Applications of Differential Geometry to Physics

Part III Lent 2020

Lectures by Maciej Dunajski

Report typos to: uco21@cam.ac.uk
More notes at: uco21.user.srcf.net

March 11, 2020

Contents

1 Differential Geometry

1.1 Introduction: Kepler / Newton Orbits

The importance of the role of geometry in physics in undeniable. For example, consider Kepler orbits $\mathbf{r}(t) \in \mathbb{R}^3$ obeying

$$\ddot{\mathbf{r}} = -\frac{GMv}{r^3}\mathbf{r}. \tag{1.1}$$

The solutions are conic sections, such as the one illustrated in Fig. 1.1.

A general conic section can be written as the set of solutions of the equation

$$ax^2 + by^2 + cxy + dx + ey + f = 0, \tag{1.2}$$

where $a, b, c, d, e, f \in \mathbb{R}$. Apollonius of Perga asked 'what is the unique conic thorugh five points, no three of which are co-linear?' Since multiplying all coefficients by a real constant $y \neq 0$ gives the same conic, the space of conics is the projective 5-space $\mathbb{RP}^5 = (\mathbb{R}^6 \setminus \{0\})/\sim$, where $x \sim y$ iff $\exists \gamma \in \mathbb{R} \setminus \{0\}$ s.t. $x = \gamma y$. In particular, we have homogeneous coordinates

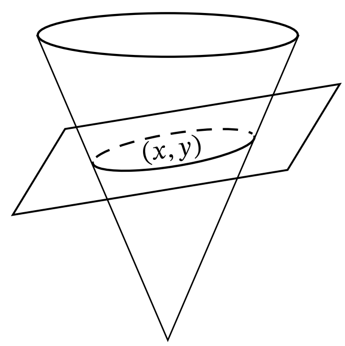

Figure 1.1: Conic section

$$[a, b, c, d, e, f] \sim [\gamma a, \gamma b, \gamma c, \gamma d, \gamma c, \gamma f], \gamma \in \mathbb{R}^*, \tag{1.3}$$

where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. This is an application of geometry, rather than an application of differential geometry.

Remark: Apolonius proved this geometrically.

Equations like (1.2) are studied in what we now call *algebraic geometry*. In this course however, we will be interested in the physical applications of *differential geometry* and will look at the following:

1) Hamiltonian mechanics (mid 19^{th}). This is an elegant way of reformulating Newton's mechanics, turning second order differential equations into first order differential equations with

the use of a function $H(p,q)$. The system of ODEs is

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{q} = -\frac{\partial H}{\partial q} \tag{1.4}$$

This led to the development of symplectic geometry (1960s). The connection is that the phase-space to which p and q belong has a 2 -form $dp \wedge dq$. Using the Hamiltonian function, one can find a vector field

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} \tag{1.5}$$

and looks for a one-parameter group of transformations, called symplectomorphisms, generated by this vector field. Under these symplectomorphisms, the 2 -form is unchanged meaning that the area illustrated in F2 is preserved. Details of this are going to come within the course.

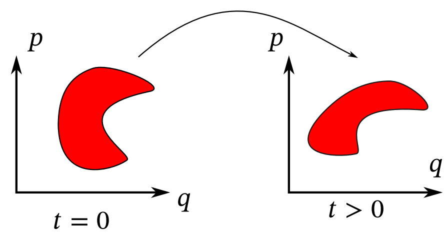

Figure 1.2

2) General Relativity (1915) ← Riemannian Geometry (1850)

3) Gauge theory (Maxwell, Yang Mills) ↔ Connection on Principal Bundle (U(1) (Maxwell), SU(2), SU(3))

$$A_+ = A_- + dg \qquad g = \psi_+ - \psi_- \qquad \omega = \begin{cases} A_+ + d\psi_+ \\ A_- + d\psi_- \end{cases} \tag{1.6}$$

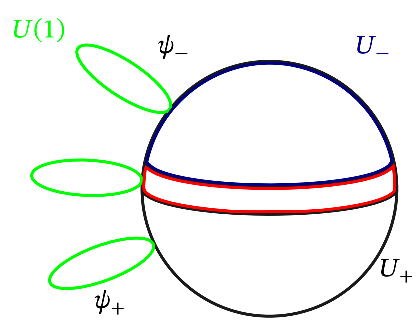

Figure 1.3

This course: cover 1, 2, 3 in some detail. Unifying feature: Lie groups.

- Prove some theorems, *lots of* examples (often instead of proofs)
- Want to be able to do calculations; compute characteristic classes etc.

We will assume that you took either Part III General Relativity, or Part III Differential Geometry, or some equivalent course.

1.2 Manifolds

Definition 1 (manifold)**:** An n -dimensional *smooth manifold* is a set M and a collection[1] of open sets U_α , labelled by $\alpha = 1, 2, 3, \ldots$, called *charts* such that

- U_α cover M
- $\exists$ 1-1 maps $\phi_\alpha : U_\alpha \to V_\alpha \in \mathbb{R}^n$ such that

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{1.7}$$

  is a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

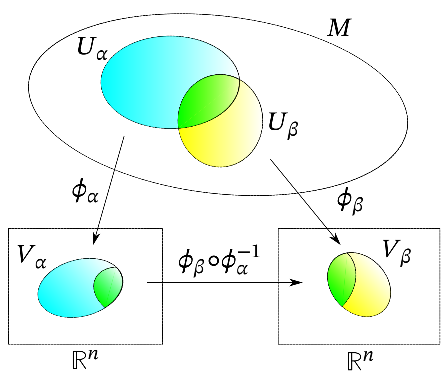

Figure 1.4: Manifold

As such, manifolds are topological spaces with additional structure, allowing us to do calculus.

Example 1.2.1 ($M = \mathbb{R}^n$)**:** There is the *trivial manifold*, which can be covered by only one open set. There are other possibilities. In fact, there are infinitely many smooth structures on $\mathbb{R}^4$ (Proof by Donaldson in 1984 in his PhD. He used Gauge theory).

[1] In all examples that we will look at, there will be finitely α.

Example 1.2.2 (sphere $S^n = \{\mathbf{r} \in \mathbb{R}^{n+1}, |\mathbf{r}| = 1\}$)**:** Intuitively, the n-sphere S^n is an n-dimensional manifold. To show this, we will construct a map $\phi : S^n \to \mathbb{R}^n$ by projecting the north pole $N : (0, 0 \ldots, 0, 1) \in \mathbb{R}^{n+1}$, through the point $p \in S^n$ onto the hyperplane defined by $r_{n+1} = 0$. This is illustrated for S^2 in Fig. 1.5. From this figure, we can already see that this projection map is ill-

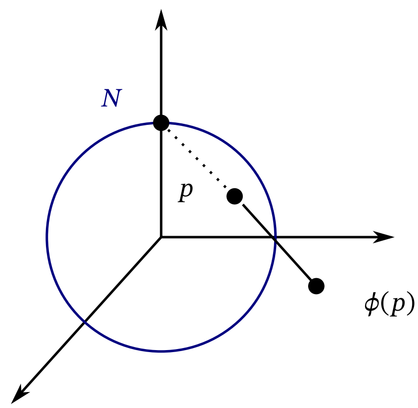

Figure 1.5: The map ϕ projects the north pole N through a point $p \in S^2$ onto $\phi(p)$, which lies in the R^2-plane defined by $z = 0$. This map is not defined for the north-pole itself.

defined at the north pole itself. However, the definition of a manifold allows us to work with multiple charts. Let us cover S^n with the following two open sets

$$U = S^n / \{0, 0, 0, , \ldots, 0, 1\}, \qquad \widetilde{U} = S^n / \{0, 0, 0, , \ldots, 0, -1\}, \tag{1.8}$$

so U does not include the north pole and $\widetilde{U}$ does not include the south pole. We can then define two maps, ϕ and $\widetilde{\phi}$, which project from the north pole and the south pole respectively. Using x_i to denote coordinates in $\mathbb{R}^n$ and r_i to denote coordinates in $\mathbb{R}^{n+1}$ we take maps

$$\text{defined on } U: \qquad \phi(r_1, \ldots, r_{n+1}) = \left(\frac{r_1}{1 - r_{n+1}}, \ldots, \frac{r_n}{1 - r_{n+1}}\right) = (x_1, \ldots, x_n) \tag{1.9}$$

$$\text{defined on } \widetilde{U}: \qquad \widetilde{\phi}(r_1, \ldots, r_{n+1}) = \left(\frac{r_1}{1 + r_{n+1}}, \ldots, \frac{r_n}{1 + r_{n+1}}\right) = (\widetilde{x}_1, \ldots, \widetilde{x}_n). \tag{1.10}$$

On the overlap $U \cap \widetilde{U}$, the coordinates are related

$$\overbrace{\frac{r_k}{1 + r_{n+1}}}^{\widetilde{x}_k} = \frac{1 - r_{n+1}}{1 + r_{n+1}} \overbrace{\frac{r_k}{1 - r_{n+1}}}^{x_k}, \qquad k = 1, \ldots, n, \tag{1.11}$$

where we can write the transition factor on the right in terms of the coordinates x_i:

$$\frac{1 - r_{n+1}}{1 + r_{n+1}} = \frac{(1 - r_{n+1})^2}{r_1^2 + r_2^2 + \cdots + r_n^2} = \frac{1}{x_1^2 + x_2^2 + \cdots + x_n^2}. \tag{1.12}$$

So on $U \cap \widetilde{U}$, the transition function is

$$\widetilde{\phi} \circ \phi^{-1} : (x_1, \ldots, x_n) \longmapsto (\widetilde{x}_1, \ldots, \widetilde{x}_n) = \left(\frac{x_1}{x_1^2 + \cdots + x_n^2}, \ldots, \frac{x_n}{x_1^2 + \cdots + x_n^2}\right), \tag{1.13}$$

which is a smooth map from $\mathbb{R}^n \to \mathbb{R}^n$.

Example 1.2.3: A Cartesian product of manifolds is a manifold, for example we have the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$.

Definition 2 (surface)**:** Let $f_1, \dots, f_k : \mathbb{R}^N \to \mathbb{R}$ be smooth functions. A surface $f_1 = 0, \dots, f_k = 0$ is a manifold of dimension dim $n = N - k$ if the rank of the matrix $\frac{\partial f_\alpha}{\partial x^i}$, $\alpha = 1, \dots, k$ and $i = 1, \dots, N$ is maximal and equal to k at all points of $\mathbb{R}^N$..

Example 1.2.4: The n-sphere S^n is a surface in $\mathbb{R}^{n+1}$ with $f_1 = 1 - |\mathbf{r}|^2$.

Theorem 1 (Whitney)**:** Every smooth manifold of dimension n is an embedded surface in $\mathbb{R}^N$, where $N \leq 2n$.

> If you enjoy using geometrical intuition and looking at surfaces, this theorem ensures that you can always do that and not loose generality.

Definition 3 (real projective space)**:** The n-dimensional *real projective space* is defined as the quotient

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim, \tag{1.14}$$

where we quotient out the equivalence classes $[X_1, \dots, X_n + 1] \sim [cX_1, \dots, cX_{n+1}]$ for all $c \in \mathbb{R}^*$. The $[X_1, \dots, X_{n+1}]$ are called *homogeneous coordinates*.

In other words, this is the space of all lines through the origin in $\mathbb{R}^{n+1}$.

Claim 1: $\mathbb{RP}^n$ is a smooth manifold of dimension n with $(n + 1)$ open sets.

Proof. Let us define our open sets with respect to the homogeneous coordinates. We define the set $U_\alpha : [X] \in \mathbb{RP}^n$ such that $X_\alpha \neq 0$ $\alpha = 1, \dots, n + 1$. We can now find local coordinates on $\phi_\alpha : U_\alpha \to V_\alpha \in \mathbb{R}^n$

$$x_1 = \frac{X_1}{X_\alpha} \quad \dots \quad x_{\alpha-1} = \frac{X_{\alpha-1}}{X_\alpha} \quad x_{\alpha+1} = \frac{X_{\alpha+1}}{X_\alpha} \quad \dots \quad x_n = \frac{X_n}{X_\alpha}. \tag{1.15}$$

□

Exercise 1.1: Prove smoothness of $\phi_\beta \circ \phi_\alpha^{-1}$.

Now it turns out that this manifold is equivalent to $\mathbb{RP}^n = S^n/\mathbb{Z}^2$. From quantum mechanics, we know that this means in particular $\mathbb{RP}^3 = SO(3)$. This is illustrated in 1.6.

1.3 Vector Fields

Let $M, \widetilde{M}$ be smooth manifolds of dimension $n, \tilde{n}$.

Definition 4 (smooth map)**:** A map $f : M \to \widetilde{M}$ is *smooth* if $\widetilde{\phi}_\beta \circ f \circ \phi_\alpha^{-1}$ is a smooth map from $\mathbb{R}^n$ to $\widetilde{\mathbb{R}}^n$ for all α, β. We call $f : M \to \mathbb{R}$ a *function*, whereas we call $f : \mathbb{R} \to M$ a *curve*.

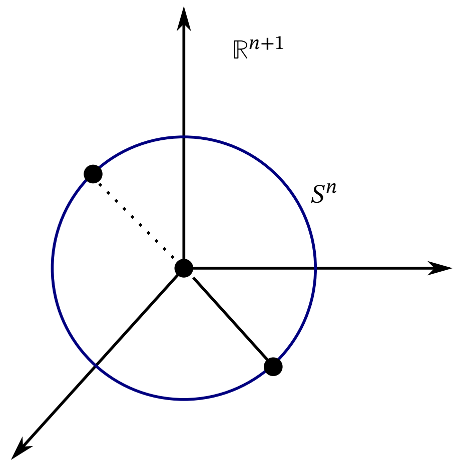

Figure 1.6: Real projective space $\mathbb{RP}^n$ is isomorphic to $S^n/\mathbb{Z}^n$, identifying antipodal points.

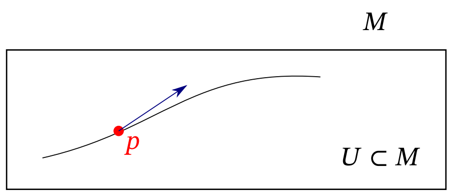

Figure 1.7

Let $\gamma : \mathbb{R} \to M$ be a curve. For some $U \in M, U \simeq \mathbb{R}^n$, we can define local coordinates $(x^1, \dots, x^n)$.

Definition 5 (tangent vector)**:** A *tangent vector* V to γ at p is

$$V|_p = \left.\frac{\mathrm{d}\psi}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \in T_pM, \tag{1.16}$$

where T_pM is the *tangent space* to M at p.

Definition 6 (tangent bundle)**:** We define the *tangent bundle* as $TM := \bigcup_{p\in M} T_pM$.

Definition 7 (vector field)**:** A *vector field* assigns a tangent vector to all $p \in M$.

