# General Relativity

Part III Michaelmas 2019

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# 1 Introducing Differential Geometry

Structure: will do only differential geometry (maths methods) in the first 12 lectures, first 4 weeks. Afterwards the connection to the physics will be made. We will develop a mathematical language that allows us to write valid equations involving vectors, similar to dimensional analysis for physicists. The physics will be introduced with the action principle.

Extra book: "Geometry, Topology, and Physics" by Nakahara.

Office Hours: Friday 4 $\to$ 5 in B2.13

Extra stuff not examinable. Lectures are what matters.

## 1.1 Manifolds

**Definition 1:** An *n-dimensional manifold* $\mathcal{M}$ is a space that locally looks like $\mathbb{R}^n$. More precisely, we require

1. For each point $p \in \mathcal{M}$, there is a map $\phi : \mathcal{O} \to U$ where $\mathcal{O} \subset \mathcal{M}$ is an open set with $p \in \mathcal{O}$ and $U \subset \mathbb{R}^n$. We will think of $\phi(p) = (x^1(p), \dots, x^n(p))$ as coordinates on $\mathcal{O} \subset \mathcal{M}$.

   This map must be a *homeomorphism*:

   (a) injective (or 1-1) $p \neq q \implies \phi(p) \neq \phi(q)$

   (b) surjective (or onto) $\phi(\mathcal{O}) = U$

   These ensure that $\phi^{-1}$ exists.

   (c) both $\phi$ and $\phi^{-1}$ are continuous

2. If $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ are two open sets with

$$\phi_\alpha : \mathcal{O}_\alpha \to U_\alpha \text{ and } \phi_\beta : \mathcal{O}_\beta \to U_\beta \tag{1.1}$$

then the *transition functions* $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \to \phi_\alpha(\mathcal{O}_\alpha \cap \mathcal{O}_\beta)$ are smooth (ie. infinitely differentiable). The $\phi_\alpha$ are called *charts*. The idea is that there may be different ways to assign coordinates to a given point $p \in \mathcal{M}$. The collection of all charts is called an *atlas*. We require that these coordinate systems are mutually compatible. This is depicted in Figure 1.1.

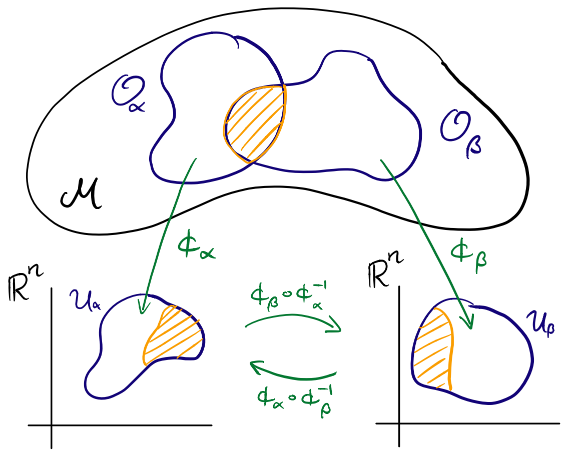


Figure 1.1: An illustration of charts on a manifold

**Example 1.1.1** ($\mathbb{R}^n$)**:** $\mathbb{R}^n$ or any open subset of $\mathbb{R}^n$ is a manifold. You only need a single chart.

**Example 1.1.2** ($S^1$)**:** We can view this as $(\cos\theta, \sin\theta) \in \mathbb{R}^2$ with $\theta \in [0, 2\pi)$. The closed set $[0, 2\pi)$ means that we cannot differentiate at 0. This is not a good chart because it is not an open set. We need at least two charts, depicted in 1.2:

$$\begin{aligned} \phi_1 : \mathcal{O}_1 &\to (0, 2\pi) & \phi_2 : \mathcal{O}_2 &\to (-\pi, \pi) \\ p &\longmapsto \theta_1 & p &\longmapsto \theta_2 \end{aligned} \tag{1.2}$$

The transition function is

$$\theta_2 = \phi_2(\phi_1^{-1}(\theta_1)) = \begin{cases} \theta_1, & \theta_1 \in (0, \pi) \\ \theta_1 - 2\pi, & \theta_1 \in (\pi, 2\pi) \end{cases} \tag{1.3}$$

**Remark:** The fact that the coordinates go bad in the case of closed sets, similar to spherical polar coordinates, does not bother us too much for physical applications.

Since we can map $\mathcal{M} \to \mathbb{R}^n$ (at least locally), anything we can do on $\mathbb{R}^n$, we can now also do on $\mathcal{M}$ (e.g. differentiation).

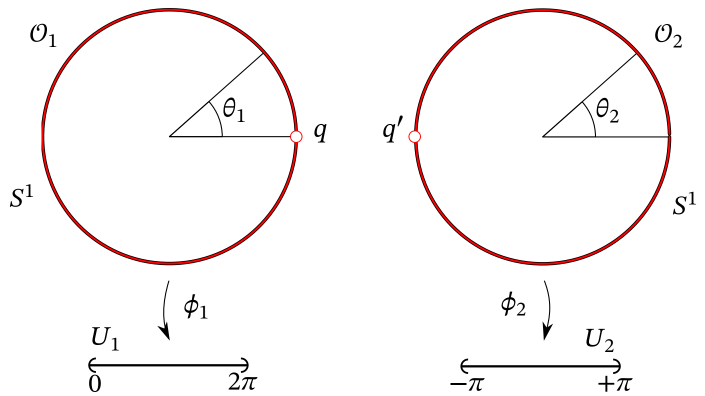


Figure 1.2: The two charts $\phi_1, \phi_2$ form an atlas of the manifold $S^1$.

> **Remark:** Note that at the moment, the distance in $\mathbb{R}^n$ cannot be translated back to the manifold. This is because the maps $\phi_\alpha$ are arbitrary.

**Definition 2** (Diffeomorphism)**:** A *diffeomorphism* is a smooth homeomorphism $f : \mathcal{M} \to \mathcal{N}$ between two manifolds. I.e. two manifolds are diffeomorphic if the map $\psi \circ f \circ \phi^{-1} : U \to V$ is smooth for all charts $\phi : \mathcal{M} \to U \subset \mathbb{R}^n$ and $\psi : N \to V \subset \mathbb{R}^n$.

> **Remark:** There are interesting properties of $S^7$ and $\mathbb{R}^4$. You can find two atlasses of these manifolds such that the two atlasses are not diffeomorphic to each other. These are called *exotic* manifolds; they are homeomorphic but not diffeomorphic to their respective Euclidean counterparts. As far as we know, there is yet no application of this in physics.

> **Remark:** These manifolds have intrinsic meaning and do not depend on an embedding. Physically, the 3 + 1-dimensional spacetime is a manifold that we do not think is embedded in any higher dimensional space.

## 1.2 Tangent Spaces

So far, we have the idea of a manifold. It is a space that looks locally like $\mathbb{R}^n$, with a set of charts that allow us to map patches to $\mathbb{R}^n$ and differentiate (and later, integrate).

Let $C^\infty(\mathcal{M})$ denote the set of all smooth functions that assign each point in the manifold $\mathcal{M}$ a real number in $\mathbb{R}$. We know how to differentiate on $\mathbb{R}^n$. Smooth functions allow us to differentiate in $\mathbb{R}^n$ before we map back onto the manifold.

**Definition 3** (tangent vector)**:** A *tangent vector* $X_p$ is an object that differentiates functions at some point $p \in \mathcal{M}$ on the manifold. Specifically, it is a map $X_p : C^\infty(\mathcal{M}) \to \mathbb{R}$ with certain properties:

1. linearity: $X_p(f+g) = X_p(f) + X_p(g)$ for all $f, g \in C^\infty(\mathcal{M})$
2. constants vanish: $X_p(f) = 0$ when $f$ is constant
3. Leibniz rule: $X_p(fg) = f(p)X_p(g) + X_p(f)g(p)$ for all $f, g \in C^\infty(\mathcal{M})$

**Definition 4** (tangent space)**:** The set of all tangent vectors at $p$ is the *tangent space* $T_p(\mathcal{M})$ at $p$.

One of the early surprises of differential geometry is thinking of vectors as differential operators. In $\mathbb{R}^3$, we used position vectors as displacement. However, this does not generalise to curved spaces. The second idea of a vector in physics is the velocity of a particle. This change in time is analogous to the definition that we use here in differential geometry. By differentiating, the tangent vector tells us how things change when moving in a particular direction.

**Claim:** In a chart $\phi = (x^1, \cdots, x^n)$, we can write a tangent vector as

$$X_p = X^\mu \partial_\mu |_p, \tag{1.4}$$

with $X^\mu = x_p(x^\mu)$ and $\partial_\mu f = \frac{\partial(f \circ \phi^{-1})}{\partial x^\mu}$ for all functions $f \in C^\infty(\mathcal{M})$. This definition of $\partial_\mu f$ uses the fact that we can differentiate on $\mathbb{R}^n$, which we map to via $f \circ \phi^{-1}$.

*Proof.* The idea is that given a function $f$, we use coordinates to push it to $\mathbb{R}^n$, and then we differentiate there. A detailed proof is in the notes. □

> **Remark:** We are going to use the summation convention with indices up/down. The object with coefficients as superscripts are tangent vectors, different from objects with subscript coefficients. Hence, the position of the indices matter even more than in Special Relativity.

Writing $X_p = X^\mu \partial_\mu$ clearly depends on coordinates $x^\mu$. What would happen if we chose different coordinates? If we picked another set of coordinates $\widetilde{x}^\mu$, we could write

$$X_p = X^\mu \left.\frac{\partial}{\partial x^\mu}\right|_p = \widetilde{X}^\mu \left.\frac{\partial}{\partial \widetilde{x}^\mu}\right|_p. \tag{1.5}$$

**Notation:** We write $\partial_\mu := \frac{\partial}{\partial x^\mu}$ and drop the implicit action of going to $\mathbb{R}^n$ via $\phi^{-1}$.

Acting on a function $f$, we can use the chain rule

$$X_p(f) = X^\mu \left.\frac{\partial f}{\partial x^\mu}\right|_p = X^\mu \left.\frac{\partial \widetilde{x}^\nu}{\partial x^\mu}\right|_{\phi(p)} \left.\frac{\partial f}{\partial \widetilde{x}^\nu}\right|_p. \tag{1.6}$$

We can view this as a change of basis of $T_p(\mathcal{M})$

$$\left.\frac{\partial}{\partial x^\mu}\right|_p = \left.\frac{\partial \widetilde{x}^\nu}{\partial x^\mu}\right|_{\phi(p)} \left.\frac{\partial}{\partial \widetilde{x}^\nu}\right|_p . \tag{1.7}$$

This is a *covariant* transformation. Alternatively, we can think of this as a change of components

$$\widetilde{X}^\nu = X^\mu \left.\frac{\partial \widetilde{x}^\nu}{\partial x^\mu}\right|_{\phi(p)} . \tag{1.8}$$

This is called a *contravariant* transformation; the position of the index tells us which way things transform.

To see why $X_p$ is called a tangent vector, we are going to consider a path $\sigma(t) : \mathbb{R} \to \mathcal{M}$ in the manifold such that $\sigma(0) = p$. We can think of the parameter $t$ as time for example. Given a coordinate chart $\phi$, this becomes a path $x^\mu(t)$ in $\mathbb{R}^n$. The tangent in $\mathbb{R}^n$ is $X^\mu = \left.\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t}\right|_{t=0}$. We can then use this to define $X_p \in T_p(\mathcal{M})$ as

$$X_p = \left.\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t}\right|_{t=0} \left.\frac{\partial}{\partial x^\mu}\right|_p . \tag{1.9}$$

In this sense, $T_p(\mathcal{M})$ is the space of all tangent vectors at $p$. By considering all possible paths, one can describe all possible tangent vectors. Physically, we can think about $T_p(\mathcal{M})$ as the space of all possible velocity vectors $X_p$ of a particle traversing a point $p \in \mathcal{M}$, as illustrated in Figure 1.3.

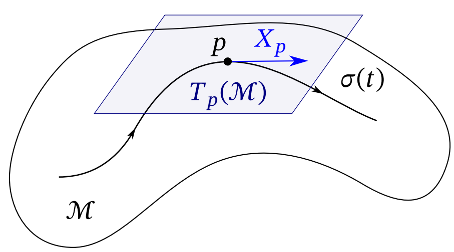


Figure 1.3: The tangent vector $X_p \in T_p(\mathcal{M})$ to a path $\sigma(t)$.

> **Remark:** Tangent spaces at two different points are two different vector spaces. It is meaningless to try to add vectors of two different tangent spaces ...for now.

## 1.3 Vector Fields

**Definition 5** (vector field)**:** A *vector field* $X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ is a smooth assignment of tangent vectors $X_p$ at each point $p \in \mathcal{M}$ with $(Xf)(p) = X_p(f)$. In a given coordinate chart we can write

$$X = X^\mu(x)\partial_\mu . \tag{1.10}$$

**Notation:** We will denote the space of all vector fields as $\mathfrak{X}(\mathcal{M})$.

**Remark:** We will make the notion of *smooth* more precise later. Intuitively, we want neighbouring tangent spaces to have similar directions.

Given two vector fields $X, Y \in \mathfrak{X}(\mathcal{M})$, we can define the *commutator*.

**Definition 6** (commutator)**:** The *commutator* $[X, Y] \in \mathfrak{X}(\mathcal{M})$ is a map

$$[X, Y]f = X(Yf) - Y(Xf). \tag{1.11}$$

**Remark:** Both $X, Y$ are first order differential operator. Thus, each term on the RHS is a second order operator. However, by subtracting these terms from each other, the second order terms cancel by commutation of partial derivatives. Thus, the commutator is a first order differential operator.

**Exercise 1.1:** In a coordinate basis, you can check

$$[X, Y] = \left( X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu} \right) \frac{\partial}{\partial x^\nu}. \tag{1.12}$$

**Remark:** As the course progresses, we will see connections with the *Symmetries, Field and Particles* course and its treatment of Lie algebras though these commutators.

**Exercise 1.2:** Check that the *Jacobi identity* holds:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{1.13}$$

### 1.3.1 Integral Curves

There is a relationship between vector fields and streamlines, also known as flows, on a manifold $\mathcal{M}$.

**Definition 7** (flow)**:** A *flow* on $\mathcal{M}$ is a one-parameter family of diffeomorphisms $\sigma_t : \mathcal{M} \to \mathcal{M}$, labelled by $t \in \mathbb{R}$, such that $\sigma_{t=0} = id$ and $\sigma_s \circ \sigma_t = \sigma_{s+t}$.

This defines *flow lines* on $\mathcal{M}$ as depicted in Figure 1.4. For each point on the manifold, there is a unique such streamline that flows through it. Each line has coordinates $x^\mu(t)$.

We can obtain a vector field $X$ by taking the tangent vector at each point:

$$X^\mu(x(t)) = \frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t}. \tag{1.14}$$

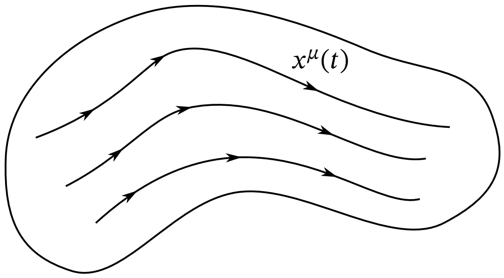


Figure 1.4: Flow lines

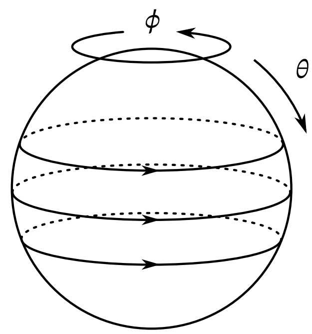


Figure 1.5: Integral curves of $X = \frac{\partial}{\partial \phi}$ on $S^2$.

**Definition 8** (integral curves)**:** Conversely, given a vector field $X$, we can integrate (1.14) to find $x^\mu(t)$. These flow lines are called *integral curves*.

**Example 1.3.1:** On $S^2$, with $(\theta, \phi)$ polar coordinates, consider $X = \frac{\partial}{\partial \phi}$. From (1.14), $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}\phi}{\mathrm{d}t} = 1$. This means that we have constant $\theta = \theta$.

$$\phi = \phi_0 + t \implies \sigma_t(\theta, \phi) \to (\theta, \phi + t) \tag{1.15}$$

As shown in Fig. 1.5, the integral curves are lines of constant lattitude.

## 1.4 Lie Derivatives

We have learned that vector fields differentiate scalar functions. But how can we differentiate vector fields? Given $X, Y \in \mathfrak{X}(\mathcal{M})$, is there some way to know how $Y$ changes in the direction of $X$? There is a problem here. For a function $f : \mathbb{R} \to \mathbb{R}$, differentiation is defined as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{h\to 0}\left\{\frac{f(t+h) - f(t)}{h}\right\}. \tag{1.16}$$

Similarly, to differentiate $Y \in \mathfrak{X}(\mathcal{M})$, we need to compare $Y_p \in T_p\mathcal{M}$ with some neighbouring $Y_q \in T_q\mathcal{M}$. The problem is that these are in different vector spaces! As each vector space has different coordinates, simply working with component values gives us a bad definition of a derivative; one which is dependent of coordinate choice. We cannot just add or subtract one vector from another. To differentiate $Y$, we need to understand how to map vectors in $T_q(\mathcal{M})$ to vectors in $T_p(\mathcal{M})$, so we can compare them.

### 1.4.1 Push-Forward and Pull-Back

First, we think more generally about maps of the form $\varphi : \mathcal{M} \to \mathcal{N}$, where $\mathcal{N}$ and $\mathcal{M}$ can be manifolds of different dimensions.

**Definition 9** (pull-back and push-forward)**:** Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a smooth map between manifolds, $f : \mathcal{N} \to \mathbb{R}$ a smooth function on $\mathcal{N}$, and $Y : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ a vector field on $\mathcal{M}$.

The *pull-back* of $f$ by $\varphi$ is $(\varphi^* f) = f \circ \varphi : \mathcal{M} \to \mathbb{R}$, a smooth function on $\mathcal{M}$.

The *push-forward* of $Y$ by $\varphi$ is $(\varphi_* Y) : C^\infty(\mathcal{N}) \to C^\infty(\mathcal{N})$ is a vector field on $\mathcal{N}$ defined by

$$\varphi^* \left[(\varphi_* Y)(f)\right] = Y(\varphi^* f) \tag{1.17}$$

Let $x^\mu$ be coordinates on $\mathcal{M}$ and $y^a$ be coordinates on $Y$ such that $\varphi(x) = y^a(x)$. Let $Y : Y^\mu \partial_\mu \in \mathfrak{X}(\mathcal{M})$ be a vector field on $\mathcal{M}$, then we can write the push-forward of $Y$ to $\mathcal{N}$ as

$$(\varphi_* Y)(f) = Y^\mu \frac{\partial f(y(x))}{\partial x^\mu} = Y^\mu \frac{\partial y^a}{\partial x^\mu} \frac{\partial f(y)}{\partial y^a}. \tag{1.18}$$

In components:

$$(\varphi_* Y)^a = Y^\mu \frac{\partial y^a}{\partial x^\mu}. \tag{1.19}$$

Some objects are naturally pulled back and some are pushed forward. Note that if $\varphi$ is a diffeomorphism, then we can go both ways by using the inverse $\varphi^{-1}$.

### 1.4.2 Differentiation by Lie Derivative

Now we can use this to differentiate. The idea is that given a vector field $X \in \mathfrak{X}(\mathcal{M})$, we can define a flow $\sigma_t : \mathcal{M} \to \mathcal{M}$. We use this to push forward vectors at $T_p\mathcal{M}$ to $T_{\sigma_t(p)}\mathcal{M}$. This is called the *Lie derivative*, denoted $\mathcal{L}_X$. Let us first look at functions:

$$\mathcal{L}_X f = \lim_{t \to 0} \left\{ \frac{f(\sigma_t(x)) - f(x)}{t} \right\} \tag{1.20}$$

$$= \left. \frac{\mathrm{d} f(\sigma_t(x))}{\mathrm{d}t} \right|_{t=0} = \left. \frac{\partial f}{\partial x^\mu} \, \frac{\partial x^\mu}{\partial t} \right|_{t=0} \tag{1.21}$$

$$= X^\mu \, \frac{\partial f}{\partial x^\mu} = X(f) \tag{1.22}$$

As $X^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}$, we have

$$\mathcal{L}_X f = X^\mu(x) \, \frac{\partial f}{\partial x^\mu} = X(f). \tag{1.23}$$

The action of the Lie derivative $\mathcal{L}_X$ on a function is the same as the action of the vector field $X$. Now we differentiate vector fields: The Lie derivative of a vector field $Y$ is defined by

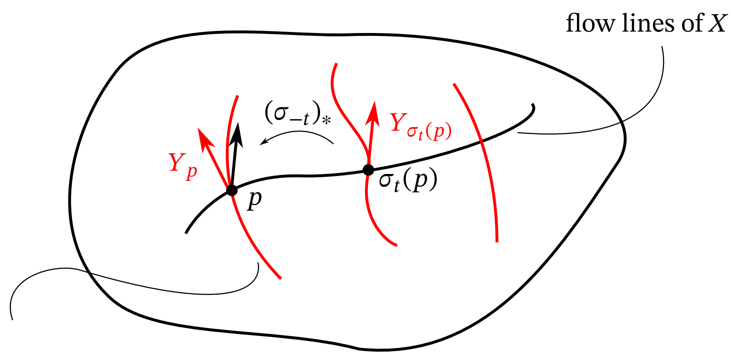


Figure 1.6: The Lie derivative uses the push-forward $(\sigma_{-t})_*$ to be able to compare vectors at a point $p \in \mathcal{M}$ and a bit further along the flow at $\sigma_t(p)$ defined by the vector field $X$.

$$\mathcal{L}_X Y = \lim_{t\to 0}\left\{\frac{((\sigma_{-t})_* Y) - Y}{t}\right\}. \tag{1.24}$$

The minus sign is needed to push with the inverse flow map, $\sigma_{-t} : \sigma_t(p) \longmapsto p$ as illustrated in Fig. 1.6. Let us calculate the action of the Lie derivative on a basis $\{\partial_\mu = \partial/\partial x^\mu\}$:

$$\mathcal{L}_X \partial_\mu = \lim_{t\to 0}\left\{\frac{(\sigma_{-t})_*\partial_\mu - \partial_\mu}{t}\right\}. \tag{1.25}$$

The components of the push-forward are

$$(\varphi_* f)^\alpha = Y^\mu \frac{\partial y^\alpha}{\partial x^\mu} \tag{1.26}$$

Now, we use the expression for the push-forward of a tangent vector as given by Equation (1.19), except that we replace the coordinates $y^a$ by the infinitesimal coordinate change induced by the flow $\sigma_t$:

$$y^\mu = x^\mu - tX^\mu + ... \tag{1.27}$$

Therefore, we have for small $t$:

$$(\sigma_{-t})_*\partial_\mu = (\delta^\nu_\mu - t\frac{\partial X^\nu}{\partial x^\mu} + ...)\partial_\nu. \tag{1.28}$$

And finally, we see that the Lie derivative acts on a coordinate basis as

$$\mathcal{L}_X \partial_\mu = -\frac{\partial X^\nu}{\partial x^\mu}\partial_\nu. \tag{1.29}$$

To find out how the $\mathcal{L}_X$ acts on a general vector field $Y = Y^\mu\partial_\mu$, we can use Leibniz' formula

$$\mathcal{L}_X Y = \mathcal{L}_X(Y^\mu\partial_\mu) = \mathcal{L}_X(Y^\mu)\partial_\mu + Y^\mu\mathcal{L}_x\partial_\mu \tag{1.30}$$

$$= X^\nu\frac{\partial Y^\mu}{\partial x^\nu}\partial_\mu - Y^\mu\frac{\partial X^\nu}{\partial x^\mu}\partial_\nu \tag{1.31}$$

$$= [X, Y]. \tag{1.32}$$

This is a surprising fact! We can think of the commutator as describing how $Y$ changes in the direction of $X$.

**Corollary:** From the Jacobi identity of commutators, we have for vector fields $X, Y, Z$:

$$[\mathcal{L}_X, \mathcal{L}_Y]Z = \mathcal{L}_{[X,Y]}Z. \tag{1.33}$$

## 1.5 Tensors

### Dual Spaces

**Definition 10** (dual space)**:** Let $V$ be a vector space. The *dual vector space* $V^*$ is the space of all linear maps from $V$ to $\mathbb{R}$.

The easiest way to define the dual space is to work with a given basis $\{e_\mu\}$, $\mu = 1, \dots, n$ of $V$. We can then introduce a dual basis $\{f^\mu\}$, $\mu = 1, \dots, n$ for $V^*$, defined by

$$f^\nu(e_\mu) = \delta^\nu_\mu. \tag{1.34}$$

The dual basis elements $f^\mu$ are linear maps which pick out their corresponding basis element $e_\mu$ and send it to unity. By definition, a general vector $X \in V$ can be decomposed in this basis as $X = X^\mu e_\mu$. Since the maps are linear, the dual basis elements act on a general vector hence as $f^\nu(X) = X^\mu f^\nu(e_\mu) = X^\nu$. For a given basis, the correspondence between the bases provides an isomorphism between $V$ and $V^*$. In particular, this means that $\dim V = \dim V^*$. However, this map $e_\mu \longmapsto f^\mu$ depends on the basis and is thus a bad isomorphism to work with in a coordinate invariant theory. We can repeat the construction to show that there is a natural isomorphism $(V^*)^* \to V$, which is independent of the choice of basis.

### 1.5.1 One-Forms

**Definition 11** (cotangent space and vector)**:** The *cotangent space* $T^*_p(\mathcal{M})$ at a point $p \in \mathcal{M}$ is the dual space to the tangent space $T_p(\mathcal{M})$ at $p$. Elements of the cotangent space are called *cotangent vectors* or *covectors*.

Given a basis $\{e_\mu\}$ of the tangent space $T_p(\mathcal{M})$, we can introduce a dual basis $\{f^\mu\}$ for $T^*_p(\mathcal{M})$ and expand any covector as $\omega = \omega_\mu f^\mu$.

**Definition 12** (one-form)**:** A *one-form* or *cotangent field* is a smooth assignment of cotangent vectors taken from the cotangent spaces across all points in the manifold. These one-forms are maps from vector fields to real numbers.

