# Applications of Differential Geometry to Physics

Part III Lent 2020

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## 0.1 Kepler / Newton Orbits

$$\ddot{\mathbf{r}} = -\frac{GMv}{r^3}\mathbf{r} \quad \leftrightarrow \quad \text{conic sections} \tag{1}$$

General conic section is

$$ax^2 + by^2 + cxy + dx + ey + f = 0 \tag{2}$$

This is nowadays more generally studied in what we now call *algebraic geometry* rather than differential geometry.

Apolonius of Penge (?) asked 'what is the unique conic thorugh five points, no three of which are co-linear?'

The space of conics is $\mathbb{R}^6 - \{0\}/n = \mathbb{RP}^5$ (projective 5-space).

$$[a,b,c,d,e,f] \sim [\gamma a, \gamma b, \gamma c, \gamma d, \gamma c, \gamma f], \gamma \in \mathbb{R}^* \tag{3}$$

This is an application of geometry, rather than an application of differential geometry.

**Remark:** Apolonius proved this geometrically.

In this course however, we will look at the following.

1) Hamiltonian mechanics ( mid $19^{\text{th}}$). This is an elegant way of reformulating Newton's mechanics, turning second order differential equations into first order differential equations with the use of a function $H(p,q)$. The system of ODEs is

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{q} = -\frac{\partial H}{\partial q} \tag{4}$$

This led to the development of symplectic geometry ( 1960s). The connection is that the phase-space to which $p$ and $q$ belong has a 2 -form $dp \wedge dq$ . Using the Hamiltonian function, one can find a vector field

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} \tag{5}$$

and looks for a one-parameter group of transformations, called symplectomorphisms, generated by this vector field. Under these symplectomorphisms, the 2 -form is unchanged meaning that the area illustrated in F2 is preserved. Details of this are going to come within the course.

2) General Relativity (1915) $\leftarrow$ Riemannian Geometry ( 1850)

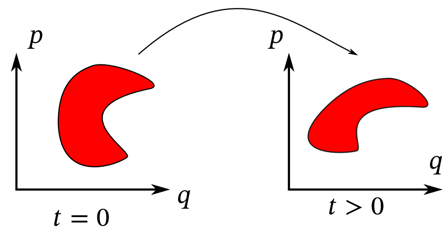


Figure 1

3) Gauge theory (Maxwell, Yang Mills) ↔ Connection on Principal Bundle (U(1) (Maxwell), SU(2), SU(3))

$$A_+ = A_- + dg \qquad g = \psi_+ - \psi_- \qquad \omega = \begin{cases} A_+ + d\psi_+ \\ A_- + d\psi_- \end{cases} \tag{6}$$

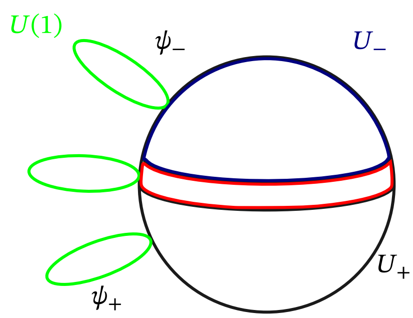


Figure 2

This course: cover 1, 2, 3 in some detail. Unifying feature: Lie groups.

- Prove some theorems, *lots of* examples (often instead of proofs)
- Want to be able to do calculations; compute characteristic classes etc.

We will assume that you took either Part III General Relativity, or Part III Differential Geometry, or some equivalent course.

# 1 Manifolds

**Definition 1** (manifold)**:** An $n$ -dimensional *smooth manifold* is a set $M$ and a collection[2] of open sets $U_\alpha$ , labelled by $\alpha = 1, 2, 3, \ldots$ , called *charts* such that

- $U_\alpha$ cover $M$
- $\exists$ 1-1 maps $\phi_\alpha :\ U_\alpha \to V_\alpha \in \mathbb{R}^n$ such that

$$\phi_\beta \circ \phi_\alpha^{-1} :\ \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{1.1}$$

  is a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^n$ .

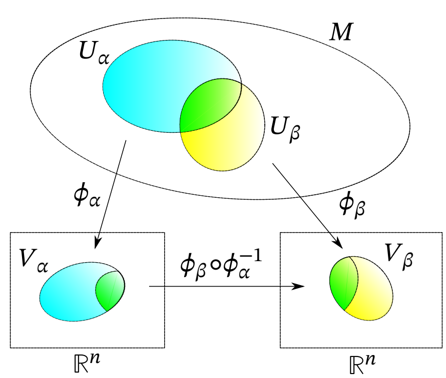


Figure 1.1: Manifold

As such, manifolds are topological spaces with additional structure, allowing us to do calculus.

**Example 1.0.1** ($M = \mathbb{R}^n$)**:** There is the *trivial manifold*, which can be covered by only one open set. There are other possibilities. In fact, there are infinitely many smooth structures on $\mathbb{R}^4$ (Proof by Donaldson in 1984 in his PhD. He used Gauge theory).

[2] In all examples that we will look at, there will be finitely $\alpha$.

**Example 1.0.2** (sphere $S^n = \left\{\mathbf{r} \in \mathbb{R}^{n+1}, |\mathbf{r}| = 1\right\}$ )**:** Intuitively, the $n$-sphere $S^n$ is an $n$-dimensional manifold. To show this, we will construct a map $\phi : S^n \to \mathbb{R}^n$ by projecting the north pole $N : (0, 0 \dots, 0, 1) \in \mathbb{R}^{n+1}$, through the point $p \in S^n$ onto the hyperplane defined by $r_{n+1} = 0$. This is illustrated for $S^2$ in Fig. 1.2. From this figure, we can already see that this projection map is ill-

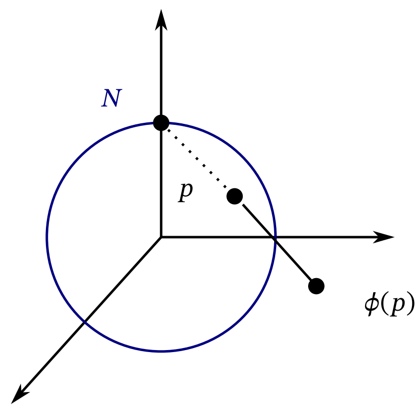


Figure 1.2: The map $\phi$ projects the north pole $N$ through a point $p \in S^2$ onto $\phi(p)$, which lies in the $R^2$-plane defined by $z = 0$. This map is not defined for the north-pole itself.

defined at the north pole itself. However, the definition of a manifold allows us to work with multiple charts. Let us cover $S^n$ with the following two open sets

$$U = S^n / \{0, 0, 0, , \dots, 0, 1\}, \qquad \widetilde{U} = S^n / \{0, 0, 0, , \dots, 0, -1\}, \tag{1.2}$$

so $U$ does not include the north pole and $\widetilde{U}$ does not include the south pole. We can then define two maps, $\phi$ and $\widetilde{\phi}$, which project from the north pole and the south pole respectively. Using $x_i$ to denote coordinates in $\mathbb{R}^n$ and $r_i$ to denote coordinates in $\mathbb{R}^{n+1}$ we take maps

$$\text{defined on } U\text{:} \qquad \phi(r_1, \dots, r_{n+1}) = \left(\frac{r_1}{1 - r_{n+1}}, \dots, \frac{r_n}{1 - r_{n+1}}\right) = (x_1, \dots, x_n) \tag{1.3}$$

$$\text{defined on } \widetilde{U}\text{:} \qquad \widetilde{\phi}(r_1, \dots, r_{n+1}) = \left(\frac{r_1}{1 + r_{n+1}}, \dots, \frac{r_n}{1 + r_{n+1}}\right) = (\widetilde{x}_1, \dots, \widetilde{x}_n). \tag{1.4}$$

On the overlap $U \cap \widetilde{U}$, the coordinates are related

$$\overbrace{\frac{r_k}{1 + r_{n+1}}}^{\widetilde{x}_k} = \frac{1 - r_{n+1}}{1 + r_{n+1}} \overbrace{\frac{r_k}{1 - r_{n+1}}}^{x_k}, \qquad k = 1, \dots, n, \tag{1.5}$$

where we can write the transition factor on the right in terms of the coordinates $x_i$:

$$\frac{1 - r_{n+1}}{1 + r_{n+1}} = \frac{(1 - r_{n+1})^2}{r_1^2 + r_2^2 + \cdots + r_n^2} = \frac{1}{x_1^2 + x_2^2 + \cdots + x_n^2}. \tag{1.6}$$

So on $U \cap \widetilde{U}$, the transition function is

$$\widetilde{\phi} \circ \phi^{-1} : (x_1, \dots, x_n) \longmapsto (\widetilde{x}_1, \dots, \widetilde{x}_n) = \left(\frac{x_1}{x_1^2 + \cdots + x_n^2}, \dots, \frac{x_n}{x_1^2 + \cdots + x_n^2}\right), \tag{1.7}$$

which is a smooth map from $\mathbb{R}^n \to \mathbb{R}^n$.

