Advanced Quantum Field Theory

Part III Lent 2019

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Course Outline

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1 Path Integrals in QM

Goal: Schrödinger's equation → path integral

Consider a Hamiltonian in one dimension $\hat{H} = H(\hat{x}, \hat{p})$, where position and momentum operators satisfy the common commutation relations $[\hat{x}, \hat{p}] = i\hbar$. Assume the it takes the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{1.1}$$

Schrödinger's equation then says that the time evolution of a state $|\psi(t)\rangle$ is given by

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle . \tag{1.2}$$

This has a formal solution, giving us the time-evolution operator

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle . \tag{1.3}$$

In the Schrödinger picture, the states are evolving in time whereas operators and their eigenstates are constant in time.

Definition 1 (wavefunction)**:** $\Psi(x,t) := \langle x|\psi(t)\rangle$

The Schrödinger equation then becomes

$$\langle x| \hat{H} |\psi(t)\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi(x,t). \tag{1.4}$$

We will turn this differential equation into an integral equation, where we will sum over particle paths—a path integral. We can introduce an integral by inserting a complete set of states $1 = \int dx_0 |x_0\rangle \langle x_0|$.

$$\begin{aligned}
\Psi(x,t) &= \langle x| e^{-i\hat{H}t/\hbar} |\psi(0)\rangle && (1.5)\\
&= \int_{-\infty}^{\infty} \mathrm{d}x_0 \, \langle x| e^{-i\hat{H}t/\hbar} |x_0\rangle \langle x_0|\psi(0)\rangle && (1.6)\\
&:= \int_{-\infty}^{\infty} \mathrm{d}x_0 \, \underbrace{K(x,x_0;t)}_{\text{`kernel'}} \Psi(x_0,0) && (1.7)
\end{aligned}$$

We repeat this insertion for n intermediate times and positions.

Notation: Let $0 := t_0 < t_1 < \cdots < t_n < t_{n+1} := T$

And we also want to factor the exponential into n terms:

$$e^{i\hat{H}T/\hbar} = e^{-\frac{i}{\hbar}\hat{H}(t_{n+1}-t_n)} \cdots e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)}. \tag{1.8}$$

Then

$$K(x, x_0; T) = \int_{-\infty}^{\infty} \left[\prod_{r=1}^{n} \mathrm{d}x_r \, \langle x_{r+1}| \, e^{-\frac{i}{\hbar}\hat{H}(t_{r+1}-t_r)} \, |x_r\rangle\right] \langle x_1| \, e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)} \, |x_0\rangle \tag{1.9}$$

Integrals are over all possible position eigenstates at times $t_r, r = 1, \ldots, n$.

Free Theory

Consider the "free" theory, with $V(\hat{x}) = 0$. We will now play a similar but different trick to what we did before. Let us insert a complete set of momentum eigenstates $1 = \int_{-\infty}^{\infty} đp \, |p\rangle \langle p|$. We also note that these momentum eigenstates are plane waves $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$.

Definition 2:

$$đp := \frac{\mathrm{d}p}{2\pi\hbar}. \tag{1.10}$$

The corresponding kernel is

$$K_0(x, x'; t) = \langle x| \exp\left(\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t\right)\left|x'\right\rangle. \tag{1.11}$$

$$= \int_{-\infty}^{\infty} đp \, e^{-ip^2t/2m\hbar} e^{ip(x-x')/\hbar} \tag{1.12}$$

$$= \sqrt{\frac{m}{2\pi i\hbar t}} e^{\frac{im(x-x')^2}{2\hbar t}}. \tag{1.13}$$

Remark:

$$\lim_{t\to 0}\{K_0(x, x'; t)\} = \delta(x - x'). \tag{1.14}$$

As expected from $\left\langle x\middle|x'\right\rangle = \delta(x - x')$.

From the Baker-Campbell-Hausdorff formula, we know that

$$e^{\epsilon\hat{A}}e^{\epsilon\hat{B}} = \exp\left(\epsilon\hat{A} + \epsilon\hat{B} + \frac{\epsilon^2}{2}[\hat{A}, \hat{B}] + \ldots\right) \neq e^{\epsilon(\hat{A}+\hat{B})} \tag{1.15}$$

$$\text{for small } \epsilon: \qquad e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}}e^{\epsilon\hat{B}}(1 + O(\epsilon^2)) \tag{1.16}$$

Letting $\epsilon = 1/n$ and raising the above to the n^{th} power[1] gives

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty} \left\{ e^{\hat{A}/n} e^{\hat{B}/n} \right\}^n . \tag{1.17}$$

We will use this to separate kinetic and potential terms.

