Black Holes

Part III Lent 2020

Lectures by Harvey Reall

Report typos to: uco21@cam.ac.uk
More notes at: uco21.user.srcf.net

March 11, 2020

Contents

Administrative

- Office hours: Fridays 2pm, B2.09
- Lecture notes: www.damtp.cam.ac.uk/user/hsr1000 and /examples
  - everything in notes is examinable
- Conventions: $G = c = 1$, ignore Λ (negligible for Black holes)
- indices: $\mu, \nu, \ldots$ refer to *specific* basis,

  $a, b, c, \ldots$ 'abstract indices' (Penrose) refer to *any* basis

$$\text{e.g.} \qquad \Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}(g^{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}) \qquad R = g^{ab} R_{ab} \tag{1}$$

- Books listed in lecture notes (Wald etc)

1 Spherical Stars

1.1 Cold stars

Gravitational force, which wants the star to contract, is balanced by pressure of nuclear reactions. If we wait long enough, star will exhaust nuclear fuel and the star will contract. What happens next? Any new source of pressure will have to be non-thermal, since time will cause the star to cool down. There is one such source of pressure coming from the Pauli principle. If you have a gas of fermions, it will resist compression. This is called 'degeneracy pressure'. This is entirely a quantum effect, which is not thermal.

Definition 1: A *white dwarf* is a star in which gravity is balanced by electron degeneracy pressure.

This is a very dense star: a white dwarf with the same mass as our sun, $M = M_{\odot}$ has a radius $R \sim \frac{1}{100} R_{\odot}$.

However, not all stars can end their life this way. The maximum mass of a white dwarf is the Chandrasekhar limit $M_{wd} \leq 1.4 M_{\odot}$.

If matter is sufficiently dense, we have inverse β-decay, which turns the protons in the star into neutrons. We therefore get a second class of star:

Definition 2: A *neutron dwarf* is a star in which gravity is balanced by neutron degeneracy pressure.

These are tiny: taking a neutron star with $M \sim M_{\odot}$, then $R \sim 10$ km. Compare this with the radius of our sun, which is $R_{\odot} \simeq 7 \times 10^5$ km. Because they are so dense, their gravitational force on the surface is very strong. In terms of Newtonian gravity, we have $|\Phi| \sim 0.1$ at the surface. General relativity becomes negligible if $|\Phi| \ll 1$. So here, general relativity is important.

We will show that for any cold star there is a maximal mass around five solar masses. This bound will be independent of our ignorance of the properties of matter at such high densities.

In order to make this problem tractable, we will assume that the star is spherically symmetric and time independent.

1.2 Spherical Symmetry

Definition 3: The *unit round metric* on S^2 is $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$.

Roughly speaking, spherical symmetry is the isometry group of this metric. The isometry group in this case is $SO(3)$.

Definition 4: A spacetime is *spherically symmetric* if its isometry group contains an $SO(3)$ subgroup, whose orbits are 2-spheres.

Pick a point and act on it with all $SO(3)$ elements. It will then fill out a sphere with unit round metric.

Definition 5: In a spherically symmetric spacetime (M, g) , the *area radius function* is

$$\begin{aligned} r: M &\to \mathbb{R} \\ p &\longmapsto r(p) = \sqrt{\frac{A(p)}{4\pi}} \end{aligned} \tag{1.1}$$

where $A(p)$ is the area of the S^2 orbit through p.

You can think of r as the radial coordinate. Instead of defining r in terms of distance from the origin (which does not exist on S^2), we define it here via the area.

Remark: The S^2 has induced metric $r(p)^2 d\Omega^2$.

1.3 Time-independence

Definition 6 (stationary)**:** The spacetime (M, g) is *stationary* if there exists a Killing vector field (KVF) k^a, which is everywhere timelike ($g_{ab}k^a k^b < 0$).

Our spacetime has a time-translation symmetry.

Pick some hypersurface Σ transverse to k^a . We can then pick coordinates x^i , $i = 1, 2, 3$ on Σ.

We assign coordinates (t, x^i) to point parameter distance t along an integral curve k^a through a point on Σ with coordinates x^i . This implies that $k = \partial/\partial t$, implying that the metric in independent of t (since k^a is Killing).

$$ds^2 = g_{00}(x^k)dt^2 + 2g_{0i}(x^k)dt dx^i + g_{ij}(x^k)dx^i dx^j \tag{1.2}$$

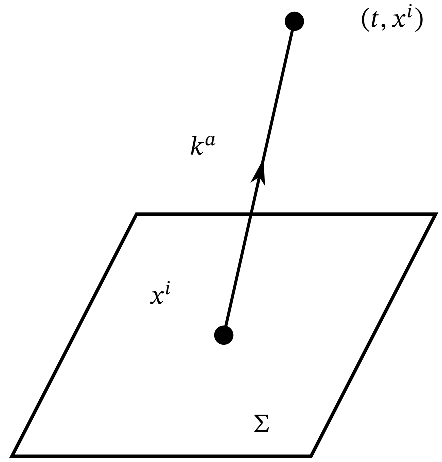


Figure 1.1

with $g_{00} < 0$. Conversely, any metric of this form is stationary.

This is the weakest notion of time-independence we can use. There is also a more refined notion. Before we can introduce that, we need to talk about hypersurface orthogonality.

Claim 1: Let Σ be a hypersurface of constant $f = 0$ on Σ where $f : M \to \mathbb{R}$ a smooth function where $df \neq 0$ on Σ. Then df is normal to Σ.

Proof. Let t^a be a vector that is tangent to Σ. Then

$$df(t) = t(f) = t^\mu \partial_\mu f = 0 \tag{1.3}$$

since f is constant on Σ. □

> Normals to a surface are not unique. For example, we can rescale f to get another normal on Σ. In fact, we can also add something that vanishes on Σ.

Claim 2: If n is also normal to Σ, then $n = gdf + fn'$, where $g \neq 0$ on Σ and n' is a smooth 1-form.

Proof. By the rules of the exterior derivative,

$$dn = dg \wedge df + df \wedge n' + fdn' \tag{1.4}$$

Evaluating this on Σ gives

$$dn|_\Sigma = (dg - n') \wedge df \ \Rightarrow \ n \wedge dn|_\Sigma = 0, \tag{1.5}$$

as $n \propto df$ on Σ. □

This is very useful since there is also a converse of this statement:

Theorem 1 (Frobenius)**:** If $n \neq 0$ is a 1-form such that $n \wedge dn \equiv 0$, then $\exists$ functions g, f such that $n = gdf$. So n is normal to surfaces of constant f. We say that n is *hypersurface orthogonal*.

Definition 7 (static)**:** A spacetime (M, g) is *static* if there is a hypersurface-orthogonal timelike Killing vector field.

Remark: This is a refinement since static $\Rightarrow$ stationary.

By Frobenius' theorem, we can choose $\Sigma \perp k^a$ when defining (t, x^i) (since k^a is hypersurface-orthogonal). But Σ is $t = 0$, normal to Σ is dt. Therefore, $k_\mu\big|_{t=0} \propto (1, 0, 0, 0)$. In particular, the spatial components in these coordinates are $k_i\big|_{t=0} = 0$, but $k_i = g_{0i}(x^k)$. Therefore, $g_{0i}(x^k) = 0$. In a static spacetime, the off-diagonal elements of the metric are zero.

$$ds^2 = g_{00}(x^i)dt^2 + g_{ij}(x^k)dx^i dx^j \qquad (g_{00} < 0.) \tag{1.6}$$

There is now an additional symmetry present. We have a discrete time-reversal symmetry $(t, x^i) \to (-t, x^i)$.

Roughly speaking, static means 'time-independent and invariant under time-reversal'.

Example 1.3.1: A rotating star can be stationary, but not static.

> Static means non-rotating.

1.4 Static, spherically symmetric spacetimes

Let us talk about a spherical, non-rotating star. More formally, we will assume the isometry group $\mathbb{R} \times SO(3)$.

> $SO(3)$ are the spatial rotations. $\mathbb{R}$ are the time-translations associated to the timelike Killing vector field k^a.

Claim 3: This implies that the spacetime is static (rotation breaks spherical symmetry).

On Σ, choose coordinates $x^i = (r, \theta, \varphi)$, where r is defined via the area-radius. A consequence of the spherical symmetry is that the metric must take the following form on Σ

$$ds^2\big|_\Sigma = e^{2\Psi(r)}dr^2 + r^2 d\Omega^2 \tag{1.7}$$

(this is because $drd\theta$ or $drd\varphi$ break spherical symmetry.)

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Psi(r)}dt^2 + r^2 d\Omega^2. \tag{1.8}$$

The choice of g_{00} is inspired by the Newtonian limit.

At the moment there is no origin. There is no reason to think of r as the distance to the origin. In fact, it is not the distance to the origin.

For a static, spherically symmetric star, we use the metric (1.8). To find Φ and Ψ, we need to solve the Einstein equations. In order to find those, we need to determine what matter the star contains.

We will model the matter inside the star as a perfect fluid with energy-momentum tensor

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}, \tag{1.9}$$

where u_a is the velocity of the fluid, obeying $g_{ab}u^a u^b = -1$. The quantities ρ and p are, respectively, the energy density and the pressure in the fluid's rest frame.

Time-independence implies that $u^a = e^{-\Phi}(\frac{\partial}{\partial t})^a$. The velocity is fixed by the symmetry assumptions, which also imply that $\rho = \rho(r)$ and $p = p(r)$ can only be functions of r. Also $\rho, p = 0$ for $r > R$, where R is the radius of the star.

1.5 Tolman–Oppenheimber–Volkoff Equations

Solving the Einstein equation imposes the fluid equations, so we do not separately need to deal with those. However, in what follows, it will actually be slightly easier to derive one of the following equations by using the fluid equation $\nabla_\mu T^{\mu\nu} = 0$ instead of some components of the Einstein equations.

By symmetry, there are only really 3 equations to solve. To write these down more concisely, we define $m(r)$ by the relation

$$e^{2\Psi(r)} = (1 - \frac{2m(r)}{r})^{-1} \overset{\text{LHS>0}}{\Rightarrow} m(r) < r/2. \tag{1.10}$$

Exercise 1.1 (Sheet 1)**:** Using the $(\mu\nu)$ component of the Einstein equation, we can derive

$$(tt): \qquad \frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho \tag{TOV 1}$$

$$(rr): \qquad \frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{m + 4\pi r^3 \rho}{r(r-2m)} \tag{TOV 2}$$

$$\nabla_\mu T^{\mu\nu} = 0 \qquad \frac{\mathrm{d}p}{\mathrm{d}r} = -(p+\rho)\frac{m + 4\pi r^3 p}{r(r-2m)} \tag{TOV 3}$$

where instead of using a third Einstein equation, it is easiest to use $\nabla_\mu T^{\mu\nu} = 0$ to derive (TOV 3).

We have three equations, but four unknowns (m, Φ, ρ, π) . However, luckily we have some extra information coming from *thermodynamics*.

A cold star has $T = 0$ but $T = T(\rho, p)$, so $T = 0$ fixes some relation $p = p(\rho)$. This is known as a "barotopic equation of state".

We will not need much information about this relation. However, we will assume that $\rho, p > 0$ and $\frac{\mathrm{d}p}{\mathrm{d}\rho} > 0$.

> Else, we have an unstable fluid: an increase in density $\delta\rho > 0$ would cause a decrease in pressure $\delta p < 0$, which causes more fluid to flow into a given volume, causing in turn an even bigger increase in density.

1.6 Outside a Star: Schwarzschild Solution

Outside the star, at $r > R$, we have no matter and therefore $\rho = p = 0$.

From (TOV 1), we then find that $m(r) = M$ is a constant. One can then integrate (TOV 2) to find that $\Phi(r) = \frac{1}{2}\ln\left(1 - \frac{2M}{r}\right) + \Phi_0$, where Φ_0 is some constant of integration.

However, Φ_0 is not physical: as $r \to \infty$, $\Phi(r) \to \Phi_0$, so $g_{tt} \to e^{-2\Phi_0}$ as $r \to \infty$. This means that we can eliminate Φ_0 by absorbing it into the time coordinate via the coordinate transform $A' = e^{\Phi_0}t$. Without loss of generality, we may therefore set $\Phi_0 = 0$. The resulting metric is the *Schwarzschild solution*

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{1.11}$$

We interpret M to be the mass of the star.

At $r = 2M$, the "Schwarzschild radius", the metric components $g_{\mu\nu}$ (in a coordinate basis) are singular. Since in our derivation every step was sound, this singularity must be inside the star, where the metric is not valid. The star must have

$$R > 2M. \tag{1.12}$$

Remark: To get from GR to Newtonian physics, we take the limit of $c \to \infty$. The inequality

$$R > 2M \quad \xleftrightarrow[\text{units}]{\text{reinstate}} \quad \frac{GM}{c^2R} < \frac{1}{2} \tag{1.13}$$

then becomes trivial, meaning that there is no Newtonian analogue of this new GR effect.

> **Remark:** This is not true for black holes: they violate the assumption of static spacetime.

This inequality is certainly true for the sun, which has a Schwarzschild radius of $2M_\odot \approx 3km$ and a radius of $R_\odot \approx 7 \times 10^5$ km.

1.7 Interior Solution

From (TOV 1), we have that

$$m(r) = 4\pi \int_0^r \rho(r')r'^2 \, \mathrm{d}r' + m_*, \tag{1.14}$$

where m_* is some integration constant.

Let Σ_t denote a surface of constant time t. The metric induced on such a surface is

$$ds^2\big|_{\Sigma_t} = e^{2\Psi(r)}dr^2 + r^2 d\Omega^2. \tag{1.15}$$

We want the metric to be smooth at $r = 0$. This implies that the spacetime is locally flat at $r = 0$. For small r , this spacetime will look like Euclidean space $\mathbb{R}^3$.

As such, a point on S^2 of small radius r must be a distance r from the origin $r = 0$ (since this is true in $\mathbb{E}^3$). For small r we have

$$\therefore \quad r \approx \int_0^r e^{\Phi(r')} \, \mathrm{d}r' \approx e^{\Phi(0)}r \implies \Phi(0) = 0 \tag{1.16}$$

Else, there is some kind of singularity at the origin; the origin would not be smooth.

This means that $m(0) = 0 \implies m_* = 0$ in Eq. (1.14).

This was outside R. Continuity tells us that

$$m(r) = M = 4\pi \int_0^R \rho(r)r^2 \, \mathrm{d}r \,. \tag{1.17}$$

The fact that this is the same as in Newtonian physics is a coincidence; this is not in general true for general relativity.

More specifically, in general relativity, the total energy is obtained by integrating the energy density $\rho(r)$ over the appropriate volume form. On Σ_t, this is

$$e^{\Psi(r)} \underbrace{r^2 \sin\theta dr \wedge d\theta \wedge d\varphi}_{\text{usual volume form on } \mathbb{E}^3} . \tag{1.18}$$

The energy of matter on Σ_t is then

$$E = 4\pi \int_0^R e^{\Psi(r)}\rho(r)r^2 \, \mathrm{d}r \,. \tag{1.19}$$

Since $m > 0$, we find that $e^\Psi > 1$. Therefore $E > M$: the energy of the matter in the star is larger than the total energy of the star. This means that there is some gravitational binding energy $E - M$.

Finally, reduces to (TOV 3) $\frac{\mathrm{d}p}{\mathrm{d}r} < 0$. Together with the previously mentioned assumption that $\frac{\mathrm{d}p}{\mathrm{d}\rho} > 0$, this implies that $\frac{\mathrm{d}\rho}{\mathrm{d}r} < 0$.

Exercise 1.2 (Sheet 1)**:** One can then show that

$$\frac{m(r)}{r} < \frac{2}{9}\left[1 - 6\pi r^2 p(r) + (1 + 6\pi r^2 p(r))^{1/2}\right] \tag{1.20}$$

At $r = R$, the surface of the star, the pressure vanishes $p = 0$. This then reduces to the "Buchdahl inequality"

$$R > \frac{9}{4}M, \tag{1.21}$$

which is an improvement on Eq. (1.12).

Now (TOV 1) and (TOV 3) are coupled ordinary differential equations for m and ρ (via $p = p(\rho)$). These can be solved numerically given initial conditions. Eq. (1.14) automatically implies $m(0) = 0$, so we only need to specify $\rho(0) = \rho_c$.

In particular, (TOV 3) implies that the pressure p decreases as we move out towards higher r. We define the radius R by $p(R) = 0$ giving us $R = R(\rho_c)$. Similarly, once we have done this Eq. (1.17) fixes $M = M(\rho_c)$. Finally, we fix Φ by solving (TOV 2) in $r < R$ with initial condition

$$\Phi(R) = \frac{1}{2}\ln\left(1 - \frac{2M}{R}\right). \tag{1.22}$$

As such, for a given equation of state, cold stars form a one-parameter family labelled uniquely by the energy density ρ_c at the center of the star.

1.8 Maximum Mass of Cold Star

The maximum mass $M_{\max}$ depends on the equation of state.

> In particular, choosing the density of state for the degenerate electron gas gives the Chandrasekhar limit!

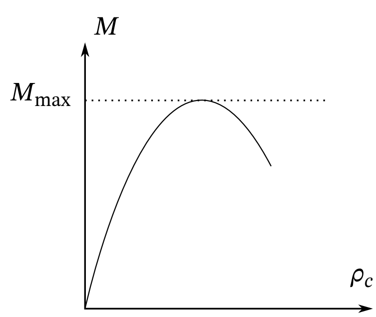


Figure 1.2

Experimentally, we can only know the equation of state up to nuclear density ρ_0 .

Claim 4: The maximum mass is always $M_{\max} \lesssim 5M_\odot$ whatever happens for $\rho > \rho_0$.

Proof. We know that ρ decreases with r. Let us now define two regions:

core region where $\rho > \rho_0$ ($r < r_0$)

envelope region where $\rho < \rho_0$ ($r_0 < r < R$)

We then define the 'core mass' to be $M_0 := m(r_0)$. Then Eq. (1.14) gives

$$M_0 > \frac{4}{3}\pi r_0^3 \rho_0. \tag{1.23}$$

The core mass has a higher density than nuclear density ρ_0.

On the other hand, eq. (1.20) for $r = r_0$ gives that $\frac{m_0}{r_0} < \frac{2}{9}\left[1 - 6\pi r_0^2 p_0 + (1 + 6\pi r_0^2 p_0)^{\frac{1}{2}}\right]$, but we know the quantity $p_0 = p(r_0)$ from the equation of state. Now the right-hand side of this is a decreasing function of p_0 , so to simplify, we can evaluate this at $p_0 = 0$. This then gives the *Bookdahl bound*

$$m_0 < \frac{4}{9} r_0, \tag{1.24}$$

which is satisfied by the core alone. We can of course get a sharper inequality by not restricting to $p_0 = 0$, but this is not needed here. Now the intersection of (1.23) and (1.24), as illustrated in

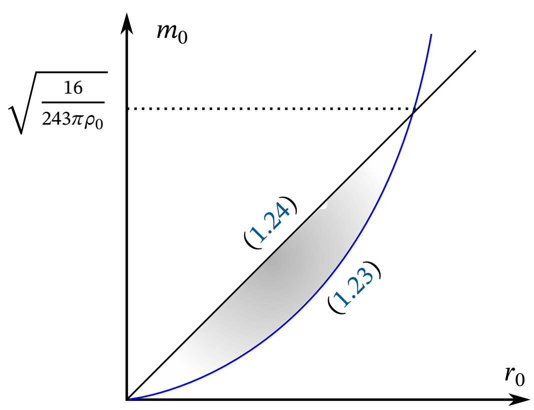


Figure 1.3

Fig. 1.3, turns out to be

$$m_0 < \sqrt{\frac{16}{243\pi\rho_0}} \simeq 5M_\odot, \tag{1.25}$$

where in evaluating this last expression we used the nuclear density ρ_0.

For any (m_0, r_0) in the allowed region, we solve (TOV 1) and (TOV 3) in the envelope region with $\rho = \rho_0$, $m = m_0$ at $r = r_0$. This fixes M in terms of (m_0, r_0). Numerically, we find that the total mass

M is maximised when the core mass m_0 is maximised. At this maximum, the envelope is small, so $M_{\text{max}} \lesssim 5M_\odot$. If we possess extra information, we can lower this bound further. However, this result as it is holds independent of the densities at $\rho > \rho_0$. $\square$

2 The Schwarzschild Black Hole

In contrast to cold stars, which cannot have masses more than a few times $M_\odot$, hot stars will undergo complete gravitational collapse to form a *black hole*. The simplest black hole solution is described by the Schwarzschild metric, which we will assume to be valid everywhere in this chapter.

2.1 Birkhoff's Theorem

In *Schwarzschild coordinates* (t, r, θ, ϕ) , the Schwarzschild metric is the one-parameter family

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.1}$$

where the parameter $M > 0$ is interpreted as a mass. This is a solution to the vacuum Einstein equations for $0 < r < r_S = 2M$, the *Schwarzschild radius*. This is spherically symmetric, but it turns out that staticity is not required.

Theorem 2 (Birkhoff)**:** Any spherically symmetric solution of the vacuum Einstein equations is isometric to the Schwarzschild solution.

Proof. See Hawking and Ellis. □

The theorem assumes only spherical symmetry, but the Schwarzschild solution has an additional isometry: $\partial/\partial t$ is a hypersurface-orthogonal Killing vector field, which is timelike for $r > 2M$, so the corresponding Schwarzschild solution is static.

Birkhoff's theorem implies that the spacetime outside any spherical body is the time-independent (exterior) Schwarzschild spacetime, even if the body itself is time-dependent. In particular, the Schwarzschild solution is a good description of the spacetime outside a spherical star during its gravitational collapse collapse.

2.2 Gravitational Redshift

Let A and B be two observers in Schwarzschild spacetime at fixed (r, θ, ϕ) with $r_B > r_A$. Now A sends two photons to B, separated by a coordinate time Δt as measured by A. Since $\partial/\partial t$ is an isometry, the two photons follow the same paths, separated by a time translation of Δt.

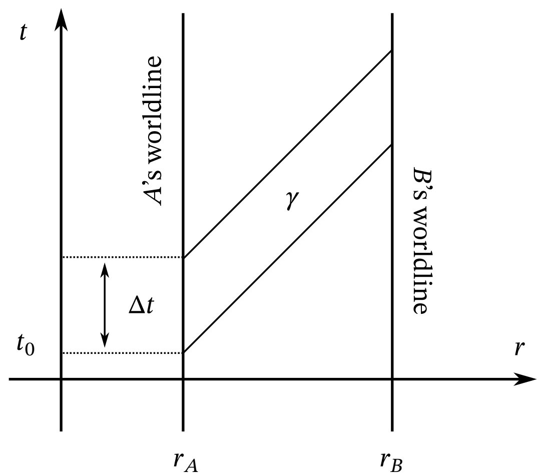


Figure 2.1

Claim 5: A measures the *proper* time between the photons to be $\Delta\tau_A = \sqrt{1 - \frac{2M}{r_A}}\Delta t$.

Proof. Proper time from a point a to b is given by $\tau = \int_a^b \sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}d\lambda$. We want to measure the proper time elapsing along the worldline of A in between sending the two photons. The relevant points are $a = (t_0, r_A, \theta, \phi)$ and $b = (t_0 + \Delta t, r_A, \theta, \phi)$. Parametrising this path with $\lambda = t$, we have

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0} = \delta^\mu_0 \quad \Rightarrow \quad \Delta\tau_A = \int_{t_0}^{t_0+\Delta t} \sqrt{g_{00}}\,\mathrm{d}t = \sqrt{1 - \frac{2M}{r_A}}\Delta t. \tag{2.2}$$

□

By the same argument, the proper time along B 's worldline is $\Delta\tau_B = \sqrt{1 - \frac{2M}{r_B}}\Delta t$. The difference here is due to the difference in metric: the curvature of the Schwarzschild spacetime at r_B is different from the curvature at r_A. Eliminating Δt gives

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \sqrt{\frac{1 - 2M/r_B}{1 - 2M/r_A}} > 1. \tag{2.3}$$

Note that this diverges as $r_A \to 2M$. We can apply this argument to two successive wavecrests of light waves propagaing from A to B to relate the period $\Delta\tau_A$ of waves emitted by A to the period $\Delta\tau_B$

of the waves received by B. For light $\Delta\tau = \lambda$ (with $c = 1$), where λ is the wavelength of the light. Hence $\lambda_B > \lambda_A$: the light becomes redshifted as it climbs out of the gravitational field. If $r_B \gg 2M$, the redshift z is given by

$$1 + z := \frac{\lambda_B}{\lambda_A} \approx \sqrt{\frac{1}{1 - 2M/r_A}}. \tag{2.4}$$

Refering back to the Buchdahl inequality (1.21), the maximum possible redshift of light emitted from the surface $r_A = R > 9M/4$ of a spherical star is therefore $z = 2$.

2.3 Geodesics of the Schwarzschild Solution

Let $x^\mu(\tau)$ be an affinely parametrised geodesic with tangent vector $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$. Since $k = \partial/\partial t$ and $m = \partial/\partial\phi$ are Killing vector fields, we have the conserved quantities

$$E = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} \qquad \text{and} \qquad h = r^2 \sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}\tau}. \tag{2.5}$$

We can interpret these quantities by evaluating the expressions at large r, where the metric is almost flat, and comparing these with analogous results from special relativity. For a timelike geodesic, chossing τ to be proper time gives E and h the interpretations of energy and angular momentum, both per unit rest mass, respectively. For a null geodesic, we can rescale the affine parameter and E and h do not have clear physical interpretations. However, the ratio h/E is invariant under this rescaling. For a null geodesic which propagates to large r, $b = |h/E|$ is the *impact parameter*.

Claim 6: We can always choose coordinates θ and ϕ so that the geodesic is confined to the equatorial plane.

Proof. Using the geodesic Lagrangian $L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, the Euler–Lagrange equation for $\theta(\tau)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\dot{\theta}\right) - r^2 \sin\theta\cos\theta\dot{\phi}^2 = 0 \tag{2.6}$$

$$r^2\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right) - h^2\frac{\cos\theta}{\sin^3\theta} = 0. \tag{2.7}$$

We can choose coordinates (θ, ϕ) on S^2 so that the geodesic initally lies in the equatorial plane $\theta(0) = \frac{\pi}{2}$ and moves tangentially to it with $\left.\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right|_{\tau=0} = 0$. For any function $r(\tau)$, Eq.(2.7) is then a second order ODE for θ with two initial conditions. One solution to this is $\theta(\tau) = \pi/2$ and uniqueness results for ODEs guarantee that this is in fact the unique solution. □

Claim 7: The radial motion of the geodesic is determined by the same equation as a Newtonian particle of unit mass and energy $E^2/2$ moving in a $1d$ potential

$$V(r) = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(\sigma + \frac{h^2}{r^2}\right), \qquad \sigma = \begin{cases} 1, & \text{timelike} \\ 0, & \text{null} \\ -1, & \text{spacelike} \end{cases}. \tag{2.8}$$

Proof. We will use the relation $g_{\mu\nu}u^\mu u^\nu = -\sigma$.

$$-\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 = -\sigma \tag{2.9}$$

$$-E^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)\frac{h^2}{r^2} = -\left(1 - \frac{2M}{r}\right)\sigma \tag{2.10}$$

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V(r) = \frac{1}{2}E^2. \tag{2.11}$$

□

2.4 Eddington–Finkelstein Coordinates

In this section we will have a closer look at radial null geodesics.

Definition 8 (radial)**:** A geodesic is *radial* if θ and ϕ are constant along it.

We evidently have $h = 0$, but for null geodesics we can also rescale the affine parameter τ so that $E = 1$. The geodesic equations are

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2M}{r}\right)^{-1} \qquad \frac{\mathrm{d}r}{\mathrm{d}\tau} = \begin{cases} +1, & \text{outgoing} \\ -1, & \text{ingoing.} \end{cases} \tag{2.12}$$

An ingoing geodesic at some $r > 2M$ will reach $r = 2M$ in finite affine parameter. Dividing gives

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\left(1 - \frac{2M}{r}\right)^{-1}. \tag{2.13}$$

This has a simple pole at $r = 2M$, so t diverges logarithmically as $r \to 2M$.

2.4.1 Coordinate Singularity

Definition 9 (Regge–Wheeler)**:** To investigate what is happening at $r = 2M$, we define the *Regge–Wheeler radial coordinate* r_* by

$$dr_* = \frac{dr}{\left(1 - \frac{2M}{r}\right)}. \tag{2.14}$$

Making a choice of integration, we get $r_* = r + 2M \ln\left|\frac{r}{2M} - 1\right|$. Note that $r_* \sim r$ for large r and $r_* \to -\infty$ as $r \to 2M$. This is illustrated in Fig. 2.2. Along a radial null geodesic we have $\frac{\mathrm{d}t}{\mathrm{d}r_*} = \pm 1$, so $t \mp r_*$ is constant.

Definition 10 (Eddington–Finkelstein)**:** The *Eddington–Finkelstein coordinates* (v, r, θ, ϕ) are obtained by defining a new coordinate $v = t + r_*$, which is constant along ingoing radial null geodesics.

We eliminate t by $t = v - r_*(r)$ and hence

$$dt = dv - \frac{dr}{\left(1 - \frac{2M}{r}\right)}. \tag{2.15}$$

In these coordinates, the metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2. \tag{2.16}$$

As a matrix, we have

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 1 & & \\ 1 & & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix}, \tag{2.17}$$

with all empty entries being zero.

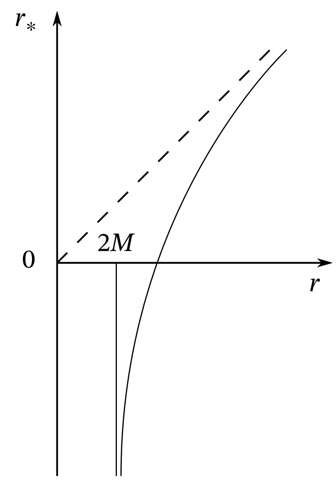


Figure 2.2: Regge–Wheeler radial coordinates.

Unlike Schwarzschild coordinates, the metric components in Eddington–Finkelstein coordinates are smooth for all $r > 0$, including $r = 2M$. The determinant $\det(g_{\mu\nu}) = -r^4 \sin^2\theta$ means that the metric is non-degenerate for all $r > 0$.[1] This means that the signature is Lorentzian for $r > 0$, since a change of signature would require an eigenvalue passing through zero.

Definition 11 (real analytic)**:** A *real analytic function* can be expanded as a convergent power series about any point.

The metric components are real analytic functions of the above coordinates. If a real analytic metric satisfies the Einstein equations in some open set, then it will satisfy them everywhere. Without encountering any problems, the Schwarzschild spacetime can therefore be *extended* though the surface $r = 2M$ to a new region with $r < 2M$. The metric (2.16) is a solution to the vacuum Einstein equations for all $r > 0$.

Remark: The new region $0 < r < 2M$ is spherically symmetric. This is consistent with Birkhoff's theorem since we can just transform back to coordinates (t, r, θ, ϕ) to obtain the Schwarzschild metric in Schwarzschild coordinates, but now with $r < 2M$.

[1]Except at $\theta = 0, \pi$, because the coordinates (θ, ϕ) are not defined at the poles.

2.4.2 Curvature Singularity

Ingoing radial null geodesics in Eddington–Finkelstein coordinates obey $\frac{\mathrm{d}r}{\mathrm{d}\tau} = -1$ and will reach $r = 0$ in finite affine parameter. Since the metric is Ricci flat, the simplest non-trivial scalar constructed from the metric is

$$R_{abcd}R^{abcd} \propto \frac{M^2}{r^6}. \tag{2.18}$$

This diverges as $r \to 0$. Since it is a scalar, it diverges in all coordinate charts. Therefore, there exists no chart for which the metric can be smoothly extended through $r = 0$, which is an example of a *curvature singularity*, where tidal forces become infinite and general relativity ceases to hold. Strictly speaking, $r = 0$ is not part of the spacetime manifold because the metric is not defined there.

