Supersymmetry

Part III Lent 2020

Lectures by David Skinner

Report typos to: uco21@cam.ac.uk
More notes at: uco21.user.srcf.net

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1 Introduction to Supersymmetry

Resources

The official typed notes can be found on the departmental lecturer's departmental website. The main book that we are going to follow is Hori & Vafa "Mirror Symmetry" (Chapters 8-16), which are available for free on the Clay Maths Insitute website.

1.1 Motivation: What is supersymmetry?

In a theory with bosons and fermions, the Hilbert space splits up into $\mathscr{H} = \mathscr{H}_B \oplus \mathscr{H}_F$, where $\mathscr{H}_{B(F)}$ has even (odd) number of fermionic excitations.

Such a theory is supersymmetric if there exists an operator Q mapping $\mathscr{H}_B \to \mathscr{H}_F$ and $\mathscr{H}_F \to \mathscr{H}_B$ such that

$$\left\{Q, Q^\dagger\right\} = 2H \qquad Q^2 = 0. \tag{1.1}$$

Here, $\{A, B\} = AB + BA$ is the *anti-commutator*, H is the Hamiltonian.

Consequences

i) The Hamiltonian and Q commute

$$\begin{aligned} [H, Q] &= \frac{1}{2}[\left\{Q, Q^\dagger\right\}, Q] & (1.2)\\ &= \frac{1}{2}[(QQ^\dagger + Q^\dagger Q)Q - Q(QQ^\dagger + Q^\dagger Q)] & (1.3)\\ &= 0. & (1.4)\end{aligned}$$

The two inner terms vanish since $Q^2 = 0$ and the two outer terms cancel identically. Therefore, the operator Q is *conserved*, and the transformations it generates will be *symmetries*. We call them *supersymmetries* because they mix bosons and fermions.

ii) All states ψ in our theory have non-negative energy

$$E = \langle\psi| H |\psi\rangle = \frac{1}{2} \langle\psi| \left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\} |\psi\rangle \tag{1.5}$$

$$= \frac{1}{2} \|\mathcal{Q} |\psi\rangle\|^2 + \frac{1}{2} \left\|\mathcal{Q}^\dagger |\psi\rangle\right\|^2 \geq 0, \tag{1.6}$$

with equality if and only if $\mathcal{Q} |\psi\rangle = 0 = \mathcal{Q}^\dagger |\psi\rangle$, meaning that the state is invariant under supersymmetry.

If we have a Lorentz invariant quantum field theory (QFT), then H is part of the momentum vector $P_m = (H, \mathbf{P})$. It is then natural to expect that there is a multiplet of $\mathcal{Q}$'s.

Indeed in $d = 4$ we have $\left\{\mathcal{Q}_\alpha, \mathcal{Q}^\dagger_{\dot{\alpha}}\right\} = 2\sigma^m_{\alpha\dot{\alpha}} P_m$, where $\sigma^m = (\mathbb{1}_{2\times 2}, \boldsymbol{\sigma})$.

For generic dimension d, this becomes $\left\{\mathcal{Q}_A, \mathcal{Q}^\dagger_B\right\} = 2\Gamma^m_{AB} P_m$.

Why study supersymmetry?

Traditionally, this question was answered by phenomenology. Supersymmetry was a promising approach to solve ongoing problems in dark matter, the unification of couplings in the standard model, as well as stabilizing the Higgs mass. The involvement of supersymmetry in the last issue was ruled out in experiments at CERN, and the above reasons will not be the motivation that drives us to study supersymmetry in this course.

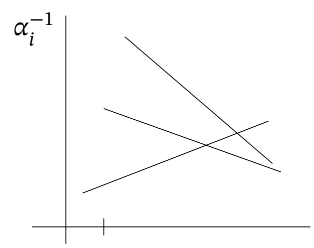

Figure 1.1: In many supersymmetric theories, the running couplings meet at the same point. This was taken to be indicative of a grand unified theory (GUT).

In this course, we will be driven by the following fact: QFT is hard! Usually, we have to study it via perturbation theory, as exemplified in 1.2. This is very different to quantum mechanics, where we first practice with exactly solvable systems. The reason for this is twofold. Firstly, they are usually better approximations to reality than a free particle; a spherically symmetric Coulomb potential is a better starting point to describe atoms than a free particle is, even though we still need to consider realistic models like (hyper)fine-structure perturbatively. Secondly, it helps us understand what quantum mechanics actually *is*.

Supersymmetry allows us to get exact results for (some observables) in QFT. This is especially true in $d < 4$, but also in $d = 4$.

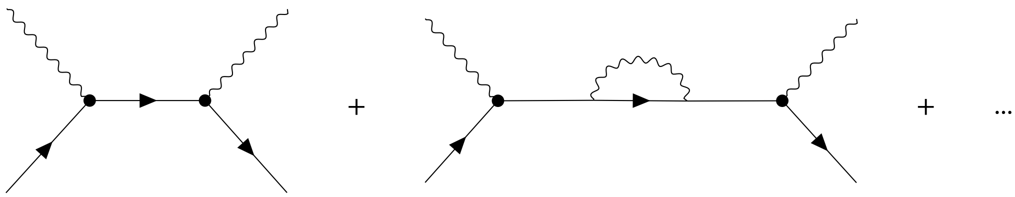

Figure 1.2: In the diagrammatic perturbation series in QFT, particles are almost always propagating freely, except at the interaction vertices.

These exact results are often closely related to deep maths, such as the Atiyah–Singer theorem (which we will meet in $d = 1$) , mirror symmetry and enumerative geometry ($d = 2$), and Donaldson–Seiberg–Witten invariants ($d = 4$).

1.2 Fermions and Super Vector Spaces

Definition 1: A $\mathbb{Z}_2$-graded vector space is of the form $V = V_0 \oplus V_1$.

Definition 2 (parity)**:** We let the *parity* $|v|$ of $v \in V$ be

$$|v| = \begin{cases} 0, & \text{if } v \in V_0 \qquad \text{(even / bosonic)} \\ 1, & \text{if } v \in V_1 \qquad (odd/fermionic/Grassman) \end{cases} \tag{1.7}$$

Notation: If $\dim_{\mathbb{R}}(V_0) = p$ and $\dim_{\mathbb{R}}(V_1) = q$, then we write $V = \mathbb{R}^{p|q}$.

As usual, the dual V^* of a $\mathbb{Z}_2$-graded vector space (over $\mathbb{R}$) is the space of linear maps $\phi :\ V \to \mathbb{R}$ with $(V^*)_{0(1)}$ being those linear maps that vanish on $V_{1(0)}$ respectively.

Unsurprisingly, the direct sum of two $\mathbb{Z}_2$-graded vector spaces is

$$\begin{aligned} V \oplus W &= (V \oplus W)_0 \oplus (V \oplus W)_1 && (1.8) \\ &= (V_0 \oplus W_0) \oplus (V_1 \oplus W_1). && (1.9) \end{aligned}$$

Likewise, we can take the tensor product

$$\begin{aligned} (V \otimes W)_0 &= (V_0 \otimes W_0) \oplus (V_1 \otimes W_1) && (1.10) \\ (V \otimes W)_1 &= (V_0 \otimes W_1) \oplus (V_1 \otimes W_0) && (1.11) \end{aligned}$$

Until now, we are just dealing with usual vector spaces, where we keep track of the fact that some elements have parity 1. To make V a super vector space, we define an unusual exchange operation.

$$\text{usually (bosonic):} \quad \begin{aligned} s: \ U \otimes U' &\to U' \otimes U \\ u \otimes u' &\longmapsto u' \otimes u \end{aligned} \tag{1.12}$$

$$\text{super vector space:} \quad \begin{aligned} s: \ V \otimes W &\to W \otimes V \\ v \otimes w &\longmapsto (-1)^{|v||w|} w \otimes v. \end{aligned} \tag{1.13}$$

Definition 3 (superalgebra)**:** Closely related is a superalgebra. This is a supervector space A with a multiplication map $\bullet: \ A \times A \to A$ with $|a \cdot b| = |a| + |b| \pmod 2$.

Definition 4 (commutative)**:** A is supercommutative (or just commutative) if $ab = (-1)^{|a||b|} ba$.

Example 1.2.1: To treat $\mathbb{R}^{p|q}$ as a superalgebra, we take

$$x^i x^j = x^j x^i, \qquad x^i \psi^a = \psi^a x^i, \qquad \text{but } \psi^a \psi^b = -\psi^b \psi^a, \tag{1.14}$$

where $x^i \in \mathbb{R}^{p|0}$ and $\psi^a \in \mathbb{R}^{0|q}$. In particular, $(\psi^a)^2 = 0$ for any fixed a.

Not all A are (super-)commutative.

Definition 5 (Lie superalgebra)**:** A *Lie superalgebra* is a supervector space $g = g_0 \oplus g_1$ with a bilinear Lie bracket operation $[,]: \ g \times g \to g$ that

- is 'graded anti-symmetric' $[X, Y] = -(-1)^{|x||y|}[Y, X]$
- obeys $[X, [Y, Z]] + (-1)^{|X|(|Y|+|Z|)}[Y, [Z, X]] + (-1)^{|Y|(|Z|+|X|)}[Z, [X, Y]] = 0$

Definition 6 (polynomials)**:** We can define polynomials on a super vector space $\mathbb{R}^{p|q} = v$ as $O(V) \simeq \mathrm{Sym}^*(V_0^*) \otimes \Lambda^*(V_1^*)$. They are of the form

$$c_{\underbrace{ijk\ldots m}_{\text{symmetric}}\underbrace{abc\ldots d}_{\text{antisymmetric}}} x^i x^j \ldots x^m \psi^a \ldots \psi^d. \tag{1.15}$$

Definition 7 (smooth functions)**:** We define smooth functions on a super vector space to be $C^\infty(V) = C^\infty(V_0) \otimes \Lambda^*(V_1^*)$, so a generic function has an expansion

$$F(x^i, \psi^a) = f(x^i) * \rho_a(x^i)\psi^a + g_{ab}(x^i)\psi^a\psi^b + \cdots + \frac{h(x)}{(\dim V_1)!}\epsilon_{ab\ldots d}\psi^a\psi^b \ldots \psi^d, \tag{1.16}$$

where the coefficients f, ρ_a, g_{ab}, …are smooth functions on V_0. We often call such functions $F(x^i, \psi^a)$ *superfields*, while the smooth functions $f, \rho_a, g_{ab}, \ldots, h$ are the *component fields*. Note that if $F(x, \psi)$ is bosonic, then the component fields with even indices $f, g_{ab} = -g_{ba}, \ldots$ are bosonic whilst the ones with odd indices $\rho_a, \ldots$ are fermionic.

Remark: This is reminiscent of a *polyform* $F \in \Omega^*(V)$

$$F(x^i, dx^i) = f(x) + \rho_i(x)dx^i + g_{ij}(x)dx^i \wedge dx^j + \cdots + h(x)dx^i \wedge \cdots \wedge dx^n. \tag{1.17}$$

There is a fundamental difference: the polyform indices run over i, while the superfield indices a need not be the same as i. However, if they have the same indexing set, then these are really similar.

1.3 Differentiation and Integration of Fermions

Definition 8 (derivation)**:** A *derivation* of a (super-)algebra A is a linear map $D : A \to A$ obeying

$$D(ab) = (Da)b + (-1)^{|a||D|}a(Db) \qquad \text{(graded Leibniz rule).} \tag{1.18}$$

Example 1.3.1: On $\mathbb{R}^{p|q}$, we have even derivatives $\frac{\partial}{\partial x^i}$ and odd derivatives $\frac{\partial}{\partial \psi^a}$, which act in the way you would expect on single fields:

$$\frac{\partial}{\partial x^i}(x^j) = \delta_i^j \qquad \frac{\partial}{\partial x^i}\psi^a = 0 \qquad \frac{\partial x^j}{\partial \psi^a} = 0 \qquad \frac{\partial}{\partial \psi^a}(\psi^b) = \delta_a^b. \tag{1.19}$$

However,

$$\frac{\partial}{\partial \psi^a}(\psi^b\psi^c) = \delta_a^b\psi^c - \psi^b\delta_a^c. \tag{1.20}$$

More generally, a (smooth) vector field on $\mathbb{R}^{p|q}$ is

$$X(x, \psi) = X_0^i(x, \psi)\frac{\partial}{\partial x^i} + X_1^a(x, \psi)\frac{\partial}{\partial \psi^a}, \tag{1.21}$$

where $X_0^i, x_1^a \in C^\infty(\mathbb{R}^{p|q})$.

For integration, since $f(\psi) = \rho + a\psi$, we only need to define $\int 1\,\mathrm{d}\psi$ and $\int \psi\,\mathrm{d}\psi$. We require our measure $\mathrm{d}\psi$ to be translation invariant[1]: if $\psi' = \psi + \eta$ for some fixed fermionic $\eta \in \mathbb{R}^{0|1}$, then we want

$$\int \psi'\,\mathrm{d}\psi' = \int (\psi + \eta)\,\mathrm{d}\psi = \int \psi\,\mathrm{d}\psi + \eta \int \mathrm{d}\psi \;\Rightarrow\; \boxed{\int 1\,\mathrm{d}\psi = 0}. \tag{1.22}$$

We then normalise our measure by defining

$$\boxed{\int \psi\,\mathrm{d}\psi = 1} \tag{1.23}$$

These rules are called *Berezin integration*.