Let $f : M \to \mathbb{R}$. The rate of change of f along γ is

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(x^a(\epsilon))|_{\epsilon=0} = \sum_a \dot{x}^a \frac{\partial f}{\partial x^a} \tag{1.17}$$

$$= \sum_a V^a \left.\frac{\partial f}{\partial x^a}\right|_{\epsilon=0}, \tag{1.18}$$

where $V^a := \dot{x}^a|_{\epsilon=0,\dots,x_n}$.

Vector fields are first order differential operators

$$V = \sum_a V^a(\mathbf{x}) \frac{\partial}{\partial x^a}. \tag{1.19}$$

The derivatives $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}\Big|_p$ form a basis of T_pM.

1.3.1 Integral Curves

Definition 8 (integral curve)**:** An *integral curve* (a *flow*) of a vector field is defined by

$$\dot{\gamma}(\epsilon) = V|_{\gamma(\epsilon)}, \tag{1.20}$$

where the dot denotes differentiation with respect to ϵ.

On n first order ODEs: $\dot{x}^a = V^a(x)$.

There exists a unique solution given initial data $X^a(0)$. Given a solution $X^a(\epsilon)$, we can expand it in a Taylor series as

$$X^a(\epsilon) = X^a(0) + V^a \cdot \epsilon + O(\epsilon^2). \tag{1.21}$$

Up to first order in ϵ, the vector field determines the flow. We call V a *generator* of its flow.

The following example illustrates how you get from a vector field to its flow.

Example 1.3.1 ($M = \mathbb{R}^2, x^a = (x, y)$)**:** Consider the vector field $V = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. The system of ODEs we solve is $\dot{x} = x$ and $\dot{y} = 1$. This gives us the integral curve $(x(\epsilon), y(\epsilon)) = (x(0)e^{\epsilon}, y(0) + \epsilon)$. From this we can see that $x(\epsilon)\cdot\exp(-y(\epsilon))$ is constant along γ. Using this we can draw the unparametrised integral curve in Fig. 1.8.

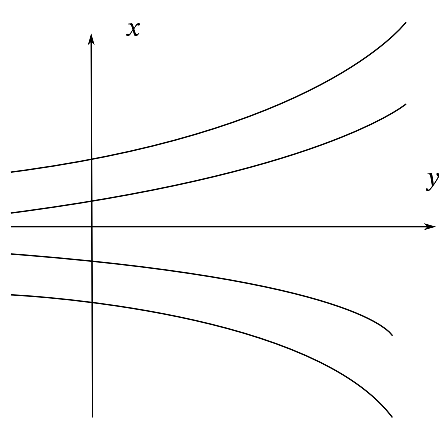

Figure 1.8

This example motivates the following definition.

Definition 9 (invariant)**:** An *invariant* of a vector field V is a function f constant along the flow of V.

$$f(x^a(0)) = f(x^a(\epsilon)) \qquad \forall \epsilon. \tag{1.22}$$

Equivalently, $V(f) = 0$.

Let us now consider an example that goes the other way: from flow to vector field.

Example 1.3.2: Consider the 1 -parameter group of rotations of a plane.

$$(x(\epsilon), y(\epsilon)) = (x_0 \cos\epsilon - y_0 \sin\epsilon, x_0 \sin\epsilon + y_0 \cos\epsilon). \tag{1.23}$$

The associated vector field is

$$V = \left(\frac{\partial y(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial y} + \frac{\partial x(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial x} \right)\bigg|_{\epsilon=0} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial y}. \tag{1.24}$$

Now you can add vector fields, but there is also another operation.

Definition 10 (Lie bracket)**:** A *Lie bracket* $[V, W]$ of two vector fields V, W is a vector field defined by

$$[V, W](f) = V(W(f)) - W(V(f)) \qquad \forall f. \tag{1.25}$$

This is indeed another vector field since the commutator of two first order operators is another first order operator.

Example 1.3.3: Let $V = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $W = \frac{\partial}{\partial x}$. We then have $[V, W] = -W$.

This is not always the case but sometimes the Lie bracket reproduces some of the vector fields. There is an interesting algebraic structure to this.

1.4 Lie Algebras

Definition 11 (Lie algebra)**:** A *Lie algebra* is a vector space $\mathfrak{g}$ with an anti-symmetric, bilinear operation $[\ ,\ ]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called a *Lie bracket*, which satisfies the *Jacobi identity*

$$[V, [U, W]] + [W, [V, U]] + [U, [W, V]] = 0 \qquad \forall U, V, W \in \mathfrak{g}. \tag{1.26}$$

We will spend some time discussing this abstractly, but then focus on the Lie algebras of vector fields in the main part of this course.

Any two vector spaces of a given dimension are isomorphic; there is nothing special other than the dimension distinguishing vector spaces. For Lie algebras this is not so.

Example 1.4.1: Even in dimension 2, which is the lowest non-trivial dimension, there are two Lie algebras (up to isomorphism)

$$a) \quad [V, W] = -W, \qquad b) \quad [V, W] = 0. \tag{1.27}$$

If the vector space underlying $\mathfrak{g}$ is finite-dimensional, and V_α, $\alpha = 1, \dots, \dim \mathfrak{g}$ is a basis of $\mathfrak{g}$, we can define the Lie algebra by specifying the brackets

$$[V_\alpha, V_\beta] = \sum_\gamma f_{\alpha\beta}{}^\gamma V_\gamma, \tag{1.28}$$

where $f_{\alpha\beta}{}^\gamma$ are the *structure constants.*

Example 1.4.2 ($\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$)**:** The vector space is given by $n \times n$ real matrices, and the Lie bracket is the matrix commutator. The dimension of this Lie algebra is $\dim \mathfrak{g} = n^2$.

Example 1.4.3 (Vector fields)**:** The set of all vector fields on a manifold M form an infinite-dimensional Lie algebra.

Example 1.4.4: Consider diff($\mathbb{R}$) or diff(S^1), vector fields on a line or on a circle respectively.

$$\text{diff}(\mathbb{R}), \quad x \in \mathbb{R}, \quad V_\alpha = -x^{\alpha+1} \frac{\partial}{\partial x} \tag{1.29}$$

$$\text{diff}(S^1), \quad \theta \in S^1, \quad V_\alpha = i e^{i\alpha\theta} \frac{\partial}{\partial \theta} \tag{1.30}$$

$$[V_\alpha, V_\beta] = (\alpha - \beta) V_{\alpha+\beta}. \tag{1.31}$$

Example 1.4.5 (Virasoro algebra)**:** The *Virasoro algebra* Vir $=$ diff(S^1) $\oplus\, \mathbb{R}$ is the central extension[1] of diff(S^1), with *central charge* $c = \mathbb{R}$.

$$\begin{cases} [V_\alpha, c] = 0 \\ [V_\alpha, V_\beta]_{\text{vir}} = (\alpha - \beta) V_{\alpha+\beta} + \dfrac{c}{12}(\alpha^3 - \alpha)\delta_{\alpha+\beta,0} \end{cases} \tag{1.32}$$

Remark:

$$[f(\theta)\frac{\partial}{\partial\theta}, g(\theta)\frac{\partial}{\partial\theta}] = \underbrace{(fg' - gf')}_{\text{Wronskian}} \frac{\partial}{\partial\theta} \tag{1.33}$$

'After Witten'.

$$[f\frac{\partial}{\partial\theta}, g\frac{\partial}{\partial\theta}]_{\text{vir}} = [f\frac{\partial}{\partial\theta}, g\frac{\partial}{\partial\theta}] + \frac{ic}{48\pi}\int_0^{2\pi} (f'''g - g'''f)\, \mathrm{d}\theta \tag{1.34}$$

[1] We will meet the concept of central extension and central charge in this term's *String Theory* course.

Theorem 2 (Ado)**:** Every finite-dimensional Lie algebra is isomorphic to some matrix Lie algebra, a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.

> **Remark:** n is not necessarily the dimension of the Lie algebra.

1.5 Lie Groups

Definition 12 (Lie group)**:** A *Lie group* is a smooth manifold G, which is also a group, such that the group operations

$$\text{multiplication} \qquad G \times G \to G, \qquad (g_1, g_2) \longmapsto g_1 \cdot g_2 \tag{1.35}$$

$$\text{inverse} \qquad G \to G, \qquad g \longmapsto g^{-1} \tag{1.36}$$

are smooth maps between manifolds.

Example 1.5.1 ($G = GL(n, \mathbb{R}) \in \mathbb{R}^{n^2}$)**:** The general linear group $GL(n, \mathbb{R})$ is defined as the set of invertible matrices $\{g \in G \mid \det g \neq 0\}$. The dimension is $\dim(G) = n^2$.

Example 1.5.2 ($G = O(n, \mathbb{R})$)**:** This is the group of orthogonal matrices, defined by $\frac{1}{2}n(n+1)$ conditions $g^T g = \mathbb{1}$. The dimension is then $\dim O(n, \mathbb{R}) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$. We also have to check that these conditions define a manifold in the sense that the associated Jacobian has maximal rank.

Definition 13 (group action)**:** A *group action* on a manifold M is a map $G \times M \to M$ mapping $(g, p) \to g(p)$ such that

$$e(p) = p, \qquad g_1(g_2(p)) = (g_1 \cdot g_2)(p) \tag{1.37}$$

for all $p \in M$ and al g_1, g_2 on G.

Definition 14 (transformation group)**:** If we have a group action, we refer to G as a group of *transformations*.

Example 1.5.3: Take $M = \mathbb{R}^2$ and $G = E(2)$, the three-dimensional Euclidean group.

$$g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \tag{1.38}$$

Take $g_\epsilon \in G$ to be a one-parameter subgroup of G. There are three such subgroups, with $\epsilon \in \{\theta, a, b\}$

$$g_\theta : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos\theta \cdot x - \sin\theta \cdot y \\ \sin\theta \cdot x + \cos\theta \cdot y \end{pmatrix} \tag{1.39}$$

$$g_a : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x + a \\ y \end{pmatrix} \tag{1.40}$$

$$g_b : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x \\ y + b \end{pmatrix}. \tag{1.41}$$

Each of these one-parameter subgroups generates a flow. We can think of this flow as being gener-

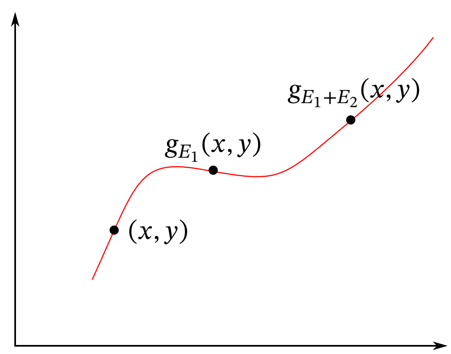

Figure 1.9

ated by a vector field $V|_p = \frac{\mathrm{d}}{\mathrm{d}\epsilon} g_\epsilon(p)\Big|_{\epsilon=0}$ according to the definition (1.16) of a tangent vector.

$$V_\theta = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{1.42}$$

$$V_a = \left(\frac{\mathrm{d}\tilde{x}}{\mathrm{d}a}\frac{\partial}{\partial \tilde{x}} + \frac{\mathrm{d}\tilde{y}}{\mathrm{d}a}\frac{\partial}{\partial \tilde{y}}\right)\Big|_{a=0} = \frac{\partial}{\partial x} \tag{1.43}$$

$$V_b = \frac{\partial}{\partial y}. \tag{1.44}$$

We define a 3-dimensional Lie algebra of $E(2)$ as

$$[V_a, V_\theta] = V_b \qquad [V_b, V_\theta] = -V_a \qquad [V_a, V_b] = 0, \tag{1.45}$$

represented by vector fields on M.

1.5.1 Geometry on Lie Groups

Definition 15 (tangent map)**:** Let $f : M \to \widetilde{M}$ be a smooth map between manifolds. We define the *tangent map* or *push forward* to be

$$\begin{aligned} f_* : T_p(M) &\to T_{f(p)}(\widetilde{M}) \\ V &\longmapsto f_*(V) = \frac{\mathrm{d}}{\mathrm{d}E} f(\gamma(E))\bigg|_{E=0} . \end{aligned} \tag{1.46}$$

This extends to the tangent bundle $T(M)$. If x^α are coordinates of $\mathcal{M} \supset M$, $(y^{\alpha'})$ coordinates on $\widetilde{\mathcal{M}} \subset M$, then

$$V = V^\alpha \frac{\partial}{\partial x^a} \qquad f_*(V) = V^\alpha \frac{\partial f^i}{\partial x^a} \frac{\partial}{\partial y^i}. \tag{1.47}$$

Definition 16 (Lie derivative)**:** Let V, W be vector fields, where V generates a flow $V = \dot{\gamma}$. The *Lie derivative* is

$$\mathcal{L}_V W|_p := \lim_{\epsilon \to 0} \frac{W(p) - \gamma(\epsilon)_* W(p)}{\epsilon} \tag{1.48}$$

We can extend this definition over the whole manifold.

Exercise 1.2: Show that $L_V W = [V, W]$.

Definition 17: On functions $f : M \to \mathbb{R}$, we define the Lie derivative as $\mathcal{L}_V(f) = V(f)$.

Claim 2 (Cartan's Magic Formula)**:** On differential forms, we can use the Leibniz rule to show that

$$\mathcal{L}_V \Omega = d(\iota_V \Omega) + \iota_V(d\Omega) = d(V \rightharpoonup \Omega) + V \rightharpoonup d\Omega. \tag{1.49}$$

Definition 18: We define the cotangent space $T_p^* M = \text{Span}\{dx^1, \dots, dx^n\}$ as the space of one-forms. The cotangent bundle is then

$$\bigcup_{p \in M} T_p^* M = T^* M. \tag{1.50}$$

Definition 19 (r-form)**:** Using the wedge product, which is anti-commutative on one-forms $dx^i \wedge dx^j = -dx^j \wedge dx^i$, we can define an r-form

$$\Omega = \frac{1}{r!} \Omega_{ij\dots k} dx^i \wedge dx^j \wedge \cdots \wedge dx^k. \tag{1.51}$$

Definition 20 (contraction)**:** We write a *contraction* as

$$\frac{\partial}{\partial x^i} \rightharpoonup dx^j = \iota_{\frac{\partial}{\partial x^i}} dx^j = \delta_i^{\ j}. \tag{1.52}$$

For a general vector field V and one-form Ω, we have

$$\iota_V \Omega = V \rightharpoonup \Omega = V^i \frac{\partial}{\partial x^i} \rightharpoonup \Omega_j dx^j = V^i \Omega_j \delta^j_{\ i} = V^i \Omega_i. \tag{1.53}$$

Remark: No metric is needed to define contraction.

Definition 21: A *Lie algebra* $\mathfrak{g}$ of a Lie group G is the tangent space $T_e G$ to G at the identity. The Lie bracket on $\mathfrak{g}$ is the commutator of vector fields on G.

