String Theory

Part III Lent 2019

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Contents

Admin Stuff

Books and Lecture Notes

- String Theory Vol. 1 — Polchinski, CUP

  Probably fits this course most closely, although there will be some things that will be in this course that are not in the book and vice-versa.

- Superstrings Vol. 1 — Green et. al, CUP
- A String Theory Primer — Schomerus, CUP

Online Resources

- Online lecture notes by the professor.
- David Tong's notes on arXiv:0908.0333
- 'Why string theory?' Conlon (CRC) for some light reading and history

1 Introduction

1.1 What is String Theory?

We do not actually know. This question still needs fleshing out to be answered well. Most likely, the actual string theory that we are working towards will be very different to the content that we will cover in these lectures.

In some sense, string theory is an attempt at quantising the gravitational field.

Naive quantisation of the Einstein-Hilbert action presents a number of problems.

Conceptual Problems

- The nature of time: Non-relativistic quantum mechanics is based on the Hamiltonian formulation. This is not necessarily a technical problem.
- How to quantise without a pre-existing causal structure?

  One of the first things we learn in QM, when going to QFT, is that it is important to know whether two operators are timelike or spacelike separated. We have a notion that all operators that are spacelike separated commute. However, if we are talking about general relativity, the metric contains information about and determines the causal structure. But this is what we want to quantise. So it is not immediately obvious what the algebra of operators should look like.
- The symmetry of general relativity is diffeomorphism invariance (coordinate reparametrisations).

  This is a gauge symmetry. One thing we will discuss (although not prove) when talking about scattering amplitudes in string theory, the fact is that there are no *local* diffeomorphism-invariant observables. As such, it is not even obvious what the quantum observables should be.

We will not be able to answer these deeper conceptual questions within the framework of string theory.

But perhaps more importantly, there are technical obstacles. You might ask, are there any assumptions that allow us to make progress and return to the difficult conceptual problems later?

Technical obstacles

Let us look at perturbation theory.

In particle physics, we often expand $g_{\mu\nu}$ about some classical solution, e.g. Minkowski spacetime:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x). \tag{1.1}$$

This means that we can use the causal structure of the *background* classical metric $\eta_{\mu\nu}$ to talk about quantisation of the fluctuations $h_{\mu\nu}$. This is what we do in *Field Theory in Cosmology* for inflation. In spirit, this is very close to what we do in string theory.

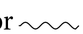
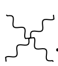
However, in some sense this split between background and perturbation is artificial and arbitrary. Nonetheless, we can take the Einstein–Hilbert action in general dimensions D

$$S[g] = \frac{1}{k_D}\int \mathrm{d}^D x\,\sqrt{-g}R(g) \tag{1.2}$$

and expand it out by choosing a gauge. Choosing a gauge wisely, we get an action

$$S[h] \approx \frac{1}{k_D}\int \mathrm{d}^D x\,\left(h_{\mu\nu}\Box h^{\mu\nu} + \dots\right). \tag{1.3}$$

Since the Ricci scalar involves inverse powers of the metric, this expansion will not terminate. We say that this expression is *non-polynomial* in $h_{\mu\nu}$.

The quadratic term gives us a propagator  and the interaction terms give us vertices  and . These are the Feynman rules.

However, loops give divergences, which cannot be dealt with using standard techniques (renormalisation, c.f. *Advanced Quantum Field Theory*).

String theory provides a way to do quantum perturbation theory of the gravitational 'field' (and much more).

String theory gives us a framework to ask meaningful questions concerning quantum gravity (although we are not able to answer all of them at the moment).

There are also other approaches out there, although string theory is most thoroughly studied / understood (although this might be because most resources in this area flow into string theory).

One of the possibilities is trying to make classical gravity consistent with the standard model without needing to quantise it.

There will be a sense in which we will learn quite a lot about QFT from studying string theory.

To a certain extend, the motivation that we will take throughout this course is not whether string theory will give us a theory of the real world, but rather whether it might give us a hint on how to reconcile quantum mechanics and gravity. We have been trying to do this for the better part of a hundred years, so any hint would be appreciated.