> **Remark:** Differentiation and integration is really the same thing. Not unlike complex variables.

Remark: These imply that

$$\int \frac{\partial}{\partial\psi}(F(\psi, \dots))\,\mathrm{d}\psi = 0. \tag{1.24}$$

In other words, when we perform integration by part for fermions, we never have to worry about boundary terms as long as we are careful about minus signs.

Suppose that we instead have a general case of n fermionic variables ψ^a . Then by iterated application of the previous rules, we define

$$\int \psi^1\psi^2 \dots \psi^n\,\mathrm{d}^n\psi = 1 \tag{1.25}$$

if they all appear, and zero otherwise. If they all appear, but not in the correct order, then we get extra minus signs

$$\int \underbrace{\psi^a\psi^b \dots \psi^c}_{n \text{ fermions}}\,\mathrm{d}^n\psi = e^{ab\dots c}. \tag{1.26}$$

> **Remark:** Note in particular that if any index appears twice, the square on the left-hand side vanishes, just like the Levi-Civita symbol on the right.

Suppose $\chi^a = N^a{}_b\psi^b$ for some $N \in GL(n, \mathbb{R})$. Then by linearity

$$\begin{aligned}
\int \chi^{a_1} \dots \chi^{a_n}\,\mathrm{d}^n\psi &= N^{a_1}{}_{b_1} \dots N^{a_n}{}_{b_n} \int \psi^{b_1} \dots \psi^{b_n}\,\mathrm{d}^n\psi &(1.27)\\
&= N^{a_1}{}_{b_1} \dots N^{a_n}{}_{b_n}\,\epsilon^{b_1b_2\dots b_n} &(1.28)\\
&= \det(N)\epsilon^{a_1\dots a_n} &(1.29)\\
&= \det(N)\int \chi^{a_1}\chi^{a_2} \dots \chi^{a_n}\,\mathrm{d}^n\chi\,. &(1.30)
\end{aligned}$$

[1] In particular, this will be necessary to derive Ward identities in QFT.

We conclude that if $\chi^a = N^a{}_b \psi^b$, then $\mathrm{d}^n \chi = \frac{1}{\det(N)} \mathrm{d}^n \psi$.

> **Remark:** This is not the same as if you were doing bosonic integration, where you do not have the inverse of the determinant.

Example 1.3.2: If $\chi = a\psi$, then $\mathrm{d}\psi = d(a\psi) = \frac{1}{a}\,\mathrm{d}\psi$.

1.4 QFT in Zero Dimensions

1.4.1 Bosonic Theory

In $d = 0$, our whole universe is just a single point $M = \{\mathrm{pt}\}$. So a bosonic field is just a map $x : M \to \mathbb{R} \simeq \{\mathrm{pt}\} \to \mathbb{R}$, which is nothing else than a real variable. With n such real fields, the space of all field configurations is $\mathcal{C} = \mathbb{R}^n$. The path-integral measure $[\mathcal{D}X]$ is just the usual Lebesgue measure $\mathrm{d}^n x$. Then the partition function becomes $Z = \int_{\mathbb{R}^n} e^{-S(x)/\hbar}\,\mathrm{d}^n x$, where $S : \mathbb{R}^n \to \mathbb{R}$ is the action.

> Compare this with today's lecture on *Advanced Quantum Field Theory*.

There cannot be any kinetic terms since the universe is just a point and there cannot be anything we differentiate with respect to. However, we can have a mass term and maybe some sort of interaction such as

$$S(x^i) = \frac{m_2}{2}\delta_{ij}x^i x^j + \frac{\lambda^{ijkl}}{4}x^i x^j x^k x^l. \tag{1.31}$$

Now in $d = 0$ this is a finite-dimensional integral. But nonetheless, it is a difficult integral! Expanding the action to quadratic order around its stationary point, we can find that in the limit $\hbar \to 0^+$, the integral is asymptotic to

$$\int_{\mathbb{R}^n} e^{-S(x)/\hbar}\,\mathrm{d}^n x \sim (2\pi\hbar)^{n/2}\frac{e^{-S(x_*)}}{(\det \partial_i \partial_j S)^{1/2}|_{x=x_*}}\left(1 + \hbar A_1 + \hbar^2 A_2 + \dots\right), \qquad \text{(steepest descent)} \tag{1.32}$$

where x_* is a minimum of $S(x)$.

This is complicated! And approximate (zero radius of convergence)!

Remark: In the whole of the QFT course we basically just computed different numerators of this. In AQFT we will go on to loop diagrams and compute the denominator as well as the first order expansion. If you end up doing a PhD in the wrong area, you might compute higher and higher terms. But what is even the point? This series doesn't even converge!

1.4.2 Fermionic Theory

Let us now consider a purely fermionic theory. We need at least two fermions. Take $S = A\psi^1\psi^2$.

$$Z = \int e^{-S(\psi)/\hbar}\,\mathrm{d}^2\psi = \int \left(1 - \frac{A}{\hbar}\psi^1\psi^2\right)\mathrm{d}^2\psi = -\frac{A}{\hbar}. \tag{1.33}$$

More generally, for $2m$ fermions ψ^a and antisymmetric matrix A_{ab} ,

$$Z = \int e^{-\frac{A_{ab}}{2\hbar}\psi^a\psi^b}\,\mathrm{d}^{2m}\psi = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2\hbar)^k k!}\int (A_{ab}\psi^a\psi^b)^k\,\mathrm{d}^{2m}\psi \tag{1.34}$$

$$= \frac{(-1)^m}{(2\hbar)^m m!}\epsilon^{a_1 b_1 \ldots a_m b_m}A_{a_1 b_1}\ldots A_{a_m b_m} \tag{1.35}$$

$$= \left(\frac{-1}{\hbar}\right)^m \mathrm{Pfaff}(A). \tag{1.36}$$

Definition 9 (Pfaffian)**:** In the preceding derivation, we stumbled across the *Pfaffian* of a $2m \times 2m$ antisymmetric matrix A, defined by

$$\mathrm{Pfaff}(A) = \frac{1}{2^m m!}e^{a_1 a_2 \ldots a_{2m}}A_{a_1 a_2}\cdots A_{a_{2m-1}a_{2m}}. \tag{1.37}$$

Exercise 1.1: For antisymmetric A, show that $(\mathrm{Pfaff}\,A)^2 = \det A$.

This means that the Gaussian integral (1.36) can be written as $\pm\sqrt{\det A}$.

> **Remark:** Again, up to a normalisation of the measure, this is the inverse to what you expect from the bosonic counterpart of the Gaussian integral.

If our action contains finitely many fermions, it is always easy to compute the fermionic integral exactly (unlike the bosonic case).

Example 1.4.1: If we have a quartic action

$$S(\psi^1, \dots, \psi^4) = A(\psi^1\psi^2 + \psi^3\psi^4) + \lambda\psi^1\psi^2\psi^3\psi^4. \tag{1.38}$$

Then $S^2 \neq 0$, but $S^3 = 0$. So the exponential measure truncates just after the second term

$$\begin{aligned} e^{-S/\hbar} &= 1 - \frac{S}{\hbar} + \frac{S^2}{2\hbar^2} && (1.39)\\ &= 1 - \frac{1}{\hbar}[A(\psi^1\psi^2 + \psi^3\psi^4) + \lambda\psi^1\psi^2\psi^3\psi^4] && (1.40)\end{aligned}$$

and hence, the integral extracts the piece

$$\int e^{-S(\psi)/\hbar}\, \mathrm{d}^4\psi = \frac{A^2}{\hbar^2} - \frac{\lambda}{\hbar}. \tag{1.41}$$

1.4.3 Supersymmetric Theory

A generic theory containing both fermions and bosons is intractable because of the bosonic integral. Even in $d = 0$, we get a complicated integral that is hard to solve. However, let us consider a theory containing one bosonic and two fermionic fields $(x, \psi, \overline{\psi})$.

> These fermionic fields can be considered $\psi = \psi^1 + i\psi^2$ and $\overline{\psi} = \psi^1 - i\psi^2$.

The most general action for these fields would be

$$S(x, \psi\overline{\psi}) = f(x) + g(x)\overline{\psi}\psi, \tag{1.42}$$

for some functions f and g. We can choose a very special relation between fermionic and bosonic fields by choosing the action

$$S(x, \psi\overline{\psi}) = \frac{1}{2}(\partial w)^2 - \overline{\psi}\psi\partial^2 w, \tag{1.43}$$

where $w = w(x)$ is a polynomial and $\partial w = \frac{\partial w}{\partial x}$.

Claim 1: This action is invariant under the flow generated by the fermionic vector fields

$$Q = \psi\frac{\partial}{\partial x} + (\partial w)\frac{\partial}{\partial \overline{\psi}} \qquad \text{and} \qquad Q^\dagger = \overline{\psi}\frac{\partial}{\partial x} - (\partial w)\frac{\partial}{\partial \psi}. \tag{1.44}$$

These are odd derivations of $\mathbb{R}^{1|2}$ with

$$\mathcal{Q}(x) = \psi \qquad \mathcal{Q}^\dagger(x) = \overline{\psi} \tag{1.45}$$

$$\mathcal{Q}(\psi) = 0 \qquad \mathcal{Q}^\dagger(\psi) = -\partial w(x) \tag{1.46}$$

$$\mathcal{Q}(\overline{\psi}) = \partial w(x) \qquad \mathcal{Q}^\dagger(\overline{\psi}) = 0 \tag{1.47}$$

Proof. We will only show this for $\mathcal{Q}^\dagger$.

$$\mathcal{Q}^\dagger(S) = \overline{\psi}\frac{\partial}{\partial x}\left(\frac{1}{2}(\partial w)^2 - \overline{\psi}\psi\partial^2 w\right) - (\partial w)\frac{\partial}{\partial \psi}\left(\frac{1}{2}(\partial w)^2 - \overline{\psi}\psi\partial^2 w\right) \tag{1.48}$$

$$= \overline{\psi}\partial w\partial^2 w - \overline{\psi}(\partial w)\partial^2 w = 0. \tag{1.49}$$

□

We say that $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ generate *supersymmetries* of this action.

We can also calculate the anti-commutation relations

$$\{\mathcal{Q},\mathcal{Q}\} = 2(\partial^2 w)\psi\frac{\partial}{\partial\overline{\psi}} \qquad \{\mathcal{Q}^\dagger,\mathcal{Q}^\dagger\} = -2(\partial^2 w)\overline{\psi}\frac{\partial}{\partial\psi}$$
$$\{\mathcal{Q},\mathcal{Q}^\dagger\} = -\partial w\left(\psi\frac{\partial}{\partial\psi} - \overline{\psi}\frac{\partial}{\partial\overline{\psi}}\right), \tag{1.50}$$

which also generate bosonic symmetries.

This supersymmetry obeys $\mathcal{Q}^2 = 0 = \mathcal{Q}^{\dagger 2}$ only up to the $\psi,\overline{\psi}$ 'equation of motion'[1]. To do better, we will need superfields.

In a supersymmetric theory (in $d = 0$) we can compute the partition function $Z = \int e^{-S/\hbar}\,\mathrm{d}x\,\mathrm{d}^2\psi$ exactly. To do this, let us rescale $w \to \lambda w$, for $\lambda \in \mathbb{R}_{\geq 0}$. Then the rescaled action $S_\lambda = \frac{\lambda^2}{2}(\partial w)^2 - \overline{\psi}\psi\lambda\partial^2 w$ is invariant under $Q_\lambda = \psi\frac{\partial}{\partial x} + \lambda(\partial w)\frac{\partial}{\partial\overline{\psi}}$ and $\mathcal{Q}_\lambda^\dagger$.

Claim 2: The key point is that $Z_\lambda = \int e^{-S_\lambda/\hbar}\,\mathrm{d}x\,\mathrm{d}^2\psi$ is independent of λ.

We will set $\hbar = 1$ from now on.[2]

Proof.

$$-\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_\lambda = \int\frac{\mathrm{d}S_\lambda}{\mathrm{d}\lambda}e^{-S_\lambda}\,\mathrm{d}x\,\mathrm{d}^2\psi = \int\left(\lambda(\partial w)^2 - \overline{\psi}\psi\partial^2 w\right)e^{-S_\lambda}\,\mathrm{d}x\,\mathrm{d}^2\psi\,. \tag{1.51}$$

Observe that $\mathcal{Q}_\lambda^\dagger(\psi\partial w) = \overline{\psi}\psi\partial^2 w - \lambda(\partial w)^2 = -\frac{\mathrm{d}S_\lambda}{\mathrm{d}\lambda}$. Hence, we can write this as

$$\frac{\mathrm{d}Z_\lambda}{\mathrm{d}\lambda} = \int\mathcal{Q}_\lambda^\dagger(\psi\partial w)e^{-S^\lambda}\,\mathrm{d}x\,\mathrm{d}^2\psi = \int Q_\lambda^\dagger\left(\psi\partial w e^{-S_\lambda}\right)\mathrm{d}x\,\mathrm{d}^2\psi \tag{1.52}$$

where we used that the action itself is invariant under $Q_\lambda^\dagger$. Since $Q_\lambda^\dagger = \overline{\psi}\frac{\partial}{\partial x} - \lambda(\partial w)\frac{\partial}{\partial\psi}$, the second term does not survive the integral over $\mathrm{d}^2\psi$. The first term is a total derivative in x, so it dies under integration over $\mathrm{d}x$[3]. We conclude that $\frac{\mathrm{d}Z_\lambda}{\mathrm{d}\lambda} = 0$. □

[1] Of course we are in $d = 0$ and there is no time, so we do not have any time-evolution.

