lambda \le 1$ and action

$$S = -m \int d\tau \equiv -m \int d\lambda \, \sqrt{-g_{\mu\nu}(x(\lambda))\dot{x}^\mu \dot{x}^\nu}$$

where τ is proper time and a dot denotes a derivative with respect to λ.

i) Show that varying this action with respect to $x^\mu(\lambda)$ leads to the non-affinely parameterised geodesic equation.

$$\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho \dot{x}^\sigma = \frac{1}{L}\frac{dL}{d\lambda}\dot{x}^\mu$$

Explain why we can choose a parameterisation so that $dL/d\sigma = 0$. [*Hint*: You may want to look at chapter 1 of the lecture notes to refresh your geodesic knowledge.]

ii) Show that the energy-momentum tensor of the particle in any chart is

$$T^{\mu\nu}(x) = \frac{m}{\sqrt{-g(x)}} \int d\tau \; u^\mu(\tau) u^\nu(\tau) \delta^4(x - x(\tau))$$

where u^μ is the 4-velocity of the particle.

iii) Conservation of the energy-momentum tensor is equivalent to the statement that

$$\int_R d^4x \, \sqrt{-g}\, v_\nu \nabla_\mu T^{\mu\nu} = 0$$

for any vector field v^μ and region R. By choosing v^μ to be compactly supported in a region intersecting the particle worldline, show that conservation of $T^{\mu\nu}$ implies that test particles move on geodesics. (This is an example of how the "geodesic postulate" of GR is a consequence of energy-momentum conservation.)

12. Physically reasonable matter with energy-momentum tensor $T^{\mu\nu}$ is expected to satisfy the *weak energy condition*, i.e.

$$T_{\mu\nu}u^\mu u^\nu \ge 0$$

for all timelike u^μ. Give a physical interpretation for this condition. You measure the components of $T^\mu{}_\nu$ in some basis and determine its eigenvalues λ and eigenvectors v^μ satisfying

$$T^\mu{}_\nu v^\nu = \lambda v^\mu$$

You find that it has precisely one timelike eigenvector with eigenvalue $-\rho$ and three spacelike eigenvectors with eigenvalues $P_{(i)}$. Under which necessary and sufficient condition on these eigenvalues is the weak energy condition satisfied?