**Notation:** The set of all one-forms on $\mathcal{M}$ is denoted $\Lambda^1(M)$.

**Definition 13** (differential)**:** The *differential* $df \in \Lambda^1(\mathcal{M})$ of a smooth function $f$ on $\mathcal{M}$ is a special one-form. It is defined by its action on a vector field $X$:

$$\begin{aligned} df : \mathfrak{X} &\to \mathbb{R} \\ X &\longmapsto X(f). \end{aligned} \tag{1.35}$$

**Claim 1:** These differentials can be used to find a basis for the space $\Lambda^1(\mathcal{M})$ of one-forms on $\mathcal{M}$.

*Proof.* Let $\{x^\mu\}$ be coordinates on $\mathcal{M}$. This defines a coordinate basis $\{\frac{\partial}{\partial x^\mu} = \partial_\mu\}$ of vector fields. Then, for each value of $\mu$, we take the function $f$, which we use to build the differential, to simply be that coordinate $x^\mu$. This gives a one-form $dx^\mu$ for each value $\mu$. By definition, they act on the basis of vector fields as

$$dx^\mu(\partial_\nu) = \partial_\nu(x^\mu) = \delta^\mu_\nu. \tag{1.36}$$

Therefore, the set $\{dx^\mu\}$ is dual to the coordinate basis $\{\partial_\mu\}$ and provides a basis for $T^*(\mathcal{M}) = \Lambda^1(\mathcal{M})$. □

**Corollary:** In general, an arbitrary one-form $\omega \in \Lambda^1(\mathcal{M})$ can hence be expanded as $\omega = \omega_\mu dx^\mu$.

**Claim 2:** In this basis, the coefficients $\omega_\mu$ are the partial derivatives; the differential can be expanded as

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu. \tag{1.37}$$

*Proof.* We can check that this expansion satisfies Equation (1.35) by acting with it on a vector field $X$:

$$df(X) = \frac{\partial f}{\partial x^\mu} dx^\mu(X^\nu \partial_\nu) = X^\mu \frac{\partial f}{\partial x^\mu} = X(f). \tag{1.38}$$

□

How do one-forms transform under a change of coordinates? Recall that given two charts $\phi = (x^1, \dots, x^n)$ and $\widetilde{\phi} = (\tilde{x}^1, \dots, \tilde{x}^n)$, the partial derivatives, which form the basis of vector fields, are related via the chain rule:

$$\frac{\partial}{\partial \tilde{x}^\mu} = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \frac{\partial}{\partial x^\nu}. \tag{1.39}$$

Since the differentials form the corresponding dual basis, they transform in the inverse manner

$$d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu. \tag{1.40}$$

**Claim 3:** This transformation law ensures that the new basis $\{d\tilde{x}^\mu\}$ is also dual to the new chart $\widetilde{\phi}$.

*Proof.* This is an exercise in definitions and symbol manipulation: We need to show that

$$d\tilde{x}^\mu\left(\frac{\partial}{\partial\tilde{x}^\nu}\right) = \delta^\mu_\nu. \tag{1.41}$$

Starting from the left hand side, we first use the two transformation laws

$$d\tilde{x}^\mu\left(\frac{\partial}{\partial\tilde{x}^\nu}\right) = \frac{\partial\tilde{x}^\mu}{\partial x^\rho}dx^\rho\left(\frac{\partial x^\sigma}{\partial\tilde{x}^\nu}\frac{\partial}{\partial x^\sigma}\right) \tag{1.42}$$

Then we can pull out the factor $\partial x^\sigma/\partial\tilde{x}^\nu$ since the one-form $dx^\rho$ only acts on the vector field $\partial^\sigma$:

$$\cdots = \frac{\partial\tilde{x}^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial\tilde{x}^\nu}dx^\rho\left(\frac{\partial}{\partial x^\sigma}\right) = \frac{\partial\tilde{x}^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial\tilde{x}^\nu}\delta^\rho_\sigma = \frac{\partial\tilde{x}^\mu}{\partial x^\rho}\frac{\partial x^\sigma}{\partial\tilde{x}^\rho}. \tag{1.43}$$

Lastly, we use the fact that these change of variable matrices are inverses

$$\cdots = \frac{\partial\tilde{x}^\mu}{\partial x^\rho}\frac{\partial x^\rho}{\partial\tilde{x}^\nu} = \delta^\mu_\nu. \tag{1.44}$$

□

Therefore, we can expand a one-form $\omega$ in the new basis as well

$$\omega = \omega_\mu dx^\mu = \tilde{\omega}_\mu d\tilde{x}^\mu \quad \text{with} \quad \tilde{\omega}_\mu = \frac{\partial x^\nu}{\partial\tilde{x}^\mu}\omega_\nu. \tag{1.45}$$

We say that these transformations of covectors are *covariant* transformations.

### 1.5.2 Lie Derivative of One-Forms

Given a map $\varphi : \mathcal{M} \to \mathcal{N}$, we can push forward a vector field $X$ on $\mathcal{M}$ to a vector field $\varphi_* X$ on $\mathcal{N}$. One-forms work the other way around; we can pull back a one-form $\omega$ on $\mathcal{N}$ to a one-form $\varphi^*\omega$ on $\mathcal{M}$, defined by

$$(\varphi^*\omega)(X) = \omega(\varphi_* X). \tag{1.46}$$

If $x^\mu$ are coordinates on $\mathcal{M}$ and $y^\alpha$ coordinates on $\mathcal{N}$, then the coordinates of the pull-back are

$$(\varphi^*\omega)_\mu = \omega_\alpha\frac{\partial y^\alpha}{\partial x^\mu}. \tag{1.47}$$

To define the Lie derivative $\mathcal{L}_X$, we use the vector field $X$ to generate a flow $\sigma_t : \mathcal{M} \to \mathcal{M}$. Using the pull-back, we can then compare one-forms at different points in the manifold. Writing the cotangent vector as $\omega(p) = \omega_p$, the Lie derivative of a one-form $\omega$ is defined as

$$\mathcal{L}_X\omega := \lim_{t\to 0}\left\{\frac{(\sigma_t^*\omega)_p - \omega_p}{t}\right\}. \tag{1.48}$$

Infinitesimally, $\sigma_t$ acts on coordinates as $x^\mu(t) = x^\mu(0) + tX^\mu + \ldots$. From (1.46), the pull-back of a basis vector $dx^\mu$ is

$$\sigma_t^* dx^\mu = \left(\delta^\mu_\nu + t\frac{\partial X^\mu}{\partial x^\nu} + \ldots\right)dx^\nu. \tag{1.49}$$

The Lie derivative of the basis vector is then

$$\mathcal{L}_X(dx^\mu) = \frac{\partial X^\mu}{\partial x^\nu} dx^\nu, \tag{1.50}$$

which differs by a minus sign from (1.29) for $\mathcal{L}_X \partial_\mu$. This is because we use $\sigma_t$ to pull back one-forms, whereas we pushed forward vector fields with $\sigma_{-t}$. The Lie derivative of a general one-form $\omega = \omega_\mu dx^\mu$ can then be deduced via the Leibniz rule

$$\mathcal{L}_X \omega = (\mathcal{L}_X \omega_\mu) dx^\mu + \omega_\nu \mathcal{L}_X(dx^\nu) \tag{1.51}$$

$$= (X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu) dx^\mu. \tag{1.52}$$

### 1.5.3 Tensors and Tensor Fields

**Definition 14:** A *tensor of rank* $(r, s)$ at a point $p \in \mathcal{M}$ is defined to be a multi-linear map

$$T : \overbrace{T_p^*(\mathcal{M}) \times \cdots \times T_p^*(\mathcal{M})}^{r} \times \overbrace{T_p(\mathcal{M}) \times \cdots \times T_p(\mathcal{M})}^{s} \to \mathbb{R}. \tag{1.53}$$

We say that $T$ has *total rank* $r + s$.

A cotangent vector in $T_p^*(\mathcal{M})$ is a tensor of type $(0, 1)$ , while a tangent vector in $T_p(\mathcal{M})$ is a tensor of type $(1, 0)$ (since $T_p^{**}(\mathcal{M}) = T_p(\mathcal{M})$).

**Definition 15:** A *tensor field* is a smooth assignment of an $(r, s)$ tensor to every point $p \in \mathcal{M}$.

Given a basis $\{e_\mu\}$ for vector fields and a dual basis $\{f^\mu\}$ for one-forms, the components of the tensor are

$$T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} = T(f^{\mu_1}, \ldots f^{\mu_r}, e_{\nu_1}, \ldots, e_{\mu_s}). \tag{1.54}$$

On a manifold of dimension $n$, there are $n^{r+s}$ such components. For a tensor field, each of these is a smooth function over $\mathcal{M}$.

> **Remark:** We often say 'tensor' or 'vector' when we really mean 'tensor field' or 'vector field'.

**Example 1.5.1:** A $(2, 1)$ tensor acts on the one-forms $\omega, \eta \in \Lambda^1(\mathcal{M})$ on the manifold as

$$T(\omega, \eta; X) = T(\omega_\mu f^\mu, \eta_\mu f^\nu; X^\rho e_\rho) = \omega_\mu \eta_\nu X^\rho T^{\mu\nu}{}_\rho, \tag{1.55}$$

where $X \in \mathfrak{X}(\mathcal{M})$. Consider a coordinate transformation $\tilde{e}_\nu = A^\mu{}_\nu e_\mu$ with $A^\mu{}_\nu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}$ and $\tilde{f}^\rho = B^\rho{}_\sigma f^\sigma$ with $B^\rho{}_\sigma = \frac{\partial \tilde{x}^\rho}{\partial x^\sigma}$. Then a $(2, 1)$ tensor transforms as

$$\tilde{T}^{\mu\nu}{}_\rho = B^\mu{}_\sigma B^\nu{}_\tau A^\lambda{}_\rho T^{\sigma\tau}{}_\lambda. \tag{1.56}$$

This transformation law is often taken as the definition of a tensor. Here, it is just a consequence of the tensor being a *linear* map.

## Motivation

We care about tensors since we want the physical equations to be independent of the coordinates we are using. To actually do calculations, we cannot use this abstract notation, but we really need to introduce coordinates. However, we need to make sure that the results do not depend on coordinates, but are physically meaningful. This is why tensors are important for us.

### 1.5.4 Tensor operations

There are a number of operations that we can perform on tensors:

- vector space: can add / subtract or multiply by functions
- tensor product: if $S$ has rank $(p,q)$ and $T$ has rank $(r,s)$, then $S \otimes T$ has rank $(p+r, q+s)$ defined by

$$\begin{aligned} S \otimes T(\omega_1, \dots, \omega_p, \eta_1, \dots, \eta_r; X_1, \dots, X_q, Y_1, \dots Y_s) \\ = S(\omega_1, \dots, \omega_p; X_1, \dots, X_q) T(\eta_1, \dots, \eta_r; Y_1, \dots, Y_s). \end{aligned} \tag{1.57}$$

  In components, this is

$$(S \otimes T)^{\mu_1 \dots \mu_p \nu_1 \dots \nu_r}{}_{\rho_1 \dots \rho_s \sigma_1 \dots \sigma_s} = S^{\eta_1 \dots \eta_p}{}_{\rho_1 \dots \rho_q} T^{\nu_1 \dots \nu_r}{}_{\sigma_1 \dots \sigma_s}. \tag{1.58}$$

- contraction: We can turn an $(r,s)$-tensor into an $(r-1, s-1)$-tensor. If $T$ is a $(2,1)$ tensor, then

$$S(\omega) = T(\omega, f^\mu; e_\mu) \tag{1.59}$$

  Since we sum over $\mu$, this is basis independent. In components, this is

$$S^\mu = T^{\mu\nu}{}_\nu. \tag{1.60}$$

  **Remark:** This is typically different from $(S')^\mu = T^{\nu\mu}{}_\nu$.

- (anti)-symmetrisation: For example, given a $(0,2)$-tensor $T$, we can define two new $(0,2)$-tensors

$$S(X,Y) = \frac{1}{2}\left[T(X,Y) + T(Y,X)\right] \qquad A(X,Y) = \frac{1}{2}\left[T(X,Y) - T(Y,X)\right]. \tag{1.61}$$

  In terms of components, we write

$$S_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) \qquad A_{\mu\nu} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) \tag{1.62}$$

> **Remark:** This is similar to how we can decompose a matrix into its symmetric and anti-symmetric components.

**Notation:** We write

$$T_{(\mu\nu)} := \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu}) \qquad T_{[\mu\nu]} := \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}). \tag{1.63}$$

We can also (anti)-symmetrise over multiple indices.

**Example 1.5.2:**

$$T^{\mu}{}_{(\nu\rho\sigma)} = \frac{1}{3!}(T^{\mu}{}_{\nu\rho\sigma} + 5 \text{ permutations}) \tag{1.64}$$

$$T^{\mu}{}_{[\nu\rho\sigma]} = \frac{1}{3!}(T^{\mu}{}_{\nu\rho\sigma} + \text{sign(perm)} \times \text{ permutations}) \tag{1.65}$$

$$\tag{1.66}$$

In general, we divide by $p!$, where $p$ is the number of indices we (anti)-symmetrise over.

- We can define a Lie derivative $\mathcal{L}_X$ on a tensor field.

The next part is $\varepsilon$ more interesting than all these preceding definitions. But the good thing is that $\varepsilon > 0$.

## 1.6 Differential Forms

**Definition 16** (p-forms)**:** Totally anti-symmetric $(0, p)$-tensors are called *p-forms*. The space of all $p$-forms is denoted $\Lambda^p(\mathcal{M})$.

**Example 1.6.1:** 0-forms are simply functions.

In general, if the dimension of the manifold is $\dim M = n$, then $p$-forms have $\begin{pmatrix} n \\ p \end{pmatrix}$ independent components. $n$-forms are called *top-forms*.

**Definition 17** (wedge product)**:** Given $\omega \in \Lambda^p(\mathcal{M})$ and $\eta \in \Lambda^q(\mathcal{M})$, we can form a $(p+q)$-form by taking the tensor product and anti-symmetrising. This is the *wedge product*

$$(\omega \wedge \eta)_{\mu_1 \dots \mu_p \nu_1 \dots \nu_q} = \frac{(p+q)!}{p!q!} \omega_{[\mu_1 \dots \mu_p} \eta_{\nu_1 \dots \nu_q]} \tag{1.67}$$

**Example 1.6.2:** For one-forms, we have $(\omega \wedge \eta)_{\mu\nu} = \omega_\mu \eta_\nu - \omega_\nu \eta_\mu$.

**Exercise 1.3:** Let $\omega \in \Lambda^p(\mathcal{M})$ and $\eta \in \Lambda^q(\mathcal{M})$. Show that

$$\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega. \tag{1.68}$$

**Corollary:** The wedge product of a $p$-form with itself is $\omega \wedge \omega = 0$ for odd $p$, but not necessarily for even $p$.

**Example 1.6.3** (A wedge we have seen before)**:** For $\mathcal{M} = \mathbb{R}^3$ and $\omega, \eta \in \Lambda^1(\mathcal{M})$, pick some coordinates $x_1, x_2, x_3$ and expand the one-form in the coordinate basis as

$$(\omega \wedge \eta) = (\omega_1 dx^1 + \omega_2 dx^2 + \omega_3 dx^3) \wedge (\eta_1 dx^1 + \eta_2 dx^2 + \eta_3 dx^3) \tag{1.69}$$

$$= (\omega_1 \eta_2 - \eta_1 \omega_2) dx^1 \wedge dx^2 + (\omega_2 \eta_3 - \eta_2 \omega_3) dx^2 \wedge dx^3 + (\omega_3 \eta_1 - \eta_3 \omega_1) dx^3 \wedge dx^1 \tag{1.70}$$

These are the components of the *cross-product* in $\mathbb{R}^3$! The cross-product is really a wedge product between forms. We thus find out that we really always got back a two-form by taking the cross-product between two one-forms. However, as we will see later, there is a natural correspondence between two-forms and one-forms.

In a coordinate basis, we write

$$\omega = \frac{1}{p!} \omega_{\eta_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}. \tag{1.71}$$

### 1.6.1 The Exterior Derivative

Given a function $f$, we can construct a 1-form $df = \frac{\partial f}{\partial x^\mu} dx^\mu$. The *exterior derivative* generalises the differential to all $p$-forms. This is the second of three different derivatives we will meet, the first having been the Lie derivative.[1] There is also a (horrendous) coordinate-free definition, but it is easiest to work in coordinates.

**Definition 18:** The *exterior derivative* is a map $d : \Lambda^p(\mathcal{M}) \to \Lambda^{p+1}(\mathcal{M})$ defined as

$$d\omega = \frac{1}{p!} \frac{\partial \omega_{\mu_1 \dots \mu_p}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \tag{1.72}$$

or

$$(d\omega)_{\mu_1 \dots \mu_{p+1}} = (p+1) \partial_{[\mu_1} \omega_{\mu_2 \dots \mu_{p+1}]}. \tag{1.73}$$

**Claim 4:** The exterior derivative has the following properties:

- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\omega$

[1]The third one will be the *covariant derivative*. With the Lie derivative and the covariant one, we get back an object that is the same as the one we feed in. Here, we get an object that is one dimension higher.

- for pull-backs: $d(\varphi^*\omega) = \varphi^* d\omega$
- $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X\omega)$
- $d(d\omega) = d^2\omega = 0$

**Definition 19:** A $p$-form is *closed* if $d\omega = 0$ everywhere.

**Definition 20:** A $p$-form is *exact* if $\omega = d\eta$ everywhere for some $\eta$.

**Corollary:** The statement that $d^2 = 0$ implies that exact $p$-forms are automatically closed.

$$\text{exact} \quad \Rightarrow \quad \text{closed} \tag{1.74}$$

**Claim 5** (Poincaré's Lemma)**:** On $\mathbb{R}^n$, or locally on $\mathcal{M}$,

$$\text{closed} \quad \Rightarrow \quad \text{exact.} \tag{1.75}$$

> **Remark:** Whenever there is a second derivative, even with such a simple equation, there are beautiful things that follow. Think of Laplace's equation and holonomic function in complex analysis. We will see a hint of how this relates to differential forms in Sec. 2.1.2.

**Example 1.6.4** (one-form on $\mathbb{R}^3$)**:** Consider the 1-form $\omega = \omega_\mu(x)dx^\mu$. Then

$$(d\omega)_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu \quad \text{or} \quad d\omega = \frac{1}{2}(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu)dx^\mu \wedge dx^\nu. \tag{1.76}$$

On $\mathbb{R}^3$, we have

$$d\omega = (\partial_1\omega_2 - \partial_2\omega_1)dx^1 \wedge dx^2 + (\partial_2\omega_3 - \partial_3\omega_2)dx^2 \wedge dx^3 + (\partial_3\omega_1 - \partial_1\omega_3)dx^3 \wedge dx^1. \tag{1.77}$$

Here we have used the property of the wedge product that $dx^i \wedge dx^i = 0$. These three terms are the components of the curl $\boldsymbol{\nabla} \times \boldsymbol{\omega}$. The correct way to think about the curl is as an exterior derivative as a one-form, which coincidentally has three components as well.

**Example 1.6.5** (two form on $\mathbb{R}^3$)**:** Consider $B \in \Lambda^2(\mathbb{R}^3)$ with

$$B = B_1(x)dx^2 \wedge dx^3 + B_2(x)dx^3 \wedge dx^1 + B_3(x)dx^1 \wedge dx^2. \tag{1.78}$$

Taking the exterior derivative of this two-form, we get a three form. However, this is a top-form in $\mathbb{R}^3$, so we only have once component.

$$dB = (\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3)dx^1 \wedge dx^2 \wedge dx^3. \tag{1.79}$$

We have seen this before as well. These are the components of the divergence $\boldsymbol{\nabla} \cdot \mathbf{B}$.

**Example 1.6.6:** In electromagnetism, the gauge field $A^\mu$ should be thought of as a one-form $A \in \Lambda^1(\mathbb{R}^4)$. In components, this is $A = A_\mu dx^\mu$. Taking the exterior derivative we get

$$F = dA = F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu. \tag{1.80}$$

Gauge transformations act as $A \to A + d\alpha$, where $\alpha \in \Lambda^0(\mathcal{M}) = C^\infty(\mathcal{M})$. Under this transformation, the field strength is invariant since

$$F = dA \to d(A + d\alpha) = dA. \tag{1.81}$$

Moreover, since $F = dA$ is exact, we have $dF = d^2 A = 0$. This is the *Bianchi identity*, which is equivalent to two of the Maxwell equations. We need one more ingredient to write the other two Maxwell equations in terms of differential forms.

### 1.6.2 Integration on Manifolds

On a manifold, we integrate with the use of differential forms.

**Definition 21** (volume form)**:** A *volume form* $v$, also called an *orientation*, is a nowhere-vanishing top-form. Locally, it can be written as

$$v = v(x)\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^\mu, \tag{1.82}$$

with $v(x) \neq 0$ everywhere.

**Definition 22:** If a volume form exists, $\mathcal{M}$ is said to be *orientable*.

> **Remark:** There are a bunch of subtleties here; not all manifolds are orientable, e.g. the Möbius strip, or real projective space $\mathbb{RP}^{2n}$. However, these are not useful in GR; which is why we will brush these subtleties under the carpet.

**Definition 23** (integration)**:** Given a volume form, we can integrate any function $f \in C^\infty(\mathcal{M})$ over $\mathcal{M}$. To do this, we map it to $\mathbb{R}$ via a chart, and then integrate over $\mathbb{R}^n$ as we would usually do. In a coordinate patch $\mathcal{O} \subset \mathcal{M}$, we have

$$\int_{\mathcal{O}} f v := \int_U f(x) v(x) dx^1 \ldots dx^\mu. \tag{1.83}$$

We then sum over patches $\mathcal{O}$ to integrate over $\mathcal{M}$.

> **Remark:** In the language of integration, the volume form is the *measure*, which tells us how to weight functions on the manifold. This is because we have no notion of distance between points on the manifold without it.

**Definition 24** (submanifold)**:** A manifold $\Sigma$ of dimension $k < n$ is a *submanifold* of $\mathcal{M}$ if there exists a bijection $\phi : \Sigma \to \mathcal{M}$, which embeds $\Sigma$ into $\mathcal{M}$, such that $\phi_* : T_p(\Sigma) \to T_p(\mathcal{M})$ is also a bijection.

These definitions make sure that everything is nice and smooth. We can then integrate any $\omega \in \Lambda^k(\mathcal{M})$ over the submanifold $\Sigma$ by

$$\int_{\phi(\Sigma)} \omega = \int_{\Sigma} \phi^* \omega, \tag{1.84}$$

where $\omega^*$ is the pull-back.

**Example 1.6.7:** Let $\sigma$ be a map $\sigma : C \subset \mathbb{R} \to \mathcal{M}$. This defines a non-intersecting curve in $\mathcal{M}$, parametrised by $\tau$. Then, if $A \in \Lambda^1(\mathcal{M})$, in coordinates $x^\mu$, the integral is

$$\int_{\sigma(C)} A = \int_C \sigma^* A = \int \mathrm{d}\tau \, A_\mu(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}. \tag{1.85}$$

The action in Minkowski space is obtained by pulling back the one-form in Minkowski space $\mathbb{M}^4$ to the real line, and then integrating over this line.

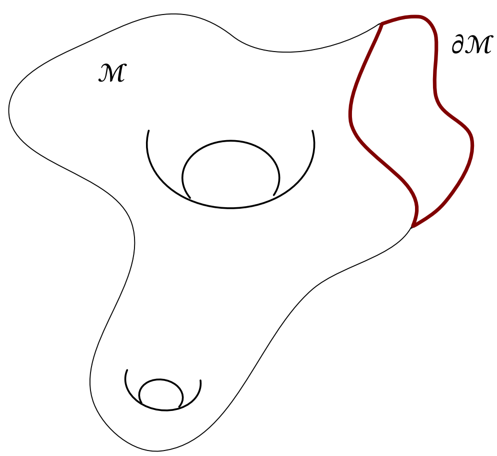


Figure 1.7: Manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$.

**Theorem 1** (Stokes' Theorem)**:** Let $\mathcal{M}$ be a manifold with boundary $\partial M$, as illustrated in 1.7. If $\omega \in \Lambda^{n-1}(\mathcal{M})$, then

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega. \tag{1.86}$$

**Example 1.6.8:** Let $\mathcal{M}$ be the one-dimensional interval $I$ with $x \in [a, b]$. The zero-form $\omega(x)$ is a function, and $d\omega = \frac{\mathrm{d}\omega}{\mathrm{d}x} dx$ is a one-form. Stokes' theorem says that

$$\int_{\mathcal{M}} d\omega = \int_a^b \frac{\mathrm{d}\omega}{\mathrm{d}x} \mathrm{d}x \quad \text{and} \quad \int_{\partial\mathcal{M}} \omega = \omega(b) - \omega(a) \tag{1.87}$$

where the minus sign is related to the subtleties at the boundary, which we have previously swept under the carpet. This is the *fundamental theorem of calculus*.

**Example 1.6.9:** Let $\mathcal{M} \subset \mathbb{R}^2$ and $\omega \in \Lambda^1(\mathcal{M})$. Then

$$\int_{\mathcal{M}} d\omega = \int_{\mathcal{M}} \left( \frac{\partial \omega_2}{\partial x^1} - \frac{\partial \omega_1}{\partial x^2} \right) dx^1 \wedge dx^2 \tag{1.88}$$

$$\text{and} \quad \int_{\partial\mathcal{M}} \omega = \int_{\partial\mathcal{M}} \omega_1 dx^1 + \omega_2 dx^2. \tag{1.89}$$

The equality between left and right hand sides is *Green's theorem* in the plane.

**Example 1.6.10:** Let $\mathcal{M} \subset \mathbb{R}^3$ and $\omega \in \Lambda^2(\mathcal{M})$. Then

$$\int_{\mathcal{M}} d\omega = \int dx^1 dx^2 dx^3 \left( \partial_1 \omega_1 + \partial_2 \omega_2 + \partial_3 \omega_3 \right) \tag{1.90}$$

$$\text{and} \quad \int_{\partial\mathcal{M}} \omega = \int_{\partial\mathcal{M}} \omega_1 dx^2 dx^3 + \omega_2 dx^3 dx^1 + \omega_3 dx^1 dx^2. \tag{1.91}$$

The equality between these two sides is *Gauss' divergence theorem*.

# 2 Introducing Riemannian Geometry

Riemannian geometry is differential geometry with a metric. You can tell that the metric is important, because it literally changes the name of the whole subject. As we will see, we should really be differentiating between Riemannian and Lorentzian manifolds, but one should be aware that people often just refer to a manifold with metric as a Riemannian manifold.