[3] We might need to worry about large x behaviour, but ∂w is some polynomial, so the exponential decay of e^{-S_λ} will always dominate at large x.

In particular, $Z(1) = \lim_{\lambda\to\infty} Z(\lambda)$. This is useful because it is easy to compute $Z(\lambda)$ at large λ: As $\lambda \to \infty$, the term $e^{\frac{\lambda^2}{2}(\partial w)^2}$ suppresses all contributions, except near critical points x_*, where $w'(x_*) = 0$. Suppose $w(x)$ is a generic polynomial of degree D with isolated[1], non-degenerate[2] critical points x_* . Then near any critical point,

$$w(x) = w(x_*) + \frac{c_*}{2}(x - x_*)^2 + \dots , \tag{1.53}$$

where $c_* = \partial^2 w(x_*)$. Hence, near x_*,

$$S(x, \psi, \overline{\psi}) = \frac{c_*^2}{2}(x - x_*)^2 - \overline{\psi}\psi c_* + \dots \tag{1.54}$$

The higher order terms in $\delta x = x - x_*$ will be negligible as $\lambda \to \infty$. Hence, near the critical point,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int e^{-S(x,\psi,\overline{\psi})}\, \mathrm{d}x\, \mathrm{d}^2\psi &= \frac{1}{\sqrt{2\pi}} \int e^{-\frac{c_*}{2}(x-x_*)^2} \left[-1 + c_*\overline{\psi}\psi\right] \mathrm{d}x\, \mathrm{d}^2\psi && (1.55)\\
&= \frac{c_*}{\sqrt{2\pi}} e^{-\frac{c_*}{2}(x-x_*)^2}\, \mathrm{d}x && (1.56)\\
&= \frac{c_*}{\sqrt{2\pi}} \sqrt{\frac{2\pi}{c_*^2}} = \frac{c_*}{|c_*|} && (1.57)\\
&= \mathrm{sgn}(\partial^2 w|_{x_*}). && (1.58)
\end{aligned}$$

Summing over all critical points,

$$Z = \sum_{x_* : \partial w|_{x_*} = 0} \mathrm{sgn}(\partial^2 w|_{x_*}). \tag{1.59}$$

As seen in Figure XXX, the partition function only really cares about the degree of the polynomials.

Let us think about what would have happened if we calculated this with the use of Feynman diagrams. If you did that, you would see that to all orders in the loop expansions, the diagrams cancel exactly. The reason for this is really this localisation.

We are just counting the number of things, this is the first sniff at some sort of index theorem.

[1]For each critical point, there is always some open neighbourhood around it that does not contain any others.
[2]This means that the second derivative of w does *not* vanish at such a point.

1.5 The Duistermaat–Heckmann Theorem

Definition 10 (symplectic manifold)**:** A *symplectic manifold* (M, ω) is a smooth manifold M of dimension $\dim_{\mathbb{R}}(M) = 2n$ on which we have a 2-form ω, which is

- closed: $d\omega = 0$,
- non-degenerate: $\omega(X, Y) = 0$ for all vector fields Y iff $X = 0$.

Equivalently, the non-degeneracy condition can be expressed as $\omega^n = \det(\omega_{ab}) dx^1 \wedge \cdots \wedge dx^{2n}$ is non-vanishing, and therefore provides a (Liouville) volume form.

Let (M, ω) be a symplectic manifold. Suppose X is a vector field on (M, ω) with ω invariant along the flow generated by X. This means that the Lie derivative vanishes

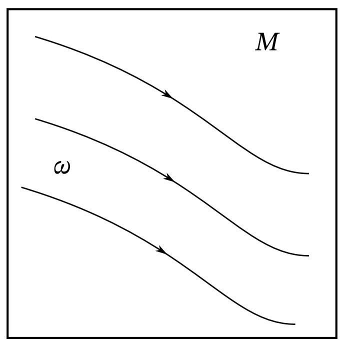

Figure 1.3

$$0 = \mathcal{L}_X \omega = (\iota_X d + d\iota_X)\,\omega = d(\iota_X \omega), \tag{1.60}$$

where in the last equation we used that $d\omega = 0$.

Definition 11 (Hamiltonian)**:** We say X is a *Hamiltonian vector field* if there exists a map $h :\ M \to \mathbb{R}$ such that $\iota_X \omega = -dh$.

Example 1.5.1: Let $M = \mathbb{R}^{2n}$ and $\omega = dp_i \wedge dq^i$. Take $X = \frac{\partial}{\partial q^i}$. Then $\iota_X \omega = -dp_i$. So translations are Hamiltonian with p_i as the Hamiltonian function.

We will be interested in compact (M, ω) with $\partial M = \emptyset$. We will also require that X generates a $U(1)$[1] action on M, meaning that generic orbits of X are circles.

Example 1.5.2: Consider $M = S^2$ with $\omega = \sin\theta d\theta \wedge d\phi$. Take $X = \frac{\partial}{\partial \phi}$. There are two fixed points at the north and south poles, as illustrated in 1.4.

[1]Strictly speaking it does not need to be $U(1)$. It can be other groups as well, but this is the simplest case.

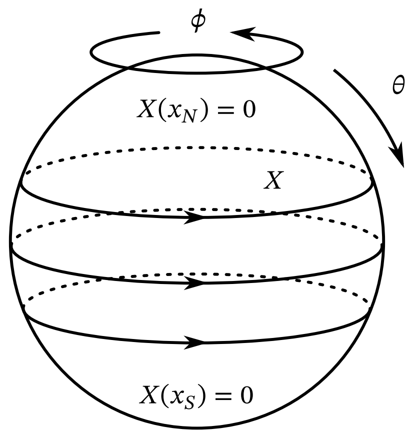

Figure 1.4: Integral curves on S^2.

Then the Duistermaat–Heckmann theorem states that for any $\alpha \in \mathbb{R}$

$$\int_M e^{i\alpha h(x)} \frac{\omega^n}{n!} \tag{1.61}$$

localises to fixed points $x_* \in M$, where $X(x_*) = 0$.

Example 1.5.3 $((M, \omega) = (S^2, \sin\theta d\theta \wedge d\phi))$**:** Have $X = -\frac{\partial}{\partial \phi}$ and $h = \cos\theta$. The integral can simply be done without using a fancy localisation theorem. Use the substitution $z = \cos\theta$:

$$\int_{S^2} e^{i\alpha\cos\theta} \sin\theta d\theta \wedge d\phi = 2\pi \int_{-1}^{+1} e^{i\alpha z}\, \mathrm{d}z = \frac{2\pi}{i\alpha}\left[e^{i\alpha} - e^{-i\alpha}\right], \tag{1.62}$$

which is simply the value of $e^{i\alpha h(x_*)}$ at the north and south poles.

Proof. We can derive this using supersymmetry. Our 'fields' are (x^a, ψ^b), where the ψ^a transform as vectors on M. Thus the space of fields is $\mathcal{C} = \Pi TM$. A generic superfield is

$$F(x, \psi) = f(x) + \rho_a(x)\psi^a + g_{ab}(x)\psi^a\psi^b + \cdots + h(x)\psi^1\psi^2 \cdots \psi^{2n}. \tag{1.63}$$

The Π is just a notation that reminds us that the ψ^a are anti-commuting.

As before, we can identify the space of smooth functions $C^\infty(\Pi TM) = \Omega^*(M)$ to be the space of polyforms on the manifold.

On any coordinate patch we have a supervector space given by $\mathbb{R}^{2n|2n}$. For a general curved manifold we might need to worry about what happens on the overlaps of the coordinate patches, but we are not going to.

We choose our action to have 2 parts. Firstly,

$$S_0 = -i\alpha \left(h(x) + \omega_{ab}(x)\psi^a\psi^b \right). \tag{1.64}$$

Claim 3: This is invariant under supersymmetry transformations generated by the vector field

$$\mathcal{Q} = \psi^a \frac{\partial}{\partial x^a} + X^a(x)\frac{\partial}{\partial \psi^a}. \tag{1.65}$$

Proof.

$$\frac{i}{\alpha}Q(S_0) = \psi^a\partial_a h + \partial_a\omega_{bc}\psi^b\psi^b + 2X^a\omega_{ab}\psi^b \tag{1.66}$$

Now the second part vanishes since $d\omega = 0$.

$$\cdots = \psi^a(\partial_a h + 2X^b\omega_{ba}) = 0 \tag{1.67}$$

since $\iota_X\omega = -dh$ by Def. 11 □

We can write "$\mathcal{Q} = d + \iota_X$", giving

$$\frac{1}{2}\{\mathcal{Q}, \mathcal{Q}\} = (d + \iota_X)^2 = d\iota_X + \iota_X d = \mathcal{L}_X. \tag{1.68}$$

So $\mathcal{Q}^2 = 0$ on forms that are invariant along the flow of X. We now deform S by picking a positive definite metric g on M. Then for a constant $\lambda \in \mathbb{R}$ that measures the deformation, we have

$$S_\lambda = S_0 + \lambda\mathcal{Q}(g(X, \psi)). \tag{1.69}$$

Provided the metric is invariant under the flow, $\mathcal{Q}(S_\lambda) = \lambda Q^2(g_{ab}X^a\psi^b) = 0$. However, this corresponds to $\lambda\mathcal{L}_X(g_{ab}X^a dx^b)$.

The partition function of this is as always

$$Z = \int_{\Pi TM} e^{-S_\lambda}\, \mathrm{d}^{2n}x\, \mathrm{d}^{2n}\psi \tag{1.70}$$

Exercise 1.2: Check that the measure $\mathrm{d}^{2n}x\,\mathrm{d}^{2n}\psi$ is invariant under Diff(M).

Again, if we differentiate this with respect to the parameter λ, we have

$$-\frac{\mathrm{d}Z}{\mathrm{d}\lambda} = \int \mathcal{Q}(g(X, \psi))e^{-S_\lambda}\, \mathrm{d}^{2n}x\, \mathrm{d}^{2n}\psi = \int \mathcal{Q}(g(X, \psi)e^{-S_\lambda})\, \mathrm{d}^{2n}x\, \mathrm{d}^{2n}\psi = 0 \tag{1.71}$$

Because out whole action is supersymmetric, we were able to make $\mathcal{Q}$ act on everything in the first equality. Hence Z_λ is, as before, independent of λ.

In particular, this is useful since

$$Z(\lambda = 0) = \int_{\Pi TM} e^{-S_0}\,\mathrm{d}^{2n}x\,\mathrm{d}^{2n}\psi = (i\alpha)^n \int_M e^{i\alpha h(x)}\mathrm{Pfaff}(\omega_{ab})\,\mathrm{d}^{2n}x \tag{1.72}$$

$$= (i\alpha)^n \int_M e^{i\alpha h(x)}\frac{\omega^n}{n!}, \tag{1.73}$$

which is the Duistermaat–Heckmann integral.

▌ We have recast the problem into supersymmetric language.

Remark: The deformation term has two pieces $\mathcal{Q}(g(X,\psi)) = (\partial_c g_{ab}X^a)\psi^c\psi^b + \mathrm{g}(X,X)$. The second term is the important bit; it is purely bosonic and positive definite. Hence as we scale λ to a very large value $\lambda \to \infty$, we only get contributions from a neighbourhood of any critical points, where the vector field has zero length and thus vanishes $X(x_*) = 0$.