Definition 22 (Left translations)**:** For all $\mathfrak{g} \in G$, we define the *left translations*

$$\begin{aligned} L_g : G &\to G \\ h &\longmapsto g \cdot h \end{aligned} \tag{1.54}$$

Definition 23 (left invariant vector fields)**:** Using the left translation maps, we define their push forward $(L_g)_* : \mathfrak{g} \equiv T_e G \to T_g G$, which maps $V \in \mathfrak{g}$ to vector fields $(L_g)_*(V)$ on G. This defines *left invariant vector fields* as sections $V \in TG$ such that $(L_g)_* V = V$ for all $g \in G$. Therefore,

$$[(L_g)_* V, (L_g)_* W] = (L_g)_*[V, W]_{\mathfrak{g}}. \tag{1.55}$$

Remark: It is important to understand the notation!

Left-invariant vector fields form a basis of $\mathfrak{g}$, meaning that $\dim(G)$ is the number of global, non-vanishing vector fields on G.

Definition 24 (parallelisable)**:** A manifold M is *parallelisable* if there exists a set of vector fields $\{V_i\}$, where $i = 1, \dots, \dim M$, such that $\forall p \in M$, the tangent vectors $\{V_i(p)\}$ form a basis for the tangent space $T_p M$.

Claim 3: Lie groups are parallelisable manifolds.

Claim 4: The converse is not true.

Proof. S^1, S^3, S^7 are the only parallelisable spheres.

The first two are indeed manifolds:

$$S^1 = U(1), \qquad S^3 = SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| \, |a|^2 + |b|^2 = 1 \right\}. \tag{1.56}$$

However, S^7 is not a Lie group. □

▌ This has introduced the field of K-theory.

Claim 5: Let $\{L_\alpha\}$, $\alpha = 1, \dots, \dim \mathfrak{g}$ be a basis of left invariant vector fields with $[L_\alpha, L_\beta] = \sum_\gamma f_{\alpha\beta}{}^\gamma L_\gamma$. Let σ^α be a dual basis of left-invariant one-forms, meaning that $L_\alpha \rightharpoonup \sigma^\beta = \delta_\alpha{}^\beta$. Then

$$d\sigma^\alpha + \frac{1}{2} f^\alpha{}_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma = 0. \tag{1.57}$$

Proof (Sheet 1). Use the identity

$$d\Omega(V, W) = V(\Omega(W)) - W(\Omega(V)) - \Omega([V, W]). \tag{1.58}$$

□

▌ Watch out for signs and factors in the upcoming derivations! Things can easily go wrong.

Definition 25 (Maurer–Cartan 1-form)**:** Assume that G is a matrix Lie group. The *Maurer–Cartan one-form* on G is then defined as

$$\rho := g^{-1} dg. \tag{1.59}$$

Claim 6: The Maurer–Cartan 1-form

1. is left invariant,
2. takes values in the Lie algebra,
3. obeys the *Maurer–Cartan equation*

$$d\rho + \rho \wedge \rho = 0. \tag{1.60}$$

Proof. 1. With $g_0 \in G$ we have

$$(g_0 g)^{-1} d(g_0 g) = g^{-1} dg. \tag{1.61}$$

2. Take C a smooth curve $g(s) \subset G$.

$$g^{-1}(s) g(s+\epsilon) = \underbrace{\epsilon}_{\mathbb{1}} + \epsilon \underbrace{g^{-1}\frac{\mathrm{d}g}{\mathrm{d}\epsilon}}_{\in T_e G \simeq \mathfrak{g}}|_{\epsilon=0} + O(\epsilon^2). \tag{1.62}$$

So $g^{-1}dg = \sum_\alpha \sigma^\alpha \otimes T_\alpha$, where T_α are matrices with $[T_\alpha, T_\beta] = \sum_\gamma f_{\alpha\beta}{}^\gamma T_\gamma$.

3. Consider first the exterior derivative term

$$d\rho = \sum_\alpha d\sigma^\alpha \cdot T_\alpha = -\frac{1}{2} f^\alpha{}_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma \cdot T_\alpha. \tag{1.63}$$

The wedge product term is

$$\rho \wedge \rho = \sigma^\alpha T_\alpha \wedge \sigma^\beta T_\beta = \frac{1}{2}\sigma^\alpha \wedge \sigma^\beta [T_\alpha, T_\beta] = \frac{1}{2}\sigma^\alpha \wedge \sigma^\beta f_{\alpha\beta}{}^\gamma T_\gamma. \tag{1.64}$$

□

Example 1.5.4 (Heisenberg group)**:** The *Heisenberg group* (sometimes just called *Nil*) is the group of upper-triangular matrices

$$g = \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} = \mathbb{1} + xT_1 + yT_2 + zT_3, \tag{1.65}$$

where $\{T_i\}$ are also the generators of the Lie algebra. Explicitly, we have

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.66}$$

and so their commutation relations are

$$[T_1, T_2] = T_3, \quad [T_1, T_3] = 0 = [T_2, T_3]. \tag{1.67}$$

We can interpret $T_1 = \hat{x}$ as position, $T_2 = \hat{p}$ as momentum, and $T_3 = i\hbar\hat{\mathbb{1}}$ as the identity.

Remark: Examples such as the lecture today will be important for the exam! If someone gives you a matrix Lie group, you will proceed in this order.

Taking the inverse of (1.65), we construct the Maurer–Cartan 1-form

$$\rho = g^{-1}dg = \begin{pmatrix} 1 & -x & -z+xy \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} & dx & dz \\ & & dy \\ & & \end{pmatrix} \tag{1.68}$$

$$= T_1 dx + T_2 dy + T_3(dz - x dy). \tag{1.69}$$

We define the dual basis of left-invariant one-forms

$$\sigma^1 = dx, \qquad \sigma^2 = dy, \qquad \sigma^3 = dz - x dy \tag{1.70}$$

$$d\sigma^1 = 0 \qquad d\sigma^2 = 0 \qquad d\sigma^3 = -dx \wedge dy = d\sigma^1 \wedge \sigma^2. \tag{1.71}$$

From these we find the left-invariant vector fields

$$L_1 = \frac{\partial}{\partial x}, \qquad L_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad L_3 = \frac{\partial}{\partial z}. \tag{1.72}$$

We find the Lie algebra by computing the brackets

$$[L_1, L_2] = L_3, \qquad [L_1, L_3] = 0, \qquad [L_2, L_3] = 0. \tag{1.73}$$

This is the same as (1.67), the Lie algebra of the Heisenberg group. We have a choice of representing the Lie algebra either in terms of matrices, T_i or left-invariant vector fields L_i.

1.5.2 Right-Invariance

We could have defined the *right translations* $R_g(h) = h \cdot g$, and associated right-invariant vector fields and one-forms.

Claim 7: The commutation relations for right-invariant vector fields differs by a minus sign from left-invariant vector fields and they commute

$$[R_\alpha, R_\beta] = -f_{\alpha\beta}{}^{\gamma} R_\gamma, \qquad [R_\alpha, L_\beta] = 0. \tag{1.74}$$

Example 1.5.5 (Nil)**:** For the Heisenberg group, using $dg \cdot g^{-1}$

$$R_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \qquad R_2 = \frac{\partial}{\partial y}, \qquad R_3 = \frac{\partial}{\partial z}. \tag{1.75}$$

Notation (2)**:** Right-invariant vector fields are said to generate left-translations and vice-versa.

1.6 Metrics on Lie Groups

We can do some local differential geometry by defining a left-invariant metric on G.

Definition 26 (left-invariant metric)**:** A *left-invariant metric* h on a Lie group G is of the form

$$h = g_{\alpha\beta}\sigma^\alpha \odot \sigma^\beta, \qquad \alpha, \beta = 1, \dots, \dim G, \tag{1.76}$$

where $g_{\alpha\beta}$ is a non-degenerate constant matrix.

Given the group and left-invariant metric, the right-invariant vector fields are generating isometries for this metric, which means that they are Killing vectors. Recall that the Maurer–Cartan 1-form is $\rho = g^{-1}dg = \sigma^\alpha \otimes T_\alpha$, where σ^α are invariant under $g \to g_0 g$. Therefore

$$\mathcal{L}_{R_\alpha}\sigma^\beta = 0 \qquad \forall \alpha, \beta. \tag{1.77}$$

Therefore, right invariant vector fields are Killing vectors for h, meaning that

$$\mathcal{L}_{R_\alpha}(h) = 0. \tag{1.78}$$

Example 1.6.1 (Nil)**:** The metric is defined as

$$h = \delta_{\alpha\beta}\sigma^\alpha \cdot \sigma^\beta = dx^2 + dy^2 + (dz - xdy)^2. \tag{1.79}$$

How do we find the isometries?

- We can see that the metric components do not involve z , so it is invariant under $z \to z + \omega$, which is generated by $\frac{\partial}{\partial z} = R_3$.
- Similarly, we can see the same for $y \to y + \epsilon$, which is generated by $\frac{\partial}{\partial y} = R_2$.
- Finally, let us consider what happens for $x \to x + \epsilon$. As it stands, this is not an isometry. The parenthesis includes a term δdy, which we can get rid off by introducing another transformation $z \to z + \epsilon y$. This is generated by $\frac{\partial}{\partial x} + y\frac{\partial}{\partial z} = R_1$.

These agree with the right-invariant vector fields of Eq. (1.75).

1.6.1 Kaluza–Klein Interpretation

Consider the motion of a charged particle on the space of orbits of $R_3 = \frac{\partial}{\partial z}$.

This was introduced as a way to combine the two known forces at the time: gravity and electromagnetism. This is done by introducing extra dimensions, by looking at gravity in dimension $d = 5$, which reduces to gravity and electromagnetism in dimension $d = 4$. The higher dimension is taken to be compactified into a circle of very large radius, so it is not detected by experiment. This idea is still stuck with us today in *String Theory*.

Example 1.6.2: For the Heisenberg group, the metric is independent of z, so we can take our manifold to be periodic in z.

To find the equations of motion, it is useful to write down the geodesic Lagrangian

$$\mathscr{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + (\dot{z} - x\dot{y})^2), \tag{1.80}$$

where the dot denotes $\bullet = \frac{\mathrm{d}}{\mathrm{d}s}$. The Euler–Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial \mathscr{L}}{\partial \dot{x}^i} = \frac{\partial \mathscr{L}}{\partial x^i} \tag{1.81}$$

$$\ddot{x} = -\dot{y}(\dot{z} - x\dot{y}) \tag{1.82}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\dot{y} - x(\dot{z} - x\dot{y})) = 0 \tag{1.83}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\dot{z} - x\dot{y}) = 0. \tag{1.84}$$

Using the final equation to introduce a constant $c = \dot{z} - x\dot{y}$, the first two equations reduce to

$$\ddot{x} = -c\dot{y} \qquad \& \qquad \ddot{y} = c\dot{x}. \tag{1.85}$$

Compare this with geodesic motion in a magnetic field. Let the spacetime be the Riemannian manifold $(M, g = g_{ij}dx^i \odot dx^j)$ and the magnetic field be the closed 2-form $F = \frac{1}{2}F_{ij}dx^i \wedge dx^j$. The components of the Levi–Civita connection associated to g are

$$\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right). \tag{1.86}$$

The geodesic equation of motion is then given by

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = cF^i{}_j\dot{x}^j, \tag{1.87}$$

where we see $F^i{}_j$ as an endomorphism rather than a 2-form. This is the general form of geodesic motion in a magnetic field. We want to compare this to (1.85), so we take $M = \mathbb{R}^2$ and $g_{ij} = \delta_{ij}$, which gives $\Gamma^k_{ij} = 0$. Moreover, we take $F = -dx \wedge dy$, meaning that $F_{ij} = -\epsilon_{ij}$ is the volume-form. So geodesics of the left-invariant metric h on $G = \mathrm{Nil}$ projects to the trajectories of a charged particle in a constant magnetic field. We think of this as in Fig. 1.10.

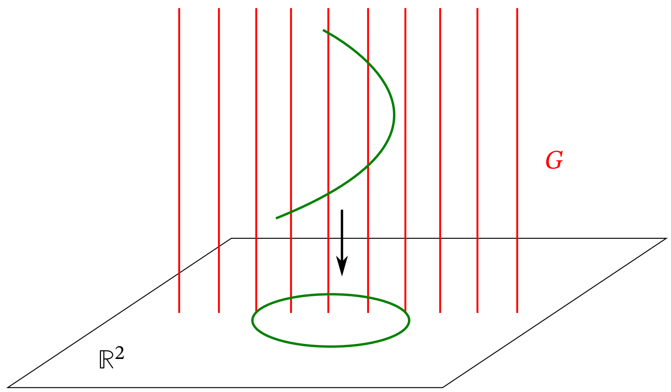

Figure 1.10: Kaluza–Klein reduction.

For the general case of Kaluza–Klein reduction, we have

$$h = (dz + A)^2 + g_{ij} dx^i \odot dx^j. \tag{1.88}$$

Take coordinates (z, x^i), where z is the extra dimension. Moreover, A is a one-form on M and we define $F = dA$. The geodesic equation is

$$\dot{z} + A_i \dot{x}^i = c \qquad \text{(conserved charge)} \tag{1.89}$$

Assume z is periodic. On G, the Laplacian is

$$\nabla^2 = L_1^2 + L_2^2 + L_3^2. \tag{1.90}$$

The Schrödinger equation is then $-\nabla^2\phi = E\phi$.

$$\phi_{xx} + \phi_{zz} + (\partial_y + xd_z)^2\phi = -E\phi. \tag{1.91}$$

Changing variables to $\phi = \Psi(x, y)e^{iez}$, we have

$$\Psi_{xx} - e^2\Psi + (\partial_y + ixe)^2\Psi = -E\Psi. \tag{1.92}$$

We can package some of the terms together to obtain a Maxwell potential. Recall that $F = dy \wedge dx = dA$. The equation for a charged particle moving in a magnetic field is

$$(\partial_x - ieA_x)^2\Psi + (\partial_y - ieA_y)^2\Psi = -(E - e^2)\Psi. \tag{1.93}$$

Comparing these equations, find that $A_x = 0$ and $A_y = -x$, which means that we have

$$A = -xdy, \qquad dA = -dx \wedge dy = F. \tag{1.94}$$

Landau problem; $0 \leq z < 2\pi L$. Charge quantisation $e \cdot L \in \mathbb{Z}$.

1.6.2 Killing Metric

Recall the Maurer–Cartan one-form $\rho = g^{-1}dg = \sigma^\alpha \otimes T_\alpha$. Define a metric to be

$$h := -\operatorname{Tr}\big(g^{-1}dg \odot g^{-1}dg\big) \tag{1.95}$$

$$= -\operatorname{Tr}\big(T_\alpha \cdot T_\beta\big)\sigma^\alpha \odot \sigma^\beta \tag{1.96}$$

$$= h_{\alpha\beta}\sigma^\alpha \odot \sigma^\beta \tag{1.97}$$

$$= -\operatorname{Tr}\big(dg \cdot g^{-1} \odot dgg^{-1}\big). \tag{1.98}$$

This is both left-invariant and right-invariant. We say that the metric is *bi-invariant*.