**Example 1.0.3:** A Cartesian product of manifolds is a manifold, for example we have the $n$-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$.

**Definition 2** (surface)**:** Let $f_1, \dots, f_k : \mathbb{R}^N \to \mathbb{R}$ be smooth functions. A surface $f_1 = 0, \dots, f_k = 0$ is a manifold of dimension $\dim n = N - k$ if the rank of the matrix $\frac{\partial f_\alpha}{\partial x^i}$, $\alpha = 1, \dots, k$ and $i = 1, \dots, N$ is maximal and equal to $k$ at all points of $\mathbb{R}^N$..

**Example 1.0.4:** The $n$-sphere $S^n$ is a surface in $\mathbb{R}^{n+1}$ with $f_1 = 1 - |\mathbf{r}|^2$.

**Theorem 1** (Whitney)**:** Every smooth manifold of dimension $n$ is an embedded surface in $\mathbb{R}^N$, where $N \leq 2n$.

> If you enjoy using geometrical intuition and looking at surfaces, this theorem ensures that you can always do that and not loose generality.

**Definition 3** (real projective space)**:** The $n$-dimensional *real projective space* is defined as the quotient

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim, \tag{1.8}$$

where we quotient out the equivalence classes $[X_1, \dots, X_n + 1] \sim [cX_1, \dots, cX_{n+1}]$ for all $c \in \mathbb{R}^*$. The $[X_1, \dots, X_{n+1}]$ are called *homogeneous coordinates.*

In other words, this is the space of all lines through the origin in $\mathbb{R}^{n+1}$ .

**Claim 1:** $\mathbb{RP}^n$ is a smooth manifold of dimension $n$ with $(n + 1)$ open sets.

*Proof.* Let us define our open sets with respect to the homogeneous coordinates. We define the set $U_\alpha : [X] \in \mathbb{RP}^n$ such that $X_\alpha \neq 0 \ \alpha = 1, \dots, n + 1$ . We can now find local coordinates on $\phi_\alpha : U_\alpha \to V_\alpha \in \mathbb{R}^n$

$$x_1 = \frac{X_1}{X_\alpha} \quad \dots \quad x_{\alpha-1} = \frac{X_{\alpha-1}}{X_\alpha} \quad x_{\alpha+1} = \frac{X_{\alpha+1}}{X_\alpha} \quad \dots \quad x_n = \frac{X_n}{X_\alpha}. \tag{1.9}$$

□

**Exercise 1.1:** Prove smoothness of $\phi_\beta \circ \phi_\alpha^{-1}$.

Now it turns out that this manifold is equivalent to $\mathbb{RP}^n = S^n / \mathbb{Z}^2$. From quantum mechanics, we know that this means in particular $\mathbb{RP}^3 = SO(3)$. This is illustrated in 1.3.

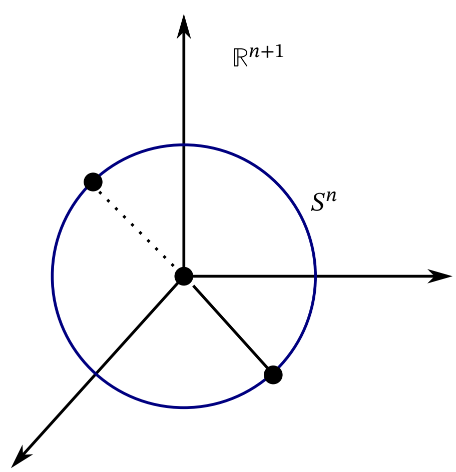


Figure 1.3: Real projective space $\mathbb{RP}^n$ is isomorphic to $S^n/\mathbb{Z}^n$, identifying antipodal points.

# 2 Vector Fields

Let $M, \widetilde{M}$ be smooth manifolds of dimension $n, \tilde{n}$.

**Definition 4** (smooth map)**:** A map $f : M \to \widetilde{M}$ is *smooth* if $\widetilde{\phi}_\beta \circ f \circ \phi_\alpha^{-1}$ is a smooth map from $\mathbb{R}^n$ to $\widetilde{\mathbb{R}}^n$ for all $\alpha, \beta$. We call $f : M \to \mathbb{R}$ a *function*, whereas we call $f : \mathbb{R} \to M$ a *curve*.

Let $\gamma : \mathbb{R} \to M$ be a curve. For some $U \in M$, $U \simeq \mathbb{R}^n$ , we can define local coordinates $(x^1, \dots, x^n)$

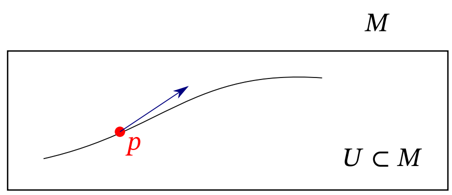


Figure 2.1

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**Definition 5** (tangent vector)**:** A *tangent vector* $V$ to $\gamma$ at $p$ is

$$V|_p = \left.\frac{\mathrm{d}\psi}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \in T_pM, \tag{2.1}$$

where $T_pM$ is the *tangent space* to $M$ at $p$.

**Definition 6** (tangent bundle)**:** We define the *tangent bundle* as $TM := \bigcup_{p\in M} T_pM$.

**Definition 7** (vector field)**:** A *vector field* assigns a tangent vector to all $p \in M$ .

Let $f : M \to \mathbb{R}$ . The rate of change of $f$ along $\gamma$ is

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(x^a(\epsilon))|_{\epsilon=0} = \sum_a \dot{x}^a \frac{\partial f}{\partial x^a} \tag{2.2}$$

$$= \sum_a V^a \left.\frac{\partial f}{\partial x^a}\right|_{\epsilon=0}, \tag{2.3}$$

where $V^a := \dot{x}^a|_{\epsilon=0,\dots,x_n}$.

Vector fields are first order differential operators

$$V = \sum_a V^a(\mathbf{x}) \frac{\partial}{\partial x^a}. \tag{2.4}$$

The derivatives $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}\Big|_p$ form a basis of $T_pM$.

## 2.1 Integral curves

**Definition 8** (integral curve)**:** An *integral curve* (a *flow*) of a vector field is defined by

$$\dot{\gamma}(\epsilon) = V|_{\gamma(\epsilon)}, \tag{2.5}$$

where the dot denotes differentiation with respect to $\epsilon$.

On $n$ first order ODEs: $\dot{x}^a = V^a(x)$ .

There exists a unique solution given initial data $X^a(0)$ . Given a solution $X^a(\epsilon)$ , we can expand it in a Taylor series as

$$X^a(\epsilon) = X^a(0) + V^a \cdot \epsilon + O(\epsilon^2). \tag{2.6}$$

Up to first order in $\epsilon$, the vector field determines the flow. We call $V$ a *generator* of its flow.

The following example illustrates how you get from a vector field to its flow.

**Example 2.1.1** ($M = \mathbb{R}^2, x^a = (x, y)$)**:** Consider the vector field $V = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ . The system of ODEs we solve is $\dot{x} = x$ and $\dot{y} = 1$ . This gives us the integral curve $(x(\epsilon), y(\epsilon)) = (x(0)e^{\epsilon}, y(0) + \epsilon)$ . From this we can see that $x(\epsilon)\cdot\exp(-y(\epsilon))$ is constant along $\gamma$. Using this we can draw the unparametrised integral curve in Fig. 2.2.

This example motivates the following definition.