We know that Z_λ is independent of λ, so the original integral must localise. We can evaluate $\lim_{\lambda\to\infty} Z_\lambda$ by using steepest descent.

$$Z_\lambda = \lim_{\lambda\to\infty}\{Z_\lambda\} \sim \frac{(2\pi)^n}{(i\alpha)^n}\sum_{\substack{x_*\in M\\ X(x_*)=0}} e^{i\alpha h(x_*)}\left.\frac{\epsilon^{a_1b_1\dots a_nb_n}(\partial_{a_1}X_{b_1})\dots(\partial_{a_n}X_{b_n})}{\sqrt{\det\partial_a\partial_b \mathrm{g}(X,X)}}\right|_{x=x_*} \tag{1.74}$$

where $X_b = g_{bc}X^c$. Localisation tells us that this result is exact. □

Example 1.5.4: Let use the localisation theorem to recompute the answer (1.62) that we got for S^2. Critical points are the north and south poles. Near these, we can find (Darboux) coordinates such that $\omega = dq \wedge dp$. In these coordinates,

$$X = k\left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right), \tag{1.75}$$

for some $k \in \mathbb{Z}$. Again, we refer to the illustration in Fig. 1.4. The associated Hamiltonian is

$$h(q,p) = \frac{1}{2}k(q^2+p^2) \tag{1.76}$$

and our $U(1)$-invariant positive definite metric

$$g = dq^2 + dp^2, \qquad g(X,X) = k^2(q^2+p^2). \tag{1.77}$$

We then have

$$\epsilon^{ab}\partial_a(g_{bc}X^c) = \partial_q X_p - \partial_p X_q = 2k. \tag{1.78}$$

$$k^4\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4k^4 \Rightarrow \sqrt{\det \partial_a\partial_b g(X,X)} = 2k^2 \tag{1.79}$$

Now $k_N = +1$ and $k_S = -1$, because we rotate anticlockwise at N and clockwise at S.

$$\lim_{\lambda\to\infty} Z_\lambda = (2\pi)\sum_{x_*=N,S}\frac{e^{i\alpha h(x_*)}}{k_*}2\pi\left(e^{i\alpha} - e^{-i\alpha}\right) \tag{1.80}$$

Therefore, the integral is

$$\int_{S^2} e^{i\alpha\cos\theta}\sin\theta\, d\theta \wedge d\phi = \frac{2\pi}{i\alpha}\left(e^{i\alpha} - e^{-i\alpha}\right). \tag{1.81}$$

2 Quantum Mechanics (QFT in $d = 1$)

Consider a single (free) particle in $\mathbb{R}^n$. In quantum mechanics, we describe this by some state $\Psi \in \mathscr{H} \simeq L^2(\mathbb{R}^n, d^n x)$ at any time t. As time evolves, our state Ψ changes under the action of some unitary operator $U : \mathscr{H} \to \mathscr{H}$, with $U(t) = e^{-iHt}$ if the Hamiltonian H is time-independent. We will often Wick rotate to Euclidean time $t \to -i\tau$, where the operator becomes $U(\tau) = e^{-H\tau}$, which is not unitary, but well behaved for Hermitian H.

If our particle is definitely at some $y_0 \in \mathbb{R}^n$ at $\tau = 0$, the amplitude to find it at $y_1 \in \mathbb{R}^n$ at $\tau = \beta$ is

$$\langle y_1 | \, e^{-\beta H} \, | y_0 \rangle = K_\beta(y_1, y_0), \tag{2.1}$$

which is called the *heat kernel* (or *propagator* in the non-Euclidean version). Explicitly, if our particle has $m = 1$ and $V(x) = 0$, then the Hamiltonian is $H = -\frac{1}{2}\nabla^2$ and

$$K_\tau(y_0, y_1) = \frac{1}{(2\pi\tau)^{n/2}} e^{-\frac{1}{2}|y_0 - y_1|^2}. \tag{2.2}$$

As illustrated in Fig. 2.1, we break up our time interval into N pieces with $\Delta\tau = \beta/N$. The heat

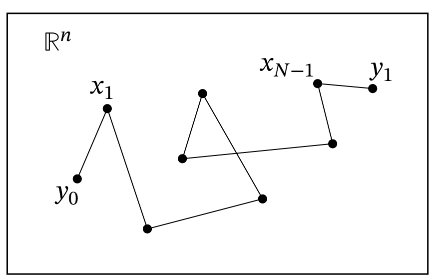

Figure 2.1: Constructing the path integral.

kernel can then be expanded by inserting identity operator expansions in between each time step

$$\langle y_1| e^{-\beta H} |y_0\rangle = \langle y_1| e^{-\Delta\tau H} e^{-\Delta\tau H} \dots e^{-\Delta\tau H} |y_0\rangle \tag{2.3}$$

$$= \int \langle y_1| e^{-\Delta t H} |x_{N-1}\rangle \langle x_{N-1}| e^{-\Delta t H} |x_{N-2}\rangle \dots \langle x_1| e^{-\Delta t H} |y_0\rangle \, \mathrm{d}^n x_1 \, \mathrm{d}^n x_2 \dots \mathrm{d}^n x_{N-1} \tag{2.4}$$

$$= \int K_{\Delta\tau}(y_1, x_{N-1}) \dots K_{\Delta\tau}(x_2, x_1) K_{\Delta\tau}(x_1, y_0) \prod_{i=1}^{N-1} \mathrm{d}^n x_1 \,. \tag{2.5}$$

If we set $S_N(x) = \frac{1}{2}\sum_{i=0}^{N-1} \frac{|x_{i+1}-x_i|^2}{(\Delta\tau)^2}\Delta\tau$ and $D_N x = \frac{1}{(2\pi\Delta\tau)^{nN/2}}\prod_{i=1}^{N-1} \mathrm{d}^n x_i$ then we can define the path integral as

$$\int e^{-S[x]}\, \mathcal{D}x := \int \lim_{N\to\infty} \left(e^{-S_N(x)} D_N x\right). \tag{2.6}$$

> Although the limit of the action and the measure separately does not exist, the product of them does exist in quantum mechanics. This is not true in quantum field theory, which is why we always need to explain how we take this limit; either by putting it on a lattice, imposing a cutoff, or some other regularisation procedure. The job of renormalisation is to explain how our answer depends on the way we take this limit.

Heuristically, the limit of the action is understood as $\lim_{N\to\infty} S_N(x) = \frac{1}{2}\int_0^\beta |\dot{x}|^2 \, \mathrm{d}\tau = S[x]$. Hence, as a path integral

$$\langle y_1| e^{-H\beta} |y_0\rangle = \int_{C_\beta[y_0,y_1]} e^{-S[x]}\, \mathcal{D}x, \tag{2.7}$$

where $C_\beta[y_0, y_1]$ is the space of continuous[1] maps $x : [0, \beta] \to \mathbb{R}^n$ starting at $x(0) = y_0$ and ending at $x(\beta) = y_1$.

Definition 12 (partition function)**:** Closely related to this is the *partition function* of QM, given by the trace over the Hilbert space $\mathscr{H}$

$$\mathcal{Z}(\beta) = \mathrm{tr}_{\mathscr{H}}(e^{-\beta H}). \tag{2.8}$$

We have

$$\mathcal{Z}(\beta) = \int_{\mathbb{R}^n} \langle y| e^{-\beta H} |y\rangle \, \mathrm{d}^n y = \int_{\mathbb{R}^n} \left(\int_{C_\beta[y,y]} e^{-S[x]}\, \mathcal{D}x \right) \mathrm{d}^n y \,. \tag{2.9}$$

Hence

$$\mathcal{Z}(\beta) = \int_{C_{S^1}} e^{-S[x]}\, \mathcal{D}x \tag{2.10}$$

over the space $C_{S^1} = \mathrm{Maps}(S^1_\beta, \mathbb{R})$ of continuous maps $x : S^1 \to \mathbb{R}^n$, where $S[x] = \oint_{S^1} \frac{1}{2}|\dot{x}|^2 \, \mathrm{d}\tau$ and the *worldline* is now a *worldcircle*, since the trace causes the integration to come back to the same point.

[1]The path in Fig. 2.1 is clearly continuous, but not necessarily smooth!

2.1 A Free Particle on a Circle

Let $S[x] = \oint \frac{1}{2}\dot{x}\,\mathrm{d}\tau$ be the action for a free particle, where $x(\tau) \sim x(\tau) + 2\pi R$, so the target space is $S^1_{2\pi R}$. The Hamiltonian for this free particle is, after canonical quantisation

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2}, \tag{2.11}$$

the Laplacian on a circle. Consequently, a basis of $\hat{H}$-eigenstates is $\phi_n(x) = e^{inx/R}$, where $n \in \mathbb{Z}$, with corresponding $\hat{H}$ -eigenvalues $E_n = n^2/2R^2$. Quantisation arises here because the target space is compact. Thus the partition function is

$$Z(\beta) = \mathrm{Tr}_{\mathscr{H}}(e^{-\beta H}) = \sum_{n\in\mathbb{Z}} e^{\beta n^2/2R^2}. \tag{2.12}$$

We can recast this using the Poisson resummation identity:

$$\sum_{n\in\mathbb{Z}} e^{-\frac{a}{2}(2\pi n)^2} = \int e^{-\frac{ax^2}{2}} \sum_{n\in\mathbb{Z}}^{\infty} \delta(x+2\pi n)\,\mathrm{d}x \tag{2.13}$$

$$= \frac{1}{2\pi}\int e^{-\frac{ax^2}{2}} (\sum_{m\in\mathbb{Z}} e^{imx})\,\mathrm{d}x \tag{2.14}$$

$$= \frac{1}{\sqrt{2\pi a}} \sum_{m\in\mathbb{Z}} e^{-\frac{m^2}{2a}}, \tag{2.15}$$

where in going to the second line we used that $\sum_{n\in\mathbb{Z}} \delta(x+2\pi n) = \sum_{m\in\mathbb{Z}} e^{imx}\frac{1}{2\pi}$. In our case,

$$Z(\beta) = \sum_{n\in\mathbb{Z}} e^{-\frac{\beta n^2}{2R^2}} = \sqrt{\frac{2\pi R^2}{\beta}} \sum_{m\in\mathbb{Z}} e^{-2\pi^2 m^2 R^2/\beta}. \tag{2.16}$$

Let us now recompute this using a path integral over all continuous maps $S'_\beta \to S'_{2\pi R}$. Such maps are classified by their winding number $m \in \mathbb{Z}$, as illustrated in Fig. 2.2. We can write this set of

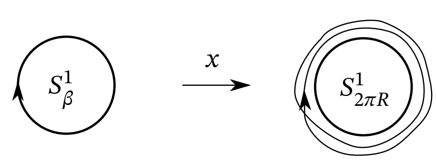

Figure 2.2: A map which winds around $S^1_{2\pi R}$ twice.

maps as the disjoint union

$$\mathrm{Maps}(S^1, S^1) = \bigsqcup_{m\in\mathbb{Z}} \mathrm{Maps}_m(S^1, S^1). \tag{2.17}$$

We take the path integral to include a sum $\sum_{m\in\mathbb{Z}}$ over these different topological sectors. To do this, let $x(\tau) = y(\tau) + 2\pi R m\tau/\beta$, where $y(\tau+\beta) = y(\tau)$ is periodic, so $\oint \dot{y}(\tau)\,\mathrm{d}\tau = 0$. Then

$$\int \mathcal{D}x\, e^{-S[x]} = \sum_{m\in\mathbb{Z}} \int \mathcal{D}y\, e^{-S[y+2\pi R m\tau/\beta]}. \tag{2.18}$$

For winding number m , the action is

$$S_m[x] = \frac{2m^2\pi^2R^2}{\beta} - \frac{1}{2}\oint_{S^1_\beta} y\ddot{y}\,\mathrm{d}\tau\,. \tag{2.19}$$

Since $y(\tau)$ is periodic, it has a Fourier series

$$y(\tau) = \frac{y_0}{\sqrt{\beta}} + \sum_{n=1}^{\infty}\left[y_n\sqrt{\frac{2}{\beta}}\cos\!\left(\frac{2\pi n\tau}{\beta}\right) + \widetilde{y}_n\sqrt{\frac{2}{\beta}}\sin\!\left(\frac{2\pi n\tau}{\beta}\right)\right] \tag{2.20}$$

and we take the path integral measure to be

$$\mathcal{D}y = \frac{\mathrm{d}y_0}{\sqrt{2\pi}}\prod_{n=1}^{\infty}\frac{\mathrm{d}y_n\,\mathrm{d}\widetilde{y}_n}{2\pi}. \tag{2.21}$$

Then, inserting this into the action, formally we find

$$Z(\beta) = \sum_{m\in Z} e^{-2m^2\pi^2R^2/\beta}\left(2\pi R\sqrt{\frac{\beta}{2\pi}}\right)\prod_{n=1}^{\infty}\left(\frac{\beta}{2\pi n}\right)^2. \tag{2.22}$$

Exercise 2.1: Do the Gaussian integrals to check this!

2.1.1 ζ-function Regularisation

This is formal because we have an infinite product, which requires regularisation. A nice way to do this is to use ζ -function regularisation. The Riemann ζ -function is defined for $\mathrm{Re}(s) > 1$ by an infinite sum

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{2.23}$$

It is then extended by analytic continuation to $s \in \mathbb{C} \setminus \{1\}$. In particular, $\sigma(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$. We will borrow this for our regularisation. In our case, we have a modified function

$$\widetilde{\zeta}(s) = \sum_{n=1}^{\infty}\left(\frac{2\pi n}{\beta}\right)^{-2s} = \left(\frac{\beta}{2\pi}\right)^{2s}\zeta(2s). \tag{2.24}$$

Differentiating term-by-term, we have

$$\widetilde{\zeta}'(0) = 2\zeta(0)\ln\!\left(\frac{\beta}{2\pi}\right) + 2\zeta'(0) \tag{2.25}$$

$$= -\ln\!\left(\frac{\beta}{2\pi}\right) - \ln 2\pi = -\ln\beta. \tag{2.26}$$

Thus, with ζ-function regularisation of the infinite product, we have

$$Z(\beta) = \sum_{m\in\mathbb{Z}} e^{-2\pi^2R^2m^2/\beta}\sqrt{\frac{2\pi R^2}{\beta}}, \tag{2.27}$$

in agreement with the canonical quantisation result.