Recall that $k = \partial/\partial t$ is a Killing vector field of the Schwarzschild solution for $r > 2M$. In ingoing Eddington–Finkelstein coordinates x^μ , this is

$$k = \frac{\partial}{\partial t} = \frac{\partial x^\mu}{\partial t}\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial v}, \tag{2.19}$$

since the Eddington–Finkelstein coordinates are independent of t except for $v = t + r_*(r)$. This equation can be used to extend the definition of k to $r \leq 2M$. Since $k^2 = g_{vv}$, k is null at $r = 2M$ and spacelike for $0 < r < 2M$. Hence the extended Schwarzschild solution is static only in the $r > 2M$ region.

2.5 Finkelstein Diagram

Definition 12: *Outgoing* radial null geodesics in $r \gg M$ have $t - r_* =$ constant. This means that

$$v = 2r + 4M \ln \left| \frac{r}{2M} - 1 \right| + \text{const.} \tag{2.20}$$

Exercise 2.1: Consider radial null geodesics in ingoing Eddington–Finkelstein coordinates. Show that these are either (i) ingoing $v =$ constant or (ii) 'outgoing'—either (2.20) or $r \equiv 2M$.

Plot:

In $r < 2M$, *both* families have decreasing r and reach $r = 0$ in finite τ.

2.6 Gravitational Collapse

Eventually the star collapses through the Schwarzschild surface, forming a black hole. It continues to collapse until it hits the curvature singularity in finite time, which marks its end. In fact, that time is very short.

Exercise 2.2 (Sheet 1)**:** Proper time along any timelike curve in the region $r \leq 2M$ cannot exceed πM.

Taking the Sun $M = M_\odot$, then you get to live 10^{-5}s before you are destroyed in a singularity.

Assume it is not you falling into the black hole, just your friend who is going to be destroyed. You will see that the light is gradually more reshifted and time slows down. An observer at $r > 2M$ never sees the star cross $r = 2M$. They see an ever-increasing redshift causing the star to fade away.

2.7 Black Hole Region

Definition 13 (causal)**:** A vector is *causal* if it is timelike or null (and therefore not the zero-vector). A curve is causal if its tangent vector is not causal.

Definition 14 (time-orientability)**:** A spacetime (M, g) is *time-orientable* if there exists a *time-orientation*: a causal vector field T^a.

Given such a vector field, every point in spacetime either lives in the future or past lightcone with respect to it. However, this is not always possible since there can be obstructions to this. If the spacetime is time-orientable, there are only two inequivalent choices.

Definition 15: A causal vector is future-directed (past-directed) if it lies in the same (opposite) light cone as T^a.

For the Schwarzschild solution with $r > 2M$, the obvious choice is the Killing vector field $k = \partial/\partial t$ as a time-orientation. In Eddington–Finkelstein coordinates, $k = \partial/\partial v$ works for $r > 2M$ but becomes spacelike for $r < 2M$. However, the component $g_{rr} = 0$, the vector fields $\pm\partial/\partial r$ are null and therefore causal. Which one do we choose?

Claim 8: Choosing $-\partial/\partial r$ gives an equivalent time-orientation as k for $r > 2M$.

Proof. Let us take the inner product

$$k \cdot (-\partial/\partial r) = -g_{vr} = -1. \tag{2.21}$$

If the product of two timelike vector fields is negative, they are in the same lightcone. Therefore, $-\partial/\partial r$ is in the same cone as k for $r > 2M$ and defines the time orientation for $r > 0$ (tangent to ingoing radial null geodesic). □

Claim 9: Let $x^\mu(\lambda)$ be a future-directed causal curve such that initially $r(\lambda_0) \leq 2M$. Then $r(\lambda) \leq 2M$ for all $\lambda > \lambda^0$.

Proof. The tangent vector $V^\mu = \frac{\partial x^\mu}{\partial \lambda}$ is future-directed causal. Therefore, since $-\partial/\partial r$ is also future-directed causal, their inner product is non-positive. Evaluating this gives

$$0 \geq \left(-\frac{\partial}{\partial r}\right) \cdot V = -g_{r\mu}V^\mu = -V^v = -\frac{\mathrm{d}v}{\mathrm{d}\lambda} \tag{2.22}$$

$$\therefore \frac{\mathrm{d}v}{\mathrm{d}\lambda} \geq 0 \tag{2.23}$$

$$\Rightarrow \quad -2\frac{\mathrm{d}v}{\mathrm{d}\lambda}\frac{\mathrm{d}r}{\mathrm{d}\lambda} = \underbrace{-V^2}_{\geq 0} + \underbrace{\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}\lambda}\right)^2}_{\geq 0 \text{ in } r \leq 2M} + \underbrace{r^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda}\right)^2}_{r \geq 0} \geq 0 \text{ in } r \leq 2M \tag{2.24}$$

$$\Rightarrow \quad \frac{\mathrm{d}v}{\mathrm{d}\lambda}\frac{\mathrm{d}r}{\mathrm{d}\lambda} \leq 0 \text{ in } r \leq 2M. \tag{2.25}$$

Assume for contradiction that $\frac{\mathrm{d}r}{\mathrm{d}\lambda} > 0$ at each point in $r \leq 2M$. Therefore $\frac{\mathrm{d}v}{\mathrm{d}\lambda} \leq 0$. But (2.23) then means that $\frac{\mathrm{d}v}{\mathrm{d}\lambda} = 0$. Then (2.24) implies that $-V^2 = 0$ or $\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\lambda}\right)^2 = 0$. Thus, the only non-zero component of V^μ is $V^r = \frac{\mathrm{d}r}{\mathrm{d}\lambda} > 0$. This means that V^μ is a positive multiple of $\partial/\partial r$, meaning that V^μ is past-directed. This is a contradiction.

Therefore, $\frac{\mathrm{d}r}{\mathrm{d}\lambda} \leq 0$ in $r \leq 2M$. With the initial condition $r(\lambda_0) \leq 2M$, we have that $r(\lambda) \leq 2M$ $\forall \lambda \geq \lambda_0$. □

Definition 16 (black hole)**:** A *black hole* is a region of spacetime from which no signal can reach infinity[1].

We have shown that for $r \leq 2M$ of the Schwarzschild ingoing Eddington–Finkelstein coordinates is a black hole.

Definition 17 (event horizon)**:** The boundary $r = 2M$ is called the *event horizon*.

2.8 Detecting Black Holes

There are two key properties of black holes

- There is no upper bound on the mass of a black hole (unlike for a cold star).
- Black holes are very small.

In practice, we infer the existence of black holes by looking at their gravitational effect on nearby orbiting stars. This is what makes us confident that there is a $4 \times 10^6 M_\odot$ supermassive black hole at the centre of our galaxy. It is still unknown how supermassive black holes (with $M \geq 10^6 M_\odot$) can form in the first place.

[1] We will define infinity more rigorously later in the course, but at the moment the intuitive notion is satisfactory.

Definition 18: A *solar mass black hole* has a mass $M \lesssim 100M_\odot$. These are formed by the gravitational collapse of a star.

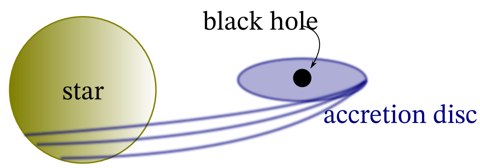


Figure 2.3

Approximate disc particles as following circular orbits.

As the energy decreases (say by friction), the radius slowly decreases. A particle from the disc reaches the ISCO. If it has energy $E = \sqrt{8/9}$ it falls into the hole. A fraction of $1 - \sqrt{8/9} \approx 6\%$ of the rest mass is lost to friction. This colossal amount of energy is converted to electromagnetic radiation.

2.9 White Holes

Consider the $r > 2M$ Schwarzschild solution.

Definition 19 (outgoing EF coords)**:** Now define $u := t - r_*$, which is constant along *outgoing* radial null geodesics. Then (u, r, θ, ϕ) define *outgoing Eddington–Finkelstein coordinates*.

The metric in these coordinates is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2 d\Omega^2. \tag{2.26}$$

Just as in the ingoing case, we can extend this though $r = 2M$ to $r \leq 2M$ until the curvature singularity at $r = 0$.

Claim 10: This is not the same as the previous $r \leq 2M$ region!

Proof. Consider for example the outgoing radial null geodesics $u =$ const. and $\frac{\mathrm{d}r}{\mathrm{d}\tau} = +1$. In this $r < 2M$ region, r is increasing, so it cannot be the same region as before. □

Exercise 2.3: Repeat the calculation as for the ingoing case to show $k = \frac{\partial}{\partial u}$ in ingoing EF coordinates and $\frac{\partial}{\partial r}$ is the time-orientation equivalent to k in $r > 2M$.

> The fundamental confusion of calculus: $\frac{\partial}{\partial r}$ in the ingoing coordinates is not the same as in the outgoing coordinates, since we are holding different coordinates fixed.

Definition 20 (white hole)**:** A *white hole* is a region that cannot receive a signal from ∞.

The $r \leq 2M$ region is a *white hole*!

A white hole is essentially a time-reverse of a black hole. If we substitute $u = -v$, we recover the metric from the ingoing coordinates. Therefore, $u = -v$ is an isometry mapping the white hole to the black hole, which reverses the time orientation.

White holes are unphysical[1], since there is no mechanism for forming them; you would have to start with the singularity at $r = 0$ and get the white hole emerging from it. Black holes are stable; small perturbations will decay. Since white holes are time-reversals of black holes, they are unstable objects.

2.10 Kruskal Extension

Definition 21: For $r > 2M$, take the *Kruskal–Szekeres* coordinates (U, V, θ, ϕ) with $0 > U = -e^{-u/4M}$ and $0 < V = +e^{v/4M}$.

We then have

$$UV = -e^{r_*/2M} = -e^{r/2M}\left(\frac{r}{2M} - 1\right). \tag{2.27}$$

The right hand side is monotonic. Therefore, if we know U and V , we can determine $r = r(U, V)$ uniquely. Similarly,

$$\frac{V}{U} = -e^{t/2M} \tag{2.28}$$

fixes $t(U, V)$.

Exercise 2.4: Show that in these coordinates the metric is

$$ds^2 = -\frac{32M^3}{r(U,V)}e^{-r(U,V)/2M}dUdV + r(U,V)^2 d\Omega^2. \tag{2.29}$$

We can smoothly extend this metric to a larger range of U and V , since it remains smooth and invertible. We can now use (2.27) to define $r(U, V)$ for $U \geq 0$ or $V \leq 0$. The metric can then be analytically extended with $\det g_{\mu\nu} \neq 0$ to new regions, where either $U > 0$ or $V < 0$.

[1]As discussed in *General Relativity*, our universe (emerging from the big bang singularity) looks a bit like the inside of a white hole in 5 dimensions.

What does the Schwarzschild radius $r = 2M$ correspond to? We have $UV = 0$, which corresponds either to $U = 0$ or $V = 0$. In fact, this is not one surface but two!

What about the curvature singularity at $r = 0$? Equation (2.27) gives $UV = 1$, a hyperbola. Radial

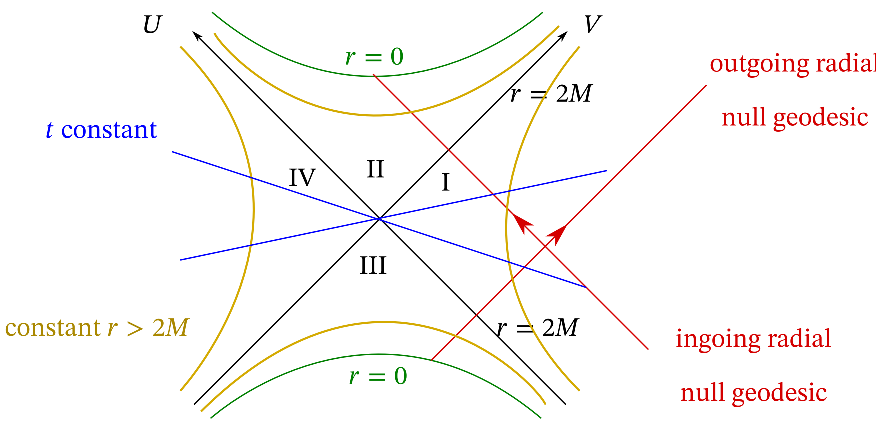


Figure 2.4: Kruskal diagram

null geodesics correspond to constant U or V.

We have four regions

I: $r > 2M$ Schwarzschild

II: Black hole region

III: White hole region

IV: new region with $r > 2M$ isometric to I via $(U, V) \to (-U, -V)$

Ingoing EF cover I and II, while outgoing EF cover I and III.

Exercise 2.5: Show

$$k = \frac{1}{4M}\left(V\frac{\partial}{\partial V} - U\frac{\partial}{\partial U}\right) \qquad k^2 = -\left(1 - \frac{2M}{r}\right), \tag{2.30}$$

timelike in I, IV, spacelike in II, III, and null at $U = 0$ or $V = 0$.

$\{U = 0\}$ and $\{V = 0\}$ are fixed by k. $k = 0$ on 'bifurfaction 2-sphere' $U = V = 0$ (also fixed by k).

Recall that every point on the diagram represents a suppressed two-sphere.

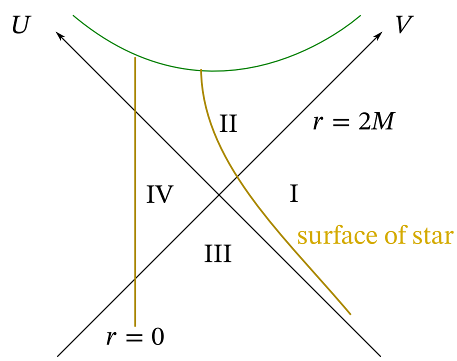


Figure 2.5: A star collapsing to form a black hole. The interior of the star covers up III and IV.

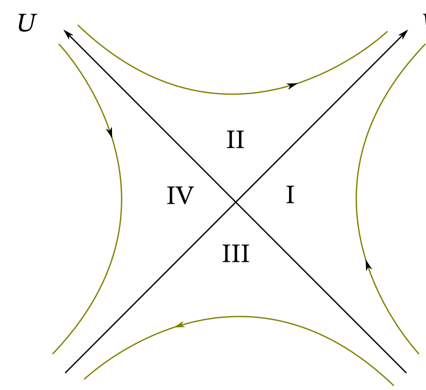


Figure 2.6: Orbits (integral curves) of k.

2.11 Einstein–Rosen Bridge

Taking t constant in I corresponds to V/U being constant. This extends into IV.

Let $r = \rho + M + \frac{M^2}{4\rho}$ and choose $\rho > \frac{M}{2}$ in I and $0 < \rho < \frac{M}{2}$ in IV.

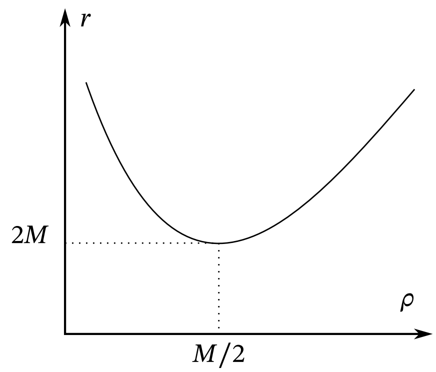


Figure 2.7

Exercise 2.6: Show that the Schwarzschild metric in *isotropic coordinates* (t, ρ, θ, ϕ) is

$$ds^2 = -\frac{\left(1 - \frac{M}{2\rho}\right)^2}{\left(1 + \frac{M}{2\rho}\right)^2} dt^2 + \left(1 + \frac{M}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\Omega^2). \tag{2.31}$$

Taking $\rho \to \frac{M^2}{4\pi}$ is an isometry I $\leftrightarrow$ IV.

For t constant, we have

$$ds^2 = \left(1 + \frac{M}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\Omega^2), \tag{2.32}$$

which is smooth $\forall \rho > 0$.

This gives us an *Einstein–Rosen bridge* connecting two far-away regions of spacetime.

2.12 Extendibility

Definition 22 (extendible)**:** A Riemannian manifold (M, g) is *extendible* if it is isometric to a proper subset of another spacetime (M', g'), which is called an *extension* of (M, g).

Example 2.12.1: Let (M, g) be the $r > 2M$ Schwarzschild spacetime. Then (M', g') can be taken to be the Kruskal extension of (M, g). Kruskal itself is *inextendible*—it is a *maximal analytic extension* of (M, g).

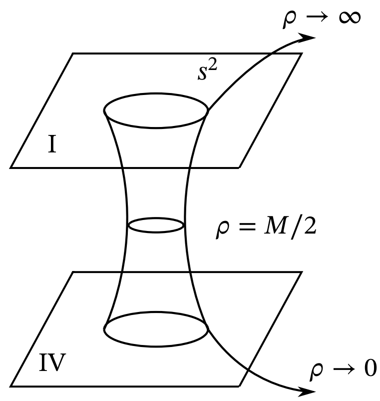


Figure 2.8: Einstein–Rosen bridge

2.13 Singularities

Definition 23 (singular)**:** The metric $g_{\mu\nu}$ is *singular* if it is not smooth or $\det g_{\mu\nu} = 0$ somewhere.

There are different kinds of singularity:

Coordinate singularities can be eliminated via a change of coordinates (e.g. at $r = 2M$ in Schwarzschild coordinates). This is not a physical singularity.

Scalar curvature singularities are points where the scalar built from $R^a{}_{bcd}$ diverges (e.g. at $r = 0$ in Schwarzschild spacetime).

Non-curvature singularities

2.13.1 Conical Singularity

Let us give an example of a non-curvature singularity. Let $M = \mathbb{R}^2$ be a manifold with metric $g = dr^2 + \lambda^2 r^2 d\phi^2$ in polar coordinates (r, ϕ), with the identification $\phi \sim \phi + 2\pi$.

For $\lambda > 0$, we have a singularity $\det g_{\mu\nu} = 0$ at $r = 0$. If $\lambda = 1$, then this is just Euclidean space $\mathbb{E}^2$ and we can switch to Cartesian coordinates, where the metric has $\det g_{\mu\nu} = 1$; in this case, $r = 0$ is a coordinate singularity. However, whenever $\lambda \neq 1$, changing coordinates $\phi' = \lambda\phi$ gives the metric $g = dr^2 + r^2 d\phi'^2$, which is locally isometric to $\mathbb{E}^2$; in this case $r = 0$ is clearly not a coordinate singularity. Moreover, the curvature tensor $R^a{}_{bcd} = 0$ vanishes everywhere, which means that the singularity cannot be a curvature singularity either.

Now the change of coordinates means that the new angular coordinates has a different period

$\phi' \sim \phi' + 2\pi\lambda$, so this is spacetime is not globally isometric to $\mathbb{E}^2$. Take a circle with radius $r = \epsilon$. The ratio of circumference and radius is

$$\frac{\text{circumference}}{\text{radius}} = \frac{2\pi\lambda\epsilon}{\epsilon} = 2\pi\lambda \not\to 2\pi \text{ as } \epsilon \to 0. \tag{2.33}$$

As such, the manifold is not locally flat at $r = 0$ and the metric cannot be smoothly extended to $r = 0$. We call this a *conical singularity*.

2.13.2 Geodesic Compleneness

Definition 24 (future endpoint)**:** A point $p \in M$ is a *future endpoint* of a future-directed causal curve $\gamma : (a, b) \to M$ if, for any neighbourhood $\mathcal{O}$ of p, there exists a value t_0 such that $\gamma(t) \in \mathcal{O}$ for all $t > t_0$.

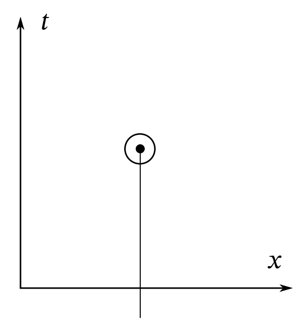


Figure 2.9: Future endpoint

Definition 25 (future-inextendible)**:** A curve γ is *future-inextendible* if it has no future endpoint.

Example 2.13.1: Take Minkowski space $\mathbb{M}^4$ with the curve $\gamma : (-\infty, 0) \to M$ defined by $\gamma(t) = (t, 0, 0, 0)$. Then $(0, 0, 0, 0)$ is a future endpoint. Taking the manifold $\mathbb{M}^4 \setminus \{(0, 0, 0, 0)\}$, γ becomes future-inextendible.

Definition 26 (complete)**:** A geodesic is *complete* if an affine parameter extends to $\pm\infty$.

Definition 27 (geodesically complete)**:** A Riemannian manifold (M, g) is *geodesically complete* iff all inextendible causal geodesics are complete.

Example 2.13.2: Minkowski spacetime with a static spherical star is geodesically complete.

Example 2.13.3: Kruskal is geodesically incomplete—geodesics reach $r = 0$ in finite affine parameter.

An extendible spacetime is geodesically incomplete in a boring way; we can just make it into a bigger spacetime. This motivates the following definition:

Definition 28 (singular)**:** A spacetime (M, g) is *singular* if it is inextendible and geodesically incomplete.

Example 2.13.4: Kruskal is singular.

3 The Initial Value Problem

3.1 Predictability

Definition 29 (partial Cauchy surface)**:** Let (M, g) be time-orientable. A *partial Cauchy surface* Σ is a hypersurface such that no two points are connected by a causal curve in M.

Definition 30 (domain of dependence)**:** The *future (past) domain of dependence* of Σ, denoted $D^{+(-)}(\Sigma)$, is the set of points $p \in M$ such that every past-(future-)inextendible causal curve through p intersects Σ. The full domain of dependence is $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)$.

Any causal geodesic in $D(\Sigma)$ must intersect Σ . Therefore, it is determined uniquely by a tangent vector on Σ.

▌ We can predict particle trajectories, say, on $D(\Sigma)$ by specifying initial data on Σ.

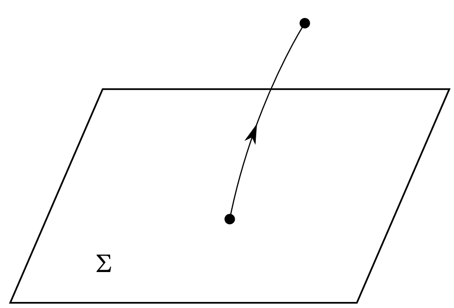


Figure 3.1

Definition 31 (hyperbolic PDEs)**:** Take fields $T^{(i)ab\dots}{}_{cd\dots}$, where $i = 1, \dots, N$, which obey equations of motion

$$g^{ef} \nabla_e \nabla_f T^{(i)ab\dots}{}_{cd\dots} = \dots . \tag{3.1}$$

The right hand side depends on g and its derivatives. It depends *linearly* on $T^{(i)}$ and their *first* derivatives.

Example 3.1.1: The Klein–Gordon, and Maxwell equations in Lorentz gauge are Hyperbolic PDEs. This is a large class of equations encompassing most physical equations of motion.

Solutions of such equations are uniquely determined in $D(\Sigma)$ from initial data on Σ.

Example 3.1.2: Take $\mathbb{M}^2$ and Σ to be the positive x-axis. This is illustrated in 3.2. Taking Σ' to be the

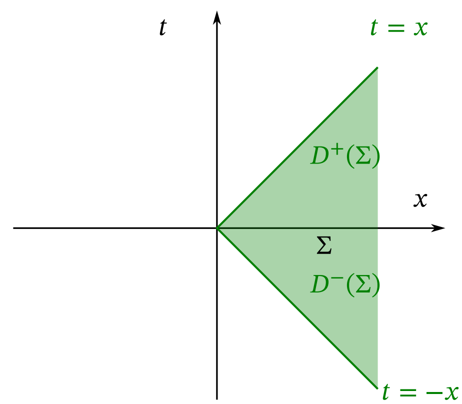


Figure 3.2

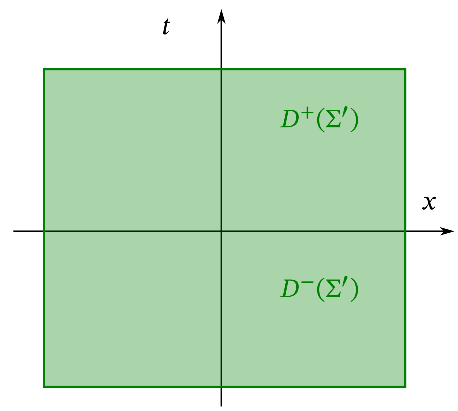


Figure 3.3

whole x-axis, we have the case illustrated in 3.3.

Consider the wave equation $\nabla^a \nabla_a \psi = -\partial_t^2 \psi + \partial_x^2 \psi = 0$. The solutions in $D(\Sigma)$ (M) are uniquely determined by data in $(\psi, \partial_t \psi)$ on Σ (Σ').

Generally, if $D(\Sigma) \neq M$, then physics in $M \setminus D(\Sigma)$ is not determined by data on Σ.

Definition 32: A spacetime (M, g) is *globally hyperbolic* if there exists a *Cauchy surface*—a partial Cauchy surface such that $D(\Sigma) = M$.

Definition 33: The *Cauchy horizon* is the boundary of $D(\Sigma)$ in M.

A spacetime (M, g) is globally hyperbolic iff no Cauchy horizon for Σ.

Examples of globally hyperbolic spacetimes include:

- Minkowski spacetime (t = const. is Cauchy)
- Kruskal spacetime

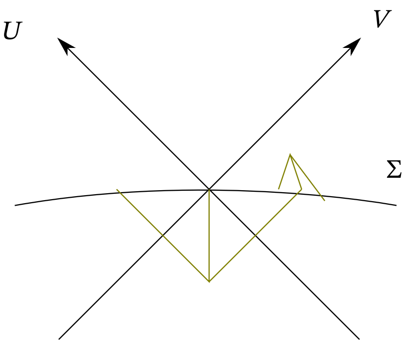


Figure 3.4

- spherical gravitational collapse

Non-globally hyperbolic spacetimes are:

- Minkowski with the origin removed: There is no Cauchy surface

Globally hyperbolic spacetimes are *nice* in the following sense:

Theorem 3: Let (M, g) be globally hyperbolic. Then

(i) There exists a *global time function* $t : M \to \mathbb{R}$ such that $-(dt)^a$ is future-directed timelike.

(ii) The constant-t surfaces are Cauchy and all have the same topology Σ.

(iii) M has topology $\mathbb{R} \times \Sigma$.

Exercise 3.1: Show that the function $U + V$ is a global time function for the Kruskal spacetime.

In particular, the surface $U + V = 0$, a straight line through the origin in Kruskal spacetime, is an Einstein–Rosen bridge. As illustrated in Fig. 2.8, we can think of this as a cylinder $\Sigma \simeq \mathbb{R} \times S^2$. Therefore, the manifold is $M \simeq \mathbb{R}^2 \times S^2$.

Let x^i be coordinates on the $t = 0$ surface Σ, and let T^a be an arbitrary timelike vector field. If we now pick an arbitrary $p \in M$, an integral curve of T^a through p intersects Σ at a unique point. Let the coordinates of that point be $x^i(p)$. This defines a map $x^i : M \to \mathbb{R}$, shown in Fig. 3.5.

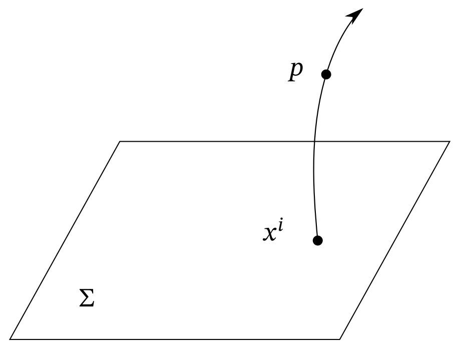


Figure 3.5

Use (t, x^i) as coordinates on M. We write the metric as

$$ds^2 - N - N^2 dt^2 h_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{3.2}$$

We call $N(t,x)$ the *lapse function* and $N^i(t,x)$ the *shift vector*. Finally, $h_{ij}(t,x)$ is the metric on the surface of constant t.

3.2 Extrinsic Curvature

Definition 34 (spacelike surface)**:** A hypersurface Σ is *spacelike* if its normal n_a is everywhere timelike.

If X^a is a vector that is tangent to the surface, then by definition $n_a X^a = 0$ and if n_a is timelike, then X^a is spacelike. Any vector tangent to the spacelike hypersurface is spacelike.

Assume that we have normalised the normal to be a unit vector, $n_a n^a = -1$. Then define

$$h^a{}_b := \delta^a_b + n^a n_b. \tag{3.3}$$

Lowering indices gives $h_{ab} = g_{ab} + n_a n_b$, so it is a symmetric tensor.

If X^a , Y^a are tangent vectors, then

$$h_{ab} X^a Y^b = g_{ab} X^a Y^a. \tag{3.4}$$

Therefore, h_{ab} is the *induced metric* on Σ (the pull-back of g_{ab}).

Since $h^a{}_b n^b = 0$ and $h^a{}_c h^c{}_b = h^a{}_b$, the $h^a{}_b$ is a *projection* onto Σ.

Using this tensor, we can decompose any vector into a parallel and perpendicular component

$$X^a = \delta^a_b X^b = \underbrace{h^a{}_b X^b}_{X^a_\parallel} - \underbrace{n^a n_b X^b}_{X^a_\perp}. \tag{3.5}$$

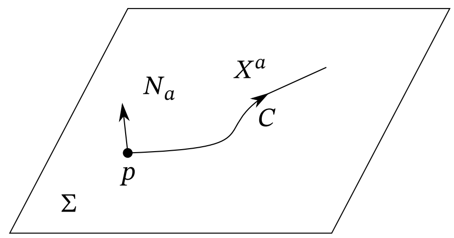


Figure 3.6

Let N_a be perpendicular to Σ at p. Parallel transport N_a along $C:\ X^b\nabla_b N_a = 0$. Does N_a remain perpendicular to Σ? Let Y^a be tangent to Σ. T hen

$$X(N\cdot Y) = X^b\nabla_b(Y^aN_a) = N_aX^b\nabla_bY^a. \tag{3.6}$$

If $N\cdot Y\equiv 0$, then $(\nabla_X Y)_\perp = 0$.

Definition 35 (extrinsic curvature tensor)**:** Extend n_a to a neighbourhood of Σ, with $n_an^a = -1$. The *extrinsic curvature tensor* K_{ab} is defined for $p\in\Sigma$ by $K(X,Y) = -n_a(\nabla_{X_\parallel}Y_\parallel)^a$.

Lemma 4: Independent of the extension of n_a, we have

$$K_{ab} = h_a{}^ch_b{}^d\nabla_cn_d. \tag{3.7}$$

Proof. Using the definition of the parallel components,

$$-n_dX^c_\parallel\nabla_cY^d_\parallel = -X^c_\parallel\nabla_c(\cancel{n_dY^d_\parallel}) + X^c_\parallel Y^d_\parallel\nabla_cn_d = (h_a{}^ch_b{}^d\nabla_cn_d)X^aY^b. \tag{3.8}$$

□

Remark: $n^b\nabla_cn_b = \frac{1}{2}\nabla_c(n_bn^b) = 0$, so the second index is automatically a tangential index. We have

$$K_{ab} = h_a{}^c\nabla_cn_b. \tag{3.9}$$

Lemma 5: The extrinsic curvature is a symmetric tensor

$$K_{ab} = K_{ba}. \tag{3.10}$$

Proof. Let Σ be a surface of constant f, with $df|_\Sigma\neq 0$. Then there is some g (fixed by $n_an^a = -1$) such that $N_a|_\Sigma = g(df)_a$. Use this to extend n_a off Σ.

$$\nabla_cn_d = g\nabla_c\nabla_df + \nabla_cg\underbrace{\nabla_df}_{g^{-1}n_d} \;\Rightarrow\; K_{ab} = gh_a{}^ch_b{}^d\nabla_c\nabla_df. \tag{3.11}$$

□

Lemma 6: There is also a definition in terms of the Lie derivative

$$K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}, \tag{3.12}$$

with respect to n^a.

Proof. Exercise sheet 2. □

We think of the extrinsic curvature intuitively as how the manifold is bending. Of course we also have the intrinsic curvature defined by the metric. The question of how they are related is the matter of the following section.

3.3 Gauss–Codacci Equations

Proofs not examinable and will be skipped but can be found in lecture notes.