**Definition 9** (invariant)**:** An *invariant* of a vector field $V$ is a function $f$ constant along the flow of $V$.

$$f(x^a(0)) = f(x^a(\epsilon)) \qquad \forall \epsilon. \tag{2.7}$$

Equivalently, $V(f) = 0$.

Let us now consider an example that goes the other way: from flow to vector field.

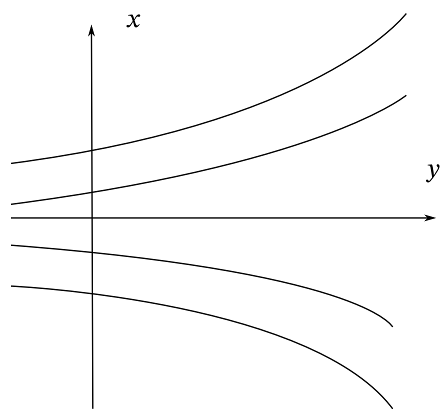


Figure 2.2

**Example 2.1.2:** Consider the 1 -parameter group of rotations of a plane.

$$(x(\epsilon), y(\epsilon)) = (x_0 \cos\epsilon - y_0 \sin\epsilon, x_0 \sin\epsilon + y_0 \cos\epsilon). \tag{2.8}$$

The associated vector field is

$$V = \left( \frac{\partial y(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial y} + \frac{\partial x(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial x} \right) \bigg|_{\epsilon=0} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial y}. \tag{2.9}$$

Now you can add vector fields, but there is also another operation.

**Definition 10** (Lie bracket)**:** A *Lie bracket* $[V, W]$ of two vector fields $V, W$ is a vector field defined by

$$[V, W](f) = V(W(f)) - W(V(f)) \qquad \forall f. \tag{2.10}$$

This is indeed another vector field since the commutator of two first order operators is another first order operator.

**Example 2.1.3:** Let $V = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $W = \frac{\partial}{\partial x}$ . We then have $[V, W] = -W$.

This is not always the case but sometimes the Lie bracket reproduces some of the vector fields. There is an interesting algebraic structure to this.

**Definition 11** (Lie algebra)**:** A *Lie algebra* is a vector space $\mathfrak{g}$ with an anti-symmetric, bilinear operation $[\ ,\ ] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called a *Lie bracket*, which satisfies the *Jacobi identity*

$$[V, [U, W]] + [W, [V, U]] + [U, [W, V]] = 0 \qquad \forall U, V, W \in \mathfrak{g}. \tag{2.11}$$

We will spend some time discussing this abstractly, but then focus on the Lie algebras of vector fields in the main part of this course.

Any two vector spaces of a given dimension are isomorphic; there is nothing special other than the dimension distinguishing vector spaces. For Lie algebras this is not so.

**Example 2.1.4:** Even in dimension 2, which is the lowest non-trivial dimension, there are two Lie algebras (up to isomorphism)

$$a)\quad [V,W] = -W, \qquad b)\quad [V,W] = 0. \tag{2.12}$$

If the vector space underlying $\mathfrak{g}$ is finite-dimensional, and $V_\alpha$, $\alpha = 1, \dots, \dim \mathfrak{g}$ is a basis of $\mathfrak{g}$, we can define the Lie algebra by specifying the brackets

$$[V_\alpha, V_\beta] = \sum_\gamma f_{\alpha\beta}{}^\gamma V_\gamma, \tag{2.13}$$

where $f_{\alpha\beta}{}^\gamma$ are the *structure constants.*

**Example 2.1.5** ($\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$)**:** The vector space is given by $n \times n$ real matrices, and the Lie bracket is the matrix commutator. The dimension of this Lie algebra is $\dim \mathfrak{g} = n^2$.

**Example 2.1.6** (Vector fields)**:** The set of all vector fields on a manifold $M$ form an infinite-dimensional Lie algebra.

**Example 2.1.7:** Consider diff($\mathbb{R}$) or diff($S^1$), vector fields on a line or on a circle respectively.

$$\mathrm{diff}(\mathbb{R}), \quad x \in \mathbb{R}, \quad V_\alpha = -x^{\alpha+1} \frac{\partial}{\partial x} \tag{2.14}$$

$$\mathrm{diff}(S^1), \quad \theta \in S^1, \quad V_\alpha = i e^{i\alpha\theta} \frac{\partial}{\partial \theta} \tag{2.15}$$

$$[V_\alpha, V_\beta] = (\alpha - \beta) V_{\alpha+\beta}. \tag{2.16}$$

**Example 2.1.8** (Virasoro algebra)**:** The *Virasoro algebra* Vir $= \mathrm{diff}(S^1) \oplus \mathbb{R}$ is the central extension[1] of diff($S^1$), with *central charge* $c = \mathbb{R}$.

$$\begin{cases} [V_\alpha, c] = 0 \\ [V_\alpha, V_\beta]_{\mathrm{vir}} = (\alpha - \beta) V_{\alpha+\beta} + \dfrac{c}{12}(\alpha^3 - \alpha)\delta_{\alpha+\beta,0} \end{cases} \tag{2.17}$$

**Remark:**

$$[f(\theta)\frac{\partial}{\partial\theta}, g(\theta)\frac{\partial}{\partial\theta}] = \underbrace{(fg' - gf')}_{\text{Wronskian}} \frac{\partial}{\partial\theta} \tag{2.18}$$

'After Witten'.

$$[f\frac{\partial}{\partial\theta}, g\frac{\partial}{\partial\theta}]_{\mathrm{vir}} = [f\frac{\partial}{\partial\theta}, g\frac{\partial}{\partial\theta}] + \frac{ic}{48\pi}\int_0^{2\pi} (f'''g - g'''f)\,\mathrm{d}\theta \tag{2.19}$$

**Theorem 2** (Ado)**:** Every finite-dimensional Lie algebra is isomorphic to some matrix Lie algebra, a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.

[1] We will meet the concept of central extension and central charge in this term's *String Theory* course.

**Remark:** $n$ is not necessarily the dimension of the Lie algebra.

# 3 Lie Groups

**Definition 12** (Lie group)**:** A *Lie group* is a smooth manifold $G$, which is also a group, such that the group operations

$$\text{multiplication} \qquad G \times G \to G, \qquad (g_1, g_2) \longmapsto g_1 \cdot g_2 \tag{3.1}$$

$$\text{inverse} \qquad G \to G, \qquad g \longmapsto g^{-1} \tag{3.2}$$

are smooth maps between manifolds.

**Example 3.0.1** ($G = GL(n, \mathbb{R}) \in \mathbb{R}^{n^2}$)**:** The general linear group $GL(n, \mathbb{R})$ is defined as the set of invertible matrices $\{g \in G \mid \det g \neq 0\}$. The dimension is $\dim(G) = n^2$.

**Example 3.0.2** ($G = O(n, \mathbb{R})$)**:** This is the group of orthogonal matrices, defined by $\frac{1}{2}n(n+1)$ conditions $g^T g = \mathbb{1}$. The dimension is then $\dim O(n, \mathbb{R}) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$. We also have to check that these conditions define a manifold in the sense that the associated Jacobian has maximal rank.

**Definition 13** (group action)**:** A *group action* on a manifold $M$ is a map $G \times M \to M$ mapping $(g, p) \to g(p)$ such that

$$e(p) = p, \qquad g_1(g_2(p)) = (g_1 \cdot g_2)(p) \tag{3.3}$$

for all $p \in M$ and al $g_1, g_2$ on $G$.

**Definition 14** (transformation group)**:** If we have a group action, we refer to $G$ as a group of *transformations*.