1.2 Worldsheets and Embeddings

Popular science books and textbooks alike usually start off a discussion of string theory by starting with the assumption that particles are not point-like, but rather tiny little strings.

The starting point is to consider a worldsheet Σ , a two-dimensional surface swept out by a string. This is analogous to a worldline swept out by a pointlike particle moving through spacetime, as illustrated in Fig. 1.1.

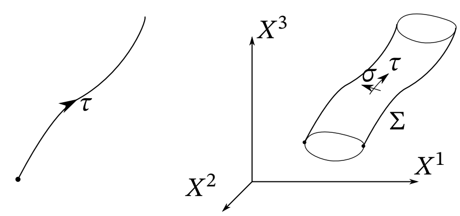


Figure 1.1: Particle worldline and string worldsheet

We put coordinates (σ, τ) on Σ (at least locally) and we define an embedding of Σ in the background spacetime M by the functions $X^\mu(\sigma, \tau)$, where the X^μ are coordinates on M, i.e. $X : \Sigma \to M$.

There are rules (which we shall investigate) for glueing such worldsheets together, in a way that is consistent with the symmetries of the theory.

We shall see that diagrams such as Fig. 1.2 are in one-to-one correspondence with correlation functions in some quantum theory. It is natural to interpret such diagrams as Feynman diagrams in a perturbative expansion of some theory about a vacuum.

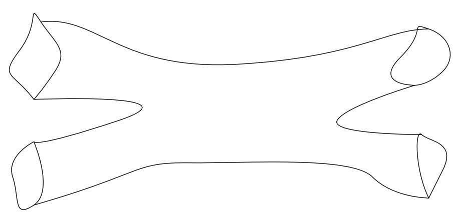


Figure 1.2

2 The Classical Particle and String

In this section we will start systematically constructing a theory of quantum strings. As a precursor to this discussion, we will need to talk a bit about non-relativistic quantum mechanics.

In non-relativistic QM, we treat time (t) as a parameter and position ($\hat{x}$) as an operator. Obviously, this kind of distinction should not survive in a relativistic theory. This means there are some choices to be made.

Second quantisation: Both x^i and t are considered parameters. They are not the fundamental objects we quantise. Instead, we quantise quantum fields $\phi(\mathbf{x}, t)$, which are the basic objects of our theory. We require that the fields transform in appropriate ways under Lorentz transformations. This is historically overwhelmingly the most useful way to do it, seeing its success in quantum field theory and the standard model.

First quantisation: Here we make the other choice: we elevate t to being an operator. This is the natural framework for describing the relativistic embedding of worldlines (-sheets, -volumes) into spacetime. Here, $X^\mu = (x^i, t)$ is an operator, the fundamental object we quantise, and there will be some other natural parameter entering the theory.

There is such a thing as string field theory, which employs the viewpoint of second quantisation. However, apart from a few exceptions, there is not much that you do not also obtain from first quantisation string theory. However, first quantisation string theory has made significant advances on problems that have not yet be solved with second quantisation.

2.0.1 Worldlines and Particles

We consider the embedding of a worldline $\mathcal{L}$ into spacetime M. The basic field is the embedding $X^\mu : \mathcal{L} \to M$. An action that describes this embedding might be

$$S[X] = -m \int_{x_2}^{x_1} \mathrm{d}s = -m \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \sqrt{-\eta^{\mu\nu} \dot{X}^\mu \dot{X}^\nu}, \tag{2.1}$$

where τ (a parameter) is the proper time and $X^\mu(\tau_2) = x_2^\mu$, $X^\mu(\tau_1) = x_1^\mu$ are end points of the world line.

