Black Holes

Part III Lent 2019

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Contents

Administrative

- Office hours: Fridays 2pm, B2.09
- Lecture notes: www.damtp.cam.ac.uk/user/hsr1000 and /examples
  - everything in notes is examinable
- Conventions: $G = c = 1$, ignore Λ (negligible for Black holes)
- indices: $\mu, \nu, \ldots$ refer to *specific* basis,

  $a, b, c, \ldots$ 'abstract indices' (Penrose) refer to *any* basis

$$\text{e.g.} \qquad \Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}(g^{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}) \qquad R = g^{ab} R_{ab} \tag{1}$$

- Books listed in lecture notes (Wald etc)

1 Spherical Stars

1.1 Cold stars

Gravitational force, which wants the star to contract, is balanced by pressure of nuclear reactions. If we wait long enough, star will exhaust nuclear fuel and the star will contract. What happens next? Any new source of pressure will have to be non-thermal, since time will cause the star to cool down. There is one such source of pressure coming from the Pauli principle. If you have a gas of fermions, it will resist compression. This is called 'degeneracy pressure'. This is entirely a quantum effect, which is not thermal.

Definition 1: A *white dwarf* is a star in which gravity is balanced by electron degeneracy pressure.

This is a very dense star: a white dwarf with the same mass as our sun, $M = M_\odot$ has a radius $R \sim \frac{1}{100} R_\odot$.

However, not all stars can end their life this way. The maximum mass of a white dwarf is the Chandrasekhar limit $M_{wd} \leq 1.4 M_\odot$.

If matter is sufficiently dense, we have inverse β-decay, which turns the protons in the star into neutrons. We therefore get a second class of star:

Definition 2: A *neutron dwarf* is a star in which gravity is balanced by neutron degeneracy pressure.

These are tiny: taking a neutron star with $M \sim M_\odot$, then $R \sim 10$ km. Compare this with the radius of our sun, which is $R_\odot \simeq 7 \times 10^5$ km. Because they are so dense, their gravitational force on the surface is very strong. In terms of Newtonian gravity, we have $|\Phi| \sim 0.1$ at the surface. General relativity becomes negligible if $|\Phi| \ll 1$. So here, general relativity is important.

We will show that for any cold star there is a maximal mass around five solar masses. This bound will be independent of our ignorance of the properties of matter at such high densities.

In order to make this problem tractable, we will assume that the star is spherically symmetric and time independent.

1.2 Spherical Symmetry

Definition 3: The *unit round metric* on S^2 is $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$.

Roughly speaking, spherical symmetry is the isometry group of this metric. The isometry group in this case is $SO(3)$.

Definition 4: A spacetime is *spherically symmetric* if its isometry group contains an $SO(3)$ subgroup, whose orbits are 2-spheres.

> Pick a point and act on it with all $SO(3)$ elements. It will then fill out a sphere with unit round metric.

Definition 5: In a spherically symmetric spacetime (M, g) , the *area radius function* is

$$\begin{aligned} r : \; M &\to \mathbb{R} \\ p &\longmapsto r(p) = \sqrt{\frac{A(p)}{4\pi}} \end{aligned} \tag{1.1}$$

where $A(p)$ is the area of the S^2 orbit through p.

> You can think of r as the radial coordinate. Instead of defining r in terms of distance from the origin (which does not exist on S^2), we define it here via the area.

Remark: The S^2 has induced metric $r(p)^2 d\Omega^2$.

1.3 Time-independence

Definition 6 (stationary)**:** The spacetime (M, g) is *stationary* if there exists a Killing vector field (KVF) k^a, which is everywhere timelike ($g_{ab}k^a k^b < 0$).

> Our spacetime has a time-translation symmetry.

Pick some hypersurface Σ transverse to k^a . We can then pick coordinates x^i , $i = 1, 2, 3$ on Σ.