Example 1.6.3 ($G = SU(2)$)**:** As a manifold, $SU(2)$ is the three-dimensional sphere S^3. The Killing metric will be the round metric on S^3. Its isometry group is $SO(4)$. It fits into $SO(3) \rtimes SO(3)$; one of these generates the left-invariant vector fields and the other the right-invariant ones.

2 Hamiltonian Mechanics and Symplectic Geometry

Let M be a $2n$-dimensional manifold, which we refer to as the *phase space*. It does not come equipped with a metric, but there is another structure on it.

Definition 27 (Poisson bracket)**:** If $f, g : M \to \mathbb{R}$ are functions on the phase space, then their *Poisson bracket* is

$$\left\{f, g\right\}_{\mathrm{PB}} = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a}, \tag{2.1}$$

where (q^a, p_a), with $a = 1, \dots, n$ is a local coordinate system on M.

Definition 28 (Hamiltonian)**:** The *Hamiltonian* is a function $H : M \to \mathbb{R}$ such that Hamilton's equations hold:

$$\dot{p}_a = -\frac{\partial H}{\partial q^a}, \qquad \dot{q}^a = \frac{\partial H}{\partial p_a}. \tag{2.2}$$

Definition 29 (Hamiltonian vector field)**:** The *Hamiltonian vector field* is

$$X_H = \frac{\partial H}{\partial p_a}\frac{\partial}{\partial q^a} - \frac{\partial H}{\partial q^a}\frac{\partial}{\partial p_a}, \tag{2.3}$$

whose integral curves are $t \to (\mathbf{p}(t), \mathbf{q}(t))$.

A more general framework is kicked off by the following definition.

Definition 30 (Poisson manifold)**:** Let M be phase space and $\omega^{ij} = \omega^{[ij]}$, for $i, j = 1, \dots, \dim M = m$, be a tensor field on M. We call $(M, \omega^{ij} = \omega)$ a *Poisson manifold* and ω a *Poisson structure*, if the Poisson bracket

$$\left\{f, g\right\}_{\mathrm{PB}} = \omega^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} \tag{2.4}$$

is such that the Jacobi identity

$$\left\{f, \left\{g, h\right\}_{\mathrm{PB}}\right\}_{\mathrm{PB}} + \left\{h, \left\{f, g\right\}_{\mathrm{PB}}\right\}_{\mathrm{PB}} + \left\{g, \left\{h, f\right\}_{\mathrm{PB}}\right\}_{\mathrm{PB}} = 0, \tag{2.5}$$

holds for all functions f, g, h on M.

Not any anti-symmetric ω^{ij} satisfies this; the Jacobi identity gives conditions on ω^{ij}.

Remark: We do not distinguish between position and momenta x^i.

In the following, we will drop the subscript 'PB' to denote the Poisson bracket.

Example 2.0.1: Let $M = \mathbb{R}^3$ and $\omega^{ij} = \epsilon^{ijk}x^k$. The Poisson brackets are

$$\{x^1, x^2\} = x^3, \qquad \{x^3, x^1\} = x^2, \qquad \{x^2, x^3\} = x^4. \tag{2.6}$$

We can then define the Casimir

$$f(r) = r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{2.7}$$

This Poisson commutes with the x^i, meaning that $\{f(r), x^i\} = 0$. For example, we have

$$\{(x^1)^2 + (x^2)^2 + (x^3)^2, x^1\} = 2x^2 \overbrace{\{x^2, x^1\}}^{-x^3} + 2x^3 \overbrace{\{x^3, x^1\}}^{x^2} = 0. \tag{2.8}$$

Take the Hamiltonian to be

$$H = \frac{1}{2}\left(\frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + \frac{(x^3)^2}{a_3}\right). \tag{2.9}$$

Then the time-evolution is given by

$$\dot{g} = \{g, H\}, \qquad \underbrace{\dot{x}^i = \omega^{ij}\frac{\partial H}{\partial x^j}}_{\text{Hamilton's equations}}. \tag{2.10}$$

Writing Hamilton's equations out explicitly gives Euler's equations for a rigid body

$$\dot{x}^1 = \frac{a_3 - a_2}{a_2 a_3}x^2 x^3, \qquad \dot{x}^2 = \frac{a_1 - a_3}{a_1 a_3}x^1 x^3, \qquad \dot{x}^3 = \frac{a_2 - a_1}{a_2 a_1}x^2 x^1. \tag{2.11}$$

Example 2.0.2: Let us restrict the Poisson structure from the first example to $S^2 \subset \mathbb{R}^3$.

$$x^1 = \sin\theta\cos\phi, \qquad x^2 = \sin\theta\sin\phi, \qquad x^3 = \cos theta. \tag{2.12}$$

So θ, ϕ are functions of $\mathbb{R}^3$. Get

$$\{\theta, \phi\} = \frac{1}{\sin\theta} \tag{2.13}$$

(Exercise) and a Poisson structure on S^2, which is non-degeneate,

$$(\omega^{-1})_{ij}dx^i \wedge dx^j = \underbrace{\sin\theta d\theta \wedge d\phi}_{\text{symplectic structure}}, \tag{2.14}$$

where $x^1 = (\theta, \phi)$.

Last time, we discussd Poisson structures, which we can now specialise.

Definition 31 (symplectic manifold)**:** A *symplectic manifold* is a smooth manifold M of dimension $2n$ with a closed 2-form $\omega \in \Lambda^2(\mathcal{M})$, which is non-degenerate, meaning that

$$\underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n} \neq 0, \tag{2.15}$$

or ω is a 2×2 matrix of maximal rank.

The symplectic form ω provides an isomorphism between TM and T^*M as

$$v \in TM \longmapsto v \rightharpoonup \omega \in T^*M. \tag{2.16}$$

However, we tave to be more careful as this is now antisymmetric. If $f : M \to \mathbb{R}$, then df is a 1-form and is naturally associated to a Hamiltonian vector field X_f,

$$X_f \rightharpoonup \omega = -df, \tag{2.17}$$

where f is the Hamiltonian.

Claim 8: We can define a Poisson bracket by

$$\{f, g\}_{\text{PB}} := X_g(f) = \omega(X_g, X_f) \tag{2.18}$$

Remark: Note that this is antisymmetric since $\omega(X_g, X_f) = -\omega(X_f, X_g)$.

In local coordinates, the Poisson bracket is

$$\{f, g\} = \sum_{i,j=1}^{2n} \omega^{ij} \frac{\partial f}{\partial x^j} \frac{\partial g}{\partial x^i}. \tag{2.19}$$

Exercise 2.1: The Jacobi identity follows from the closure $d\omega = 0$.

If we compute the Lie bracket of two vector fields X_f, X_g, we find the *anti-homomorphism*

$$[X_f, X_g] = -X_{\{f,g\}}. \tag{2.20}$$

Hamiltonian vector fields preserve the symplectic form. The finite way of saying this is that they generate a flow under which the symplectic form is invariant. Infinitesimally this is expressed with the Lie derivative as

$$\mathcal{L}_{X_f}\omega = d(X_f \rightharpoonup \omega) + X_f \rightharpoonup \cancel{d\omega} = -d(df) = 0. \tag{2.21}$$

Theorem 3 (Darboux)**:** Let (M, ω) be a $2n$ -dimensional symplectic manifold. There exist *local* coordinates $x^1 = q^1, \dots, x^n = q^n, x^{n+1} = p_1, \dots, x^{2n} = p_n$ around any point in M, such that

$$\omega = \sum_{a=1}^{n} dp_a \wedge dq^a, \tag{2.22}$$

and the Poisson bracket takes the standard form.

Proof. The proof proceeds by induction with respect to half of the dimension of the symplectic manifold. We first choose an arbitrary function $p_1 : M \to \mathbb{R}$ (which will later become the first momentum coordinate). Given this function, we search for another function $q^1 : M \to \mathbb{R}$ such that

$$X_{p_1}(q^1) = 1. \tag{2.23}$$

We denote this as $\dot{q}^1 = 1$. This is an ordinary differential equation, which will have a solution[1] subject to some initial condition.

The second step is as follows. Consider the surface $M_1 = \{x \in M, p_1 = \text{const.}, q^1 = \text{const.}\}$. This is a submanifold $M_1 \subset M$, which we can prove from the embedding theorem; the maximal rank condition of the Jacobian is encoded in (2.23). This M_1 is locally symplectic with symplectic form $\omega_1 \equiv \omega|_{p_1, q^1 \text{ const.}}$

Now look for p_2, q^2 and so on. A full proof is given in the book by Arnold. □

Claim 9: Let Q, which will play the role of configuration space, be an n-dimensional manifold. The cotangent bundle T^*Q admits a global symplectic structure.

Proof. Our goal is to describe what this symplectic structure is. We have a projection $\pi : T^*Q \to Q$ acting as $\pi(q, p) = q$.

Definition 32 (pull-back)**:** Given a map $f : M \to N$, the *pull-back* $f^* : T^*_{f(p)}N \to T^*_p M$ defined as

$$f^*(p)(V) = p(f_* V). \tag{2.24}$$

In local coordinates, taking x^i, with $i = 1, \dots, \dim M$ coordinates on M and y^a with $a = 1, \dots, \dim N$ coordinates on N, we can use the chain rule to write explicitly

$$f^*(dy^a) = \sum_i \frac{\partial f^a}{\partial x^i} dx^i. \tag{2.25}$$

The pull-back of π is a map

$$\pi^* : T^*(Q) \to T^*(T^*Q). \tag{2.26}$$

[1]The assumption for the existence theorem would be that we work in a Lipshitz-class of functions.

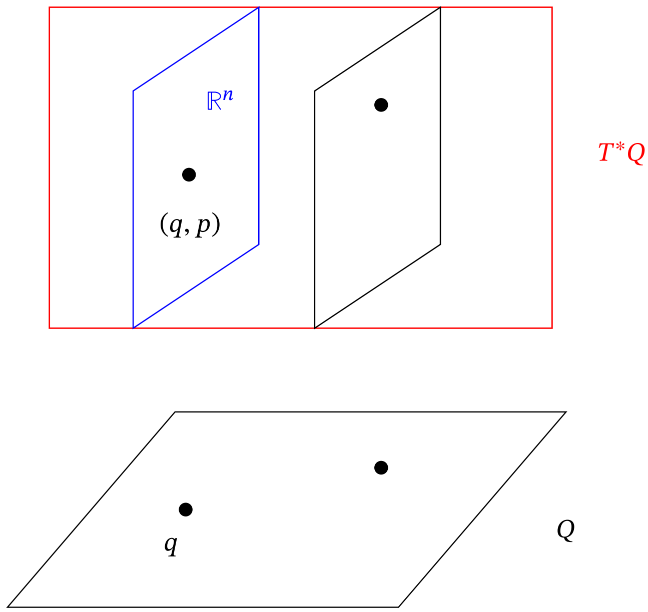

Figure 2.1

Say that $p \in T^*(Q)$ is a one-form on Q. We can define $\theta = \pi^*(p)$ and from this we get our canonical symplectic form

$$\omega = d\theta. \tag{2.27}$$

This is manifestly closed. This construction does not depend on coordinates, but usually in $(p_{\dot{a}}, q^{\dot{a}})$ coordinates we have

$$\theta = p_{\dot{a}} dq^{\dot{a}}, \qquad \omega = d\theta = dp_a \wedge dq^a. \tag{2.28}$$

□

We can define yet another structure that symplectic geometry gives us.

Definition 33 (Canonical transformations)**:** Let (M, ω) be a $2n$-dimensional symplectic manifold. An endomorphism $f : M \to M$ is called *canonical* when $f^*(\omega) = \omega$.

The one-parameter groups of canonical transformations are generated by Hamiltonian vector fields. For dimension $n = 1$, these are area preserving maps.

Consider the canonical transformation $(p_a, q^a) \xrightarrow{f} (P_a, Q^a)$.

$$d(\mathbf{p} \cdot d\mathbf{q}) = -d(\mathbf{Q} \cdot d\mathbf{P}), \tag{2.29}$$

where $\mathbf{P} = P(p,q)$ and $Q = Q(p,q)$. Then

$$d(\mathbf{p} \cdot d\mathbf{q} + \mathbf{Q} d\mathbf{P}) = 0. \tag{2.30}$$

So the one-form $\mathbf{p} \cdot d\mathbf{q} + \mathbf{Q} \cdot d\mathbf{P}$ is a closed-one form. Locally, this implies that it is exact, meaning that there is some *generating function* $S = S(\mathbf{q}, \mathbf{P})$ such that

$$\mathbf{p} \cdot d\mathbf{q} + \mathbf{Q} d\mathbf{P} = dS. \tag{2.31}$$

From this function we can define Q and p as

$$Q^a = \frac{\partial S}{\partial p_a}, \qquad p_a = \frac{\partial S}{\partial q^a}. \tag{2.32}$$

This gives $\mathbf{P}(q, p)$ and $Q(q, p)$.

2.1 Geodesics, Killing vectors, Killing tensors

We will explore the connection between Riemannian and symplectic geometry. Let (M, g) be a (pseudo-)Riemannian manifold of dimension n. In coordinates, we have

$$g = g_{ij}(x) dx^i dx^j. \tag{2.33}$$

Then we know from the *General Relativity* course, that there exists at unique Levi-Civita connection Γ^i_{jk} such that for geodesics $x^i = x^i(\tau)$, we have

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^i \dot{x}^j = 0. \tag{2.34}$$

Geodesics on M are integral curves of a Hamiltonian vector field on (T^*M, ω).

As illustrated in Fig. 2.2, we are looking for a curve in (T^*M, ω) specified by a single point (x^i, p_i) which projects down to the geodesic in M.

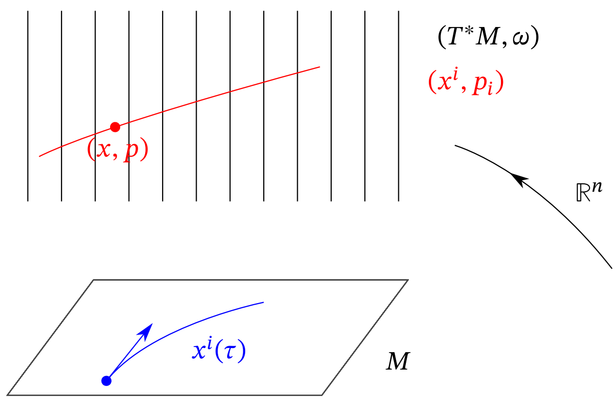

Figure 2.2

$$\dot{x}^i = p^i = g^{ij} p_j = X_H(x^i) \tag{2.35}$$

$$\dot{p}^i = -\Gamma^i_{jk} p^i p^k = X_H(p^i) \tag{2.36}$$

The Hamiltonian vector field is

$$X_H = g^{ij} p_i \frac{\partial}{\partial x^j} - \Gamma^i_{jk} p^j p^k \frac{\partial}{\partial p^i}, \tag{2.37}$$

where $H = \frac{1}{2} g^{ij}(x) p_i p_j$.