**Example 3.0.3:** Take $M = \mathbb{R}^2$ and $G = E(2)$, the three-dimensional Euclidean group.

$$g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \tag{3.4}$$

Take $G_E \in G$ to be a one-parameter subgroup of $G$. There are three such subgroups

$$G_\theta : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos\theta \cdot x - \sin\theta \cdot y \\ \sin\theta \cdot x + \cos\theta \cdot y \end{pmatrix} \tag{3.5}$$

$$G_a : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x + a \\ y \end{pmatrix} \tag{3.6}$$

$$G_b : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x \\ y + b \end{pmatrix}. \tag{3.7}$$

Each of these one-parameter subgroups generates a flow. We can think of this flow as being gener-

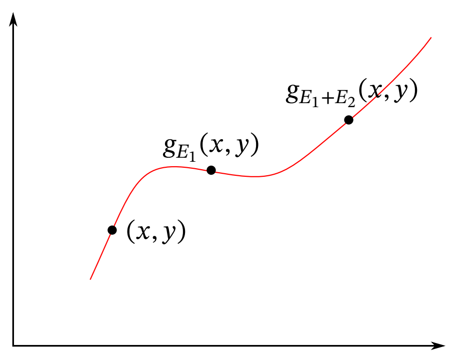


Figure 3.1

ated by a vector field $V|_p = \frac{\mathrm{d}}{\mathrm{d}E} g_E(p)\Big|_{E=0}$.

$$V_\theta = d\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{3.8}$$

$$V_a = \left(\frac{\mathrm{d}\tilde{x}}{\mathrm{d}a}\frac{\partial}{\partial \tilde{x}} + \frac{\mathrm{d}\tilde{y}}{\mathrm{d}a}\frac{\partial}{\partial \tilde{y}}\right)\Big|_{a=0} = \frac{\partial}{\partial x} \tag{3.9}$$

$$V_b = \frac{\partial}{\partial y}. \tag{3.10}$$

We define a 3-dimensional Lie algebra of $E(2)$ as

$$[V_a, V_\theta] = V_b \qquad [V_b, V_\theta] = -V_a \qquad [V_a, V_b] = 0, \tag{3.11}$$

represented by vector fields on $M$.

## 3.1 Geometry on Lie Groups

**Definition 15** (tangent map)**:** Let $f : M \to \widetilde{M}$ be a smooth map between manifolds. We define the *tangent map* or *push forward* to be

$$\begin{aligned} f_* : T_p(M) &\to T_{f(p)}(\widetilde{M}) \\ V &\longmapsto f_*(V) = \frac{\mathrm{d}}{\mathrm{d}E} f(\gamma(E))\Big|_{E=0}. \end{aligned} \tag{3.12}$$

This extends to the tangent bundle $T(M)$ . If $x^\alpha$ are coordinates of $\mathcal{M} \supset M$ , $(y^{\alpha'})$ coordinates on $\widetilde{\mathcal{M}} \subset M$ , then

$$V = V^\alpha \frac{\partial}{\partial x^a} \qquad f_*(V) = V^\alpha \frac{\partial f^i}{\partial x^a} \frac{\partial}{\partial y^i}. \tag{3.13}$$

**Definition 16** (Lie derivative)**:** Let $V, W$ be vector fields, where $V$ generates a flow $V = \dot{\gamma}$. The *Lie derivative* is

$$\mathcal{L}_V W|_p := \lim_{\epsilon \to 0} \frac{W(p) - \gamma(\epsilon)_* W(p)}{\epsilon} \tag{3.14}$$

We can extend this definition over the whole manifold.

**Exercise 3.1:** Show that $L_V W = [V, W]$.

**Definition 17:** On functions $f : M \to \mathbb{R}$, we define the Lie derivative as $\mathcal{L}_V(f) = V(f)$.

**Claim 2** (Cartan's Magic Formula)**:** On differential forms, we can use the Leibniz rule to show that

$$\mathcal{L}_V \Omega = d(\iota_V \Omega) + \iota_V(d\Omega) = d(V \rightharpoonup \Omega) + V \rightharpoonup d\Omega. \tag{3.15}$$

**Definition 18:** We define the cotangent space $T_p^* M = \operatorname{Span}\{dx^1, \ldots, dx^n\}$ as the space of one-forms. The cotangent bundle is then

$$\bigcup_{p \in M} T_p^* M = T^* M. \tag{3.16}$$

**Definition 19** ($r$-form)**:** Using the wedge product, which is anti-commutative on one-forms $dx^i \wedge dx^j = -dx^j \wedge dx^i$, we can define an $r$-form

$$\Omega = \frac{1}{r!} \Omega_{ij\ldots k} dx^i \wedge dx^j \wedge \cdots \wedge dx^k. \tag{3.17}$$

**Definition 20** (contraction)**:** We write a *contraction* as

$$\frac{\partial}{\partial x^i} \rightharpoonup dx^j = \iota_{\frac{\partial}{\partial x^i}} dx^j = \delta_i^{\ j}. \tag{3.18}$$

For a general vector field $V$ and one-form $\Omega$, we have

$$\iota_V \Omega = V \rightharpoonup \Omega = V^i \frac{\partial}{\partial x^i} \rightharpoonup \Omega_j dx^j = V^i \Omega_j \delta^j_{\ i} = V^i \Omega_i. \tag{3.19}$$

**Remark:** No metric is needed to define contraction.

**Definition 21:** A *Lie algebra* $\mathfrak{g}$ of a Lie group $G$ is the tangent space $T_e G$ to $G$ at the identity. The Lie bracket on $\mathfrak{g}$ is the commutator of vector fields on $G$.

**Definition 22** (Left translations)**:** For all $\mathfrak{g} \in G$, we define the *left translations*

$$\begin{aligned} L_g : G &\to G \\ h &\longmapsto g \cdot h \end{aligned} \tag{3.20}$$

**Definition 23** (left invariant vector fields)**:** Using the left translation maps, we define their push forward $(L_g)_* : \mathfrak{g} \equiv T_e G \to T_g G$, which maps $V \in \mathfrak{g}$ to vector fields $(L_g)_*(V)$ on $G$. This defines *left invariant vector fields* as sections $V \in TG$ such that $(L_g)_* V = V$ for all $g \in G$. Therefore,

$$[(L_g)_* V, (L_g)_* W] = (L_g)_* [V, W]_{\mathfrak{g}}. \tag{3.21}$$

**Remark:** It is important to understand the notation!

Left-invariant vector fields form a basis of $\mathfrak{g}$, meaning that $\dim(G)$ is the number of global, non-vanishing vector fields on $G$.

**Definition 24** (parallelisable)**:** A manifold $M$ is *parallelisable* if there exists a set of vector fields $\{V_i\}$, where $i = 1, \dots, \dim M$, such that $\forall p \in M$, the tangent vectors $\{V_i(p)\}$ form a basis for the tangent space $T_pM$.

**Claim 3:** Lie groups are parallelisable manifolds.

**Claim 4:** The converse is not true.

*Proof.* $S^1, S^3, S^7$ are the only parallelisable spheres.

The first two are indeed manifolds:

$$S^1 = U(1), \qquad S^3 = SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| \, |a|^2 + |b|^2 = 1 \right\}. \tag{3.22}$$

However, $S^7$ is not a Lie group. □

This has introduced the field of $K$-theory.

**Claim 5:** Let $\{L_\alpha\}$, $\alpha = 1, \dots, \dim \mathfrak{g}$ be a basis of left invariant vector fields with $[L_\alpha, L_\beta] = \sum_\gamma f_{\alpha\beta}{}^\gamma L_\gamma$. Let $\sigma^\alpha$ be a dual basis of left-invariant one-forms, meaning that $L_\alpha \rightharpoonup \sigma^\beta = \delta_\alpha{}^\beta$. Then

$$d\sigma^\alpha + \frac{1}{2} f^\alpha{}_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma = 0. \tag{3.23}$$

*Proof (Sheet 1).* Use the identity

$$d\Omega(V, W) = V(\Omega(W)) - W(\Omega(V)) - \Omega([V, W]). \tag{3.24}$$

□

Watch out for signs and factors in the upcoming derivations! Things can easily go wrong.

**Definition 25** (Maurer–Cartan 1-form)**:** Assume that $G$ is a matrix Lie group. The *Maurer–Cartan one-form* on $G$ is then defined as

$$\rho := g^{-1} dg. \tag{3.25}$$

**Claim 6:** The Maurer–Cartan 1-form

1. is left invariant,
2. takes values in the Lie algebra,
3. obeys the *Maurer–Cartan equation*

$$d\rho + \rho \wedge \rho = 0. \tag{3.26}$$

*Proof.* 1. With $g_0 \in G$ we have

$$(g_0 g)^{-1} d(g_0 g) = g^{-1} dg. \tag{3.27}$$

2. Take $C$ a smooth curve $g(s) \subset G$ .

$$g^{-1}(s) g(s+\epsilon) = \underbrace{\epsilon}_{\mathbb{1}} + \epsilon \underbrace{g^{-1} \frac{\mathrm{d}g}{\mathrm{d}\epsilon}}_{\in T_e G \simeq \mathfrak{g}} |_{\epsilon=0} + O(\epsilon^2). \tag{3.28}$$

So $g^{-1} dg = \sum_\alpha \sigma^\alpha \otimes T_\alpha$, where $T_\alpha$ are matrices with $[T_\alpha, T_\beta] = \sum_\gamma f_{\alpha\beta}{}^\gamma T_\gamma$.