Definition 36 (invariant)**:** A tensor at $p \in \Sigma$ is *invariant under projection* $h^a{}_b$ if

$$T^{a_1 \dots a_r}{}_{b_1 \dots b_s} = h^{a_1}{}_{c_1} \dots h^{a_r}{}_{c_r} h^{d_1}{}_{b_1} \dots h^{d_s}{}_{b_s} T^{c_1 \dots c_r}{}_{d_1 \dots d_s} \tag{3.13}$$

can be identified with tensors on Σ.

Definition 37 (covariant derivative)**:** The *covariant derivative* on Σ is

$$D_a T^{b_1 \dots b_r}{}_{c_1 \dots c_s} = h_a{}^d h^{b_1}{}_{e_1} \dots h^{b_r}{}_{e_r} h^{f_1}{}_{c_1} \dots h^{f_s}{}_{c_s} \nabla_d T^{e_1 \dots e_r}{}_{f_1 \dots f_s} \tag{3.14}$$

Lemma 7: The covariant derivative D_a is the Levi–Civita connection associated with h_{ab}.

Claim 11: The Riemann tensor of D_a is given by *Gauss' equation*

$$R'^a{}_{bcd} = h^a{}_e h^f{}_b h^g{}_c h^h{}_d R^e{}_{fgh} - 2K_{[c}{}^a K_{d]b}. \tag{3.15}$$

Lemma 8: The Ricci scalar of D_a is

$$R' = R + 2R_{ab} n^a n^b - K^2 + K^{ab} K_{ab}, \tag{3.16}$$

where $K = K^a{}_a = g^{ab} K_{ab} = h^{ab} K_{ab}$, where the contraction can be done with either metric since it is purely tangential.

Claim 12: *Codacci's equation*

$$D_a K_{bc} - D_b K_{ac} = h_a{}^d h_b{}^e h_c{}^f n^g R_{defg}. \tag{3.17}$$

Lemma 9: Taking a contraction of this, we get

$$D_a K^a{}_b - D_b K = h^c{}_b R^{cd} n^d. \tag{3.18}$$

3.4 The Constraint Equations

Take Einstein's equations $G_{ab} = 8\pi T_{ab}$. Contract with n^a to get

$$G_{ab}n^a n^b = R_{ab}n^a n^b + \frac{1}{2}R. \tag{3.19}$$

Observe that we get this in Eq. (3.16). Can rewrite this as

$$G_{ab}n^a n^b = \frac{1}{2}(R' - K^{ab}K_{ab} + K^2), \tag{3.20}$$

rewritten in terms of intrinsic and extrinsic curvature of our hypersurface.

Then considering the right-hand side gives the *Hamiltonian constraint*

$$R' - K^{ab}K_{ab} + K^2 = 16\pi\rho, \tag{3.21}$$

where $\rho = T_{ab}n^a n^b$ is the energy density seen by an observer that moves with velocity n^a. We see K like a time derivative, so this is not an evolution equation. Instead, it is a constraint. This comes from the normal-normal components of the Einstein equation. Considering instead the

$$8\pi h_a{}^b T_{bc}n^c = h_a{}^b G_{bc}n^c = h_a{}^b R_{bc}n^c, \tag{3.22}$$

then Eq. (3.16) now tells us that

$$D_b K^b{}_a - D_a K = 8\pi h_a{}^b T_{bc}n^c, \tag{3.23}$$

where $h_a{}^b T_{bc}n^c$ is (minus) the momentum density seen by an observer with velocity n^a. This can be seen as an equation involving a space and a time derivative, rather than two time derivatives. It is therefore not an evolution equation. We call this the *momentum constraint*. The remaining equations have two time derivatives and give the time-evolution.

3.5 Initial Value Problem for GR

In the following section we will assume throughout that we have a triple of initial data (Σ, h_{ab}, K_{ab}), where Σ is a three-manifold, h_{ab} a Riemannian metric on Σ, and K_{ab} a symmetric tensor on Σ.

Theorem 10: Given such date in vacuum ($T_{ab} = 0$), there exists a (up to diffeomorphism) unique spacetime (M, g) , called the *maximal Cauchy development* of (Σ, h_{ab}, K_{ab}) , such that

i) (M, g) obeys the vacuum Einstein equation

ii) (M, g) is globally hyperbolic with Cauchy surface Σ, which means that we can predict the physics from the initial date

iii) the induced metric and extrinsic curvature of Σ are h_{ab} and K_{ab}

iv) any other spacetime that obeys conditions (i - iii) is isometric to a subset of (M, g).

Example 3.5.1: Let $\Sigma = \mathbb{R}^3$ and choose coordinates such that $h_{\mu\nu} = \delta_{\mu\nu}$ and $K_{\mu\nu} = 0$. This satisfies the constraints in vacuum. We can view Σ as a surface of constant time in $\mathbb{M}^4$, which is its maximal Cauchy development.

The spacetime (M, g) so obtained may be extendible. By property (iv), Σ cannot be Cauchy for the extended spacetime (M', g'). Therefore,

$$M = D(\Sigma) \supset M'. \tag{3.24}$$

This means that there exist Cauchy horizons for Σ in M'. We cannot predict g'_{ab} in $M' \setminus D(\Sigma)$ from data on Σ.

Example 3.5.2: Let $\Sigma = \{(x, y, z) \mid x > 0\}$ with $h_{\mu\nu} = \delta_{\mu\nu}$ and $K_{\mu\nu} = 0$. We have cut Σ in half compared to Example 3.5.1. The maximal Cauchy development is illustrated in Fig. 3.7. This is

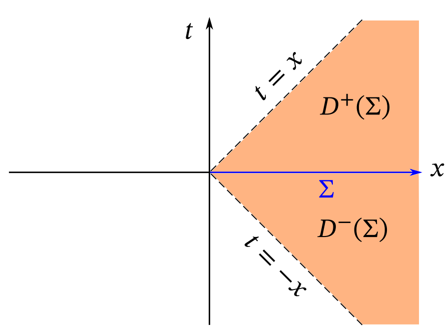


Figure 3.7: Maximal Cauchy development of the positive x-axis.

extendible to infinitely many spacetimes, all of which are flat in $D(\Sigma)$ but may disagree outside, for example by allowing the propagation of gravitational waves that do not enter $D(\Sigma)$. In this example, the *initial data* is extendible (to $x < 0$).

Example 3.5.3: Consider $M < 0$ Schwarzschild spacetime with metric

$$ds^2 = -\left(1 + \frac{2|M|}{r}\right) dt^2 + \left(1 + \frac{2|M|}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \tag{3.25}$$

This has a curvature singularity at $r = 0$, but no coordinate singularity. There is no black hole. Take $\Sigma = \{t = 0\}$, The induced metric is

$$h = \left(1 + \frac{2|M|}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \tag{3.26}$$

The extrinsic curvature can be calculated to vanish: $K_{ab} = 0$. Since there is a singularity at $r = 0$, (Σ, h_{ab}) is not geodesically complete. The maximal Cauchy development is *not* all of $M < 0$ Schwarzschild: it is not globally hyperbolic. The outgoing (ingoing) radial null geodesics is the

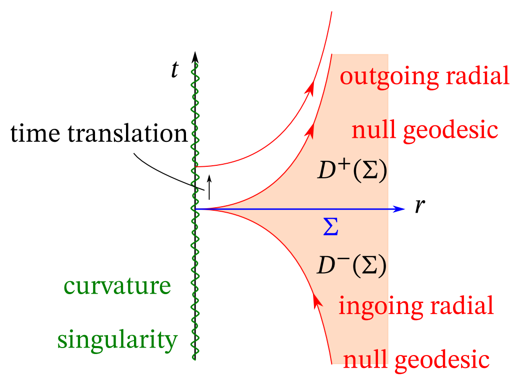


Figure 3.8

future (past) Cauchy horizon. Again the maximal Cauchy development is extendible, this time because the initial date is singular. The solution outside $D(\Sigma)$ need not be spherically symmetric (unlike $M < 0$ Schwarzschild).

Example 3.5.4: Take Σ to be the hyperboloid

$$\{-t^2 + x^2 + y^2 + z^2 = -1 \mid t < 0\} \subset \mathbb{M}^4. \tag{3.27}$$

Let h_{ab} and K_{ab} be the induced metric and extrinsic curvature. The situation is illustrated in Fig. 3.9 Again this is extendible. The solution does not need to be Minkowski outside $D(\Sigma)$. The reason for the extendibility this time is that Σ is *asymptotically null*, meaning that it is asymptotic to the orange lightcone. Information can arrive from infinity without intersecting Σ.

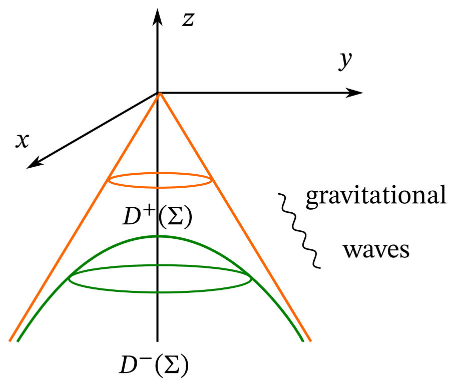


Figure 3.9

3.6 Asymptotically Flat Initial Data

Definition 38 (asymptotically flat ends)**:** (a) Initial data (Σ, h_{ab}, K_{ab}) is an *asymptotically flat end* if

(i) $\Sigma \simeq \mathbb{R}^3 \setminus B$, where B a closed ball centered on the origin in $\mathbb{R}^3$

(ii) if we pull-back $\mathbb{R}^3$ coordinates, we get coordinates x^i on Σ. The $h_{ij} = \delta_{ij} + O(\frac{1}{r})$ and $K_{ij} = O(\frac{1}{r^2})$ as $r \to \infty$ where $r = \sqrt{x^i x^i}$.

(iii) derivatives of these conditions hold. For example, $h_{ij,k} = O(1(r^2))$ and so forth.

(b) Initial data is *asymptotically flat with N ends* if it is the union of a compact set with N asymptotically flat ends.

Exercise 3.2: Consider $M > 0$ Schwarzschild with Σ a surface of constant t and $r > 2M$. Take h_{ab} and K_{ab} to be the induced data. This is an asymptotically flat end.

Example 3.6.1 (Einstein–Rosen bridge)**:** Consider Kruskal with Σ a surface of constant t, together with the induced data. Here, Σ is the union of two copies of Exercise 3.2 with compact set $\{U = V = 0\}$. This is asymptotically flat with two ends.

3.7 Strong Cosmic Censorship (CSS)

Conjecture 1 (Strong Cosmic Censorship)**:** Let (Σ, h_{ab}, K_{ab}) be geodesically complete, asymptotically flat initial data for the vacuum Einstein equation with N ends. Then, generically[1], the maximum Cauchy development is inextendible.

[1] Later we will find that rotating, charged black holes violate this. 'Generically' means that small perturbations to these special cases give inextendible Cauchy developments.

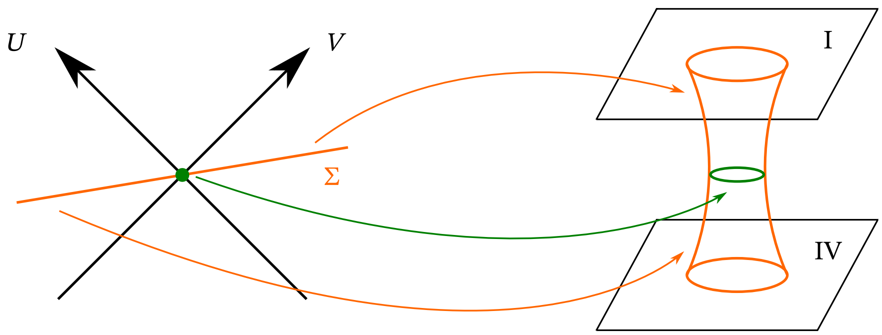


Figure 3.10

Christodoulou and Klainerman proved in 1994 that this holds for (Σ, h_{ab}, K_{ab}) close to data of the constant t surface in Minkowski. (The spacetime 'settles down' to Minkowski.)

4 The Singularity Theorem

Definition 39 (null hypersurface)**:** A hypersurface $\mathcal{N}$ is *null* if its normal n_a is null everywhere.

Example 4.0.1: Consider a surface of constant r in Schwarzschild spacetime. Its normal is $n = dr$. In ingoing Eddington–Finkelstein coordinates, we have

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 - \frac{2M}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix} \tag{4.1}$$

Now the norm of n_a is

$$g^{\mu\nu} n_\mu n_\nu = g^{rr} = 1 - \frac{2M}{r}. \tag{4.2}$$

Thus we can see that the hypersurface defined by $r = 2M$ has a normal n_a that is null everywhere; $r = 2M$ defines a null hypersurface.

$$n^\mu = g^{\mu\nu} n_\nu = g^{\mu r} \;\Rightarrow\; n^a|_{r=2M} = \left(\frac{\partial}{\partial v}\right)^a. \tag{4.3}$$

Claim 13: The integral curves of n^a are null geodesics, the *generators* of $\mathcal{N}$.

Proof. Assume $\mathcal{N}$ is a surface of constant f with $df|_{\mathcal{N}} \neq 0$. Therefore $n = hdf$ for some h. Write $N = df$, which has the same integral curves as n. Then $N_a N^a|_{\mathcal{N}} = 0$, which implies that

$$\nabla_a(N_b N^b)|_{\mathcal{N}} = 2\alpha N_a \tag{4.4}$$

for some $\alpha : \mathcal{N} \to \mathbb{R}$.

$$\nabla_a N_b = \nabla_a \nabla_b f = \nabla_b \nabla_a f = \nabla_b N_a. \tag{4.5}$$

$$\therefore \nabla_a(N_b N^b) = 2N^b \nabla_a N_b = 2N^b \nabla_b N_a. \tag{4.6}$$

So (4.4) is the geodesic equation

$$N^b \nabla_b N_a|_{\mathcal{N}} = \alpha N_a. \tag{4.7}$$

□

Example 4.0.2: Consider Kruskal with $N = dU$. This is *globally null* ($g^{UU} = 0$) normal to a *family* of null hypersurfaces with constant U. In this case we can get a stronger result than we obtained in Cl. 13.

$$N^b = \nabla_b N_a = N^b \nabla_b \nabla_a U = N^b \nabla_a \nabla_b U = N^b \nabla_a N_b = \frac{1}{2}\nabla_a(\underbrace{N_b N^b}_{\equiv 0}) = 0. \tag{4.8}$$

This means that N^a is tangent to affinely parametrised geodesic. For example, we have

$$N^a = \frac{r}{16M^3} e^{\frac{r}{2M}} \left(\frac{\partial}{\partial V}\right)^a. \tag{4.9}$$

Let $\mathcal{N}$ be a surface of constant $U = 0$. Then $r = 2M$ gives $N^a|_{\mathcal{N}}$ constant multiple of $(\frac{\partial}{\partial V})^a$. This means that V is an affine parameter for generators of $\{U = 0\}$. Similarly, U is an affine parameter for generators of $\{V = 0\}$.

4.1 Geodesic Deviation

Consider a one-parameter family of geodesics $x^\mu(s, \lambda)$, where s labels the geodesics and λ is the affine parameter.

Let $U^\mu = \frac{\partial x^\mu}{\partial \lambda}$ be the tangent vectors to the geodesics and $S^\mu = \frac{\partial x^\mu}{\partial s}$ be the deviation vector. We have $[S, U] = 0$, which means that $U^b \nabla_b S^a = S^b \nabla_b U^a$. This gives the geodesic deviation equation

$$U^c \nabla_c(U^b \nabla_b S^a) = R^a{}_{bcd} U^b U^c S^d, \tag{4.10}$$

which we have already seen in last term's *General Relativity* course.

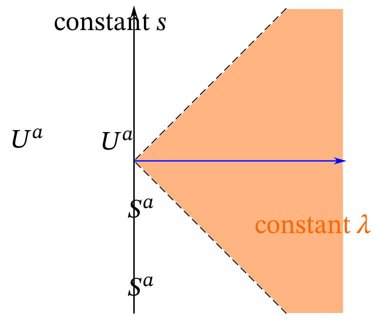


Figure 4.1: One-parameter family of geodesics.

4.2 Geodesic Congruences

Definition 40: Let $\mathcal{U} \subset M$. A *geodesic congruence* in $\mathcal{U}$ is a family of geodesics such that exactly one passes through each point $p \in \mathcal{U}$.

Assume all geodesics are of the same time (timelike/spacelike/null). Then $U^2 = \pm 1$ or $U^2 \equiv 0$ since U^a is tangent to the geodesics.

The one-parameter family in a congruence satisfies

$$U^b \nabla_b S^a = B^a{}_b S^b, \qquad B^a{}_b = \nabla_b U^a. \tag{4.11}$$

Then $B^a{}_b = 0$ since U^b is affinely parametrised.

$$U_a B^a{}_b = U_a \nabla_b U^a = \frac{1}{2} \nabla_b (\underbrace{U_a U^a}_{\text{const}}) = 0 \tag{4.12}$$

$$U \cdot \nabla(U \cdot S) = \cancel{(U \nabla U^a)} S_a + U^a U \cdot \nabla S_a = U^a B_{ab} S^b = 0. \tag{4.13}$$

Therefore, $U \cdot S$ is constant along each geodesic in the congruence. Let $\lambda' = \lambda - a(s)$, then $S'^a = S^a + \frac{\mathrm{d}a}{\mathrm{d}s} U^a$.

Exercise 4.1: The deviation vector gives the same geodesic; we have a gauge-freedom in choosing our deviation vector.

$$U \cdot S' = U \cdot S + \frac{\mathrm{d}a}{\mathrm{d}s} U^2 \tag{4.14}$$

For the case of timelike/spacelike, we have $U^2 = \pm 1$. We can then choose $a(s)$ such that the right-hand side vanishes at $\lambda = 0$ on each geodesic. This means that $U \cdot S' = 0$ at $\lambda = 0$, but $U \cdot S'$ is constant. Thus $U \cdot S' \equiv 0$.

For the null congruences we have to work harder.

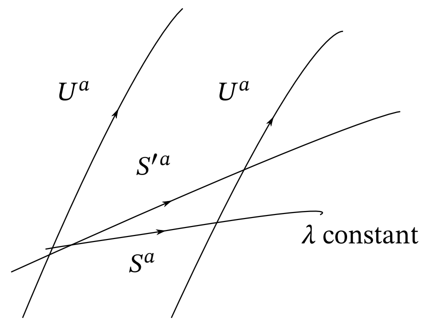


Figure 4.2

4.3 Null Geodesic Congruences

Pick N^a such that $N^2|_\Sigma = 0$, $N \cdot U|_\Sigma = -1$. Extend N^a off Σ by demanding parallel transport

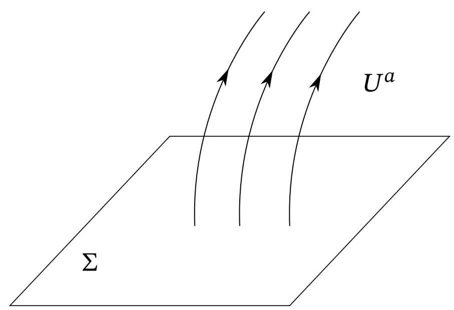


Figure 4.3

$U \cdot \nabla N^a = 0$.

Exercise 4.2: This gives $N^2 \equiv 0$ and $N \cdot U \equiv -1$.

However, N is not uniquely defined. Now

$$S^a = \alpha U^a + \beta N^a + \hat{S}^a, \tag{4.15}$$

where $U \cdot \hat{S} = N \cdot \hat{S} = 0$, meaning that $\hat{S}^a$ is spacelike. Let us now look at the scalar product

$$U \cdot S = -\beta. \tag{4.16}$$

This means that β is constant along each geodesic. Therefore, we can rearrange our expression to be

$$S^a = \underbrace{\alpha U^a + \hat{S}^a}_{\perp U^a} + \overbrace{\beta N^a}^{\text{parallel transported}} \tag{4.17}$$

Example 4.3.1: Suppose we have a congruence containing the generators of a null hypersurface $\mathcal{N}$.

4.4 Expansion and Shear of Null Hypersurface

As before, let us look at a null geodesic congruence that contains the generators of our hypersurface $\mathcal{N}$, where $\hat{\omega}_{ab}|_{\mathcal{N}} = 0$.

Since this is a three-dimensional hypersurface, we will draw Fig. 4.4.

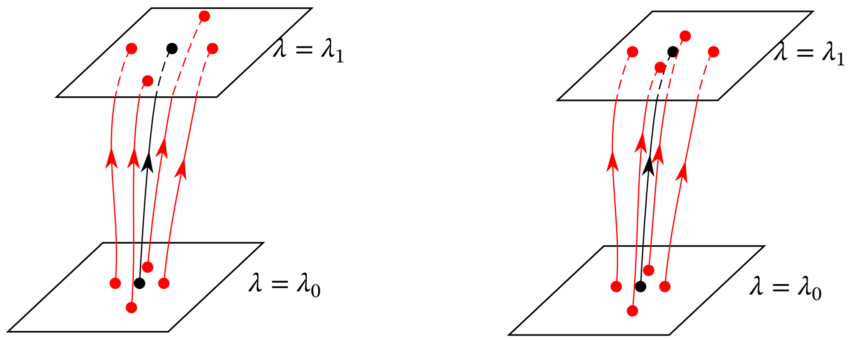


Figure 4.4: The expansion effect of $\theta > 0$ (left) and the shearing effect of $\hat{\sigma}_{ab}$ (right).

4.4.1 Gaussian Null Coordinates

Construct *Gaussian null coordinates* near $\mathcal{N}$ in the following way. Let S be a spacelike surface inside $\mathcal{N}$ with coordinates y^i, $i = 1, 2$. Assign coordinates (λ, y^i) to a point, which lies an affine parameter distance λ along the geodesic of $\mathcal{N}$ that intersects S at y^i. This gives coordinates λ, y^i on $\mathcal{N}$ such that $U^a = \left(\frac{\partial}{\partial\lambda}\right)^a$.

Pick a null V^a on $\mathcal{N}$ such that $V \cdot \frac{\partial}{\partial y^i} = 0$ and $V \cdot U = 1$. Assign coordinates (r, λ, y^i) to a point lying an affine parameter r along the null geodesic starting at $(\lambda, y^i) \in \mathcal{N}$ with tangent V^a there.

By definition, $\mathcal{N}$ is the surface of $r = 0$. Then

$$U^a|_{\mathcal{N}} = \left(\frac{\partial}{\partial\lambda}\right)^a, \qquad V^a|_{\mathcal{N}} = \left(\frac{\partial}{\partial r}\right)^a, \tag{4.18}$$

and $\frac{\partial}{\partial r}$ is tangent to affinely parametrised null geodesics. Thus $g_{rr} = 0$.

Exercise 4.3: Show that the geodesic equation for $\frac{\partial}{\partial r}$ implies $g_{r\mu,r} = 0, \forall\mu$.

On $\mathcal{N}$, where $r = 0$,

$$g_{r\lambda} = U \cdot V = 1 \qquad g_{ri} = V \cdot \frac{\partial}{\partial y^i} = 0. \tag{4.19}$$

Since these equations do not depend on r , they must hold everywhere! We write this as

$$g_{r\lambda} \equiv 1 \qquad g_{ri} \equiv 0. \tag{4.20}$$

Moreover, the following components of the metric vanish

$$g_{\lambda\lambda} = U^2 = 0 \qquad g_{\lambda i} = U \cdot \frac{\partial}{\partial y^i} = 0. \tag{4.21}$$

Thus, we find that

$$g_{\lambda\lambda} = rF \qquad g_{\lambda i} = rh_i, \tag{4.22}$$

where F and h_I are smooth. Putting everything together, we find that the metric is

$$ds^2 = 2drd\lambda + rFd\lambda^2 + 2rh_i d\lambda y^i + h_{ij}dy^i dy^j. \tag{4.23}$$

These are the *Gaussian null coordinates.*

These coordinates are very nice when working on $\mathcal{N}$, since

$$g|_{\mathcal{N}} = 2drd\lambda + h_{ij}dy^i dy^j. \tag{4.24}$$

On our hypersurface, $U^\mu|_{\mathcal{N}} = (0,1,0,0)$ and $U_\mu|_{\mathcal{N}} = (1,0,0,0)$.

Since $U \cdot B = B \cdot U = 0$, these expressions give $B^r{}_\mu = B^\mu{}_\lambda = 0$ on $\mathcal{N}$. We can now calculate the expansion on $\mathcal{N}$ to be

$$\theta|_{\mathcal{N}} = B^\mu{}_\mu = B^i{}_i = \nabla_i U^i = \partial_i U^i + \Gamma^i_{i\mu} U^\mu = \Gamma^i_{i\lambda}, \tag{4.25}$$

where the last equality holds since ∂_i is tangential and U^i vanishes on $\mathcal{N}$. The definition of the Christoffel symbols is

$$\Gamma = \frac{1}{2} g^{i\mu}(g_{\mu i,\lambda} + g_{\mu\lambda,i} - g_{i\lambda,\mu}). \tag{4.26}$$

Now $g^{i\mu} = 0$ unless $\mu = j$. Therefore, the inverse is $g^{ij}|_{\mathcal{N}} = h^{ij}$, the inverse of h_{ij}. Therefore, we find

$$\theta|_{\mathcal{N}} = \frac{1}{2} h^{ij}(g_{ji,\lambda} + \cancel{g_{j\lambda,i}} - \cancel{g_{i\lambda,j}}) = \frac{1}{2} h^{ij} h_{ij,\lambda} = \frac{1}{\sqrt{h}} \partial_\lambda \sqrt{h}, \tag{4.27}$$

where we write as usual $h = \det h_{ij}$.

We interpret the equation

$$\frac{\partial}{\partial\lambda}\sqrt{h} = \theta\sqrt{h} \tag{4.28}$$

by recognising that $\sqrt{h}$ is the area element on surfaces of constant λ inside $\mathcal{N}$. Hence θ measures the rate of increase of area with respect to the affine parameter.

4.5 Trapped Surfaces

Consider a two-dimensional spacelike surface S. If $p \in S$, we can find two future-directed null vectors $U^a_{1,2}$, defined up to an overall factor due to the freedom in choosing the affine parameter, orthogonal to S . Suppressing one of the dimensions of this two-dimensional surface, we draw this as Fig. 4.5 As illustrated, this gives two families of null geodesics starting on S perpendicular to S.

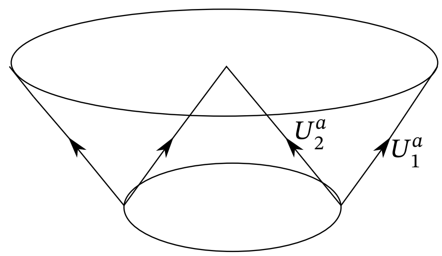


Figure 4.5

Hence we have two null hypersurfaces $\mathcal{N}_{1,2}$.

Example 4.5.1: Let S be the two-sphere and $U = U_0, V = V_0$ in Kruskal spacetime.. The generators of N_i are radial null geodesics. $\mathcal{N}_1$ has tangent $U_1 \propto dU$, which means $U^a_i = re^{\frac{r}{2M}}\left(\frac{\partial}{\partial V}\right)^a$. Similarly, $U^a_2 = re^{\frac{r}{2M}}\left(\frac{\partial}{\partial V}\right)^a$.

$$\theta_1 = \nabla_a U^a_1 = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}U^\mu_1) = r^{-1}e^{\frac{r}{2M}}\partial_V\left(re^{-\frac{r}{2M}}re^{\frac{r}{2M}}\right) = 2e^{\frac{r}{2M}}\partial_V r. \tag{4.29}$$

Now $r = r(U, V)$, so

$$\theta_1 = -\frac{8M^2}{r}U \qquad \text{and similarly} \qquad \theta_2 = -\frac{8M^2}{r}V. \tag{4.30}$$

On S, we have $U = U_0, V = V_0$. For region $I \supset S$, we have $\theta_1 > 0$ and $\theta_2 < 0$. The *outgoing* null geodesics are expanding and the *ingoing* contracting. For Region $IV \supset S$, $\theta_1 < 0$ and $\theta_2 > 0$, where ingoing and outgoing are flipped. In $II \supset S$, $\theta_1, \theta_2 < 0$ and both families are converging. Similarly for $III \supset S$ $\theta_1, \theta_2 > 0$ both families are expanding.

Definition 41 (traped)**:** A compact, orientable spacelike 2-surface S is *trapped* if both families of null geodesics perpendicular to S have $\theta < 0$ everywhere on S (*marginally trapped* for $\theta \leq 0$).

Example 4.5.2 (Kruskal)**:** The surface of $U = U_0, V = V_0$ in II is trapped. The event horizon $U_0 = 0$ and $V_0 > 0$ is marginally trapped.

4.6 Raychandhuri Equation

Claim 14: The derivative of the expansion with respect to the affine parameter is

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^{ab}\hat{\sigma}_{ab} + \hat{\omega}^{ab}\hat{\omega}_{ab} - R_{ab}U^aU^b \tag{4.31}$$

Proof. By the definition of θ,

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = U \cdot \nabla({B^a}_b {P^b}_a). \tag{4.32}$$

We can pull out P since it is parallelly transported

$${P^b}_a U \cdot \nabla {B^a}_b = P^b_a U^c \nabla_c \nabla_b U^a. \tag{4.33}$$

Using the definition of the Riemann tensor, we can commute the derivatives

$$\begin{aligned} \ldots &= P^b_a U^c (\nabla_b \nabla_c U^a + {R^a}_{dcb} U^d) & (4.34)\\ &= {P_a}^b \left[\nabla_b (U^c \nabla_c U^a) - (\nabla_b U^c)\nabla_c U^a\right] + P^b_a {R^a}_{dcb} U^c U^d. & (4.35) \end{aligned}$$

Now the first term in the bracket vanishes identically $U^c\nabla_c U^a \equiv 0$. The remaining terms can be rewritten using the definition of P as

$$\cdots = -{B^c}_b {P^b}_a {B^a}_c - R_{cd} U^c U^d. \tag{4.36}$$

Using the definition of $\hat{B}$, we can show (exercise) that

$$\cdots = -{\hat{B}^c}{}_a {\hat{B}^a}{}_c - R_{ab} U^a U^b. \tag{4.37}$$

Using the expansion of $\hat{B}$ in terms of expansion, rotation and shear, we find the Raychandhuri equation. □

4.7 Energy Conditions

The energy-momentum tensor should reflect physically reasonable matter. What do we mean by this? Any observer measures an *energy-momentum current* $j^a = -{T^a}_b u^b$, where u^b is the observer's 4-velocity. A natural criterion is that no observer seeing energy moving faster than the speed of light.

Dominant energy condition: Contracting the energy momentum tensor with a future-directed timelike vector field V^a, we have $-{T^a}_b V^a$, which should be future-direct causal (or zero).

Claim 15: If $T_{ab} = 0$ in some subset S of the initial data surface Σ, then the dominant energy condition implies that $T_{ab} \equiv= 0$ in the domain of dependence $D^+(S)$.

Proof. Hawking and Ellis. □

Weak energy condition: $T_{ab}V^aV^b \geq 0$ for all causal V^a.

Null energy condition: $T_{ab}V^aV^b \geq 0$ for all null V^a.

$$\text{DEC} \;\Rightarrow\; \text{WEC} \;\Rightarrow\; \text{NEC} \tag{4.38}$$

Strong energy condition: $(T_{ab} - \frac{1}{2}T^c{}_c g_{ab})V^aV^b \geq 0$ for all causal V^a.

If this is imposed and the Einstein equation is satisfied, then $R_{ab}V^aV^b \geq 0$. This is the statement that "gravity is attractive"; geodesics converge. The SEC does not imply the DEC.

4.8 Penrose Singularity Theorem

Lemma 11: Suppose the spacetime (M, g) satisfies the Einstein equation and the null energy condition. The generators of a null hypersurface obey

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \leq -\frac{1}{2}\theta^2. \tag{4.39}$$

Proof. This is a consequence of the Raychandhuri equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^{ab}\hat{\sigma}_{ab} + \hat{\omega}^{ab}\hat{\omega}_{ab} - R_{ab}U^aU^b. \tag{4.40}$$

Since the expansion of $\hat{\sigma}$ is in terms of a spacelike basis $T_\perp$, we have $\hat{\sigma}^{ab}\hat{\sigma}_{ab} \geq 0$. And $\hat{\omega}_{ab} = 0$.