2.2 Fermionic Quantum Systems

Take n $\mathbb{C}$-fermions $\psi^a(\tau)$ with action

$$S[\overline{\psi}, \psi] = \int \left[i\overline{\psi}_a \dot{\psi}^a - V(\overline{\psi}, \psi) \right] \mathrm{d}\tau. \tag{2.28}$$

The conjugate momentum to ψ^a is $\pi_a = \frac{\delta L}{\delta \dot{\psi}^a} = i\overline{\psi}_a$. Hence in canonical quantisation, we have

$$\{\hat{\psi}^a, \hat{\psi}^b\} = \hat{0} = \{\hat{\overline{\psi}}_b, \hat{\overline{\psi}}_a\}, \tag{2.29}$$

$$\{\hat{\psi}^a, \hat{\pi}_b\} = i\delta^a{}_b \quad \text{or equivalently} \quad \{\hat{\psi}^a, \hat{\overline{\psi}}_b\} = \delta^a{}_b. \tag{2.30}$$

These relations are reminiscent of the raising and lowering operators which obey $[A^a, A_b^\dagger] = \delta^a{}_b$ of a simple harmonic oscillator with $H = \hbar\omega(A_a^\dagger A^a + \frac{n}{2})$. We also define the fermion number operator $\hat{F} = \hat{\overline{\psi}}_a \hat{\psi}^a$, which obeys

$$[\hat{F}, \hat{\psi}^a] = -\hat{\psi}^a \qquad \& \qquad [\hat{F}, \hat{\overline{\psi}}_a] = +\hat{\overline{\psi}}_a. \tag{2.31}$$

This suggests that we treat $\hat{\overline{\psi}}_a$ as a raising and $\hat{\psi}^a$ as a lowering operator.

We define the vacuum $|0\rangle$ by being annihilated $\hat{\psi}^a |0\rangle = 0$ for all a. The Hilbert space is spanned by states ($|\{\psi^a\}| = n$)

$$\{|0\rangle, \hat{\overline{\psi}}_a |0\rangle, \hat{\overline{\psi}}_a \hat{\overline{\psi}}_b |0\rangle, \dots, \hat{\overline{\psi}}_{a_1} \dots \hat{\overline{\psi}}_{a_n} |0\rangle\}. \tag{2.32}$$

We can split the Hilbert space $\mathscr{H}$ into $\mathscr{H} = \mathscr{H}_B \oplus \mathscr{H}_F$, where $\mathscr{H}_B$ ($\mathscr{H}_F$) contain states with an even (odd) number of $\hat{\overline{\psi}}$'s.

We also have

$$(-1)^F |\Psi\rangle = \begin{cases} + |\Psi\rangle, & \text{if } \Psi \in \mathscr{H}_B \\ - |\Psi\rangle, & \text{if } |\Psi\rangle \in \mathscr{H}_F, \end{cases} \tag{2.33}$$

so $(-1)^F$ gives the *parity* of our state. Finally, we give $\mathscr{H}$ an inner product by declaring

$$\left(\hat{\overline{\psi}}_a \hat{\overline{\psi}}_b \dots \hat{\overline{\psi}}_c |0\rangle \right)^\dagger = \langle 0| \hat{\psi}^c \dots \hat{\psi}^b \hat{\psi}^a \tag{2.34}$$

and $\langle 0|0\rangle = 1$.

Example 2.2.1: The inner product between $\hat{\overline{\psi}}_a |0\rangle$ and $\hat{\overline{\psi}}_b |0\rangle$ is

$$\langle 0| \hat{\psi}^b \hat{\overline{\psi}}_a |0\rangle = \langle 0| \left(\{\hat{\psi}^b, \hat{\overline{\psi}}_a\} - \hat{\overline{\psi}}_a \hat{\psi}^b \right) |0\rangle = \delta^b{}_a \langle 0|0\rangle = \delta^b{}_a. \tag{2.35}$$

In fact, all states are orthonormal.

Just as for the harmonic oscillator, we have fermionic *coherent states* labelled by a fixed Grassmann parameter η^a defined as

$$|\eta\rangle = e^{-\eta^a \hat{\overline{\psi}}_a} |0\rangle\,, \tag{2.36}$$

which are eigenstates of the lowering operators $\hat{\psi}^a |\eta\rangle = \eta^a |\eta\rangle$. These are analogous to position eigenstates $\hat{x}^a |x'\rangle = x'^a |x'\rangle$ in the bosonic system. As such, they are useful building blocks for the path integral.

Claim 4:

- The unit operator on the Hilbert space can be expanded as

$$1_{\mathcal{H}} = \int e^{-\overline{\eta}_a \eta^a} |\eta\rangle \langle\overline{\eta}| \,\mathrm{d}\eta \tag{2.37}$$

- If $\hat{A} : \mathcal{H}_B \oplus \mathcal{H}_F \to \mathcal{H}_B \oplus \mathcal{H}_F$, then the supertrace is

$$\mathrm{Str}_{\mathcal{H}}(\hat{A}) := \mathrm{Tr}_{\mathcal{H}_B}(\hat{A}) - \mathrm{Tr}_{\mathcal{H}_F}(\hat{A}) = \int e^{-\overline{\eta}_a \eta^a} \langle\overline{\eta}| \hat{A} |\eta\rangle \,\mathrm{d}^2\eta\,, \tag{2.38}$$

whereas the normal trace is

$$Tr_{\mathcal{H}_B}(\hat{A}) + \mathrm{Tr}_{\mathcal{H}_F}(\hat{A}) = \int e^{-\overline{\eta}_a \eta^a} \langle-\overline{\eta}| \hat{A} |\eta\rangle \,\mathrm{d}^2\eta\,, \tag{2.39}$$

where $\langle\overline{\eta}| = \langle 0| \exp\big(-\hat{\psi}^a \overline{\eta}_a\big)$ is conjugate to $|\eta\rangle$.

Proof. Exercise. □

Exercise 2.2: Check signs!

2.3 Path Integral for Fermions

With the action $S = \int i\overline{\psi}_a \dot{\psi}^a - V(\overline{\psi}, \psi)\,\mathrm{d}\tau$, then $H = V(\overline{\psi}, \psi)$. Taking anticommutators if needed, we can choose to bring the $\hat{\psi}$ operators to the right of all $\hat{\overline{\psi}}$ operators in $\hat{H}$.

Let $|\chi\rangle$, $|\chi'\rangle$ be fermionic coherent states. As before, we define the heat kernel, the propagator between $|\chi\rangle$ and $|\chi'\rangle$ to be

$$\langle\overline{\chi}'| e^{-\beta H} |\chi\rangle = \langle\overline{\chi}'| e^{-\Delta\tau H} e^{-\Delta\tau H} \dots e^{-\Delta\tau H} |\chi\rangle \tag{2.40}$$

$$= \langle\overline{\chi}'| e^{-\Delta\tau H} |\eta_{N-1}\rangle \dots \langle\overline{\eta}_2| e^{-\Delta\tau H} |\eta_1\rangle \langle\overline{\eta}_1| e^{-\Delta\tau H} |\chi\rangle \prod_{i=1}^{N-1} e^{-\overline{\eta}_i \eta_i} \,\mathrm{d}^{2n}\eta_i\,. \tag{2.41}$$

We have for very small time intervals, $\Delta\tau$ infinitesimal,

$$\langle\overline{\eta}_{i+1}| e^{-\Delta\tau H(\hat{\overline{\psi}}, \hat{\psi})} |\eta_i\rangle = e^{-\Delta\tau H(\overline{\eta}_{i+1}, \eta_i)} \langle\overline{\eta}_{i+i}|\eta_i\rangle = e^{-\Delta\tau H(\overline{\eta}_{i+1}, \eta_i)} e^{\overline{\eta}_{i+1}\eta_i} \tag{2.42}$$

where we just took the linear term from the right hand side. The normal ordered Hamiltonian can then be replaced (to linear order) by the value on its eigenfunctions.

Thus

$$\left\langle \overline{\chi}' \right| e^{-\beta H} \left| \chi \right\rangle = \lim_{N\to\infty} \int \exp\left(\sum_{i=1}^{N} \overline{\eta}_i \eta_{i-1} - \Delta\tau V(\overline{\eta}_i, \eta_{i-1}) \right) \prod_{i=1}^{N-1} e^{-\overline{\eta}_i \eta_i} \, \mathrm{d}^{2n}\eta_i \tag{2.43}$$

$$= \lim_{N\to\infty} \int \exp\left(-\sum_{i=1}^{N} \left[\overline{\eta}_i \frac{\eta_i - \eta_{i-1}}{\Delta\tau} - V(\overline{\eta}_i, \eta_{i-1}) \right] \Delta\tau \right) e^{\overline{\eta}_N \eta_N} \prod_{i=1}^{N-1} \mathrm{d}^{2n}\eta_i \,, \tag{2.44}$$

where $\eta_0 := \chi$ and $\eta_N := \chi'$. The argument of exp(...) is a discretised version of the Euclidean action

$$S[\eta, \overline{\eta}] = \int_0^\beta \left[\overline{\eta}\dot{\eta} + V(\overline{\eta}, \eta) \right] \mathrm{d}\tau \,. \tag{2.45}$$

Thus, formally

$$\left\langle \overline{\chi}' \right| e^{-\beta H} \left| \chi \right\rangle = \int e^{-S[\overline{\psi}, \psi]} e^{\overline{\psi}(\beta)\psi(\beta)} \, \mathcal{D}\psi \, \mathcal{D}\overline{\psi}, \tag{2.46}$$

where $\psi(0) = \chi$ and $\psi(\beta) = \chi'$.

To construct the partition function, we must first choose whether ψ is periodic, meaning $\psi(\tau+\beta) = \psi(\tau)$, or antiperiodic $\psi(\tau+\beta) = -\psi(\tau)$. The antiperiodic version is allowed since each term in the action $S[\overline{\psi}, \psi]$ must contain an even number of fermions.

Let us look at both of these cases separately.

periodic This gives the supertrace, since

$$\mathrm{Str}_{\mathcal{H}}(e^{-\beta H}) = \left\langle \overline{\chi} \right| e^{-\beta H} \left| \chi \right\rangle e^{-\overline{\chi}\chi} \, \mathrm{d}^{2n}\chi = \int_{\text{periodic}} e^{-S[\overline{\psi}, \psi]} \, \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \,. \tag{2.47}$$

antiperiodic We now obtain the trace, since

$$\mathrm{Tr}\big(e^{-\beta H}\big) = \int \left\langle -\overline{\chi} \right| e^{-\beta H} \left| \chi \right\rangle e^{-\overline{\chi}\chi} \, \mathrm{d}^{2n}\chi = \int_{\text{antiperiodic}} e^{-S[\overline{\psi}, \psi]} \, \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \,. \tag{2.48}$$

Remark: Often, the ‘usual’ trace is called the *partition function* of the theory, whereas the supertrace is (in SUSY theories) known as the *Witten index*.

2.4 Supersymmetric Quantum Mechanics (SQM)

There are two different (closely related) types, depending on whether we have complex fermions ($\mathcal{N} = 2$ SQM) or real fermions ($\mathcal{N} = 1$ SQM).

For complex $\mathbb{C}$-fermions, the simplest action is

$$S[x, \overline{\psi}, \psi] = \int \frac{1}{2}\delta_{ab}\dot{x}^a\dot{x}^b + i\delta_{ab}\overline{\psi}^a\dot{\psi}^b \, \mathrm{d}t \,, \tag{2.49}$$

where we have Minkowski time on the worldline. We have canonical momenta $p_a = \delta_{ab}\dot{x}^b$ and $\pi = i\overline{\psi}_a$. Quantising this theory leads to canonical commutation relations for the bosons and anti-commutation relations for the fermions,

$$[\hat{p}_a, \hat{x}^b] = -i\delta_a^b, \qquad \{\hat{\overline{\psi}}_a, \hat{\psi}^b\} = \delta_a^b, \tag{2.50}$$

just as it did for the two theories separately.

It is natural to take the Hilbert space to be $\mathcal{H} = \Omega^*(\mathbb{R}^n, \mathbb{C})$ the space of all ($\mathbb{C}$-valued) forms on $\mathbb{R}^n$. Explicitly, if $\hat{\psi}^a \left|0\right\rangle = 0$ for all $a = 1, \ldots, n$, then

$$f^{ab\ldots c}(x)\hat{\overline{\psi}}^a\hat{\overline{\psi}}^b \ldots \hat{\overline{\psi}}^c \left|0\right\rangle \leftrightarrow f(x) = f_{ab\ldots c}(x)dx^a \wedge dx^b \wedge \cdots \wedge dx^c. \tag{2.51}$$

If f and g are forms of the same degree, then we have

$$(f, g) = \int_{\mathbb{R}^n} \overline{f} \wedge \star g = \int_{\mathbb{R}^n} \overline{f^{ab\ldots c}(x)} g_{ab\ldots c}(x) \, \mathrm{d}^n x \,. \tag{2.52}$$

If their degree differs, then $(f, g) = 0$. We also require the norm to be finite $\|f\|^2 = (f, f) < \infty$.