3. Consider first the exterior derivative term

$$d\rho = \sum_\alpha d\sigma^\alpha \cdot T_\alpha = -\frac{1}{2} f^\alpha{}_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma \cdot T_\alpha. \tag{3.29}$$

The wedge product term is

$$\rho \wedge \rho = \sigma^\alpha T_\alpha \wedge \sigma^\beta T_\beta = \frac{1}{2} \sigma^\alpha \wedge \sigma^\beta [T_\alpha, T_\beta] = \frac{1}{2} \sigma^\alpha \wedge \sigma^\beta f_{\alpha\beta}{}^\gamma T_\gamma. \tag{3.30}$$

□

**Example 3.1.1** (Heisenberg group)**:** The *Heisenberg group* (sometimes just called *Nil*) is the group of upper-triangular matrices

$$g = \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} = \mathbb{1} + x T_1 + y T_2 + z T_3, \tag{3.31}$$

where $\{T_i\}$ are also the generators of the Lie algebra. Explicitly, we have

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.32}$$

and so their commutation relations are

$$[T_1, T_2] = T_3, \quad [T_1, T_3] = 0 = [T_2, T_3]. \tag{3.33}$$

We can interpret $T_1 = \hat{x}$ as position, $T_2 = \hat{p}$ as momentum, and $T_3 = i\hbar\hat{\mathbb{1}}$ as the identity.

**Remark:** Examples such as the lecture today will be important for the exam! If someone gives you a matrix Lie group, you will proceed in this order.

Taking the inverse of (3.31), we construct the Maurer–Cartan 1-form

$$\rho = g^{-1}dg = \begin{pmatrix} 1 & -x & -z+xy \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} & dx & dz \\ & & dy \\ & & \end{pmatrix} \tag{3.34}$$

$$= T_1 dx + T_2 dy + T_3(dz - xdy). \tag{3.35}$$

We define the dual basis of left-invariant one-forms

$$\sigma^1 = dx, \qquad \sigma^2 = dy, \qquad \sigma^3 = dz - xdy \tag{3.36}$$

$$d\sigma^1 = 0 \qquad d\sigma^2 = 0 \qquad d\sigma^3 = -dx \wedge dy = d\sigma^1 \wedge \sigma^2. \tag{3.37}$$

From these we find the left-invariant vector fields

$$L_1 = \frac{\partial}{\partial x}, \qquad L_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad L_3 = \frac{\partial}{\partial z}. \tag{3.38}$$

We find the Lie algebra by computing the brackets

$$[L_1, L_2] = L_3, \qquad [L_1, L_3] = 0, \qquad [L_2, L_3] = 0. \tag{3.39}$$

This is the same as (3.33), the Lie algebra of the Heisenberg group. We have a choice of representing the Lie algebra either in terms of matrices, $T_i$ or left-invariant vector fields $L_i$.

### 3.1.1 Right-Invariance

We could have defined the *right translations* $R_g(h) = h \cdot g$ , and associated right-invariant vector fields and one-forms.

**Claim 7:** The commutation relations for right-invariant vector fields differs by a minus sign from left-invariant vector fields and they commute

$$[R_\alpha, R_\beta] = -f_{\alpha\beta}{}^{\gamma} R_\gamma, \qquad [R_\alpha, L_\beta] = 0. \tag{3.40}$$

**Example 3.1.2** (Nil)**:** For the Heisenberg group, using $dg \cdot g^{-1}$

$$R_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \qquad R_2 = \frac{\partial}{\partial y}, \qquad R_3 = \frac{\partial}{\partial z}. \tag{3.41}$$

**Notation** (2)**:** Right-invariant vector fields are said to generate left-translations and vice-versa.

## 3.2 Metrics on Lie Groups

We can do some local differential geometry by defining a left-invariant metric on $G$.

**Definition 26** (left-invariant metric)**:** A *left-invariant metric* $h$ on a Lie group $G$ is of the form

$$h = g_{\alpha\beta}\sigma^\alpha \odot \sigma^\beta, \qquad \alpha, \beta = 1, \dots, \dim G, \tag{3.42}$$

where $g_{\alpha\beta}$ is a non-degenerate constant matrix.

Given the group and left-invariant metric, the right-invariant vector fields are generating isometries for this metric, which means that they are Killing vectors. Recall that the Maurer–Cartan 1-form is $\rho = g^{-1}dg = \sigma^\alpha \otimes T_\alpha$, where $\sigma^\alpha$ are invariant under $g \to g_0 g$. Therefore

$$\mathcal{L}_{R_\alpha}\sigma^\beta = 0 \qquad \forall \alpha, \beta. \tag{3.43}$$

Therefore, right invariant vector fields are Killing vectors for $h$, meaning that

$$\mathcal{L}_{R_\alpha}(h) = 0. \tag{3.44}$$

**Example 3.2.1** (Nil)**:** The metric is defined as

$$h = \delta_{\alpha\beta}\sigma^\alpha \cdot \sigma^\beta = dx^2 + dy^2 + (dz - x dy)^2. \tag{3.45}$$

How do we find the isometries?

- We can see that the metric components do not involve $z$ , so it is invariant under $z \to z + \omega$, which is generated by $\frac{\partial}{\partial z} = R_3$.
- Similarly, we can see the same for $y \to y + \epsilon$ , which is generated by $\frac{\partial}{\partial y} = R_2$.
- Finally, let us consider what happens for $x \to x + \epsilon$. As it stands, this is not an isometry. The parenthesis includes a term $\delta dy$, which we can get rid off by introducing another transformation $z \to z + \epsilon y$. This is generated by $\frac{\partial}{\partial x} + y\frac{\partial}{\partial z} = R_1$.

These agree with the right-invariant vector fields of Eq. (3.41).

### 3.2.1 Kaluza–Klein Interpretation

Consider the motion of a charged particle on the space of orbits of $R_3 = \frac{\partial}{\partial z}$.

This was introduced as a way to combine the two known forces at the time: gravity and electromagnetism. This is done by introducing extra dimensions, by looking at gravity in dimension $d = 5$, which reduces to gravity and electromagnetism in dimension $d = 4$. The higher dimension is taken to be compactified into a circle of very large radius, so it is not detected by experiment. This idea is still stuck with us today in *String Theory*.

**Example 3.2.2:** For the Heisenberg group, the metric is independent of $z$, so we can take our manifold to be periodic in $z$.

To find the equations of motion, it is useful to write down the geodesic Lagrangian

$$\mathscr{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + (\dot{z} - x\dot{y})^2), \tag{3.46}$$

where the dot denotes $\bullet = \frac{\mathrm{d}}{\mathrm{d}s}$. The Euler–Lagrange equations are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial \mathscr{L}}{\partial \dot{x}^i} &= \frac{\partial \mathscr{L}}{\partial x^i} && (3.47)\\ \ddot{x} &= -\dot{y}(\dot{z} - x\dot{y}) && (3.48)\\ \frac{\mathrm{d}}{\mathrm{d}s}\left(\dot{y} - x(\dot{z} - x\dot{y})\right) &= 0 && (3.49)\\ \frac{\mathrm{d}}{\mathrm{d}s}(\dot{z} - x\dot{y}) &= 0. && (3.50) \end{aligned}$$

Using the final equation to introduce a constant $c = \dot{z} - x\dot{y}$ , the first two equations reduce to

$$\ddot{x} = -c\dot{y} \qquad \& \qquad \ddot{y} = c\dot{x}. \tag{3.51}$$

Compare this with geodesic motion in a magnetic field. Let the spacetime be the Riemannian manifold $(M, g = g_{ij}dx^i \odot dx^j)$ and the magnetic field be the closed 2-form $F = \frac{1}{2}F_{ij}dx^i \wedge dx^j$ . The components of the Levi–Civita connection associated to $g$ are

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right). \tag{3.52}$$

The geodesic equation of motion is then given by

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = cF^i{}_j\dot{x}^j, \tag{3.53}$$

where we see $F^i{}_j$ as an endomorphism rather than a 2-form. This is the general form of geodesic motion in a magnetic field. We want to compare this to (3.51), so we take $M = \mathbb{R}^2$ and $g_{ij} = \delta_{ij}$ , which gives $\Gamma^k_{ij} = 0$ . Moreover, we take $F = -dx \wedge dy$ , meaning that $F_{ij} = -\epsilon_{ij}$ is the volume-form. So geodesics of the left-invariant metric $h$ on $G = \mathrm{Nil}$ projects to the trajectories of a charged particle in a constant magnetic field. We think of this as in Fig. 3.2.