## 2.1 The Metric

The metric defines an inner product on the space of tangent vectors. It has a few properties, which guarantee that the metric is a good inner product.

**Definition 25** (metric)**:** A *metric* $g$ is a $(0,2)$-tensor that is

- symmetric: $g(X,Y) = g(Y,X)$
- non-degenerate: if $g(X,Y)_p = 0$ for all $Y_p \in T_p(\mathcal{M})$, then $X_p = 0$

> **Remark:** The non-degeneracy ensures that the metric is invertible.

In a particular coordinate basis, the metric gets two indices $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$, with $g_{\mu\nu} = g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$.

**Notation:** We often write this as a *line element* $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

**Definition 26** (signature)**:** If we diagonalise $g_{\mu\nu}$, it has positive and negative elements (none are zero, since it is non-degenerate). The number of negative elements is called the *signature* of the metric.

> **Remark:** There is a theorem (Sylvester's law of inertia) in matrix algebra that says that the signature is unchanged under a change of basis.

There are two metrics that are going to be of interest to us.

## Riemannian Metrics

**Definition 27:** A *Riemannian manifold* is a manifold with metric with signature $(+ + \cdots +)$.

**Definition 28:** *Euclidean space* $\mathbb{E}^n$ is $\mathbb{R}^n$ with metric components $g_{ij} = \delta_{ij}$

$$g = dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n. \tag{2.1}$$

A metric gives us a way to measure

- the length of a vector
$$|X| = \sqrt{g(X, X)}, \tag{2.2}$$
- the angle between vectors
$$g(X, Y) = |X||Y| \cos\theta, \tag{2.3}$$
- the distance between two points $p$ and $q$ along a curve $\sigma : [a, b] \to \mathcal{M}$, with end points $\sigma(a) = p$ and $\sigma(b) = q$
$$\text{distance} = \int_a^b \mathrm{d}t \sqrt{g(X, X)|_{\sigma(t)}}, \tag{2.4}$$
where $X$ is tangent to the curve. If the curve has coordinates $x^\mu(t)$, this is
$$\text{distance} = \int_a^b \mathrm{d}t \sqrt{g_{\mu\nu}(t) \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \frac{\mathrm{d}x^\nu}{\mathrm{d}t}} \tag{2.5}$$

## Lorentzian Metrics

**Definition 29:** A *Lorentzian manifold* is a manifold equipped with a metric of signature $(- + + \cdots +)$.

**Definition 30:** *Minkowski space* $\mathbb{M}^n$ is $\mathbb{R}^n$ with metric components $\eta_{\mu\nu} = \text{diag}(-1, +1, \ldots, +1)$

$$\eta = -dx \otimes dx^0 + dx^1 \otimes dx^1 + \cdots + dx^{n-1} \otimes dx^{n-1} \tag{2.6}$$

Because of this minus sign in the metric, 'lengths' and 'distances'—in the sense of inner products of vectors—can be negative. We classify vectors $X \in T_p(\mathcal{M})$ as

$$g(X,X)\begin{cases} < 0 & \text{timelike} \\ = 0 & \text{null} \\ > 0 & \text{spacelike} \end{cases}. \tag{2.7}$$

At each point $p \in \mathcal{M}$, we can draw null tangent vectors called *lightcones*.

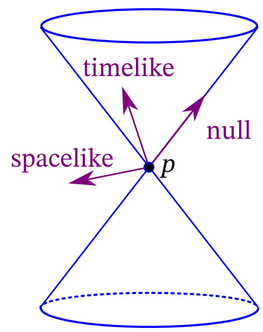


Figure 2.1: The causal structure around a point $p$ in a Lorentzian manifold.

**Definition 31:** A curve is called *timelike* if its tangent vector is everywhere timelike.

**Definition 32** (proper time)**:** For timelike curves, we can measure the distance between two points

$$\tau = \int_a^b \mathrm{d}t \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}}. \tag{2.8}$$

This is the *proper time* between $a$ and $b$.

> **Remark:** The parametrisation $t$ is arbitrary.

### 2.1.1 The Joys of a Metric

The metric gives a natural (basis independent) isomorphism at every point

$$\begin{aligned} g: T_p(\mathcal{M}) &\to T_p^*(\mathcal{M}) \\ X &\longmapsto g(X,\bullet). \end{aligned} \tag{2.9}$$

Given a vector field $X \in \mathfrak{X}(\mathcal{M})$, we can construct $g(X,\bullet) \in \Lambda^1(\mathcal{M})$. If $X = X^\mu\partial_\mu$, then the corresponding 1-form is

$$g_{\mu\nu}X^\mu dx^\nu := X_\nu dx^\nu. \tag{2.10}$$

Because $g$ is non-degenerate, there is an inverse. In components, $g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}_{\rho}$. This defines a rank $(2,0)$ tensor $\hat{g} = g^{\mu\nu}\partial_\mu \otimes \partial_\nu$. We use this to raise indices

$$X^\mu := g^{\mu\nu}X_\nu. \tag{2.11}$$

**Remark:** As physicists, we often say that we can use the metric to *raise* and *lower* indices. This is what we really mean with that; the covariant and contravariant vectors really live in different mathematical spaces, and the *lowering* of an index is the statement that the metric provides a natural isomorphism. This isomorphism allowed us to jump between vectors and 1-forms without even worrying about the fact that they are different objects. In spaces other than Euclidean space, their difference becomes important.

The metric also gives a natural volume form.

- On a Riemannian manifold,
$$v = \sqrt{\det g_{\mu\nu}}dx^1 \wedge \cdots \wedge dx^n. \tag{2.12}$$
We usually write $g = \det g_{\mu\nu}$.
- On a Lorentzian manifold,
$$v = \sqrt{-g}dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{n-1}. \tag{2.13}$$

**Claim 6:** This is independent of coordinates.

*Proof.* In new coordinates, $dx^\mu = A^\mu{}_\nu d\tilde{x}^\nu$ with $A^\mu{}_\nu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}$,

$$dx^1 \wedge \cdots \wedge dx^n = A^1{}_{\mu_1} \dots A^1{}_{\mu_n} \underbrace{d\tilde{x}^{\mu_1} \wedge \cdots \wedge d\tilde{x}^{\mu_n}}_{\text{rearrange to } d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n} \tag{2.14}$$

$$= \sum_{\text{perm } \pi} \text{sign}(\pi) A^1{}_{\pi(1)} \dots A^n{}_{\pi(n)} d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n \tag{2.15}$$

$$= \det(A) d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n. \tag{2.16}$$

The determinant $\det A > 0$ if the coordinate change preserves orientation. Meanwhile,

$$g_{\mu\nu} = \frac{\partial \tilde{x}^\rho}{\partial x^\mu}\frac{\partial \tilde{x}^\sigma}{\partial x^\nu}\tilde{g}_{\rho\sigma} \tag{2.17}$$

$$= (A^{-1})^\rho{}_\mu (A^{-1})^\sigma{}_\nu \tilde{g}_{\rho\sigma} \tag{2.18}$$

$$\Rightarrow \det g_{\mu\nu} = (\det A^{-1})^2 \det \tilde{g}_{\rho\sigma} = \frac{\det \tilde{g}_{\rho\sigma}}{(\det A)^2} \tag{2.19}$$

$$\Rightarrow v = \sqrt{|\tilde{g}|} d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n. \tag{2.20}$$

□

In components, $v = \frac{1}{n!} v_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$, with

$$v_{\mu_1 \dots \mu_n} = \sqrt{|g|} \varepsilon_{\mu_1 \dots \mu_n}, \tag{2.21}$$

where $\varepsilon$ is the totally anti-symmetric tensor. We can then integrate any function as

$$\int_{\mathcal{M}} f v = \int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|} f(x). \tag{2.22}$$

So the metric provides the measure in the form of $\sqrt{|g|}$.

### 2.1.2 A Glimpse of Hodge Theory

There are a few more things we can define with a metric.

**Definition 33:** The *Hodge dual* is a map $\star : \Lambda^p(\mathcal{M}) \to \Lambda^{n-p}(\mathcal{M})$, where $n = \dim \mathcal{M}$, defined as

$$(\star \omega)_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \sqrt{|g|} \varepsilon_{\mu_1 \dots \mu_{n-p} \nu_1 \dots \nu_p} \omega^{\nu_1 \dots \nu_p}. \tag{2.23}$$

> **Remark:** For $\mathbb{R}^3$, this is what is really going on when we were able to conflate 1-forms and 2-forms.

**Exercise 2.1:** You can check that $\star(\star \omega) = (-1)^{p(n-p)} s \omega$, where the sign depends on the signature of the metric; $s = +1$ for Riemannian and $s = -1$ for Lorentzian metrics.

**Corollary:** The inverse of $\star$ is given by $\star^{-1} \omega = (-1)^{p(n-p)} s \star \omega$

**Definition 34:** The *codifferential* $d^\dagger = (-1)^k \star^{-1} d \star$ is a map from $p$-forms to $(p-1)$-forms.

**Definition 35:** The *inner product* of two $p$-forms $\omega, \eta \in \Lambda^p(\mathcal{M})$ is defined as

$$\langle \eta, \omega \rangle = \int_{\mathcal{M}} \underbrace{\eta \wedge \star \omega}_{\text{this is a top-form}} . \tag{2.24}$$

**Claim 7:** For $\omega \in \Lambda^p(\mathcal{M})$ and $\alpha \in \Lambda^{p-1}(\mathcal{M})$, we then have

$$\langle d\alpha, \omega \rangle = \langle \alpha, d^\dagger \omega \rangle. \tag{2.25}$$

*Proof.* On a closed manifold, Stokes' theorem implies that

$$0 = \int_{\mathcal{M}} d(\alpha \wedge \star \omega) = \int_{\mathcal{M}} \left[ d\alpha \wedge (\star \omega) + (-1)^{p-1} \alpha \wedge d(\star w) \right] \tag{2.26}$$

$$= \langle d\alpha, \omega \rangle + (-1)^{p-1} s \langle \alpha, \star d \star \omega \rangle \tag{2.27}$$

$$= \langle d\alpha, \omega \rangle - \left\langle \alpha, d^\dagger \omega \right\rangle \tag{2.28}$$

□

**Definition 36:** The *Laplace-deRham* operator is $\Delta = (d^\dagger + d)^2 = d^\dagger d + dd^\dagger$.

Like the Laplacian, it is the divergence of the gradient in the language of differential forms. It is self adjoint, which gives $(\Delta\omega, \alpha) = (\omega, \Delta\alpha)$, and the study of *harmonic forms*, which satisfy $\Delta\omega = 0$ lies at the heart of Hodge theory.

> **Remark:** Compare this to the Hermitian conjugate of an operator in quantum mechanical inner products on a Hilbert space. Most of the time, we do not play with forms in QM. However, there is a close relationship between forms in differential geometry and fermions in QFT; this relationship will be fleshed out in the Lent term's *Supersymmetry* course.

## 2.2 Connections

We have met two derivatives so far. We will now meet the final, and ultimately most useful one.

**Definition 37:** A *connection* is a map $\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ We write this as $\nabla(X, Y) = \nabla_X Y$ to make it look more like differentiation. Here, $\nabla_X$ is called the *covariant derivative* and it satisfies the following properties:

- linearity on the first argument: $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$ for all functions $g, g$
- some linearity on the second argument: $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$
- but functions in the second argument obey the Leibniz rule: $\nabla_X(fY) = f\nabla_X Y + (\nabla_X f)Y$, where $\nabla_X f := X(f)$.

> **Remark:** This last term in the Leibniz rule is the reason why this is not a linear map, and thus not a tensor.

> **Remark:** We often refer to the abstract object and also its components both simply as the *connection*.

On a basis $\{e_\mu\}$ of vector fields, not necessarily a coordinate basis, we write

$$\nabla_{e_\rho} e_\nu = \Gamma_{\rho\nu}{}^{\mu} e_\mu. \tag{2.29}$$

This defines Γ, but it is simply the most general form it can be. We also use the notation $\nabla_{e_\mu} = \nabla_\mu$ to make the connection look like a partial derivative. Then for a general vector

$$\nabla_X Y = \nabla_X(Y^\mu e_\mu) = X(Y^\mu)e_\mu + Y^\mu \nabla_X e_\mu. \tag{2.30}$$

$$= X^\nu e_\nu(Y^\mu)e_\mu + Y^\mu X^\nu \nabla_\nu e_\mu = X^\nu(e_\nu(Y^\mu) + \Gamma^\mu{}_{\nu\rho}Y^\rho)e_\mu. \tag{2.31}$$

Because the components $X^\nu$ sits out front, we can write $\nabla_X Y = X^\nu \nabla_n Y$ with

$$\nabla_n Y = (e_\nu(Y^\mu) + \Gamma^\mu{}_{\nu\rho}Y^\rho)e_\mu, \tag{2.32}$$

or with a slightly slippery notation $(\nabla_\nu Y)^\mu := \nabla_\nu Y^\mu = e_\nu(Y^\mu) + \Gamma^\mu{}_{\nu\rho}Y^\rho$.

> **Remark:** For $\mathcal{L}_X$ depends on $X$ and $\partial X$, so we cannot write something like "$\mathcal{L}_X = X^\mu \mathcal{L}_\mu$". This is ultimately why the covariant derivative is more useful. The flip side is that the Lie derivative comes for free, while the numbers Γ are extra structure that needs to be specified on the manifold.

If we take a coordinate basis $\{e_\mu\} = \{\partial_\mu\}$, then

$$\nabla_\nu Y^\mu = \partial_\nu Y^\mu + \Gamma^\mu{}_{\nu\rho}Y^\rho. \tag{2.33}$$

> **Remark:** This should be familiar from previous courses in GR or fluid dynamics.

**Notation:** We often write the covariant derivative as $\nabla_\nu Y^\mu := Y^\mu{}_{;\nu}$ with a semicolon and the partial derivative with a comma as $\partial_\nu Y^\mu := Y^\mu{}_{,\nu}$.

> **Remark:** The name *connection* hints that we can use this to connect tangent spaces at different points in the manifold.

**Claim 8:** The connection is *not* a tensor.

*Proof.* Consider a change of basis $\tilde{e}_\nu = A^\mu{}_\nu e_\mu$ with $A^\mu{}_\nu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}$ for a coordinate bases. Then

$$\nabla_{\tilde{e}_\rho}\tilde{e}_\nu = \tilde{\Gamma}^\mu{}_{\rho\nu}\tilde{e}_\mu \tag{2.34}$$

$$= \nabla_{(A^\sigma{}_\rho e_\sigma)}(A^\lambda{}_\nu e_\lambda) \tag{2.35}$$

$$= A^\sigma{}_\rho \nabla_\sigma(A^\lambda{}_\nu e_\lambda) \tag{2.36}$$

Using the Leibniz rule we have

$$\ldots = A^\sigma{}_\rho(A^\lambda{}_\nu \Gamma^\tau{}_{\sigma\lambda}e_\tau + e_\lambda \partial_\sigma A^\lambda{}_\nu) \tag{2.37}$$

$$= A^\sigma{}_\rho(A^\lambda{}_\nu \Gamma^\tau{}_{\sigma\lambda} + \partial_\sigma A^\tau{}_\nu)e_\tau, \qquad e_\tau = (A^{-1})^\mu{}_\tau \tilde{e}_\mu \tag{2.38}$$

$$\Rightarrow\ \tilde{\Gamma}^\mu{}_{\rho\nu} = (A^{-1})^\mu{}_\tau A^\sigma{}_\rho A^\lambda{}_\nu \Gamma^\tau{}_{\sigma\lambda} + \underbrace{(A^{-1})^\mu{}_\tau A^\sigma{}_\rho \partial_\sigma A^\tau{}_\nu}_{\text{this is why it is not a tensor}}\ . \tag{2.39}$$

□

> **Remark:** There are other objects, where things transform as something → something + $\partial$other. The Maxwell tensor and the Yang-Mills gauge potential are examples of these. Mathematicians actually call both of these objects *connections*, precisely because of this property. The connections will become more obvious in the Lent term's *Applications of Differential Geometry to Physics* course.

We can also use the connection to differentiate other tensors. We simply ask that it obeys the Leibniz rule.

**Example 2.2.1:** We can use the covariant derivative to differentiate a function. Let us think of a one-form $\omega \in \Lambda^1(\mathcal{M})$ acting on a vector field $Y \in \mathfrak{X}(\mathcal{M})$ as that function. Using the Leibniz rule,

$$\nabla_X(\omega(Y)) = X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y) \tag{2.40}$$

$$\Rightarrow \ (\nabla_X \omega)(Y) = (\omega(Y)) - \omega \nabla_X(Y). \tag{2.41}$$

In terms of coordinates, we have

$$X^\mu(\nabla_\mu \omega_\nu)Y^\nu = X^\mu \partial_\mu(\omega_\nu X^\nu) - \omega_\nu X^\mu(\partial_\mu Y^\nu + \Gamma^\nu{}_{\mu\rho} Y^\rho \tag{2.42}$$

$$= X^\mu(\partial_\mu \omega_\rho - \Gamma^\nu{}_{\mu\rho}\omega_\nu)Y^\rho \tag{2.43}$$

$$\Rightarrow \ \nabla^\mu \omega_\rho = \partial_\mu \omega_\rho - \Gamma^\nu{}_{\mu\rho}\omega_\nu. \tag{2.44}$$

For a general $(p, q)$-tensor, we have one term for each index:

$$\nabla_\rho T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} = \partial_\rho T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} + \Gamma^{\mu_1}{}_{\rho\sigma} T^{\sigma\mu_2 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q} + \cdots - \Gamma^\sigma_{\rho\nu_1} T^{\mu_1 \ldots \mu_p}{}_{\sigma\nu_2 \ldots \nu_q} - \ldots . \tag{2.45}$$

**Claim 9:** Given a connection, we can construct two tensors. Let $\omega \in \Lambda^1(\mathcal{M})$ and $X, Y \in \mathfrak{X}(\mathcal{M})$, then

**Torsion** is a rank $(1, 2)$-tensor (field)

$$T(\omega; X, Y) := \omega(\nabla_X Y - \nabla_Y X - [X, Y]) \tag{2.46}$$

We can also think of $T$ as a map

$$\begin{aligned} T:\ \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) &\rightarrow \mathfrak{X}(\mathcal{M}) \\ (X, Y) &\longmapsto T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \end{aligned} \tag{2.47}$$

**Curvature** is a rank $(1, 3)$-tensor

$$R(\omega; X, Y, Z) = \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \tag{2.48}$$

This is the *Riemann tensor*. We can also think of it as a map

$$\begin{aligned} R:\ \mathfrak{X}(\mathcal{M})\times\mathfrak{X}(\mathcal{M}) &\to \text{differential operator on } \mathfrak{X}(\mathcal{M}) \\ (X,Y) &\longmapsto R(X,Y)=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]} \end{aligned} \tag{2.49}$$

*Proof.* To show that these are tensors, we need to check linearity in all arguments.

**Example 2.2.2:**

$$T(\omega;fX,Y)=\omega(\nabla_{fX}Y-\nabla_Y(fX)-[fX,Y]) \tag{2.50}$$

$$=\omega\left[f\nabla_XY-f\nabla_YX-Y(f)\cdot X-(f[X,Y]-Y(f)X)\right] \tag{2.51}$$

$$=f\omega(\nabla_XY-\nabla_YX-[X,Y])=fT(\omega;X,Y) \tag{2.52}$$

Similar calculations for $X,Y,Z$ for torsion and curvature show that these are all linear. □

In a coordinate basis $\{e_\mu=\partial_\mu\}$ and $\{f^\mu=dx^\mu\}$, we have

$$T^\rho{}_{\mu\nu}=T(f^\rho;e_\mu,e_\nu) \tag{2.53}$$

$$=f^\rho(\nabla_\mu e_\nu-\nabla_\nu e_\mu-\underbrace{[e_\mu,e_\nu]}_{\text{vanishes in coord. basis}}) \tag{2.54}$$

$$=\Gamma^\rho_{\mu\nu}-\Gamma^\rho_{\nu\mu}. \tag{2.55}$$

**Remark:** Therefore, we learn that although $\Gamma$ is not a tensor, the anti-symmetrisation of it is!

**Remark:** In this lecture, since we never raise or lower indices on the Christoffel symbols, we do not care about the position of its indices.

A connection with $\Gamma^\rho_{\mu\nu}=\Gamma^\rho_{\nu\mu}$ has $T^\rho{}_{\mu\nu}=0$ and is said to be *torsion free*. We also have

$$R^\sigma{}_{\rho\mu\nu}=R(f^\sigma;e_\mu,e_\nu,e_\rho) \tag{2.56}$$

**Remark:** This is a slightly odd ordering on the right hand side.

$$\ldots=f^\sigma\left(\nabla_\mu\nabla_\nu e_\rho-\nabla_\nu\nabla_\mu e_\rho-\nabla_{[e_\mu,e_\nu]}e_\rho\right) \tag{2.57}$$

$$=f^\sigma(\nabla_\mu(\Gamma^\lambda_{\nu\rho}e_\lambda)-\nabla_\nu(\Gamma^\lambda_{\mu\rho}e_\lambda) \tag{2.58}$$

$$R^\sigma{}_{\rho\mu\nu}=\partial_\mu\Gamma^\sigma_{\nu\rho}-\partial_\nu\Gamma^\sigma_{\mu\rho}+\Gamma^\lambda_{\nu\rho}\Gamma^\sigma_{\mu\lambda}-\Gamma^\lambda_{\mu\rho}\Gamma^\sigma_{\nu\lambda}. \tag{2.59}$$

> **Remark:** We mentioned that $\Gamma$ transforms similarly to the gauge potential. We have a similar thing in Yang-Mills theory. This Riemann tensor is to GR and curvature $R$ what the field strength $F$ is to Maxwell theory.

Clearly, we have anti-symmetry in the last indices:

$$R^{\sigma}{}_{\rho\mu\nu} = -R^{\sigma}{}_{\rho\nu\mu}. \tag{2.60}$$

There are a few more subtle symmetry properties of $R$, which we will prove in the upcoming sections.

So far, the connection is completely independent of the metric. However, it turns out that, given a metric, we can define a natural connection from it.

### 2.2.1 Levi-Civita Connection

**Theorem 2** (Fundamental Theorem of Riemannian Geometry)**:** There exists a unique, torsion-free connection with the property $\nabla_X g = 0$ for all vector fields $X \in \mathfrak{X}(\mathcal{M})$.

*Proof.* Suppose that the connection exists. We then follow a series of manipulation which gives us the desired result. Consider the object $X(g(Y,Z))$. Since $g(Y,Z)$ is a function, we have

$$X(g(Y,Z)) = \nabla_X[g(Y,Z)] \tag{2.61}$$

By the Leibniz rule, we have

$$\begin{aligned} ... &= \nabla_X g(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z) && (2.62) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) && (2.63) \end{aligned}$$

However, the torsion vanishes by assumption. Therefore we have

$$\nabla_X Y - \nabla_Y X = [X,Y]. \tag{2.64}$$

We use this equation in (2.63) to find that

$$X(g(Y,Z)) = g(\nabla_Y X, Z) + g(\nabla_X Z, Y) + g([X,Y],Z). \tag{2.65}$$

Now we repeat these exact same calculations twice over, cyclically permuting $X, Y$, and $Z$. These give

$$\begin{aligned} Y(g(Z,X)) &= g(\nabla_Z Y, X) + g(\nabla_Y X, Z) + g([Z,Y],X), && (2.66) \\ Z(g(X,Y)) &= g(\nabla_X Z, Y) + g(\nabla_Z Y, X) + g([Z,X],Y). && (2.67) \end{aligned}$$

Then, taking (2.65) + (2.66) − (2.67), we have

$$g(\nabla_Y X, Z) = \frac{1}{2}\big[X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ - g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)\big]. \quad (2.68)$$

In a coordinate basis $\{e_\mu = \partial_\mu\}$, this becomes

$$g(\nabla_\nu e_\mu, e_\rho) = \Gamma^\lambda_{\nu\mu} g_{\lambda\rho} = \frac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \quad (2.69)$$

$$= \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \quad (2.70)$$

This is the *Levi-Civita connection* and the explicit representation $\Gamma^\lambda_{\mu\nu}$ are the *Christoffel symbols*.

**Exercise 2.2:** We showed that this is the unique form of the connection if it exists. We now still have to show that it is actually a connection, by considering how it transforms.

□

We basically only introduced torsion to show that given a metric, a torsion-free connection is this unique one. From now on, we will talk about this object when we talk about a connection.

**Theorem 3** (Divergence theorem)**:** Consider a Riemannian manifold $\mathcal{M}$ with metric $g$ and boundary $\partial\mathcal{M}$. Let $n^\mu$ be an outward-pointing unit vector orthogonal to (any tangent vectors in) the boundary. Then, for any $X^\mu$,

$$\int_{\mathcal{M}} \mathrm{d}^n x \sqrt{|g|} \nabla_\mu X^\mu = \int_{\partial\mathcal{M}} \mathrm{d}^{n-1} x \sqrt{|\gamma|} n_\mu X^\mu, \quad (2.71)$$

where $\gamma_{ij}$ is the pull-back of $g$ onto $\partial\mathcal{M}$. On a Lorentzian manifold, this also holds with $\sqrt{g} \to \sqrt{-g}$ provided that $\partial\mathcal{M}$ is purely timelike or purely spacelike.

> **Remark:** Note that this follows on from Stokes' theorem. However, we did not prove Stokes' theorem, and since we will use the divergence theorem over and over again, we will prove this explicitly.