Take $t_{r+1} - t_r = \delta t$ with $\delta t \ll T$ and n large such that $n\delta t = T$.

$$e^{-\frac{i}{\hbar}\hat{H}\delta t} = \exp\left(-\frac{i\hat{p}^2\delta t}{2m\hbar}\right)\exp\left(-\frac{iV(\hat{x})\delta t}{\hbar}\right)[1 + O(\delta t^2)] \tag{1.18}$$

Using the result (1.13),

$$\begin{aligned} \langle x_{r+1}| \exp\left(-\frac{i\hat{H}}{\hbar}\delta t\right) |x_r\rangle &= e^{-iV(x_r)\delta t/\hbar} K_0(x_{r+1}, x_r; \delta t) && (1.19)\\ &= \sqrt{\frac{m}{2\pi i\hbar\delta t}} \exp\left[\frac{im}{2\hbar}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2 \delta t - \frac{i}{\hbar}V(x_r)\delta t\right] && (1.20) \end{aligned}$$

With $T = n\delta t$

$$K(x, x_0; T) = \int \left[\prod_{r=1}^{n} \mathrm{d}x_r\right] \left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{n+1}{2}} \exp\left\{ i \sum_{r=0}^{n} \left[\frac{m}{2\hbar}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2 - \frac{1}{\hbar}V(x_r)\right]\delta t \right\} \tag{1.21}$$

In the limit $n \to \infty$, $\delta t \to 0$ with $n\delta t = T$ fixed, the exponent becomes

$$\frac{1}{\hbar}\int_0^T \mathrm{d}t \left[\frac{1}{2}m\dot{x}^2 - V(x)\right] = \int_0^T \mathrm{d}t\, L(x, \dot{x}), \tag{1.22}$$

where L is the classical Lagrangian, the Legendre transformation of the classical Hamiltonian. The classical action is $S = \int \mathrm{d}t\, L(x, \dot{x})$.

The main result therefore is that the path integral for the kernel is

$$\boxed{K(x, x_0; t) := \langle x| e^{-i\hat{H}t/\hbar} |x_0\rangle = \int \mathcal{D}x\, e^{\frac{i}{\hbar}S}} \tag{1.23}$$

Definition 3 (functional integral)**:**

$$\mathcal{D}x = \lim_{\substack{\delta t\to 0 \\ n\delta T \text{ fixed}}} \left\{ (\sqrt{\ldots}) \prod_{r=1}^{n} (\sqrt{\ldots} dx_r) \right\} \tag{1.24}$$

We do not need to care about normalization factors.

Remark: In the limit $\hbar \to 0$, the interference of amplitudes is dominated by the ones close to the extremal path where δS. This leads to Hamilton's principle of least action.

[1] This step is sometimes called Susuki–Trotter (?) decomposition.

We may analytically continue this to imaginary time. Let $\tau = it$. In terms of this imaginary time, we have

$$\langle x| \, e^{-\hat{H}\tau/\hbar} \, |x_0\rangle = \int \mathcal{D}x \, e^{-S/\hbar}. \tag{1.25}$$

Mathematically, it makes these integrals much more well-defined, clearly convergent. Here the least-action principle is really evident from the $\hbar = 0$ argument. We also see the connection to statistical physics, interpreting $e^{-S/\hbar}$ as the Boltzmann factor $e^{-\beta H}$.

Quantum mechanics is quantum field theory in 0+1 dimensions. We are treating space differently from time: $\hat{x}(t)$ is a field, whereas t is a variable. However, Lorentz invariance forces us to put x and t on the same footing. In QFT, we solve this problem by demoting x from a field to another label. We then talk about fields $\phi(x, t)$ and want to know about the behaviour of these in all of spacetime.

▌ String theory gives another ansatz to this problem by promoting again.

2 Integrals and their Diagrammatic Expansions

In QFT, we are interested in correlation functions. The following discussion will be very similar to what we have seen in the *Statistical Field Theory* course in Michaelmas term, also no knowledge from that course will be assumed here.

For simplicity, consider a 0 -dimensional field $\varphi \in \mathbb{R}$. As if we are in imaginary time, let

$$Z = \int_{\mathbb{R}} \mathrm{d}\varphi \, e^{-\frac{S(\varphi)}{\hbar}}. \tag{2.1}$$

Assume that the action $S(\varphi)$ is an even polynomial and $S(\varphi) \to \infty$ as $\varphi \to \pm\infty$.

We will be interested in expectation values

$$\langle f \rangle = \frac{1}{Z} \int \mathrm{d}\varphi \, f(\varphi) e^{-S/\hbar}. \tag{2.2}$$

Again, assume f does not grow too fast as $\varphi \to \pm\infty$. Usually, f is polynomial in φ.

2.1 Free Theory

Say we have N scalar fields (in $0 + 1$ dimensions we should really just say 'variables') φ_a with $a = 1, \dots, N$, with action

$$S_0(\varphi) = \frac{1}{2} m_{ab} \varphi_a \varphi_b = \frac{1}{2} \varphi^T m \varphi, \tag{2.3}$$

where m is an $N \times N$ symmetric, positive definite ($\det m > 0$) matrix.