The $\mathcal{N} = 2$ SQM action is invariant under SUSY transformations

$$\delta x^a = \epsilon\overline{\psi}^a - \overline{\epsilon}\psi^a, \qquad \delta\psi^a = i\epsilon\dot{x}^a, \qquad \delta\overline{\psi}^a = -i\overline{\epsilon}\dot{x}^a, \tag{2.53}$$

where $\epsilon, \overline{\epsilon}$ are constant Grassmann parameters. Unlike in $d = 0$, here we have Noether charges Q and $\overline{Q}$, which generate these (by Poisson brackets).

$$Q = +i\delta_{ab}\dot{x}^a\overline{\psi}^b, \qquad \overline{Q} = -i\delta_{ab}\dot{x}^a\psi^b, \tag{2.54}$$

where $\delta = \overline{\epsilon}Q + \epsilon\overline{Q}$.

As usual, we have a Hamiltonian

$$H = p_a\dot{x}^a + \pi_a\dot{\psi}^a - L = \frac{1}{2}\delta^{ab}p_a p_b. \tag{2.55}$$

Claim 5: The Poisson brackets of our charges turn out to be

$$\{Q, \overline{Q}\}_{\mathrm{PB}} = -2iH, \tag{2.56}$$

realising the SUSY algebra in $d = 1$.

Representing the Hilbert space in terms of polyforms $\mathscr{H} \simeq \Omega^*(\mathbb{R}^n, \mathbb{C})$, then the supercharge becomes

$$\hat{Q} = i\hat{p}_a\hat{\overline{\psi}}^a \to dx^a \wedge \frac{\partial}{\partial x^a} = d, \tag{2.57}$$

the exterior derivative. The interpretation of $\hat{\overline{Q}}$ is slightly more subtle.

The adjoint of the supercharge is expected to strip off a form

$$Q^\dagger = i\hat{p}_a\hat{\psi}^a \to \iota_{\partial/\partial x^a}\frac{\partial}{\partial x^a} = d^\dagger, \tag{2.58}$$

where the contraction $\iota_{\frac{\partial}{\partial x^a}}(dx^b) = \delta^b{}_a$.

The operator $d^\dagger$ is the adjoint of d with respect to the inner product on $\mathscr{H}$ and can be also be written as $d^\dagger = (-1)^{n(p+1)+1} \star d \star$ acting on $\Omega^p(\mathbb{R}^n)$, where $\star : \Omega^p \to \Omega^{n-p}$ is the *Hodge star*. This follows from the adjoint statement:

$$\int \alpha \wedge \star d^\dagger \beta = (\alpha, d^\dagger\beta) = (d\alpha, \beta) = \int_{\mathbb{R}^n} d\alpha \wedge \star\beta, \tag{2.59}$$

and integration by part. Note also that $d^\dagger : \Omega^p \to \Omega^{p-1}$ as expected for a contraction.

Our action is also invariant under the (bosonic) $U(1)$ transformations that leave x unchanged but rotate the phase of ψ and $\overline{\psi}$ opposite ways

$$x^a \to x^a, \qquad \psi^a \to e^{i\alpha}\psi^\alpha, \qquad \overline{\psi} \to e^{-i\alpha}\overline{\psi}. \tag{2.60}$$

The associated Noether current is $F = \delta_{ab}\overline{\psi}^a\psi^b$ quantised as the Fermion number operator. This allows us to treat the space of polyforms $\mathscr{H} \simeq \Omega^*(\mathbb{R}^n, \mathbb{C})$ as a supervector space, with the bosonic and fermionic parts of the Hilbert space being

$$\mathscr{H}_B = \bigoplus_{p \text{ even}} \Omega^p(\mathbb{R}^n, \mathbb{C}) \qquad \text{and} \qquad \mathscr{H}_F = \bigoplus_{p \text{ odd}} \Omega^p(\mathbb{R}^n, \mathbb{C}). \tag{2.61}$$

Indeed F simply reads off the homogeneity of the form.

Remark: If $\omega \in \Omega^p$ and $\rho \in \Omega^q$, then $\omega \wedge \rho = (-1)^{pq}\rho \wedge \omega$.

2.5 $\mathcal{N} = 1$ SQM and Spinors

$\mathcal{N} = 1$ SQM is a theory of n real fermions with action

$$S[x, \psi] = \frac{1}{2}\int \delta_{ab}\dot{x}^a\dot{x}^b + i\delta_{ab}\psi^a\dot{\psi}^b \,\mathrm{d}\tau\,. \tag{2.62}$$

Remark: Note that the fermionic part of the action would not make sense if ψ^a were bosonic, since integration by part would induce a minus sign; this form of an action would therefore vanish for bosonic fields. For fermionic fields, the minus sign is cancelled by the change in order due to the integration by parts.

This is invariant under the supersymmetry

$$\delta x^a = \epsilon\psi^a \qquad \delta\psi^a = i\epsilon\dot{x}^a, \tag{2.63}$$

which is generated by the supercharge

$$Q = i\delta_{ab}\dot{x}^a\psi^b. \tag{2.64}$$

However, not that the bosonic $U(1)$ is broken to a $\mathbb{Z}_2$ subgroup; we cannot change the phase of ψ , only the sign.

Upon quantisation, we have standard commutation relations between $\hat{p}_b$ and $\hat{x}^a$ as $[\hat{p}_b, \hat{x}^a] = i\delta^a{}_b$ as before. But since

$$\pi_a = \frac{\delta L}{\delta\dot{\psi}^a} = \frac{i}{2}\psi^b d_{ab}, \tag{2.65}$$

the fermionic field is its own conjugate momentum! It obeys the standard anticommutation relations, which become

$$\{\hat{\psi}^a, \hat{\psi}^b\} = 2\delta^{ab}. \tag{2.66}$$

To find a representation of this $\mathscr{H}$, we need to split the $\hat{\psi}^a$'s into raising and lowering operators. We can do this in the language of forms. However, the $\hat{\psi}$'s are most naturally interpreted as γ-matrices acting on the space of spinors, since they obey (2.66), which is exactly the Clifford algebra.

First suppose n is even. Over $\mathbb{C}$, construct $\frac{n}{2}$ raising and lowering operators as

$$\gamma^i_\pm = \frac{1}{2}\left(\gamma^{2i} \pm i\gamma^{2i+1}\right), \tag{2.67}$$

for $i = 1, \ldots, \frac{n}{2}$. These obey the algebra of creation and annihilation operators

$$\{\gamma^i_{pm}, \gamma^j_\pm\} = 0 \qquad \{\gamma^i_+, \gamma^j_-\} = \delta^{ij}, \tag{2.68}$$

just as in $\mathscr{N} = 2$ SQM, but in half the dimension.

Starting from a spinor χ that obeys $\gamma^i_-\chi = 0$ for all i, we construct a basis of the space of spinors by acting with each raising operator γ^i_+ at most once (since they anti-commute, as before). Hence we obtain a representation of the spin group Spin(n) , with dimension $2^{n/2}$; for each value of i, there is a choice of whether to use or not use the raising operator.

Example 2.5.1: For $n = 4$, we have $2^{4/2} = 4$ components. This is the Dirac spinor that we are familiar with from QED.

The generators of Spin(n) act on this representation by $\Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$. These generators obey

$$[\Sigma^{ab}, \Sigma^{cd}] = \delta^{bc}\Sigma^{ad} + \delta^{ad}\Sigma^{bc} - \delta^{ac}\Sigma^{bd} - \delta^{bd}\Sigma^{ac}. \tag{2.69}$$

of Spin(n) $\simeq SO(n)$. Similarly to the familiar story in $n = 4$, since Σ^{ab} are quadratic in the γ 's, states with an odd / odd number of creation operators ψ^i_+ 's acting on the vacuum χ do not mix under $SO(n)$ transformations. Hence, the representation is *reducible*; there are two invariant non-trivial subspaces.

Definition 13 (chirality matrix)**:** We define the *chirality matrix* as the product of all the γ matrices

$$\gamma = (i)^{n/2}\gamma^1\gamma^2 \dots \gamma^n = \frac{i^{n/2}}{n!}\epsilon_{a_1 a_2 \dots a_n}\gamma^{a_1}\gamma^{a_2} \dots \gamma^{a_n}. \tag{2.70}$$

Remark: This is the generalisation of γ^5 in $d = 4$. We could call this γ^{2n+1}, but we will just call it γ to save writing large indices.

As in $d = 4$, we have

$$\gamma^2 = 1, \qquad \{\gamma, \gamma^a\} = 0, \qquad [\gamma, \Sigma^{ab}] = 0. \tag{2.71}$$

Since $\gamma^2 = 1$, its eigenvalues are ± 1.

Definition 14: We say a spinor is *chiral* if it is in the $+1$ eigenspace of γ and *antichiral* if in the -1 eigenspace.

Hence, the space of spinors splits into spinors of definite chirality

$$S = S^+ \oplus S^-. \tag{2.72}$$

Remark: In $d = 4$, this is the statement that we can decompose the 4-component Dirac spinor into two 2-component Weyl spinors.

Here, γ plays the role of $(-1)^F$. We do not have an operator F in this case, since we do not have the $U(1)$ symmetry; however, we do have the operator $(-1)^F$.

Remark: For n odd, we can construct a $2^{\lfloor \frac{n}{2} \rfloor}$-dimensional spin representation as before. However, now this representation is irreducible as γ^n appears in some of the Σ^{ab}.

To summarise, we represent $\mathscr{H}$ in $\mathcal{N} = 1$ SQM as the space $\Gamma(\mathbb{R}^n, S)$ of (L^2-integrable) Dirac spinors on $\mathbb{R}^n$. When acting on this space, the supercharge becomes

$$Q = i\psi^a p_a \to \not{\partial}, \tag{2.73}$$

the Dirac operator. When n is even, we can split this

$$\not{\partial} = \not{\partial}\left(\frac{1+\gamma}{2}\right) + \not{\partial}\left(\frac{1-\gamma}{2}\right) = \not{\partial}_+ + \not{\partial}_-, \tag{2.74}$$

where the $\frac{1}{2}(1 \pm \gamma)$ project onto $S^\pm$, the spaces of definite chirality. Due to the anticommutation relation $\{\gamma^a, \gamma\} = 0$, the chirality operator anticommutes with $\not{\partial}$, so

$$\not{\partial}_+ : \Gamma(\mathbb{R}^n, S^+) \to \Gamma(\mathbb{R}^n, S^-) \quad \text{and} \quad \not{\partial}_- : \Gamma(\mathbb{R}^n, S^-) \to \Gamma(\mathbb{R}^n, S^+). \tag{2.75}$$

In particular, $\not{\partial}_+^2 = \not{\partial}_-^2 = 0$.

3 Nonlinear Sigma Models

This is a theory of maps $x : [0, \beta] \to (M, g)$ to a Riemannian manifold. As illustrated in Fig. 3.1 the

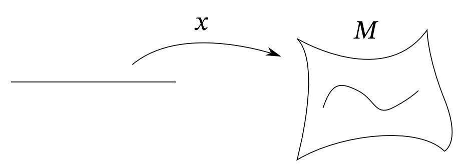

Figure 3.1: A map $x : [0, \beta] \to (M, g)$.

target space, in which we worldline lives, may now be curved.

The natural action in the bosonic case is

$$S[x] = \frac{1}{2} g(\dot{x}, \dot{x})\, \mathrm{d}\tau = \frac{1}{2} \int g_{ab} \dot{x}^a \dot{x}^b \, \mathrm{d}\tau \tag{3.1}$$

as we know from general relativity. Classically, trajectories are geodesics in M.

Remark: This is an interacting theory on $[0, \beta]$. If we consider perturbing around the constant map $x([0, \beta]) = x_0 \in M$, use Riemann normal coordinates to write $g_{ab}(x_0 + \delta x) = \delta_{ab} - \frac{1}{3} R_{abcd}(x_0) \delta x^c \delta x^d + O(\delta x^2)$ and the δx terms gives us interactions on the worldline.

We can try to quantise as usual by finding the canonical momenta

$$p_a = \frac{\delta L}{\delta \dot{x}^a} = g_{ab}(x) \dot{x}^b \tag{3.2}$$

and then imposing canonical commutation relations

$$[\hat{p}_a, \hat{x}^b] = i \delta^b_a. \tag{3.3}$$

If we represent the Hilbert space as $\mathscr{H} = L^2(M, \sqrt{g}\, \mathrm{d}^n x)$, then the momentum operator is a derivative operator $\hat{p}_a = -i \frac{\partial}{\partial x^a}$. However, the Hamiltonian is ambiguous: classically, $H = \dot{x}^a p_a - L = \frac{1}{2} g^{ab}(x) p_a p_b$, but we have to decide how to order the p's vs the x's in the curved metric $g^{ab}(x)$. Once we turn p into a derivative operator all these choices of positioning matter. There are a number of things we could require.

- $\hat{H}$ is at most second order in derivatives acting on $\mathscr{H}$.
- We could ask that $\hat{H}$ is compatible with Diff(M), so that $\hat{H}$ is some sort of covariant operator.
- Reduces to the usual Laplacian $-\frac{1}{2}\partial^a\partial_a$ in flat space when $g = \delta$.