**Lemma 4:**

$$\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{g}} \partial_\nu \sqrt{g} \quad (2.72)$$

*Proof of Lemma 4.* Introducing the notation that $\hat{g}$ is a matrix, we have

$$\begin{aligned}
\Gamma^{\mu}_{\mu\nu} &= \frac{1}{2} g^{\mu\rho}\partial_\nu g_{\mu\rho} = \frac{1}{2}\operatorname{tr}\big(\hat{g}^{-1}\partial_\nu\hat{g}\big) && (2.73)\\
&= \frac{1}{2}\operatorname{tr}(\partial_\nu \log\hat{g}) && (2.74)\\
&= \frac{1}{2}\partial_\nu \log\det\hat{g}, \qquad \operatorname{tr}\log A = \log\det A && (2.75)\\
&= \frac{1}{2}\frac{1}{\det\hat{g}}\partial_\nu \det\hat{g} && (2.76)\\
&= \frac{1}{\sqrt{\det\hat{g}}}\partial_\nu\sqrt{\det\hat{g}} && (2.77)\\
&= \frac{1}{\sqrt{g}}\partial_\nu\sqrt{g} && (2.78)
\end{aligned}$$

□

*Proof of Theorem 3.* Using Lemma 4, we have that

$$\begin{aligned}
\sqrt{g}\nabla_\mu X^\mu &= \sqrt{g}(\partial_\mu X^\mu + {\Gamma^\mu}_{\mu\nu}X^\nu) && (2.79)\\
&= \sqrt{g}(\partial_\mu X^\mu + X^\nu\frac{1}{\sqrt{g}}\partial_\nu\sqrt{g}) && (2.80)\\
&= \partial_\mu(\sqrt{g}X^\mu). && (2.81)
\end{aligned}$$

Let us now restrict our discussion to a particular metric

$$g_{\mu\nu} = \begin{pmatrix} \gamma_{ij} & 0 \\ 0 & N^2 \end{pmatrix} \tag{2.82}$$

where the boundary $\partial\mathcal{M}$ is a surface with $x^n = \text{const.}$. We then use standard integration by parts over this boundary to give

$$\begin{aligned}
\int_{\mathcal{M}} \mathrm{d}^n x\,\sqrt{g}\nabla_\mu X^\mu &= \int_{\mathcal{M}} \mathrm{d}^n x\,\partial_\mu(\sqrt{g}X^\mu) && (2.83)\\
&= \int_{\partial\mathcal{M}} \mathrm{d}^{n-1} x\,\sqrt{\gamma N^2}X^n. && (2.84)
\end{aligned}$$

The unit normal vector is chosen $n^\mu = (0, 0, \dots, 0, 1/N)$, so that when measured with respect to the metric, we indeed satisfy the definition for a unit vector: $g_{\mu\nu}n^\mu n^\nu = 1$. This means that when we lower with the metric we have $n_\mu = (0, 0, \dots, N)$ and

$$\int_{\mathcal{M}} \mathrm{d}^n x\,\partial_\mu(\sqrt{g}X^\mu) = \int_{\partial\mathcal{M}} \mathrm{d}^{n-1} x\,\sqrt{\gamma}n_\mu X^\mu. \tag{2.85}$$

The final expression is covariant, and so holds in all coordinate systems. □

### 2.2.2 Parallel Transport

The general story is that differentiation requires us to compare the vectors at two 'neighbouring' points. But these points have different tangent spaces, and we cannot simply add vectors in these different tangent spaces. The connection allows us to define a map between vector spaces that live at different points. This map is called *parallel transport*.

**Definition 38** (parallel transport)**:** Take a vector field $X$ and an integral curve $C$ with the property

$$X^\mu|_C = \frac{\mathrm{d}x^\mu(\tau)}{\mathrm{d}\tau}. \tag{2.86}$$

A tensor $T$ is said to be *parallel transported* along C if

$$\nabla_X T = 0. \tag{2.87}$$

**Example 2.2.3:** A vector field $Y \in \mathfrak{X}(\mathcal{M})$ is parallel transported along $C$ if it obeys $\nabla_X Y = 0$. In components, this is

$$X^\nu(\partial_\nu Y^\mu + \Gamma^\mu_{\nu\rho} Y^\rho) = 0. \tag{2.88}$$

In particular, if we restrict to the curve $C$, consider $Y^\mu(x(\tau))$, which, from obeys (2.88)

$$\frac{\mathrm{d}Y^\mu}{\mathrm{d}\tau} + X^\nu \Gamma^\mu_{\nu\rho} Y^\rho = 0. \tag{2.89}$$

Given an initial condition $Y(x(\tau = 0)) \in T_p(\mathcal{M})$, where $p = x(\tau = 0)$, equation (2.89) determines a unique vector at each point along $C$. This is simply because it is a first order differential equation.

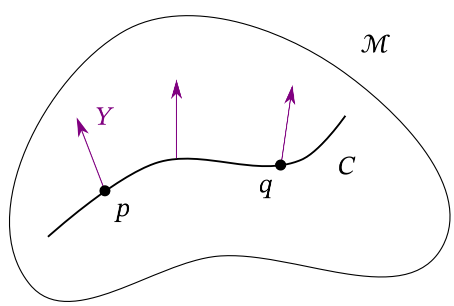


Figure 2.2: Parallel transport

> **Remark:** We can define parallel transport with only one curve. However, we need a vector field to cover the whole manifold. The parallel transport property depends on the path along which we parallel transport. Different paths will in general give different results.

### 2.2.3 Geodesics

**Definition 39:** A *geodesic* is a curve tangent to $X \in \mathfrak{X}(\mathcal{M})$ that is parallel transported along itself: it obeys $\nabla_X X = 0$.

From (2.89), the coordinates $x^\mu(\tau)$ along the curve obey the (affinely parametrised) geodesic equation:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\rho\nu}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0. \tag{2.90}$$

The geodesic equation also arises as the equation of motion from the action

$$S = \int \mathrm{d}\tau\, g_{\mu\nu}(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}. \tag{2.91}$$

> **Remark:** If you are ever given a metric and you need to compute the Christoffel symbols, it gets quite difficult quickly. Do not compute the long way round The quick way round is to plug it into the action (2.91) and then work out the equations of motion. Comparing this with the geodesic equation (2.90) gives the Christoffel symbols $\Gamma^\mu_{\rho\nu}$. There is a small subtlety with symmetrisation.

## 2.3 Normal coordinates

**Claim 10** (Riemann Normal Coordinates)**:** For any point $p \in \mathcal{M}$, we can always find *normal coordinates* in which the metric components are

$$g_{\mu\nu}(p) = \delta_{\mu\nu} \quad \text{and} \quad g_{\mu\nu,\rho}(p) = 0. \tag{2.92}$$

*Motivation.* To see that this is possible, start with coordinates $\tilde{x}^\mu$ and change to $x^\mu$, such that $\tilde{x}^\mu(p) = x^\mu(p) = 0$. Then

$$\tilde{g}_{\rho\sigma}\frac{\partial\tilde{x}^\rho}{\partial x^\mu}\frac{\partial\tilde{x}^\sigma}{\partial x^\nu} = g_{\mu\nu} \tag{2.93}$$

Taylor expanding we get

$$\tilde{x}^\rho \; \frac{\partial\tilde{x}^\rho}{\partial x^\mu}\bigg|_{x=0} x^\mu + \frac{1}{2}\,\frac{\partial^2\tilde{x}^\rho}{\partial x^\mu\partial x^\nu}\bigg|_{x=0} x^\mu x^\nu + \ldots \tag{2.94}$$

At leading order, we want

$$\tilde{g}_{\rho\sigma}\,\frac{\partial\tilde{x}^\rho}{\partial x^\mu}\bigg|_{x=0}\,\frac{\partial\tilde{x}^\sigma}{\partial x^\nu}\bigg|_{x=0} = \delta_{\mu\nu}. \tag{2.95}$$

We have $\frac{1}{2}n(n+1)$ conditions, but $n^2$ coefficients in $\partial\tilde{x}^\rho/\partial x^\mu$. We can do it with $\frac{1}{2}n(n-1) = \dim SO(n)$ left over. At second order, we want to set $g_{\mu\nu,\rho} = 0$. This gives $\frac{1}{2}n(n+1)\times n$ conditions.

Meanwhile, $\partial^2\tilde{x}^\rho / \partial x^\mu \partial x^\nu$ has $n \times \frac{1}{2}n(n+1)$ components. In terms of degrees of freedom, this suggests that we can do it. To actually prove it we will have to do some more work. However, the usefulness of this counting comes in when we consider the second derivative of the metric: If we try to go further, we have $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)$ components in $g_{\mu\nu,\rho\sigma}$. And the third derivative of the coordinate transformation $\partial^3\tilde{x}^\rho / \partial x^\mu \partial x^\nu \partial x^\lambda$ has $n \times \frac{1}{3!}n(n+1)(n+2)$ components. We find that we do not have enough degrees of freedom to do this; we are short by $\frac{1}{12}n^2(n^2-1)$. This is precisely the number of independent coordinates of the Riemann tensor $R^\sigma{}_{\rho\mu\nu}$.

Let us now actually give the coordinates that satisfy the claim. We can construct normal coordinates using geodesics and the *exponential map* (c.f. *Symmetries, Fields, and Particles*).

$$\mathrm{Exp}: T_p(\mathcal{M}) \to \mathcal{M} \tag{2.96}$$

We 'follow along' the geodesic such that $d\tilde{x}^\mu / d\tau \,|_{\tau=0} = \tilde{X}^\mu$ for distance $\tau = 1$. Now, pick an or-

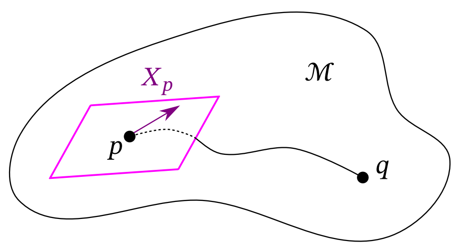


Figure 2.3

thonormal basis $\{e_\mu\}$ of $T_p(\mathcal{M})$ such that $X_p = X^\mu e_\mu$. Normal coordinates are $x^\mu(q) = X^\mu$. The completion of the proof is in the printed notes. □

> **Remark:** This is a mathematical version of the equivalence principle.

**Corollary:** In normal coordinates, $\Gamma^\mu_{\nu\rho}(p) = 0$.

## 2.4 Path Dependence and Curvature

How does parallel transport depend on the path taken? Let $X, Y \in \mathfrak{X}(\mathcal{M})$ be vector fields such that $[X, Y]$. Consider two curves generated by $X$ and $Y$. Now consider a vector $Z_p$ living in the tangent space $T_p(\mathcal{M})$ at point $p$. As illustrated in Figure 2.4, the result of parallel transporting this vector from $p$ to another point $r$ on the manifold will depend on the path taken. The expression for their difference is most clearly expressed in the case of infinitesimal translations. Moreover, life will be much easier if we pick normal coordinates at $p$ with $x^\mu = (\tau, \lambda, 0, 0, \dots)$. Along $p \to q$, the parallel

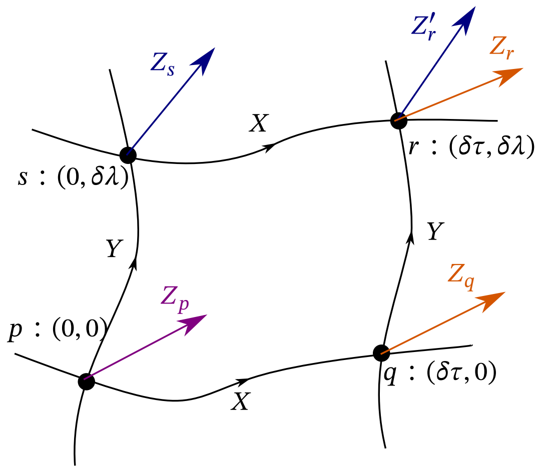


Figure 2.4: Path dependence of parallel transport

transport equation is a first order differential equation:

$$\frac{\mathrm{d}Z^\mu}{\mathrm{d}\tau} + X^\nu \Gamma^\mu_{\rho\nu} Z^\rho = 0. \tag{2.97}$$

Let us first see which vector $Z_r$ we end up with when taking the path

$$p \xrightarrow{\delta\tau} q \xrightarrow{\delta\lambda} r. \tag{2.98}$$

The first step is to Taylor expand $Z_q$ about the point $p : (\tau = 0, \lambda = 0, \dots)$. Let

$$Z^\mu_q = Z^\mu_p + \left.\frac{\mathrm{d}Z^\mu}{\mathrm{d}\tau}\right|_{\tau=0} d\tau + \frac{1}{2} \left.\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\tau^2}\right|_{\tau=0} d\tau^2 + \dots \tag{2.99}$$

$$\left.\frac{\mathrm{d}Z^\mu}{\mathrm{d}\tau}\right|_p = -(X^\nu Z^\rho \Gamma^\mu_{\rho\nu})_p = 0. \tag{2.100}$$

The last expression vanishes since $\Gamma^\mu_{\rho\nu}(p) = 0$ in normal coordinates. Moreover, we have

$$\left.\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\tau^2}\right|_{\tau=0} = -\left( X^\nu Z^\rho \frac{\mathrm{d}\Gamma^\mu_{\rho\nu}}{\mathrm{d}\tau} + \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} Z^\rho \Gamma^\mu_{\rho\nu} + X^\nu \frac{\mathrm{d}Z^\rho}{\mathrm{d}\tau} \Gamma^\mu_{\rho\nu} \right)_p . \tag{2.101}$$

By the previous reasoning, the latter two summands vanish. Using $\frac{\mathrm{d}}{\mathrm{d}\tau} = X^\sigma \partial_\sigma$, we obtain

$$\left.\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\tau^2}\right|_{\tau=0} = -\left( X^\nu X^\sigma Z^\rho \Gamma^\mu_{\rho\nu,\sigma} \right)_p . \tag{2.102}$$

Hence, we have

$$Z^\mu_q = Z^\mu_p - \frac{1}{2} (X^\nu X^\sigma Z^\rho \Gamma^\mu_{\rho\nu,\sigma})_p d\tau^2 + \dots \tag{2.103}$$

Next, we will go from $q \to r$:

$$Z^\mu_r = Z^\mu_q + \left.\frac{\mathrm{d}Z^\mu}{\mathrm{d}\lambda}\right|_q d\lambda + \frac{1}{2} \left.\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\lambda^2}\right|_q d\lambda^2 + \dots \tag{2.104}$$

For the first summand, we use the earlier result, while for the second term we use parallel transport

$$\left.\frac{\mathrm{d}Z^\mu}{\mathrm{d}\lambda}\right|_q = -\left(Y^\nu Z^\rho \Gamma^\mu_{\rho\nu}\right)_q \tag{2.105}$$

$$= -\left(Y^\nu Z^\rho \frac{\mathrm{d}\Gamma^\mu_{\rho\nu}}{\mathrm{d}\tau}\right)_p d\tau + \dots \tag{2.106}$$

$$= -\left(Y^\nu Z^\rho X^\sigma \Gamma^\mu_{\rho\nu,\sigma}\right) d\tau + \dots \tag{2.107}$$

Similarly, the second derivative is

$$\left.\frac{\mathrm{d}^2 Z^\mu}{\mathrm{d}\lambda^2}\right|_q = -\left(Y^\nu Y^\sigma Z^\rho \Gamma^\mu_{\rho\nu,\sigma}\right)_q + \dots . \tag{2.108}$$

The higher order terms $(\dots)$ include $\Gamma^\mu_{\rho\nu}(q) \sim \frac{\mathrm{d}}{\mathrm{d}t}\Gamma^\mu_{\rho\nu} d\tau$. Putting all of this together, the components of the vector $Z$ at $r$ is

$$Z^\mu_r = Z^\mu_p - \frac{1}{2}(\Gamma^\mu_{\rho\nu,\sigma})_p \left[X^\nu X^\sigma Z^\rho d\tau^2 + 2Y^\nu Z^\rho X^\sigma d\tau d\lambda + Y^\nu Y^\sigma Z^\rho d\lambda^2\right]_p + \dots . \tag{2.109}$$

If we go the other way, we have

$$(Z')^\mu_r = Z^\mu_p - \frac{1}{2}(\Gamma^\mu_{\rho\nu,\sigma})_p \left[X^\nu X^\sigma Z^\rho d\tau^2 + 2X^\nu Z^\rho Y^\sigma d\lambda d\tau + Y^\nu Y^\sigma Z^\rho d\lambda^2\right] + \dots \tag{2.110}$$

The difference between the two is

$$\Delta Z^\mu_r = Z^\mu_r - (Z')^\mu_r \tag{2.111}$$

$$= \left(\Gamma^\mu_{\rho\nu,\sigma} - \Gamma^\mu_{\rho\sigma,\nu}\right)_p (Y^\nu Z^\rho X^\sigma)_p \, d\lambda d\tau + \dots \tag{2.112}$$

$$= (R^\mu{}_{\rho\nu\sigma} Y^\nu X^\rho X^\sigma)_p d\lambda d\tau + \dots \tag{2.113}$$

We picked very special coordinates. However, the final expression is a tensor equation, which has to hold in all coordinate systems, even though we used special coordinates to derive it.

> **Remark:** If a tensor evaluates to zero in one coordinate system, it does so in all coordinate systems. This is not true of the Christoffel symbols.

> **Remark:** There is a whole area of geometry, called *holonomy*, which deals with the possible rotations of the vector that can be obtained when going along certain paths along a manifold. Calabi-Yau manifolds are related to *special holonmies*.

### 2.4.1 Geodesic Deviation

Consider the one-parameter family of geodesics $x^\mu(\tau; s)$; for a fixed $s$, $x^\mu(\tau; s)$ is a geodesic with affine parameter $\tau$. Figure 2.5 depicts these geodesics, which we can think of being integral curves

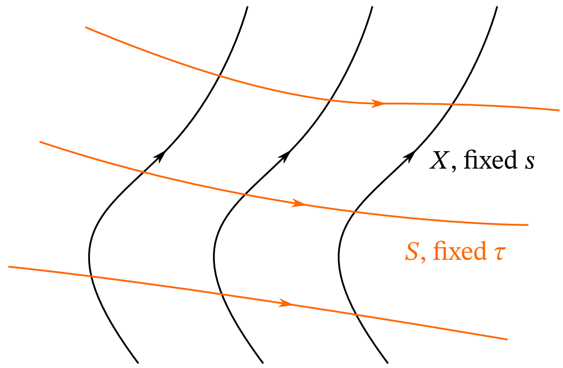


Figure 2.5

generated by a vector field $X^\mu = \partial x^\mu/\partial\tau|_s$. Moreover, the curves of fixed $\tau$ can be thought of as being generated by a vector fields $S^\mu = \partial x^\mu/\partial s|_\tau$ We will not prove it, but we can always pick $\tau, s$ such that $[S, X] = 0$.

**Claim 11:**

$$\nabla_X \nabla_X S = R(X, S)X \tag{2.114}$$

$$\text{or} X^\nu \nabla_\nu (X^\rho \nabla_\rho S^\mu) = R^\mu{}_{\nu\rho\sigma} X^\nu X^\rho S^\sigma \tag{2.115}$$

$$\text{or } \frac{\mathrm{d}^2 S^\mu}{\mathrm{d}\tau^2} = R^\mu{}_{\nu\rho\sigma} X^\nu X^\rho S^\sigma \tag{2.116}$$

*Proof.* We are dealing with the Levi-Civita connection, which is torsion free. For a torsion-free connection, we have

$$[X, S] = 0 \Rightarrow \nabla_X S = \nabla_S X. \tag{2.117}$$

We use this in the left hand side of (2.115) to find

$$\nabla_X \nabla_X S = \nabla_X \nabla_S X = \nabla_S \nabla_X X + R(X, S)X. \tag{2.118}$$

Now $\nabla_S \nabla_X X = 0$ as $X$ is a geodesic. □

> **Remark:** In the physics-part of this lecture course, we will see this again when we talk about gravitational waves.

## 2.5 More on the Riemann Tensor

Recall that the definition of the Riemann tensor in terms of the Christoffel symbols is

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\rho\nu} - \partial_\nu \Gamma^\sigma_{\rho\mu} + \Gamma^\lambda_{\nu\rho}\Gamma^\sigma_{\mu\lambda} - \Gamma^\lambda_{\mu\rho} - \Gamma^\lambda_{\mu\rho}\Gamma^\sigma_{\nu\lambda}. \tag{2.119}$$

This satisfies certain symmetry properties which are more evident once we lower the first index.

**Claim 12:** $R_{\sigma\rho\mu\nu} = g_{\sigma\lambda}R^{\lambda}{}_{\rho\mu\nu}$ obeys

- $R_{\sigma\rho\mu\nu} = -R_{\sigma\rho\nu\mu}$
- $R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\mu\nu}$
- $R_{\sigma\rho\mu\nu} = R_{\mu\nu\sigma\rho}$
- $R_{\sigma[\rho\mu\nu]} = 0$, the *first Bianchi identity*

*Proof.* Use normal coordinates

$$R_{\sigma\rho\mu\nu} = g_{\sigma\lambda}(\partial_\mu\Gamma^\lambda_{\nu\rho} - \partial_\nu\Gamma^\lambda_{\mu\rho}) \tag{2.120}$$
$$= \frac{1}{2}(\partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\nu\partial_\rho g_{\mu\sigma} + \partial_\nu\partial_\sigma g_{\mu\rho}) \tag{2.121}$$

Now stare at this. □

**Claim 13** (The Second Bianchi Identity)**:**

$$\nabla_{[\lambda}R_{\sigma\rho]\mu\nu} = 0 \quad \text{or equivalently} \quad R_{\sigma\rho[\mu\nu;\lambda]} = 0 \tag{2.122}$$

*Proof.* The Riemann tensor is, schematically, $R \sim \partial\Gamma + \Gamma\Gamma$ and so its derivative is $\partial R \sim \partial^2\Gamma + 2\Gamma\partial\Gamma$. In normal coordinates at a point, we have $\Gamma = 0$ (but $\partial\Gamma \neq 0$ in general), so the last summand vanishes and the partial derivative of the Riemann tensor is

$$R_{\sigma\rho\mu\nu,\lambda} = \frac{1}{2}(g_{\nu\sigma,\rho\mu\lambda} - g_{\nu\rho,\sigma\mu\lambda} - g_{\mu\sigma,\rho\nu\lambda} + g_{\mu\rho,\sigma\nu\lambda}) \tag{2.123}$$

The partial derivatives $\partial_\mu\partial_\lambda$ in the first term commute: $g_{\nu\sigma,\rho\mu\lambda} = g_{\nu\sigma,\rho\mu\lambda}$. Therefore, anti-symmetrising over $\mu$ and $\lambda$ makes this term vanish. In the same way, the other three terms vanish when we anti-symmetriese over $\mu, \nu$, and $\lambda$. Thus $R_{\sigma\rho[\mu\nu,\lambda]} = 0$. In normal coordinates, this is the same as $R_{\sigma\rho[\mu\nu;\lambda]} = 0$, which is a tensor equation and therefore holds in all coordinate systems. □

> **Remark:** The Bianchi identity here implies a similar identity on the Ricci tensor, which we will see soon. There is a very elegant proof, which we will discuss, of that identity from the action principle of GR.

We can easily construct new tensors from the Riemann tensor. The Riemann tensor has four indices; to build a new one, we just raise index and contract it with one of the lower ones. However, due to the symmetry properties, we have to contract one of the first pair indices to one index from the second pair.

**Definition 40:** The *Ricci tensor* $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$. This obeys $R_{\mu\nu} = R_{\nu\mu}$.

**Definition 41:** The *Ricci scalar* $R = g^{\mu\nu}R_{\mu\nu}$.

We are loosing information every time we contract indices like this. However, in some sense the information that is left over is the most essential part of the information. We can also apply the Bianchi identity to get

$$\nabla^{\mu}R_{\mu\nu} = \frac{1}{2}\nabla_{\nu}R. \tag{2.124}$$

**Definition 42:** The *Einstein tensor* $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$.

The whole point of building this tensor is that, using the Riemann tensor's Bianchi identity, these two tensors cancel to give a Bianchi identity for the Einstein tensor:

$$\nabla^{\mu}G_{\mu\nu} = 0. \tag{2.125}$$

Proving the Bianchi identity for the Riemann tensor is a pain, but in Section 3.2 we will show that the Bianchi identity of the Einstein tensor follows from the diffeomorphism invariance of the Einstein-Hilbert action.

## 2.6 Connection One-Forms

These objects are, as we will see, closely related to two-forms. Connection 1-forms are a technology designed to make it easier to calculate the Riemann tensor for a specific metric. It is not a slick trick like the one that we saw to find the Levi-Civita connection components.

Given a coordinate basis $\{e_{\mu}\} = \{\partial_{\mu}\}$. The idea is that we will take a different basis of tangent vectors so that things start to look simpler. Given the coordinate basis, we can always introduce a different basis $\{\hat{e}_a = e_a{}^{\mu}\partial_{\mu}\}$ . There is a particularly clever basis to choose: On a Riemannian / Lorentzian manifold, we pick a basis such that when we evaluate the metric on this new metric, we get

$$g(\hat{e}_a, \hat{e}_b) = g_{\mu\nu}e_a{}^{\mu}e_b{}^{\nu} = \begin{cases} \delta_{ab} & \text{Riemannian} \\ \eta_{ab} & \text{Lorentzian} \end{cases} \tag{2.126}$$

Note that this is not the metric in a set of coordinates (such as in Riemann normal coordinates). This is rather a diagonalised version of the metric. The components $e_a{}^{\mu}$ are called *vielbeins*. In general, having an $n$-component basis is called an $n$-bein (where we use the German word for $n$, e.g. $n = 2$: zweibeins). We raise and lower greek indices with $g_{\mu\nu}$ and latin indices using $\delta_{ab}$. The basis of one-forms $\left\{\hat{\theta}^a\right\}$ obey

$$\hat{\theta}^a(\hat{e}_b) = \delta^a_b. \tag{2.127}$$

The right hand side is a Krönicker delta for both Lorentzian and Riemannian metrics. They are $\hat{\theta}^a = e^a{}_\mu dx^\mu$ with $e^a{}_\mu e_b{}^\mu = \delta^a_b$ and $e^a{}_\mu e_a{}^\nu = \delta^\nu_\mu$. The metric is

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = \delta_{ab}\hat{\theta}^a \otimes \hat{\theta}^b \implies g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \delta_{ab}. \tag{2.128}$$

**Example 2.6.1:** Consider the metric

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2(d\theta^2 + \sin^2(\theta) d\phi^2) \tag{2.129}$$

This is one of the most important class of metric in General relativity, since for a particular choice of $f$ we obtain the *Schwarzschild solution*. In our new notation, this is $ds^2 = \eta_{ab}\hat{\theta}^a \otimes \hat{\theta}^b$ with non-coordinate 1-forms

$$\hat{\theta}^0 = f dt \quad \hat{\theta}^1 = f^{-1} dr \quad \hat{\theta}^2 = r d\theta \quad \hat{\theta}^3 = r \sin\theta d\phi. \tag{2.130}$$

> **Remark:** Note that this is a slightly annoying convention; $\hat{\theta}$ has nothing to do with $\theta$ or $d\theta$.

In the basis $\{\hat{e}_a\}$, the components of the connection are by definition

$$\nabla_{\hat{e}_c}\hat{e}_b := \Gamma^a_{cb}\hat{e}_a. \tag{2.131}$$

> **Remark:** Another slightly annoying convention: the type of indices (Roman or Greek) determine which object $\Gamma$ we are looking at. $\Gamma^\mu{}_{\rho\nu} \neq \Gamma^b{}_{cb}$.

