Advanced Quantum Field Theory

Part III Lent 2020

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1 Path Integrals in QM

Goal: Schrödinger's equation → path integral

Consider a Hamiltonian in one dimension $\hat{H} = H(\hat{x}, \hat{p})$, where position and momentum operators satisfy the common commutation relations $[\hat{x}, \hat{p}] = i\hbar$. Assume the it takes the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{1.1}$$

Schrödinger's equation then says that the time evolution of a state $|\psi(t)\rangle$ is given by

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle . \tag{1.2}$$

This has a formal solution, giving us the time-evolution operator

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle . \tag{1.3}$$

In the Schrödinger picture, the states are evolving in time whereas operators and their eigenstates are constant in time.

Definition 1 (wavefunction)**:** $\Psi(x,t) := \langle x|\psi(t)\rangle$

The Schrödinger equation then becomes

$$\langle x| \hat{H} |\psi(t)\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi(x,t). \tag{1.4}$$

We will turn this differential equation into an integral equation, where we will sum over particle paths—a path integral. We can introduce an integral by inserting a complete set of states $1 = \int dx_0 |x_0\rangle \langle x_0|$.

$$\begin{aligned}
\Psi(x,t) &= \langle x| e^{-i\hat{H}t/\hbar} |\psi(0)\rangle && (1.5)\\
&= \int_{-\infty}^{\infty} \mathrm{d}x_0 \, \langle x| e^{-i\hat{H}t/\hbar} |x_0\rangle \langle x_0|\psi(0)\rangle && (1.6)\\
&:= \int_{-\infty}^{\infty} \mathrm{d}x_0 \, \underbrace{K(x, x_0; t)}_{\text{`kernel'}} \Psi(x_0, 0) && (1.7)
\end{aligned}$$

We repeat this insertion for n intermediate times and positions.

Notation: Let $0 := t_0 < t_1 < \cdots < t_n < t_{n+1} := T$

And we also want to factor the exponential into n terms:

$$e^{i\hat{H}T/\hbar} = e^{-\frac{i}{\hbar}\hat{H}(t_{n+1}-t_n)} \cdots e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)}. \tag{1.8}$$

Then

$$K(x, x_0; T) = \int_{-\infty}^{\infty} \left[\prod_{r=1}^{n} \mathrm{d}x_r \, \langle x_{r+1}| \, e^{-\frac{i}{\hbar}\hat{H}(t_{r+1}-t_r)} \, |x_r\rangle \right] \langle x_1| \, e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)} \, |x_0\rangle \tag{1.9}$$

Integrals are over all possible position eigenstates at times $t_r, r = 1, \ldots, n$.

Free Theory

Consider the "free" theory, with $V(\hat{x}) = 0$. We will now play a similar but different trick to what we did before. Let us insert a complete set of momentum eigenstates $1 = \int_{-\infty}^{\infty} đp \, |p\rangle \langle p|$. We also note that these momentum eigenstates are plane waves $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$.

Definition 2 (barred differential)**:** We define the normalised differential in Fourier space to be

$$đp := \frac{\mathrm{d}p}{2\pi\hbar}. \tag{1.10}$$

In higher dimensional QFT, this generalises to

$$đ^n p := \frac{\mathrm{d}^n p}{(2\pi\hbar)^n} \tag{1.11}$$

The corresponding kernel is

$$K_0(x, x'; t) = \langle x| \exp\left(\frac{i}{\hbar} \frac{\hat{p}^2}{2m} t \right) \left|x'\right\rangle . \tag{1.12}$$

$$= \int_{-\infty}^{\infty} đp \, e^{-ip^2 t/2m\hbar} e^{ip(x-x')/\hbar} \tag{1.13}$$

$$= \sqrt{\frac{m}{2\pi i\hbar t}} e^{\frac{im(x-x')^2}{2\hbar t}} . \tag{1.14}$$

Remark:

$$\lim_{t\to 0} \left\{ K_0(x, x'; t) \right\} = \delta(x - x'). \tag{1.15}$$

As expected from $\left\langle x \middle| x' \right\rangle = \delta(x - x')$.