Canonical symplectic form $\omega = dp_i \wedge dx^i$.

$$H \to X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i} \tag{2.38}$$

$$= g^{ij} p_j \frac{\partial}{\partial x^i} - \frac{1}{2}\frac{\partial g^{jk}}{\partial x^i} p_j p_k \frac{\partial}{\partial p_i}. \tag{2.39}$$

Now use

$$\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^m_{ki} g_{jm} - \Gamma^m_{kj} g_{im} = 0 \tag{2.40}$$

Definition 34 (Killing vector)**:** *Killing vectors K* satisfy

$$\mathcal{L}_K g = 0 \iff \nabla_{(i} K_{j)} = 0. \tag{2.41}$$

These correspond to first integrals of the Hamiltonian flow which are linear in the momenta. Poisson commuting with H.

$$\{\underbrace{\kappa^i p_i}_{\kappa}, H\}_{\text{PB}} = \frac{\partial \kappa^i}{\partial x^j} p_i g^{jk} p_k - \frac{1}{2}\kappa^i \frac{\partial g_{jk}}{\partial x^i} p^j p^k \tag{2.42}$$

$$\overset{(2.40)}{=} \nabla_{(i} K_{j)} p^i p^j = 0 \tag{2.43}$$

Killing vectors are symmetries of the Hamiltonian flow.

2.1.1 Killing Tensors

Definition 35 (Killing tensor)**:** Write

$$\kappa = K^{ij\ldots k} \underbrace{p_i p_j \ldots p_k}_{r}. \tag{2.44}$$

We assume nothing about κ except that it Poisson-commutes with the Hamiltonian H. We find that this corresponds to

$$\{H, \kappa\}_{\text{PB}} = 0 \iff \nabla_{(i} \kappa_{jk\ldots l)} = 0. \tag{2.45}$$

A κ satisfying this is called a rank-r *Killing tensor*.

We will see that we can think of Killing tensors as higher / hidden symmetries, since we can see them in the cotangent bundle but not in the manifold itself.

The associated Hamiltonian vector field is

$$X_\kappa = \frac{\partial \kappa}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial \kappa}{\partial x^i}\frac{\partial}{\partial p_i} \tag{2.46}$$

$$= rK^{i_1\ldots i_r} p_{i_1} \ldots p_{i_{r-1}} \frac{\partial}{\partial x^{i_r}} - \frac{\partial K^{i_1\ldots i_r}}{\partial x^k} p_{i_1} \ldots p_{i_r} \frac{\partial}{\partial p_k}. \tag{2.47}$$

Projecting this down onto the manifold gives

$$\pi_k(X_\varkappa) = \begin{cases} 0, & \text{if } r > 1 \\ K^i \frac{\partial}{\partial x^i}, & \text{if } r = 1. \end{cases} \tag{2.48}$$

If $r > 1$, there is no 'geometric' symmetry on M, but $\varkappa$ is constant along geodesic.

Example 2.1.1: The Kerr black hole does not have enough Killing vectors to solve the geodesic equations, but using the Killing tensors allows us to find the geodesics.

2.2 Integrability

Intuitively, a Hamiltonian system (e.g. geodesic motion) is *integrable* if there exist sufficiently many first integrals, i.e. functions constant along the flow of the Hamiltonian vector field X_H.

Definition 36 (integrable system)**:** An *integrable system* is a symplectic manifold (M, ω) of dimension $\dim M = 2n$, together with n functions $f_i : M \to \mathbb{R}$, $i = 1, \dots, n$, with the properties of

involution: $\{f_i, f_j\} = 0$ for all i, j,

independence: $df_i \wedge df_j \wedge \cdots \wedge df_n \neq 0$.

This is a strange definition because it does not specify any dynamics, any equations of motion.

The point is that with such a system any f_i in our system can be declared to be a Hamiltonian. The corresponding Hamilton's equations will be solvable!

Theorem 4 (Arnold–Liouville)**:** Let (M, ω, f_i) form an integrable system with the Hamiltonian chosen to be $H = f_1$. Then

- The level set $M_f = \{x \in M \mid f_1 = c_1, \dots, f_n = c_n\}$ (if connected[1]) is diffeomorphic to $\mathbb{R}^k \times T^{n-k}$ for some $0 \leq k \leq n$.[2]
- There exists a canonical transformation to action-angle variables

$$\phi_1, \dots, \phi_n, I_1, \dots, I_n \tag{2.49}$$

  such that in a neighbourhood of M_f in M, ϕ_i are the coordinates, called *angles*, on M_f, and I_i are first integrals, called the *actions*.

[1] If M_f is not connected, the theorem applies to each connected component.

[2] Usually this theorem is stated with M_f *compact*. In that case, we have $k = 0$ and we just have a torus T^n.

- Hamilton's equations are solvable by quadratures
$$\dot{I}_i = \frac{\partial H}{\partial \phi_i} = 0 \qquad \dot{\phi}_i = \frac{\partial H}{\partial I_i} = \Omega_i(I_1, \dots, I_n), \tag{2.50}$$
so $I_i(t) = I_i(0)$ and $\phi_i(t) = \phi_i(0) + \Omega_i t$.

Proof. The df_k for $k = 1, \dots, n$ are independent, so M_f is a manifold of dimension n. Any function f_k gives rise to a Hamiltonian vector field X_{f_k}. If we contract this with any differential, we have
$$X_{f_k} \rightharpoonup df_j = X_{f_k}(f_j) = -\{f_k, f_j\}_{\text{PB}} = 0, \qquad \forall j, k. \tag{2.51}$$
If df_k are normal to M_f, then the corresponding Hamiltonian vector fields X_{f_j} are tangent to M_f. $\square$

Proof. Then using (2.20) we have the anti-homomorphism $[X_{f_i}, X_{f_j}] = -X_{\{f_i,f_j\}} = 0$. The X_f generate the action of n-dimensional Abelian group $\mathbb{R}^n$ on M and on M_f. Let $p_0 \in M$ and consider the lattice Γ of vectors in $\mathbb{R}^n$, which preserve p_0. This $\Gamma \subset \mathbb{R}^n$ is a discrete subgroup. Therefore, by the orbit-stabiliser theorem,

$$M_f \simeq \mathbb{R}^n/\Gamma = \begin{cases} T^n, & \text{if compact} \\ T^{n-k} \times \mathbb{R}^k, & \text{otherwise} \end{cases}. \tag{2.52}$$

Assume M_f is compact, so that

$$M_f = T^n := \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}. \tag{2.53}$$

Different constants c_i give us different choices. For every choice, we can find a torus. Through every point $p_0 \in M$, there is exactly one torus. We say that M is *foliated* by tori (different choices of c_i).

Moreover, $M_f \subset M$ is a *Lagrangian* submanifold, meaning that $\omega|_{M_f} = 0$. This is because we proved that the n Hamiltonian vector fields are tangent to M_f. These are independent, so at each point $p \in M_f$ they span the tangent space T_pM_f. It is thus sufficient to show that

$$\omega(X_{f_i}, X_{f_j}) = 0, \tag{2.54}$$

which is indeed the case since $\{f_i, f_j\} = 0$.

Now let us construct the action-angle variables (I, ϕ). Local argument in a neighbourhood of M_f in M. $\omega = d\theta$.

Taking any closed contractable curve C, which does not wind around the whole of the torus, as shown in Fig. 2.3, we have

$$\iint_{D \in M_f} \omega = \oint_C \theta = 0. \tag{2.55}$$

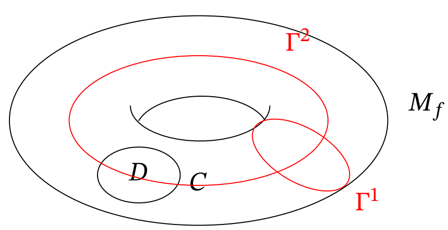

Figure 2.3

missed something

The canonical transformation is

$$(\mathbf{p}, \mathbf{q}) \to (I, \phi) \qquad \omega = d\theta = d(\mathbf{p} \cdot d\mathbf{q}) \tag{2.56}$$

In analogy to last time, we define the generating function to be

$$S(q, I) = \int_{q_0}^{q} \theta. \tag{2.57}$$

The canonical transformations obtained from this generating function are independent of the path taken. This is because we can use Stokes' theorem to integrate to zero around the closed loop obtained by concatenating two different paths. Their difference is thus zero. If the path winds around the whole in the torus, we can add a bit to the action, which will not impact the derivatives with respect to the action, and thus will not impact the canonical transformations.

We define the angles ϕ_k for $k = 1, \dots, n$ as

$$\phi_k = \frac{\partial S}{\partial I_k}. \tag{2.58}$$

These obey

$$\{\phi_k, \phi_j\} = 0, \qquad \{\phi_k, I_j\} = \delta_{kj}. \tag{2.59}$$

The Hamiltonian $H = f_i = H(I_1, \dots, I_n)$ gives time-evolution:

$$\dot{I}_k = -\frac{\partial H}{\partial \phi_k} = 0, \qquad \dot{\phi}_k = \frac{\partial H}{\partial I_k} = \text{const.} \tag{2.60}$$

As in A-L. □

Example 2.2.1 ($M = \mathbb{R}^2$)**:** All Hamiltonian systems are integrable. Take $H = \frac{1}{2}(p^2 + \omega_0^2 q^2)$, where (p, q) are coordinates on $\mathbb{R}^2$ and ω_0 is a constant frequency. Construct

$$M_f = \{(p, q) \in \mathbb{R}^2 \mid H = E\} = \Gamma. \tag{2.61}$$

Imposing one condition on $\mathbb{R}^2$ gives a curve. So M_f is a one-dimensional torus, which has just one cycle Γ. The integral is

$$I = \frac{1}{2\pi} \oint_\Gamma \phi d\phi. \tag{2.62}$$

We can solve this in two ways. Either ??. Or we can use Stokes' theorem, by considering Fig. 2.4. We have

$$I = \frac{1}{2\pi} \oint_\Gamma \phi d\phi = \frac{1}{2\pi} \iint_\mathcal{E} dp dq = \frac{E}{\omega}. \tag{2.63}$$

So $I = \frac{H}{\omega}$ and $\dot{\phi} = \frac{\partial H}{\partial I} = \omega$. Therefore, $\phi(t) = \phi(0) + \omega(t)$.

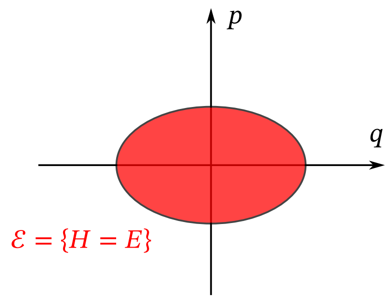

Figure 2.4

2.3 Geodesics in Non-Riemannian Geometries

We want to look at trajectories in Newtonian physics. These are governed by Newton's equations

$$\ddot{\mathbf{x}} = -\nabla V, \qquad V : \mathbb{R}^3 \to \mathbb{R}^3, \tag{2.64}$$

where the dot represents derivative with respect to an absolute time t. These are invariant under the Galilean transformations. We can reformulate (after Cartan) (2.64) as geodesic motion. Newtonian spacetime, like the spacetime of GR, is also four-dimensional. However, the geometry is different. We assemble time and space into a vector $x^a = (t, x^i)$ and look for a connection such that the geodesic equation $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$ is the same as (2.64). The unparametrised solution (i.e. parametrised by a coordinate t) is

$$\Gamma^i_{00} = \delta^{ij}\frac{\partial V}{\partial x^j}, \tag{2.65}$$

and all other components vanishing. Importantly, this is not a Levi-Civita connection, which means that there is no metric underlying it! We call this a *non-Riemannian* connection. Cartan asked what geometry underlies Newtonian physics.

2.4 Newton–Cartan Geometry

Newton–Cartan geometry in n dimensions is a triple (h, θ, ∇), where

- $h = h^{ab}\frac{\partial}{\partial x^a} \odot \frac{\partial}{\partial x^b}$ is a degenerate rank-$(n-1)$ symmetric tensor
- θ is a closed 1-form, called the *clock*, in the kernel of h. This means

$$h^{ab}\theta_a = 0 \qquad d\theta = 0 \to \theta = dt, \tag{2.66}$$

  where t is *global time*.

- torsion-free connection ∇ preserving θ and h, meaning that

$$\nabla_a h^{bc} = 0 \qquad \nabla_a \theta_b = 0. \tag{2.67}$$

In our example, $h = \mathrm{diag}(0, 1, \dots, 1)$. The connection ∇ has symbols

$$\Gamma^i_{00} = h^{ij} \frac{\partial V}{\partial x^j}. \tag{2.68}$$

The Newton–Cartan spacetime is drawn in Fig. 2.5.

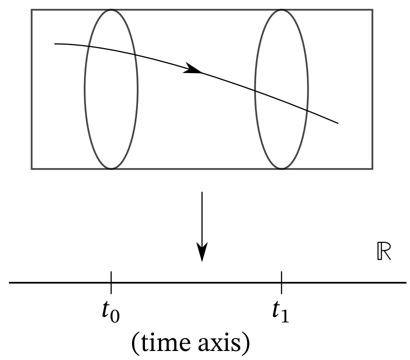

Figure 2.5: Newton–Cartan spacetime. The fibres define a notion of *simultaneity*. A geodesic of ∇ cuts through the fibres. A contravariant metric on the fibre is given by h.

3 Topological Charges

3.1 Kinks

Consider Minkowski space $\mathbb{M}^2$ with metric $ds^2 = dt^2 - dx^2$, and let $\phi : \mathbb{M}^2 \to \mathbb{R}$ be a scalar field with Lagrangian

$$L = \int_{\mathbb{R}} \left(\frac{1}{2}(\phi_t^2 - \phi_x^2) - U(\phi) \right) \mathrm{d}x \,. \tag{3.1}$$

where we denote the partial derivatives $\phi_t = \frac{\partial \phi}{\partial t}$ and $\phi_x = \frac{\partial \phi}{\partial x}$.

$$L = T - V, \qquad T = \frac{1}{2} \int \phi_t^2 \, \mathrm{d}x \tag{3.2}$$

and $T + V$ is the total energy. The Euler–Lagrange equations give the equation of motion

$$\phi_{tt} - \phi_{xx} = -\frac{\mathrm{d}U}{\mathrm{d}\phi}. \tag{3.3}$$

Assume $U(\phi) \geq U_0$ for stability and normalise $U_0 = 0$. Assume further that $U^{-1}(0) = \{\phi_1, \phi_2, \dots\}$ is a discrete set with at least two elements. We can approach this problem in various different ways.

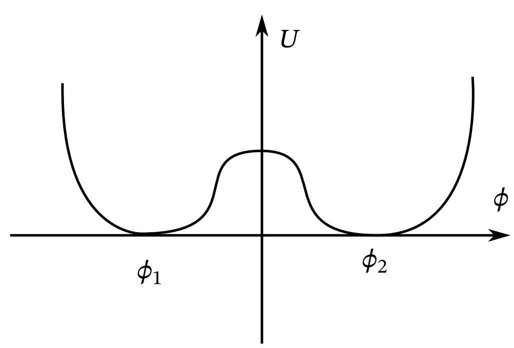

Figure 3.1

Perturbation theory: Choose a ground state ϕ_1 and let $\phi \approx \phi_1 + \delta\phi$, where $\delta\phi$ is small. The Euler–Lagrange quation becomes the Klein–Gordon equation for a scalar boson

$$(\Box + m^2)\delta\phi = 0. \tag{3.4}$$

Solitons: non-singular, static, finite energy solutions to EL. In general, $\phi \to \varphi \in U^{-1}(0)$.