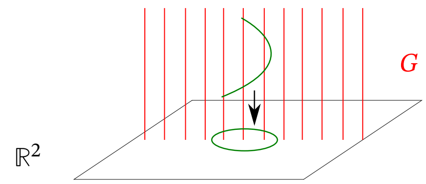


Figure 3.2: Kaluza–Klein reduction.

For the general case of Kaluza–Klein reduction, we have

$$h = (dz + A)^2 + g_{ij} dx^i \odot dx^j. \tag{3.54}$$

Take coordinates $(z, x^i)$, where $z$ is the extra dimension. Moreover, $A$ is a one-form on $M$ and we define $F = dA$. The geodesic equation is

$$\dot{z} + A_i \dot{x}^i = c \qquad \text{(conserved charge)} \tag{3.55}$$

Assume $z$ is periodic. On $G$, the Laplacian is

$$\nabla^2 = L_1^2 + L_2^2 + L_3^2. \tag{3.56}$$

The Schrödinger equation is then $-\nabla^2\phi = E\phi$.

$$\phi_{xx} + \phi_{zz} + (\partial_y + x d_z)^2\phi = -E\phi. \tag{3.57}$$

Changing variables to $\phi = \Psi(x, y)e^{iez}$ , we have

$$\Psi_{xx} - e^2\Psi + (\partial_y + ixe)^2\Psi = -E\Psi. \tag{3.58}$$

We can package some of the terms together to obtain a Maxwell potential. Recall that $F = dy \wedge dx = dA$ . The equation for a charged particle moving in a magnetic field is

$$(\partial_x - ieA_x)^2\Psi + (\partial_y - ieA_y)^2\Psi = -(E - e^2)\Psi. \tag{3.59}$$

Comparing these equations, find that $A_x = 0$ and $A_y = -x$ , which means that we have

$$A = -xdy, \qquad dA = -dx \wedge dy = F. \tag{3.60}$$

Landau problem; $0 \leq z < 2\pi L$ . Charge quantisation $e \cdot L \in \mathbb{Z}$ .

### 3.2.2 Killing Metric

Recall the Maurer–Cartan one-form $\rho = g^{-1}dg = \sigma^\alpha \otimes T_\alpha$ . Define a metric to be

$$h := -\operatorname{Tr}\big(g^{-1}dg \odot g^{-1}dg\big) \tag{3.61}$$
$$= -\operatorname{Tr}\big(T_\alpha \cdot T_\beta\big)\sigma^\alpha \odot \sigma^\beta \tag{3.62}$$
$$= h_{\alpha\beta}\sigma^\alpha \odot \sigma^\beta \tag{3.63}$$
$$= -\operatorname{Tr}\big(dg \cdot g^{-1} \odot dgg^{-1}\big). \tag{3.64}$$

This is both left-invariant and right-invariant. We say that the metric is *bi-invariant*.

**Example 3.2.3** ($G = SU(2)$)**:** As a manifold, $SU(2)$ is the three-dimensional sphere $S^3$. The Killing metric will be the round metric on $S^3$. Its isometry group is $SO(4)$. It fits into $SO(3) \rtimes SO(3)$; one of these generates the left-invariant vector fields and the other the right-invariant ones.

# 4 Hamiltonian Mechanics and Symplectic Geometry

Let $M$ be a $2n$-dimensional manifold, which we refer to as the *phase space*. It does not come equipped with a metric, but there is another structure on it.

**Definition 27** (Poisson bracket)**:** If $f, g : M \to \mathbb{R}$ are functions on the phase space, then their *Poisson bracket* is

$$\left\{f, g\right\}_{\mathrm{PB}} = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a}, \tag{4.1}$$

where $(q^a, p_a)$, with $a = 1, \dots, n$ is a local coordinate system on $M$.

**Definition 28** (Hamiltonian)**:** The *Hamiltonian* is a function $H : M \to \mathbb{R}$ such that Hamilton's equations hold:

$$\dot{p}_a = -\frac{\partial H}{\partial q^a}, \qquad \dot{q}^a = \frac{\partial H}{\partial p_a}. \tag{4.2}$$

**Definition 29** (Hamiltonian vector field)**:** The *Hamiltonian vector field* is

$$X_H = \frac{\partial H}{\partial p_a}\frac{\partial}{\partial q^a} - \frac{\partial H}{\partial q^a}\frac{\partial}{\partial p_a}, \tag{4.3}$$

whose integral curves are $t \to (\mathbf{p}(t), \mathbf{q}(t))$.

A more general framework is kicked off by the following definition.

**Definition 30** (Poisson manifold)**:** Let $M$ be phase space and $\omega^{ij} = \omega^{[ij]}$, for $i, j = 1, \dots, \dim M = m$, be a tensor field on $M$. We call $(M, \omega^{ij} = \omega)$ a *Poisson manifold* and $\omega$ a *Poisson structure*, if the Poisson bracket

$$\left\{f, g\right\}_{\mathrm{PB}} = \omega^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} \tag{4.4}$$

is such that the Jacobi identity

$$\left\{f, \left\{g, h\right\}_{\mathrm{PB}}\right\}_{\mathrm{PB}} + \left\{h, \left\{f, g\right\}_{\mathrm{PB}}\right\}_{\mathrm{PB}} + \left\{g, \left\{h, f\right\}_{\mathrm{PB}}\right\}_{\mathrm{PB}} = 0, \tag{4.5}$$

holds for all functions $f, g, h$ on $M$.

Not any anti-symmetric $\omega^{ij}$ satisfies this; the Jacobi identity gives conditions on $\omega^{ij}$.

**Remark:** We do not distinguish between position and momenta $x^i$.

In the following, we will drop the subscript 'PB' to denote the Poisson bracket.

**Example 4.0.1:** Let $M = \mathbb{R}^3$ and $\omega^{ij} = \epsilon^{ijk} x^k$. The Poisson brackets are

$$\{x^1, x^2\} = x^3, \qquad \{x^3, x^1\} = x^2, \qquad \{x^2, x^3\} = x^4. \tag{4.6}$$

We can then define the Casimir

$$f(r) = r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{4.7}$$

This Poisson commutes with the $x^i$, meaning that $\{f(r), x^i\} = 0$. For example, we have

$$\{(x^1)^2 + (x^2)^2 + (x^3)^2, x^1\} = 2x^2 \overbrace{\{x^2, x^1\}}^{-x^3} + 2x^3 \overbrace{\{x^3, x^1\}}^{x^2} = 0. \tag{4.8}$$

Take the Hamiltonian to be

$$H = \frac{1}{2}\left(\frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + \frac{(x^3)^2}{a_3}\right). \tag{4.9}$$

Then the time-evolution is given by

$$\dot{g} = \{g, H\}, \qquad \underbrace{\dot{x}^i = \omega^{ij}\frac{\partial H}{\partial x^j}}_{\text{Hamilton's equations}}. \tag{4.10}$$

Writing Hamilton's equations out explicitly gives Euler's equations for a rigid body

$$\dot{x}^1 = \frac{a_3 - a_2}{a_2 a_3} x^2 x^3, \qquad \dot{x}^2 = \frac{a_1 - a_3}{a_1 a_3} x^1 x^3, \qquad \dot{x}^3 = \frac{a_2 - a_1}{a_2 a_1} x^2 x^1. \tag{4.11}$$

**Example 4.0.2:** Let us restrict the Poisson structure from the first example to $S^2 \subset \mathbb{R}^3$.

$$x^1 = \sin\theta\cos\phi, \qquad x^2 = \sin\theta\sin\phi, \qquad x^3 = \cos theta. \tag{4.12}$$

So $\theta, \phi$ are functions of $\mathbb{R}^3$ . Get

$$\{\theta, \phi\} = \frac{1}{\sin\theta} \tag{4.13}$$

(Exercise) and a Poisson structure on $S^2$, which is non-degeneate,

$$(\omega^{-1})_{ij} dx^i \wedge dx^j = \underbrace{\sin\theta d\theta \wedge d\phi}_{\text{symplectic structure}}, \tag{4.14}$$

where $x^1 = (\theta, \phi)$.