From the Baker-Campbell-Hausdorff formula, we know that

$$e^{\epsilon\hat{A}}e^{\epsilon\hat{B}} = \exp\left(\epsilon\hat{A} + \epsilon\hat{B} + \frac{\epsilon^2}{2}[\hat{A},\hat{B}] + \ldots\right) \neq e^{\epsilon(\hat{A}+\hat{B})} \tag{1.16}$$

$$\text{for small } \epsilon: \qquad e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}}e^{\epsilon\hat{B}}(1 + O(\epsilon^2)) \tag{1.17}$$

Letting $\epsilon = 1/n$ and raising the above to the n^{th} power[1] gives

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty}\left\{e^{\hat{A}/n}e^{\hat{B}/n}\right\}^n. \tag{1.18}$$

We will use this to separate kinetic and potential terms.

Take $t_{r+1} - t_r = \delta t$ with $\delta t \ll T$ and n large such that $n\delta t = T$.

$$e^{-\frac{i}{\hbar}\hat{H}\delta t} = \exp\left(-\frac{i\hat{p}^2\delta t}{2m\hbar}\right)\exp\left(-\frac{iV(\hat{x})\delta t}{\hbar}\right)[1 + O(\delta t^2)] \tag{1.19}$$

Using the result (1.14),

$$\langle x_{r+1}|\exp\left(-\frac{i\hat{H}}{\hbar}\delta t\right)|x_r\rangle = e^{-iV(x_r)\delta t/\hbar}K_0(x_{r+1}, x_r; \delta t) \tag{1.20}$$

$$= \sqrt{\frac{m}{2\pi i\hbar\delta t}}\exp\left[\frac{im}{2\hbar}\left(\frac{x_{r+1} - x_r}{\delta t}\right)^2\delta t - \frac{i}{\hbar}V(x_r)\delta t\right] \tag{1.21}$$

With $T = n\delta t$

$$K(x, x_0; T) = \int\left[\prod_{r=1}^{n}\mathrm{d}x_r\right]\left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{n+1}{2}}\exp\left\{i\sum_{r=0}^{n}\left[\frac{m}{2\hbar}\left(\frac{x_{r+1} - x_r}{\delta t}\right)^2 - \frac{1}{\hbar}V(x_r)\right]\delta t\right\} \tag{1.22}$$

In the limit $n \to \infty$, $\delta t \to 0$ with $n\delta t = T$ fixed, the exponent becomes

$$\frac{1}{\hbar}\int_0^T \mathrm{d}t\left[\frac{1}{2}m\dot{x}^2 - V(x)\right] = \int_0^T \mathrm{d}t\, L(x, \dot{x}), \tag{1.23}$$

where L is the classical Lagrangian, the Legendre transformation of the classical Hamiltonian. The classical action is $S = \int \mathrm{d}t\, L(x, \dot{x})$.

The main result therefore is that the path integral for the kernel is

$$\boxed{K(x, x_0; t) := \langle x|\, e^{-i\hat{H}t/\hbar}\, |x_0\rangle = \int \mathcal{D}x\, e^{\frac{i}{\hbar}S}} \tag{1.24}$$

Definition 3 (functional integral)**:**

$$\mathcal{D}x = \lim_{\substack{\delta t\to 0\\ n\delta T\text{ fixed}}}\left\{(\sqrt{\ldots})\prod_{r=1}^{n}(\sqrt{\ldots}dx_r)\right\} \tag{1.25}$$

We do not need to care about normalization factors.

Remark: In the limit $\hbar \to 0$, the interference of amplitudes is dominated by the ones close to the extremal path where δS. This leads to Hamilton's principle of least action.

[1] This step is sometimes called Suzuki–Trotter decomposition.

We may analytically continue this to imaginary time. Let $\tau = it$. In terms of this imaginary time, we have

$$\langle x| e^{-\hat{H}\tau/\hbar} |x_0\rangle = \int \mathcal{D}x \, e^{-S/\hbar}. \tag{1.26}$$

Mathematically, it makes these integrals much more well-defined, clearly convergent. Here the least-action principle is really evident from the $\hbar = 0$ argument. We also see the connection to statistical physics, interpreting $e^{-S/\hbar}$ as the Boltzmann factor $e^{-\beta H}$.

Quantum mechanics is quantum field theory in 0+1 dimensions. We are treating space differently from time: $\hat{x}(t)$ is a field, whereas t is a variable. However, Lorentz invariance forces us to put x and t on the same footing. In QFT, we solve this problem by demoting x from a field to another label. We then talk about fields $\phi(x,t)$ and want to know about the behaviour of these in all of spacetime.