However, these conditions are obeyed by the covariant Laplacian *plus* any multiple of the scalar curvature:

$$\hat{H}\Psi = -\frac{1}{2}\nabla^a\nabla_a\Psi + \alpha R\Psi = -\frac{1}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b\Psi) + \alpha R\Psi, \tag{3.4}$$

for any $\alpha \in \mathbb{R}$. Exactly which α we find depends on our normal ordering / path integral (from different regularisation procedures). This ambiguity is annoying and we would like to do better. We *can* do better in the $\mathcal{N} = 1$ supersymmetric version.

We have the action

$$S[x,\psi] = \frac{1}{2}\int g(\dot{x},\dot{x}) + ig(\psi,\nabla_\tau\psi)\,\mathrm{d}\tau \tag{3.5}$$

$$= \frac{1}{2}\int g_{ab}(x)\dot{x}^a\dot{x}^b + ig_{ab}(x)\psi^a\left(\dot{\psi}^b + \dot{x}^c\Gamma^b_{cd}\psi^d\right)\mathrm{d}\tau\,, \tag{3.6}$$

where $\Gamma = \Gamma(x(\tau))$. This action is invariant under the same supersymmetric transformations (2.63) as it was in flat space $(M,g) = (\mathbb{R}^n,\sigma)$,

$$\delta x^a = \epsilon\psi^a \qquad \delta\psi^a = i\epsilon\dot{x}^a. \tag{3.7}$$

In fact, not only is the action supersymmetric, but we also have

$$\mathcal{Q}\left(-\frac{i}{2}\int g_{ab}\psi^a\dot{x}^b\,\mathrm{d}\tau\right) = \frac{1}{2}\int\left(g_{ab}\dot{x}^a\dot{x}^b + ig_{ab}\psi^a\dot{\psi}^b - i\partial_c g_{ab}\psi^c\psi^a\dot{x}^b\right)\mathrm{d}\tau\,. \tag{3.8}$$

Comparing this to (3.6), we recognise the first two terms, but it is not obvious that the final term is the Christoffel symbol. In fact it is: we have

$$\partial_c g_{ab}\psi^c\psi^a = -\frac{1}{2}\left(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ba}\right)\psi^c\psi^a = -g_{cd}\Gamma^d_{ab}\psi^c\psi^a. \tag{3.9}$$

So the action itself is $S = \mathcal{Q}(\dots)$, we say it is $\mathcal{Q}$*-exact*. This will be crucial for localisation of the path integral.

It will be useful to do the canonical quantisation first. We are expecting to get some sort of spinors in curved space, like we did in flat space. We expect the Dirac operators that we got in the flat case to turn into some sort of covariant Dirac operators.

The momenta (p_a,π_a) are

$$p_a = \frac{\delta L}{\delta\dot{x}^a} = g_{ab}\dot{x}^b + ig_{bc}\psi^b\Gamma^c_{ad}\psi^d, \qquad \pi_a = \frac{\delta L}{\delta\psi^a} = ig_{ab}\psi^b. \tag{3.10}$$

Therefore, upon quantisation we find our usual commutation relations for the x and p: $[\hat{x}^a, \hat{p}_b] = i\delta^c_b$, and similarly for the π's and ψ's, but the anticommutation relation for the fermions is now

$$\{\hat{\psi}^a, \hat{\psi}^b\} = 2g^{ab}(x). \tag{3.11}$$

The fact that the $\hat{\psi}$ anticommutators involve the field x is a little awkward. How can it be that the commutators of the different fields know about each other?

To do better, we introduce at each point $x \in M$ an orthonormal set of basis vectors $\{e_i = e_i^a(x)\frac{\partial}{\partial x^a}\}$ on the tangent space T_xM, often called a *frame*. Orthonormality here means that

$$g(e_i, e_j) = g_{ab}e^a{}_i e^b{}_j = \delta_{ij}. \tag{3.12}$$

We have a dual basis of 1-forms, often called *vielbeins*, unfortunately conventionally denoted by the same letter $\{e^i = e^i{}_a dx^a\}$, which are dual in the sense $e_i(e^j) = e_i{}^a e_a{}^j = \delta^i_j$.

Since the e_i are a basis, we have completeness relations

$$g^{-1} = \delta^{ij} e_i \otimes e_j \qquad\qquad g = \delta_{ij} e^i \otimes e^j \tag{3.13}$$

$$\text{i.e.}\quad g^{ab} = e^a{}_i e^b{}_j \delta^{ij} \qquad\qquad \text{i.e.}\quad g_{\alpha\beta} = e^i{}_a e^j{}_b \delta_{ij}. \tag{3.14}$$

Remark: You might interpret this by saying that the e_j are the 'square root' of the metric.

Expanding our fermions in this basis, we have $\psi^a = \psi^i e^a{}_i$ or $\psi^i = e^i{}_a \psi^a$, where the $\hat{\psi}^i$'s obey

$$\{\hat{\psi}^i, \hat{\psi}^j\} = \{e^i{}_a \hat{\psi}^a, e^j{}_b \hat{\psi}^b\} = e^i{}_a e^j{}_b \{\hat{\psi}^a, \hat{\psi}^b\} = 2e^i{}_a e^j{}_b g^{ab} = 2\delta^{ij}. \tag{3.15}$$

In the vielbein basis, the $\hat{\psi}$'s obey exactly the same anticommutation relations as in flat space.

So far, we have an orthonormal frame $\{e_i(x)\}$ at each point $x \in M$, and in general this choice may vary over M. To compare, we introduce a connection ∇ on the tangent bundle TM by

$$\nabla(e_i) = de_i + \omega_i{}^j e_j, \tag{3.16}$$

where the connection 1-form $\omega^i{}_j = dx^a \omega_a(x)^i{}_j$ is the *spin connection*.

Remark: In (3.16) we supressed the spacetime index a.

Each of the spin connection components is an antisymmetric matrix $\omega^{ij} = \delta^{jk}\omega^i{}_k = -\omega^{ji}$, since it preserves orthonormality. A priori, ω has nothing to do with Γ, but we usually (basically always) impose compatibility by the torsion-free condition on the frames

$$\nabla_a e^b{}_i = \partial_a e^b{}_i + \Gamma^b{}_{ac} e^c{}_i + \omega_a{}^j{}_i e^b{}_j = 0 \tag{3.17}$$

$$\text{i.e.}\quad (\omega_a)^i{}_j = e^i{}_b \Gamma^b{}_{ac} e^c{}_j + e^i{}_b \partial_a e^b{}_j. \tag{3.18}$$

We get exactly the same expression from our fermion action:

$$g(\psi, \nabla_\tau \psi) = g_{ab}\psi^a \nabla_\tau \psi^b = g_{ab} e^a{}_i \psi^i \left(\frac{\partial}{\partial \tau}(e^b{}_j \psi^j) + \dot{x}^c \Gamma^b{}_{cd} e^d{}_j \psi^j \right) \tag{3.19}$$

$$= \delta_{ij}\psi^i \partial_\tau \psi^j + e_{bi}\dot{x}^a \partial_a e^b{}_j \psi^i \psi^j + e_{ib}\dot{x}^a \Gamma^b{}_{ac} e^c{}_j \psi^i \psi^j \tag{3.20}$$

$$= \delta_{ij}\psi^i \left(\partial_\tau \psi^j + \dot{x}^a (\omega_a)^j{}_k \psi^k \right) \tag{3.21}$$

$$= \delta(\psi, \nabla_\tau \psi) \tag{3.22}$$

where $e_{bi} = g_{ab} e^a{}_i$.

The result (3.18) is just what we would get by changing coordinates $x^a \longmapsto x^i$, where $e^i{}_a = \frac{\partial y^i}{\partial x^a}$. In particular, $R[\omega]^i{}_j = R[\Gamma]^c{}_d e^i{}_c e^d{}_j$ where $R[\omega] = d\omega + \frac{1}{2}[\omega, \omega]$ is the curvature 2-form of the spin connection and $R[\Gamma]^c{}_d = R_{ab}{}^c{}_d dx^a \wedge dx^b = (d\Gamma + \frac{1}{2}[\Gamma, \Gamma])^c{}_d$ is the Riemann curvature tensor.

In terms of the ψ^i's ($\psi^i = e^i{}_a \psi^a$), our supercharge becomes

$$Q = ig_{ab}\dot{x}^a\psi^b = e^a{}_i\psi^i(ip_a + \frac{1}{2}\omega_{a\,jk}\psi^j\psi^k). \tag{3.23}$$

There is now no longer any ordering ambiguity between the ψ's and p's. We teased this ambiguity out into the frames $e^a{}_i$, and the ψ's and p's simply commute.

So under quantisation with $\hat{p}_a \longmapsto -i\partial_a$ and $\hat{\psi}^i \rightarrow \gamma^i$, we find that $\hat{Q}$ is the *covariant Dirac operator*

$$\hat{Q} = e^a{}_i\gamma^a(\partial_a + \frac{1}{2}\omega_{a\,jk}\gamma^j\gamma^k) = e^a{}_i\gamma^i(\underbrace{\partial_a + \omega_{a\,jk}\Sigma^{jk}}_{\nabla_a})) = \not{\nabla}, \tag{3.24}$$

where $\Sigma^{jk} = \frac{1}{4}[\gamma^j, \gamma^k]$ are the generators of $so(n)$ in Dirac spinor representations.

Furthermore, the normal ordering ambiguity in $\hat{H}$ is resolved in the $\mathcal{N} = 1$ model, since our supersymmetry algebra tells us that

$$\hat{H} = -\hat{Q}^2 = \not{\nabla}\not{\nabla} = \gamma^i e^a{}_i\nabla_a\left(\gamma^j e^b{}_j\nabla_b\right) = \gamma^i\gamma^j e^a{}_i e^b{}_j\nabla_a\nabla_b \tag{3.25}$$

$$= \left(\frac{1}{2}\{\gamma^i, \gamma^j\} + \frac{1}{2}[\gamma^i, \gamma^j]\right)e^a{}_i e^b{}_j\nabla_a\nabla_b \tag{3.26}$$

$$= g^{ab}\nabla_a\nabla_b + \Sigma^{ij}e^a{}_i e^b{}_j[\nabla_a, \nabla_b] \tag{3.27}$$

$$= \Delta + \Sigma^{ij}e^a{}_i e^b{}_j R^{abkl}\Sigma^{kl} \overset{\text{ex.}}{=} \Delta + R, \tag{3.28}$$

where showing the last equality, with R being the Ricci scalar curvature, is an exercise. The fact that we want our Hamiltonian to be $\hat{H} = \hat{Q}^2$ fixes the amount of scalar curvature, which we we previously had some freedom in, as demonstrated in Eq. (3.4).

3.1 The Atiyah–Singer Theorem

We work with $n = \dim M$ even. Let us now compute the supersymmetric partition function of our $\mathcal{N} = 1$ NLSM. In the canonical framework, this is $\mathrm{Str}_{\mathscr{H}}\left(e^{-\beta\hat{H}}\right)$. It is easy to see we only get contributions from states with $E = 0$: suppose $|\Phi\rangle \in \mathscr{H}_B$ and $\hat{H}|\Phi\rangle = E|\Phi\rangle$ with $E \neq 0$. Then we can write the state as

$$|\Phi\rangle = -\frac{1}{E}Q^2|\Phi\rangle = Q|\chi\rangle, \tag{3.29}$$

where $|\chi\rangle = -\frac{1}{E}Q|\Phi\rangle \in \mathscr{H}_F$ is a fermionic state. And since $[\hat{H}, \hat{Q}] = 0$ we have $\hat{H}|\chi\rangle = E|\chi\rangle$. Thus states with $E \neq 0$ come in boson/fermion pairs, and their contribution cancels in $\mathrm{Str}(e^{-\beta\hat{H}})$.

Also, since $H = -Q^2$, if $Q\left|\Phi\right\rangle = 0$, then $\hat{H}\left|\Phi\right\rangle = 0$, so $\ker(\hat{H}) \supset \ker(\hat{Q})$. Furthermore, if $\hat{H}\left|\chi\right\rangle = 0$, then

$$0 = \left\langle\chi\right|\hat{H}\left|\chi\right\rangle = -\left\langle\chi\right|Q^2\left|\chi\right\rangle = -\|Q\left|\chi\right\rangle\|^2, \tag{3.30}$$

since $Q^\dagger = Q$ with real fermions ψ^a. Therefore, $Q\left|\chi\right\rangle = 0$ and $\ker(Q) \supset \ker(H)$. Combining these, we have $\ker(Q) = \ker(H)$. Therefore, since $\gamma = (-1)^F$, the supersymmetric partition function is

$$\mathrm{STr}_{\mathcal{H}}(e^{-\beta H}) = \mathrm{Tr}_{\mathcal{H}}(e^{-\beta H}\gamma) = \mathrm{Tr}_{\mathcal{H}}(e^{-\beta H}\frac{1+\gamma}{2}) - \mathrm{Tr}_{\mathcal{H}}(e^{-\beta H}\frac{1-\gamma}{2}). \tag{3.31}$$

However, we know that we will just get contributions of the ground state, so this counts the number of chiral minus the number of antichiral ground states.

$$\mathrm{STr}_{\mathcal{H}}(e^{-\beta H}) = \dim\ker(\not{\nabla}_+) - \dim\ker(\not{\nabla}_-) := \mathrm{ind}(\not{\nabla}), \tag{3.32}$$

where $\not{\nabla}_\pm = \not{\nabla}\left(\frac{1\pm\gamma}{2}\right)$ and $\mathrm{ind}(\not{\nabla})$ is called the *index* of the Dirac operator on (M, g). We got this by canonical quantisation and understanding the spinor states on the Hilbert space.