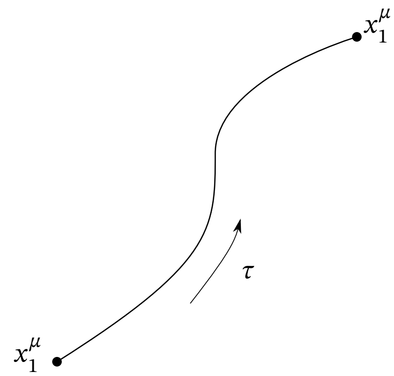


Figure 2.1

The momentum conjugate to $X^\mu(\tau)$ is $P_\mu(\tau) = -m\frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}}$. This satisfies $\dot{X}^2 = \eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu$. This satisfies $P^2 + m^2 = 0$ identically, an *on-shell condition*.

Symmetries

Rigid symmetry: $X^\mu(\tau) \to \Lambda^\mu{}_\nu X^\nu(\tau) - a^\mu$, where $\Lambda^\mu{}_\nu$ is a Lorentz transformation and a^μ is a (constant) displacement.

Reparameterisation invariance: $\tau \to \tau + \xi(\tau)$. The embedding X^μ changes as

$$X^\mu(\tau) \to X^\mu(\tau + \xi) = X^\mu(\tau) + \xi\dot{X}^\mu(\tau) + \dots. \tag{2.2}$$

To first order, $\delta X^\mu(\tau) = \xi\dot{X}^\mu(\tau)$.

There is a rewriting of this action (2.1) that makes life a bit easier. We do this by introducing a new auxiliary field $e(\tau)$, which is a one-form, on the worldline $\mathcal{L}$.

$$S[X, e] = \frac{1}{2}\int_{\mathcal{L}} \mathrm{d}\tau \left(e^{-1}\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu - em^2\right). \tag{2.3}$$

If you like, you can think of e as something like a one-dimensional metric, which sets a scale for distances on the line.

We will be able to do something analogous for strings, which will be very useful!

The equations of motion for $X^\mu(\tau)$ and $e(\tau)$ are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(e^{-1}\dot{X}^\mu\right) = 0. \tag{2.4}$$

However, the equation of motion for e will not depend on $\dot{e}$, but be purely algebraic:

$$\dot{X}_2 + e^2 m^2 = 0. \tag{2.5}$$

As such, e can be thought of as a Lagrange multiplier, enforcing a constraint. What is the constraint that it enforces?

The momentum conjugate to X^μ is

$$P_\mu = e^{-1}\dot{X}_\mu. \tag{2.6}$$

Combining (2.5) and (2.6), we can eliminate e to find precisely the on-shell condition $P^2 + m^2 = 0$.

> The auxiliary field e basically enforces energy-momentum conservation on the worldline.

Exercise 2.1: We can write $e^{-1} = m/\left|\dot{X}\right|$. Plug this into the action (2.3) to find that $S[X, e]$, subject to the equation of motion for $e(\tau)$ gives precisely the action (2.1).

There are two reasons why we should consider this new action instead. Firstly, in the $m \to 0$ limit, it will be easy to show that it describes null worldlines. Secondly, since there is no square root in this theory, it will be easier to quantise. This is at the cost of having to introduce a new non-dynamical field e.

The action $S[X, e]$ has the following symmetries:

- Poincaré invariance (where e is invariant)
- Reparameterisation invariance:

$$\delta X^\mu = \xi \dot{X}^\mu \qquad \delta e = \frac{\mathrm{d}}{\mathrm{d}\tau}(\xi e) \tag{2.7}$$

  provided these variations vanish on the endpoints.

> Note that this transformation law under diffeomorphisms shows that e needs to transform like a one-form.

> Locally, we can think of e as a pure gauge.

Remark: We could generalise $\eta_{\mu\nu} \to g_{\mu\nu}\big(X(\tau)\big)$. The model becomes highly non-linear. We will look at this later in the context of strings.