**Definition 43:** Given this, we define the *connection 1-form* or *spin connection* $\omega^a{}_b = \Gamma^a_{cb}\hat{\theta}^c$.

> **Remark:** The reason that this is called the spin-connection is that we have to use this object when we want to couple to Dirac spinors.

**Claim 14** (First Cartan Structure Equation)**:** Defining $\omega_{ab} = \delta_{ac}\omega^c{}_b$, we have

$$d\hat{\theta}^a + \omega^a{}_b \wedge \hat{\theta}^b = 0 \tag{2.132}$$

*Proof.* No. □

**Claim 15:** For the Levi-Civita connection, the lowered spin connection is anti-symmetric:

$$\omega_{ab} = -\omega_{ba}. \tag{2.133}$$

**Definition 44:** In the vielbein basis, $R^a{}_{bcd} = R(\hat{\theta}^a; \hat{e}_c, \hat{e}_d, \hat{e}_b)$ (again, the indices tell us what kind of object this is), with $R^a{}_{bcd} = -R^a{}_{bdc}$. We define the *curvature 2-form*

$$\mathcal{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\hat{\theta}^c \wedge \hat{\theta}^d. \tag{2.134}$$

> **Remark:** This has the full information of the Riemann tensor. Note that when we write this in components $\hat{\theta} = e^a{}_\mu dx^\mu$, we find that the components of the curvature 2-form are $(\mathcal{R}^a{}_b)_{\mu\nu}$.

**Claim 16** (Second Cartan Structure Equation)**:**

$$\mathcal{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{2.135}$$

*Proof.* No. □

If we are given a metric, the quickest way to compute the Riemann tensor is to calculate the curvature one-forms $\hat{\theta}$. Then we can easily take the exterior derivative, which allows us to use the first Cartan structure equation. Plugging this into the second structure equation, we can find the components of the Riemann tensor.

## Back to the Example

Compute $d\hat{\theta}^a$:

$$\hat{\theta}^0 = f' dr \wedge dt \qquad d\hat{\theta}^1 = f' dr \wedge dr = 0 \tag{2.136}$$

$$d\hat{\theta}^2 = dr \wedge d\theta \qquad d\hat{\theta}^3 = \sin\theta dr \wedge d\phi - r\cos\theta d\theta \wedge d\phi. \tag{2.137}$$

Use $d\hat{\theta}^0 = -\omega^0{}_b \wedge \hat{\theta}^b$ to convince yourself that $\omega^0{}_1 = f' f dt = f'\hat{\theta}^0$ and $\omega^0{}_1 = -\omega_{01} = +\omega_{10} = \omega^1{}_0$, where we used the Minkowski metric in the first equality.

**Check** $d\hat{\theta}^1 = -\omega^1{}_b \wedge \hat{\theta}^b = -\omega^1{}_0 \wedge \hat{\theta}^0 + \cdots = -f'\hat{\theta}^0 \wedge \hat{\theta}^0 + \ldots$, where the $\hat{\theta}^0 \wedge \hat{\theta}^0$ vanishes. ✓

Proceed in the same way to find that the only non-vanishing ones are

$$\omega^0{}_1 = \omega^1{}_0 = f'\hat{\theta}^0 \tag{2.138}$$

$$\omega^2{}_1 = -\omega^1{}_2 = \frac{f}{r}\hat{\theta}^2 \tag{2.139}$$

$$\omega^3{}_1 = -\omega^1{}_3 = \frac{f}{r}\hat{\theta}^3 \tag{2.140}$$

$$\omega^3{}_2 = -\omega^2{}_3 = \frac{\cot\theta}{r}\hat{\theta}^3 \tag{2.141}$$

Now the curvature tensor is

$$\begin{aligned}
\mathcal{R}^0{}_1 &= d\omega^0{}_1 + \overbrace{\omega^0{}_c \wedge \omega^c{}_1}^{\text{In this case, these vanish}} && (2.142)\\
&= f' d\hat{\theta}^0 + f'' dr \wedge \hat{\theta}_0 && (2.143)\\
&= ((f')^2 + f'' f) dr \wedge dt && (2.144)\\
&= -((f')^2 + f'' f)\hat{\theta}^0 \wedge \hat{\theta}^1 && (2.145)\\
\Rightarrow \ R_{0101} &= f f'' + (f')^2 && (2.146)
\end{aligned}$$

We convert back to Greek indices using the vielbeins:

$$R_{\mu\nu\rho\sigma} = e^a{}_\mu e^b{}_\nu e^c{}_\rho e^d{}_\sigma R_{abcd} \tag{2.147}$$

In our case, we find $R_{trtr} = f f'' + (f')^2$.

# 3 The Einstein Equations

It is time to do some physics! All of the mathematics that we developed will pay off quickly in that we will quickly understand why we have to write down the Einstein equations.

We should think about gravity in much the same way as we think about other forces. Spacetime is a manifold $\mathcal{M}$, equipped with a Lorentzian metric $g$. This metric is a field, which tells us how gravity is to behave. This is similar to the fields in electromagnetism or other forces. The right way to describe a fundamental force in physics is with an action.

## 3.1 The Einstein-Hilbert Action

The dynamics is governed by the *Einstein-Hilbert action*,

$$S = \int \mathrm{d}^4x \sqrt{-g} R \tag{3.1}$$

**Remark:** This is where we see the payoff: There is not very much that we can write down. From differential geometry, we know that we need to cook up a top-form. Thankfully, given the metric, there is a natural volume form $\sqrt{-g}$ (minus sign for Lorentzian metric).

We then integrate over a scalar function. The simplest function—a constant function—gives us the volume of the manifold, but it is not quite the right thing to give dynamics. The next simplest thing that we can write down is the Ricci scalar $R$. Note that $R \sim \partial\Gamma + \Gamma\Gamma$ and $\Gamma \sim g^{-1}\partial g$. Therefore, the Ricci scalar $R$ has two derivatives. This is what we have come to expect from an action for a bosonic scalar field, or the Maxwell action.

There are other choices, like the square of the Riemann tensor, or others. However, these would be four-derivative terms. It turns out that with four derivatives we have exactly three options and at six derivatives we have even more. We will only consider two-derivatives, like in the other action principles we use in theoretical physics.

**Remark:** This has to be the Levi-Civita connection, since we just have the metric. Other than $g$ and $\mathcal{M}$, there is nothing else to play with in this theory.

To derive the equations of motion, we vary the field, which in this case is simply the metric,

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \tag{3.2}$$

in the usual way. We can also consider how the inverse of the metric changes

$$\begin{aligned} g^{\mu\nu} g_{\rho\mu} = \delta^\nu_\rho \quad &\Rightarrow \quad g^{\mu\nu}\delta g_{\rho\mu} + g_{\rho\mu}\delta g^{\mu\nu} = 0 && (3.3) \\ &\Rightarrow \quad \delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma}\delta g_{\rho\sigma}. && (3.4) \end{aligned}$$

It turns out that it is actually marginally simpler to look at how the inverse changes. We have

$$\delta S = \int \mathrm{d}^4 x \left[ \delta(\sqrt{-g}) g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}\left(\delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right)\right]. \tag{3.5}$$

**Claim 17:** $\delta\sqrt{-g} = \frac{-1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}$.

*Proof.* We use $\log\det A = \operatorname{tr}\log A$

$$\Rightarrow \frac{1}{\det A}\delta(\det A) = \operatorname{tr}\big(A^{-1}\delta A\big). \tag{3.6}$$

**Remark:** We vary the log of a matrix as

$$B = \log A \;\Rightarrow\; A = e^B \tag{3.7}$$

$$\begin{aligned} &\Rightarrow \operatorname{Tr}(\delta A) = \operatorname{Tr}\big(e^B \delta B\big) = \operatorname{Tr}(A\delta B) && (3.8) \\ &\Rightarrow \operatorname{Tr}[\delta(\log A)] = \operatorname{Tr}\big(A^{-1}\delta A\big) && (3.9) \end{aligned}$$

$$\begin{aligned} \Rightarrow \delta\sqrt{-g} &= \frac{1}{2}\frac{1}{\sqrt{-g}}(-g) g^{\mu\nu}\delta g_{\mu\nu} && (3.10) \\ &= \frac{1}{2}\sqrt{-g} g^{\mu\nu}\delta g_{\mu\nu} && (3.11) \end{aligned}$$

□

**Claim 18:**

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\nabla_\mu \delta g_{\sigma\nu} + \nabla_\nu \delta g_{\sigma\mu} - \nabla_\sigma \delta g_{\mu\nu}) \tag{3.12}$$

*Proof.* First, note that $\Gamma^{\mu}_{\rho\nu}$ is *not* a tensor, but the variation $\delta\Gamma^{\mu}_{\rho\nu}$ is the difference between two connections and *is* a tensor. In normal coordinates, at some point,

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu \delta g_{\sigma\nu} + \partial_\nu \delta g_{\sigma\mu} - \partial_\sigma \delta g_{\mu\nu}) \tag{3.13}$$

The $\delta g^{\rho\sigma}$ multiplies $\partial g$, which is $\partial g = 0$ in normal coordinates. Moreover, in normal coordinates we have $\Gamma = 0$ and therefore $\partial \to \nabla$.

$$\cdots = \frac{1}{2}g^{\rho\sigma}(\nabla_\mu \delta g_{\sigma\nu} + \nabla_\nu \delta g_{\sigma\mu} - \nabla_\sigma \delta g_{\mu\nu}). \tag{3.14}$$

This is a tensor equation, and is therefore true in all coordinates. □

**Claim 19:**

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^{\rho}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\rho}_{\mu\rho} \tag{3.15}$$

*Proof.* In normal coordinates where $\Gamma = 0$,

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\nu\rho} - \partial_\nu \Gamma^{\sigma}_{\mu\rho} \tag{3.16}$$

$$\Rightarrow \delta R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \delta\Gamma^{\sigma}_{\nu\rho} - \partial_\nu \delta\Gamma^{\sigma}_{\mu\nu} \tag{3.17}$$

$$= \nabla_\mu \delta\Gamma^{\sigma}_{\nu\rho} - \nabla_\nu \delta\Gamma^{\sigma}_{\mu\nu} \tag{3.18}$$

□

Therefore, the variation in the Einstein-Hilbert action is

$$\delta S = \int \mathrm{d}^4 x \sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right]\delta g^{\mu\nu} + \nabla_\mu X^\mu, \tag{3.19}$$

with $X^\mu = g^{\rho\nu}\delta\Gamma^{\mu}_{\rho\nu} - g^{\mu\nu}\delta\Gamma^{\rho}_{\nu\rho}$. This last term is a total derivative and we will just drop it.

> **Remark:** Unlike in other field theories, you do not have to look too far to find situations where these boundary terms become important. You can introduce the Gibbons-Hawking term to deal with these boundary terms. These might well become important in next term's *Black Holes* course.

Requiring $\delta S = 0$ for all variations $\delta g^{\mu\nu}$, we obtain the *Einstein equations*:

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 0 \tag{3.20}$$

In fact, we can simplify this by contracting on the right with $g^{\mu\nu}$: we get $R = 0$ which gives us

$$R_{\mu\nu} = 0. \tag{3.21}$$

These are the *Einstein vacuum equations*.

> **Remark:** Note that this does not say that spacetime has no curvature! It is the more subtle type of curvature, which the Riemann tensor has and the Ricci scalar has not, that is relevant for spacetime.

## Dimensional Analysis

The action should have exactly the same dimension as $\hbar$, which is $(energy) \times (time) = ML^2T^{-1}$. Meanwhile $[\mathrm{d}^4x] = L^4$ (or $L^3T$) and $[\sqrt{-g}] = 0$ (which is sort-of-true) and $[R] = L^{-2}$.

> **Remark:** We insist that in $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, the dimensions are hiding in the $dx^\mu$, and $g_{\mu\nu}$ is dimensionless.

Therefore, for dimensional consistency, we have

$$S = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}R. \tag{3.22}$$

At the moment, this does not change the equations of motion. However, once we introduce matter, this will have an effect.

**Definition 45:** The *Planck mass* is $M_{\mathrm{pl}}^2 = \frac{\hbar c}{8\pi G}$ and $M_{\mathrm{pl}} \sim 10^{18}$ GeV.

We work in units with $c = 1$ and $\hbar = 1$. In this case, the action can be written as

$$S = \frac{1}{2}M_{\mathrm{pl}}^2\int \mathrm{d}^4x\,\sqrt{-g}R. \tag{3.23}$$

Depending on the physics that we are interested in, we use these different forms of the action. We will also keep $G$ in all the formulae; other relativists tend to set it to one, but we will try to keep gravity on an equal footing with the other forces in the universe.

## The Cosmological Constant

We could add a further term to the action

$$S = \frac{1}{2}M_{\mathrm{pl}}^2\int \mathrm{d}^4x\,\sqrt{-g}\,(R - 2\Lambda) \tag{3.24}$$

We can think of this constant $\Lambda$ as a potential term for gravity. Doing the same variations as usual, we obtain

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\Lambda g_{\mu\nu} \;\Rightarrow\; R = 4\Lambda \;\Rightarrow\; R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{3.25}$$

We will study gravity by itself first. Only later we will try to understand how gravity couples to other fields, like masses.

> **Remark:** Einstein called $\Lambda$ to be his biggest mistake, since he picked it to balance out the expansion of the universe in the solutions he found, when compared against the redshift observed by Slipher and Hubble.

## 3.2 Diffeomorphisms Revisited

In part, we revisit this section to count the number of degrees of freedom of these equations. The metric has $\frac{1}{2}\times 4\times 5 = 10$ components. But two metrics related by a change of coordinates $x^\mu \to \tilde{x}^\mu(x)$ are physically equivalent. Therefore, the metric contains $10 - 4 = 6$ degrees of freedom. You might worry that the 10 equations of (3.25) over-determine the metric; we will see that this is not true. The change of coordinates can be viewed as a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$.

> **Remark:** This is a bit like the active vs passive transformation issue.

Such diffeomorphisms (diffeos) are the "gauge symmetry" of GR. We will look at diffeomorphism of the action. However, the action does not depend on coordinates we use; the change of coordinates is like a gauge theory of the action. Similarly to Noether's theorem, where symmetries give rise to conserved quantities, this gauge symmetry will give rise to something. We will try to find out what that something is.

**Claim 20:** Acting with an infinitesimal diffeomorphism along a vector field $X \in \mathfrak{X}(\mathcal{M})$, the metric changes as the Lie derivative

$$\delta g_{\mu\nu} = (\mathcal{L}_X g)_{\mu\nu}. \tag{3.26}$$

*Proof.* Consider a diffeomorphism that take points $x^\mu$ to nearby points $\tilde{x}^\mu$ as

$$x^\mu \to \tilde{x}^\mu(x) = x^\mu + \delta x^\mu. \tag{3.27}$$

We also know that a vector field on a manifold induces a flow; we will use this to take us to this nearby point. As such, we can think of the associated coordinate change as being generated by a vector field $X^\mu = -\delta x^\mu$. The metric transforms as

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x). \tag{3.28}$$

The Jacobian associated with the change of coordinate $\tilde{x}^\mu = x^\mu - X^\mu(x)$ can be inverted:

$$\frac{\partial \tilde{x}^\mu}{\partial x^\rho} = \delta^\mu_\rho - \partial_\rho X^\mu \quad \Rightarrow \quad \frac{\partial x^\rho}{\partial \tilde{x}^\mu} = \delta^\rho_\mu + \partial_\mu X^\rho + \dots, \tag{3.29}$$

where the higher order terms can be neglected for infinitesimal $X^\mu$. We then have

$$\tilde{g}_{\mu\nu}(\tilde{x}) = (\delta^\rho_\mu + \partial_\mu X^\rho)(\delta^\sigma_\nu + \partial_\nu X^\sigma) g_{\rho\sigma}(x) \tag{3.30}$$

$$= g_{\mu\nu}(x) + g_{\mu\rho}(x)\partial_\nu X^\rho + g^{\nu\rho}(x)\partial_\mu X^\rho \tag{3.31}$$

Meanwhile, we may Taylor expand $\tilde{g}_{\mu\nu}(\tilde{x})$ on the left-hand side around $x$ to give

$$\widetilde{g}_{\mu\nu}(\tilde{x}) = \tilde{g}_{\mu\nu}(x + \delta x) = \widetilde{g}_{\mu\nu}(x) - X^\lambda \partial_\lambda g_{\mu\nu}(x). \tag{3.32}$$

Thus, we find the change $\delta g_{\mu\nu}(x)$ of the metric at the point $x$ to be

$$\delta g_{\mu\nu} := \tilde{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = X^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\rho}\partial_\nu X^\rho + g_{\nu\rho}\partial_\mu X^\rho \tag{3.33}$$

$$= (\mathcal{L}_X g)_{\mu\nu}. \tag{3.34}$$

□

Alternatively, we can write

$$\delta g_{\mu\nu} = \partial_\mu X_\nu + \partial_\nu X_\mu + X^\rho \underbrace{(\partial_\rho g_{\mu\nu} - \partial_\mu g_{\rho\nu} - \partial_\nu g_{\mu\rho})}_{2g^{\rho\sigma}\Gamma^\sigma_{\mu\nu}} \tag{3.35}$$

$$= \nabla_\mu X_\nu + \nabla_\nu X_\mu. \tag{3.36}$$

Now look at the action

$$\delta S = \int \mathrm{d}^4 x \sqrt{-g} G^{\mu\nu} \delta g_{\mu\nu}. \tag{3.37}$$

The symmetries of the action are those for which the change in the action is zero. If we restrict to these changes of coordinates, it must be that for all choices of $X_\nu$, we have

$$\cdots = 2 \int \mathrm{d}^4 x \sqrt{-g} G^{\mu\nu} \nabla_\mu X_\nu = 0 \tag{3.38}$$

because changing coordinates is a gauge symmetry. Integrating by parts, and using the divergence theorem,

$$\dots = -2 \int \mathrm{d}^4 x \sqrt{-g} (\nabla_\mu G^{\mu\nu}) X_\nu \tag{3.39}$$

$$\Rightarrow \ \nabla_\mu G^{\mu\nu} = 0. \tag{3.40}$$

But we know that this is true for any metric. This is the Bianchi identity. This is essentially another derivation of the Bianchi identity using the path integral, using the diffeomorphism invariance of the action.

> **Remark:** The metric only actually has 6 pieces of information in it. However, the Einstein equations $G_{\mu\nu} = 0$ are 10 equations. You might think that this over-determines the metric, but the Bianchi identity saves us. In other words: $\nabla_\mu G^{\mu\nu} = 0$ are 4 conditions, which mean that the Einstein equations $G_{\mu\nu} = 0$ are really only 6 equations, which is the right number to determine the metric $g_{\mu\nu}$.

## 3.3 Some Simple Solutions

There are three cases that we should consider, depending on the cosmological constant: zero, positive and negative. We will spend a lot of time understanding the differences between these solutions. Even the solutions that look completely trivial—such as the one we will write down shortly—have interesting subtleties about them.

### 3.3.1 $\Lambda = 0$: Minkowski Spacetime

Need to solve $R_{\mu\nu} = 0$. The solution $g_{\mu\nu}$ is not allowed!

> **Remark:** Usually, the simplest solution that solves a field theory is always $\phi = 0$. On geometrical grounds, it is obvious that the metric cannot vanish, since we have to have an inverse. Nonetheless, this restriction, that none of the eigenvalues cross zero, is a very weird constraint to put on a dynamical field. What this might be telling us is that the metric is not a fundamental field of nature; the metric might emerge. For example, the other case where a field like this pops up is in fluid mechanics. When we write down the Navier-Stokes equations, we do not want the energy-density to cross zero, since the approximations that lead to the fluid equations break down otherwise. There are lots of hints that gravity is like fluid mechanics.
>
> Note that there are infinitely many solutions of these. We are nowhere close to understanding the properties of the general solution.

The simplest solution is *Minkowski spacetime*.

$$ds^2 = -dt^2 + d\mathbf{x}^2 \tag{3.41}$$

### 3.3.2 $\Lambda > 0$: De Sitter Spacetime

There are many solutions, many of them very hard to solve since these are coupled second order differential equations. We will look for solutions to $R_{\mu\nu} = \Lambda g_{\mu\nu}$ of the form

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2(d\theta^2 + \sin^2(\theta) d\phi^2). \tag{3.42}$$

We can compute $R_{\mu\nu}$ (using for example the curvature two-forms) to find

$$R_{tt} = -f^4 R_{rr} = f^3(f'' + \frac{2f'}{r} + \frac{(f')^2}{f}) \tag{3.43}$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta} = (1 - f^2 - 2ff'r)\sin^2\theta. \tag{3.44}$$

With $R_{\mu\nu} = \Lambda g_{\mu\nu}$, this becomes

$$\mu,\nu = tt, rr \;\Rightarrow\; f'' + \frac{2f'}{r} + \frac{(f')^2}{f} = -\frac{\Lambda}{f} \tag{3.45}$$

$$\mu,\nu = \theta\theta, \phi\phi \;\Rightarrow\; 1 - 2ff'r - f^2 = \Lambda r^2 \tag{3.46}$$

Both equations are solved by

$$f(r) = \sqrt{1 - \frac{r^2}{R^2}}, \qquad \text{with } R^2 = \frac{3}{\Lambda}. \tag{3.47}$$

**Definition 46:** The *de Sitter* (dS) spacetime is

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right) dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{3.48}$$

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the *round metric* on $S^2$.

> **Remark:** $r \in [0, R]$ but the metric appears to be singular at $r = R$.

Let us now build up some intuition of what it is like to live in a universe determined by metric (3.48). We can look at geodesics. With $\sigma$ denoting proper time, and $(\dot{\ldots}) = \frac{\mathrm{d}(\ldots)}{\mathrm{d}\sigma}$, we have the action:

$$S = \int \mathrm{d}\sigma \left[ -f(r)^2(\dot{t})^2 + f(r)^{-2}(\dot{r})^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \right]. \tag{3.49}$$

Given an action it is really tempting to find the equations of motion. However, in this case this is the wrong thing to do; the right thing to do is to consider the symmetries. There are two conserved quantities:

$$\text{angular momentum} \qquad l = \frac{1}{2}\frac{\partial L}{\partial \dot{\phi}} = r^2 \sin^2\theta\dot{\phi} \tag{3.50}$$

$$\text{energy} \qquad E = -\frac{1}{2}\frac{\partial L}{\partial \dot{t}} = f(r)^2 \dot{t}. \tag{3.51}$$

For the action without the square root, we should also enforce a constraint that tells us whether things are timelike, spacelike, or null. For massive particles, we require that the trajectory is timelike. With $\theta$ being proper time, this means that the Lagrangian itself should be $-1$:

$$-f^2\dot{t}^2 + f^{-2}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = -1. \tag{3.52}$$

For spacelike or timelike trajectories, we would have 1 or 0 on the right hand side respectively. Let us look for geodesics with $\theta = \pi/2$ and $\dot{\theta} = 0$. This is similar to the Kepler problem, where we say that angular momentum is conserved, so all the motion has to be in the plane normal to the angular momentum vector. Having done this, we do not look at the equations of motion, but instead see that the constraint becomes

$$\Rightarrow \dot{r}^2 + V_{\text{eff}}(r) = E^2, \tag{3.53}$$

with

$$V_{\text{eff}}(r) = \left(1 + \frac{l^2}{r^2}\right)\left(1 - \frac{r^2}{R^2}\right). \tag{3.54}$$

Here our intuition of relativistic Newtonian mechanics starts to kick in. This is the usual angular momentum barrier, illustrated in Figure 3.1. Notice that the potential is unbounded from below,

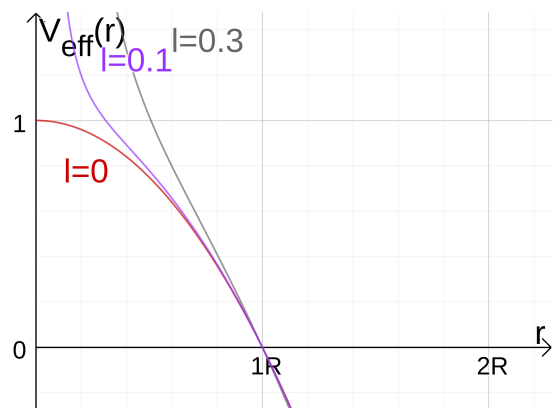


Figure 3.1

but it does so after the point at which we cannot trust the metric; it is singular at $r = R$ and so we should think of the particle to only be moving at $r < R$. This is a bit like an inside-out black hole! We get pushed outwards. For $l = 0$, we have

$$r(\sigma) = R\sqrt{E^2 - 1}\sinh\left(\frac{\sigma}{R}\right). \tag{3.55}$$

We can look for weird things that happen at the singularity $r = R$. However, nothing weird happens. In fact, this hits $r = R$ in some finite proper time $\sigma$. Meanwhile, we can ask what happens to coordinate time. Coordinate time solves

$$\frac{\mathrm{d}t}{\mathrm{d}\sigma} = E\left(1 - \frac{r^2}{R^2}\right)^{-1}. \tag{3.56}$$

**Claim 21:** Solutions to this have $t \to \infty$ as $r \to R$.

*Proof.* To see this, suppose $r(\sigma_*) = R$ and expand $\sigma = \sigma_* - \epsilon$. Find that

$$\frac{\mathrm{d}t}{\mathrm{d}\epsilon} \approx -\frac{-\alpha}{\epsilon} \tag{3.57}$$

for some constant $\alpha$. This implies that $t \sim -\alpha \log(\epsilon/R) \to \infty$ as $\epsilon \to 0$. □

This means that it takes a particle a finite proper time $\sigma$, but infinite coordinate time $t$, to move outwards from the origin and reach the singular point $r = R$.

> **Remark:** Again, this looks very similar to an inside-out black hole. Ultimately, we will find out that this is where we live; our universe is like an inside-out black hole…

**Claim 22:** In fact, de Sitter spacetime can be embedded in five-dimensional Minkowski space, which is $\mathbb{M}^5 = (\mathbb{R}^{1,4}, g)$ with metric

$$ds^2 = -(dX^0)^2 + \sum_{i=1}^{4}(dX^i)^2 \tag{3.58}$$

as the surface $-(X^0)^2 + \sum_{i=1}^4 (X^i)^2 = R^2$. The metric gets pulled back onto this surface.