Definition 37 (soliton)**:** *Solitons* are non-singular, static, finite energy solutions to the Euler–Lagrange equations.

Example 3.1.1 (Kink)**:** The kink solution has the asymptotic behaviour $\phi \to \phi_1$ as $x \to -\infty$ and $\phi \to \phi_2$ as $x \to +\infty$. This is illustrated in Fig. 3.2.

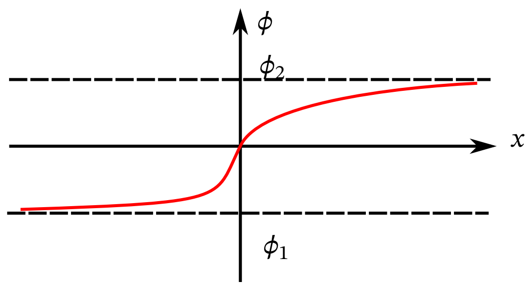

Figure 3.2: Scalar kink

There is no way we can obtain this in perturbation theory, since ϕ_2 is finite distance away from ϕ_1. This kink is inherently a non-perturbative configuration.

Let us try and get an analytical handle on this. Assume

$$U = \frac{1}{2}\left(\frac{\mathrm{d}W}{\mathrm{d}\phi}\right), \tag{3.5}$$

for some function $W(\phi)$, which is often called the *superpotential*, although there is no supersymmetry here.

$$E = \frac{1}{2}\int_{\mathbb{R}} \left[\phi_t^2 + \phi_x^2 + (W_\phi)^2\right] \mathrm{d}x \tag{3.6}$$

$$= \frac{1}{2}\int_{\mathbb{R}} \left[\phi_t^2 + (\phi_x \mp W_\phi)^2\right] \mathrm{d}x \pm \int_{\mathbb{R}} \underbrace{\frac{\mathrm{d}W}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}x}\,\mathrm{d}x}_{\mathrm{d}W} \tag{3.7}$$

$$\underbrace{\geq W(\phi(\infty)) - W(\phi(-\infty))}_{\text{Bogomolny bound}} \tag{3.8}$$

The B-bound is saturated for the *Bogomolny equations*

$$\boxed{\frac{\partial\phi}{\partial t} = 0, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}x} = \pm\frac{\mathrm{d}W}{\mathrm{d}\phi}} \tag{3.9}$$

From a variational perspective, they are absolute minimisers of the action. Moreover, finding these absolute minima can be obtained by solving first order Bogomolny equations rather than second order Euler–Lagrange equations. We will find that this is often possible even when the theories are not integrable.

$$x - x_0 = \pm\int^{\phi} \frac{1}{\sqrt{2U}}\,\mathrm{d}\phi\,. \tag{3.10}$$

Let us consider again the boundary term that we managed to integrate in (3.7).

Definition 38 (topological charge)**:** Define $\phi_\pm = \lim_{x\to\pm\infty}\phi$. The *topological charge* N is the difference

$$N = \phi_+ - \phi_- = \int_{-\infty}^{+\infty}\frac{\mathrm{d}\phi}{\mathrm{d}x}\,\mathrm{d}x\,. \tag{3.11}$$

To define N, we do not need to know the field equations for ϕ. We just need to know about their limiting behaviour. It depends on boundary conditions and is conserved if the energy is finite. However, this is not a Noether conservation law, since it does not involve the field equations.

If the field is continuous, we would need to have a continuous deformation of the red curve in Fig. 3.2. Then ??. This would imply that the energy is infinite.

There are three possibilities for the topological charge of solitons, depicted in Fig. 3.3.

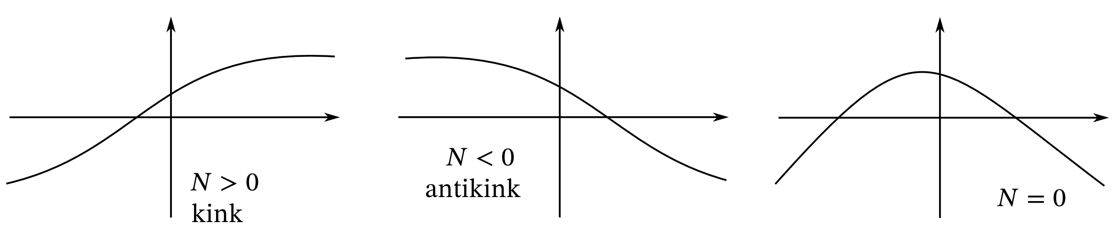

Figure 3.3: Topological charges N of solitons.

3.2 Degree of a Map

Definition 39 (topological degree)**:** Let M, M' be oriented, compact manifolds without boundary of the same dimension. Let $f : M \to M'$ be a smooth map. The topological degree def(f) is a number defined by

$$\int_M f^*(\mathrm{vol}(M')) = \deg(f) \int_{M'} \mathrm{vol}(M'). \tag{3.12}$$

Theorem 5: The topological degree is an integer, given by

$$\deg(f) = \sum_{x \in f^{-1}(y)} \mathrm{sign}(J(x)), \tag{3.13}$$

with $J = \det\left(\frac{\partial y^i}{\partial x^j}\right)$ and y is regular. It does not depend on the volume form vol(M').

Example 3.2.1: Take both M and M' to be circles. We then have $f : S^1 \to S^1$. The degree of f is then the winding number. For the function illustrated in Fig. 3.4, the topological degree is

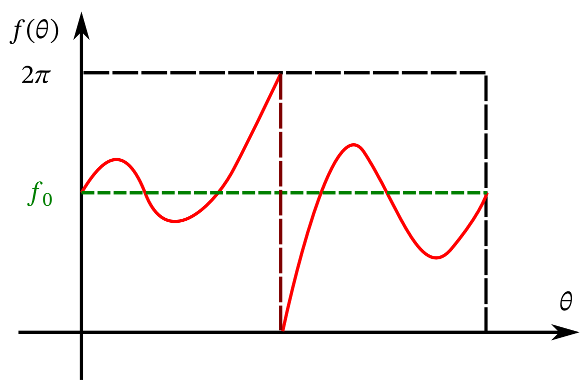

Figure 3.4: A function $f : S^1 \to S^1$.

$$\deg(f) = \sum_{\theta\,:\, f(\theta)=f_0} \mathrm{sign}\left(\frac{\partial f}{\partial \theta}\right) = 1 - 1 + 1 + 1 - 1 = 1, \tag{3.14}$$

where we only include the identified point at $\theta = 0 \sim \theta = 2\pi$ once. The fact that $\mathrm{def}(f) = 1$ shows that the function wraps up once. Equivalently,

$$\deg(f) = \frac{1}{\mathrm{vol}(S^1)} \int_{S^1} df = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}f}{\mathrm{d}\theta}\, \mathrm{d}\theta\,. \tag{3.15}$$

If we view S^1 as a set of complex numbers with modulus 1, and let $f(z) = z^k$, then $\deg(f) = k$.

Example 3.2.2: Consider maps between two-spheres: $f : S^2 \to S^2$. Let us think about this S^2 concretely as a round sphere in $\mathbb{R}^3$. The coordinates on the image sphere are,

$$f^a = f^a(\rho^i) \in \mathbb{R}^3, \qquad |\mathbf{f}| = 1, \tag{3.16}$$

where the coordinates on the domain are $\rho^i = (\theta, \phi)$, $i = 1, 2$. The volume (i.e. area) form on a sphere is

$$\sin\theta d\theta \wedge d\phi = \frac{1}{r^3}[xdy \wedge dz + ydz \wedge dy + zdx \wedge dy]. \tag{3.17}$$

To obtain the degree, we take (3.12) and divide both sides by $\mathrm{vol}(S^2)$:

$$\deg(f) = \frac{1}{\mathrm{vol}(S^2)} \int \epsilon^{abc} f^a df^b \wedge df^c \tag{3.18}$$

$$= \frac{1}{8\pi} \int \epsilon^{ij} \epsilon^{abc} f^a \frac{\partial f^b}{\partial \rho^i} \frac{\partial f^c}{\partial \rho^j} \, \mathrm{d}^2\rho \,. \tag{3.19}$$

Example 3.2.3: Let $f : M \to SU(2)$, where $\dim(M) = 3$ (compact, no boundary) and $SU(2) \simeq S^3$.

Exercise 3.1:

$$\deg(f) = \frac{1}{24\pi^2} \int_M \mathrm{Tr}\big[(f^{-1}df)^3\big] \tag{3.20}$$

f is an $SU(2)$-valued matrix. Now $f^{-1}df$ is a Lie algebra valued one-form. Taking the (exterior) group makes it still Lie algebra valued. Taking the trace gives a three-form that we can integrate.

3.3 Applications of Topological Degree in Physics

3.3.1 Sigma Model Lumps

We will talk about a scalar Lagrangian field theory of the following type. Our scalar will be a map

$$\phi : \mathbb{R}^{2,1} \to S^{N_1} \subset \mathbb{R}^N. \tag{3.21}$$

We will assume the potential $U(\phi) = 0$. However, the theory is still non-linear. This non-linearity enters on a more fundamental level because the target, on which the fields are constrained to live, is a non-linear space. We take the Lagrangian to be

$$\mathscr{L} = \frac{1}{2}\eta^{\mu\nu} \frac{\partial \phi^a}{\partial x^\mu} \frac{\partial \phi^b}{\partial x^\nu} \delta_{ab} \tag{3.22}$$

with metric $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ and the constraint

$$\sum_{a=1}^{N} \phi^a \phi^a = 1. \tag{3.23}$$

Lagrange Multiplier

To incorporate this constraint, we may modify the Lagrangian by introducing a Lagrange multiplier λ

$$\mathscr{L}' = \mathscr{L} - \frac{1}{2}\lambda(1 - \sum_a |\phi|^2). \tag{3.24}$$

The resulting equation of motion is

$$\Box\phi^a - \lambda\phi^a = 0, \qquad \Box = \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}. \tag{3.25}$$

As it stands, this equation is not what we want yet; we need to solve for λ and substitute it back into this to obtain the constrained equations of motion. We then obtain the non-linear equations

$$\Box\phi^a - (\phi^b\Box\phi^b)\phi^a = 0. \tag{3.26}$$

Moreover, the non-linearity is of a particularly nasty type; the non-linear term arises with the highest derivative, unlike quasi-linear equations like Einstein's equations.

Without Lagrange Multiplier

Alternatively, we may solve $|\phi|^2 = 1$ for the final component

$$\phi^N = \pm\sqrt{1 - \phi^p\phi^p}, \qquad p = 1, \dots, N-1. \tag{3.27}$$

We obtain

$$\mathscr{L} = \frac{1}{2}g_{pq}(\phi)\eta^{\mu\nu}\partial^\mu\phi^p\partial^\nu\phi^q, \tag{3.28}$$

where,

$$g_{pq} = \delta_{pq} + \frac{\phi_p\phi_q}{1 - \sum_{i=1}^{N-1}\phi_i\phi_i} \tag{3.29}$$

is the round metric on S^{N-1}. We are not going to work much with the Lagrangian (3.28), but we want to point it out since it provides a broad framework for

$$(\Sigma, \eta) \to (M, g). \tag{3.30}$$

It is useful in a variety of situations:

1) Bosonic sector of superstring theory. In this framework, (Σ, η) the string worldsheet (2D Riemann surface) and (M, g) is a 10-dimensional space-time. Then (3.28) gives the basic scalar Lagrangian.

2) Point particle moving on a curved space-time. Take $\Sigma = \mathbb{R}$ and (M, g) to be space-time (or any manifold). Then (3.28) is the geodesic Lagrangian.

3) If $\dim \Sigma = k$, then (3.28) describes the behaviour of "$(k-1)$ branes".

These examples are all scalar theories, but we can also add gauge fields to this Lagrangian.

3.3.2 Solitons

GO back to $\Sigma = \mathbb{R}^2$ and $M = S^{N-1}$. Take $N = 3$, so that the target space is a two-sphere. We will be interested in solitons — static solutions to the Euler–Lagrange equations with finite energy. The total energy for static solutions is

$$E = \int_{\mathbb{R}^2} (\partial_i \phi^a \cdot \partial_i \phi^a) dx dy < \infty, \tag{3.31}$$

where $x^i = (x, y)$ and $\partial_i = \frac{\partial}{\partial x^i}$. The metric is $\eta = dt^2 - dx^2 - dy^2$, but $\phi = \phi(x^i)$.

We know that an integral of r^{-p} is logarithmically divergent for $p = 1$, but not for $p > 1$. For the integral to be finite, we choose the following boundary conditions

$$r|\nabla \phi^a| \to 0 \quad \text{as} \quad r \to \infty. \tag{3.32}$$

Therefore, we also have $\nabla \phi^a \to 0$, so $\phi \to \phi^\infty$ for some constant ϕ^∞ at spatial infinity. Choose $\phi^\infty = (0, 0, 1)$, which is the north pole of S^2.

We may reinterpret this boundary condition as saying that static, finite energy configurations ϕ extend to the one-point compactification of $\mathbb{R}^2$, which is a two-sphere $S^2 = \mathbb{R}^2 + \{\infty\}$, with $\phi(\{\infty\}) = (0, 0, 1)$. We can view

$$\phi : S^2 \to S^2. \tag{3.33}$$

These maps are classified by the degree $\deg(\psi)$ (topological charge). This is only a partial classification, as fields can have dive different energies in the same topological sector.

Claim 10: The energy

$$E = \frac{1}{2} \int_{S^2} \partial_i \phi^a \partial_i \phi^a \, \mathrm{d}^2 x \geq 4\pi |\deg(\phi)|, \tag{3.34}$$

is bounded from below, with equality when the first order Bogomolny equations

$$\boxed{\partial_i \phi^a = \pm \epsilon_{ij} \epsilon^{abc} \phi^b \partial_j \phi^c} \tag{3.35}$$

are satisfied. Here $i, j = 1, 2$, $\epsilon_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $|\phi| = 1$.

In this case, non-integrable second-order equations reduce to integrable first-order equations.