The null energy condition implies

$$0 \leq 8\pi T_{ab}U^aU^b = (R_{ab} - \frac{1}{2}Rg_{ab})U^aU^b = R^{ab}U^aU^b, \tag{4.41}$$

since U is null. □

Corollary: If $\theta = \theta_0 < 0$ at a point p on the generator γ of the null hypersurface, then $\theta \to -\infty$ along γ within affine parameter distance $2/|\theta_0|$ (provided the generator extends this far).

Proof. Choose an affine parameter that has $\lambda = 0$ at p. Equation (4.39) gives an inequality, which we can integrate with respect to λ

$$\frac{\mathrm{d}\theta^{-1}}{\mathrm{d}\lambda} \geq \frac{1}{2} \;\Rightarrow\; \theta^{-1}\theta_0^{-1} \geq \frac{1}{2}\lambda \;\Rightarrow\; \theta \leq \frac{\theta_0}{1 + \lambda\theta_0/2}. \tag{4.42}$$

For $\theta_0 < 0$, the right-hand side diverges to $-\infty$ as $\lambda \to 2/|\theta_0|$. □

Theorem 12 (Penrose 1965)**:** Let (M, g) be a globally hyperbolic spacetime with a non-compact Cauchy surface Σ. Assume the Einstein equation and the null energy condition. Assume further that there exist a trapped surface $T \subset M$. Let $\theta_0 < 0$ be the maximum value of the expansion of both families of null geodesics orthogonal to T.

Then at least one of these geodesics is future-inextendible with affine length $\leq 2/|\theta_0|$.

Let us discuss the consequences of this. Trapped surfaces are very common. There is a numerical argument, but also a mathematical argument called 'Cauchy stability'.

Remark: This theorem basically tells us that whenever we have a trapped surface we expect (assuming SCC) a singularity to show up. It does not tell us anything about the nature of this singularity, or anything about black holes. There is also the Weak Cosmic Censorship conjecture that states that any singularity forms inside a black hole.

5 Asymptotic Flatness

We have discussed asymptotic flatness of initial data; we now want to discuss asymptotic flatness of spacetime.

5.1 Conformal Compactification

On (M, g) let $\overline{g} = \Omega^2 g$, where $\Omega : M \to \mathbb{R}$ and $\Omega > 0$. We want to choose Ω to understand the structure of g near infinity. We want $\Omega \to 0$ at infinity.

Choose Ω such that $(M, \overline{g})$ is extendible to $(\overline{M}, \overline{g})$. The boundary ∂M of M in $\overline{M}$ is such that

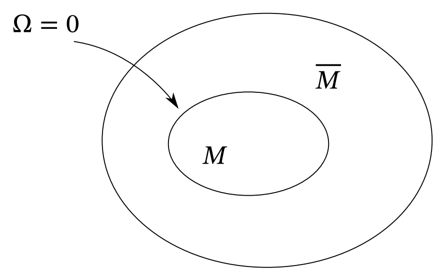


Figure 5.1

$\Omega|_{\partial M} = 0$.

Example 5.1.1: Consider $\mathbb{M}^4$ with $g = -dt^2 + dr^2 + r^2 d\omega^2$, where $d\omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Then $(\overline{M}, \overline{g})$ is the Einstein static universe $\mathbb{R} \times S^3$ with

$$\overline{g} = -dT^2 + d\chi^2 + \sin^2\chi d\omega^2. \tag{5.1}$$

Suppress S^2 to obtain the Penrose diagram. Each point is a two-sphere S^2, boundary is axis of symmetry ($r = 0$) or at ∞ with respect to g (or singular). Radial null geodesics are lines at 45° .

Penrose diagram for the Kruskal diagram: Let $P = P(U)$, $Q = Q(V)$ such that $P, Q \in (-\frac{\pi}{2}, \frac{\pi}{2})$, say.

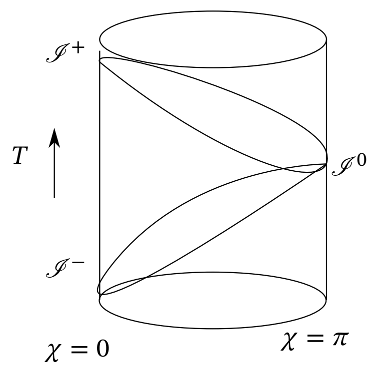


Figure 5.2

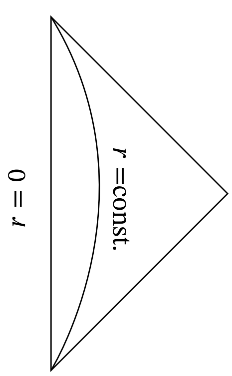


Figure 5.3: Penrose diagram of Minkowski space $\mathbb{M}^4$ (GEODESICS MISSING).

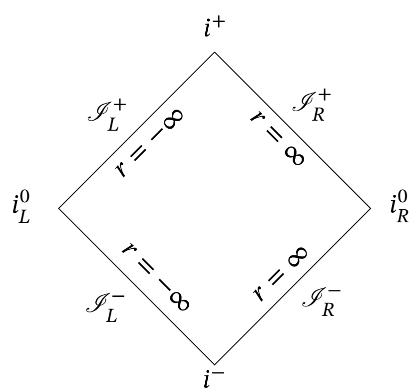


Figure 5.4: Penrose diagram of $\mathbb{M}^2$.

Find Ω such that $(M, \overline{g})$ is extendible to $(\overline{M}, \overline{g})$. The boundary ∂M as 4 components P or Q is $\pm\frac{\pi}{2}$ (U or V is $\pm\infty$). Thus we have $\mathscr{I}^+, \mathscr{I}^-$ in I and $\mathscr{I}^+, \mathscr{I}^-$ in IV.

The Penrose diagram is depicted in Fig. 5.5.

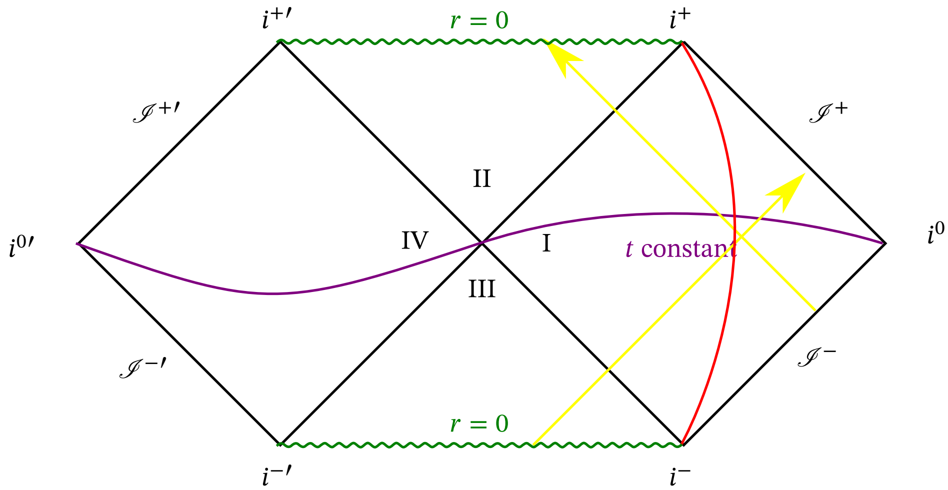


Figure 5.5: The surfaces of constant r are depicted in red, while radial null geodesics are in yellow.

The extended metric $\overline{g}$ is singular at $\mathscr{I}^{\pm}, \mathscr{I}^{\pm\prime}$ and not smooth at $\mathscr{I}^0, \mathscr{I}^{0\prime}$.

5.2 Asymptotic Flatness

Definition 42 (manifold with boundary)**:** A *manifold with boundary* has charts $\phi : M \to \mathbb{R}^n/2$. The boundary ∂M is the set of points with $x' = 0$ in some chart.

$$\partial M = \{(x^1, \dots, x^n) \mid x^1 \leq 0\}. \tag{5.2}$$

Definition 43 (asymptitically flat at null infinity)**:** A time orientable manifold (M, g) is *asymptotically flat at null infinity* if there is $(\overline{M}, \overline{g})$ such that

1. There is $\Omega : M \to \mathbb{R}$ with $\Omega > 0$ such that $(\overline{M}, \overline{g})$ is an extension of $(M, \Omega^2 g)$. (We regard $M < \overline{M}$ and $\overline{g} = \Omega^2 g$ on M.)
2. Can extend M within $\overline{M}$ to obtain a manifold with boundary $M \cup \partial M$.
3. Ω extends to a function on $\overline{M}$ such that $\Omega|_{\partial M} = 0$ and $d\Omega|_{\partial M} \neq 0$.
4. ∂M is the disjoint union[1] of $\mathscr{I}^+, \mathscr{I}^-$, each of which are diffeomorphic to $\mathbb{R} \times S^2$.
5. No past (future) directed causal curve starting in M intersects $\mathscr{I}^+$ ($\mathscr{I}^-$).
6. $\mathscr{I}^\pm$ are "complete".

Remark: Points $1 - 3$ say that there is a conformal compactification of the manifold, ensuring that the spacetime approaches Minkowski at the appropriate rate. The other points ensure that the spacetime has the same structure as $\mathbb{M}$, with $\mathscr{I}^+$ lying to the future and $\mathscr{I}^-$ lying to the past of M.

Example 5.2.1 (Schw.)**:** For $\mathscr{I}^+$, use outgoing Eddington–Finkelstein coordinates (u, r, θ, ϕ). $\mathscr{I}^+$ is $r \to \infty$, u finite.

$$r = \frac{1}{x} \implies g = -(1 - 2Mx)du^2 + \frac{2dudx}{x^2} + \frac{1}{x^2}d\omega^2. \tag{5.3}$$

Let $\Omega = x$ to multiply by x^2:

$$\overline{g} = -x^2(1 - 2Mx)du^2 + 2dudx + d\omega^2. \tag{5.4}$$

This extends across $\{x = 0\} := \mathscr{I}^+$ parametrised by $\{u, \theta, \phi\}$. This means that $\mathscr{I}^+$ is indeed diffeomorphic to $\mathbb{R} \times S^2$. Similarly for $\mathscr{I}^-$ we use the ingoing EF coordinates. Importantly, the area radius function r is the same. Therefore, exactly the same choice of Ω gives us an extension to $\mathscr{I}^-$.

Exercise 5.1 (Sheet 2)**:**

$$R_{ab} = \overline{R}_{ab} + 2\Omega^{-1}\overline{\nabla}_a\overline{\nabla}_b\Omega + \overline{g}_{ab}\overline{g}^{cd}\left(\Omega^{-1}\overline{\nabla}_c\overline{\nabla}_c\Omega - 3\Omega^{-2}\partial_c\Omega\partial_d\Omega\right). \tag{5.5}$$

[1] This just means that their intersection is empty.

Assume $R_{ab} = 0$. Then

$$0 = \Omega\overline{R}_{ab} + 2\overline{\nabla}_a\overline{\nabla}_b\Omega + \overline{g}_{ab}\overline{g}^{cd}\left(\overline{\nabla}_c\overline{\nabla}_c\Omega - 3\Omega^{-1}\partial_c\Omega\partial_d\Omega\right). \tag{5.6}$$

Since the first three summands are smooth at $\Omega = 0$, i.e. at $\mathscr{I}^{\pm}$, the fourth term

$$\Omega^{-1}\overline{g}^{cd}\partial_c\Omega\partial_d\Omega \tag{5.7}$$

is also smooth at $\Omega = 0$. Therefore, at $\mathscr{I}^{\pm}$, we have

$$\overline{g}^{cd}\partial_c\Omega\partial_d\Omega = 0 \tag{5.8}$$

But $d\Omega$ is normal to $\mathscr{I}^{\pm}$. Therefore, $g^{\pm}$ are *null hypersurfaces* in the unphysical spacetime $(\overline{M}, \overline{g})$.

The point 6. "complete" means that the generators of $\mathscr{I}^{\pm}$ are complete.

Example 5.2.2: Consider the null generators of (5.4). Let $n = d\Omega = dx$ be the normal to $\mathscr{I}^{+}$.

Exercise 5.2: Show that on $\mathscr{I}^{+}$, n is tangent to affinely parametrised geodesics

$$n^b\nabla_b n_a|_{x=0} = 0 \qquad n^a|_{x=0} = \left(\frac{\partial}{\partial u}\right)^a . \tag{5.9}$$

This implies that u is an affine parameter along the generators of $\mathscr{I}^{+}$. This extends to $\pm\infty$, which means it is complete.

5.3 Definition of a Black Hole

(We are going to use some definitions that we did not introduce in these lectures. They can be found in the printed notes in Section 4.11.)

Definition 44: Let (M, g) be time orientable and pick some subset $U \subset M$. The *choronological future* of U is the set of points $I^+(U)$ that you can reach along a future-directed timelike curve from U to p.

Definition 45: The *causal future* $J^+(U)$ is the set of points reachable along a future-directed causal curve from U, but it also includes all of U as well.

We define similarly the *chronological / causal past* $I^-(U), J^-(U)$.

We also need to introduce some topological notions

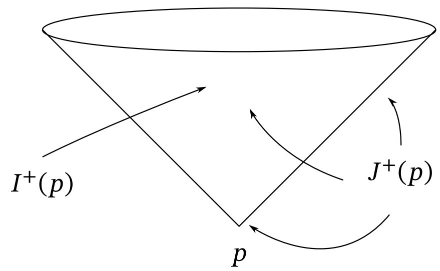


Figure 5.6

Definition 46 (open)**:** A set $S \subset M$ is *open* if $\forall p \in S$ there exists a neighbourhood[1] U of p such that $V \subset S$ $I^{\pm}(U)$ are open.

Definition 47 (closure)**:** The *closure* $\overline{S}$ of S is the union of S with all its limit points.

Example 5.3.1 (Mink)**:** In Minkowski space, $\overline{I^{\pm}(p)} = J^{\pm}(p)$, so $J^{\pm}(p)$ is closed. This is not true in general, as is illustrated in Fig. 5.7 for $\mathbb{M}^2$.

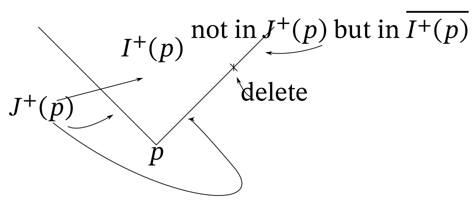


Figure 5.7: 2d Minkowski.

Definition 48 (interior point)**:** A point $p \in S$ is an *interior point* of S if $\exists$ a neighbourhood V of p such that $V \subset S$

$$\text{int}(S) = \{\text{interior points of } S\} \qquad S \text{ open} \iff S = \text{int}(S) \tag{5.10}$$

Definition 49 (boundary)**:** The *boundary* of S is

$$\dot{S} = \overline{S} \setminus \text{int}(S). \tag{5.11}$$

Remark: This is a different notion of boundary than we had for manifolds.

For $\mathscr{I}^+ \supset \overline{M}$ then we can define $J^-(\mathscr{I}^+) \subset \overline{M}$. Region of M that can send signal to $\mathscr{I}^+$ is $M \cap J^-(\mathscr{I}^+)$.

Definition 50 (black hole)**:** Let (M, g) be asymptotically flat at null ∞. The *black hole region* is

$$\mathcal{B} = M \setminus \left[M \cap J^-(\mathscr{I}^+)\right], \tag{5.12}$$

[1] defined using the chart

where $J^-(\mathscr{I}^+)$ defined with respect to $(\overline{M}, \overline{g})$. Its boundary $\mathcal{H}^+ = \dot{\mathcal{B}}(= M \cap \dot{J}^-(\mathscr{I}^+))$ is the *future event horizon*.

Definition 51 (white hole)**:** Analogously we define the *white hole region* $\mathcal{W} = M \setminus \left[M \cap J^+(\mathscr{I}^-)\right]$ with *past event horizon* $\mathcal{H}^- = \dot{\mathcal{W}}(= M \cap \dot{J}^+(\mathscr{I}^-))$.

Example 5.3.2 (Kruskal)**:** The black hole region is $\mathcal{B} = II \cup IV$ (including $U = 0$) with future event horizon $\mathcal{H}^+ = \{U = 0\}$. The white hole region is $\mathcal{W} = III \cup IV$ (including $V = 0$) with future event horizon $\mathcal{H}^- = \{V = 0\}$.

Claim 16: Can show that $\mathcal{H}^\pm$ are null hypersurfaces. Furthermore, the generators of $\mathcal{H}^+$ cannot have future end points; light rays skimming the future end horizon cannot leave it. However, they can have past endpoints, an example of which is shown in Fig. 5.8.

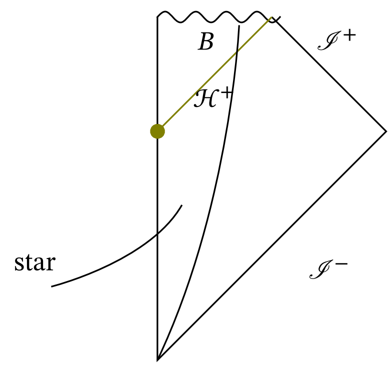


Figure 5.8: The marked point is the past endpoint of generators of $\mathcal{H}^+$.

Definition 52: An *asymptotically flat spacetime* (M, g) is *strongly asymptotically predictable* if there exists and open unphysical $\bar{V} \subset \overline{M}$ such that the closure

$$\overline{M \cap J^-(\mathscr{I}^+)} \supset \bar{V} \tag{5.13}$$

and $(\overline{V}, \overline{g})$ is globally hyperbolic.

Remark: This means that "the physics is predictable on and outside $\mathcal{H}^+$".

Theorem 13: Let (M, g) be strongly asymptotically predictable. Let Σ_1 and Σ_2 be surfaces for $\overline{V}$ such that $\Sigma_2 \subset I^+(\Sigma_1)$. Let B be a connected component of $\mathcal{B} \cap \Sigma_1$. Then $J^+(B) \cap \Sigma_2$ is contained within a connected component of $\mathcal{B} \cap \Sigma_2$.

Remark: $\mathcal{B} \cap \Sigma_i$ is a single "Black hole region at time Σ_1". This theorem says that black holes do not bifurcate.

Proof. Global hyperbolicity implies that every causal curve from Σ_1 intersects Σ_2, since they are both Cauchy surfaces. By definition, the causal future of any point in the black hole region must be in the black hole region, so $J^+(B) \subset \mathcal{B}$. Let us look at the intersection $J^+(B) \cap \Sigma_2 \subset \mathcal{B} \cap \Sigma_2$. We claim that the left-hand side lies entirely in a single connected component of the right. Assume for contradiction that this is not the case. Then, as shown in Fig. 5.9, there exist open $\mathcal{O}, \mathcal{O}'$ such that

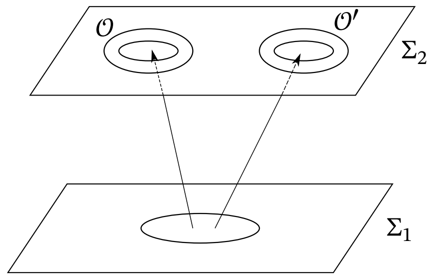


Figure 5.9

$$J^+(B) \cap \Sigma_2 \subset \mathcal{O} \cup \mathcal{O}', \qquad \mathcal{O} \cap \mathcal{O}' = \emptyset \tag{5.14}$$

$$\text{and} \qquad J^+(B) \cap \mathcal{O} \neq \emptyset \qquad J^+(B) \cap \mathcal{O}' \neq \emptyset \tag{5.15}$$

$$B \cap I^-(\mathcal{O}) \neq \emptyset \qquad B \cap I^-(\mathcal{O}') \neq \emptyset \tag{5.16}$$

$$B \subset I^-(\mathcal{O}) \cup I^-(\mathcal{O}') \tag{5.17}$$

If $p \in B \cap I^-(\mathcal{O})$ and $p \in B \cap I^-(\mathcal{O}')$, then we can divide future-directed timelike geodesics from p into 2 sets according to whether they go to $\mathcal{O}$ or $\mathcal{O}'$. Therefore, we can divide future-directed timelike vectors at p into two disjoint open sets. This contradicts the connectedness of the future light-cone at p. Thus, $B \cap I^-(\mathcal{O})$ and $B \cap I(\mathcal{O}')$ have empty intersection. Hence, $B = [B \cap I^-(\mathcal{O})] \cup [B \cap I^-(\mathcal{O}')]$ is a disjoint union, which contradicts the connectedness of B. □

Definition 53 (future Cauchy horizon)**:** The *future Cauchy horizon* of a partial Cauchy surface Σ is $H^+(\Sigma) = \overline{D^+(\Sigma)} \setminus I^-[D^+(\Sigma)]$.

We similarly define $H^-(\Sigma)$ and $\dot{D}(\Sigma) = H^+(\Sigma) \cup H^-(\Sigma)$. The $H^\pm(\Sigma)$ are null hypersurfaces.

5.4 Weak Cosmic Censorship

We have already seen three different kinds of singularity: These are all naked singularities.

But the solution beyond $H^+(\Sigma)$ is not determined by data on Σ. We drew one possible extension in Fig. 5.11, but there are infinitely many others. We should only really draw the maximal Cauchy development as in Fig. 5.12. However, this has two problems:

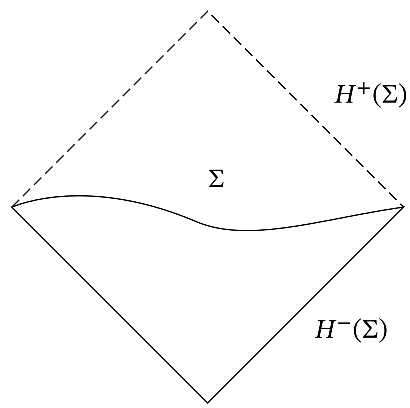


Figure 5.10

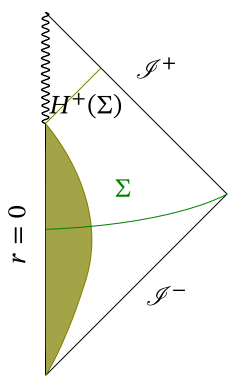


Figure 5.11: Collapse to a naked singularity.

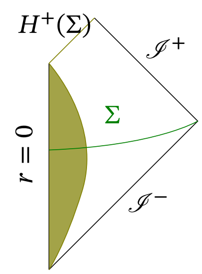


Figure 5.12

- It is extendible across $H^+(\Sigma)$. This violates SCC.
- $\mathscr{I}^+$ is incomplete (violates asymptotic flatness).

Weak Cosmic Censorship conjecture: Let (Σ, h_{ab}, K_{ab}) be geodesically complete, asymptotically flat data. Suppose matter fields obey hyperbolic equations and the DEC. Then *generically*, the maximal Cauchy development is an asymptotically flat spacetime ($\Rightarrow$ complete $\mathscr{I}^+$) that is strongly asymp. predictable.

The 'generic' statement in the conjecture is explained by examining the following example.

Example 5.4.1 (gravity and massless scalar, sph. sym.)**:** There exists initial data labelled by a parameter p such that

$$p < p_* \longrightarrow \text{scalar disperses} \tag{5.18}$$

$$p > p_* \longrightarrow \text{collapse to a black hole} \tag{5.19}$$

Finte tuned data $p = p_*$ gives incomplete $\mathscr{I}^+$, but this is non-generic.

Despite the name, the strong and weak cosmic censorship conjectures are logically independent. This can be shown by considering the following Penrose diagrams. The diagram in Fig. 5.13 satisfies

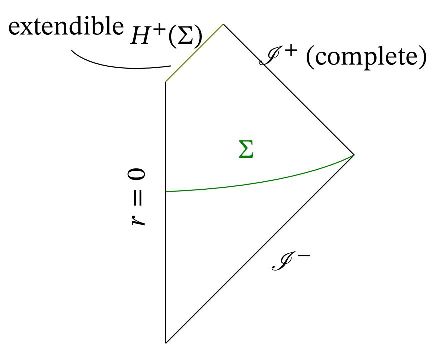


Figure 5.13

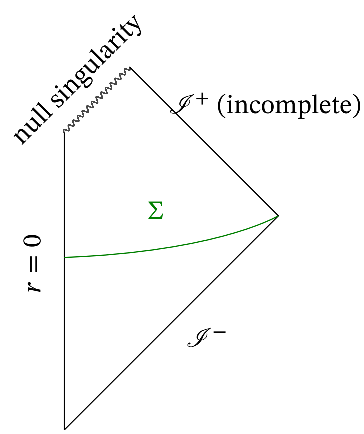


Figure 5.14

WCC but not SCC, while the diagram Fig. 5.14 satisfies SCC but not WCC.

Moreover, a spherically symmetric system of gravity and (unphysical) pressureless fluid ("dust") violates both SCC and WCC. Gravity and a massless scalar, with spherical symmetry satisfies both SCC and WCC.

Numerical simulations provide a lot of evidence that these singularities are true; singularities seem to always form inside black holes.

5.5 Apparent Horizon

Theorem 14: Let T be a trapped surface in a strongly asymptotiacally predictable spacetime (M, g) obeying the null energy condition. Then $T \subset B$.

Strong. asymp. pred. foliate with Cauchy surfaces $\Sigma_t = \{t = const.\}$, where t is a time function. We would like to call the "black hole region at time t " $B_t = \mathcal{B} \cap \Sigma_t$ and the "event horizon at time t " $H_t = \mathcal{H}^+ \cap \Sigma_t$. This motivates the following definition:

Definition 54 (trapped region)**:** Let Σ_t be a Cauchy surface. The *trapped region* $\mathcal{T}_t$ of Σ_t is

$$\mathcal{T}_t := \{p \in \Sigma_t \mid \exists \text{ trapped } S \text{ s.t. } p \in S, S \subset \Sigma_t\}. \tag{5.20}$$

Definition 55: The *apparent horizon* $\mathcal{A}_t = \dot{\mathcal{T}}_t$.

The basic idea is that you want to regard the trapped region as an approximation to the black hole region and the apparent horizon an approximation to the event horizon. WCC implies that $\mathcal{T} \subset \mathcal{B}$ and so $\mathcal{A}_t \subset \mathcal{B}$. Thus $\mathcal{A}_t$ is inside / on H_t.

However, it is important to note that A_t depends on choice of time function and Σ_t.

Example 5.5.1 (Kruskal)**:** In a spherically symmetric Σ_t, we have $A_t = H_t$.

In general, we expect $\mathcal{A}_t$ to be marginally trapped. In fact, this is how it is determined in numerical simulations.

6 Charged Black Holes

These are not very relevant in nature, because we do not get large imbalances in nature. If you did form one, it would attract opposite charges and the charge would equilibrate. However, they are a nice warm-up in discussing the rotating Kerr black hole.

6.1 Reissner–Nordström Solution

This is the simplest kind of charged black hole. Let us write the Einstein–Maxwell action as

$$S = \frac{1}{16\pi}\int \mathrm{d}^4x\,\sqrt{-g}\left(R - F_{ab}F^{ab}\right), \tag{6.1}$$

where $F = dA$, so $dF = 0$. The normalisation of the Maxwell action might look different to its presentation in other courses, but we do this only to make the solution as simple as possible. The Einstein–Maxwell equations are

$$R_{ab} - \frac{1}{2}Rg_{ab} = 2\left(F_a{}^cF_{bc} - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right) \qquad \nabla^b F_{ab} = 0. \tag{6.2}$$

There is a generalisation of Birchhoff's theorem to understand maxwell theory:

Theorem 15: The unique spherically symmetric solution of the Einstein–Maxwell equations with non-constant area radius r is the Reissner–Nordström (NS) solution

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dt^2 + r^2 d\Omega^2, \tag{6.3}$$

with

$$A = -\frac{Q}{r}dt - P\cos\theta d\phi \qquad e = \sqrt{Q^2 + P^2}. \tag{6.4}$$

We interpret M as mass, Q and P as electric and magnetic charge respectively.

This admits a static timelike KVF $k^a = \left(\frac{\partial}{\partial t}\right)^a$ and is asymp. flat at null ∞.

Notation: We define the quadratic polynomial

$$\Delta := r^2 - 2Mr + e^2 = (r - r_+)(r - r_-), \qquad r_\pm = M \pm \sqrt{M^2 - e^2}. \tag{6.5}$$

The metric can then be written

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2 d\Omega^2. \tag{6.6}$$

If $M < e$, then $\Delta > 0$ for $r > 0$. We have a curvature singularity at $r = 0$. This is a naked singularity like $M < 0$ Schwarzschild. Naked singularities are excluded physically by the WCC. What would happen to a ball of charged matter with $M < e$? It could not collapse to $r = 0$. An elementary particle like an electron has $M > e$; however, they are described quantum mechanically not classically, so there is no sense in which this describes the gravitational field of an elementary particle.

6.2 Eddington–Finkelstein Coordinates

For $M > e$,Δ two real roots $r_\pm > 0$. In exactly the same way as we did for Schwarzschild, we start in a region $r > r_+$ and introduce coordinates

$$dr_* = \frac{r^2}{\Delta}dr \implies r_* = r + \frac{1}{2\kappa_+}\ln\left|\frac{r - r_+}{r_+}\right| + \frac{1}{2\kappa_-}\ln\left|\frac{r - r_-}{r_-}\right| + \text{const.} \tag{6.7}$$

where we introduced two constants

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2}. \tag{6.8}$$

We define $u = t - r_*$ and $v = t + r_*$. The ingoing Eddington–Finkelstein coordinates are (v, r, θ, ϕ) with metric

$$ds^2 = -\frac{\Delta}{r^2}dv^2 + 2dvdr + r^2 d\Omega^2. \tag{6.9}$$

These are smooth and Lorentzian for all $r > 0$. Again we analytically continue to $0 < r < r_+$ to the curvature singularity at $r = 0$.

A constant-r surface has normal $n = dr$. This is null when $g^{rr} = \frac{\Delta}{r^2} = 0$, i.e. at $r = r_\pm$. Therefore $\{r = r_\pm\}$ are null hypersurfaces.

Exercise 6.1: Show that r decreases along any future-directed causal curve in $r_- < r < r_+$.

So the region $r \leq r_+$ is the black hole region with event horizon being the null hypersurface $\mathcal{H}^+ = \{r = r_*\}$.

Using the outgoing EF coordinates (u, r, θ, ϕ), we have

$$ds^2 = -\frac{\Delta}{r^2}du^2 - 2dudr + r^2 d\Omega^2. \tag{6.10}$$

This, as for Schwarzschild, defines a white hole.

6.3 Kruskal-like coordiates

For future use, we will define two sets of U and V coordinates:

$$U^{\pm} = -e^{-\kappa_{\pm}u} \qquad V^{\pm} = \pm e^{\kappa_{\pm}v}, \tag{6.11}$$

where κ is defined in (6.8). The reason for this sign choice will become apparent shortly.

For $r > r_+$ in Kruskal coordinates (U^+, V^+, θ, ϕ), the metric is

$$ds^2 = -\frac{r - +r_-}{\kappa_+^2 r^2} e^{-2\kappa_+ r} \left(\frac{r - r_-}{r_-}\right)^{1+\kappa_+/|\kappa_-|} dU^+ dV^+ + r^2 d\Omega^2, \tag{6.12}$$

where $r(U^+, V^+)$ is defined by

$$-U^+V^+ = e^{2\kappa_+ r}\left(\frac{r - r_+}{r_+}\right)\left(\frac{r_-}{r - r_-}\right)^{\kappa_+/|\kappa_-|}, \tag{6.13}$$

which is monotonically increasing for $r > r_-$.

Initially, $U^+ < 0$ and $V^+ > 0$. Can now analytically continue to $U^+ \geq 0$ or $V^+ \leq 0$.