2.1 Classical Strings: The Nambu–Goto Action

Our starting point will be the *Nambu-Goto Action*. We use units where $\hbar = c = 1$ throughout. The

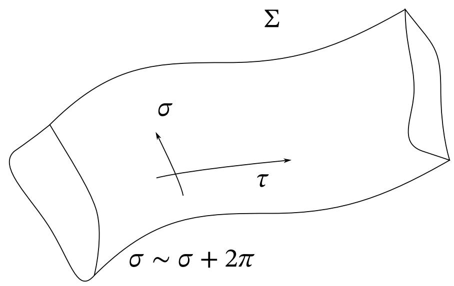


Figure 2.2

fundamental degree of freedom is

$$X : \sigma \to M, \tag{2.8}$$

where M is called the *target space*.

Definition 1 (Nambu–Goto Action)**:** The Nambu–Goto action is

$$S[X] = -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}\tau\,\mathrm{d}\sigma \sqrt{-\det\left(\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\right)}, \tag{2.9}$$

where α' is a constant with dimensions of (spacetime) area and $\sigma^a = (\tau, \sigma)$ and $\partial_a = \frac{\partial}{\partial\sigma^a}$.

Remark: One often speaks of the *string length* $l_2 = 2\pi\sqrt{\alpha'}$.

Definition 2 (string tension)**:** We introduce the *string tension* $T = \frac{1}{2\pi\alpha'}$.

> This turns out to be a good starting point; quantising this puts us on the right track. However, it is horrendously difficult to actually perform this quantisation. Instead, we will consider another action.

Definition 3 (Polyakov Action)**:** The Polyakov action is

$$S[X, h] = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h} h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu. \tag{2.10}$$

The new feature as compared to the Nambu–Goto action is the new field h_{ab}, which is a metric on Σ. It is non-dynamical; there are no Einstein-Hilbert terms and similarly to e previously it can be thought of as enforcing a constraint. We will find that h_{ab} is extremely important.

> This form is quite suggestive. All this is, when quantised, is a two-dimensional massless Klein–Gordon field in three dimensions. It is difficult to find an easier theory than this.

Let us look at the equations of motion for the Polyakov action.

Exercise 2.2: If we vary the worldsheet metric $h_{ab} \to h_{ab} + \delta h_{ab}$, then the action changes as

$$\delta S = -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h} T_{ab} \delta h^{ab}, \tag{2.11}$$

with *stress tensor* $T_{ab} = \frac{1}{\alpha'} \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab} h^{cd} \partial_c X^\mu \partial_d X_\mu$.

> **Remark:** Indices are raised and lowered with the Minkowski metric.

So the h_{ab} equation of motion is

$$\boxed{T_{ab} = 0} \tag{2.12}$$

Remark:

- The trace vanishes $h^{ab} T^{ab} = 0$, because $h_{ab} h^{ab} = 2$.
- Symmetric $T_{ab} = T_{ba}$ because $h_{ab} = h_{ba}$.
- The X^μ equation of motion is

$$\frac{1}{\sqrt{-h}} \partial_a \left(\sqrt{-h} h^{ab} \partial_b X^\mu \right) = 0. \tag{2.13}$$

If the $h_{ab} = \mathrm{diag}(-1, 1)$, then this is the wave equation.

2.2 Classical Equivalence of Polyakov and Nambu–Goto

It is useful to define $G_{ab} = \eta_\mu \partial_a X^\mu \partial_b X^\nu$. The Nambu–Goto action is then

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-\det(G_{ab})}. \tag{2.14}$$

By imposing the metric's equation of motion, the vanishing of T_{ab} tells us that

$$G_{ab} - \frac{1}{2} h_{ab} \underbrace{G_{cd} h^{cd}}_{G := \mathrm{tr}(G_{ab})} = 0. \tag{2.15}$$

The determinant is

$$\det(G_{ab}) = \frac{1}{4} G^2 \det(h_{ab}) = \frac{1}{4} G^2 h. \tag{2.16}$$

So $\sqrt{-h} G = 2\sqrt{-\det G_{ab}}$ and

$$\frac{1}{2} \sqrt{-h} h^{cd} \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \sqrt{-\det(G_{ab})}. \tag{2.17}$$

Remark: This is because h_{ab} is not appearing dynamically in the action (no derivatives), so its equation of motion is just a constraint.