We assign coordinates (t, x^i) to point parameter distance t along an integral curve k^a through a point on Σ with coordinates x^i . This implies that $k = \partial/\partial t$, implying that the metric in independent of t (since k^a is Killing).

$$ds^2 = g_{00}(x^k)dt^2 + 2g_{0i}(x^k)dt dx^i + g_{ij}(x^k)dx^i dx^j \tag{1.2}$$

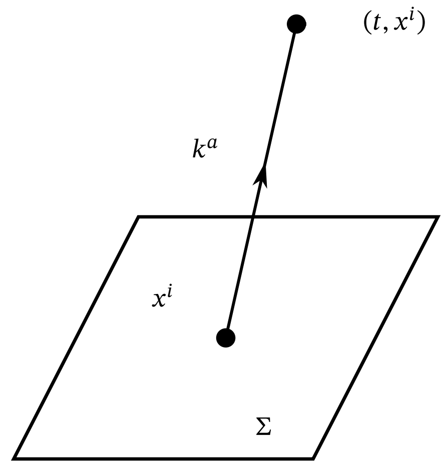

Figure 1.1

with $g_{00} < 0$. Conversely, any metric of this form is stationary.

This is the weakest notion of time-independence we can use. There is also a more refined notion. Before we can introduce that, we need to talk about hypersurface orthogonality.

Claim 1: Let Σ be a hypersurface of constant $f = 0$ on Σ where $f : M \to \mathbb{R}$ a smooth function where $df \neq 0$ on Σ. Then df is normal to Σ.

Proof. Let t^a be a vector that is tangent to Σ. Then

$$df(t) = t(f) = t^\mu \partial_\mu f = 0 \tag{1.3}$$

since f is constant on Σ. □

> Normals to a surface are not unique. For example, we can rescale f to get another normal on Σ. In fact, we can also add something that vanishes on Σ.

Claim 2: If n is also normal to Σ, then $n = gdf + fn'$, where $g \neq 0$ on Σ and n' is a smooth 1-form.

Proof. By the rules of the exterior derivative,

$$dn = dg \wedge df + df \wedge n' + fdn' \tag{1.4}$$

Evaluating this on Σ gives

$$dn|_\Sigma = (dg - n') \wedge df \;\Rightarrow\; n \wedge dn|_\Sigma = 0, \tag{1.5}$$

as $n \propto df$ on Σ. □

This is very useful since there is also a converse of this statement:

Theorem 1 (Frobenius)**:** If $n \neq 0$ is a 1-form such that $n \wedge dn \equiv 0$, then $\exists$ functions g, f such that $n = gdf$. So n is normal to surfaces of constant f. We say that n is *hypersurface orthogonal*.

Definition 7 (static)**:** A spacetime (M, g) is *static* if there is a hypersurface-orthogonal timelike Killing vector field.

Remark: This is a refinement since static $\Rightarrow$ stationary.

By Frobenius' theorem, we can choose $\Sigma \perp k^a$ when defining (t, x^i) (since k^a is hypersurface-orthogonal). But Σ is $t = 0$, normal to Σ is dt. Therefore, $k_\mu\big|_{t=0} \propto (1,0,0,0)$. In particular, the spatial components in these coordinates are $k_i\big|_{t=0} = 0$, but $k_i = g_{0i}(x^k)$. Therefore, $g_{0i}(x^k) = 0$. In a static spacetime, the off-diagonal elements of the metric are zero.

$$ds^2 = g_{00}(x^i)dt^2 + g_{ij}(x^k)dx^i dx^j \qquad (g_{00} < 0.) \tag{1.6}$$

There is now an additional symmetry present. We have a discrete time-reversal symmetry $(t, x^i) \rightarrow (-t, x^i)$.

Roughly speaking, static means 'time-independent and invariant under time-reversal'.

Example: A rotating star can be stationary, but not static.

> Static means non-rotating.

1.4 Static, spherically symmetric spacetimes

Let us talk about a spherical, non-rotating star. More formally, we will assume the isometry group $\mathbb{R} \times SO(3)$.

> $SO(3)$ are the spatial rotations. $\mathbb{R}$ are the time-translations associated to the timelike Killing vector field k^a.

Claim 3: This implies that the spacetime is static (rotation breaks spherical symmetry).