Last time, we discussd Poisson structures, which we can now specialise.

**Definition 31** (symplectic manifold)**:** A *symplectic manifold* is a smooth manifold $M$ of dimension $2n$ with a closed 2-form $\omega \in \Lambda^2(\mathcal{M})$ , which is non-degenerate, meaning that

$$\underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n} \neq 0, \tag{4.15}$$

or $\omega$ is a $2 \times 2$ matrix of maximal rank.

The symplectic form $\omega$ provides an isomorphism between $TM$ and $T^*M$ as

$$v \in TM \longmapsto v \rightharpoonup \omega \in T^*M. \tag{4.16}$$

However, we tave to be more careful as this is now antisymmetric. If $f : M \to \mathbb{R}$, then $df$ is a 1-form and is naturally associated to a Hamiltonian vector field $X_f$,

$$X_f \rightharpoonup \omega = -df, \tag{4.17}$$

where $f$ is the Hamiltonian.

**Claim 8:** We can define a Poisson bracket by

$$\{f,g\}_{\text{PB}} := X_g(f) = \omega(X_g, X_f) \tag{4.18}$$

**Remark:** Note that this is antisymmetric since $\omega(X_g, X_f) = -\omega(X_f, X_g)$.

In local coordinates, the Poisson bracket is

$$\{f,g\} = \sum_{i,j=1}^{2n} \omega^{ij} \frac{\partial f}{\partial x^j} \frac{\partial g}{\partial x^i}. \tag{4.19}$$

**Exercise 4.1:** The Jacobi identity follows from the closure $d\omega = 0$.

If we compute the Lie bracket of two vector fields $X_f, X_g$, we find the *anti-homomorphism*

$$[X_f, X_g] = -X_{\{f,g\}}. \tag{4.20}$$

Hamiltonian vector fields preserve the symplectic form. The finite way of saying this is that they generate a flow under which the symplectic form is invariant. Infinitesimally this is expressed with the Lie derivative as

$$\mathcal{L}_{X_f}\omega = d(X_f \rightharpoonup \omega) + X_f \rightharpoonup \cancel{d\omega} = -d(df) = 0. \tag{4.21}$$

**Theorem 3** (Darboux)**:** Let $(M, \omega)$ be a $2n$ -dimensional symplectic manifold. There exist *local* coordinates $x^1 = q^1, \dots, x^n = q^n, x^{n+1} = p_1, \dots, x^{2n} = p_n$ around any point in $M$, such that

$$\omega = \sum_{a=1}^{n} dp_a \wedge dq^a, \tag{4.22}$$

and the Poisson bracket takes the standard form.

*Proof.* The proof proceeds by induction with respect to half of the dimension of the symplectic manifold. We first choose an arbitrary function $p_1 : M \to \mathbb{R}$ (which will later become the first momentum coordinate). Given this function, we search for another function $q^1 : M \to \mathbb{R}$ such that

$$X_{p_1}(q^1) = 1. \tag{4.23}$$

We denote this as $\dot{q}^1 = 1$. This is an ordinary differential equation, which will have a solution[1] subject to some initial condition.

The second step is as follows. Consider the surface $M_1 = \{x \in M, p_1 = \text{const.}, q^1 = \text{const.}\}$. This is a submanifold $M_1 \subset M$, which we can prove from the embedding theorem; the maximal rank condition of the Jacobian is encoded in (4.23). This $M_1$ is locally symplectic with symplectic form $\omega_1 \equiv \omega|_{p_1, q^1 \text{ const.}}$

Now look for $p_2, q^2$ and so on. A full proof is given in the book by Arnold. □

**Claim 9:** Let $Q$, which will play the role of configuration space, be an $n$-dimensional manifold. The cotangent bundle $T^*Q$ admits a global symplectic structure.

*Proof.* Our goal is to describe what this symplectic structure is. We have a projection $\pi : T^*Q \to Q$ acting as $\pi(q, p) = q$.

**Definition 32** (pull-back)**:** Given a map $f : M \to N$, the *pull-back* $f^* : T^*_{f(p)}N \to T^*_p M$ defined as

$$f^*(p)(V) = p(f_* V). \tag{4.24}$$

In local coordinates, taking $x^i$, with $i = 1, \dots, \dim M$ coordinates on $M$ and $y^a$ with $a = 1, \dots, \dim N$ coordinates on $N$, we can use the chain rule to write explicitly

$$f^*(dy^a) = \sum_i \frac{\partial f^a}{\partial x^i} dx^i. \tag{4.25}$$

The pull-back of $\pi$ is a map

$$\pi^* : T^*(Q) \to T^*(T^*Q). \tag{4.26}$$

[1]The assumption for the existence theorem would be that we work in a Lipshitz-class of functions.

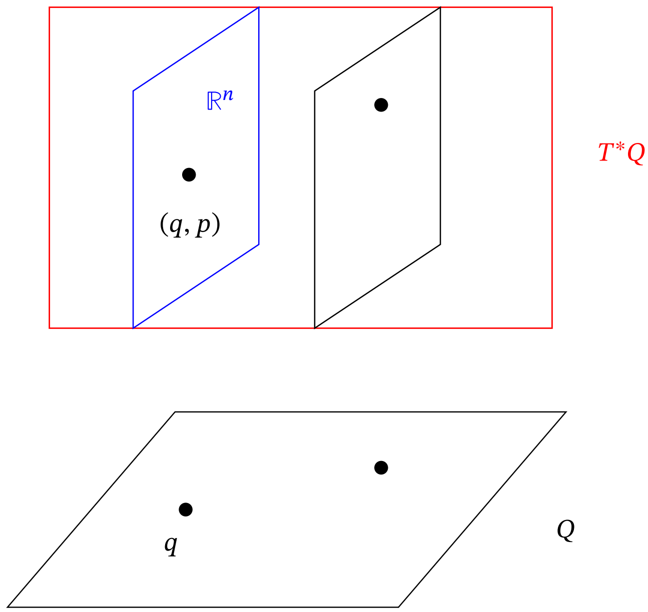


Figure 4.1

Say that $p \in T^*(Q)$ is a one-form on $Q$. We can define $\theta = \pi^*(p)$ and from this we get our canonical symplectic form

$$\omega = d\theta. \tag{4.27}$$

This is manifestly closed. This construction does not depend on coordinates, but usually in $(p_{\dot{a}}, q^{\dot{a}})$ coordinates we have

$$\theta = p_{\dot{a}} dq^{\dot{a}}, \qquad \omega = d\theta = dp_a \wedge dq^a. \tag{4.28}$$

□

We can define yet another structure that symplectic geometry gives us.

**Definition 33** (Canonical transformations)**:** Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold. An endomorphism $f : M \to M$ is called *canonical* when $f^*(\omega) = \omega$.

The one-parameter groups of canonical transformations are generated by Hamiltonian vector fields. For dimension $n = 1$, these are area preserving maps.

Consider the canonical transformation $(p_a, q^a) \xrightarrow{f} (P_a, Q^a)$ .

$$d(\mathbf{p} \cdot d\mathbf{q}) = -d(\mathbf{Q} \cdot d\mathbf{P}), \tag{4.29}$$

where $\mathbf{P} = P(p, q)$ and $Q = Q(p, q)$ . Then

$$d(\mathbf{p} \cdot d\mathbf{q} + \mathbf{Q} d\mathbf{P}) = 0. \tag{4.30}$$

So the one-form $\mathbf{p} \cdot d\mathbf{q} + \mathbf{Q} \cdot d\mathbf{P}$ is a closed-one form. Locally, this implies that it is exact, meaning that there is some *generating function* $S = S(\mathbf{q}, \mathbf{P})$ such that

$$\mathbf{p} \cdot d\mathbf{q} + \mathbf{Q} d\mathbf{P} = dS. \tag{4.31}$$

From this function we can define $Q$ and $p$ as

$$Q^a = \frac{\partial S}{\partial p_a}, \qquad p_a = \frac{\partial S}{\partial q^a}. \tag{4.32}$$

This gives $\mathbf{P}(q, p)$ and $Q(q, p)$.