2.3 The Polyakov Action

$$S[X,h] = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}h^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu.$$

Can we generalise this action?

- We could replace $\eta_{\mu\nu}$ with a general $g_{\mu\nu}(x)$ (more later).
- What about a 2-D Einstein–Hilbert term, to make h_{ab} dynamical?

  $$\frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}\,R(h) = \chi. \tag{2.18}$$

  In 2-D, this is a topological invariant: the Euler characteristic χ of the worldsheet.
- What about a cosmological constant on Σ?

  $$\Lambda\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h} \tag{2.19}$$

  The equations of motion for h_{ab} will be of the form

  $$T_{ab} \propto \Lambda h_{ab}. \tag{2.20}$$

  However, we know that T_{ab} is traceless. We therefore have $h^{ab}T_{ab} \propto 2\Lambda$, so we need $\Lambda = 0$.
- We could include background fields in the spacetime.

  For example, there could be a 2-form field $B(X) = \frac{1}{2}B_{\mu\nu}dX^\mu \wedge dX^\nu$. We could include the term

  $$-\frac{1}{2\pi\alpha'}\int_\Sigma B = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}\,\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}. \tag{2.21}$$

  Or we might have a scalar field $\phi(X)$. The sort of term we might have is something like

  $$\frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}\,\phi(X)R(h). \tag{2.22}$$

2.3.1 Symmetries

- Poincaré invariance: $X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu$, where $\Lambda^\mu{}_\nu$ and a^μ are independent of (σ,τ). (These are rigid / global symmetries, as opposed to gauge / local symmetries). The metric $h^{ab} \to h^{ab}$ does not change.

- Diffeomorphism invariance.

  Infinitesimally: $\sigma^a \to \sigma^a + \xi^a$. Then

$$\delta X^\mu = \xi^a \partial_a X^\mu \tag{2.23}$$

$$\delta h_{ab} = \xi^c \partial_c h_{ab} + (\partial_a \xi^c) h_{bc} + (\partial_b \xi^c) h_{ac}. \tag{2.24}$$

- Weyl invariance: $X^\mu \to X^\mu$ does not change. But the worldsheet is rescaled by some position dependent factor $h_{ab} \to e^{2\Lambda(\sigma,\tau)} h_{ab}$.

  Infinitesimally:

$$\delta X^\mu = 0, \qquad \delta h_{ab} = 2\Lambda h_{ab}. \tag{2.25}$$

2.3.2 Classical Solutions

The two-dimensional metric h_{ab} has three degrees of freedom. We can use the diffeomorphism invariance to fix two of the degrees of freedom in h_{ab} (at least locally) and write it as

$$h_{ab} = e^{2\Phi} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.26}$$

Moreover, Weyl invariance means that the factor $e^{2\phi}$ drops out, eliminating the final degree of freedom.

The action then becomes

$$S[X] = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \left(-\dot{X}^2 + (X')^2\right), \qquad \dot{X}^\mu := \frac{\partial X^\mu}{\partial \tau}, \quad (X')^\mu := \frac{\partial X^\mu}{\partial \sigma}. \tag{2.27}$$

The equation of motion is

$$\Box X^\mu = 0 \qquad \Box = -\partial_\tau^2 + \partial_\sigma^2. \tag{2.28}$$

Solutions are split between left- and right-movers.

$$X^\mu(\sigma,\tau) = X_R^\mu(\tau+\sigma) + X_L^\mu(\tau+\sigma). \tag{2.29}$$

Introduce the Fourier modes α_n^μ and $\overline{\alpha}_n^\mu$ (which are not complex conjugate to each other, but it will be useful to think of them as if they were). The solutions then become

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in(\tau-\sigma)} \tag{2.30}$$

$$X_R^\mu(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2}p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in(\tau+\sigma)}. \tag{2.31}$$

The x^μ and p^μ are the centre of mass position and momentum of the string in spacetime.