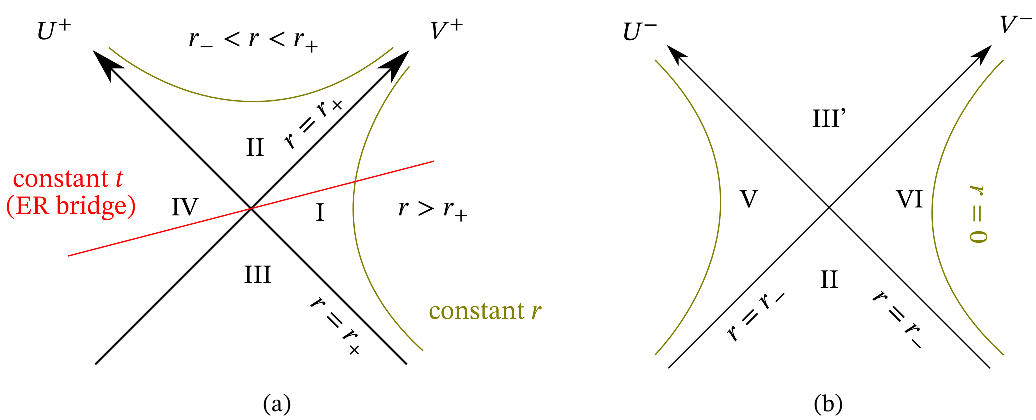


Figure 6.1: Reissner–Nordström solution in $(U^{\pm}, V^{\pm})$ coordinates.

Have $r > r_-$ everywhere. $k^a = 0$ at $U^+ = V^+ = 0$ (bifurcation 2-sphere).

Ingoing EF: ingoing radial null geo. reaches $r = r_-$ in finite affine parameter ($U^+V^+ \to -\infty$).

In region II, we start with ingoing EF (v, r, θ, ϕ) and reintroduce static coordinates. Let $t = v - r_*$. Converting the metric back to (t, r, θ, ϕ) to get

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dt^2 + r^2 d\Omega^2, \tag{6.14}$$

which is now defined in region II.

Let $u = t - r_* = v - 2r_*$. We can use the definition (6.11) to define $U^- < 0$ and $V^- < 0$ in region II. In these coordinates, the metric takes the form

$$ds^2 = -\frac{r_+ r_-}{\kappa_-^2 r^2} r^{2|\kappa_-|r} \left(\frac{r_+ - r}{r_+}\right)^{1+|\kappa_-|/\kappa_+} dU^- dV^+ r^2 d\Omega^2, \tag{6.15}$$

where $r(U^-, V^-)$ is defined by

$$U^- V^- = e^{-2|\kappa_-|r} \left(\frac{r - r_-}{r_-}\right)\left(\frac{r_+}{r_+ - r}\right)^{|\kappa|/\kappa_+}, \tag{6.16}$$

which is monotonic for $r < r_+$.

We can now analytically continue this to $U^- \geq 0$ or $V^- \geq 0$. We can draw another Kruskal diagram, illustrated in Fig. 6.1b. We need to introduce new regions V and VI, and for reasons to become apparent we wrote one of them as III'. The region $r = 0$ corresponds to $U^- V^- = -1$.

When we talked about the white hole, we said that the black and white holes are isometric, however reversing time orientation. Here, III' is isometric to III, and the isometry preserves the time-orientation; they are indistinguishable, except that III' is in the white hole region! We can define $(U^{+\prime}, V^{+\prime})$ in III' analytically continue. This gives another diagram 6.2, and we can keep going on

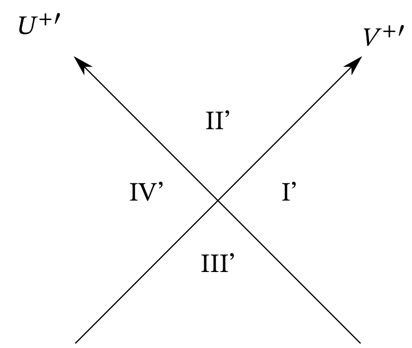


Figure 6.2

and on. As such, there are infinitely many regions and the Penrose diagram looks something like 6.3, which repeats infinitely many times. As always, radial null geodesics are lines at 45°.

6.4 Cauchy Horizons

We have a geodesically complete Σ, which is asymptotically flat with 2 ends. There exists a Cauchy horizons $H^\pm(\Sigma) : r = r_-$. The solution beyond $H^\pm(\Sigma)$ is not deterred by data on Σ. (e.g. need not be spherically symmetric or analytic.) This appears to violate CSS. However there is a get-out: the word "generic". If this violation is only for single solution we have no problem. We need to perturb around this.

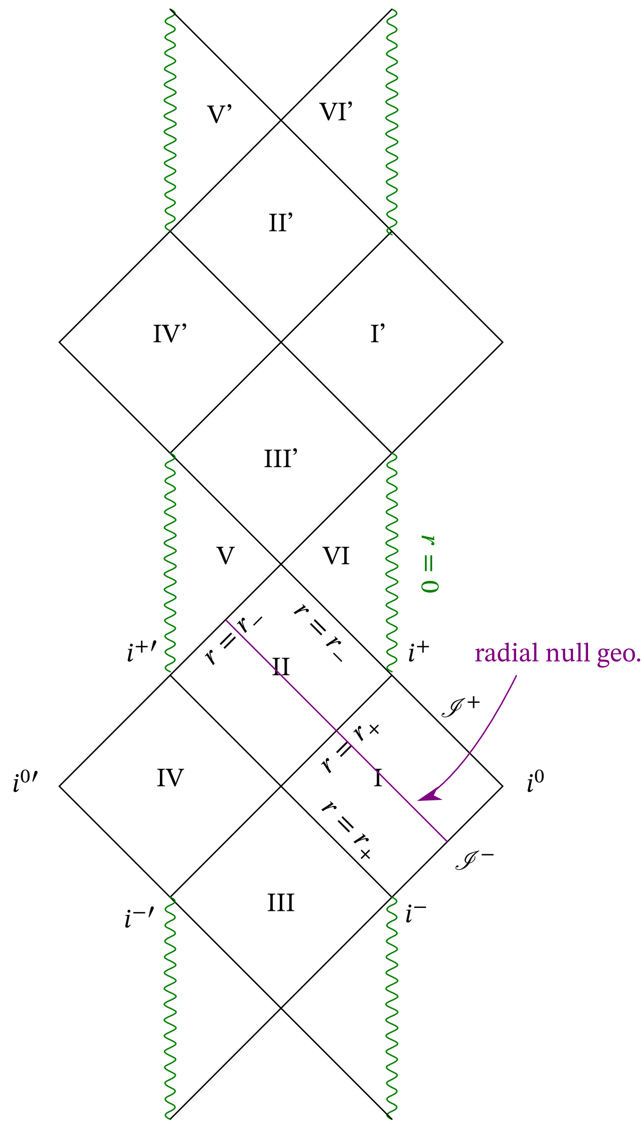


Figure 6.3: Penrose diagram of the maximally extended Reissner–Nordström solution.

Consider two observers Alice A and Bob B, as illustrated in 6.4. Bob is immortal; he lives forever. He is also sensible and stays out of the black hole region. Alice is more adventurous. For reassurance, Bob sends infinitely many light signals to Alice, one every second. Alice receives all signals, infinitely many of them, as she crosses the Cauchy horizon $H^+(\Sigma)$. This gives an infinite blueshift; she measures the gradient associated with these waves to be very large. We get infinite energy (as measured by A). If the energy is diverging, it gives a large gravitational effect.

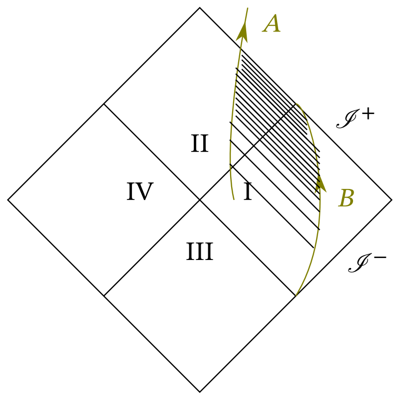


Figure 6.4

This suggests that a small portion in I causes a large gravitational backreaction at $H^+(\Sigma)$. This is an instability! So this is indeed non-generic. Expect $H^+(\Sigma)$ to be replaced by a curvature singularity in perturbed spacetime.

6.5 Extreme RN

The Reissner–Nordström solution with $M = e$ is called *extreme RN*. The metric is

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dt^2 + r^2 d\Omega^2. \tag{6.17}$$

Again we define, starting in the region $r > M$,

$$dr_* = \frac{dr}{\left(1 - \frac{M}{r}\right)^2} \quad \to \quad r_* = r + 2M \ln\left|\frac{r - M}{M}\right| - \frac{M^2}{r - M}. \tag{6.18}$$

We then introduce ingoing EF coordinates $v = t + r_*$ in which the metric becomes

$$ds^2 = -\left(1 - \frac{M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2. \tag{6.19}$$

We can go through the whole spiel analytically continuing to $0 < r < M$ giving the black and white hole regions and so on. Finally, we obtain the Penrose diagram 6.5.

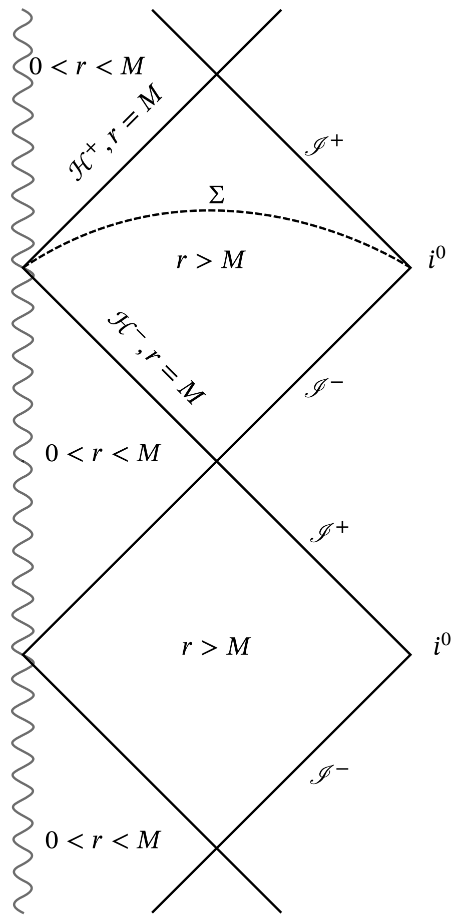


Figure 6.5: Penrose diagram of the extreme RN solution.

Remark: $H\mathcal{S}^{\pm} = H^{\pm}(\Sigma)$.

Consider a surface Σ of constant t, drawn in Fig. 6.5. Let us calculate the proper length of a line of constant t, θ, ϕ from $r = r_0 > M$ to $r = M$:

$$\int_M^{r_0} \frac{dr}{1 - M/r} = \infty. \tag{6.20}$$

So the end points are actually points at infinity. Approaching this point on an Einstein–Rosen bridge corresponds geometrically to going down the infinite throat in Fig. 6.6.

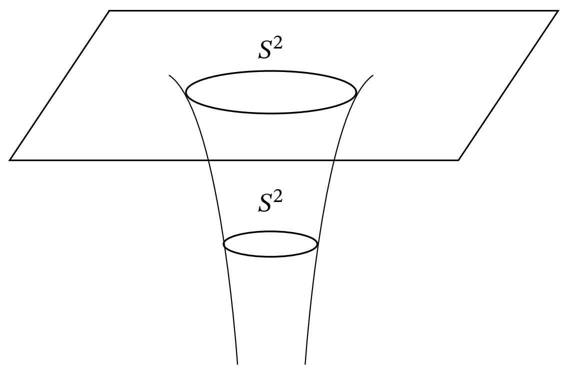


Figure 6.6: Infinite throat on a constant-t surface Σ of extreme RN.

The geometry of this throat is quite interesting. Let $r := M(1+\lambda)$. To leading order in λ, we have the *Robinson–Bertotti metric*

$$ds^2 \approx -\lambda^2 dt^2 + M^2 \frac{d\lambda^2}{\lambda^2} + M^2 d\Omega^2 \qquad (\mathrm{AdS}_2 \times S^2). \tag{6.21}$$

7 Rotating Black Holes

In this chapter we will be dealing with one of the, if not the single most important solution in general relativity. To motivate why we want to study this *Kerr black hole*, we will examine a set of theorems.

7.1 Uniqueness Theorems

We have to relax our definition of stationarity slightly, since otherwise the Kerr black hole will not be stationary.

Definition 56 (stationary)**:** A spacetime (M, g) asymptotically flat at null-infinity is *stationary* if there exists a Killing vector field k^a such that k^a is timelike in a neighbourhood of $\mathscr{I}^+$.

Definition 57 (static)**:** The spacetime is *static* if it is stationary and k^a is hypersurface orthogonal.

We refer to our previous definition, Def. 6 (k^a timelike everywhere), as "strictly stationary" (e.g. Minkowski). Kruskal is static but not strictly static (k^a spacelike in II, III).

In the case of a rotating black hole, we can no longer assume spherical symmetry. However, we can have axisymmetry.

Definition 58 (stationary and axisymmetric)**:** A spacetime (M, g) asymptotically flat at null infinity is *stationary and axisymmetric* if (i) it is stationary, and (ii) there exists a Killing vector field m^a, spacelike near $\mathscr{I}^+$, which (iii) generates a 1-parameter family of isometries isomorphic to $U(1)$, the group of rotations about an axis. Finally, (iv) the two Killing vector fields should be compatible, meaning $[k, m] = 0$.

If we have a spacetime satisfying these definitions, we can choose coordinates such that $k = \frac{\partial}{\partial t}$ and $m = \frac{\partial}{\partial \phi}$, with $\phi \sim \phi + 2\pi$.

Theorem 16 (Israel '67, Bunting–Masood '87)**:** If (M, g) is a static, asymptotically flat vacuum black hole spacetime suitably regular[2] on and outside the event horizon $\mathcal{H}^+$, then (M, g) is isometric to

[2] Stating these theorems precisely requires quite a lot of technology, which we do not have time to develop in this course.

the Schwarzschild spacetime.

There is an Einstein–Maxwell generalisation of this result.

Theorem 17 (Hawing '73, Wald '92)**:** Let (M, g) be a stationary, non-static, asymptotically flat, analytic[1] solution of the Eintein–Maxwell equations, suitably regular on and outside $\mathcal{H}^+$, then (M, g) is stationary and axisymmetric.

This is often taken to be that stationary implies axisymmetric for black holes. Now, mathematicians are quite concerned about the analyticity assumption, since a single point determines the behaviour of the whole spacetime, which is quite unphysical.

Theorem 18 (Carter '71, Robinson '75)**:** Let (M, g) be a stationary and axisymmetric, asymptotically flat vacuum spacetime suitably regular on and outside a connected $\mathcal{H}^+$, then (M, g) is a member of the Kerr (1963) family of solutions, which are parametrised by mass (M) and angular momentum (J).

The final state of gravitational collapse, if stationary and axisymmetric, is fully determined by its mass and angular momentum.

The Einstein–Maxwell version of this theorem replaces the Kerr family with the Kerr–Newman family with four parameters M, J, Q, P, which adds electric and magnetic charges.

7.2 The Kerr–Newman Solution

This is usually written down in the *Beyer–Lindquist coordinates*

$$ds^2 = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma}dt^2 - 2a\sin^2\theta\frac{(r^2+a^2-\Delta)}{\Sigma}dtd\phi + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dt^2 + \Sigma d\theta^2, \quad (7.1)$$

$$A = -\frac{Qr(dt - a\sin^2\theta d\phi) + P\cos\theta(adt - (r^2+a^2)d\phi)}{\Sigma} \quad (7.2)$$

$$\Sigma = r^2 + a^2\cos^2\theta \quad (7.3)$$

$$\Delta = r^2 - 2Mr + a^2 + e^2, \qquad e = \sqrt{Q^2+P^2} \quad (7.4)$$

At large r, $(t, r, \theta, \phi) \sim$ polar coordinates on $\mathbb{M}^4$. The (θ, ϕ) parametrise S^2 with $0 < \theta < \pi$ and $\phi \sim \phi + 2\pi$.

[1] All the components are analytic functions, so they converge in a power series.

The KN solution is asymptotically flat at null infinity. It is also stationary and axisymmetric with $k = \frac{\partial}{\partial t}$ and $m = \frac{\partial}{\partial \phi}$. It also admits an isometry $t \to -t, \phi \to -\phi$.

The parameter M is identified with the mass, Q, P are the electric and magnetic charge respectively, $a = \frac{J}{M}$ and J is the angular momentum. Taking $a = 0$ reduces the KN to RN. Taking $\phi \to -\phi$ is equivalent to taking $a \to -a$. Thus, without loss of generality, we take $a \geq 0$.

7.3 Kerr Solution

As we argued before, charged black holes are not physical. Let us therefore consider $Q = P = 0$. The RN was a warm-up to Kerr and we proceed in the same way by factorising Δ as

$$\Delta = (r - r_+)(r - r_-), \qquad r_\pm = M \pm \sqrt{M^2 - a^2}. \tag{7.5}$$

For $M^2 < a^2$ we have a naked singularity. Therefore, assuming WCC, we have $M^2 > a^2$. The metric is singular when $\Delta = 0$ (at $r = r_\pm$) and at $\Sigma = 0$ ($r = 0, \theta = \frac{\pi}{2}$).

Consider first the case where $r > r_+$. The (outgoing) Kerr coordinates are (v, r, θ, χ), where the new coordinates are defined as

$$dv = dt + \frac{r^2 + a^2}{\Delta} dr, \qquad d\chi = d\phi + \frac{a}{\Delta} dr. \tag{7.6}$$

Since χ and ϕ are linearly related, it follows that $\chi \sim \chi + 2\pi$. Moreover, since v and t are simply related by a shift, we have $k = \frac{\partial}{\partial v}$. Similarly, we have $m = \frac{\partial}{\partial \chi}$. The Kerr metric in Kerr coordinates is

$$\begin{aligned} ds^2 = -\frac{(\Delta - a^2 \sin^2\theta)}{\Sigma} dv^2 + 2dvdr - 2a\sin^2\theta \frac{(r^2 + a^2 - \Delta)}{\Sigma} dvd\chi - 2a\sin^2\theta d\chi dr \\ + \left(\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \right) \sin^2\theta d\chi^2 + \Sigma d\theta^2. \end{aligned} \tag{7.7}$$

You can check that this is smooth and Lorentzian when Δ vanishes at $r = r_+$. Hence, we can analytically continue this to $0 < r \leq r_+$.

Claim 17: The surface $r = r_+$ is a null hypersurface with normal $\xi^a = k^a + \Omega_H m^a$, where $\Omega_H = \frac{a}{r_+^2 + a^2}$.

Proof. Exercise: Calculate ξ_μ. Show that $\xi_\mu dx^\mu|_{r=r_+} \propto dr$. From this exercise, we find that χ_a is normal to $r = r_+$. Moreover, it is null $\chi^\mu \chi_\mu|_{r=r_+} = 0$ as $\xi^r = 0$. Hence, $r = r_+$ is a null hypersurface. □

For $r \leq r_+$ inside the black hole region, $r = r_*$ is part of $\mathcal{H}^+$. Take black hole coordinates $\xi = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi}$. Then $\chi^\mu \partial_\mu (\phi - \Omega_H t) = 0$. Therefore, $\phi = \Omega_H t +$ const. on integral curves of ξ. Since ϕ is constant on integral curves of k, it is also constant on trajectories of stationary observers, which have $u^a \propto k^a$. Therefore, orbits of ξ rotate with angular velocity Ω_H with respect to a stationary observer. In particular, we may take a stationary observer at infinity. But χ^a is tangent to generators of $\mathcal{H}^+$. This means that these generators rotate at angular velocity Ω_H with respect to a stationary observer at infinity. In other words, the black hole is rotating with angular velocity Ω_H.

7.4 Maximally Analytic Extension

We can also introduce outgoing Kerr coordinates, and Kruskal-like coordinates, to recover the black hole and other regions. The whole 'Penrose diagram' will look like RN. Strictly speaking it does not have a Penrose diagram since it is not spherically symmetric. However, we can look at submanifolds.

We can draw a Penrose diagram for the axis of symmetry ($\theta = 0$ or $\theta = \pi$) or the equatorial plane ($\theta = \frac{\pi}{2}$). Both of these submanifolds are examples of *totally geodesic* submanifolds.

Definition 59 (totally geodesic)**:** Any geodesic that is initially tangent to a *totally geodesic submanifold* remains tangent.

The Cauchy horizons $H^\pm(\Sigma)$ are unstable for RN. Again, as for RN, most of the diagram 7.1 is unphysical.

Kerr is not the spacetime outside a rotating star, only describes the final state of gravitational collapse. $M = a$ is extreme Kerr similar to extreme RN.

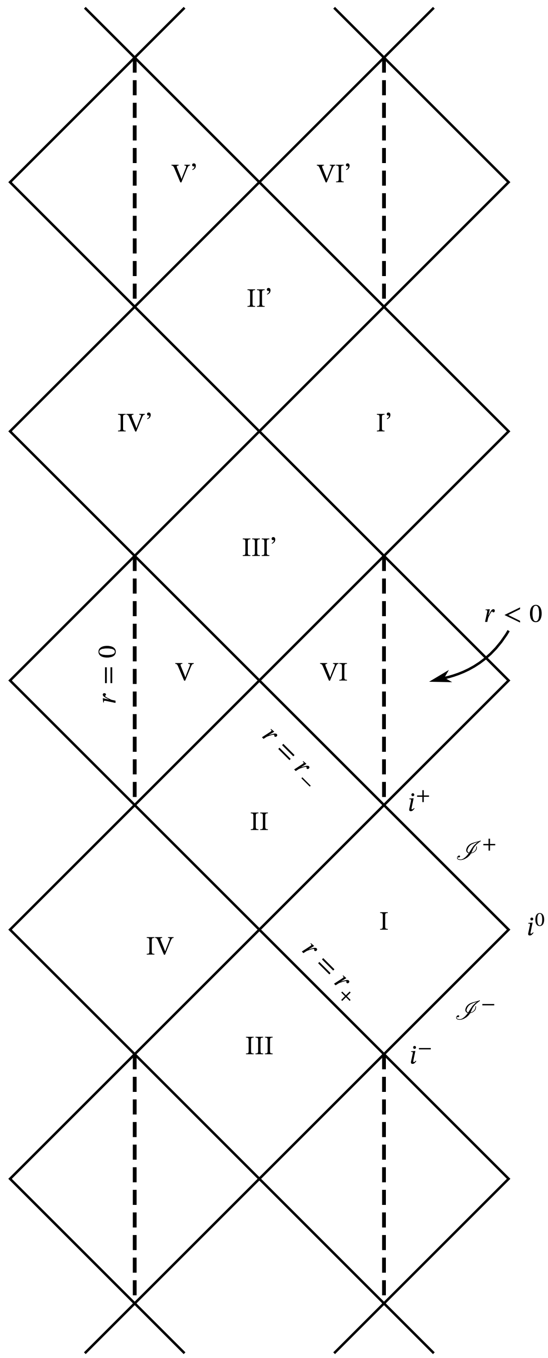


Figure 7.1: Penrose diagram of the maximally extended Kerr solution. The dotted lines denote the "ring" singularity at $r = 0, \theta = \pi/2$.

7.5 Ergosphere & Penrose Process

Let us look, in BL coorinates, at the norm of the Killing field:

$$k^2 = g_{tt} = -\frac{(\Delta - a^2\sin^2\theta)}{\Sigma} = \leftrightarrow (1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}). \tag{7.8}$$

k is timelike when $r^2 - 2Mr + a^2\cos^2\theta > 0$. This happens for $r > M + \sqrt{M^2 - a^2\cos^2\theta}$. This is greater than r_+. This means that there is a region $r_+ = M + \sqrt{M^2 - a^2} < r < M + \sqrt{M^2 - a^2\cos^2\theta}$ outside the event horizon $\mathcal{H}^+$ where k is spacelike. This region, depicted in Fig. 7.2, is called the

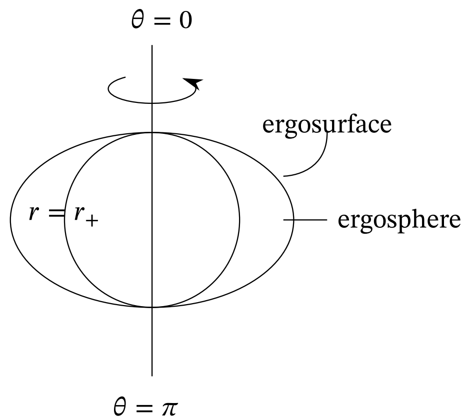
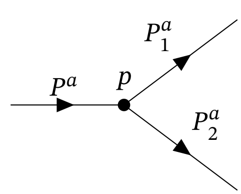


Figure 7.2: Ergosphere of a Kerr black hole.

ergosphere. Its boundary is the *ergosurface*. A stationary observer ($u^a \parallel k^a$) cannot exist in the ergosphere. Any causal curve in the ergosphere must rotate relative to ∞, in the same direction as the black hole.

Consider a particle with 4-momentum $P^a = \mu u^a$, where μ is its rest mass and u^a its 4-velocity. Along a geodesic $E = -k \cdot P$ is conserved. This is the energy according to an observer at infinity. Assume the particle decays inside the ergosphere.



(7.9)

The 4-momentum is conserved at p:

$$P^a = P_1^a + P_2^a \;\Rightarrow\; E = E_1 + E_2, \qquad E_i = -k \cdot P_i. \tag{7.10}$$

Since k is spacelike in the ergosphere, we can have $E_1 < 0$, which means that we can have $E_2 = E + |E_1| > E$. Particle 1 must fall into the hole, but 2 can escape to ∞, carrying more energy than the initial particle! Particle 1 carries negative energy into the hole, so the energy (mass) of the hole decreases. This extraction of energy from the black hole is called the *Penrose process*.

Particle crossing $\mathcal{H}^+$:

$$-P \cdot \xi \geq 0, \tag{7.11}$$

since both P^a, ξ^a are future-directed causal.

$$\xi = k + \Omega_H m \qquad E - \Omega_H L \geq 0 \qquad L = mcP, \tag{7.12}$$

where L is the angular momentum of the particle. Therefore

$$\therefore L \leq E/\Omega_H \qquad \delta M = E, \quad \delta J = L \tag{7.13}$$

$$\therefore \delta J \leq \delta M/\Omega_H = \frac{2M(M^2 + \sqrt{M^4 - J^2})}{J}\delta M. \tag{7.14}$$

Exercise 7.1: Show that this inequality is equivalent to $\delta M_{\text{irr}} \geq 0$, where

$$M_{\text{irr}} = \left[\frac{1}{2}(M^2 + \sqrt{M^4 - J^2})\right]^{\frac{1}{2}} \tag{7.15}$$

is the *irreducible mass*. In other words, in the Penrose process M_{irr} cannot decrease.

Inverting this gives

$$M^2 = M_{\text{irr}}^2 + \frac{J^2}{4M_{\text{irr}}^2} \geq M_{\text{irr}}^2. \tag{7.16}$$

The Penrose process cannot reduce M below the initial value of M_{irr}; it gives an upper bound on the energy we can extract.

Exercise 7.2: The quantity $A = 16\pi M_{\text{irr}}^2$ is the *area of the event horizon* (area of $\mathcal{H}^+ \cap \Sigma$, where Σ is a partial Cauchy surface, e.g. $v = \text{const.}$)

This means $\delta A \geq 0$ in the Penrose process (special case of the 2^{nd} law of black hole mechanics).

$$A = 8\pi(M^2 + \sqrt{M^4 - J^2}). \tag{7.17}$$

8 Charge, Mass, and Angular Momentum

8.1 Charges in Curved Spacetime

Definition 60: Let (Σ, h_{ab}, K_{ab}) be an asymptotically flat end. The *electric and magnetic charges* associated to this end are

$$Q = \frac{1}{4\pi} \lim_{r\to\infty} \int_{S_r^2} \star F, \qquad P = \frac{1}{4\pi} \lim_{r\to\infty} \int_{S_r^2} F, \tag{8.1}$$

where S_r^2 is the sphere $x^i x^i = r^2$ where x^i are the coordinates in the definition of the asymptotically flat end.

> We can have a non-zero charge even though no matter is present. In a sense, the topology of the spacetime can support charge, such as in the RN solution.

8.2 Komar Integrals

In a stationary spacetime (M, g), we can define $J_a = -T_{ab}k^b$, the conserved energy-momentum current.

$$\nabla_a J^a = 0 \qquad \Longleftrightarrow \qquad d \star J = 0. \tag{8.2}$$

Given this conserved current, we can define the related conserved charge, interpreted as the total energy of matter on a spacelike hypersurface Σ

$$E[\Sigma] = -\int_\Sigma \star J. \tag{8.3}$$

It is conserved in the sense that if we have Fig. 8.1, then

$$E[\Sigma'] - E[\Sigma] = -\int_{\partial R} \star J = -\int_R d \star J = 0. \tag{8.4}$$

Consider the 1-form

$$(\star d \star k)_a = -\nabla^b (dk)_{ab}, \tag{8.5}$$

using the identity

$$\epsilon^{a_1\dots a_p c_{p+1}\dots c_n}\epsilon_{b_1\dots b_p c_{p+1}\dots c_p} = -p!(n-p)!\delta^{a_1}_{[b_1}\dots\delta^{a_p}_{b_p]}$$
$$\Rightarrow \ (\star d\star X)_{a_1\dots a_{p-1}} = -(-1)^{p(n-p)}\nabla^b X_{a_1\dots a_{p-1}b}. \quad (8.6)$$

Using the Killing vector field, we have

$$(\star d\star dk)_a = -\nabla^b\nabla_a k_b + \nabla^b\nabla_b k_a = 2\nabla^b\nabla_\beta k_a. \quad (8.7)$$

Lemma 19: For a Killing vector field k^a, we have

$$\nabla_a\nabla_b k^c = R^c{}_{bad}k^d. \quad (8.8)$$

Thus, using the Einstein equation, and assuming that T is the energy-momentum tensor, we have

$$(\star d\star dk)_a = -2R_{ab}k^b = 8\pi J'_a, \qquad J'_a = -2(T_{ab} - \frac{1}{2}T^c{}_c g_{ab})k^b. \quad (8.9)$$

Since $d\star dk = 8\pi\star J'$, we find that $\star J'$ is *exact* ($\Rightarrow$ conserved $d\star J' = 0$). Consider a surface Σ with a single asymptotically flat end,

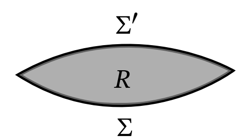


Figure 8.1

$$\therefore -\int_\Sigma \star J' = -\frac{1}{8\pi}\int_\Sigma d\star dk = -\frac{1}{8\pi}\int_{\partial\Sigma}\star dk. \quad (8.10)$$

This equation tells us something about the energy of matter.

Exercise 8.1: Consider a static, spherically symmetric perfect fluid star. Take $\Sigma = \{t = \text{const.} \mid r \le r_0\}$, $r_0 > R$. Show that the RHS of (8.10) is M. Show that in the Newtonian limit, where $P \ll \rho$, $|\Phi| \ll 1$, $|\Psi| \ll 1$, the LHS really is the total mass of the fluid.

Definition 61 (Komar mass)**:** Let (Σ, h_{ab}, K_{ab}) be an asymp. flat end in a stationary spacetime. The *Komar mass* (or energy) is

$$M_{\text{Komar}} = -\frac{1}{8\pi}\lim_{r\to\infty}\int_{S_r^2}\star dk. \quad (8.11)$$

This is entirely analogous to the definition 60 of electric and magnetic charges. This is the total mass of matter and gravity in the spacetime. We can do this for any Killing vector field. In particular, we could have done this for an axisymmetric spacetime.

Definition 62 (Komar angular momentum)**:** Let (Σ, h_{ab}, K_{ab}) be an asymp. flat end in an axisymmetric spacetime. The *Komar angular momentum* is

$$J_{\text{Komar}} = \frac{1}{16\pi}\lim_{r\to\infty}\int_{S_r^2}\star dm. \quad (8.12)$$

The normalisation ensures that this reduces to Newtonian angular momentum in the appropriate limit.

It is not obvious why this should be the definition of energy or angular momentum. Classically, the energy is the value of the Hamiltonian. Surely the 'correct' way to understand energy then is to understand the Hamiltonian treatment of general relativity. As we will see, this 'correct' definition agrees with the above in a stationary spacetime, and these are ultimately the more useful for calculations.