*Proof.* Let $r^2 = \sum_{i=1}^3 (X^i)^2$ and $X^0 = \sqrt{R^2 - r^2}\sinh(t/R)$, $X^4 = \sqrt{R^2 - r^2}\cosh(t/R)$. In particular, this means that $(X^4)^2 - (X^0)^2 = R^2 - r^2$.

**Exercise 3.1:** You can check that if you compute $dX^0 = f\,dR + g\,dt$ and $dX^4 = \ldots$, and plug these into the Minkowski spacetime metric (3.58), you recover the de Sitter metric.

□

These coordinates are not particularly symmetric, meaning that $X^4$ has been singled out from $X^{1,2,3}$, and, moreover, cover only $X^4 \geq 0$. A better choice of coordinates is

$$X^0 = R\sinh\Big(\frac{\tau}{R}\Big) \quad \text{and} \quad X^i = R\cosh\Big(\frac{\tau}{R}\Big)y^i, \tag{3.59}$$

where $y^i$ parametrise a 3-sphere, i.e. $\sum_{i=1}^4 (y^i)^2 = 1$. These are called *global coordinates* on dS.

**Exercise 3.2:** Another small calculation you can do is to look at the variation again and plug it into (3.58) to give

$$ds^2 = -d\tau^2 + R^2\cosh^2\left(\frac{\tau}{R}\right)d\Omega_3^2, \tag{3.60}$$

where $d\Omega_3^2$ is the metric on $S^3$. This is called the *FRW* form of the metric.

We know that this metric must also solve the Einstein equation with the same cosmological constant (as it must!), since there is a coordinate transformation that takes us from de Sitter (`dS`) spacetime to this; they describe the same spacetime. This form of the metric has a natural cosmological interpretation in terms of an initially contracting and later expanding universe. For the past three billion years or so, this has been a fairly good approximation to our universe.

> **Remark:** Note that the first metric is the one that de Sitter found very soon after Einstein's theory was published. He found that immoving two observers move away from each other, and also calculated the redshift that emerged when those two observers sent light rays between each other. In particular, the first observations of Hubble and Slipher of the redshift were called the de Sitter effect. However, Hubble never believed in the expanding universe, which everyone credited him with, because the first metric seems to be independent of time. The second metric is more closely related to our notion of time, in which the universe actually (first contracts and then) expands with increasing $\tau$.

### 3.3.3 $\Lambda < 0$: Anti-de Sitter Spacetime

Let us look at some more solutions to the Einstein equations. Specifically, let us consider a negative cosmological constant $\Lambda < 0$. Again, we look for solutions of the kind

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2 d\Omega_2^2. \tag{3.61}$$

Exactly the same calculation as before gives us $f(r) = \sqrt{1 + r^2/R^2}$ with $R^2 = -3/\Lambda$. This is *anti-de Sitter* (`AdS`) spacetime. The plus sign rather than minus sign means that there is no coordinate singularity. Repeating the geodesic calculations, we find that this time, massive geodesics obey $\dot{r}^2 + V_{\text{eff}}(r) = E^2$ (designed to look like Newtonian mechanics) with $V_{\text{eff}}(r) = (1 + l^2/r^2)(1 + r^2/R^2)$. $l$ is to be interpreted as angular momentum. This is illustrated in 3.2. In particular, massive particles are confined to the centre of `AdS`. With a finite amount of energy, there is only so far we can get before we fall back towards low $r$. Massless particles follow null geodesics

$$-f^2 \dot{t}r^2 + f^{-2}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0. \tag{3.62}$$

The right hand side is now zero. At $\theta = \pi/2$, $\dot{\theta} = 0$, we have

$$\dot{r}^2 + V_{\text{null}}(r) = E^2 \qquad V_{\text{null}} = \frac{l^2}{2r^2}\left(1 + \frac{r^2}{R^2}\right). \tag{3.63}$$

This qualitatively changes the behaviour of the potential, as illustrated in Figure 3.3. There is nothing that stops the light rays to go as far as they like to $r \to \infty$.

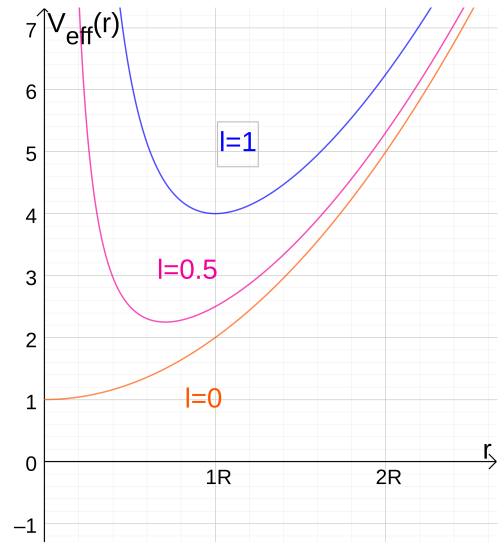


Figure 3.2

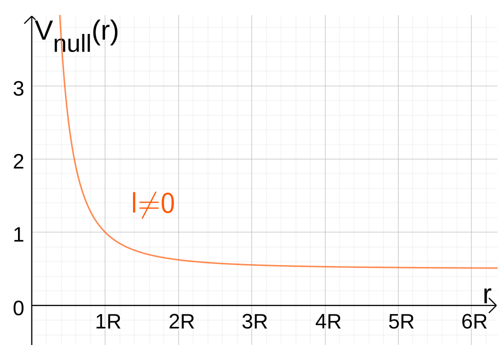


Figure 3.3

Introduce new coordinates $r = R\sinh\rho$. The $\mathtt{AdS}$ metric becomes

$$ds^2 = -\cosh^2\rho\, dt^2 + R^2\left[d\rho^2 + \sinh^2\rho\, d\Omega_2^2\right]. \tag{3.64}$$

The sinh is chosen to simplify the radial coordinate. The null geodesic equation (3.63) is

$$R\dot{\rho} = \pm\frac{E}{\cosh\rho} \qquad \Rightarrow \qquad R\sinh\rho = E(\sigma - \sigma_0), \tag{3.65}$$

where $\sigma$ is an affine parameter. Massless particles hit $\rho \to \infty$ as $\sigma \to \infty$. However, going back to $E = \cosh^2\rho\dot{t}$, the equation which defines the energy, we have $R\tan(t/R) = E(\sigma - \sigma_0)$. So $t \to \frac{\pi R}{2}$ as $\sigma \to \infty$. Massless particles reach infinity of $\mathtt{AdS}$ in finite coordinate time.

**Claim 23:** Like $\mathtt{dS}$, $\mathtt{AdS}$ can be viewed as a hyperboloid in $\mathbb{R}^{2,3}$

$$-(X^0)^2 - (X^4)^2 + \sum_{i=1}^{3}(X^i)^2 = R^2. \tag{3.66}$$

*Proof.* To see this, let

$$X^0 = R\cosh\rho\sin\left(\frac{t}{R}\right) \tag{3.67a}$$
$$X^4 = R\cosh\rho\cos\left(\frac{t}{R}\right) \tag{3.67b}$$
$$X^i = Ry^i\sinh\rho, \tag{3.67c}$$

where $\sum(y^i)^2 = 1$ defines a two-sphere and we recover the $\mathtt{AdS}$ metric in the form (3.64). □

There is one last set of coordinates, which $\mathtt{AdS}$ is most often written in in the physics literature:

$$X^i = \frac{\tilde{r}}{R}x^i, \qquad i = 0, 1, 2 \tag{3.68a}$$
$$X^4 - X^3 = \tilde{r} \tag{3.68b}$$
$$X^4 + X^3 = \frac{R^2}{\tilde{r}} + \frac{\tilde{r}}{R^2}\eta_{ij}x^i x^j \tag{3.68c}$$

Computing the metric from this parametrisation gives

$$ds^2 = R^2\frac{d\tilde{r}^2}{\tilde{r}^2} + \frac{\tilde{r}^2}{R^2} + \frac{\tilde{r}^2}{R^2}\eta_{ij}dx^i dx^j. \tag{3.69}$$

This is called the *Poincaré patch*. It does not cover the whole of $\mathtt{AdS}$. Unlike $\mathtt{dS}$ spacetime, $\mathtt{AdS}$ is not connected to our universe (as far as we can tell); however, it is an important spacetime in which we understand the behaviour of quantum gravity, which makes it worth studying!

## 3.4 Symmetries

There are three basic spacetimes in GR. We have intuition of why Minkowski is important; the other two have so far just been pulled out of thin air. The thing that makes these spaces special are the symmetries. They are the *maximally symmetric* spacetimes in general relativity.

Let us first define what a symmetry is. Any intuition you have about this is probably correct. Consider a sphere; it has the symmetry $SO(3)$. A rugby ball on the other hand, with the same topology, has only the symmetry $SO(2)$. (In general, this would be $O(3)$, but in these lectures we will only consider groups which are continuously connected to the identity.)

Consider a one-parameter family of diffeomorphisms $\sigma_t : \mathcal{M} \to \mathcal{M}$. Recall that this is associated to a vector field

$$K^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}, \tag{3.70}$$

which is tangent to the flow lines. We will define a symmetry by flowing from one point to a closeby one; if the point looks the same as the point we came from, this is a symmetry. More specifically, this flow is called an *isometry* if the metric looks the same at each point along the flow. From Claim 20, we know that this translates to

$$\mathcal{L}_K g = 0 \quad\iff\quad \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0. \tag{3.71}$$

This is the *Killing equation*. Any $K$ obeying this is called a *Killing vector*. These describe the symmetries of the metric.

Killing vector fields generate isometries when flowing along their integral curves. Infinitesimally, $K$ takes a point with coordinates $x$ to the point $x'$ with

$$x'^\mu = x^\mu + \epsilon K^\mu. \tag{3.72}$$

> **Remark:** $\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X = \mathcal{L}_{[X,Y]}$. This means that there is a Lie algebra structure emerging from continuous symmetries of metrics.

**Example 3.4.1** (Minkowski Space)**:** The Killing equation (3.71) becomes $\partial_\mu K_\nu + \partial_\nu K_\mu = 0$. The solutions to this are

$$K_\mu = c_\mu + \omega_{\mu\nu}x^\nu, \tag{3.73}$$

where $c_\mu$ are translations and $\omega_{\mu\nu} = -\omega_{\nu\mu}$ correspond to rotations or boosts. These are the components of the Killing vector. Similarly, we can define the Killing vectors themselves as

$$P_\mu = \frac{\partial}{\partial x^\mu} \quad \text{and} \quad M_{\mu\nu} = \eta_{\mu\rho}x^\rho\frac{\partial}{\partial x^\nu} - \eta_{\nu\rho}x^\rho\frac{\partial}{\partial x^\mu}. \tag{3.74}$$

Once we have two Killing vectors, we can compute their commutator. For the isometries of Minkowski space, these become

$$\begin{aligned}[][P_\mu, P_\nu] = 0 \qquad [M_{\mu\nu}, P_\sigma] = -\eta_{\mu\sigma}P_\nu + \eta_{\sigma\nu}P_\mu \\ [M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}\end{aligned} \tag{3.75}$$

These are the commutation relations of the Poincaré algebra.

**Example 3.4.2:** The isometries of dS and AdS are inherited from the $5d$ embedding.

**dS** has isometry group $SO(1,4)$

**AdS** has isometry group $SO(2,3)$

**Note:** both groups have dim(10), just like the Poincaré group of $\mathbb{M}^4$. Each of these three spaces is just as symmetric as the other two.

In $5d$, the Killing vectors are

$$M_{AB} = \eta_{AC}X^C \frac{\partial}{\partial X^B} - \eta_{BC}X^C \frac{\partial}{\partial X^A}, \tag{3.76}$$

where $\eta = (-+++)$ in dS and $\eta = (--++)$ in AdS. And $X^A$ with $A = 0, 1, 2, 3, 4$. The flows induced by $M_{AB}$ map the embedding hyperboloid to itself. This implies that these are isometries of (A)dS.

**Claim 24:** If the metric $g_{\mu\nu}(x)$ does not depend on some coordinate $y$, then $K = \frac{\partial}{\partial y}$ is a Killing vector.

*Proof.* This is easy to see when we look at what the Lie derivative does:

$$(\mathcal{L}_{\frac{\partial}{\partial y}} g)_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial y} = 0. \tag{3.77}$$

□

**Example 3.4.3** (de Sitter in static patch coordinates)**:** For the static patch, we expect $\frac{\partial}{\partial t}$ to be a Killing vector. We had $X^0 = \sqrt{R^2 - r^2}\sinh\left(\frac{t}{R}\right)$ and $X^4 = \sqrt{R^2 - r^2}\cosh\left(\frac{t}{R}\right)$. Look at $\frac{\partial}{\partial t} = \frac{\partial X^a}{\partial t}\frac{\partial}{\partial X^A} = \frac{1}{R}(X^4 \frac{\partial}{\partial X^0} + X^0 \frac{\partial}{\partial X^4})$.

**Comment:** Timelike Killing vectors, such that $g_{\mu\nu}K^\mu K^\nu < 0$, are used to define energy. Both $\mathbb{M}^4$ and AdS have such objects. And dS has such an object in the static patch, but *not* globally! Consider the Killing vector $K = X^4 \frac{\partial}{\partial X^0} + X^0 \frac{\partial}{\partial X^4}$. The first term increases the timelike direction $X^0$ when $X^4 > 0$ and decreases $X^0$ when $X^4 < 0$. The Killing vector is positive and timelike only in the static patch. Elsewhere, it is spacelike. Energy is a subtle concept in dS!

**Remark:** Timelike geodesics do not take us out of the static patch; the problems with energy only arise when we think more globally than what a single particle is doing.

### 3.4.1 Conserved Quantities

**Claim 25:** Consider a particle moving on a geodesic $x^\mu(\tau)$ in a spacetime with some Killing vector $K^\mu$. Then $Q = K_\mu \frac{dx^\mu}{d\tau}$ is conserved.

*Proof.* To see this,

$$\begin{aligned} \frac{dQ}{d\tau} &= \partial_\nu K_\mu \frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} + K_\mu \frac{d^2x^\mu}{d\tau^2} && (3.78)\\ &= \partial_\nu K_\mu \frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} - K_\mu \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} && (3.79)\\ &= \nabla_\nu K_\mu \frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} \overset{(3.71)}{=} 0. && (3.80)\end{aligned}$$

□

We can also rederive the Killing equation from the action

$$S = \int d\tau\, g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu, \tag{3.81}$$

with the symmetry transformation $\delta x^\mu(\tau) = K^\mu(x)$. We have

$$\delta S = \int d\tau \left[\partial_\rho g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}K^\rho + 2g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dK^\nu}{d\tau}\right]. \tag{3.82}$$

Using

$$\begin{aligned} g_{\mu\nu}\frac{dK^\nu}{d\tau} &= \frac{dK_\mu}{d\tau} - \frac{dg_{\mu\nu}}{d\tau}K^\rho && (3.83)\\ &= (\partial_\nu K_\mu - \partial_\nu g_{\mu\nu}K^\rho)\frac{dx^\nu}{d\tau} && (3.84)\end{aligned}$$

in the second argument, we have

$$\to \delta S = \int d\tau\, 2\nabla_\mu K_\nu \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}, \tag{3.85}$$

where the $\Gamma$ 's in $\nabla$ come from the $\partial g$ 's.

**Exercise 3.3:** Check this!

This means that we have

$$\delta S = 0 \quad \iff \quad \nabla_{(\mu}K_{\nu)} = 0. \tag{3.86}$$

In other words, the transformation $\delta x^\mu = K^\mu$ is a symmetry of the action if and only if $K^\mu$ obeys the Killing equation.

## 3.5 Asymptotics of Spacetime

Given a spacetime $\mathcal{M}$, with metric $g_{\mu\nu}(x)$, we consider the *conformal transformation*

$$\widetilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x), \tag{3.87}$$

where $\Omega$ is smooth, non-zero. $g_{\mu\nu}$ and $\widetilde{g}_{\mu\nu}$ describe different spacetimes, but they have the same causal structure since

$$g_{\mu\nu} X^\mu X^\nu = 0 \quad \Longleftrightarrow \quad \widetilde{g}_{\mu\nu} X^\mu X^\nu = 0. \tag{3.88}$$

So null / spacelike / timelike in $g_{\mu\nu}$ maps to null / spacelike / timelike in $\widetilde{g}_{\mu\nu}$.

## 3.6 Penrose Diagrams

The idea is to use conformal transformations to bring infinity a little closer.

> **Note:** We will try to draw a finite picture of the whole manifold $\mathcal{M}$ by stretching an squeezing our coordinates. But we still want the picture to accurately depict the causal structure of the spacetime $(\mathcal{M}, g)$.

### 3.6.1 Minkowski Space

Start with $\mathbb{R}^{1,1}$ and metric $ds^2 = -dt^2 + dx^2$. Introduce lightcone coordinates

$$u = t - x \qquad v = t + x \tag{3.89}$$

in which the metric is $ds^2 = -du dv$ with $u, v \in (-\infty, \infty)$. We now map this to a finite range

$$u = \tan \widetilde{u} \quad \text{and} \quad v = \tan \widetilde{v}, \tag{3.90}$$

with $\widetilde{u}, \widetilde{v} \in (-\frac{\pi}{2}, +\frac{\pi}{2})$.

$$ds^2 = -\frac{1}{\cos^2 \widetilde{u} \cos^2 \widetilde{v}} d\widetilde{u} d\widetilde{v} \tag{3.91}$$

Consider the new metric

$$d\widetilde{s}^2 = \cos^2 \widetilde{u} \cos^2 \widetilde{v} ds^2 = -d\widetilde{u} d\widetilde{v}. \tag{3.92}$$

> **Note:** The reason we use lightcone coordinates is the following: if we just proceed with $t = \tan \tilde{t}$ and $x = \tan \tilde{x}$, then we end up with $ds^2 = -\frac{1}{\cos^2 \tilde{t}} d\tilde{t} + \frac{1}{\cos^2 \tilde{x}} d\tilde{x}$. And trying to pull out the time prefactor, we get an ugly factor $\frac{cos^2 \tilde{t}}{cos^2 \tilde{x}}$ in front of $d\tilde{x}$.

There is a bit of a technicality there. Strictly speaking the points at the edge of the spacetime were not included in the first place, whereas $\tilde{u}, \tilde{v} \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$. Adding the points $\pm\frac{\pi}{2}$, which used to be $\pm\infty$, is called a *conformal compactification*.

> **Remark:** There is a theorem by Penrose that shows that this is essentially the unique way of doing conformal complexification.

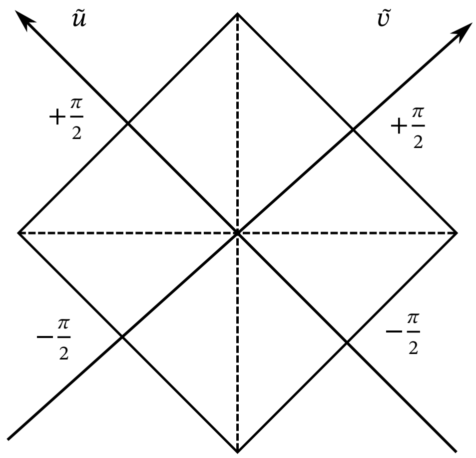


Figure 3.4

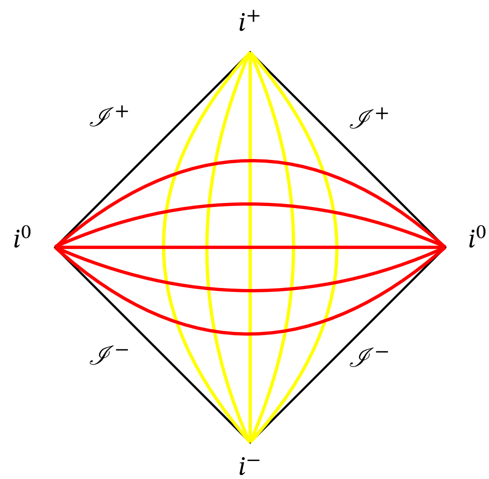


Figure 3.5

Figure 3.6: Penrose Diagrams

Now we draw the spacetime with light rays at 45° and time vertical. These diagrams, as depicted in Figure 3.4 are called *Penrose diagrams*. Note that we cannot trust distances in these diagrams; things that look close might be very far apart in Minkowski space. However, we can trust the causal structure. We can draw various geodesics on this diagram. In particular, we can draw timelike geodesics (constant x) and spacelike geodesics (constant t) as in Figure 3.5.

**Note:** All timelike geodesics start at $i^- : [-\frac{\pi}{2}, -\frac{\pi}{2}]$ and end at $i^+ : [+\frac{\pi}{2}, +\frac{\pi}{2}]$. These are called *past / future timelike infinity*.

- All spacelike geodesics start / end at two points $i^0 : [-\frac{\pi}{2}, +\frac{\pi}{2}]$ or $[+\frac{\pi}{2}, -\frac{\pi}{2}]$. These are *spacelike infinity*.
- All null curves start at $\mathscr{I}^-$ 'scri-minus' and end at $\mathscr{I}^+$ 'scri-plus'.[1] These are *past / future null infinity*.

[1]The strange names may be related to the fact that the symbol $\mathscr{I}$ is produced by the command `\mathscr{I}` in LaTeX.

**Note:** If these names do not make sense, consider that we write $\mathscr{I}$ as `\mathscr{I}` in LaTeX.

**Remark:** In some sense, most of infinity is given by the diagonal lines $\mathscr{I}$.

The Penrose diagram immediately tells us basic things about the spacetime. For example, and two points in the spacetime have a common future and a common past. For any two points, draw 45° lines, as illustrated in Figure 3.7.

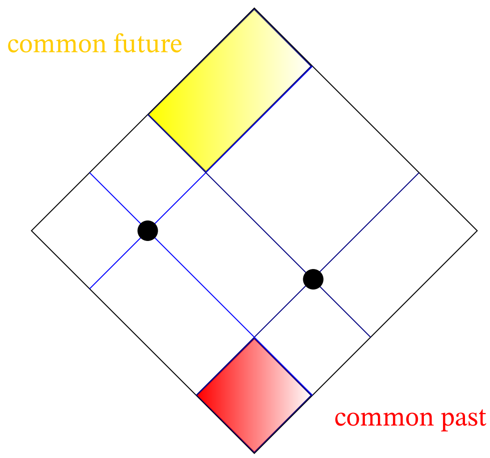


Figure 3.7: Any two points, even spacelike, share a common future and common past—spacetime regions in which their lightcones overlap.

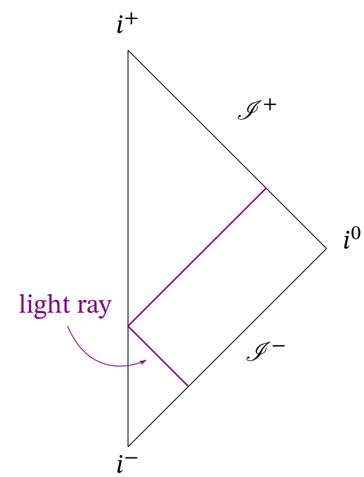


Figure 3.8: Penrose diagram for $\mathbb{R}^{1,3}$ in polar coordinates. Since $r \geq 0$, we only draw the half where $\tilde{v} \geq \tilde{u}$.

Since we can trust the causal structure of Penrose diagrams, this illustrates clearly that any two points, even if they are spacelike separated, they have a common future and common past since their lightcones overlap. If they follow timelike geodesics, they will eventually end up in their common future.

For $\mathbb{R}^{1,3}$, we do something similar. The metric is best written in polar coordinates

$$ds^2 = -dt^2 + dt^2 + r^2 d\Omega_2^2, \tag{3.93}$$

where $d\Omega_2^2$ is the metric on the two-sphere $S^2$. Let $u = t - r = \tan\widetilde{u}$ and $v = t + r = \tan\widetilde{v}$. Then

$$\begin{aligned} ds^2 &= -du dv + \frac{1}{4}(u-v)^2 d\Omega_2^2 && (3.94)\\ &= \frac{1}{4\cos^2\widetilde{u}\cos^2\widetilde{v}}\left(-4d\widetilde{u}d\widetilde{v} + \sin^2(\widetilde{u}-\widetilde{v})d\Omega_2^2\right). && (3.95)\end{aligned}$$

**Note:** Recall the identity $\sin(\tilde{v})\cos(\tilde{u}) = \frac{1}{2}(\sin(\tilde{v}+\tilde{u}) + \sin(\tilde{v}-\tilde{u}))$ to show that

$$\tan(\tilde{v}) - \tan(\tilde{u}) = \frac{\sin(\tilde{v}-\tilde{u})}{\cos(\tilde{v})\cos(\tilde{u})}. \tag{3.96}$$

There is one other subtlety, which stems from the coordinates. Unlike in the $2d$ case, the radial coordinate $r$ has to be non-negative $r \geq 0$. This means that $v \geq u$ and therefore

$$-\frac{\pi}{2} \leq \tilde{u} \leq \tilde{v} \leq \frac{\pi}{2}. \tag{3.97}$$

We drop the $S^2$ and draw the Penrose diagram, which is now only the half for which $\tilde{v} \geq \tilde{u}$. This is shown in 3.8. The 'boundary' on the left is *not* a boundary of spacetime! It is $\tilde{u} = \tilde{v}$, so $r = 0$, and the two-sphere $S^2$ shrinks to zero size. In particular, notice the behaviour of a light ray.

### 3.6.2 de Sitter

We are going to work in global coordinates, since they cover the whole of spacetime:

$$ds^2 = -d\tau^2 + R^2\cosh^2\left(\frac{\tau}{R}\right)d\Omega_3^2. \tag{3.98}$$

We now introduce what cosmologists call *conformal time* $\eta \in (-\frac{\pi}{2}, +\frac{\pi}{2})$.

$$\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = \frac{1}{R\cosh(\tau/R)} \quad \Rightarrow \quad \cos\eta = \frac{1}{\cosh(\tau/R)}. \tag{3.99}$$

The purpose of conformal time is to pull out the factor of $R^2\cosh^2$ to the front of the metric. Plugging this into (3.98), we have

$$ds^2 = \frac{R^2}{\cos^2\eta}(-d\eta^2 + d\Omega_3^2). \tag{3.100}$$

Here, the metric on the three-sphere is $d\Omega_3^2 = d\chi^2 + \sin^2\chi d\Omega_2^2$, where $\chi \in [0,\pi]$. This means that de Sitter is conformal to

$$ds^2 = -d\eta^2 + d\chi^2 + \sin^2\chi d\Omega_2^2. \tag{3.101}$$

The difference is that these are not lightcone coordinates. This means that the Penrose diagram for dS is just a square, as in Figure 3.9. The vertical lines are not boundaries of the spacetime. They correspond to the north pole of $S^3$ and south pole of $S^3$ respectively. The horizontal lines do correspond the boundaries. We see that the boundary of dS is spacelike. There is no spatial infinty in dS; the spatial regions are compact manifolds without asymptotic region. Rotating by 45° has changed the physics dramatically. None of the above statements about $\mathbb{M}$ are true. In particular, no matter how long you wait, you cannot see the whole space; nor can you influence the whole space.