Notation: We also introduce the notation

$$\alpha_0^\mu = \overline{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu \tag{2.32}$$

X^μ is real, so we also require that $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$.

2.4 Classical Hamiltonian Dynamics of the String

We continue to work in conformal gauge, where $h_{ab} = e^{\phi}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Definition 4 (canonical momentum)**:** We define the *canonical momentum field*, conjugate to X^μ as the functional derivative

$$P_\mu(\sigma,\tau) = \frac{\delta S(x)}{\delta X^\mu(\sigma,\tau)} = \frac{1}{2\pi\alpha'}\dot{X}_\mu. \tag{2.33}$$

Definition 5: Given the Lagrangian density $\mathscr{L}$, the Hamiltonian density is

$$\mathscr{H} = P_\mu \dot{X}^\mu - \mathscr{L} = \frac{1}{4\pi\alpha'}(\dot{X}^2 + X'^2). \tag{2.34}$$

Definition 6 (Poisson brackets)**:** We introduce Poisson brackets $\{\cdot,\cdot\}_{\text{PB}}$. For a particle theory, where our coordinates $x^\mu(\tau)$ and momenta $p_\mu(\tau)$ are our fundamental variables, it is useful to define

$$\{f,g\}_{\text{PB}} = \frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial x^\mu}. \tag{2.35}$$

Example: We have for example, $\{x^\mu, p_\nu\} = \delta^\mu_\nu$.

The Hamiltonian $\mathscr{H}$ is the generator of time translations

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \frac{\partial f}{\partial \tau} + \{f,\mathscr{H}\}. \tag{2.36}$$

Our field theoretic generalisation requires

$$\{X^\mu(\sigma,\tau), P_\mu(\sigma',\tau)\} = \delta^\mu_\nu \delta(\sigma-\sigma'). \tag{2.37}$$

Recall that the $X^\mu(\sigma,\tau)$ can be written in terms of Fourier modes α^μ_n and $\overline{\alpha}^\mu_n$, where $\sigma \sim \sigma + 2\pi$ is periodic, which is the reason why n takes on discrete values.

Claim 1: The Poisson bracket relationship between X^μ and P_μ requires

$$\{\alpha^\mu_m, \alpha^\nu_n\}_{\text{PB}} = -im\eta^{\mu\nu}\delta_{m+n,0} \tag{2.38a}$$

$$\{\overline{\alpha}^\mu_m, \overline{\alpha}^\nu_n\}_{\text{PB}} = -im\eta^{\mu\nu}\delta_{m+n,0} \tag{2.38b}$$

$$\{\alpha^\mu_m, \overline{\alpha}^\nu_n\}_{\text{PB}} = 0 \tag{2.38c}$$

Proof. Without loss of generality, let us check this at $\tau = 0$:

$$X^\mu(\sigma) = x^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^\mu e^{+in\sigma} + \overline{\alpha}_n^\mu e^{-in\sigma}\right), \tag{2.39}$$

$$P^\mu(\sigma) = \frac{p^\mu}{2\pi} + \frac{1}{2\pi}\frac{1}{\sqrt{2\alpha'}} \sum_{n\neq 0}\left(\alpha_n^\mu e^{+in\sigma} + \overline{\alpha}_n^\mu e^{-in\sigma}\right). \tag{2.40}$$

$$\tag{2.41}$$

The Poisson bracket is

$$\left\{X^\mu(\sigma), P^\nu(\sigma')\right\}_{\text{PB}} = \frac{1}{2\pi}\left\{x^\mu, p^\nu\right\}_{\text{PB}} + \frac{i}{4\pi}\sum_{m,n\neq 0}\frac{1}{m}\left(\left\{\alpha_m^\mu, \alpha_n^\nu\right\}_{\text{PB}} e^{i(m\sigma + n\sigma')} + \left\{\overline{\alpha}_m^\mu, \overline{\alpha}_n^\nu\right\}_{\text{PB}} e^{-i(m\sigma+n\sigma')}\right) \tag{2.42}$$