Proof. Consider the identity

$$\int (\partial_i \phi^a \pm \epsilon_{ij} \epsilon^{abc} \phi^b \partial_j \phi^c) \cdot (\partial_i \phi^a \pm \epsilon_{ik} \epsilon^{ade} \phi^d \partial_k \phi^e) \, \mathrm{d}^2 x \geq 0. \tag{3.36}$$

This is a square of scalar quantities, which is therefore non-negative. □

Proof (continued). The indices run from $i, j = 1, 2$ and $a, b, c = 1, 2, 3$. We make use of the following relations

$$\epsilon_{ij}\epsilon_{ik} = \delta_{jk}, \qquad \epsilon^{abc}\epsilon^{ade} = \delta^{bd}\delta^{ce} - \delta^{be}\delta^{cd}. \tag{3.37}$$

We will also use that ϕ takes values in S^2, so if we differentiate,

$$\phi^a \partial_j \phi^a = 0. \tag{3.38}$$

Let us now expand (3.36) as

$$\int \left[\partial_i \delta^a \partial_i \phi^a + \delta_{jk}(\delta^{bd}\delta^{ce} - \cancel{\delta^{be}\delta^{cd}})(\phi^b\phi^d\partial_j\phi^c\partial_k\phi^e) \pm 2\epsilon_{ij}\partial_i\phi^a\epsilon^{abc}\phi^b\partial_j\phi^c\right] \mathrm{d}^2x$$
$$= 2E + 2E \pm 16\pi \deg(\phi) \geq 0, \tag{3.39}$$

where we think of ϕ as maps $\phi : S^2 \to S^2$. This can only be an equality if (3.36) is an equality. Since that is a total square, each term must vanish identically, giving the Bogomolny equations. □

The solutions to the Bogomolny equations are critical points of the energy functional, so they are also solutions to the 2nd order Euler–Lagrange equations.

4 Gauge Theory

We will head towards the Yang–Mills equations in Euclidean signature, so we will pay particular attention to the signs.

4.1 Hodge Duality

We work in $\mathbb{R}^n$ with metric $\eta = \mathrm{d}\mathbf{x}^2 - \mathrm{d}\mathbf{t}^2$ with signature $(n-t, t)$. We have coordinates $x^a = (\mathbf{x}, t)$ in which the metric has components η_{ab}.

Definition 40 (inner product)**:** We define an *inner product on p-forms* as

$$(\alpha, \beta) = \frac{1}{p!}\alpha^{a_1\ldots a_p}\beta_{a_1\ldots a_p} = (\beta, \alpha), \tag{4.1}$$

where we raised the indices on α with the inverse metric η^{ab}.

Definition 41 (volume form)**:** The volume form

$$\frac{1}{n!}\epsilon_{a_1\ldots a_n} dx^{a_1} \wedge \cdots \wedge dx^{a_n}. \tag{4.2}$$

Definition 42 (hodge star)**:** The *Hodge operator* $\star : \Lambda^p \to \Lambda^{n-p}$ maps $\lambda \in \Lambda^p$ to $\star\lambda$, defined by

$$\lambda \wedge \eta = (\star\lambda, \eta)\mathrm{vol}, \qquad \forall \eta \in \Lambda^{n-p}. \tag{4.3}$$

In components,

$$(\star\lambda)^{b_1\ldots b_q} = \frac{(-1)^t}{p!}\epsilon^{a_1\ldots a_p b_1\ldots b_q}\lambda_{a_1\ldots a_p}. \tag{4.4}$$

Claim 11: Applying the Hodge operator twice on a differential form $\lambda \in \Lambda^p$ gives

$$\star\star\lambda = (-1)^t(-1)^{p(n-p)}\lambda, \tag{4.5}$$

which depends on the signature t and dimension n of spacetime.

Example 4.1.1 ($n = 4, t = 1, p = 2$)**:** For the Maxwell 2-form,

$$\star \star F = -F. \tag{4.6}$$

We can check this by writing $\epsilon_{ijk0} = \epsilon_{ijk}$. The volume form is then

$$\mathrm{vol} = dx \wedge dy \wedge dz \wedge dt. \tag{4.7}$$

The Hodge duals are

$$\star\, dt \wedge dx = -dy \wedge dz, \qquad \star dy \wedge dz = dt \wedge dx. \tag{4.8}$$

The Maxwell tensor is

$$F = F_{0i} dt \wedge dx^i + \frac{1}{2} F_{ij} dx^i \wedge dx^j \tag{4.9}$$

$$= -E_i dt \wedge dx^i + B_1 dx^2 \wedge dx^3 + ... \tag{4.10}$$

$$\star F = B_i dt \wedge dx^i + E_1 dx^2 \wedge dx^3 + ... \tag{4.11}$$

$$\star(\mathbf{E}, \mathbf{B}) = (-\mathbf{B}, \mathbf{E}), \tag{4.12}$$

using that $B_i = \frac{1}{2}\epsilon_{ijk} F_{jk}$.

Let

$$A = -\phi dt + A_i dx^i. \tag{4.13}$$

Then

$$F = dA = \left(-\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} \right) dx^i \wedge dt + \partial_{[k} A_{i]} dx^k \wedge dx^i. \tag{4.14}$$

Alternatively,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi, \qquad \mathbf{B} = \boldsymbol{\nabla} \wedge \mathbf{A}. \tag{4.15}$$

Then since $d^2 = 0$, we have the Bianchi identity $dF = d^2 A = 0$. The field equations are

$$d \star F = 0 \tag{4.16}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0, \qquad \boldsymbol{\nabla} \times \mathbf{B} = \dot{\mathbf{E}}. \tag{4.17}$$

Example 4.1.2 (self-duality, $n = 4, t = 0$)**:** On two-forms, we have

$$\star^2 F = F. \tag{4.18}$$

So if $F = dA$ is exact and F is *self-dual*, meaning

$$\boxed{F = \star F}, \tag{4.19}$$

then by the Bianchi identity

$$d \star F = dF = 0. \tag{4.20}$$

The second order Maxwell equations follow from the first order self-duality condition on F.

Remark: We can write any two-form as

$$F = \frac{1}{2}(F + \star F) + \frac{1}{2}(F - \star F) = F_+ + F_-, \tag{4.21}$$

which is the sum of its self-dual and anti-self-dual parts

$$\star F_+ = F_+, \qquad \star F_- = -F_-. \tag{4.22}$$

This splits the 6-dimensional vector space into two 3-dimensional subspaces

$$\Lambda^2(\mathbb{R}^4) = \Lambda^2_+ \oplus \Lambda^2_-. \tag{4.23}$$

This would be more complicated in non-Euclidean signatures.

4.2 Yang–Mills Equations

Again we consider the manifold $(\mathbb{R}^n, \eta, \mathrm{vol})$, where the metric η has unspecified signature. Let us now consider the Lie algebra $\mathfrak{g}$ of a Lie group G. Let A be a $\mathfrak{g}$-valued 1-form, called the *gauge potential*. In coordinates,

$$A = A_a dx^a = A^\alpha_a T_\alpha dx^a \tag{4.24}$$

where the components A_a are not ordinary functions but take values in the Lie algebra $\mathfrak{g}$. The T_α, with indices $\alpha = 1, \dots, \dim(\mathfrak{g})$, are the generators of the Lie algebra, obeying

$$[T_\alpha, T_\beta] = c_{\alpha\beta\gamma} T_\gamma. \tag{4.25}$$

Definition 43: We define the *gauge field* as

$$F = \frac{1}{2} F_{ab} dx^a \wedge dx^b = \boxed{dA + A \wedge A}. \tag{4.26}$$

In components, we find

$$F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b] \tag{4.27}$$

$$= [D_a, D_b], \tag{4.28}$$

where defined the *covariant derivative*

$$D = d + A. \tag{4.29}$$

4.2.1 Gauge Transformations

We will identify (A, A') and (F, F') if

$$A' = gAg^{-1} - dgg^{-1}, \qquad F' = gFg^{-1}, \tag{4.30}$$

where $g = g(x^a) \in G$ is *local*, meaning that it is allowed to depend on the position x^a. This property will make the connection to geometry, as we will see shortly.

Let us compute the covariant derivative on F:

$$\begin{aligned} DF &:= dF + [A, F] && (4.31) \\ &= d^2A + dA \wedge A - A \wedge dA + A \wedge dA + A^3 - dA \wedge A - A^3 = 0. && (4.32) \end{aligned}$$

This is the *Bianchi identity*.

We will mostly be concerned about the case where $G = SU(2)$. The Lie algebra $\mathfrak{su}(2)$ is generated by

$$[T^\alpha, T_{\beta]} = -\epsilon_{\alpha\beta\gamma} T_\gamma, \qquad \mathrm{Tr}\big(T_\alpha T_\beta\big) = -\frac{1}{2}\delta_{\alpha\beta}. \tag{4.33}$$

Then $X \in \mathfrak{su}(2)$ are anti-Hermitian metrics.

$$-\,\mathrm{Tr}(F \wedge \star F) = -\frac{1}{2}\,\mathrm{Tr}\big(F_{ab}F^{ab}\big) = \frac{1}{4}(F_{ab})^\alpha (F^{cb})^\alpha \mathrm{vol}. \tag{4.34}$$

4.3 Yang–Mills Instantons

Let A be the gauge potential, a $\mathfrak{g}$-valued 1-form, and

$$F = dA + A \wedge A, \tag{4.35}$$

a $\mathfrak{g}$-valued 2-form. Taking $G = SU(2)$, minus the trace is positive definite, so we have the action

$$S[A] = -\int_{\mathbb{R}^n} \mathrm{Tr}(F \wedge \star F). \tag{4.36}$$

Varying with respect to the gauge potential and requiring $\delta S = 0$, we obtain the Yang–Mills equations

$$D \star F = 0. \tag{4.37}$$

We also have the Bianchi identity

$$DF = 0. \tag{4.38}$$

Take now $n = 4$, $\eta_{ab} = \delta_{ab}$. This might be viewed as originating from Lorentzian geometry by having transformed a Wick rotation.

Definition 44 (instantons)**:** *Instantons* are non-singular solutions of the classical Euler–Lagrange equations in Euclidean space, with finite action.

These are not localised only in space, but also at an instant in time, hence the name.

The motivation for physicists comes from Euclidean quantum gravity. To understand the quasi-classical limit in the WKB approximation, the largest contribution comes from solutions with imaginary time. There is a heuristic argument that claims that this is also valid for quantum field theory, which motivates the study of instantons. However, it is not clear whether they have a direct connection with physics.

In $n = 4$, we have

$$S = -\int_{\mathbb{R}^4} \mathrm{Tr}(F \wedge \star F), \tag{4.39}$$

with the following boundary conditions: Defining $r^2 = \delta_{ab} x^a x^b$, we want

$$F_{ab}(x) \sim O\left(\frac{1}{r^3}\right), \qquad \text{as} \quad r \to \infty. \tag{4.40}$$

The motivation for this is that the Jacobian in 3 Euclidean dimensions in spherical polar coordinates is of the form $r^2 \sin\theta dr d\theta d\phi$. For 3 dimensional Euclidean space, we have instead a volume form containing $r^3 dr$. If F was r^{-2}, then $F^2 \sim r^{-4}$ and the integral would diverge logarithmically. If you were an analyst, you could prove that r^{-3} is the lowest bound that we can have for a convergent integral. For the gauge potential, we want

$$A_a(x) \sim -(\partial_a g)g^{-1} + O\left(\frac{1}{r^2}\right) \qquad \text{as} \quad r \to \infty. \tag{4.41}$$

The gauge transformation g needs only to be defined asymptotically at $\partial R^4 = S^3_\infty$. So $g : S^3_\infty \to SU(2) \simeq S^3$. This map is partially classified by the topological degree, which we will look out for in the upcoming calculations.

4.3.1 Aside: Gauge Theory on $\mathbb{R}^n$

F is a matrix, so we can take its determinant. Take the wedge product to be the multiplication operation to calculate the determinant

$$C(F) = \det\left(\mathbb{1} + \frac{i}{2\pi}F\right), \tag{4.42}$$

$$= 1 + C_1(F) + C_2(F) + \dots . \tag{4.43}$$

Then $C_p(F)$ is a $2p$-form (polynomial in traces of powers of F), called the p^{th} *Chern form*. These are

- gauge invariant
$$C(gFg^{-1}) = C(F) \tag{4.44}$$
- closed (by the Bianchi identity)

Let us compute the first two Chern forms. In $\frac{SU}{2}$ gauge theory,

$$C_1(F) = \frac{i}{2\pi}\,\mathrm{Tr}(F) = 0 \tag{4.45}$$

$$C_2(F) = \frac{1}{8\pi^2}\big(\mathrm{Tr}(F \wedge F) - \mathrm{Tr}(F) - \mathrm{Tr}(F)\big) = \frac{1}{8\pi^2}\,\mathrm{Tr}(F \wedge F). \tag{4.46}$$

The first Chern form plays a big role in $U(1)$ gauge theory of magnetic monopoles, but we will be more interested in the second Chern form.

Let us explicitly check the closure on the second Chern form

$$dC_2 = \frac{1}{4\pi^2}\,\mathrm{Tr}(dF \wedge F) \tag{4.47}$$

$$= \frac{1}{4\pi^2}\,\mathrm{Tr}(DF \wedge F - A \wedge F \wedge F + F \wedge A \wedge F) = 0, \tag{4.48}$$

where the first term vanishes by the Bianchi identity and the second and third cancel due to cyclic permutation of the trace.

Since $\mathbb{R}^n$ is contractable, every globally defined closed form is globally exact. Therefore, $C_2 = dY_3$, where Y_3 is called the *Chern–Simons* 3-*form*

$$Y_3 = \frac{1}{8\pi^2}\operatorname{Tr}\left(dA \wedge A + \frac{2}{3}A^3\right). \tag{4.49}$$

Exercise 4.1: Check this. Use $\operatorname{Tr}(A^4) = 0$.

There is another way to rewrite the Chern–Simons 3-form using the Bianchi identity. Rewrite (4.35) to find $dA = F - A \wedge A$, giving

$$Y_3 = \frac{1}{8\pi^2}\operatorname{Tr}\left(F \wedge A - \frac{1}{3}A^3\right). \tag{4.50}$$

4.3.2 Back to $\mathbb{R}^4$: Chern Number

Definition 45 (Chern number)**:** The p^{th} *Chern number* is the integral over the p^{th} Chern class.

The second Chern number is

$$c_2 = \int_{\mathbb{R}^4} C_2 = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} dY_3. \tag{4.51}$$

$$= \frac{1}{8\pi^2}\int_{S^3_\infty} \operatorname{Tr}\left(F \wedge A - \frac{1}{3}A^3\right) \qquad \text{(instantons)} \tag{4.52}$$

$$= -\frac{1}{24\pi^2}\int_{S^3_\infty} \operatorname{Tr}(A^3) \in \mathbb{Z} \tag{4.53}$$

as $A_\infty = -(dg)g^{-1}, F_\infty = 0$,

$$c_2 = \frac{1}{24\pi^2}\int_{S^3_\infty} \operatorname{Tr}(dg \cdot g^{-1}) = \deg(g). \tag{4.54}$$

If we calculate the Chern number for any solution to Yang–Mills on $\mathbb{R}^4$, then we have no reason to expect it to be an integer. However, for instantons, it is an integer that agrees with the topological degree of the map $g: S^3_\infty \to S^3 = SU(2)$.

We have not yet used the Yang–Mills equations, but we will do so now.