8.3 Hamilonian Formulation of GR

Set $16\pi G = 1$. In a $3+1$ split, we define the lapse function and the shift and we have metric

$$ds^2 = N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{8.13}$$

$$S = \int \mathrm{d}^4 x \sqrt{-g} R = \int \mathrm{d}t \, \mathrm{d}^3 x \sqrt{h} N({}^{(3)}R + K_{ij}K^{ij} - K^2) = \int \mathrm{d}t \, \mathrm{d}^3 x \, \mathscr{L}. \tag{8.14}$$

Exercise 8.2 (Sheet 2)**:** A formula for the extrinsic curvature tensor is

$$K_{ij} = \frac{1}{2N}(\dot{h}_{ij} - D_i N_j - D_j N_i), \tag{8.15}$$

where the dot denotes time-derivative.

Remark: S is independent of the time derivatives of the lapse and the shift, $\dot{N}, \dot{N}^i$. Hamilton's equations are constraints

$$\frac{\delta S}{\delta N} \to \text{Ham. constraint}, \qquad \frac{\delta S}{\delta N^i} \to \text{momentum constraint} \tag{8.16}$$

They are not really dynamical variables; they are a gauge choice that we can choose to whatever is convenient. On the other hand $\frac{\delta S}{\delta h_{ij}}$ gives evolution equations for h_{ij}.

The conjugate momenta of the lapse and shift are zero, since there are no time derivatives. We are left with

$$\pi^{ij} = \frac{\delta S}{\delta \dot{h}_{ij}} = \sqrt{h}(K^{ij} - K h^{ij}), \tag{8.17}$$

where we used that $\dot{h}_{ij}$ only lives within K_{ij}. Because of the square root of the determinant, this is a tensor density rather than a tensor.

The Hamiltonian is given by the Legendre transform

$$H = \int \mathrm{d}^3x\,(\pi^{ij}\dot{h}_{ij} - \mathscr{L}) = \int \mathrm{d}^3x\,\sqrt{h}(N\mathcal{H} + N^i\mathcal{H}). \tag{8.18}$$

In doing this calculation, we need to integrate by parts and neglect surface terms. Upon doing this, we get

$$\mathcal{H} = -^{(3)}R + h^{-1}\pi^{ij}\pi_{ij} - \frac{1}{2}h^{-1}\pi^2, \qquad \pi = h^{ij}\pi^{ij}. \tag{8.19}$$

$$\mathcal{H}_i = -2h_{ik}D_j(h^{-1/2}\pi^{jk}). \tag{8.20}$$

The N, N^i have the interpretation as Lagrange multipliers: varying the action $\frac{\delta S}{\delta N} = \frac{\delta S}{\delta N^i} = 0$ gives constraints $\mathcal{H} = \mathcal{H}_i = 0$. We have rewritten the Hamiltonian and momentum constraints in terms of π^{ij} and h_{ij}.

Hamilton's principle says that the time derivative of h_{ij} and π^{ij} are given by

$$\dot{h}_{ij} = \frac{\delta H}{\delta \pi^{ij}} \qquad \dot{\pi}^{ij} = -\frac{\delta H}{\delta h_{ij}}. \tag{8.21}$$

The equation for $\dot{h}_{ij}$ recovers the definition of π^{ij}.

Given the Hamiltonian (8.18), we would like to define the energy to be the value of the Hamiltonian. But there is a problem; the only terms involved in the Hamiltonian are constraint equations, which vanish if the constraints are satisfied. So the energy of the spacetime is always zero with this definition.

The resolution of this problem is that we have been rather cavalier about neglecting surface terms. When calculating variational derivatives, you generate surface terms that are generally non-zero and will contribute to H. We can trust this conclusion if there are no surface terms. When the surface of constant t is compact (i.e. for a closed universe), then the energy vanishes exactly.

However, we are mostly interested in the case of black holes. Assume that our constant-t surfaces are asymptotically flat with one end. Therefore, asymptotically,

$$h_{ij} = \delta_{ij} + \mathcal{O}\left(\frac{1}{r}\right) \qquad \pi^{ij} = \mathcal{O}\left(\frac{1}{r^2}\right). \tag{8.22}$$

We would like to discuss variations $\delta_{ij} = \mathcal{O}\left(\frac{1}{r}\right)$ and $\delta\pi^{ij} = \mathcal{O}\left(\frac{1}{r^2}\right)$. Assume $N = 1 + \mathcal{O}\left(\frac{1}{r}\right)$ and $N^i \to 0$ as $r \to \infty$.

Vary the Hamiltonian in the region inside S^2_r, the two-sphere of radius r. From the variation $\frac{\delta H}{\delta \pi^{ij}}$, we obtain a surface term that vanishes as $r \to \infty$. The variation $\frac{\delta H}{\delta h_{ij}}$ gives two surface terms $S_1 + S_2$. It turns out that $S_2 \to 0$ as $r \to \infty$. However,

$$\lim_{r\to\infty} S_1 = -\lim_{r\to\infty} \int_{S^2_r} \mathrm{d}A\, n_i(\partial_j \delta h_{ij} - \partial_i \delta h_{ij}), \tag{8.23}$$

where dA is the area element on S_r^2 and n_i the outward unit normal to S_r^2. The nice thing about this formula is that we can pull the variation outside of the integral:

$$\lim_{r\to\infty} S_1 = -\delta E_{\mathrm{ADM}}, \tag{8.24}$$

where

$$E_{\mathrm{ADM}} = \lim_{r\to\infty} \int_{S_r^2} dA n_i(\partial_j h_{ij} - \partial_i h_{jj}), \tag{8.25}$$

so the surface term S_1 is itself the variation of something. Let $H' = H + E_{\mathrm{ADM}}$, so that the variations cancel. In other words, this quantity H' is the correct Hamiltonian of GR for asymptotically flat data.

8.4 ADM Energy

Let us revert back to units $G = 1$, rather than $16\pi G = 1$, so we will get some new factors appearing.

Definition 63 (ADM energy)**:** The *ADM energy* of an asymptotically flat end is

$$E_{\mathrm{ADM}} = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r^2} \mathrm{d}A\, n_i(\partial_j h_{ij} - \partial_i h_{jj}). \tag{8.26}$$

If we have multiple asymptotically flat ends, we have for each end an associated ADM energy.

For a stationary spacetime,

$$E_{\mathrm{ADM}} = M_{\mathrm{Komar}}, \tag{8.27}$$

at least if t-constant surfaces are orthogonal to k^a as $r \to \infty$. For the Kerr–Newmann spacetime,

$$E_{\mathrm{ADM}} = M, \tag{8.28}$$

which is why we called M the mass in the first place.

This gives us a satisfactory notion of energy for any asymptotically flat end.

We can define a notion of energy, but also something analogous to the three-momentum:

Definition 64 (ADM 3-momentum)**:** The *ADM 3-momentum* is

$$P_i = \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r^2} \mathrm{d}A\,(K_{ij} n_j - K n_i). \tag{8.29}$$

This gives a non-zero value if we have objects moving, i.e. are not working in the object's rest frame.

The energy of the Newtonian gravitational field is negative, so you might wonder whether the E_{ADM} might be negative. Yes it can! In fact, taking $M < 0$ Schwarzschild, we get $E_{\text{ADM}} < 0$. However, we have already discussed how the spacetime is pathological; it has a naked singularity at $r = 0$ and is not geodesically complete. Another way you can get negative E_{ADM} is by considering negative-energy matter.

The interesting case is if we have the dominant energy condition and the spacetime is non-singular. Can the E_{ADM} be negative in that case?

Theorem 20 (Positive Energy Theorem (Shoen–Yau '79, Witten '81))**:** Let (Σ, h_{ab}, K_{ab}) be geodesically complete, asymptotically flat initial data. Assume dominant energy condition. Then

$$E_{\text{ADM}} \geq \sqrt{P_i P_i}, \tag{8.30}$$

with equality iff (Σ, h_{ab}, K_{ab}) is surface in Minkowski.

If we had a black hole, we might not want to assume what happens inside a black hole. There is another version of the theorem which has an inner boundary on Σ, corresponding to the black hole horizon.

Definition 65 (ADM mass)**:** Finally, we can regard (E_{ADM}, P_i) as a 4-vector at spatial infinity i^0. We can then look at the norm of this. The *ADM* mass is

$$M_{\text{ADM}} = \sqrt{E_{\text{ADM}}^2 - P_i P_i} \geq 0. \tag{8.31}$$

9 Black Hole Mechanics

9.1 Killing Horizons and Surface Gravity

Definition 66 (Killing horizon)**:** A null hypersurface $\mathcal{N}$ is a *Killing horizon* if there exists a Killing vector field ξ^a in a neighbourhood of $\mathcal{N}$ such that ξ^a is normal to $\mathcal{N}$.

Theorem 21 (Hawking '72)**:** In a stationary, analytic, asymptotically flat vacuum black hole spacetime, the event horizon $\mathcal{H}^+$ is a Killing horizon.

Remark: In fact, this is why Killing horizons are important.

For the Kerr black hole, the event horizon $\mathcal{H}^+$ is a Killing horizon of $\xi^a = k^a + \Omega_H m^a$, not k^a. We can fix the freedom $\xi^a \to c\xi^a$ by $\xi^a = k^a + \Omega_H m^a$ with $k^2 \to -1$ at ∞.

Looking at the normal of the Killing field ξ on the hypersurface $\mathcal{N}$, we have

$$\xi^a \xi_a|_{\mathcal{N}} = 0, \tag{9.1}$$

so $\mathcal{N}$ is a surface of constant ξ^2. This means that

$$d(\xi^a \xi_a)|_{\mathcal{N}} \perp \mathcal{N}. \tag{9.2}$$

Thus, since ξ^a is the normal to $\mathcal{N}$, we must have

$$\nabla_a(\xi^b \xi_b)|_{\mathcal{N}} = -2\kappa \xi_a, \tag{9.3}$$

for some $\kappa : \mathcal{N} \to \mathbb{R}$ called the *surface gravity* of $\mathcal{N}$.

The left-hand side of (9.3) is

$$2\xi^b \nabla_a \xi_b \overset{\text{KVF}}{=} -2\xi^b \nabla_b \xi_a. \tag{9.4}$$

Therefore, we obtain the geodesic equation

$$\xi^b \nabla_b \xi^a|_{\mathcal{N}} = \kappa \xi^a. \tag{9.5}$$

In other words, κ measures the failure of ξ to follow an affinely parametrised geodesic.

Let n^a be tangent to the affinely parametrised generators of $\mathcal{N}$. Then $\xi^a|_{\mathcal{N}} = fn^a$ for some $f : \mathcal{N} \to \mathbb{R}$.

$$\therefore \xi^b \nabla_b \xi^a = fn^b \nabla_b (fn^a) = f^2 \cancel{n^b \nabla_b n^a} + fn^a n^b \nabla_b f = f^{-1} \xi^a \xi^b \partial_b f. \tag{9.6}$$

From this we can read off

$$\kappa = \xi \cdot \partial \log |f|, \tag{9.7}$$

where f might be negative, which is why we used the modulus here.

Example 9.1.1 (RN, ingoing EF)**:** The metric is

$$ds^2 = -\frac{\Delta}{r^2} dv^2 + d2vdr + r^2 d\Omega^2, \tag{9.8}$$

with $\Delta = (r - r_+)(r - r_-)$ with $r_\pm = M \pm \sqrt{M^2 - e^2}$ and $k = \frac{\partial}{\partial v}$.

$$k_a|_{r=r_\pm} = (dr)_a. \tag{9.9}$$

Now $\Delta = 0$, therefore

$$k^a \perp r = r_\pm, \tag{9.10}$$

so this gives Killing horizons.

$$d(k^b k_\beta) = d\left(-\frac{\Delta}{r^2}\right) = \left(-\frac{\Delta'}{r^2} + \frac{2\Delta}{r^3}\right) dr \overset{r=r_\pm}{=} -\frac{r_\pm - r_\mp}{r_\pm^2} dr = -\frac{(r_\pm - r_\mp)}{r_\pm^2} k|_{r=r_\pm} \tag{9.11}$$

Therefore, we have

$$\kappa = \kappa_\pm \equiv \frac{r_\pm - r_\mp}{2r_\pm^2} \tag{9.12}$$

which is why we called it $\kappa_\pm$ before; this is the surface gravity.

Example 9.1.2 (Schw.)**:** This is obtained from $e = 0$, giving

$$\kappa = \frac{1}{4M}. \tag{9.13}$$

Example 9.1.3 (extreme RN)**:** Here $r_+ = r_-$, so $\kappa = 0$.

Exercise 9.1 (Kruskal)**:** Show that the horizons $\mathcal{H}^+ = \{U = 0\}$ and $\mathcal{H}^- = \{V = 0\}$ are Killing horizons of

$$k = \frac{1}{4M}\left(V \frac{\partial}{\partial V} - U \frac{\partial}{\partial U}\right). \tag{9.14}$$

Show further that the surface gravity is $\kappa = \pm \frac{1}{4M}$.

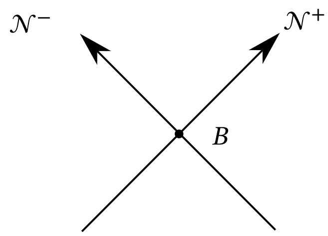


Figure 9.1: Bifurcate Killing horizon

Bifurcate Killing Horizon

Let $N^{\pm}$ be Killing horizons of the *same* ξ^a. The bifurcation surface $B = \mathcal{N}^+ \cap \mathcal{N}^-$. On it, $\xi^a|_B = 0$.

$$X^a \parallel B \;\Rightarrow\; X^a \parallel \mathcal{N}^+ \text{ and } \mathcal{N}^- \;\Rightarrow\; X^a \text{ spacelike.} \tag{9.15}$$

However, since it cannot be normal to both $\mathcal{N}^{\pm}$ at B, we have that B is a spacelike surface.

Example 9.1.4 (Kruskal)**:** For the Kruskal spacetime, this is the 2-sphere $B = \{U = V = 0\}$.

9.2 Interpretation of Surface Gravity κ

The main reason why κ is important is its relation to temperature and Hawking radiation, which we will meet in a later chapter. However, κ is also of classical interest. Consider a static, asymp. flat black hole spacetime. Assume there is a unit mass particle on orbit of k^a, with $u^a \parallel k^a$. Illustrated in

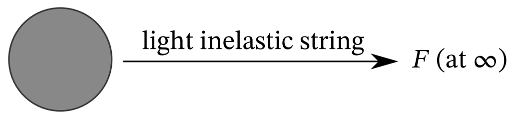


Figure 9.2

Fig. 9.2, the force F to hold a particle at this orbit tends to κ at infinity. Take four-velocity $k^a/\sqrt{-k^2}$. Then the proper acceleration is

$$A^a = u \cdot \nabla u^a = \frac{k \cdot \nabla k^a}{-k^2} + \frac{k^a}{2(-k^2)^2} k \cdot \nabla(k^2). \tag{9.16}$$

We now evaluate these terms using that k^a is a Killing vector field. The first term is

$$k^b \nabla_b k_a = -k^b \nabla_a k_b = -\nabla_a(\frac{1}{2}k^2). \tag{9.17}$$

The second term vanishes due to symmetry:

$$k \cdot \nabla(k^2) = 2k^a k^b \nabla_a k_b = 0. \tag{9.18}$$

Therefore, we can write the 4-acceleration of this particle as the logarithmic derivative

$$A_a = \frac{\partial_a(-k^2)}{2(-k^2)} = \frac{1}{2}\partial_a \ln(-k^2). \tag{9.19}$$

Thus, since $k^2 \to 0$ at the horizon, we have $A_a \to \infty$.

Example 9.2.1 (Schw)**:** We have the one-form

$$A = \frac{1}{2} d \ln\left(1 - \frac{2M}{r}\right) = \frac{M}{r^2\left(1 - \frac{2M}{r}\right)} \, \mathrm{d}r \,. \tag{9.20}$$

The norm of this one-form is

$$|A| = \sqrt{g^{ab} A_a A_b} = \sqrt{\frac{M^2}{r^4(1 - \frac{2M}{r})}} = \frac{M}{r^2\sqrt{1 - \frac{2M}{r}}}, \tag{9.21}$$

which indeed diverges $|A| \to \infty$ as $r \to 2M$. Therefore, the force needed will also diverge, so that in practice the string breaks.

9.3 The Zero$^{\text{th}}$ Law of Black Hole Mechanics

The laws are named in analogy to the laws of thermodynamics.

Claim 18: Consider a null geodesic congruence containing generators of Killing horizon $\mathcal{N}$. Then the expansion, rotation, and shear vanish $\theta = \hat{\sigma} = \hat{\omega} = 0$ on $\mathcal{N}$.

Proof. Let U^a be tangent to affinely parametrised generators of $\mathcal{N}$. Since $U \perp \mathcal{N}$, the rotation $\hat{\omega}|_{\mathcal{N}} = 0$. Moreover, there must be some $\xi \perp \mathcal{N}$ with $\xi^a|_{\mathcal{N}} = hU^a$. Take some $h : \mathcal{N} \to \mathbb{R}$. Let $\mathcal{N}$ have equation $f = 0$.

$$U^a = h^{-1}\xi^a + fV^a, \tag{9.22}$$

in other words, U and ξ are proportional on $\mathcal{N}$, but not when we move off $\mathcal{N}$. Now

$$B_{ab} = \nabla_b U_a = \partial_b h^{-1}\xi_a + h^{-1}\nabla_b\xi_a + \partial_b f V_a + f\nabla_b V_a. \tag{9.23}$$

To calculate the shear and expansion, we need to symmetrise and evaluate on $\mathcal{N}$:

$$B_{(ab)}\big|_{\mathcal{N}} = \left[\xi_{(a}\partial_{b)}h^{-1} + V_{(a}\partial_{b)}f\right], \tag{9.24}$$

with $\xi_a, \partial_a f \parallel U_a$ on $\mathcal{N}$. Therefore,

$$\hat{B}_{(ab)}|_{\mathcal{N}} = P_a{}^c B_{(cd)} P^d{}_a|_{\mathcal{N}} = 0, \qquad (Pu = 0). \tag{9.25}$$

Thus, $\theta = \hat{\sigma}_{ab} = 0$ on $\mathcal{N}$. □

Theorem 22 (Zero$^{\text{th}}$ law)**:** κ is constant on $\mathcal{H}^+$ of a stationary black hole, assuming the DEC.

Proof. The event horizon $\mathcal{N} = \mathcal{H}^+$ is a Killing horizon (Hawking) of some Killing vector field ξ^a. Let us use the Raychandhuri equation for the generators of $\mathcal{H}^+$:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^2 + \hat{\omega}^2 - R_{ab}U^aU^b. \tag{9.26}$$

All terms except the last vanish

$$R_{ab}U^aU^b|_{\mathcal{H}^+} = 0. \tag{9.27}$$

Which means that

$$\therefore 0 = R_{ab}\xi^a\xi^b|_{\mathcal{H}^+} = 8\pi(T_{ab} - \frac{1}{2}T^c{}_c g_{ab})\xi^a\xi^b|_{\mathcal{N}} = 8\pi T_{ab}\xi^a\xi^b|_{\mathcal{N}}. \tag{9.28}$$

And therefore $J \cdot \xi|_{\mathcal{N}} = 0$, where $J^a = -T_{ab}\xi^b$. Now the DEC means that J^a is future-directed causal (or zero). Now if ξ is future-directed causal, then $J \cdot \xi|_{\mathcal{N}} = 0$ implies that

$$J|_{\mathcal{N}} \parallel \xi. \tag{9.29}$$

Using the Einstein equations, we can write the following in terms of the Ricci tensor

$$0 = \xi_{[a}J_{b]}|_{\mathcal{N}} = -\xi_{[a}J_{b]}\xi^c|_{\mathcal{N}} \tag{9.30}$$

$$= -\frac{1}{8\pi}\xi_{[a}R_{b]c}\xi^c|_{\mathcal{N}}. \tag{9.31}$$

Exercise 9.2 (Sheet 4)**:**

$$\Rightarrow \xi_{[a}\partial_{b]}\kappa|_{\mathcal{H}^+} = 0. \tag{9.32}$$

This means that $\partial_a\kappa \parallel \xi_a$. Taking a tangent vector t^a tangent to $\mathcal{H}^+$, we have $t \cdot \partial\kappa = 0$. This means (assuming a single black hole) that the surface gravity κ is constant on the event horizon $\mathcal{H}^+$. □

9.4 First Law

Consider the change of parameters $M \to M + \delta M$, $a \to \alpha + \delta a$ in Kerr. The linearised change δA in the event horizon area is related to the change δM of mass and δJ of angular momentum as

$$\frac{\kappa}{8\pi}\delta A = \delta M - \Omega_H \delta J. \tag{9.33}$$

It turns out that this is true not only for the change of parameters above, but for any change of parameters.

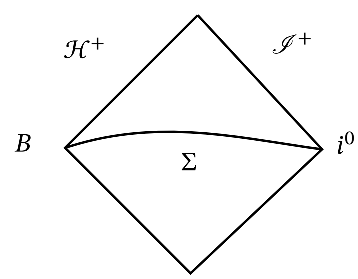


Figure 9.3

Let Σ be asymp. orthogonal to k^a near i^0. Let $(\Sigma \setminus B, h_{ab}, K_{ab})$, where h_{ab}, K_{ab} is the induced data, be an asymptotically flat end. Then

$$h_{ab} \to h_{ab} + \delta h_{ab}, \qquad K_{ab} \to K_{ab} + \delta K_{ab}. \tag{9.34}$$

If $\delta h_{ab}, \delta K_{ab}$ obey linearised constraint equations, then (9.33) holds where δA is the change in area of B, δM the change of ADM energy, and δJ the change in ADM angular momentum (which we have not defined). (Proof by Sudarsky and Wald in '92.)

Can generalise this to Einstein–Maxwell, introducing $-\Phi_H \delta Q$ on RHS of (9.33). We interpret Φ_H as the potential difference between $\mathcal{H}^+$ and ∞. This is a particular version of the first law of black hole mechanics, called the *equilibrium state* version. We are actually comparing two different spacetimes when writing down this formula.

There is also another version as follows: Let $T_{ab} = \mathcal{O}(\epsilon)$, and let $J^a = -T^a{}_b k^b$ as well as $L^a = T^a{}_b m^b$ be energy and angular momentum currents respectively. Due to backreaction on the metric, these are not exactly conserved. However, they are of order ϵ. The backreaction is also of order ϵ, so the failure of these to be conserved will be a second order effect in ϵ. In other words, the divergence of these currents is

$$\nabla_a J^a = \nabla_a L^a = \mathcal{O}(\epsilon^2), \tag{9.35}$$

so we can safely ignore them. Consider matter crossing over the null hypersurface $\mathcal{N}$ as shown in Fig. 9.4.

$$\delta M = -\int_{\mathcal{N}} \star J, \tag{9.36}$$

$$\delta J = -\int_{\mathcal{N}} \star L \tag{9.37}$$

Use Gaussian null coordinates (r, λ, y^i) on $\mathcal{H}^+$. We may choose $\lambda = 0$ on B. The event horizon is $\mathcal{H}^+$ is $r = 0$ and $\mathcal{N}$ is $r = 0, \lambda > 0$.

Order y^1, y^2 such that the volume form at $\mathcal{H}^+$ is $\eta = \sqrt{h} d\lambda \wedge dr \wedge dy^1 \wedge dy^2$. An orientation for $\mathcal{N}$ is obtained by viewing it as the boundary of $\{r > 0\}$. Finally, Stokes' theorem fixes the orientation $d\lambda \wedge dy^1 \wedge dy^2$. On $\mathcal{N}$,

$$(\star J)_{\lambda 12} = \sqrt{h} J^r = \sqrt{h} J_\lambda = \sqrt{h} U \cdot J, \tag{9.38}$$

where $U = \frac{\partial}{\partial \lambda}$ is the tangent to affinely parametrised geodesics. Therefore,

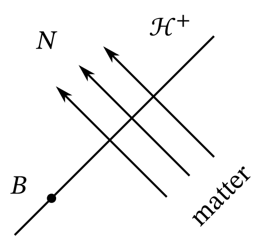


Figure 9.4

$$\delta M = -\int_{\mathcal{N}} \mathrm{d}\lambda \, \mathrm{d}^2 y \sqrt{h} U \cdot J. \tag{9.39}$$

Similarly, we find

$$\delta J = -\int_{\mathcal{N}} \mathrm{d}\lambda \, \mathrm{d}^2 y \sqrt{h} U \cdot L. \tag{9.40}$$

Since $J^a, L^a = \mathcal{O}(\epsilon)$, we can evaluate δM and δJ to $\mathcal{O}(\epsilon)$ using the Kerr metric.

Take $\mathcal{N}$ to be the Killing horizon of $\xi = k + \Omega_H M$. Now $\xi = fU$ for some function f. We saw previously that $\xi \cdot \partial \ln |f| = \kappa$, so

$$U \cdot \partial f = \kappa \Rightarrow \frac{\partial f}{\partial \lambda} = \kappa \tag{9.41}$$

Therefore

$$f = \kappa\lambda + f_0(y). \tag{9.42}$$

We also know that $\xi|_B = 0$, so $f = 0$ on B, meaning that $\lambda = 0$. Therefore,

$$f_0 = 0 \implies \xi^a = \kappa\lambda U^a \tag{9.43}$$

on $\mathcal{N}$. Substituting this into our definition $J^a = -T^a{}_b k^b$ and the formula for δM, we have

$$\delta M = \int_{\mathcal{N}} \mathrm{d}\lambda\, \mathrm{d}^2 y\, \sqrt{h} T_{ab} U^a k^b = \int_{\mathcal{N}} \mathrm{d}\lambda\, \mathrm{d}^2 y\, \sqrt{h} T_{ab} U^a (\xi^b - \Omega_H M^b) \tag{9.44}$$

$$= \int_{\mathcal{N}} \mathrm{d}\lambda\, \mathrm{d}^2 y\, \sqrt{h} T_{ab} U^a U^b \kappa\lambda - \Omega_H \underbrace{\int_{\mathcal{N}} \mathrm{d}\lambda\, \mathrm{d}^2 y\, \sqrt{h} U \cdot L}_{-\delta J}. \tag{9.45}$$

Finally, let us use the Einstein equation $8\pi T_{ab} U^a U^b = R_{ab} U^a U^b$, we have

$$\implies \delta M - \Omega_H \delta J = \frac{\kappa}{8\pi} \int \mathrm{d}\lambda\, \mathrm{d}^2 y\, \sqrt{h} \lambda R_a U^a U^b. \tag{9.46}$$

We will show that this is equal to the change in the area of the event horizon.

To calculate the integral, we look at Raych. equation (which is essentially one of the components of Eintein's eqaution). We are looking at perturbed Kerr, but the generators of the null hypersurface $\mathcal{N}$ still have vanishing rotation $\hat{\omega}|_{\mathcal{N}} = 0$. However, $\theta, \hat{\sigma} = \mathcal{O}(\epsilon)$ are not zero. Therefore,

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -R_{ab}U^a U^b + \mathcal{O}(\epsilon^2). \tag{9.47}$$

Conveniently, this is exactly the Ricci term in the integral. Substituting and integrating by parts with respect to λ gives

$$\delta M - \Omega_H \delta J = -\frac{\kappa}{8\pi}\int \mathrm{d}^2 y \int_0^\infty \sqrt{h}\lambda \frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\,\mathrm{d}\lambda \tag{9.48}$$

$$= -\lambda(8\pi)\int \mathrm{d}^2 y \left\{[\sqrt{h}\lambda\theta]_0^\infty - \int_0^\infty \left(\sqrt{h} + \lambda\frac{\mathrm{d}\sqrt{h}}{\mathrm{d}\lambda}\right)\theta\,\mathrm{d}\lambda\right\}. \tag{9.49}$$

$$\theta = \frac{1}{\sqrt{h}}\frac{\mathrm{d}\sqrt{h}}{\mathrm{d}\lambda}, \qquad \theta\frac{\mathrm{d}\sqrt{h}}{\mathrm{d}\lambda} = \theta^2\sqrt{h} = \mathcal{O}(\epsilon^2), \tag{9.50}$$

so we can ignore that term. If the black hole reaches equilibrium as $\lambda \to \infty$, then $\sqrt{h}$ becomes finite in the limit $\lambda \to \infty$. Now

$$\int_0^\infty \sqrt{h}\theta\,\mathrm{d}\lambda = \int_0^\infty \frac{\mathrm{d}\sqrt{h}}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \delta\sqrt{h}. \tag{9.51}$$

The right-hand side is finite, which means the left-hand side must be, too. Therefore, $\theta = o(1/\lambda)$, meaning that θ tends to zero faster than $1/\lambda$ as $\lambda \to \infty$. This means that the boundary term in (9.49) vanishes.

$$\delta M - \Omega_H \delta J = \frac{\kappa}{8\pi}\,\mathrm{d}^2 y\,\delta\sqrt{h} = \frac{\kappa}{8\pi}\delta\int \mathrm{d}^2 y\,\sqrt{h} = \frac{\kappa\delta A}{8\pi}. \tag{9.52}$$

This is the first law of black hole thermodynamics.

9.5 Second Law

One of Stephen Hawking's greatest hits. Informally, the area of the event horizon of a black hole can only increase.

Theorem 23 (Hawking '72)**:** Let (M, g) be a strongly asymptotic predictable spacetime, satisfying the Einstein equation and the NEC. This means that there is $U \subset M$ a globally hyperbolic region with $\overline{J^-(\mathcal{J}^+)} \subset U$. Let Σ_1, Σ_2 be spacelike Cauchy surfaces for U such that $\Sigma_2 \subset J^+(\Sigma_1)$ and $H_i = \mathcal{H}^+ \cap \Sigma_i$. Then

$$\mathrm{area}(H_2) \geq \mathrm{area}(H_1). \tag{9.53}$$

Proof (sketch). We put an extra assumption in: we assume the generators of the future event horizon $\mathcal{H}^+$ to be future-complete ("$\mathcal{H}^+$ is non-singular"). You can relax this assumption, but it makes the proof harder.

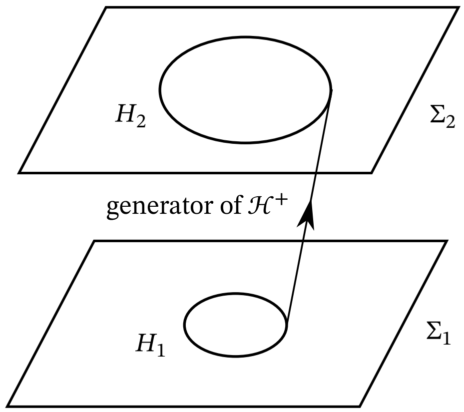


Figure 9.5: Second law of black hole mechanics, showing a horizon generator.

Claim 19: $\theta \geq 0$ on $\mathcal{H}^+$.

Proof. Assume $\theta < 0$ at $p \in \mathcal{H}^+$. Let γ be a future-inextendible generator of $\mathcal{H}^+$ through p. Then $\theta < 0 \implies \theta \to -\infty$ at $r \in \gamma$ within affine parameter $2/|\theta|$, where $|\theta|$ is the radius of θ at p. This means that the congruence is singular at r: infinitesimally nearby geodesics γ' from p intersect γ at r, as shown in Fig. 9.6. "smooth corners" $\to$ timelike curve from p to r. Now move p to p' in the

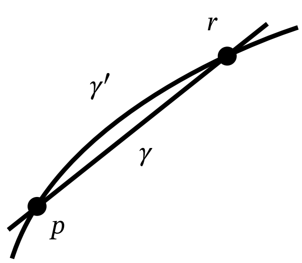


Figure 9.6

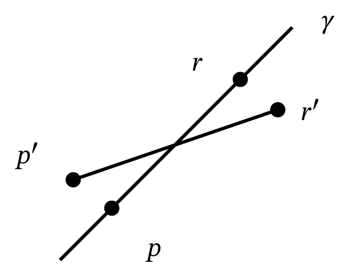


Figure 9.7

black hole region and r to r' outside $\mathcal{H}^+$ as shown in Fig. 9.7. Since we have a timelike curve from p to r, we also have a timelike curve from p' to r'. However, $r' \in J^-(\mathcal{J}^+)$ and the existence of such a curve would imply $p' \in J^-(\mathcal{J}^+)$, which contradicts the assumption of a black hole. Therefore, $\theta \geq 0$ on $\mathcal{H}^+$. $\square$

Let $\phi : H_1 \to H_2$, $p \longmapsto$ intersection of generator through p with H_2 (Σ_2 Cauchy). Therefore,

$$\text{area}(H_2) \overset{\phi(H_1)\subset H_2}{\geq} \text{area}(\phi(H_1)) \overset{\theta\geq 0}{\geq} \text{area}(H_1). \tag{9.54}$$

$\square$

Example 9.5.1 (spherically symmetric collapse)**:** Consider the Finkelstein diagram in Fig. 9.8.