> String theory gives another ansatz to this problem by promoting again, instead of demoting operators to labels.

1.1 Integrals and their Diagrammatic Expansions

In QFT, we are interested in correlation functions. The following discussion will be very similar to what we have seen in the *Statistical Field Theory* course in Michaelmas term, also no knowledge from that course will be assumed here.

For simplicity, consider a 0 -dimensional field $\varphi \in \mathbb{R}$. As if we are in imaginary time, let

$$Z = \int_{\mathbb{R}} \mathrm{d}\varphi \, e^{-\frac{S(\varphi)}{\hbar}}. \tag{1.27}$$

Assume that the action $S(\varphi)$ is an even polynomial and $S(\varphi) \to \infty$ as $\varphi \to \pm\infty$.

We will be interested in expectation values

$$\langle f \rangle = \frac{1}{Z} \int \mathrm{d}\varphi \, f(\varphi) e^{-S/\hbar}. \tag{1.28}$$

Again, assume f does not grow too fast as $\varphi \to \pm\infty$. Usually, f is polynomial in φ.

1.1.1 Free Theory

Say we have N scalar fields (in $0+1$ dimensions we should really just say 'variables') φ_a with $a = 1, \dots, N$, with action

$$S_0(\varphi) = \frac{1}{2} M_{ab} \varphi_a \varphi_b = \frac{1}{2} \varphi^T M \varphi, \tag{1.29}$$

where M is an $N \times N$ symmetric, positive definite ($\det M > 0$) matrix.

We can diagonalise this. There exists some orthogonal P such that $M = P\Lambda P^T$, where Λ is diagonal. Let $\chi = P^T\varphi$. Then the free partition function is

$$Z_0 = \int \mathrm{d}^N\varphi \exp\left(-\frac{1}{2\hbar}\varphi^T M\varphi\right) \tag{1.30}$$
$$= \int \mathrm{d}^N\chi \exp\left(-\frac{1}{2\hbar}\chi^T \Lambda\chi\right) \tag{1.31}$$
$$= \prod_{c=1}^{N}\int \mathrm{d}\chi_c\, e^{-\frac{\lambda_c}{2\hbar}\chi^2} = \sqrt{\frac{(2\pi\hbar)^N}{\det M}} \tag{1.32}$$

We want to get from the partition function to correlation functions. We can do this by introducing an N-component vector of external sources J to the action

$$S_0(\varphi) \to S_0 + J^T\varphi. \tag{1.33}$$

The partition function is then

$$Z_0(J) = \int \mathrm{d}^N\varphi \exp\left(-\frac{1}{2\hbar}\varphi^T M\varphi - \frac{1}{\hbar}\right)J^T\varphi. \tag{1.34}$$

Similar to solving an ordinary Gaussian integral, we complete the square by writing $\widetilde{\varphi} = \varphi + M^{-1}J$. One can then solve this integral to be

$$Z_0(J) = Z_0(0)\exp\left(\frac{1}{2\hbar}J^T M^{-1}J\right). \tag{1.35}$$

This is called the *generating function*[1]. Correlation functions are obtained from differentiating with respect to the auxiliary sources J and evaluating the whole expression at $J = 0$:

$$\langle\varphi_a\varphi_b\rangle = \frac{1}{Z_0(0)}\int \mathrm{d}^N\varphi\, \varphi_a\varphi_b \exp\left(-\frac{1}{2\hbar}\varphi^T M\varphi - \frac{1}{\hbar}J^T\varphi\right)\Big|_{J=0}. \tag{1.36}$$
$$= \frac{1}{Z_0(0)}\int \mathrm{d}^N\varphi \left(-\hbar\frac{\partial}{\partial J_a}\right)\left(-\hbar\frac{\partial}{\partial J_b}\right)\exp(\dots)\Big|_{J=0} \tag{1.37}$$
$$= \frac{1}{Z_0(0)}\left(-\hbar\frac{\partial}{\partial J_a}\right)\left(-\hbar\frac{\partial}{\partial J_b}\right)Z_0(J)|_{J=0} \tag{1.38}$$
$$= \hbar\,(M^{-1})_{ab} := a \,\text{———}\, b \tag{1.39}$$

Connecting this to the *Quantum Field Theory* course, we identify this as the *free propagator*.