On Σ, choose coordinates $x^i = (r, \theta, \varphi)$, where r is defined via the area-radius. A consequence of the spherical symmetry is that the metric must take the following form on Σ

$$ds^2\big|_\Sigma = e^{2\Psi(r)}dr^2 + r^2 d\Omega^2 \tag{1.7}$$

(this is because $drd\theta$ or $drd\varphi$ break spherical symmetry.)

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Psi(r)}dt^2 + r^2 d\Omega^2. \tag{1.8}$$

The choice of g_{00} is inspired by the Newtonian limit.

At the moment there is no origin. There is no reason to think of r as the distance to the origin. In fact, it is not the distance to the origin.

For a static, spherically symmetric star, we use the metric (1.8). To find Φ and Ψ, we need to solve the Einstein equations. In order to find those, we need to determine what matter the star contains.

We will model the matter inside the star as a perfect fluid with energy-momentum tensor

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}, \tag{1.9}$$

where u_a is the velocity of the fluid, obeying $g_{ab}u^a u^b = -1$. The quantities ρ and p are, respectively, the energy density and the pressure in the fluid's rest frame.

Time-independence implies that $u^a = e^{-\Phi}(\frac{\partial}{\partial t})^a$. The velocity is fixed by the symmetry assumptions, which also imply that $\rho = \rho(r)$ and $p = p(r)$ can only be functions of r. Also $\rho, p = 0$ for $r > R$, where R is the radius of the star.

1.5 Tolman–Oppenheimber–Volkoff Equations

Solving the Einstein equation imposes the fluid equations, so we do not separately need to deal with those. However, in what follows, it will actually be slightly easier to derive one of the following equations by using the fluid equation $\nabla_\mu T^{\mu\nu} = 0$ instead of some components of the Einstein equations.

By symmetry, there are only really 3 equations to solve. To write these down more concisely, we define $m(r)$ by the relation

$$e^{2\Psi(r)} = (1 - \frac{2m(r)}{r})^{-1} \overset{\text{LHS}>0}{\Rightarrow} m(r) < r/2. \tag{1.10}$$

Exercise 1.1 (Sheet 1)**:** Using the $(\mu\nu)$ component of the Einstein equation, we can derive

$$(tt): \qquad \frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho \tag{TOV 1}$$

$$(rr): \qquad \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{m + 4\pi r^3 \rho}{r(r-2m)} \tag{TOV 2}$$

$$\nabla_\mu T^{\mu\nu} = 0 \qquad \frac{\mathrm{d}p}{\mathrm{d}r} = -(p+\rho)\frac{m + 4\pi r^3 p}{r(r-2m)} \tag{TOV 3}$$

where instead of using a third Einstein equation, it is easiest to use $\nabla_\mu T^{\mu\nu} = 0$ to derive (TOV 3).

We have three equations, but four unknowns (m, Φ, ρ, π) . However, luckily we have some extra information coming from *thermodynamics*.

A cold star has $T = 0$ but $T = T(\rho, p)$, so $T = 0$ fixes some relation $p = p(\rho)$. This is known as a "barotopic equation of state".

We will not need much information about this relation. However, we will assume that $\rho, p > 0$ and $\frac{\mathrm{d}p}{\mathrm{d}\rho} > 0$.

> Else, we have an unstable fluid: an increase in density $\delta\rho > 0$ would cause a decrease in pressure $\delta p < 0$, which causes more fluid to flow into a given volume, causing in turn an even bigger increase in density.

1.6 Outside a Star: Schwarzschild Solution

Outside the star, at $r > R$, we have no matter and therefore $\rho = p = 0$.

From (TOV 1), we then find that $m(r) = M$ is a constant. One can then integrate (TOV 2) to find that $\Phi(r) = \frac{1}{2}\ln\left(1 - \frac{2M}{r}\right) + \Phi_0$, where Φ_0 is some constant of integration.

However, Φ_0 is not physical: as $r \to \infty$, $\Phi(r) \to \Phi_0$, so $g_{tt} \to e^{-2\Phi_0}$ as $r \to \infty$. This means that we can eliminate Φ_0 by absorbing it into the time coordinate via the coordinate transform $A' = e^{\Phi_0}t$. Without loss of generality, we may therefore set $\Phi_0 = 0$. The resulting metric is the *Schwarzschild solution*

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{1.11}$$

We interpret M to be the mass of the star.