$$= \frac{\eta^{\mu\nu}}{2\pi} + \frac{\eta^{\mu\nu}}{2\pi}\sum_{n\neq 0} e^{in(\sigma-\sigma')} = \frac{\eta^{\mu\nu}}{2\pi}\sum_n e^{in(\sigma-\sigma')}, \tag{2.43}$$

but $\frac{1}{2\pi}\sum_n e^{in(\sigma-\sigma')}$ is just the periodic version of the Dirac δ-function, so

$$\left\{X^\mu(\sigma), P^\nu(\sigma')\right\}_{\text{PB}} = \eta^{\mu\nu}\delta(\sigma - \sigma'). \tag{2.44}$$

□

2.5 The Stress Tensor and Witt Algebra

Let us introduce (worldsheet) light-cone coordinates $\sigma^\pm = \tau \pm \sigma$. In these coordinates, the worldsheet metric looks like $e^\phi \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ and $\partial_\pm = \frac{\partial}{\partial\sigma^\pm}$. The action and equations of motion become

$$S = -\frac{1}{2\pi\alpha'}\int \mathrm{d}\sigma^+\,\mathrm{d}\sigma^-\,\partial_+ X\cdot\partial_- X, \qquad \partial_+\partial_- X^\mu = 0. \tag{2.45}$$

The stress tensor T_{ab} is

$$T_{++} = -\frac{1}{\alpha'}\partial_+ X\cdot\partial_+ X, \qquad T_{--} = -\frac{1}{\alpha'}\partial_- X\cdot\partial_- X, \qquad \underbrace{T_{+-} = T_{-+} = 0}_{\text{effectively trace of } T_{ab}}. \tag{2.46}$$

The constraint is $T_{\pm\pm} = 0$. It is useful to introduce the Fourier modes of $T_{\pm\pm}$. We define (at $\tau = 0$) the charges

$$L_n = -\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma\, T_{--}(\sigma)e^{-in\sigma} \qquad \overline{L}_n = -\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma\, T_{++}(\sigma)e^{+in\sigma}. \tag{2.47}$$

Recall that

$$\partial_- X^\mu(\sigma^-) = \sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n e^{-in\sigma^-}, \tag{2.48}$$

where $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$. We find

$$L_n = \frac{1}{2\pi\alpha'}\int_0^{2\pi} \mathrm{d}\sigma\, \partial_- X^\mu(\sigma)\partial_- X_\mu(\sigma) \tag{2.49}$$

$$= \frac{1}{4\pi}\sum_{m,p}\alpha_m\cdot\alpha_p\int_0^{2\pi}\mathrm{d}\sigma\, e^{-i(m+p-n)\sigma} \tag{2.50}$$

$$= \frac{1}{4\pi}\sum_{m,p}\alpha_m\cdot\alpha_p 2\pi\delta_{p,n-m}, \tag{2.51}$$

and similarly for $\overline{L}_n$. We have

$$\boxed{L_n = \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m, \qquad \overline{L}_n = \frac{1}{2}\sum_m \overline{\alpha}_{n-m}\cdot\overline{\alpha}_m,} \tag{2.52}$$

The constraint can be written as $L_n = 0 = \overline{L}_n$. Using the algebra (2.38) for the α_n^μ ($\overline{\alpha}_n^\mu$), we can compute the algebra for the L_n ($\overline{L}_n$) to be

$$\left\{L_m, L_n\right\}_{\mathrm{PB}} = -i(m-n)L_{m+n}, \tag{2.53a}$$

$$\left\{\overline{L}_m, \overline{L}_n\right\}_{\mathrm{PB}} = -i(m-n)\overline{L}_{m+n}, \tag{2.53b}$$

$$\left\{L_m, \overline{L}_n\right\}_{\mathrm{PB}} = 0. \tag{2.53c}$$

This is called the *Witt algebra*. We will see that if we set $L_n = 0 = \overline{L}_n$ at a given τ, then the evolution of the system preserves $L_n = 0 = \overline{L}_n$.