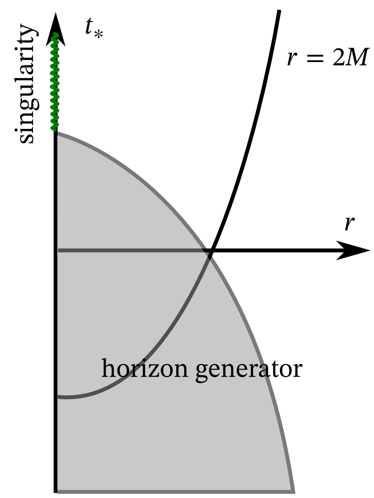


Figure 9.8: Spherically symmetric collapse.

Example 9.5.2 (Black hole merger)**:** Assume initial black holes are well separated and described by Schw. and that the final black hole is also described by Schwarzschild. The second law then tells us

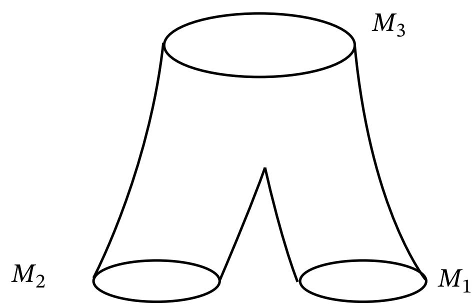


Figure 9.9: Black hole merger.

that

$$A_3 \geq A_1 + A_2 \quad \Rightarrow \quad 16\pi M_3^2 \geq 16\pi M_1^2 + 16\pi M_2^2. \tag{9.55}$$

Thus

$$M_3 \geq \sqrt{M_1^2 + M_2^2}. \tag{9.56}$$

The energy radiated in gravitational waves is

$$E = M_1 + M_2 - M_3. \tag{9.57}$$

In principle, an advanced alien civilisation would be able to extract energy from this process. The *efficiency* of the process is

$$\eta = \frac{E}{M_1 + M_2} = 1 - \frac{M_3}{M_1 + M_2} \leq 1 - \frac{\sqrt{M_1^2 + M_2^2}}{M_1 + M_2} \overset{x = M_2/M_1}{=} 1 - \frac{\sqrt{1 + x^2}}{1 + x} \leq 1 - \frac{1}{\sqrt{2}}. \tag{9.58}$$

9.5.1 Penrose Inequality

Initial data asymp. flat and contains trapped surface begin apparent horizon of area A_{app}. The initial ADM energy is E_i. By the WCC, a singularity must form, which lies inside the black hole horizon, giving a black hole spacetime. The uniqueness theorems imply that we expect this to 'settle down' to the Kerr black hole, parameters (M_f, J_f). The apparent horizon is inside $\mathcal{H}^+$ so we expect $A_{\text{app}} \leq A_i$, the initial area of $\mathcal{H}^+$. By the second law,

$$\begin{aligned} A_{\text{app}} &\leq A_i & (9.59)\\ &\leq A_{\text{Kerr}}(M_f, J_f) & (9.60)\\ &= 8\pi(M_f^2 + \sqrt{M_f^4 - J_f^2}) & (9.61)\\ &\leq 16\pi M_f^2 & (9.62)\\ &\leq 16\pi E_i^2, & (9.63) \end{aligned}$$

($M_f \leq E_i$: energy lost to gravitational waves.) This gives us a precise mathematical statement: the *Penrose inequality*:

$$E_i \geq \sqrt{\frac{A_{\text{app}}}{16\pi}}. \tag{9.64}$$

Both sides refer only to initial data. This allows us to test our assumptions (for example of weak cosmic censorship). There has not yet been found a counter example to this. In fact, the above inequality has been proved (Huisken and Ilmanen '97) for *time-symmetric* initial data ($K_{ab} = 0$) with matter obeying the weak energy condition.

Remark: The inequality can be regarded as a stronger version of the positive mass theorem.

10 QFT in Curved Spacetime

10.1 Introduction

Consider a black hole at rest $E = M$. Now consider a thermodynamic system with the same energy E and momentum J of the black hole. The first law of thermodynamics tells us

$$dE = TdS + \mu dJ, \tag{10.1}$$

where μ is the chemical potential for angular momentum. This looks very much like the first law of black hole mechanics if we make the following identifications:

$$T = \lambda\kappa \qquad S = \frac{A}{8\pi\lambda}, \qquad \mu = \Omega_H, \tag{10.2}$$

where λ is some arbitrary constant. Similarly, the zero$^{\text{th}}$ law of thermodynamics states that T is constant in thermal equilibrium. If this identification is correct, we should have κ being constant, which is exactly the zero$^{\text{th}}$ law of black hole mechanics. Similarly, the second law of thermodynamics states that the entropy S can only increase, which corresponds to the statement that the area A can only increase.

This suggests that we should think of a black hole as a thermodynamic system.

Think about a box of hot gas falling into the black hole. It seems like this violates the second law of thermodynamics, since the entropy just vanishes. However, we can remedy this by considering the black hole to have an entropy proportional to its area. However, temperature would imply that the black hole radiates (which it does not to classically).

Then Hawking in '74 found that (when including QFT considerations) black holes emit thermal radiation at the *Hawking temperature*

$$T_H = \frac{\hbar\kappa}{2\pi}, \tag{10.3}$$

which fixes λ. This means that black holes really are thermodynamic objects with a temperature. This is the reason why we are interested in QFT in curved spacetime.

10.2 Quantisation of a Free Scalar

Classically, we need to work in a globally hyperbolic spacetime, so we assume the same to be true quantum mechanically. Let (M, g) be a globally hyperbolic spacetime with metric

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{10.4}$$

where h_{ij} is a metric on Σ_t, which is a constant-t surface. Then it is obvious that $\sqrt{-g} = N\sqrt{h}$. We therefore look at a real Klein–Gordon field ϕ with action

$$S = \int_M \mathrm{d}t\, \mathrm{d}^3x \sqrt{-g}\left(-\frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi - \frac{1}{2}m^2\phi^2\right). \tag{10.5}$$

The equations of motion are the Klein–Gordon equation in curved spacetime

$$g^{ab}\nabla_a\nabla_b\phi - m^2\phi = 0. \tag{10.6}$$

To quantise, we need to find the canonical momentum

$$\pi(x) = \frac{\delta S}{\delta(\partial_t\phi(x))} = -\sqrt{-g}g^{t\mu}\partial_\mu\phi \tag{10.7}$$

$$= -N\sqrt{h}(dt)_\nu g^{\nu\mu}\partial_\mu\phi \tag{10.8}$$

$$= \sqrt{h}n^\mu\partial_\mu\phi. \tag{10.9}$$

Here, $n_a = -N(dt)_a$ is a future-directed unit normal to surfaces of constant t. Quantisation is done by promoting ϕ, π to *operators* obeying canonical commutation relations

$$[\phi(t,x), \pi(t,x')] = \delta^{(3)}(x - x'), \qquad [\phi(t,x), \phi(t,x')] = 0 = [\pi(t,x), \pi(t,x')]. \tag{10.10}$$

We use units of $\hbar = 1$ throughout.

We now want to define a Hilbert space for the operators to act on. Define S to be the space of complex solutions of

$$g^{ab}\nabla_a\nabla_b\Phi - m^2\Phi = 0. \tag{10.11}$$

By global hyperbolicity, a solution $\Phi \in S$ is uniquely specified by initial data $(\Phi, \partial_t\Phi)$ on Σ_0.

Definition 67: If we are given two solutions $\alpha, \beta \in S$, we define a sesquilinear form on S as follows

$$(\alpha,\beta) = -\int_{\Sigma_0} \mathrm{d}^3x\,\sqrt{h}n_a j^a(\alpha,\beta), \tag{10.12}$$

where the current $j(\alpha,\beta) = -i(\overline{\alpha}d\beta - \beta d\overline{\alpha})$.

Claim 20: This current is conserved.

Proof. Using (10.11)

$$\nabla^a j_a = -i(\overline{\alpha}\nabla^2\beta - \beta\nabla^2\overline{\alpha}) = -im^2(\overline{\alpha}\beta - \beta\overline{\alpha}) = 0. \tag{10.13}$$

□

Corollary: We can replace Σ_0 by Σ_t in (10.12).

Remark: This form is

(i) Hermitian: $(\alpha,\beta) = \overline{(\beta\alpha)}$

(ii) non-degenerate: if $(\alpha,\beta) = 0\ \alpha\beta \in S$, then $\alpha = 0$

(iii) $(\alpha,\beta) = -(\overline{\beta},\overline{\alpha})$. Therefore, $(\alpha,\alpha) = -(\overline{\alpha},\overline{\alpha})$. As such, the form is not positive definite and is not an inner product.

Positive Frequency Solutions

In Minkowski space, the form $(,)$ is positive definite on a subspace $S_p \subset S$, the space of *positive frequency solutions*. A basis for S_p are the positive frequency plane waves

$$\Psi_{\mathbf{p}}(x) = \frac{1}{(2\pi)^{3/2}(2p^0)^{1/2}}e^{ip\cdot x}, \tag{10.14}$$

where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$ and x represents $(t, \mathbf{x})$; we work in the Heisenberg picture.

Given $k = \frac{\partial}{\partial t}$, we have

$$\mathcal{L}_k\Psi_{\mathbf{p}} = -ip^0\Psi_{\mathbf{p}}, \tag{10.15}$$

negative imaginary eigenvalues of $\mathcal{L}_k$.

Similarly, $\overline{\Psi}_{\mathbf{p}}$ are negative frequency plane waves. Since $(\Psi_{\mathbf{p}}, \overline{\Pi}_{\mathbf{q}}) = 0$, we can orthogonally decompose

$$S = S_p \oplus \overline{S}_p. \tag{10.16}$$

For a general spacetime (M, g), there is no preferred definition of 'positive frequency'. Just pick $S_p \subset S$ such that $(,)$ is positive definite on S_p and (10.16) holds. In general, there are many ways of choosing S_p, which can be seen as the main reason why QFT in curved spacetime is different than on $\mathbb{M}^n$.

Fock Space

For $f \in S_p$, we define creation and annihilation operators

$$a(f) = (f, \Phi), \Rightarrow \; a(f)^\dagger = -(\overline{f}, \Phi), \tag{10.17}$$

where Φ is our quantum field, which we want to be Hermitian $\Phi^\dagger = \Phi$.

Example 10.2.1: If $f = \Psi_{\mathbf{p}}$, we get the standard Minkowski definition $a(f) = a_{\mathbf{p}}$.

Exercise 10.1 (Sheet 4)**:** Plugging this into our canonical commutation relations, we can show that these operators satisfy

$$[a(f), a(g)^\dagger] = (f, g), \qquad [a(f), a(g)] = [a(f)^\dagger, a(g)^\dagger = 0]. \tag{10.18}$$

Example 10.2.2: Again taking $f = \Psi_{\mathbf{p}}, g = \Psi_{\mathbf{q}}$, we find $[a_{\mathbf{p}}, a_{\mathbf{q}}] = \delta^{(3)}(\mathbf{p} - \mathbf{q})$.

From these, we build up our Hilbert space in the usual way

Definition 68 (vacuum)**:** The *vacuum state* $|0\rangle$ is defined by $a(f)\,|0\rangle = 0$ for all $f \in S_p$. We take it normalised to $\langle 0|0\rangle = 1$.

Definition 69 (particles)**:** Taking a basis $\{\Psi_i\}$ for S_p, we define $a_i = a(\Psi_i)$ and the *N-particle states* as $a^\dagger_{i_1} \dots a^\dagger_{i_N} \,|0\rangle$.

Claim 21: The Hilbert space is then the usual Fock space construction

$$\mathscr{H} = \text{vacuum} \oplus \text{1-particle states} \oplus \text{2-particle states} \oplus \dots \tag{10.19}$$

Proof. To check that this is really a Hilbert space, we need to show that we have a positive definite inner product

$$\left\| a(f)^\dagger \,|0\rangle \right\|^2 = \langle 0|\, a(f) a(f)^\dagger \,|0\rangle = \langle 0|\, [a(f), a(f)^\dagger]\, |0\rangle = (f, f) > 0, \tag{10.20}$$

as $f \in S_p$ (which is why it was so important to pick this positive frequency subspace S_p). □

If we have a different choice S'_p of 'positive frequency subspace', and $f' \in S'_p$, then (10.16) gives

$$f' = f + \overline{g}, \quad f, g \in S_p, \qquad a(f') = a(f) - a(g)^\dagger. \tag{10.21}$$

Therefore, $a(f')\,|0\rangle \neq 0$, so $|0\rangle$ is not the vacuum state if we use S'_p. The definition of $|0\rangle$, 1-particle states and so on all depend on the choice of S_p. In other words, there is no natural notion of particles in a general curved spacetime! In Minkowski space, we use the symmetry to pick out a preferred definition of positive frequency subspace.

Exercise 10.2 (Sheet 4)**:** In stationary spacetime (M, g), the Lie derivative $\mathcal{L}_k$ along the stationary Killing vector field k maps $S \to S$ and is antihermitian with respect to the sesquilinear form $(,)$.

This means that it has imaginary eigenvalues. Using this extra structure, we can make the following definition.

Definition 70: We say an eigenfunction $u \in S$ is positive frequency if

$$\mathcal{L}_k u = -i\omega u, \qquad \omega > 0. \tag{10.22}$$

Exercise 10.3 (Sheet 4)**:** This implies that $(u, u) > 0$.

Then S_p is the space spanned by such solutions. Taking $\overline{u}$ to be the negative frequencies, we split

$$S = S_p \oplus \overline{S}_p. \tag{10.23}$$

This is an orthogonal decomposition as $\mathcal{L}_k$ is antihermitian and therefore $u \perp \overline{u}$.

10.3 Bogoliubov Transforms

Let $\{\Psi_i\}$ be an orthonormal basis for S_p.

$$(\Psi_i, \Psi_j) = \delta_{ij} \quad \Rightarrow \quad (\overline{\Psi}_i, \overline{\Psi}_j) = -\delta_{ij}. \tag{10.24}$$

Then

$$S_p \perp \overline{S}_p \quad \Rightarrow \quad (\Psi_i, \overline{\Psi}_j) = 0. \tag{10.25}$$

We define the quantum field expanded in this basis as

$$\Phi = \sum_i \left(c_i \Psi_i + d_i \overline{\Psi}_i \right). \tag{10.26}$$

Then

$$a_i = a(\Psi_i) = (\Psi_i, \Phi) = c_i \tag{10.27}$$

$$a_i^\dagger = -(\Psi_i, \Phi) = d_i. \tag{10.28}$$

Therefore, the quantum field is expanded in terms of creation and annihilation operators in this basis as

$$\Phi = \sum_i \left(a_i \Psi_i + a_i^\dagger \overline{\Psi}_i \right). \tag{10.29}$$

Now take S_p' to be a different choice of positive frequency subspace, with orthonormal basis $\{\Psi_i'\}$. These are related to the first basis by the Bog. transformation

$$\Psi_i' = \sum_j \left(A_{ij} \Psi_j + B_{ij} \overline{\Psi}_j \right) \qquad \overline{\Psi}_i' = \sum_j \left(\overline{B}_{ij} \Psi_j + \overline{A}_{ij} \overline{\Psi}_j \right), \tag{10.30}$$

where A, B are Bog. coefficients.

Exercise 10.4: Show that (10.30) gives

$$a_i' = (\Psi_i', \Phi) = \sum_j \left(\overline{A}_{ij} a_j - \overline{B}_{ij} a_j^\dagger \right). \tag{10.31}$$

Exercise 10.5: Show that orthogonality of $\{\Psi_i, \overline{\Psi}_i\}$ implies

$$\sum_k \left(\overline{A}_{ik} A_{jk} - \overline{B}_{ik} B_{jk} \right) = \delta_{ij} \qquad \text{i.e.} \quad AA^\dagger - BB^\dagger = 1 \tag{10.32}$$

$$\sum_k \left(A_{ik} B_{jk} - B_{ik} A_{jk} \right) = 0, \qquad AB^T - BA^T = 0. \tag{10.33}$$

10.4 Particle Production in Non-Stationary Spacetime

Consider a spacetime

$$M = M_+ \cup M_0 \cup M_-, \tag{10.34}$$

where $M_\pm$ are stationary. Hence, on $(M_\pm, g)$ there exists a preferred choice $S_p^\pm$. Glob. hyp.: solution in $(M_\pm, g)$ extends uniquely to (M, g). Therefore S_p^+, S_p^- define two *different* choices of positive frequency subspace. Let $\{u_i^\pm\}$ be an orthonormal basis for $S_p^\pm$. This allows us to find $a_i^\pm$. For example

$$u_i^+ = \sum_j \left(A_{ij} u_j^- + B_{ij} \overline{u_j^-} \right). \tag{10.35}$$

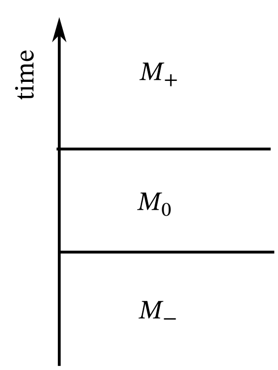


Figure 10.1

Then (10.31) gives

$$a_i^\dagger = \sum_j \left(\overline{A}_{ij} a_j^- - \overline{B}_{ij} a_j^{-\dagger} \right). \tag{10.36}$$

We have vacua $|0\pm\rangle$ with respect to $S_p^{\pm}$, which satisfy $a_i^{\pm}|0\pm\rangle = 0$. Assume no particles are present initially. Then we have the state $|0-\rangle$. The particle number operator is $N_i^+ = a_i^{+\dagger}a_i^+$. The expected number of particles of type i in M_+ (at 'late time')

$$\langle 0-| N_i^+ |0-\rangle = \langle 0-| a_i^{+\dagger}a_i^+ |0-\rangle = \sum_{j,k} \langle 0-| a_k^-(-B_{ik})(-\overline{B}_{ij})a_j^{-\dagger} |0-\rangle \tag{10.37}$$

$$= \sum_{j,k} B_{ik} \underbrace{\overline{B}_{ij} \langle 0-| a_k^- a_k^{-\dagger} |0-\rangle}_{\delta_{jk}} \tag{10.38}$$

$$= \sum_j B_{ij}\overline{B}_{ij} = (BB^\dagger)_{ii}. \tag{10.39}$$

The expected total number of particles in M_+ is $\mathrm{tr}(BB^\dagger) = \mathrm{tr}(B^\dagger B)$, which vanishes iff $B = 0$, i.e. $S_p^+ = S_p^-$. This is not true generically. Therefore, there will be particles present in M_+. Heuristically, we can think of particles being created by the gravitational field. However, if we do not have a stationary spacetime we cannot define particles uniquely and this heuristic notion is not accurate.

10.5 Rindler Spacetime

We think of *Rindler spacetime* as a toy model for the black hole horizon. Consider Schwarzschild at

$$r = 2M + \frac{x^2}{8M}. \tag{10.40}$$

Expanding in small x, the metric is

$$ds^2 \approx -k^2x^2dt^2 + dx^2 + (2M)^2d\Omega^2 + \dots, \qquad k = \frac{1}{4M}. \tag{10.41}$$

Rindler spacetime is

$$ds^2 = -k^2x^2dt^2 + dx^2, \qquad x > 0. \tag{10.42}$$

We have a coordinate singularity at $x = 0$. Let $U = -xe^{-kt}$, $V = xe^{kt}$. This gives metric

$$ds^2 = -dUdV = -dT^2 + dX^2, \tag{10.43}$$

with $U = T - X$ and $V = T + X$. So we see that Rindler space is the subspace of Minkowski with $x > 0$. In the Penrose diagram 10.2, it occupies region I.

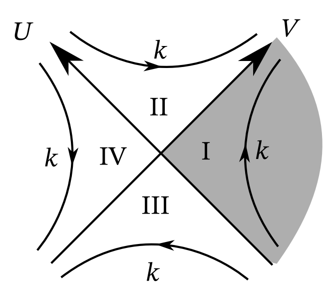


Figure 10.2: Rindler spacetime is the shaded subset of Minkowski spacetime.

We know that there is an isometry which brings us into IV, which is therefore also Rindler. We call region I the *right Rindler region R* and region IV the *left Rindler region L*.

The lines $U = 0$ and $V = 0$ correspond to a bifurcate killing horizon of

$$k = \frac{\partial}{\partial t} = \kappa(V\frac{\partial}{\partial V} - U\frac{\partial}{\partial U}) \tag{10.44}$$

with surface gravity $\pm\kappa$.

An orbit of k in Rindler space, drawn in Fig. 10.2, is a line of constant x with proper acceleration $A_a = \frac{1}{x}(dx)_a$. Therefore $|A| = \frac{1}{x}$. Such a *Rindler observer* would use k to define *positive frequency*. Then S_p is the usual Minkowski definition using $\frac{\partial}{\partial T}$.

Consider a massless scalar satisfying the wave-equation

$$(-\partial_T^2 + \partial_X^2)\phi = 0 \;\Rightarrow\; \phi = f(U) + g(V), \tag{10.45}$$

where $f(U)$ corresponds to right-moving and $g(V)$ to left-moving waves. The usual positive frequency solutions are

$$u_P(T,X) = c_p e^{-i(\omega T - pX)} = \begin{cases} c_p e^{-i\omega U}, & \text{if } p > 0 \quad \text{(R movers)} \\ c_p e^{-i\omega V}, & \text{if } p < 0 \quad \text{(L movers)} \end{cases}, \qquad \omega = |p|. \tag{10.46}$$

Solution with *Rindler frequency* σ is

$$\phi \propto e^{-i\sigma t}. \tag{10.47}$$

$$0 = \nabla^a \nabla_a \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = \frac{1}{x^2} [x \partial_x (x \partial_x \phi) + \frac{\sigma^2}{k^2} \phi] \tag{10.48}$$

$$\Rightarrow \phi = e^{-i\sigma t} x^{ip}, \qquad P = \pm \frac{\sigma}{k} \tag{10.49}$$

If $\sigma > 0$, then $p > 0$ implies that x increases with t along liens of constant phase (R movers). Conversely, $p < 0$ gives L-movers. We can write down a basis of positive Rinder-frequency solutions in the right region R

$$u_p^R = C_p e^{-i(\sigma t - P \ln x)}, \qquad \sigma = k|P|. \tag{10.50}$$

We want to find out the relation between these two definitions of positive frequency. Let us extend to all of Minkowski by defining $u_p^R = 0$ in L.

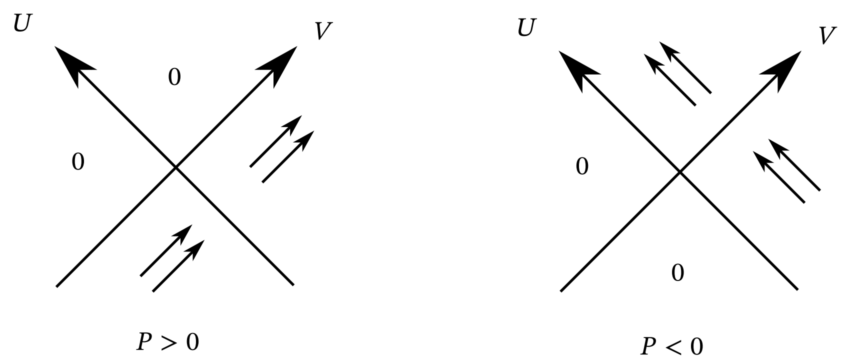


Figure 10.3: l20f2

$$u_p^R = \begin{cases} \begin{cases} c_p e^{i \frac{\sigma}{k} \ln(-U)}, & \text{if } U < 0 \\ 0, & \text{if } U > 0 \end{cases}, & \text{if } P > 0 \\ \begin{cases} 0, & \text{if } V < 0 \\ c_p e^{-i \frac{\sigma}{k} \ln V}, & \text{if } V > 0 \end{cases}, & \text{if } P < 0 \end{cases} \tag{10.51}$$

Plane waves are non-normalisable but are convenient to work with. We can make it rigorous by working with wave-packets instead. However, for convenience we will work with plane waves and assume they are normalisable (even though they are not) with the justification that we could make it rigorous by working with wave-packets.

Since $u_p^R = 0$ in region L, $\{u_p^R, \overline{u}_p^R\}$ is not a basis for Minkowski solutions. The isometry $(U, V) \to (-U, -V)$ gives solutions vanishing in R. Applying this map to (10.51) gives

$$\overline{u}_p^L = \begin{cases} \begin{cases} c_p e^{i\frac{\sigma}{k}\ln(U)}, & \text{if } U > 0 \\ 0, & \text{if } U < 0 \end{cases}, & \text{if } P > 0 \\ \begin{cases} 0, & \text{if } V > 0 \\ c_p e^{-i\frac{\sigma}{k}\ln(-V)}, & \text{if } V < 0 \end{cases}, & \text{if } P < 0 \end{cases} \tag{10.52}$$

Figure 10.3 changes to Fig. 10.4. Then $\{u_p^R, \overline{u}_p^R, u_p^L, \overline{u}_p^L\}$ gives a basis for solutions in Minkowski.

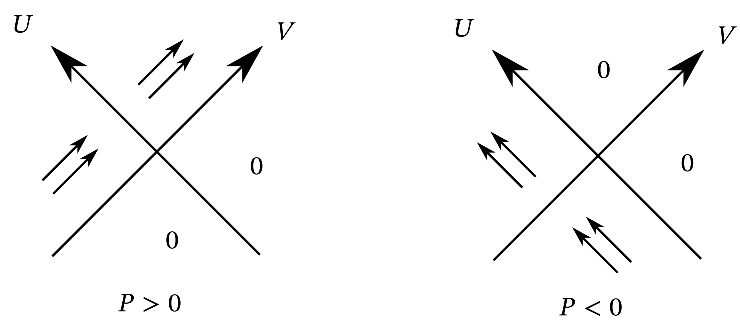


Figure 10.4

Let

$$f(U) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega U} \hat{f}(\omega), \qquad \widetilde{f}(U) = \int_{-\infty}^{\infty} \mathrm{d}U\, e^{i\omega U} f(\omega). \tag{10.53}$$

Assume $f(U)$ analytic in lower half plane.

$$\max_{\theta\in[-\pi,0]}\left|f(Re^{i\theta})\right| \to 0 \text{ as } R \to \infty. \tag{10.54}$$

Take $\omega < 0$. Then Jordan's lemma tells us that $\widetilde{f}(\omega) = 0$. Such $f(U)$ is positive frequency ($f \in S_p$) with respect to $\partial/\partial T$. Define the complex logarithm as

$$\ln z = \ln|z| + i\arg z, \qquad \arg z \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right). \tag{10.55}$$

This has a branch cut, as shown in Fig. 10.6.

For $P > 0, U > 0$,

$$\overline{u}_p^L = C_p e^{i\frac{\sigma}{\kappa}\ln(U)} = C_p e^{i\frac{\sigma}{\kappa}(\ln(-U)-i\pi)} = C_p e^{\sigma\frac{\pi}{\kappa}} e^{i\frac{\sigma}{\kappa}\ln(-U)}. \tag{10.56}$$

Since $P > 0$,

$$u_p^R + e^{-\frac{\sigma\pi}{\kappa}}\overline{u}_p^L = C_p e^{i\frac{\sigma}{\kappa}\ln(-U)} \qquad \forall U \quad (P > 0) \tag{10.57}$$

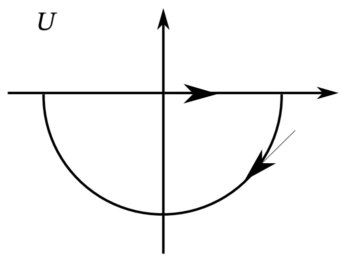


Figure 10.5

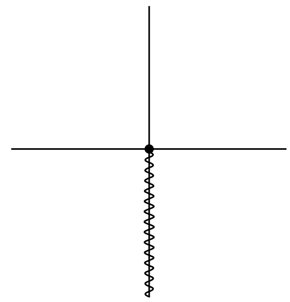


Figure 10.6: Branch cut

analytic in lower half plane and therefore in S_p.

For $P < 0$,
$$u_p^R + e^{-\frac{\pi\sigma}{\kappa}} \overline{u}_p^L = C_p e^{-\frac{\sigma\pi}{\kappa}} e^{-\frac{i\sigma}{\kappa}\ln(-V)} \tag{10.58}$$
analytic in lower half plane $\Rightarrow$ in S_p.

Similarly
$$u_p^L + e^{-\frac{\pi\sigma}{\kappa}} \overline{u}_p^R = \begin{cases} C_p e^{-\frac{\sigma\pi}{\kappa}} e^{-i\frac{\sigma}{\kappa}\ln(-U)}, & \text{if } P > 0 \\ C_p e^{i\frac{\sigma}{\kappa}\ln(-V)}, & \text{if } P < 0 \end{cases} \tag{10.59}$$
is analytic in lower half U, V-planes $\Rightarrow$ in S_p.

Therefore,
$$v_p^{(1)} = D_p^{(1)} \left(u_p^R + e^{-\frac{\sigma\pi}{\kappa}} \overline{u}_p^L \right) \qquad v_p^{(2)} = D_p^{(2)} \left(u_p^L + e^{-\frac{\sigma\pi}{\kappa}} \overline{u}_p^R \right) \tag{10.60}$$
are in S_p. The set $\{v_p^{(1)}, v_p^{(2)}, \forall p\}$ is a basis for S_p. Thus, the vacuum state $|0\rangle$ is defined by $a_p^{(i)}|0\rangle = 0$, where $a_p^{(i)}$ is the annihilation operator of $v_p^{(i)}$. This is the usual Minkowski vacuum.

$$(u_p^R, \overline{u}_p^L) = 0 \;\Rightarrow\; (v_p^{(1)}, v_p^{(1)}) = \left|D_p^{(1)}\right|^2 \left[(u_p^R, u_p^R) + e^{-2\pi\sigma/\kappa} (\overline{u}_p^L, \overline{u}_p^L) \right] \tag{10.61}$$
$$= 2\left|D_p^{(1)}\right|^2 e^{-\pi\sigma/\kappa} \sinh\!\left(\frac{\pi\sigma}{\kappa}\right) (u_p^R, r_p^R). \tag{10.62}$$

A similar expression holds for $v_P^{(2)}$. Choose the constant
$$D_P^{(i)} = \frac{e^{\pi\sigma/2\kappa}}{\sqrt{2\sinh(\pi\sigma/\kappa)}} \tag{10.63}$$
This means that $v_P^{(i)}$ is normalised in the same way as u_P^R.