At $r = 2M$, the "Schwarzschild radius", the metric components $g_{\mu\nu}$ (in a coordinate basis) are singular. Since in our derivation every step was sound, this singularity must be inside the star, where the metric is not valid. The star must have

$$R > 2M. \tag{1.12}$$

Remark: To get from GR to Newtonian physics, we take the limit of $c \to \infty$. The inequality

$$R > 2M \quad \xleftrightarrow[\text{units}]{\text{reinstate}} \quad \frac{GM}{c^2R} < \frac{1}{2} \tag{1.13}$$

then becomes trivial, meaning that there is no Newtonian analogue of this new GR effect.

> **Remark:** This is not true for black holes: they violate the assumption of static spacetime.

This inequality is certainly true for the sun, which has a Schwarzschild radius of $2M_\odot \approx 3km$ and a radius of $R_\odot \approx 7 \times 10^5$ km.

1.7 Interior Solution

From (TOV 1), we have that

$$m(r) = 4\pi \int_0^r \rho(r'){r'}^2 \,\mathrm{d}r' + m_*, \tag{1.14}$$

where m_* is some integration constant.

Let Σ_t denote a surface of constant time t. The metric induced on such a surface is

$$ds^2\big|_{\Sigma_t} = e^{2\Psi(r)}dr^2 + r^2 d\Omega^2. \tag{1.15}$$

We want the metric to be smooth at $r = 0$. This implies that the spacetime is locally flat at $r = 0$. For small r , this spacetime will look like Euclidean space $\mathbb{R}^3$.

As such, a point on S^2 of small radius r must be a distance r from the origin $r = 0$ (since this is true in $\mathbb{E}^3$). For small r we have

$$\therefore \quad r \approx \int_0^r e^{\Phi(r')}\,\mathrm{d}r' \approx e^{\Phi(0)} r \implies \Phi(0) = 0 \tag{1.16}$$

Else, there is some kind of singularity at the origin; the origin would not be smooth.

This means that $m(0) = 0 \implies m_* = 0$ in Eq. (1.14).

This was outside R. Continuity tells us that

$$m(r) = M = 4\pi \int_0^R \rho(r) r^2 \,\mathrm{d}r\,. \tag{1.17}$$

The fact that this is the same as in Newtonian physics is a coincidence; this is not in general true for general relativity.

More specifically, in general relativity, the total energy is obtained by integrating the energy density $\rho(r)$ over the appropriate volume form. On Σ_t, this is

$$e^{\Psi(r)} \underbrace{r^2 \sin\theta dr \wedge d\theta \wedge d\varphi}_{\text{usual volume form on } \mathbb{E}^3}. \tag{1.18}$$

The energy of matter on Σ_t is then

$$E = 4\pi \int_0^R e^{\Psi(r)} \rho(r) r^2 \,\mathrm{d}r\,. \tag{1.19}$$

Since $m > 0$, we find that $e^\Psi > 1$. Therefore $E > M$: the energy of the matter in the star is larger than the total energy of the star. This means that there is some gravitational binding energy $E - M$.

Finally, reduces to (TOV 3) $\frac{\mathrm{d}p}{\mathrm{d}r} < 0$. Together with the previously mentioned assumption that $\frac{\mathrm{d}p}{\mathrm{d}\rho} > 0$, this implies that $\frac{\mathrm{d}\rho}{\mathrm{d}r} < 0$.

Exercise 1.2 (Sheet 1)**:** One can then show that

$$\frac{m(r)}{r} < \frac{2}{9}\left[1 - 6\pi r^2 p(r) + (1 + 6\pi r^2 p(r))^{1/2}\right] \tag{1.20}$$

At $r = R$, the surface of the star, the pressure vanishes $p = 0$. This then reduces to the "Buchdahl inequality"

$$R > \frac{9}{4}M, \tag{1.21}$$

which is an improvement on Eq. (1.12).