Theorem 6: Within a given topological sector[1] with orientation such that

$$c_2 = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \operatorname{Tr}(F \wedge F) \geq 0, \tag{4.55}$$

[1]An integer-valued object cannot vary continuously, so it is a topological invariant.

the Yang–Mills action S is bounded from below by $8\pi^2 c_2$. The boundary is saturated if the anti-self-dual Yang–Mills (ASDYM) equations hold

$$\star F = -F \qquad \text{or} \qquad \begin{cases} F_{12} = -F_{31} \\ F_{13} = -F_{42} \\ F_{14} = -F_{23} \end{cases} \tag{4.56}$$

Remark: This is the third time we are doing this calculation. We have done it for kinks, and for sigma model lumps before. In both cases, second order equations reduce to first order Bogomolny equations.

Proof. Note $F \wedge F = \star F \wedge \star F$. Then, using that $\star \star F = 1$, write the action as

$$S = -\frac{1}{2}\int_{\mathbb{R}^4} \mathrm{Tr}[(F + \star F) \wedge \star(F + \star F)] + \int_{\mathbb{R}^4} \mathrm{Tr}(F \wedge F) \tag{4.57}$$

The cross-term is $\int \mathrm{Tr}(F \wedge F) = 8\pi^2 c_2$. As the first integral is non-negative. This gives the inequality

$$S \geq 8\pi^2 c_2. \tag{4.58}$$

This gives the Bogomolny bound for the Yang–Mills action. We obtain equality if

$$F + \star F = 0. \tag{4.59}$$

□

A similar calculation with $c_2 \leq 0$ would give the SDYM $F = \star F$. C changes the sign of the volume form interchanges ASDYM and SDYM.

If $\star F = \pm F$,

$$D \star F = \pm DF = 0 \tag{4.60}$$

YM follow from Bianchi.

An exciting current area of research is that there is evidence that all integrable systems are symmetry reductions of SDYM.

We define $\varkappa = -c_2$ to be the instanton number.

5 Fibre Bundles and Connections

> Exams: answer two out of three questions. He lectured from 2012-2016. Exam revision class in April.

We will introduce some rather formal mathematics to show that we can also think of Yang–Mills theory as a theory of connections on principal bundles with gauge group G as the fibre.

5.1 Fibre Bundles

Without getting into the definitions just yet, let us give a bit more of an overview. *Fibre bundles* are manifolds that locally look like a product of manifolds. Examples include the cylinder $S^1 \times \mathbb{R}$ and the Möbius band, which are illustrated in Fig. 5.1. These are locally equivalent, but globally different,

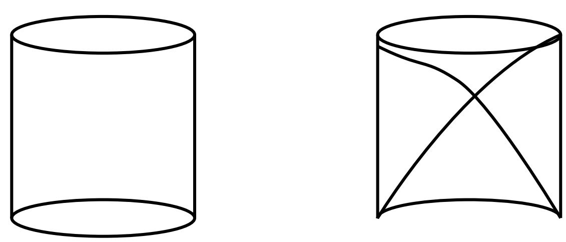

Figure 5.1: Fibre bundles over $B = S^1$ with a fibre $\mathbb{R}$.

since the Möbius band has a half twist while the cylinder is the trivial bundle. The Möbius band is non-trivial.

Definition 46 (fibre bundle)**:** A *fibre bundle* $\{E, \pi, B, F, G\}$ is a structure consisting of:

- a manifold E, called the *total space* of the bundle,
- a smooth map $\pi : E \to B$, called the *projection*, where

- B, called the *base space* is also a manifold
- $\forall x \in B$, $\exists U_\alpha \subset B$ s.t. $x \in U_\alpha$ and $\exists$ diffeomorphism

$$\phi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha), \qquad \text{s.t.} \pi(\phi_\alpha(x, f)) = x, \tag{5.1}$$

  where $(x, f) \subset U_\alpha \times F$. The manifold (!) F is called a *fibre* and ϕ_α are called *trivialisations*.

- transition functions: for $x \in U_\alpha \cap U_\beta$,

$$\phi_{\alpha\beta} := \phi_\alpha^{-1} \circ \phi_\beta : F \to F, \tag{5.2}$$

  which are elements $\phi_{\alpha\beta} \in G$, where the Lie group G is called the *structure group* of E.

  1) $\phi_{\alpha\alpha} = \mathrm{Id}$ (no summation over α)
  2) cocycle relation: for $x \in U_\alpha \cap U_\beta \cap U_\gamma$,

$$\phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma}. \tag{5.3}$$

For the example above, the total spaces E are the cylinder and the Möbius band, the base space B is a circle. The fibre F in both cases is a real line $\mathbb{R}$.

Definition 47 (principal bundle)**:** A *principal bundle* has $F = G$, and the Lie group G acts on F by left-translations.

Definition 48 (vector bundle)**:** A *vector bundle* has $F = \mathbb{R}^n$ (real vector bundle) or $F = \mathbb{C}^n$ (complex vector bundle) and $G = GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$.

Definition 49 (trivial bundle)**:** A *trivial bundle* has $E = B \times F$.

Example 5.1.1 (tangent bundle)**:** This is an example of a bundle that we have already met. The *tangent bundle*

$$\text{total space:} \qquad E = TB = \bigcup_x T_x B, \tag{5.4}$$

$$\text{fibre:} \qquad F_x = \pi^{-1}(x) = \mathbb{R}^n, \tag{5.5}$$

$$E = \{x, \mathbf{v}\} \qquad \text{(position, velocity)}, \tag{5.6}$$

$$\text{group:} \qquad G = GL(n, \mathbb{R}). \tag{5.7}$$

Example 5.1.2 (magnetic monopole bundle)**:** The *magnetic monopole bundle* is an example of a principal bundle:

$$\text{base manifold:} \qquad B = S^2 = U_+ \cap U_-, \tag{5.8}$$

$$\text{fibre:} \qquad F = U(1), \tag{5.9}$$

$$\tag{5.10}$$

with coordinate $e^{i\psi}$. Local coordinates (θ, ϕ) on S^2

$$0 \le \theta \le \pi, \qquad 0 \le \phi \le 2\pi. \tag{5.11}$$

The base manifold is covered by the open sets U_+, U_- shown in Fig. 5.2.

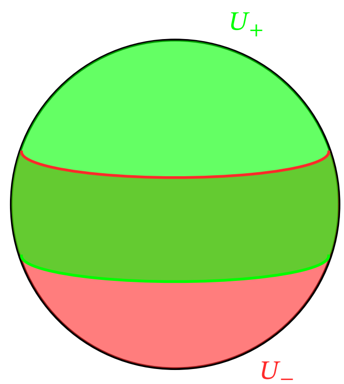

Figure 5.2

$$U_+ \times U(1): \qquad (\theta, \phi, e^{i\psi_+}) \tag{5.12}$$

$$U_- \times U(1): \qquad (\theta, \phi, e^{i\psi_-}), \tag{5.13}$$

on $U_+ \cap U_-$,

$$e^{i\psi_-} = e^{in\phi} e^{i\psi_+}. \tag{5.14}$$

Then n must be an integer for E to be a manifold. In fact, the *monopole number* n classifies $U(1)$ principal bundles over S^2.

- $n = 0$, $E = S^2 \times S^1$ (trivial bundle)
- $n = 1$, $E = S^3 \times S^1$ (Hopf fibration)

5.2 Hopf Fibration

There are two descriptions of the Hopf fibration

1. Let us take $E = SU(2)$, then
$$B = \frac{SU(2)}{U(1)} \simeq S^2, \tag{5.15}$$
where $U(1) \subset SU(2)$ diagonal. Let us think of S^3 as
$$S^3 = \left\{ (z_1, z_2) \in \mathbb{C}^2 \,\middle|\, |z_1|^2 + |z_2|^2 = 1 \right\}. \tag{5.16}$$

We take the right action of S^1 on $SU(2)$ to be

$$(z_1, z_2) \to (z_1 e^{i\alpha}, z_2 e^{i\alpha}). \tag{5.17}$$

Since the sum of the norms is still the same, this maps from $S^3 \to S^3$. This fixes the ratio

$$\frac{z_1}{z_2} \in \mathbb{CP}^1 \simeq S^2. \tag{5.18}$$

2. Any complex line

$$A_1 z_2 + A_2 z_2 = 0 \tag{5.19}$$

through $0 \in \mathbb{C}^2$, which can be seen as a two-dimensional plane in $\mathbb{R}^4$, intersects $S^3 \subset \mathbb{C}^2 \simeq \mathbb{R}^4$ in a circle S^1, which we take to be the fibre over a point $[z_2, z_2] \in \mathbb{CP}^1 \simeq S^2$.

5.3 Section

Definition 50 (section)**:** A *section* s of a bundle $\pi : E \to B$ is a map $s : B \to E$ such that $\pi \circ s = \mathrm{Id}$.

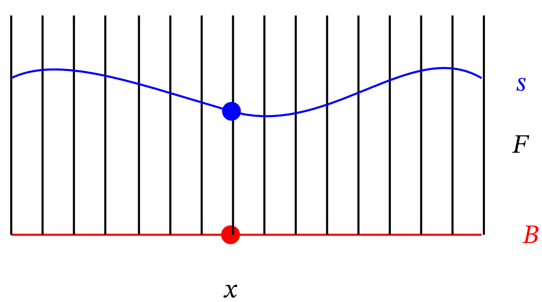

Figure 5.3: A section s of a bundle.

Claim 12: A principal bundle is trivial iff it admits a global section.

Proof. Suppose $s = s(x)$ is this section. Then $\forall p \in G_x \simeq \pi^{-1}(x)$ may be represented as $p = s(x)R_{g(p)}$, where $R_{g(p)}$ is the right action of G on G_x moving $S(x)$ to p.

$$E = \{x = \pi(p), g_g p\} \in B \times G. \tag{5.20}$$

□

5.4 The Connection and Curvature on Principal Bundles

Definition 51 (connection)**:** The *connection* on a principal bundle (E, B, π, G) is a $\mathfrak{g}$-valued 1-form ω whose vertical (fibre) component is the Maurer–Cartan 1-form on G.

In local coordinates,

$$\omega = \gamma^{-1}(A + d)\gamma, \tag{5.21}$$

where A is a $\mathfrak{g}$-valued 1-form B on $\gamma \in G$ (fibre bundle).

Say U, U' are overlapping open sets on B, $g_{UU'} = g$ is the transition function acting on G by left multiplication, i.e. $\gamma' = g\gamma$.

On $U \cap U'$.

$$\gamma^{-1}(A + d)\gamma = (\gamma')^{-1}(A' + d)\gamma'. \tag{5.22}$$

Let us compute the right-hand side:

$$\gamma^{-1}g^{-1}A'g\gamma + \underbrace{\gamma^{-1}g^{-1}(dg \cdot \gamma + gd\gamma)}_{\gamma^{-1}g^{-1}dg\gamma + \gamma^{-1}d\gamma}. \tag{5.23}$$

Comparing this to the left-hand side, the term $\gamma^{-1}d\gamma$ cancels. We are left with

$$\gamma^{-1}A\gamma = \gamma^{-1}(g^{-1}A'g + g^{-1}dg)\gamma. \tag{5.24}$$

From this we obtain

$$\boxed{A' = gAg^{-1} - dg \cdot g^{-1}} \tag{5.25}$$

In other words, A, A' are related by a gauge transformation (4.30) on B.

Definition 52 (curvature)**:** The *curvature* Ω of a connection ω is a $\mathfrak{g}$-valued 2-form on E defined by

$$\Omega = d\omega + \omega \wedge \omega. \tag{5.26}$$

Working in a local trivialisation,

$$\begin{aligned}
\Omega &= d(\gamma^{-1}A\gamma + \gamma^{-1}d\gamma) + \omega \wedge \omega && (5.27)\\
&= -\gamma^{-1}d\gamma\gamma^{-1} \wedge A\gamma + \gamma^{-1}dA\gamma - \gamma^{-1}A \wedge d\gamma - \gamma^{-1}d\gamma\gamma^{-1} \wedge d\gamma \\
&\quad + \gamma^{-1}A\gamma \wedge \gamma^{-1}A\gamma + \gamma^{-1}A\gamma \wedge \gamma^{-1}d\gamma + \gamma^{-1}d\gamma \wedge \gamma^{-1}A\gamma + \gamma^{-1}d\gamma \wedge \gamma^{-1}d\gamma && (5.28)\\
&= \gamma^{-1}(dA + A \wedge A)\gamma = \gamma^{-1}F\gamma, && (5.29)
\end{aligned}$$

where $F = dA + A \wedge A$ is a $\mathfrak{g}$-valued 2-form on B. The curvature Ω does not have the vertical (fibre) component.

Writing $\Omega = (\gamma')^{-1} F' \gamma'$ gives

$$\boxed{F' = gFg^{-1}} \tag{5.30}$$

The base-components of both ω and Ω are related by gauge transformations (4.30).

Let $\gamma : U \to G$ be any local section, $\gamma = \gamma(x)$, where $x \in U \subset B$. This can be used to pull back ω and Ω from the total space E to the base space B.

$$A = \gamma^*(\omega), \qquad F = \gamma^*(\Omega). \tag{5.31}$$

The A and F defined this way are called the *gauge potential* and *gauge field* respectively. This reveals the bundle-theoretic origin of gauge transformations. In physics we only work with the base space and say that the gauge potential is only defined locally up to gauge transformation. We see now that this gauge ambiguity is an artefact of an ambiguity in the choice of section on the fibre structure. A gauge transformation on B is the same as a change of section of E. In particular, if the bundle E is non-trivial, then a global section does not exist, and the gauge potential can only be defined locally.

Given a connection ω, we can split the tangent bundle TE into vertical and horizontal sub-bundles

$$TE = H(E) \oplus V(E), \tag{5.32}$$

where $H(E), V(E)$ are defined as

$$H(E) = \{X \in TE \mid X \rightharpoonup \omega = 0\}. \tag{5.33}$$

A basis of $H(E)$ (in a trivialisation) is

$$H(E) = \mathrm{Span}\,\{D_a\}, \qquad D_a = \frac{\partial}{\partial x^a} - A_a{}^\alpha R_\alpha, \qquad \begin{array}{l} a = 1, \ldots, \dim B \\ \alpha = 1, \ldots, \dim \mathfrak{g} \end{array} \tag{5.34}$$

where x^a are local coordinates on U and R_α are right-invariant vector fields on G with

$$[R_\alpha, R_\beta] = -f_{\alpha\beta}{}^\gamma R_\gamma. \tag{5.35}$$

The curvature is an obstruction to integrability in $H(E)$.

Exercise 5.1: Compute the Lie bracket

$$[D_a, D_b] = -F_{ab}{}^\alpha R_\alpha. \tag{5.36}$$

(Use $[T_\alpha, T_\beta] = f_{\alpha\beta}{}^\gamma T_\gamma$ and the component form of the pull-back.) The right-hand side belongs to the vertical component.

Other texts define the connection from the splitting above and then recover the 1-form from it.