Exercise 10.6: Invert (10.60) to show that
$$u_p^R = \frac{1}{\sqrt{2\sinh(\pi\sigma/\kappa)}} \left(e^{\pi\sigma/\kappa} v_P^{(1)} - e^{-\pi\sigma/\kappa} \overline{v_P^{(2)}} \right). \tag{10.64}$$

We can therefore relate the *Rindler annihilation operator* b_P^R to the usual Minkowski annihilation operators $a_P^{(i)}$ as

$$b_P^R := (u_P^R, \phi) = \frac{1}{\sqrt{2\sinh(\pi\sigma/\kappa)}}\left[e^{\pi\sigma/\kappa}(v_P^{(1)}, \phi) - e^{-\pi\sigma/\kappa}(\overline{v_P^{(2)}}, \phi)\right] \tag{10.65}$$

$$= \frac{1}{\sqrt{2\sinh(\pi\sigma/\kappa)}}\left[e^{\pi\sigma/\kappa}a_P^{(1)} + e^{-\pi\sigma/\kappa}a_P^{(2)}\right]. \tag{10.66}$$

Finally, we want to find the Rindler number operator N_P^R

$$\langle 0|\, N_P^R\, |0\rangle = \frac{e^{-\pi\sigma/\kappa}}{2\sinh(\pi\sigma/\kappa)}\langle 0|\, a_P^{(2)}a_P^{(2)\dagger}\, |0\rangle \tag{10.67}$$

Now we use

$$\langle 0|\, a_P^{(2)}a_P^{(2)\dagger}\, |0\rangle = \langle 0|\, [a_P^{(2)}, a_P^{(2)\dagger}]\, |0\rangle = (v_P^{(2)}, v_P^{(2)}) = (u_P^R, u_P^R), \tag{10.68}$$

$$\langle 0|\, N_P^R\, |0\rangle = \frac{e^{-\pi\sigma/\kappa}}{2\sinh(\pi\sigma/\kappa)}(u_P^R, u_P^R) = \frac{1}{e^{2\pi\sigma/\kappa} - 1}(u_P^R, u_P^R). \tag{10.69}$$

Now we treat u_P^R as if normalised (justification via wavepackets). Then

$$\langle 0|\, N_P^R\, |0\rangle = \frac{1}{e^{2\pi\sigma/\kappa} - 1}. \tag{10.70}$$

A Rindler observer at fixed x with

$$U^a = \frac{1}{\kappa x}\left(\frac{\partial}{\partial t}\right)^a = \frac{A}{\kappa}\left(\frac{\partial}{\partial t}\right)^a, \qquad A = \frac{1}{x}, \tag{10.71}$$

measures frequency of R-modes as

$$\hat{\sigma} = \frac{A_\sigma}{\kappa}. \tag{10.72}$$

$$\langle 0|\, N_P^R\, |0\rangle = \frac{1}{e^{2\pi\hat{\sigma}/A} - 1}. \tag{10.73}$$

This is the Planck spectrum of thermal radiation at the *Unruh temperature*

$$T_U = \frac{A}{2\pi}, \tag{10.74}$$

where we set $k_B = 1$. A uniformly accelerated observer perceives the Minkowski vacuum as thermal at temperature T_U, which is

$$T_U \approx \left(\frac{A}{10^{19}\mathrm{ms}^{-2}}\right)\mathrm{K}. \tag{10.75}$$

10.6 Wave Equation in Schwarzschild Spacetime

We are heading towards the calculation of Hawking radiation. Before that we will need to do some preliminary work on waves on Schwarzschild spacetime. In Schwarzschild coordinates, expanding our field ϕ in spherical harmonics Y_{lm},

$$\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{r} \phi_{lm}(t, r) Y_{lm}(\theta, \phi). \tag{10.76}$$

Assuming that ϕ satisfies the wave equation, we have (sheet 4)

$$\nabla^a \nabla_a \phi = 0 \qquad \Longleftrightarrow \qquad \left[\frac{\partial}{\partial t^2} - \frac{\partial}{\partial r_*^2} + V_l(r_*) \right] \psi_{lm} = 0, \tag{10.77}$$

where the potential

$$V_l(r_*) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right), \tag{10.78}$$

viewing $r = r(r_*)$. The important property is $V_l \to 0$ as $r_* \to \pm\infty$. Consider a solution describing a

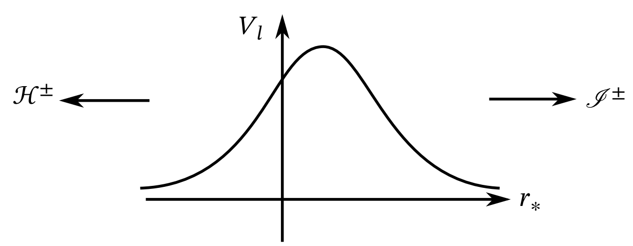


Figure 10.7: Effective potential $V_l(r_*)$ for the wave equation in the Schwarzschild spacetime.

wavepacket localised at some finite value of r_* at time t_0. At late time $t \to \infty$, we expect the solution to consist of a superposition of wavepackets propagating to the 'left' ($r_* \to -\infty$) and to the 'right' ($r_* \to \infty$). Tme reversal implies that at early time $t \to -\infty$ the solution consists of a superposition of wavepackets propagating in from the left and the right:

$$\phi_{lm}(t, r_*) \approx f_{\pm}(t - r_*) + g_{\pm}(t + r_*) \qquad \text{as } t \to \pm\infty, \tag{10.79}$$

$$= f_{\pm}(u) + g_{\pm}(v). \tag{10.80}$$

The functions $f_{\pm}, g_{\pm}$ are localised around some finite u or v. This means that the solution vanishes for $|u| \to \infty$ or $|v| \to \infty$. The solution is uniquely determined by either $\{f_+, g_+\}$ or $\{f_-, g_-\}$.

The wavepacket f_+ describes the outgoing waves propagating to $\mathscr{I}^+$. Similarly, g_+ describes ingoing waves to $\mathcal{H}^+$. In other words,

$$\phi_{lm}|_{\mathscr{I}^+} \overset{v\to\infty}{=} f_+(u), \qquad \phi_{lm}|_{\mathcal{H}^+} \overset{u\to\infty}{=} g_+(v). \tag{10.81}$$

Similarly,

$$\phi_{lm}|_{\mathscr{I}^-} = g_-(v), \qquad \phi_{lm}|_{\mathcal{H}^-} = f_-(u). \tag{10.82}$$

This is illustrated in Fig. 10.8.

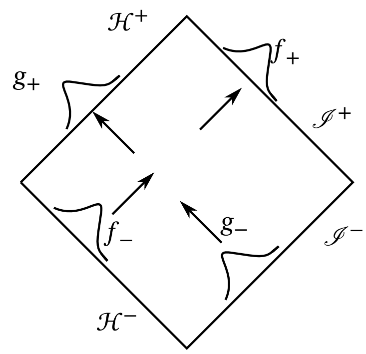


Figure 10.8

Solution is uniquely determine dby behaviour on $\mathcal{H}^+ \cup \mathscr{I}^+$ or $\mathcal{H}^- \cup \mathscr{I}^-$. We have the *out mode* $g_+ = 0$ illustrated in Fig. 10.9 as well as a *down mode* $f_+ = 0$ illustrated in Fig. 10.10. Any solution

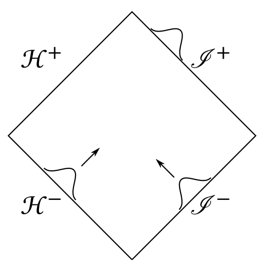


Figure 10.9: Out mode

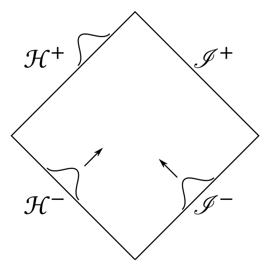


Figure 10.10: Down mode

of the type in Fig. 10.8 can be decomposed into Figs. 10.9 and 10.10. The out and down modes are *orthogonal* (out, down) $= 0$. To show this, evaluate the inner product at late time.

We also have an *in mode* $f_- = 0$, shown in Fig. 10.11 and an *up mode* $g_- = 0$, shown in Fig. 10.12.

Any solution is a unique superposition of in and up modes and (in, up) $= 0$. For example, the out mode is a superposition of in and up, up is a superposition of out and down, and so on.

Let us now look at modes with definite positive frequency with respect to $k = \frac{\partial}{\partial t}$

$$\phi_{wlm} = \frac{1}{r} \underbrace{e^{i\omega t} R_{wlm}(r)}_{\phi_{lm}} Y_{lm}(\theta, \phi) \tag{10.83}$$

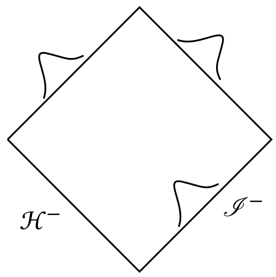


Figure 10.11: In mode

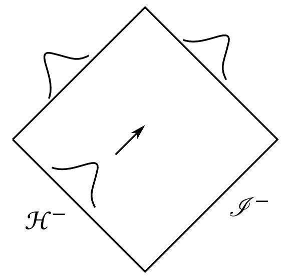


Figure 10.12: Up mode

$$\Rightarrow \left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_l(r_*)\right] R_{wlm} = \omega^2 R_{wlm}. \tag{10.84}$$

Since $V_l \to 0$ as $|r_*| \to \infty$, we have two linearly independent solutions $R_{wlm} \propto e^{\pm i\omega r_*}$ as $|r_*| \to \infty$

$$\Rightarrow \phi_{wlm} \propto e^{-i\omega(t \mp r_*)} = \begin{cases} e^{-i\omega u}, & \text{outgoing} \\ e^{-i\omega v}, & \text{ingoing} \end{cases} \qquad \text{as } |r_*| \to \infty. \tag{10.85}$$

10.7 Hawking Radiation

Consider the quantum field theory of a massless scalar field obeying $\nabla^a \nabla_a \phi = 0$ in the spacetime of a collapsing star shown in Fig. 10.13. There are no *up*, only *in* modes. The spacetime is static near $\mathscr{I}^-$, so we have a preferred definition of "positive frequency" there. We introduce a basis f_i for the positive frequency in modes at $\mathscr{I}^-$. Similarly, the spacetime is static near $\mathscr{I}^+$, so we can similarly define a basis p_i of out modes, which are positive frequency with respect to our Killing field k. There is no ambiguity in the concept of particles near future and past null infinity.

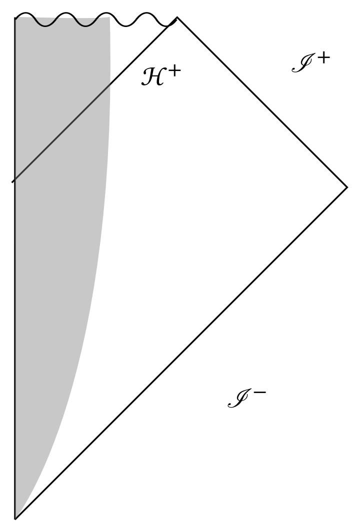


Figure 10.13

However, the neighbourhood of the event horizon is not stationary and there is no preferred notion of positive frequency and the notion of particles in ambiguous. We have to make some arbitrary choice of basis $\{q_i, \overline{q}_i\}$, since there is no canonical way to do it. We have two bases for S. An *early time* basis $\{f_i, \overline{f}_i\}$ and a *late time* basis $\{p_i, \overline{p}_i, q_i, \overline{q}_i\}$. We assume that we have chosen our norm function for these to be orthonormal

$$(f_i, f_j) = (p_i, p_j) = (q_i, q_j) = \delta_{ij}, \tag{10.86}$$

and since the out and down modes are orthonormal,

$$(p_i, q_j) = 0 = (p_i, \overline{q}_j). \tag{10.87}$$

At early time, particles are annihilated by $a_i = (f_i, \phi)$. Similarly, we can define annihilation operators near future null infinity as $b_i = (p_i, \phi)$. We want to relate particles at past and future null infinity, so let us use the fact that the f's are a basis to write

$$p_i = \sum_j (A_{ij} f_j + B_{ij} \overline{f}_j). \tag{10.88}$$

Substituting, we have

$$b_i = \sum_j (\overline{A}_{ij} a_j - \overline{B}_{ij} a_j^\dagger). \tag{10.89}$$

Let us assume that there are no particles initially at $\mathscr{I}^-$. In other words, the state is a vacuum state $|0\rangle$ with respect to the in modes, meaning $a_i |0\rangle = 0$. The expected number of particles in the i^{th} out mode is

$$\langle 0| b_i^\dagger b_i |0\rangle = (BB^\dagger)_{ii}. \tag{10.90}$$

To calculate the number of particles at late times, we need to calculate the Bogoliubov coefficient B.

This is where it gets tricky: the spacetime is time-dependent.

As we discussed in Rindler spacetime, to do things properly, we should really work with wave packets. Choose p_i to be a wave packet such that on $\mathscr{I}^+$ it is localised around a particular value $u = u_i$, containing only frequencies near $\omega_i > 0$. The picture we have in mind is Fig. 10.14.

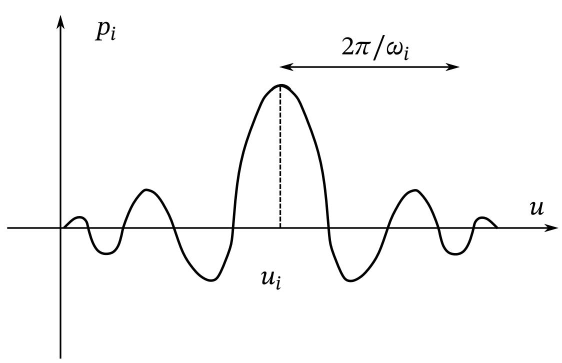
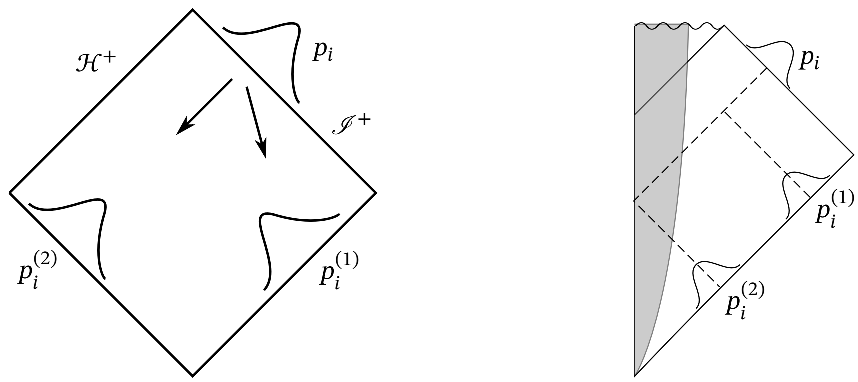


Figure 10.14

Similarly, f_i is the wave packet on $\mathscr{I}$ whose dependence on v is the same as dependence on u of p_i at $\mathscr{I}^+$.

In Kruskal, we might imagine evolving p_i backwards in time, as shown in Fig. 10.15a.



(a) Kruskal spacetime. (b) Collapse spacetime.

Figure 10.15: Backwards propagation of out modes.

$$p_i = p_i^{(1)} + p_i^{(2)}, \tag{10.91}$$

where $p_i^{(1)}$ is in and $p_i^{(2)}$ is up. Since Kruskal is stationary, these are positive frequency.

We introduce the norms of these states, called the reflection and transmission coefficients,

$$R_i = \sqrt{(p_i^{(1)}, p_i^{(1)})}, \qquad T_i = \sqrt{(p_i^{(2)}, p_i^{(2)})}. \tag{10.92}$$

Then, as $(\text{in}, \text{up}) = 0$,

$$1 = (p_i, p_i) = R_i^2 + T_i^2. \tag{10.93}$$

The reflection coefficient R_i is the fraction of p_i reflected to $\mathscr{I}^-$. Similarly, T_i is the fraction of p_i transmitted across $\mathcal{H}^-$.

The presence of a time-reversal isometry in Kruskal implies that R_i can also be interpreted as the fraction of f_i reflected to $\mathscr{I}^+$, whereas T_i is the fraction of f_i transmitted across $\mathcal{H}^+$.

For the collapsing star, consider p_i localised at *late* time u_i on $\mathscr{I}^+$. Since the geometry is static outside the star, we have exactly the same case as in Kruskal. Therefore, $p_i^{(1)}$ is the same as in Kruskal.

For $p_i^{(2)}$, propagates through time-dep. geometry inside star, out to $\mathscr{I}^-$. Mixture of positive and negative frequency

$$A_{ij} = A_{ij}^{(1)} + A_{ij}^{(2)} \qquad B_{ij} = \underbrace{B_{ij}^{(1)}}_{=0} + B_{ij}^{(2)}. \tag{10.94}$$

Even from the diagram, we can see that

$$(p_i^{(1)}, p_i^{(2)}) = 0, \tag{10.95}$$

since $p_i^{(1)}$ and $p_i^{(2)}$ are separated at $\mathscr{I}^-$. This means that $p_i^{(2)}$ has norm T_i ($T_i^2 + R_i^2 = 1$). In particular $p_i|_{\mathscr{I}^+}$ is as in Fig. 10.14 with infinitely many oscillations as $u \to \infty$. Therefore, as shown in Fig. 10.16, we have infinitely many oscillations between $u = u_i$ and $u = \infty$ (i.e. $\mathcal{H}^+$). Thus the

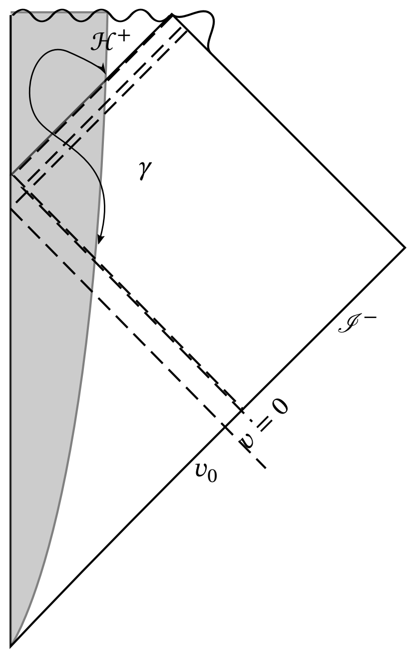


Figure 10.16

frequency of p_i measured by an infalling observer goes to infinity at $\mathcal{H}^+$. Take γ to be the generator of $\mathcal{H}^+$ extended to the past, intersecting $\mathscr{I}^-$. Without loss of generality we let $v = 0$. Then $p_i^{(2)}$ is localised around some $v_0 < 0$ on $\mathscr{I}^-$. The infinitely many oscillations are in $v_0 < v < 0$.

The phase of p_i varies rapidly near γ. Thus we can use the *geometric optics* approximation

$$\phi(x) = A(x)e^{i\lambda S(x)}, \qquad \lambda \gg 1, \qquad \nabla^2\phi = 0 \implies (\nabla S)^2 = 0 + O(\lambda). \tag{10.96}$$

Therefore, surfaces of constant phase are null hypersurfaces. Consider a null geodesic congruence containing generators of these hypersurfaces (including $\mathcal{H}^+ :\ S = \infty$). Let U^a be tangent, and take a deviation vector N^a such that $U \cdot N = -1$, $U \cdot \nabla N^a = 0$. The deviation

$$S^a = \alpha U^a + \hat{S}^a + \beta N^a. \tag{10.97}$$

The sum $(\alpha U^a + \hat{S}^a) \perp U^a$ and βN^a is parallelly transported. On $\mathcal{H}^+$, $\alpha U^a + \hat{S}^a$ is tangent to $\mathcal{H}^+$, βN^a points to a generator of nearby $S =$ const surface. Take $\beta = -\epsilon$, because we want to point out of the congruence. Then $-\epsilon N^a$ is a deviation vector from γ to generator γ' of the $S =$ const. surface.

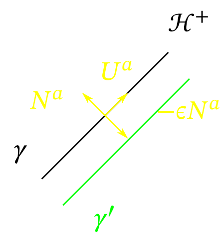


Figure 10.17: l23f2

Because of spherical symmetry, we can always choose $N^\theta = N^\phi = 0$. Outside the star (i.e. in the region of spacetime that is Kruskal), $\partial/\partial V$ is tangent to the affinely parametrised generators of the event horizon $\mathcal{H}^+$. Therefore, we can choose $U^a = (\frac{\partial}{\partial V})^a$ there.

Since N^a is null and not parallel to U^a, we must have $N^a = c(\frac{\partial}{\partial U})^a$ for some $c > 0$. ($U \cdot N = -1$ fixes c.) Thus, outside the star, the deviation vector $-\epsilon N^a$ connects γ ($U = 0$) to γ' with $U = -c\epsilon$.

The definition of the Kruskal coordinate U can be rearranged to find

$$u = -\frac{1}{\kappa}\ln(-U), \tag{10.98}$$

where κ is the surface gravity. We deduce that at late time, γ' is an outgoing radial null geodesic with $u = -\frac{1}{\kappa}\ln(c\epsilon)$.

We will use this to track the surface of the collapsing star. Define $F(u)$ to be the phase of p_i on $\mathscr{I}^+$. Since we are looking at surfaces of constant phase, the phase of p_i along γ' must be

$$S = F(-\frac{1}{\kappa}\ln(c\epsilon)). \tag{10.99}$$

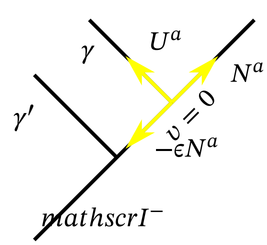


Figure 10.18: l23f3

At $\mathscr{I}^-$: γ, γ' ingoing radial null geodesics. Coordinates (u, v, θ, ϕ),

$$\Rightarrow \; U^a \propto \left(\frac{\partial}{\partial u}\right)^a, \qquad ds^2 \approx -du dv + \frac{1}{4}(u - v)^2 d\Omega^2. \tag{10.100}$$

Spherical symmetry and N^a null, not parallel to U^a implies that

$$N = D^{-1}\frac{\partial}{\partial v} \tag{10.101}$$

at $\mathscr{I}^-$, and D is a positive constant. Thus γ' intersects $\mathscr{I}^-$ at $v = -D^{-1}\epsilon$.

Then (10.99) implies that the phase of $p_i^{(2)}$ on $\mathscr{I}^-$ is $S = F(-\frac{1}{\kappa}\ln(-cD_v))$ (small $v < 0$).

$$p_i^{(2)}|_{\mathscr{I}^-} \approx \begin{cases} 0, & \text{if } v > 0 \\ A_i(v)\exp\Big[iF(-\frac{1}{\kappa}\ln(-cD_v))\Big], & \text{for small } v < 0. \end{cases} \tag{10.102}$$

Take $F(u) = -\omega u$ with $\omega > 0$. Although we can continue to work with wavepackets, we do the more pedagogical thing and fudge things just in the same way as we did before, taking $p_i|_{\mathscr{I}^+} \propto e^{i\omega u}$. Write p_ω instead of p_i

$$p_\omega^{(2)} \approx \begin{cases} 0 & v > 0 \\ A_\omega(v)\exp\Big[\frac{i\omega}{\kappa}\ln(-cDv)\Big] & \text{small } v < 0 \end{cases} \tag{10.103}$$

Take "in" modes f_σ such that $f_\sigma|_{\mathscr{I}^-} = \frac{1}{2\pi N_\sigma}e^{-i\sigma v}$ $(\sigma > 0)$. We want to write $p_\omega^{(2)}$ in terms of $\left\{f_\sigma, \overline{f_\sigma}\right\}$, which we do via the Fourier transform. To do that, we notice that $p_\omega^{(2)}$ is squeezed into small range of v near $v = 0$, which means that the Fourier transform is dominated by large frequency $|\sigma|$ coming from the rapid variation near $v = 0$, coming from the logarithm. In particular, we can use (10.103) and treat the amplitude $A_\omega(v)$ as approximately constant.

$$\widetilde{p}_\omega(\sigma = A_\omega \int_{-\infty}^{0} \mathrm{d}v\, e^{i\sigma v}\exp\left[\frac{i\omega}{\kappa}\ln(-cD_v)\right] \tag{10.104}$$

Remark: As in Rindler space, this does not converge, since we are working with plane waves instead of with wave packets, in which case it would be convergent. As before, we just pretend it converges here.

The inverse

$$p_\omega^{(2)}(v) = \underbrace{\int_{-\infty}^{+\infty}\frac{\mathrm{d}\sigma}{2\pi}e^{-i\sigma v}\widetilde{p}_\omega(\sigma) = \int_0^\infty \mathrm{d}\sigma\, N_\sigma \widetilde{p}_\omega^{(1)} f_\sigma(v)}_{\text{positive freq.}} + \underbrace{\int_0^\infty \mathrm{d}\sigma\, \overline{N_\sigma}\widetilde{p}_\omega^{(2)}(-\sigma)\overline{f_\sigma(v)}}_{\text{negative freq.}}. \tag{10.105}$$

Therefore,

$$\therefore \quad A_{\omega\sigma}^{(2)} = N_\sigma \widetilde{p}_\omega^{(2)}(\sigma), \qquad B_{\omega\sigma} = B_{\omega\sigma}^{(2)} = \overline{N_\sigma}\widetilde{p}_\omega^{(2)}(-\sigma) \qquad (\omega, \sigma > 0) \tag{10.106}$$

We defined

$$\ln z = \ln|z| + i\arg z, \qquad \arg z \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right), \tag{10.107}$$

so the integrand in (10.104) is analytic in the lower half-plane. The branch cut is as in Fig. 10.6. For $\sigma > 0$,

$$\widetilde{p}^{(2)}_{\omega}(-\sigma) = A_\omega \int_{-\infty}^{0} \mathrm{d}v\, e^{-i\sigma v} \exp(...), \tag{10.108}$$

the integrand decays for $|v| \to \infty$ in the lower half-plane. We close the contour as in Fig.

The integral over the semicircle vanishes (for wavepackets via Jordan's lemma) . Thus

$$\int_{-\infty}^{0} \mathrm{d}v = -\int_{0}^{\infty} \mathrm{d}v\,. \tag{10.109}$$

Then

$$\widetilde{p}^{(2)}_{\omega}(-\sigma) = -A_\omega \int_{0}^{\infty} \mathrm{d}v\, e^{-i\sigma v} \exp\left[\frac{i\omega}{\kappa} \ln(-cD_v)\right] \tag{10.110}$$

$$= -A_\omega \int_{0}^{\infty} \mathrm{d}v\, e^{-i\sigma v} \exp\left[\frac{i\omega}{\kappa} (\ln(-cD_v) + i\pi)\right] \tag{10.111}$$

$$\overset{v\to -v}{=} -A_\omega e^{-\omega\pi/\kappa} \int_{-\infty}^{0} \mathrm{d}v\, e^{i\sigma v} \exp\left[\frac{i\omega}{\kappa} \ln(-CD_v)\right] \tag{10.112}$$

$$= -e^{-\pi\omega/\kappa} \widetilde{p}^{(2)}_{\omega}(\sigma). \tag{10.113}$$

This relates the Bogoliubov coefficients as

$$\left|B_{\omega\sigma}\right| = e^{-\pi\omega/\kappa} \left|A^{(2)}_{\omega\sigma}\right|. \tag{10.114}$$

Switching back to wavepackets,

$$\left|B_{ij}\right| = e^{-\pi\omega_i/\kappa} \left|A^{(2)}_{ij}\right|. \tag{10.115}$$

$$T_i^2 = (p_i^{(2)}, p_i^{(2)}) \overset{p=Af+B\bar{f}}{=} \sum_j \left(\left|A_{ij}\right|^2 - \left|B_{ij}\right|^2\right) = (e^{2\pi\omega_i/\kappa} - 1) \sum_j \left|B_{ij}\right|^2 = (e^{2\pi\omega_i/\kappa} - 1)(BB^\dagger)_{ii}. \tag{10.116}$$

$$\therefore \langle 0|\, b_i^\dagger b_i \,|0\rangle = (BB^\dagger)_{ii} = \frac{\Gamma_i}{e^{2\pi\omega_i/\kappa} - 1}, \tag{10.117}$$

where $\Gamma_i = T_i^2$ is the absorption cross-section of mode f_i - fraction absorbed by BH. This is the same spectrum of a blackbody at *Hawking temperature*

$$T_H = \frac{\kappa}{2\pi}. \tag{10.118}$$

Remark: These derivations can be generalised.

The Hawking temperature is tiny, except for tiny black holes,

$$T_H = 6 \times 10^{-8} \frac{M_\odot}{M} \kappa. \tag{10.119}$$

Any realistic black hole absorbs more energy from the CMB than it loses via this Hawking radiation. This temperature is inversely proportional to the mass, so the heat capacity $\frac{\mathrm{d}M}{\mathrm{d}T} < 0$ is negative. Any homogeneous thermodynamic system with this property would be unstable, but black holes are not homogeneous systems. In fact, there are even normal stars with this property.

10.8 Black Hole Thermodynamics

Black holes really are thermodynamic systems with a temperature. In fact, the zero$^{\text{th}}$ law of black hole dynamics is exactly the same as the zero$^{\text{th}}$ law of thermodynamics for a black hole.

Let us rewrite the first law of black hole mechanics as

$$dE = T_H S_{BH} + \Omega_H dJ, \qquad S_{BH} = \frac{A}{4}, \tag{10.120}$$

which is the same as the 1$^{\text{st}}$ law of thermodynamics if a black hole has entropy S_{BH}. Reinstating the various constants, the *Beckenstein–Hawking entropy* of a black hole is

$$S_{BH} = \frac{c^3 A}{4G\hbar}. \tag{10.121}$$

The second law of black hole mechanics states that the area A of the event horizon, and therefore S_{BH} can only increase. This looks like the second law of thermodynamics. However, this is a classical statement. Quantum mechanically, Hawking radiation causes the mass of the black hole, and therefore the area and the entropy to decrease. This appears to violate the second law of thermodynamics. However, the second law is actually a statement about the total entropy of the system. In fact, we can show that the total entropy $S_{BH} + S_{\text{radiation}}$ can only increase. This leads to the *generalised 2nd law* of thermodynamics: The total entropy $S = S_{\text{matter}} + S_{BH}$ of matter and Beckenstein–Hawking entropy is non-decreasing in any physical process.

One might ask how big this black hole entropy is. For $M = M_{\odot}$, $S_{BH} \sim 10^{77}$. Since we are looking at the black hole of one solar mass, let us compare it to the entropy of our sun, which is $S_{\odot} \sim 10^{58}$. In other words, black holes carry a huge amount of entropy compared to stars with the same amount of mass.

According to statistical physics, entropy can be interpreted microscopically as $S = \ln N$, where N is the number of microstates corresponding to a given macrostate. Therefore, a black hole has $N \sim e^{A/4}$ quantum microstates.

10.9 Black Hole Evaporation

The Hawking calculation was done on a fixed black hole spacetime. Presumably, the black hole is losing energy, so if we were to solve the Einstein equations including the energy-momentum tensor of the particles moving away from the black hole. In theory we could calculate the backreaction that tells us how much mass the black hole is losing, but this is an extremely difficult calculation that has not yet been done. However, we can make a crude approximation, pretending that our black hole is just a black body in flat space. The energy loss of a blackbody of area A is then

$$\frac{\mathrm{d}E}{\mathrm{d}t} \approx -\alpha A T^4, \tag{10.122}$$

where α is some constant. Taking $E = M$ and $A \propto M^2$, $T \propto 1/M$, we have

$$\frac{\mathrm{d}M}{\mathrm{d}t} \propto -\frac{1}{M}. \tag{10.123}$$

Therefore $M \to 0$ in a time

$$\tau \sim M^3 \sim 10^{71} \left(\frac{M}{M_\odot}\right)^3 \text{ sec} \tag{10.124}$$

This is much much much longer than the age of universe. Nonetheless, this phenomenon that the black hole will eventually radiate away leads to the famous information paradox.

10.9.1 Information Paradox

Consider matter in a pure quantum state $|\psi\rangle$ that undergoes gravitational collapse and forms a black hole. After this incredibly long time, it will eventually radiate away completely and all that will be left will be the thermal Hawking radiation that is has emitted. Thermal radiation is described not by a pure quantum state but by a *mixed state*

$$\sum_i e^{-\beta E_i} |E_i\rangle \langle E_i|, \qquad \beta = \frac{1}{T}. \tag{10.125}$$

However, the (unitary) evolution from a pure state to a mixed state is impossible in standard QM! We lost information in going from the pure state to the mixed state.

Hawking's original response was that we needed a theory of quantum gravity, which does not obey the usual rules of quantum mechanics. However, the majority of physicists nowadays hold on to the unitarity of quantum mechanics and claim that there must be something wrong in this argument.

In fact, in recent years there has been a clean argument that the three following assumptions together lead to a contradiction:

1. Quantum field theory holds in a blackhole spacetime.
2. Information is not lost.
3. The event horizon is not a special place.