Now (TOV 1) and (TOV 3) are coupled ordinary differential equations for m and ρ (via $p = p(\rho)$). These can be solved numerically given initial conditions. Eq. (1.14) automatically implies $m(0) = 0$, so we only need to specify $\rho(0) = \rho_c$.

In particular, (TOV 3) implies that the pressure p decreases as we move out towards higher r. We define the radius R by $p(R) = 0$ giving us $R = R(\rho_c)$. Similarly, once we have done this Eq. (1.17) fixes $M = M(\rho_c)$. Finally, we fix Φ by solving (TOV 2) in $r < R$ with initial condition

$$\Phi(R) = \frac{1}{2}\ln\left(1 - \frac{2M}{R}\right). \tag{1.22}$$

As such, for a given equation of state, cold stars form a one-parameter family labelled uniquely by the energy density ρ_c at the center of the star.

1.8 Maximum Mass of Cold Star

The maximum mass M_{max} depends on the equation of state.

> In particular, choosing the density of state for the degenerate electron gas gives the Chandrasekhar limit!

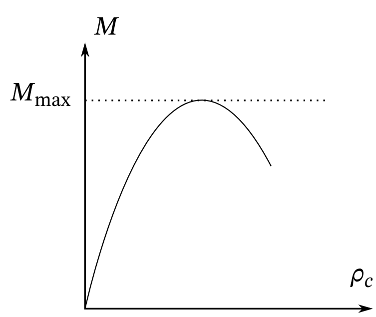

Figure 1.2

Experimentally, we can only know the equation of state up to nuclear density ρ_0 .

Claim 4: The maximum mass is always $M_{\max} \lesssim 5M_{\odot}$ whatever happens for $\rho > \rho_0$.

Proof. We know that ρ decreases with r. Let us now define two regions:

core region where $\rho > \rho_0$ ($r < r_0$)

envelope region where $\rho < \rho_0$ ($r_0 < r < R$)

We then define the 'core mass' to be $M_0 := m(r_0)$. Then Eq. (1.14) gives

$$M_0 > \frac{4}{3}\pi r_0^3 \rho_0. \tag{1.23}$$

The core mass has a higher density than nuclear density ρ_0.

On the other hand, eq. (1.20) for $r = r_0$ gives that $\frac{m_0}{r_0} < \frac{2}{9}\left[1 - 6\pi r_0^2 p_0 + (1 + 6\pi r_0^2 p_0)^{\frac{1}{2}}\right]$, but we know the quantity $p_0 = p(r_0)$ from the equation of state. Now the right-hand side of this is a decreasing function of p_0 , so to simplify, we can evaluate this at $p_0 = 0$. This then gives the *Bookdahl bound*

$$m_0 < \frac{4}{9} r_0, \tag{1.24}$$

which is satisfied by the core alone. We can of course get a sharper inequality by not restricting to $p_0 = 0$, but this is not needed here. Now the intersection of (1.23) and (1.24), as illustrated in

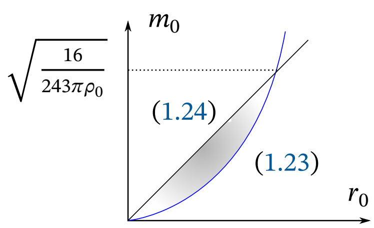

Figure 1.3

Fig. 1.3, turns out to be

$$m_0 < \sqrt{\frac{16}{243\pi\rho_0}} \simeq 5M_{\odot}, \tag{1.25}$$

where in evaluating this last expression we used the nuclear density ρ_0.

For any (m_0, r_0) in the allowed region, we solve (TOV 1) and (TOV 3) in the envelope region with $\rho = \rho_0$, $m = m_0$ at $r = r_0$. This fixes M in terms of (m_0, r_0). Numerically, we find that the total mass M is maximised when the core mass m_0 is maximised. At this maximum, the envelope is small, so $M_{\max} \lesssim 5M_{\odot}$. If we possess extra information, we can lower this bound further. However, this result as it is holds independent of the densities at $\rho > \rho_0$. □