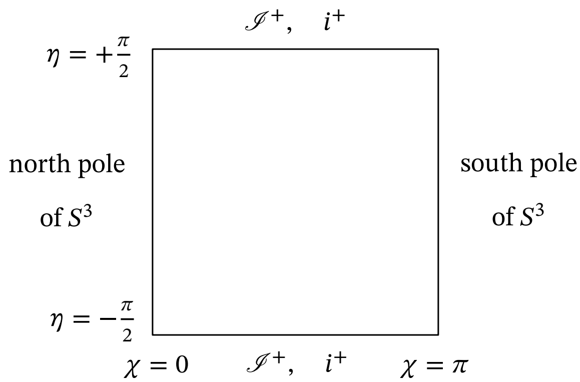


Figure 3.9

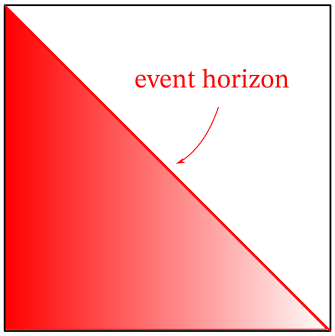


Figure 3.10

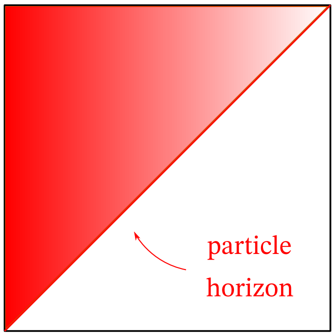


Figure 3.11

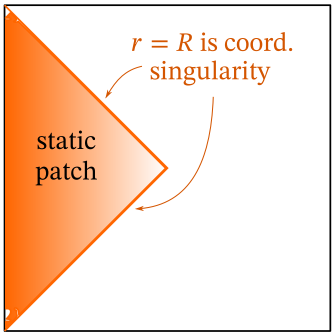


Figure 3.12

Suppose we have an observer sitting at the north pole, as illustrated in Fig. 3.10. There will be regions of dS, which she will never see; the boundary between what she can and cannot see is called the *event horizon*. Unlike in the case of black holes, every observer has a different event horizon. Similarly, if she sits on the south pole, we have the case of Figure 3.11. This in cosmology is called the *particle horizon*.

**Exercise 3.4:** Check that the static patch coordinates map to the intersection of the event and particle horizons, as depicted in Fig. 3.12.

In some sense, the static patch are the natural coordinates, since they are the region, which an observer can see and influence. However, they come with a coordinate singularity at $r = R$.

### 3.6.3 Anti-de Sitter Space

The metric in AdS is

$$ds^2 = -\cosh^2\rho\, dt^2 + R^2 d\rho^2 + R^2 \sinh^2\rho\, d\Omega_2^2, \tag{3.102}$$

where $\rho \in [0, \infty)$ is the radial coordinate. Compared to dS, the space and time components are swapped.

Let us introduce a new radial coordinate as $\frac{\mathrm{d}\psi}{\mathrm{d}\rho} = \frac{1}{\cosh\rho}$, giving $\cos\psi = \frac{1}{\cosh\rho}$. The metric becomes

$$ds^2 = \frac{1}{\cos^2\psi}\left(-d\tilde{t}^2 + d\psi^2 + \sin^2\psi d\Omega_2^2\right), \tag{3.103}$$

with $\tilde{t} = \frac{t}{R}$. AdS is conformal to a new metric

$$d\tilde{s}^2 = -d\tilde{t}^2 + d\psi^2 + \sin^2\psi d\Omega_2^2, \tag{3.104}$$

where $\tilde{t} \in (-\infty, +\infty)$ and $\psi \in [0, \frac{\pi}{2}]$. Again, we sketch the Penrose diagram in the usual way by ignoring the two-sphere.

Note that a light ray as depicted in Fig. 3.13 hits the boundary in finite coordinate time $t$.

**Remark:** There is also something strange in AdS: We cannot specify initial conditions on some spacelike surface $\Sigma$ and watch it evolve; at least not without knowledge of the boundary.

However, there are natural boundary conditions, such as saying that there will not be a random light ray coming out of the boundary at some point. In that case, we can evolve things in time!

**Remark:** It is only in AdS that we have this problem. For the cases of $\mathbb{M}$ and dS, see 3.14.

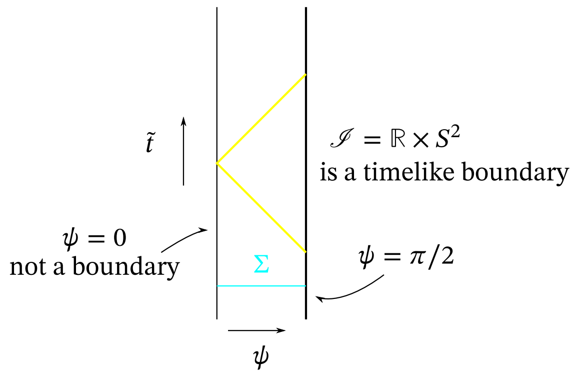


Figure 3.13

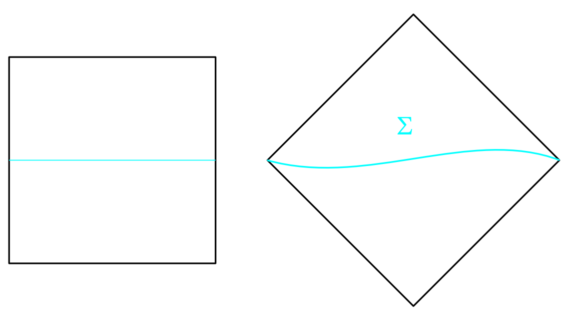


Figure 3.14: Spacelike surfaces of initial conditions in $\mathtt{dS}$ and $\mathbb{M}$.

## 3.7 Coupling Matter

In Minkowski space $\mathbb{M}$, the action for a scalar field is

$$S_{\text{scalar}} = \int \mathrm{d}^4x \left(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right). \tag{3.105}$$

In curved spacetime, we generalise this to

$$S_{\text{Scalar}} = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right) \tag{3.106}$$

**Remark:** $\nabla_\mu\phi = \partial_\mu\phi$ for a scalar. However, it is good practice to switch to covariant derivatives so that we can integrate by parts with the divergence theorem 3.

This is not quite the unique choice. We can actually add some more terms, which vanish when we go to flat space. In particular,

$$\int \mathrm{d}^4x\sqrt{-g}\frac{1}{2}\xi R\phi^2, \tag{3.107}$$

where $\xi$ is just a dimensionless constant. Varying $\phi$ gives equations of motion

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{\partial V}{\partial\phi} - \xi R\phi = 0. \tag{3.108}$$

### 3.7.1 Maxwell Theory

There are other obvious fields that we can generalise to curved space. Using the language of differential forms, where $F = dA$ is a two-form, the Maxwell action is

$$S_{\text{Maxwell}} = -\frac{1}{4}\int F\wedge\star F \tag{3.109}$$

$$= -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}, \tag{3.110}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. The last equality is true because the Christoffel symbols cancel due to the anti-symmetrisation. This is why we could differentiate forms before we had a connection; the difference cancels out.

The equation of motion—exactly the same story as in flat space, except with covariant derivatives—turn out to be

$$\nabla_\mu F^{\mu\nu} = 0. \tag{3.111}$$

This is the generalisation of the usual field theories to curved space. You can either view this as the field changing or the metric changing. The action for gravity and matter is then the sum of the Einstein-Hilbert action (3.24) with the matter action $S_m$:

$$S = \frac{1}{2}M_{pl}^2\int \mathrm{d}^4x\sqrt{-g}(R-2\Lambda) + S_m. \tag{3.112}$$

**Definition 47:** The *energy-momentum tensor*

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{3.113}$$

**Note:** Suppose we had the Lagrangian density $\mathcal{L}(x)$ differentiated by $g^{\mu\nu}(x)$. This gives a delta function, which is killed by the action $S_m$, which is why we have to vary $S_m$ and not $\mathcal{L}$.

**Remark:** There is a very slick argument: the usual $T^{\mu\nu}$ you get in QFT is a Noether current. The slick way to calculate the current (in notes) is to pretend that the parameter depends on spacetime.

In other words, take some symmetry for which $\delta S = 0$. Now take some symmetry which depends on space. The change of the action has to be of the form

$$\delta S = \int \mathrm{d}^4 x\, J^\mu \partial_\mu \epsilon(x), \tag{3.114}$$

where $\epsilon(x)$ is the symmetry parameter. Integrate by parts to get

$$\cdots = -\int \mathrm{d}^4 x\, \partial_\mu J^\mu \epsilon(x). \tag{3.115}$$

This means that $\partial_\mu J^\mu$ must vanish for constant $\epsilon$. The details are in the printed notes.

Varying the metric gives the *Einstein equation*

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{3.116}$$

Recall that if we vary the metric by a diffeomorphism, the metric changes as

$$\delta g_{\mu\nu} = (\mathcal{L}_X g)_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu. \tag{3.117}$$

The change in the matter action is

$$\delta S_m = -2\int \mathrm{d}^4 x\, \sqrt{-g} T_{\mu\nu} \nabla^\mu X^\nu. \tag{3.118}$$

Requiring $\delta S_m = 0$ by diffeomorphism invariance, it must be that $\nabla_\mu T^{\mu\nu} = 0$. In other words, the energy-momentum tensor is *covariantly conserved*.

**Remark:** In flat space, this becomes genuine conservation. In curved space, this conservation law is slightly more subtle.

**Example 3.7.1:** Let $S = \int \mathrm{d}^4 x\, \sqrt{-g}\left(\frac{1}{2}\nabla_\mu \phi \nabla^\mu \phi - V(\phi)\right)$. We want to vary this with respect to $g_{\mu\nu}$. There are two ones, one in the $\sqrt{-g}$ and one hiding in the contraction $\nabla_\mu \phi \nabla^\mu \phi$. We obtain

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu - g_{\mu\nu}(\frac{1}{2}\nabla_\rho \phi \nabla^\rho \phi + V(\phi)). \tag{3.119}$$

**Example 3.7.2:** For Maxwell theory, $S = -\frac{1}{4}\int \mathrm{d}^4 x\, \sqrt{-g} F_{\mu\nu} F^{\mu\nu}$. This gives

$$T_{\mu\nu} = g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}. \tag{3.120}$$

### 3.7.2 Fluids

If you really care about situations in which matter is backreacting on matter, the right description is in terms of a fluid, where lots of particles interact with each other. Often in GR, we model matter as a perfect fluid. These have a velocity field $u_\mu(\mathbf{x}, t)$, with $u_\mu u^\mu = -1$, which tells us that the particles are travelling on timelike geodesics. Then the stress-energy tensor is in its most general form:

$$T_{\mu\nu} = (\text{something})u_\mu u_\nu + (\text{something else})g_{\mu\nu}. \tag{3.121}$$

It turns out that we have

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \tag{3.122}$$

where $P(\mathbf{x}, t)$ is the *pressure* and $\rho(\mathbf{x}, t)$ the *energy density*. Usually, there is some relation between these two, called the *equation of state* $P = P(\rho)$.

**Remark:** This is the description of a fluid with microscopic interactions and pressure. In particular, there is no gravity yet. We could pick flat space for example. If you want to find out how they interact using gravity, you have to solve the Einstein equations.

For a fluid at rest, $u^\mu = (1, 0, 0, 0)$, and a flat metric $g_{\mu\nu} = \eta_{\mu\nu}$, then

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix}. \tag{3.123}$$

All stress tensors have to obey the conservation law.

**Exercise 3.5:** Apply the conservation law $\nabla_\mu T^{\mu\nu} = 0$ to this stress tensor. You get two equations out: mass-energy conservation and the Euler equations of fluid dynamics.

## 3.8 Energy Conservation

In flat space, currents and the energy-momentum tensor obey

$$\partial_\mu J^\mu = 0 \quad \text{and} \quad \partial_\mu T^{\mu\nu} = 0. \tag{3.124}$$

**Claim 26:** We can define conserved charges

$$Q(\Sigma) = \int_\Sigma \mathrm{d}^3x\, J^0 \quad \text{and} \quad P^\mu(\Sigma) = \int_\Sigma \mathrm{d}^3x\, T^{0\mu} \tag{3.125}$$

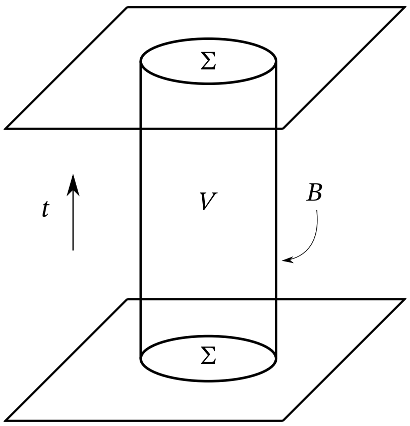


Figure 3.15: Flat space

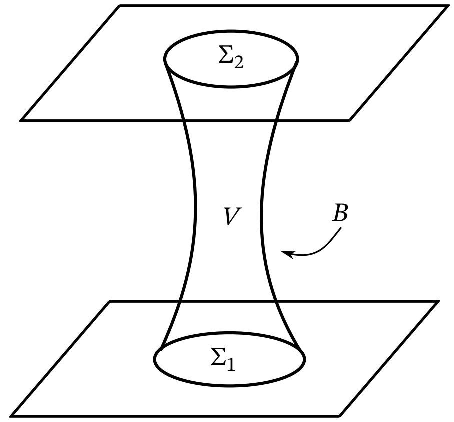


Figure 3.16: Curved space

*Proof.* There are two terms from the boundaries.

$$0 = \int_V \mathrm{d}^4x\, \partial_\mu J^\mu \tag{3.126}$$

$$= \Delta Q(\Sigma) + \int_B \mathrm{d}^3x\, n^i J_i, \tag{3.127}$$

where $n^i$ is normal to $B$.

If there is no current leaking out of $B$, meaning $n_i J^i = 0$ on $B$, then $\Delta Q(\Sigma) = 0$. Exactly the same arguments hold for $P^\mu(\Sigma)$. □

In curved spacetime, we have

$$\nabla_\mu J^\mu = 0 \quad \text{and} \quad \nabla_\mu T^{\mu\nu} = 0. \tag{3.128}$$

For $J^\mu$, we have a similar story

$$0 = \int_V \mathrm{d}^4x\, \sqrt{-g}\nabla_\mu J^\mu \tag{3.129}$$

$$= \int_{\partial V} \mathrm{d}^3x\, \sqrt{|\gamma|} n_\mu J^\mu, \tag{3.130}$$

where $\partial V = \Sigma_1 \cup \Sigma_2 \cup B$.

If $n_\mu J^\mu = 0$ on $B$, no matter which cylinder we choose in 3.16, then we have charge conservation

$$Q(\Sigma_1) = Q(\Sigma_2), \tag{3.131}$$

where $Q(\Sigma) = \int_\Sigma \mathrm{d}^3x\, \sqrt{|\gamma|} n_\mu J^\mu$ . So charge conservation in curved space is just like in flat space!

This same argument does not work for the stress-energy tensor $T^{\mu\nu}$! If we try to repeat the same argument, we now have

$$0 = \int_V \mathrm{d}^4x\, \sqrt{-g}\nabla_\mu T^{\mu\nu}. \tag{3.132}$$

There is a hanging index $\mu$. And there is no divergence theorem for objects of this type.

Instead, redoing the same derivation, we find that

$$\sqrt{-g}\nabla_\mu T^{\mu\nu} = 0 \quad \iff \quad \partial_\mu(\sqrt{-g}T^{\mu\nu}) = -\sqrt{-g}\Gamma^{\nu}_{\mu\rho}T^{\mu\rho}. \tag{3.133}$$

We have seen this before; this term looks like a driving force, which roughly speaking means that $T^{0\mu}$ is *not* conserved. Note that covariant conservation is not enough to give actual conservation.

Intuitively, the fields in a generic spacetime, with energy $T^{\mu\rho}$ slosh around. Spacetime reacts to this, emitting gravitational waves, which takes away energy from the energy in the fields. Roughly speaking, the extra term can be viewed as energy of the fields seeping into the gravitational field of spacetime itself.

There is one situation where we can make progress: Suppose we have a spacetime that has a Killing vector $K$. We can then define the following current, built from the stress-energy tensor

$$J_T^\mu := K_\nu T^{\mu\nu}. \tag{3.134}$$

This current is then conserved, since

$$\nabla_\mu J_T^\mu = (\nabla_\mu K_\nu)T^{\mu\nu} + K_\nu \nabla_\mu T^{\mu\nu} = 0. \tag{3.135}$$

The first term vanishes by the Killing equation (imprinting the symmetry of $T^{\mu\nu}$ onto $\nabla_\mu K_\nu$) and the second term is zero due to covariant conservation of $T^{\mu\nu}$ .

We can then define the conserved charge

$$Q_T(\Sigma) = \int_\Sigma \mathrm{d}^3x\, \sqrt{|\gamma|}n_\mu J_T^\mu. \tag{3.136}$$

If the Killing vector is timelike everywhere, $g_{\mu\nu}K^\mu K^\nu < 0$ , this can be interpreted as energy.

What if there is no Killing vector? Is there some generalised energy? Can we make sense of the total energy, both in matter and in the gravitational field? The short answer is: no, we cannot.

**Example 3.8.1:** Take two black holes or neutron stars. When they orbit, they will emit gravitational waves and lose energy. The orbit radii decrease in time until they hit each other. This is an example of a case in which no Killing vector exists. We should intuitively be able to define the total energy. It turns out, as we will see later, that in that we case we cannot.

There is a longer answer however. Well actually, it will be the short version of the long answer:

**Claim 27:** There is no diffeomorphism-invariant local energy-density for the gravitational field (and therefore for the gravitational waves).

In some sense we should just stop here. If it is not diffeomorphism-invariant, it is not physical. However, there are some things we can do. For example, people write the RHS of (3.133) as the Landau-Lifshitz pseudo tensor[1]—the derivative of something.

There is one case where precise statement can be made: if you are in $\mathbb{M}^n$, it makes sense to ask about the energy flux in the asymptotic part of spacetime $\mathscr{I}^+$. This is not a *local* part of spacetime, and Claim 27 does not apply. This is called the *Bondi-energy*.

This is one of the differences between general relativity and other field theories.

> **Note:** The absence of a conserved quantity called energy does not change whether or not the Einstein equations have solutions; it just means that we will have to work much harder than before to find these solutions.

[1]See Landau & Lifshitz Volume 2 on Classical Fields

# 4 When Gravity is Weak

We will solve the Einstein equations perturbatively, working in 'almost inertial coordinates'.

**Definition 48:** *Almost inertial coordinates* hold for spacetimes that are 'close' to Minkowski space in the sense that the metric can be written

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{4.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu} \ll 1$.

> **Note:** Note that the units of $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ are contained in $dx^\mu$ and $g_{\mu\nu}$ are just dimensionless numbers.

We think of gravity as a 'spin-2' field $h_{\mu\nu}$ propagating in Minkowski space. To that end, we will raise and lower indices using the Minkowski metric $\eta_{\mu\nu}$, rather than $g_{\mu\nu}$.

**Example 4.0.1:** Raising the indices of the perturbation $h_{\mu\nu}$ gives

$$h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}. \tag{4.2}$$

## 4.1 Linearised Theory

We want to build the theory of the linear ripples $h_{\mu\nu}$. Since the components $h_{\mu\nu} \ll 1$ of the perturbation are small, we work to leading order in $h_{\mu\nu}$. Then the inverse of the metric $g_{\mu\nu}$ is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{4.3}$$

where again we raise indices with the Minkowski metric. The Christoffel symbols are

$$\Gamma^\sigma_{\nu\rho} = \frac{1}{2}\eta^{\sigma\lambda}(\partial_\nu h_{\lambda\rho} + \partial_\rho h_{\nu\lambda} - \partial_\lambda h_{\nu\rho}) \tag{4.4}$$

and the Riemann tensor is

$$R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\nu} + \underbrace{\Gamma^\lambda_{\nu\rho}\Gamma^\sigma_{\mu\lambda} - \Gamma^\lambda_{\mu\rho}\Gamma^\sigma_{\nu\lambda}}_{O(h^2)}. \tag{4.5}$$

$$= \frac{1}{2}\eta^{\sigma\lambda}(\partial_\mu\partial_\rho h_{\nu\lambda} - \partial_\mu\partial_\lambda h_{\nu\rho} - \partial_\nu\partial_\rho h_{\mu\lambda} + \partial_\nu\partial_\lambda h_{\mu\rho}). \tag{4.6}$$

Working in the linearised theory is very easy; all the hard stuff disappears. The Ricci tensor is

$$R_{\mu\nu} = \frac{1}{2}(\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h), \tag{4.7}$$

and finally the Ricci scalar is

$$R = \partial^\mu\partial^\nu h_{\mu\nu} - \Box h, \tag{4.8}$$

where $\Box = \partial_\mu\partial^\mu$ is the d'Alambertian and $h = h^\mu{}_\mu = \eta^{\mu\nu}h_{\mu\nu}$ is the trace of the perturbation.

Finally, the Einstein tensor

$$G_{\mu\nu} = \frac{1}{2}\left[\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - (\partial^\rho\partial^\sigma h_{\rho\sigma} - \Box h)h_{\mu\nu}\right]. \tag{4.9}$$

This obeys the linearised Bianchi identity $\partial_\mu G^{\mu\nu} = 0$, which is the equation of motion in the absence of matter.

In this case, we did not start with the action. However, there also exists an action that encodes this equation of motion.

### 4.1.1 The Fierz-Pauli Action

The Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ follows from the following action by Fierz and Pauli, written down in the 1930s,

$$S_{\mathrm{FP}} = \int \mathrm{d}^4x\, \frac{1}{8\pi G}\left[-\frac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\nu h^{\rho\mu} + \frac{1}{4}\partial_\mu h\partial^\mu h - \frac{1}{2}\partial_\nu h^{\mu\nu}\partial_\mu h\right] + h_{\mu\nu}T^{\mu\nu} \tag{4.10}$$

The claim is twofold: Firstly, varying this action with respect to the perturbation $h_{\mu\nu}$ yields the Einstein equation. Secondly — and this is the hard part of the claim — the Fierz–Pauli action $S_{\mathrm{FP}}$ arises from the Einstein-Hilbert action $S_{\mathrm{EH}}$ by expanding to second order in the perturbation.

In particular, we have to include a lot of the second order terms (for example in the Riemann tensor).

We should think about this in very much the same way as we think of the Maxwell action; expanding out $S_{\mathrm{MW}}$ in terms of $A$ rather than $F$, and also having a term that couples to the current.

This action has a particular symmetry.

### 4.1.2 Gauge Symmetry

Under an infinitesimal diffeomorphism $x^\mu \longmapsto x^\mu - \xi^\mu(x)$, the metric changes as

$$\delta g_{\mu\nu} = (\mathcal{L}_\xi g)_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{4.11}$$

Assuming that $\xi^\mu$ are 'small', in the same sense that $h$ is small, we view this as a gauge transformation of $g^{\mu\nu}$ which, to leading order, is

$$h_{\mu\nu} \longmapsto h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \tag{4.12}$$

In particular, the covariant derivatives of the Christoffel symbols, which depend on $h$, are dropped since they are of order $O(h\xi)$.

**Exercise 4.1:** You can check that the Riemann tensor ${R^\mu}_{\rho\nu\sigma}$ given by (4.7) and therefore also $S_{\text{FP}}$ are both invariant under gauge transformations.

> **Note:** We assume that the derivatives $\partial\xi$ are also small.

**Remark:** There is a very nice story here, dating back to Feynman and Weinberg that goes beyond this: let us think about which terms we can act to this action that respect this gauge symmetry. You cannot have a mass term; you need to have derivatives of $h$. Moreover, any term you add forces you to add more terms. In the end, you will end up adding up infinitely many terms, which reconstitute the expansion of the Eintein-Hilbert action. The claim is that there is essentially a unique way to do this: the Eintein-Hilbert action.

This is the statement that a massless spin-2 particle needs gauge-invariance to make sense of the quantum theory and if you want interactions, you are obliged to come out with general relativity.

We use gauge symmetry to choose a gauge; we pick *de Donder* `(dD)` *gauge*, given by

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0. \tag{4.13}$$

This is analogous to Lorentz gauge $\partial_\mu A^\mu = 0$ in electromagnetism.

**Remark:** In the full non-linear theory, the generalisation of `dD` is something quite elegant:

$$g^{\mu\nu}\Gamma^\rho_{\mu\nu} = 0. \tag{4.14}$$

These are both 4 equations. Notice that it not a tensor equation! But this is exactly what you want for gauge fixing.

In dD gauge, the linearised Einstein equations become

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -16\pi G T_{\mu\nu} \tag{4.15}$$

**Definition 49:** Define $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$

We then have $\overline{h} = \eta_{\mu\nu}\overline{h}^{\mu\nu} = -h$ and so $h_{\mu\nu} = \overline{h}_{\mu\nu} - \frac{1}{2}\overline{h}\eta_{\mu\nu}$. And

$$\Box\overline{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{4.16}$$

### 4.1.3 The Newtonian Limit

Our aim will be to rederive Newtonian gravity in this framework. Consider stationary matter with $T_{00} = \rho(\mathbf{x})$. Then $\Box = -\partial_t^2 + \nabla^2$ and we look for solutions with $\partial/\partial t = 0$.

$$\nabla^2\overline{h}_{00} = -16\pi G\rho(\mathbf{x}) \quad \text{and} \quad \nabla^2\overline{h}_{0i} = \nabla^2\overline{h}_{ij} = 0. \tag{4.17}$$

These have the solution $\overline{h}_{0i} = \overline{h}_{ij} = 0$ and $\overline{h}_{00} = -4\Phi(\mathbf{x})$, which $\Phi$ the Newtonian gravitational potential, defined by solvint the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho(\mathbf{x}). \tag{4.18}$$

**Remark:** We are hiding the speed of light $c = 1$. By the time we put the speed of light in, the energy density $\rho$ becomes the mass density.