We can diagonalise this. There exists some orthogonal P such that $m = P \Lambda P^T$, where Λ is diagonal. Let $\chi = P^T\varphi$. Then the free partition function is

$$Z_0 = \int \mathrm{d}^N\varphi \exp\Big(-\frac{1}{2\hbar}\varphi^T m\varphi\Big) \tag{2.4}$$
$$= \int \mathrm{d}^N\chi \exp\Big(-\frac{1}{2\hbar}\chi^T \Lambda\chi\Big) \tag{2.5}$$
$$= \prod_{c=1}^{N}\int \mathrm{d}\chi_c\, e^{-\frac{\lambda_c}{2\hbar}\chi^2} = \sqrt{\frac{(2\pi\hbar)^N}{\det m}} \tag{2.6}$$

We want to get from the partition function to correlation functions. We can do this by introducing an N-component vector of external sources J to the action

$$S_0(\varphi) \to S_0 + J^T\varphi. \tag{2.7}$$

The partition function is then

$$Z_0(J) = \int \mathrm{d}^N\varphi \exp\Big(-\frac{1}{2\hbar}\varphi^T m\varphi - \frac{1}{\hbar}\Big)J^T\varphi. \tag{2.8}$$

Similar to solving an ordinary Gaussian integral, we complete the square by writing $\widetilde{\varphi} = \varphi + m^{-1}J$. One can then solve this integral to be

$$Z_0(J) = Z_0(0)\exp\Big(\frac{1}{2\hbar}J^T m^{-1}J\Big). \tag{2.9}$$

This is called the *generating function*[1]. Correlation functions are obtained from differentiating with respect to the auxiliary sources J and evaluating the whole expression at $J = 0$:

$$\langle\varphi_a\varphi_b\rangle = \frac{1}{Z_0(0)}\int \mathrm{d}^N\varphi\,\varphi_a\varphi_b \exp\Big(-\frac{1}{2\hbar}\varphi^T m\varphi - \frac{1}{\hbar}J^T\varphi\Big)\Big|_{J=0}. \tag{2.10}$$
$$= \frac{1}{Z_0(0)}\int \mathrm{d}^N\varphi\left(-\hbar\frac{\partial}{\partial J_a}\right)\left(-\hbar\frac{\partial}{\partial J_b}\right)\exp(\ldots)\Big|_{J=0} \tag{2.11}$$
$$= \frac{1}{Z_0(0)}\left(-\hbar\frac{\partial}{\partial J_a}\right)\left(-\hbar\frac{\partial}{\partial J_b}\right)Z_0(J)|_{J=0} \tag{2.12}$$
$$= \hbar(m^{-1})_{ab} = \; a \;\text{———————}\; b \tag{2.13}$$

Connecting this to the *Quantum Field Theory* course, we identify this as the *free propagator*.

More generally, let $l(\varphi)$ be a *linear* combination of φ_a.

$$l(\varphi) = \sum_{a=1}^{N} l_a\varphi_a, \qquad l_a \in \mathbb{R}. \tag{2.14}$$

Then the steps above are equivalent to swapping $l(\varphi)$ for $l(-\hbar\frac{\partial}{\partial J}) = -\hbar\sum_a l_a\frac{\partial}{\partial J_a}$.

[1]When we go to higher dimensions, where $J = J(x)$ this will be a generating functional $Z[J(x)]$.

The correlation function can again be evaluated explicitly by the introduction of an auxiliary current J:

$$\left\langle l^{(1)}(\varphi)\cdots l^{(p)}(\varphi)\right\rangle = \frac{1}{Z_0(0)}\int \mathrm{d}^N\varphi \prod_{i=1}^{p} l^{(i)}(\varphi) e^{-\frac{1}{2\hbar}\varphi^T m\varphi - \frac{1}{\hbar}J^T\varphi}\Bigg|_{J=0} . \tag{2.15}$$

$$= (-\hbar)^p \prod_{i=1}^{p} l^{(i)}\left(\frac{\partial}{\partial J}\right)\, e^{\frac{1}{2\hbar}J^T m^{-1} J}\Bigg|_{J=0} \tag{2.16}$$

In other words, if p is odd, the integrand is odd in some φ_a and the integral over $\varphi_a \in (-\infty, +\infty)$ vanishes. For $p = 2k$, the terms which are non-zero as $J \to 0$ have the following form. We need half the derivatives to bring down components of $m^{-1}J$ and half to remove the J -dependence from those terms that earlier derivatives brought down. As such, we get exactly k factors of m^{-1} . This is Wick's theorem.

Example (4-point function)**:** Consider the 4-point correlation function

$$\left\langle \varphi_b\varphi_c\varphi_d\varphi_f\right\rangle = \hbar^2\left[(m^{-1})_{bc}(m^{-1})_{df} + (m^{-1})_{bd}(m^{-1})_{cf} + (m^{-1})_{bf}(m^{-1})_{cd}\right] \tag{2.17}$$

$$= \begin{array}{cc} b & d \\ | & | \\ c & f \end{array} + \begin{array}{c} b\text{——}d \\ \\ c\text{——}f \end{array} + \begin{array}{cc} b & d \\ | & | \\ c & f \end{array} \tag{2.18}$$