## 4.1 Geodesics, Killing vectors, Killing tensors

We will explore the connection between Riemannian and symplectic geometry. Let $(M, g)$ be a (pseudo-)Riemannian manifold of dimension $n$. In coordinates, we have

$$g = g_{ij}(x)dx^i dx^j. \tag{4.33}$$

Then we know from the *General Relativity* course, that there exists at unique Levi-Civita connection $\Gamma^i_{jk}$ such that for geodesics $x^i = x^i(\tau)$, we have

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^i\dot{x}^j = 0. \tag{4.34}$$

Geodesics on $M$ are integral curves of a Hamiltonian vector field on $(T^*M, \omega)$.

As illustrated in Fig. 4.2, we are looking for a curve in $(T^*M, \omega)$ specified by a single point $(x^i, p_i)$ which projects down to the geodesic in $M$.

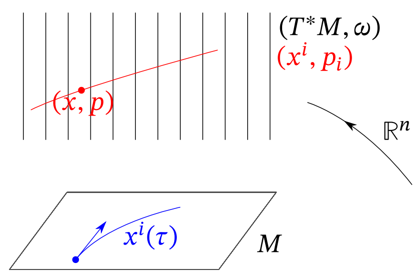


Figure 4.2

$$\dot{x}^i = p^i = g^{ij}p_j = X_H(x^i) \tag{4.35}$$
$$\dot{p}^i = -\Gamma^i_{jk}p^i p^k = X_H(p^i) \tag{4.36}$$

The Hamiltonian vector field is

$$X_H = g^{ij}p_i\frac{\partial}{\partial x^j} - \Gamma^i_{jk}p^j p^k\frac{\partial}{\partial p^i}, \tag{4.37}$$

where $H = \frac{1}{2}g^{ij}(x)p_i p_j$.

Canonical symplectic form $\omega = dp_i \wedge dx^i$.

$$H \to X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i} \tag{4.38}$$
$$= g^{ij}p_j\frac{\partial}{\partial x^i} - \frac{1}{2}\frac{\partial g^{jk}}{\partial x^i}p_j p_k\frac{\partial}{\partial p_i}. \tag{4.39}$$

Now use

$$\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^m_{ki} g_{jm} - \Gamma^m_{kj} g_{im} = 0 \tag{4.40}$$

**Definition 34** (Killing vector)**:** *Killing vectors $K$* satisfy

$$\mathcal{L}_K g = 0 \iff \nabla_{(i} K_{j)} = 0. \tag{4.41}$$

These correspond to first integrals of the Hamiltonian flow which are linear in the momenta. Poisson commuting with $H$.

$$\{\underbrace{\kappa^i p_i}_{\kappa}, H\}_{\mathrm{PB}} = \frac{\partial \kappa^i}{\partial x^j} p_i g^{jk} p_k - \frac{1}{2}\kappa^i \frac{\partial g_{jk}}{\partial x^i} p^j p^k \tag{4.42}$$

$$\overset{(4.40)}{=} \nabla_{(i} K_{j)} p^i p^j = 0 \tag{4.43}$$

Killing vectors are symmetries of the Hamiltonian flow.

### 4.1.1 Killing Tensors

**Definition 35** (Killing tensor)**:** Write

$$\kappa = K^{ij\dots k} \underbrace{p_i p_j \dots p_k}_{r}. \tag{4.44}$$

We assume nothing about $\kappa$ except that it Poisson-commutes with the Hamiltonian $H$. We find that this corresponds to

$$\{H, \kappa\}_{\mathrm{PB}} = 0 \iff \nabla_{(i} \kappa_{jk\dots l)} = 0. \tag{4.45}$$

A $\kappa$ satisfying this is called a rank-$r$ *Killing tensor*.

We will see that we can think of Killing tensors as higher / hidden symmetries, since we can see them in the cotangent bundle but not in the manifold itself.

The associated Hamiltonian vector field is

$$X_\kappa = \frac{\partial \kappa}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial \kappa}{\partial x^i}\frac{\partial}{\partial p_i} \tag{4.46}$$

$$= r K^{i_1\dots i_r} p_{i_1} \dots p_{i_{r-1}} \frac{\partial}{\partial x^{i_r}} - \frac{\partial K^{i_1\dots i_r}}{\partial x^k} p_{i_1} \dots p_{i_r} \frac{\partial}{\partial p_k}. \tag{4.47}$$

Projecting this down onto the manifold gives

$$\pi_k(X_\kappa) = \begin{cases} 0, & \text{if } r > 1 \\ K^i \frac{\partial}{\partial x^i}, & \text{if } r = 1. \end{cases} \tag{4.48}$$

If $r > 1$, there is no 'geometric' symmetry on $M$, but $\kappa$ is constant along geodesic.

**Example 4.1.1:** The Kerr black hole does not have enough Killing vectors to solve the geodesic equations, but using the Killing tensors allows us to find the geodesics.

## 4.2 Integrability

Intuitively, a Hamiltonian system (e.g. geodesic motion) is *integrable* if there exist sufficiently many first integrals, i.e. functions constant along the flow of the Hamiltonian vector field $X_H$.

**Definition 36** (integrable system)**:** An *integrable system* is a symplectic manifold $(M, \omega)$ of dimension $\dim M = 2n$, together with $n$ functions $f_i : M \to \mathbb{R}$, $i = 1, \dots, n$, with the properties of

**involution:** $\{f_i, f_j\} = 0$ for all $i, j$,

**independence:** $df_i \wedge df_j \wedge \cdots \wedge df_n \neq 0$.

This is a strange definition because it does not specify any dynamics, any equations of motion.

The point is that with such a system any $f_i$ in our system can be declared to be a Hamiltonian. The corresponding Hamilton's equations will be solvable!

**Theorem 4** (Arnold–Liouville)**:** Let $(M, \omega, f_i)$ form an integrable system with the Hamiltonian chosen to be $H = f_1$. Then

- The level set $M_f = \{x \in M \mid f_1 = c_1, \dots, f_n = c_n\}$ (if connected[1]) is diffeomorphic to $\mathbb{R}^k \times T^{n-k}$ for some $0 \leq k \leq n$.[2]
- There exists a canonical transformation to action-angle vanishes

$$\phi_1, \dots, \phi_n, I_1, \dots, I_n \tag{4.49}$$

  such that in a neighbourhood of $M_f$ in $M$, $\phi_i$ are the coordinates, called *angles*, on $M_f$, and $I_i$ are first integrals, called the *actions*.
- Hamilton's equations are solvable by quadratures

$$\dot{I}_i = \frac{\partial H}{\partial \phi_i} = 0 \qquad \dot{\phi}_i = \frac{\partial H}{\partial I_i} = \Omega_i(I_1, \dots, I_n), \tag{4.50}$$

  so $I_i(t) = I_i(0)$ and $\phi_i(t) = \phi_i(0) + \Omega_i t$.

[1] If $M_f$ is not connected, the theorem applies to each connected component.
[2] Usually this theorem is stated with $M_f$ *compact*. In that case, we have $k = 0$ and we just have a torus $T^n$.

*Proof.* The $df_k$ for $k = 1, \dots, n$ are independent, so $M_f$ is a manifold of dimension $n$. Any function $f_k$ gives rise to a Hamiltonian vector field $X_{f_k}$. If we contract this with any differential, we have

$$X_{f_k} \rightharpoonup df_j = X_{f_k}(f_j) = -\{f_k, f_j\}_{\mathrm{PB}} = 0, \qquad \forall j, k. \tag{4.51}$$

If $df_k$ are normal to $M_f$, then the corresponding Hamiltonian vector fields $X_{f_j}$ are tangent to $M_f$. □