We get an alternative expression for the $\mathrm{ind}(\not{\nabla})$ by examining the path integral

$$\mathrm{STr}(e^{-\beta H}) = \int_P \mathcal{D}x\,\mathcal{D}\psi\, e^{-S[x,\psi]}, \tag{3.33}$$

where the important thing about the action was that it was itself the supersymmetry transformation of something:

$$S[x,\psi] = Q\left(\frac{i}{2}\oint_{S^1_\beta} g(\dot{x},\psi)\,\mathrm{d}\tau\right) = \frac{1}{2}\oint_{S^1_\beta} g(\dot{x},\dot{x}) + ig(\psi,\nabla_\tau\psi)\,\mathrm{d}\tau\,. \tag{3.34}$$

The path P in (3.32) is periodic with $x^a(\tau+\beta) = x^a(\tau)$ and $\psi^a(\tau+\beta) = \psi^a(\tau)$.

Since the action is Q-exact, we can rescale $g \longmapsto \lambda g$ for $\lambda \in \mathbb{R}_+$ and the super partition function will be invariant. As $\lambda \to \infty$, it only receives contributions from a neighbourhood of *constant* maps $x(S^1_\beta) = x_0 \in M$, so $\dot{x} = 0$. Since any rescaling $\lambda \to \infty$ will give an infinitely large contribution to the action, which then suppresses its contribution in the path integral. By the same reasn, we also localise to constant fermions $\dot{\psi} = 0$. Let us expand around such maps

$$x(\tau) = x^a_0 + \delta x^a(\tau), \qquad \psi(\tau) = \psi^a_0 + \delta\psi^a(\tau), \tag{3.35}$$

where $\oint \delta x^a(\tau)\,\mathrm{d}\tau = 0 = \oint \delta\psi^a(\tau)\,\mathrm{d}\tau$ and $\psi^a_0 \in \Pi T_{x_0}M$.

To expand the action $S[x,\psi]$ to quadratic order in fluctuations, it is helpful to use Riemann normal coordinates: The metric near any point x_0 can always be written as

$$g_{ab}(x_0 + \delta x) = \delta_{ab} - \frac{1}{3}R^{acbd}(x_0)\delta x^c \delta x^d + O(\delta x^3). \tag{3.36}$$

Similarly, the connection components in this system of coordinates is

$$\Gamma^c_{ab}(x_0 + \delta x) = -\frac{1}{3}(R^c{}_{abd}(x_0) + R^c{}_{bad}(x_0))\delta x^d + O(\delta x^2). \tag{3.37}$$

Putting everything together, we find that the quadratic part of the action, expanded to quadratic order in the fluctuations, is

$$S^{(2)}[x_0+\delta x,\psi_0+\delta\psi]=\frac{1}{2}\int -\delta_{ab}\delta x^a\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\delta x^b+i\delta_{ab}\delta\psi^a\frac{\mathrm{d}}{\mathrm{d}\tau}\delta\psi^b-\frac{1}{2}R_{abcd}\psi_0^a\psi_0^b\delta x^c\frac{\mathrm{d}}{\mathrm{d}\tau}\delta x^d. \quad (3.38)$$

where we have integrated by parts to put both derivatives on one of the δx's in the first term and the last term comes from careful application of the Bianchi identity.

Performing the (Gaussian) path integrals over the fluctuations δx and $\delta\psi$, we obtain

$$\frac{\sqrt{\det'(\delta^a{}_b\partial_\tau)}}{\sqrt{\det'(-\delta^a{}_b\partial_\tau^2+R^a{}_b\partial_\tau)}}=\frac{1}{\sqrt{\det'(-\delta^a{}_b\partial_\tau+R^a{}_b)}}, \quad (3.39)$$

where $R^a{}_b=R^a{}_b(x_0,\psi_0)=R_{cd}{}^a{}_b(x_0)\,\psi_0^c\psi_0^d$ and $\det'$ is the determinant removing zero modes of the operators[1].

[1]We are going to integrate over those later.

Let us now examine this determinant. We decompose the tangent space $T_{x_0}M$ into eigenspaces of $\mathcal{R}^a{}_b$ such that restriction of $\mathcal{R}|$ to the i^{th} eigenspace, which is even-dimensional since we are on an even-dimensional manifold, looks like $\mathcal{R}_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix}$. Let D_i be the restriction of the operator we are interested in, $\delta^a{}_b\partial_\tau - \mathcal{R}^a{}_b$ to this subspace. Moreover, we write $\delta(x^a)(\tau) = \sum_{k\neq 0} \delta x^a{}_k e^{2\pi i k\tau}$ in terms of (non-zero) Fourier modes. Then the eigenvalues of D_i are $2\pi ik \pm \omega_i$, where the factor of $2\pi ik$ is picked up by the derivative. Hence, the determinant of the non-zero modes of the restricted operator is

$$\det{}'(D_i) = \prod_{k\in\mathbb{Z}\setminus\{0\}} (2\pi ik + \omega_i)(2\pi ik - \omega_i) = \prod_{k\neq 0}(-(2\pi k)^2 - w_i^2) = \prod_{k=1}^{\infty}(2\pi k)^4 \prod_{k=1}^{\infty}\left(1 + \frac{\omega_i^2}{(2\pi k)^2}\right)^2. \tag{3.40}$$

Again we can use ζ-function regularisation, from which we obtain

$$\prod_{n=1}^{\infty}(2\pi k)^2 = (4\pi^2)^{\zeta(0)} e^{-2\zeta'(0)} = 1. \tag{3.41}$$

This is us waving our hands and not looking too closely, so that we can drop the infinite divergent product.

> If we were doing a more careful job, we would have had to regularise both the fermionic and the bosonic path integrals; if we are doing the regularisation for both in the same way everything works out.

The remaining factor is the product expansion

$$\frac{\sinh(z)}{z} = \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{\pi^2 n^2}\right). \tag{3.42}$$

Comparing this to (3.40), with $z = \omega_i/2$, we find that

$$\sqrt{\det{}' D_i} = \frac{\sinh(\omega_i/2)}{(\omega_i/2)}. \tag{3.43}$$

Combining the factors from the $n/2$ eigenspaces of $\mathcal{R}^a{}_b$, we have that the term (3.39) coming from the fluctuations around the zero-map is

$$\frac{1}{\sqrt{\det{}'(\delta^a{}_b\partial_\tau - \mathcal{R}^a{}_b)}} = \prod_{i=1}^{n/2} \frac{(\omega_i/2)}{\sinh(\omega_i/2)} = \det\left(\left(\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)}\right)^a{}_b\right), \tag{3.44}$$

where the matrix inside the determinant is understood via its Taylor expansion.

This was the term coming from fluctuations about constant zero modes. Finally, we must perform the integral over the zero modes (x_0, ψ_0). The action vanishes on these zero modes, but they enter through $\mathcal{R}^a{}_{bcd}(x_0)\,\psi_0^c\psi_0^d$. We obtain

$$Z(\beta) = \int \det\left(\frac{R(x_0,\psi_0)/2}{\sinh(\mathcal{R}(x_0,\psi_0)/2)}\right) \mathrm{d}^n x_0\, \mathrm{d}^n\psi_0 \tag{3.45}$$

We expand the (slightly complicated) Taylor series until we hit the term with n fermions. Performing the fermionic integration then gives

$$Z(\beta) = \int_M \det\left(\frac{\mathcal{R}/2}{\sinh(\mathcal{R}/2)}\right)^{(n)} = \int \hat{A}(M). \tag{3.46}$$

Where the fermionic integral can be written as extracting the top-form (n-form) part of an integral over M. In other words, we now think of $\mathcal{R}^a{}_b(x_0) = \mathcal{R}^a{}_{bcd}dx^c \wedge dx^d$. This combination is often knows as the $\hat{A}$-genus of M. Explicitly, we have

$$\hat{A}(M) = 1 - \frac{1}{24}p_1(M) + \frac{7p_1^2(M) - 4p_2(M)}{5760} + \dots, \tag{3.47}$$

$$p_1(M) = -\frac{1}{2}\frac{1}{(2\pi)^2}\,\mathrm{tr}(\mathcal{R}\wedge\mathcal{R}), \tag{3.48}$$

$$p_2(M) = \frac{1}{8(2\pi)^4}\left((\mathrm{tr}\,\mathcal{R}\wedge\mathcal{R})^2 - 2\,\mathrm{tr}(\mathcal{R}\wedge\mathcal{R}\wedge\mathcal{R}\wedge\mathcal{R})\right). \tag{3.49}$$

The *Pontryagin classes* p_1 and p_2 are polynomials in traces of powers of the curvature. We see that

$$\mathrm{ind}(\not\nabla) = \int_M \hat{A}(M). \tag{3.50}$$

This is the *Atiyah–Singer* theorem for the Dirac operator. Since $z/\sinh(z)$ is an even function, we only get expansions in $z^2 \to \mathcal{R}^2$. Hence $\mathrm{ind}(\not\nabla) = 0$ whenever $\dim(M) = 4k+2$.

3.1.1 Coupling to a Vector Bundle

We may wish to describe a charged spinor (electron!) moving on M. To do this (in the Abelian case), we modify the action to include a coupling to a gauge field $A_a(x)$ on M.

$$S[x,\psi] = \int \left[\frac{1}{2}g(\dot{x},\dot{x}) + \frac{i}{2}g(\psi,\nabla_\tau\psi) + iA_a(x)\dot{x}^a + \frac{1}{2}F_{ab}(x)\psi^a\psi^b\right] \mathrm{d}\tau\,, \tag{3.51}$$

where $F = dA = \frac{1}{2}(\partial_a A_b - \partial_b A_a)dx^a \wedge dx^b$ is the EM fieldstrength, pulled back to the worldline. The bosonic term involves $\dot{x}$, so modifies the momentum operator $p_a = g_{ab}\dot{x}^b + iA_a + \text{ferms}$. Upon quantisation, the supercharge

$$Q = ig_{ab}\psi^a\dot{x}^b \to \gamma^i e^a{}_i(\partial_a + (\omega_a)_{ij}\Sigma^{ij} + iA_)). \tag{3.52}$$

The insertion

$$\exp\left(-i\oint_{S^1} A_a(x(\tau))\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\,\mathrm{d}\tau\right) \tag{3.53}$$

is the holonomy / Wilson line around $x(S^1)$, i.e. the *phase* a charged particle would acquire as it travels around a loop $x(S^1) \subset M$.

Consider Fig. 3.2. If Ψ is parallel transported from x to y along Γ, meaning that $V^a(\partial_a + iA_a)\Psi = 0$, where V is tangent to Γ,

$$\Psi(y) = \exp\left(-i\int_x^y A_a(x)V^a(x)\,\mathrm{d}\tau\right)\Psi(x). \tag{3.54}$$

Since our $\mathcal{N} = 1$ theory gives a spinor on M, we also get a contribution from its magnetic moment

$$\frac{1}{2}F_{ab}\gamma^a\gamma^b = \frac{1}{4}F_{ab}[\gamma^a,\gamma^b] = F_{ab}\Sigma^{ab}. \tag{3.55}$$

Example 3.1.1: In $n = \dim(M) = 3$, then $\Sigma^{ab} = \epsilon^{abc}\sigma^c$ and $F_{ab}\Sigma^{ab} \longmapsto \boldsymbol{\sigma}\cdot\mathbf{B}$.

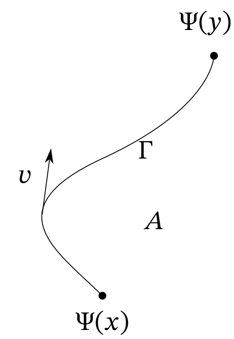

Figure 3.2: A charged fermion moving along trajectory Γ in a background field A.

The modified action is still invariant under the original supersymmetry transformations

$$\delta x^a = \epsilon\psi^a \qquad \delta\psi^a = i\epsilon\dot{x}^a, \tag{3.56}$$

with supercharge $Q = ig_{ab}\psi^a\dot{x}^b$. This is quantised as the gauge covariant Dirac operator $\not{D}$. Thus, when computing the supertrace we still get cancellation

$$\mathrm{Str}_{\mathscr{H}}(e^{-\beta\hat{H}}) = \mathrm{ind}(\not{D}). \tag{3.57}$$

What will happen on the path integral side? The new terms in $S[x,\psi]$ are independent of the target space metric g so they do not change the fact that we localise on constant maps. They also do not affect the fluctuation contributions, because they do not come with any λ. They the result is only modified by $S[x_0,\psi_0] = -\frac{1}{2}F_{ab}(x_0)\psi_0^a\psi_0^b$.

$$\Rightarrow\ \mathrm{ind}(\not{D}) = \int_M \left(\hat{A}(M)e^{-F}\right)^{(n)}. \tag{3.58}$$