**Remark:** This looks so similar to electromagnetism, since it is in some sense the only thing we could get. There are quantum mechanical arguments, but classically, gauge symmetry allows you to have a well-defined intial value problem.

This then gives $h_{00} = -2\Phi$, $h_{ij} = -2\Phi\delta_{ij}$ , $h_{0i} = 0$ . The metric then becomes

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)d\mathbf{x}^2. \tag{4.19}$$

**Example 4.1.1:** Suppose that the mass-density was a point mass $m$ sitting at the origin. We have $\Phi = -GM/r$ as the solutions to the Poisson equation. The metric then agrees with the Taylor equation of the Schwarzschild metric.

**Remark:** You can argue in intuitive terms that the factor of two in $-(1+2\Phi)dt^2$ has to be there. These arguments do not tell you about the factor in $d\mathbf{x}^2$; they follow from Einstein's equations. In the Newtonian limit, you get a lightbending answer that is off by a factor of two from the results of GR. The reason that this factor of two is the additional factor $(1-2\Phi)$ in front of $d\mathbf{x}^2$.

This fact is also how Einstein found the factor of two before the full Schwarzschild metic was found; you don't need the full Schwarzschild metric to describe lightbending.

## 4.2 Gravitational Waves

You may have heard tangentially: gravitational waves were discovered a couple of years ago. They are promising to revolutionise physics, so it might be a good idea to see how they are derived.

There are wave solutions in GR obeying

$$\Box \bar{h}_{\mu\nu} = 0. \tag{4.20}$$

The solution is

$$\bar{h}_{\mu\nu} = \mathrm{Re}(H_{\mu\nu} e^{ik_\rho x^\rho}), \tag{4.21}$$

where $H_{\mu\nu}$ is a complex, symmetric matrix that tells us about the polarisation of the gravitational waves. This solves the wave equation providing $k_\mu k^\mu = 0$. In other words, the wavevector of these waves is null, which means that gravitational waves travel at the speed of light.

To derive this equation, we had to be in `dD` gauge. We have to make sure that this solution agrees with that gauge choice.

**Claim 28:** The ansatz (4.21) satisfies the de Donder gauge condition $\partial^\mu \bar{h}_{\mu\nu} = 0$ provided that $k^\mu H_{\mu\nu} = 0$.

**Remark:** Again, analogously to electromagnetism, this means that the polarisation is transverse to the direction of propagation.

We can make further gauge transformations that leave us in `dD` gauge

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{4.22}$$

$$\Rightarrow \quad \bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \partial^\rho \xi_\rho \eta_{\mu\nu}. \tag{4.23}$$

This leaves us in *dD* gauge, provided that $\Box \xi_\mu = 0$.

**Remark:** This means that not all of these polarisations are physical.

We can take, say, $\xi_\mu = \lambda_\mu e^{ik_\rho x^\rho}$. This shifts the polarisation vector

$$H_{\mu\nu} \to H_{\mu\nu} + i(k_\mu \lambda_\nu + k_\nu \lambda_\mu - k^\rho \lambda_\rho \eta_{\mu\nu}). \tag{4.24}$$

**Claim 29:** We can choose $\lambda_\mu$ such that $H_{0\mu} = 0$ and also $H^\mu{}_\mu = 0$. This is the *transverse-traceless (TT) gauge.*

*Proof.* Problem sheet. □

This gauge has the advantage that the trace of $\bar{h}_{\mu\nu}$ vanishes, which means that $\bar{h}_{\mu\nu} = h_{\mu\nu}$.

The number of polarisations is $10 - 4 - 4 = 2$, where the last four are from the residual gauge transformations. Fortuitously, this is exactly the same number of polarisations of the photon. This is special about 4 dimensions and does not hold in arbitrary dimensions.

**Example 4.2.1:** Take a wave moving in the $z$ -direction with null wavevector

$$k^\mu = (\omega, 0, 0, \omega). \tag{4.25}$$

We can interpret $\omega$ as the frequency or wavenumber. The dD gauge tells us that the polarisation is transverse to the direction, $k^\mu H_{\mu\nu} = 0$ , which means that the polarisation has to satisfy $H_{0\nu} + H_{3\nu} = 0$ . Then, in the TT gauge, the polarisation has just two components:

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.26}$$

How do we make gravitational waves in the first place? If we have a gravitational wave, how can we tell? What effect does it have on matter? We will address the first question in a later section, but for now we will deal with the latter ones.

### 4.2.1 How to Measure a Gravitational Wave

Consider a family of geodesics $x^\mu(\tau; s)$, where $\tau$ is an affine parameter for the geodesic and $s$ labels different geodesics. In other words, we have a 4-velocity $u^\mu = \left.\frac{\partial x^\mu}{\partial \tau}\right|_s$ for each geodesic and displacement $S^\mu = \left.\frac{\partial x^\mu}{\partial s}\right|_\tau$ .

Take particles in flat space $u^\mu = (1, 0, 0, 0)$. We use the geodesic deviation equation (2.116)

$$\frac{\mathrm{d}^2 S^\mu}{\mathrm{d}\tau^2} = R^\mu{}_{\rho\sigma\nu} u^\rho u^\sigma S^\nu \quad \Rightarrow \quad \frac{\mathrm{d}^2 S^\mu}{\mathrm{d}t^2} = R^\mu{}_{00\nu} S^\nu, \tag{4.27}$$

Where we can replace $\tau$ with $t$ up to $O(h)$ . Using the previous result for linearised Riemann, we get

$$\frac{\mathrm{d}^2 S^\mu}{\mathrm{d}t^2} = \frac{1}{2} \frac{\partial^2 h^\mu{}_\nu}{\partial t^2} S^\nu. \tag{4.28}$$

So for a wave in the $z$ -direction, we have

$$\frac{\mathrm{d}^2 S^0}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 S^3}{\mathrm{d}t^2} = 0. \tag{4.29}$$

So all the action happens in the $(x - y)$-plane, which is transverse to the direction of motion. To make the equations a bit simpler, we will focus on the particular plane $z = 0$.

We have two polarisations to look at.

$H_+$ **polarisation** We set $H_\times = 0$. This means that we have

$$\frac{\mathrm{d}^2 S^1}{\mathrm{d}t^2} = -\frac{\omega^2}{2} H_+ e^{i\omega t} S^1 \tag{4.30}$$

$$\frac{\mathrm{d}^2 S^2}{\mathrm{d}t^2} = +\frac{\omega^2}{2} H_+ e^{i\omega t} S^2 \tag{4.31}$$

The solutions to linear order in $h$ are

$$S^1(t) = S^1(0)\left[1 + \frac{1}{2} H_+ e^{i\omega t} + \ldots\right] \tag{4.32}$$

$$S^2(t) = S^2(0)\left[1 - \frac{1}{2} H_+ e^{i\omega t} + \ldots\right] \tag{4.33}$$

$$\tag{4.34}$$

**Remark:** The physical solution is the real part of this.

$S^\mu$ is the displacement to neighbouring geodesics; when they move in in $x^1$, they move out in $x^2$. If we have a bunch of particles originally arranged on a circle, we obtain the time evolution depicted in 4.1.

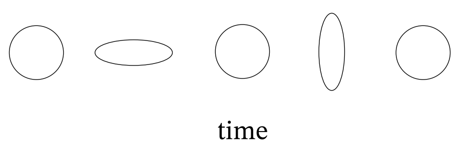


Figure 4.1: Time evolution of $H_+$ polarisation gravitational waves.

$H_\times$ **polarisation** We have $H_+ = 0$. Then

$$\frac{\mathrm{d}^2 S^1}{\mathrm{d}t^2} = -\frac{\omega^2}{2} H_\times e^{i\omega t} S^2 \tag{4.35}$$

$$\frac{\mathrm{d}^2 S^2}{\mathrm{d}t^2} = -\frac{\omega^2}{2} H_\times e^{i\omega t} S^1 \tag{4.36}$$

The solutions to linear order in $h$ are

$$S^1(t) = S^1(0) + \frac{1}{2} S^2(0) H_\times e^{i\omega t} + \ldots \tag{4.37}$$

$$S^2(t) = S^1(0) + \frac{1}{2} S^1(0) H_+ e^{i\omega t} + \ldots \tag{4.38}$$

$$\tag{4.39}$$

Define $S_\pm = S^1 \pm S^2$. It is then the same as before, but rotated by 45°. This is illustrated in

Gravitational wave detectors, such as LIGO, have two perpendicular arms. As a wave passes, the change in the length is

$$L' = L\left(1 \pm \frac{H_+}{2}\right) \quad \Rightarrow \quad \frac{\delta L}{L} = \frac{H_+}{2} \tag{4.40}$$

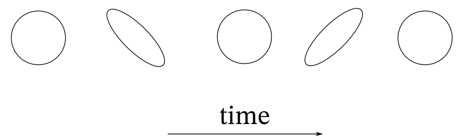


Figure 4.2: Time evolution of $H_\times$ polarisation gravitational waves.

If the wave passes at an angle $\theta$, there are various $\cos\theta$ factors we have to take into account. We will see shortly that astrophysical sources give $H_+ \sim 10^{-21}$. This is why we can do the linearised analysis in the first place. We only need quadratic order terms at precisions of $10^{-42}$. For arms of length $L \sim 3$km, the change in the length of the arms is $\delta \sim 10^{-18}$m. This is small! But this is what LIGO measured!

With LISA, LIGO will be put into space, and we will have $L \sim 3 \times 10^3$km and $\delta L = 10^{-12}$m.

### An Aside

You could ask yourself: is there actually an exact solution to the Einstein equations that contains gravitational waves? This is quite academic in nature, since the difference in predictions is so small. However, it is nice to know that there is a class of exact gravitational wave solutions, called *Brinkmann metrics*. They take the following form

$$ds^2 = -dudr + dx^a dx^a + H_{ab}(u)x^a x^b du^2, \tag{4.41}$$

where $u = t - z$ and $v = t + z$ are lightcone coordinates. $z$ will be the direction in which the wave travels. Moreover, $a = 1, 2$ and $H_{ab}(u)$ are arbitrary functions with a traceless matrix, $H^a{}_a = 0$.

This is constant if $t$ and $z$ increase at the same time; the profile is carried forward, corresponding to wave propagation.

This is in slightly different coordinates to what we have been working with, so linearising this does not quite give the metrics that we have been working with.

## 4.2.2 Making Waves

The discussion of the production of gravitational waves will very much parallel the production of electromagnetic waves. You generate electromagnetic waves by shaking charges; you generate gravitational waves by shaking mass. In fact, the punchline of the previous subsection is that this is actually inaccurate in almost all circumstances. However, for now this is a good approximation.

We want to solve

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{4.42}$$

Everything is done in linearised analysis, meaning that the right hand side must be small in an appropriate sense. As we go on we will half-see what this means. In some sense we will deal with slowly moving things with low energy.

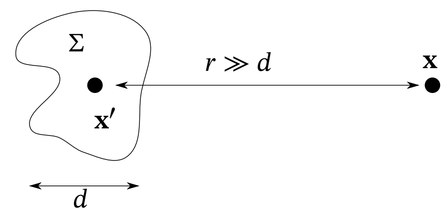


Figure 4.3

Consider some interesting region $\Sigma$ of space with an extent $d$ in which $T_{\mu\nu} \neq 0$, illustrated in Figure 4.3. For instance, this could be two neutron stars orbiting each other. Then there is a whole bunch of empty space $r \gg d$ between us and $\Sigma$.

> **Note:** This is the same setup as when we try to understand radiation in electromagnetism (far-field approximation).

We will solve this using Green's functions. Similar to the Green's function solution of electromagnetism, the solution is

$$\bar{h}_{\mu\nu}(\mathbf{x}, t) = 4G \int_\Sigma \mathrm{d}^3 x' \, \frac{T_{\mu\nu}(\mathbf{x}', t_{\text{ret}})}{|\mathbf{x} - \mathbf{x}'|}, \tag{4.43}$$

with $t_{\text{ret}} = t - \left|\mathbf{x} - \mathbf{x}'\right|$.

**Remark:** This derivative $\partial_0$ is with respect to $t$, not $t_{\text{ret}}$.

The equation (4.42) was derived in `dD` gauge. And this solution indeed obeys the `dD` gauge condition provided that $\partial_\mu T^{\mu\nu} = 0$.

For $r \gg d$,

$$\left|\mathbf{x} - \mathbf{x}'\right| \approx r - \frac{1}{r}\mathbf{x} \cdot \mathbf{x}' + \ldots \tag{4.44}$$

$$\text{and} \quad \frac{1}{|\mathbf{x} - \mathbf{x}'|} \approx \frac{1}{r} + \frac{1}{r^3}\mathbf{x} \cdot \mathbf{x}' + \ldots \tag{4.45}$$

$$\text{and} \quad T_{\mu\nu}(\mathbf{x}', t_{\text{ret}}) = T_{\mu\nu}(\mathbf{x}', t - r) + \dot{T}_{\mu\nu}(\mathbf{x}', t - r)\frac{\mathbf{x} \cdot \mathbf{x}'}{r} + \ldots, \tag{4.46}$$

where we need $\dot{T}_{\mu\nu}$ to be suitably small. At leading order,

$$\overline{h}_{\mu\nu} \approx \frac{4G}{r} \int_\Sigma \mathrm{d}^3x'\, T_{\mu\nu}(\mathbf{x}', t-r) \tag{4.47}$$

We get that

$$\Rightarrow \overline{h}_{00} \approx \frac{4GE}{r} \qquad \text{with } E = \int_\Sigma \mathrm{d}^3x'\, T_{00}(\mathbf{x}', t-r). \tag{4.48}$$

This is just recapitulating the Newtonian limit. Similarly, we have

$$\overline{h}_{0i} \approx -\frac{4GP_i}{r} \qquad \text{with } P_i = -\int_\Sigma \mathrm{d}^3x'\, T_{0i}(\mathbf{x}', t-r). \tag{4.49}$$

This is just a long-distance fall off that you would see for stationary matter, or for matter moving along way away. The interesting physics comes from the following component

$$\overline{h}_{ij} \approx \frac{4G}{r} \int_\Sigma \mathrm{d}^3x'\, T_{ij}(\mathbf{x}', t-r). \tag{4.50}$$

Unlike the other two, $T_{ij}$ is not a conserved object; it is the current part of the stress-tensor rather than the charge part.

To solve this integral, we perform the same manipulations as in electromagnetism. In particular, we write

$$T^{ij} = \partial_k(T^{ik}x^j) - (\partial_k T^{ik})x_j \tag{4.51}$$

$$= \partial_k(T^{ik}x^j) + \partial_0 T^{0i}x^j, \tag{4.52}$$

where we rewrote the second term by using the fact that the energy-momentum tensor is conserved $\partial_\mu T^{\mu\nu} = 0$. Symmetrising over $i$ and $j$,

$$T^{0(i}x^{j)} = \frac{1}{2}\partial_k(T^{0k}x^i x^j) - \frac{1}{2}(\partial_k T^{0k})x^i x^j \tag{4.53}$$

$$= \frac{1}{2}\partial_k(T^{0k}x^i x^j) + \frac{1}{2}\partial_0 T^{00}x^i x^j, \tag{4.54}$$

again using conservation of the energy-momentum tensor.

Put this into the integral $\int_\Sigma \mathrm{d}x'$ and drop any total spatial derivative. All that remains are the time derivatives $\partial_0$. What we get is that the metric where we are is given by

$$\overline{h}_{ij}(\mathbf{x}, t) \approx \frac{2G}{r}\ddot{I}_{ij}(t-r), \qquad \text{where } I_{ij} = \int \mathrm{d}^3x'\, T^{00}(\mathbf{x}', t)x'_i x'_j. \tag{4.55}$$

$I_{ij}$ is the *quadrupole* of the energy distribution in the region $\Sigma$.

**Note:** The dependence on $\mathbf{x}$ of $\overline{h}$ comes from the $r$.

**Remark:** We could now use the dD gauge condition $\partial_\mu \overline{h}^{\mu\nu} = 0$ to find the corrections to $\overline{h}_{00}$ and $\overline{h}_{0i}$ using $\overline{h}_{ij}$.

**Remark:** Shaking the matter quadrupole at some characteristic frequency $\omega$, creates waves of roughly frequency $\omega$ (possibly with some factors of two). We will see this in more detail in the next section.

**Note:** In electromagnetism there is an analogous formula: "gauge field far away = first time derivative of the dipole of the charge density". We get the quadrupole rather than the dipole, since the dipole of a mass density is related to the momentum, which is conserved and cannot shake backwards and forwards. You have to go to the quadrupole to find the gravitational waves, which is why they are weak compared to electromagnetic waves.

**Example 4.2.2** (Binary System)**:** Consider two objects, each with mass $M$, in a circular orbit in the $(x-y)$-plane at distance $R$. This is illustrated in Fig. 4.4.

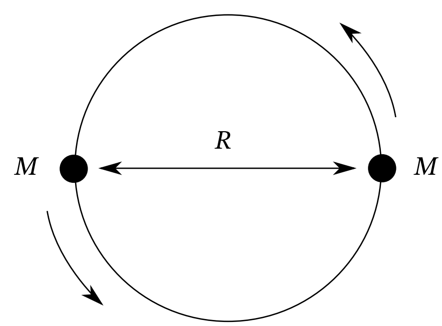


Figure 4.4: Binary system.

Newtonian gravity gives $\omega^2 = \frac{2GM}{R^3}$. Viewed as point particles,

$$T^{00}(\mathbf{x},t) = M\delta(z)\left[\delta(x-\frac{1}{2}R\cos(\omega t))\delta(y-\frac{1}{2}R\sin(\omega t)) + \delta(x+\frac{1}{2}R\cos(\omega t))\delta(y+\frac{1}{r}R\sin(\omega t))\right] \tag{4.56}$$

Compute the quadrupole $I_{ij}(t)$ to find

$$I_{ij} = \frac{MR^2}{4}\begin{pmatrix} 1+\cos(2\omega t) & \sin(2\omega t) & 0 \\ \sin(2\omega t) & 1-\cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.57}$$

where we have used the double-angle formula along the way.

This tells us that

$$\bar{h}_{ij} = -\frac{2GMR^2\omega^2}{r}\begin{pmatrix}\cos(2\omega t_{\text{ret}}) & \sin(2\omega t_{\text{ret}}) & 0\\ \sin(2\omega t_{\text{ret}}) & -\cos(2\omega t_{\text{ret}}) & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{4.58}$$

This combination of polarisation is called *circular polarisation* as in the case of light. We get circularly polarised gravitational waves travelling in the $z$-direction. Use $\omega^2 = \frac{2GM}{R^3}$ to find the magnitude

$$\left|h_{ij}\right| \sim \frac{G^2M^2}{Rr}. \tag{4.59}$$

This is what you measure in LIGO. For light, what you measure is the square of the amplitude, meaning that you can only see as $\sim 1/r^2$ far. As you start to improve LIGO, you start to see much further out than with optical telescopes!

To get a 'large' magnitude $h_{ij}$, we need compact objects nearby. The most compact objects are black holes. The closest they can come is the *Schwarzschild radius* $R_S = 2GM$. Neutron stars are not wildly different from this.

As these objects approach, the size of the gravitational wave that we will see is given by $\left|h_{ij}\right| \sim GM/r$. A black hole of a few solar masses ($R_S \sim 10$km) in our neighbouring Andromeda galaxy ($r \sim 10^{18}$km) gives a gravitational wave strength of $\left|h_{ij}\right| \sim 10^{-17}$.

### 4.2.3 Power Radiated

We want to understand how much energy is emitted in gravitational waves. This is a subtle subject; the definition of energy in GR is problematic.

To put this into perspective, we might recall how we did this calculation in Maxwell theory. In electromagnetism, we compute the power by integrating the momentum carried away (or equivalently the energy flux) by the electromagnetic field through at two-sphere.

$$\mathcal{P} = \int_{S^2} \mathrm{d}^2 S_i\, T^{0i}, \tag{4.60}$$

where $T^{0i}$ is the *Poynting vector*.

We need to defined the analogue of the Poynting vector for gravitational waves. We define an energy-momentum tensor $t_{\mu\nu}$ for the gravitational field, which, in the linearised theory, should obey the conservation law

$$\partial_\mu t^{\mu\nu} = 0. \tag{4.61}$$

Unfortunately, there is no such object that is gauge invariant! Surely, the power emitted should be independent of the coordinates we choose.

The proper way to proceed is to work in $\mathbb{M}^4$ and look at $\mathscr{I}^+$. There is a gauge invariant meaning to the amount of energy that hits $\mathscr{I}^+$ at any given time. A correct treatment this has to work asymptotically. This is called *Bondi energy*.

We will do things in a more hand-wavy and sloppy way. This is because we only really care about order-of-magnitude estimates.

There is an obvious approach to defining $t^{\mu\nu}$. From the Fierz-Pauli action, which can be viewed as a theory of the field $h_{\mu\nu}$ in a flat $\mathbb{M}^4$background, we can calculate the Noether current for translations in exactly the same way as you would for any other quantum field theory.

The result with neither be symmetric in $\mu, \nu$, nor gauge invariant. This is not a surprise; the same thing happens for Maxwell theory; however for Maxwell theory we can add terms that give us both. For the Fierz-Pauli action, we can only add a term that makes it symmetric—we can not get gauge invariance.

Let us take a shortcut. In TT gauge, with $h = 0$ and $\partial_\mu h^{\mu\nu} = 0$,the Fierz-Pauli action is

$$S_{\mathrm{FP}} = -\frac{1}{8\pi G}\int \mathrm{d}^4 x\, \frac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu}. \tag{4.62}$$

We now just pretend that this is the action for 10scalar fields $h^{\mu\nu}$. The energy density is going to be of the form

$$t^{00} \sim \frac{1}{G}\dot{h}_{\mu\nu}\dot{h}^{\mu\nu} + (\text{gradient terms}). \tag{4.63}$$

For wave solutions, the gradient term $\nabla h^{\mu\nu} \cdot \nabla h^{\mu\nu}$contributes to the same.

Previously, we had the solutions $\overline{h}_{ij} = \frac{2G}{r}\ddot{I}_{ij}$. This is not in TT gauge. If we but it in this form, we get

$$h_{ij} \sim \frac{G}{r}\ddot{\mathcal{Q}}_{ij}, \tag{4.64}$$

with $\mathcal{Q}_{ij} = I_{ij} - \frac{1}{3}I_{kk}\delta_{ij}$is the traceless part of the quadrupole. This suggests that the energy density in gravitational waves far from the source is

$$t^{00} \sim \frac{G}{r^2}\ddot{\mathcal{Q}}_{ij}^2. \tag{4.65}$$

Integrated over a sphere, this suggests that the power emitted is

$$\mathcal{P} \sim G\dddot{\mathcal{Q}}_{ij}^2. \tag{4.66}$$

In fact, the correct result is $\mathcal{P} = \frac{G}{5}\dddot{Q}_{ij}^2$, using the Bondi energy at $\mathscr{I}^+$ .

**Note:** With squared quantities, the indices are contracted.

**Remark:** The problem with this derivation is that at every single step we obtain things that depend on the coordinates. In fact, if you do things more carefully, you get this term and a bunch of other stuff that depends on the coordinates we are working with. This other stuff vanishes when you take the limit to $\mathscr{I}^+$.

**Remark:** There are multiple other ways of getting this. One is the Landau-Lifshitz pseudo-tensor.

**Remark:** The fifth is important; there was a nobel prize awarded to the Holst-Taylor binary. Long before LIGO existed, we could see the energy being emitted by the fact that neutron star orbits were decaying.

**Example 4.2.3:** A binary system, two objects that are orbiting each other, with centrifugal force $\omega^2 R \sim GM/R^2$. The quadrupole $\mathcal{Q} \sim MR^2$. This means that the third time derivative goes like $\dddot{\mathcal{Q}} \sim \omega^3 MR^2$. The power emitted is

$$\mathcal{P} \sim G\dddot{\mathcal{Q}}^2 \sim \frac{G^4M^5}{R^5}. \tag{4.67}$$

Since we want actual numbers, we have to put $c$ back in. The above formula is actually correct, however, the Schwarzschild radius for a black hole has a factor $R_S = 2GM/c^2$.

$$\mathcal{P} = \left(\frac{R_s}{R}\right)^5 L_{\text{planck}}, \tag{4.68}$$

where the Planck luminosity is $L_{\text{planck}} = \frac{c^5}{G} \approx 3.6 \times 10^{52} Js^{-1}$. To get a feel for the enormity of this, the Sun emits a luminosity $L_\odot \approx 10^{-26} L_{\text{planck}}$. All the stars in the observable universe emit $L \sim 10^{-5} L_{\text{planck}}$.

Nonetheless, when two black holes collide, their distance is roughly $R \approx R_S$ and so the power they emit for that brief fraction of a second is $L \approx L_{\text{planck}}$. To say that LIGO observed violent events in an understatement.

**Example 4.2.4:** Two objects with different masses $M^1 \gg M^2$ have a power

$$\mathcal{P} \sim \frac{G^4 M_1^3 M_2^2}{R^5}. \tag{4.69}$$

We can now start calculating the gravitational waves emitted by other things. For Jupiter, its mass is $M^2 \approx 10^{-3} M_\odot$ and its orbit is $R \approx 10^8 km$. The power emitted $\mathcal{P} \approx 10^{-44} L_{\text{planck}} = 10^{-18} L_\odot$, which is negligible. What is killing this here is the factor of 5 is the exponent of (4.68).

**Example 4.2.5:** Suppose that hit up one of the excellent nightclubs in Cambridge to celebrate the end of Michaelmas term. You give it everything you have and wave your arms wildly. How many gravitational waves do you emit? $\mathcal{Q} \approx 1kgm^2$ and $\dddot{\mathcal{Q}} \approx 1kgm^2s^{-3}$ . The power emitted while you dance is

$$\mathcal{P} \sim \frac{G\dddot{\mathcal{Q}}^2}{c^5} \approx 10^{-52} Js^{-1}. \tag{4.70}$$

This is a really low amount. To see how low, consider the next example.

**Example 4.2.6:** There is a lower bound to the gravitational wave power you can emit. A single graviton has $E = \hbar\omega$ so if $\omega = 1s^{-1}$. Therefore $E \approx 10^{-34}J$. If you want to dance until you emit a single graviton, you should wave your arms for

$$T \approx 10^{18}s \approx 10 \text{ billion years.} \tag{4.71}$$

This is one of the examples that exemplify that quantum gravity will never be important in our lifetimes.