More generally, let $l(\varphi)$ be a *linear* combination of φ_a.

$$l(\varphi) = \sum_{a=1}^{N} l_a\varphi_a, \qquad l_a \in \mathbb{R}. \tag{1.40}$$

Then the steps above are equivalent to swapping $l(\varphi)$ for $l(-\hbar\frac{\partial}{\partial J}) = -\hbar\sum_a l_a\frac{\partial}{\partial J_a}$.

[1]When we go to higher dimensions, where $J = J(x)$ this will be a generating functional $Z[J(x)]$.

The correlation function can again be evaluated explicitly by the introduction of an auxiliary current J:

$$\left\langle l^{(1)}(\varphi)\cdots l^{(p)}(\varphi)\right\rangle = \frac{1}{Z_0(0)}\int \mathrm{d}^N\varphi \prod_{i=1}^{p} l^{(i)}(\varphi) e^{-\frac{1}{2\hbar}\varphi^T M\varphi - \frac{1}{\hbar}J^T\varphi}\bigg|_{J=0} . \tag{1.41}$$

$$= (-\hbar)^p \prod_{i=1}^{p} l^{(i)}\left(\frac{\partial}{\partial J}\right) e^{\frac{1}{2\hbar}J^T M^{-1} J}\bigg|_{J=0} \tag{1.42}$$

In other words, if p is odd, the integrand is odd in some φ_a and the integral over $\varphi_a \in (-\infty, +\infty)$ vanishes. For $p = 2k$, the terms which are non-zero as $J \to 0$ have the following form. We need half the derivatives to bring down components of $m^{-1}J$ and half to remove the J -dependence from those terms that earlier derivatives brought down. As such, we get exactly k factors of M^{-1} . This is Wick's theorem.

Example 1.1.1 (4-point function)**:** Consider the 4-point correlation function. One can check that the above result gives

$$\langle \varphi_a\varphi_b\varphi_c\varphi_d\rangle = \hbar^2\left[(M^{-1})_{ab}(M^{-1})_{cd} + (M^{-1})_{ac}(M^{-1})_{bd} + (M^{-1})_{ad}(M^{-1})_{bc}\right] \tag{1.43}$$

$$= \begin{array}{ccc} a & & b \\ | & & | \\ c & & d \end{array} + \begin{array}{c} a - b \\ \\ c - d \end{array} + \begin{array}{ccc} a & & b \\ & \times & \\ c & & d \end{array} \tag{1.44}$$

We end up with three terms, one for each way of grouping the 4 fields into pairs.

In general, for $\langle\varphi_1 \ldots \varphi_{2k}\rangle$, the number of terms is the number of distinct ways of pairing the $2k$ fields. This is $(2k+1)!! = (2k)!/(2^k k!)$; the number of permutations of $2k$ fields is $(2k)!$, but we have to divide this by the 2^k permutations within the pairs and the $k!$ ways of rearranging the pairs.

Remark: For complex fields, M is Hermitian but not symmetric anymore. In that case, the order of indices of M^{-1} is important. We keep track of this by drawing the propagator with a directed line

$$\langle\phi_a\phi_b^*\rangle = \hbar\,(M^{-1})_{ab} := a \longrightarrow b \tag{1.45}$$

1.1.2 Interacting Theory

We want to go beyond the free theory. The way we are going to achieve this is by an expansion about the classical result $\hbar$. The resulting integral will end up not being convergent.

Claim 1: Integrals like $\int \mathrm{d}\phi\, f(\phi)e^{-S/\hbar}$ do not have a Taylor expansion about $\hbar = 0$.

Proof (Dyson). If the expansion about $\hbar = 0$ existed for $\hbar > 0$, then in the complex plane, there must be some open neighbourhood of $\hbar$ in which the expansion converges. For $S(\phi)$ has a minimum, the integral is divergent if $\mathrm{Re}(\hbar) < 0$. Therefore, the radius of convergence cannot be greater than zero. $\square$

So the $\hbar$-expansion is at best *asymptotic*.

Definition 4 (asymptotic)**:** A series $\sum_{n=0}^{\infty} c_n \hbar^n$ is asymptotic to a function $I(\hbar)$ if

$$\lim_{\hbar\to 0^+} \frac{1}{\hbar^N}\left| I(\hbar) - \sum_{n=0}^{N} c_n \hbar^n \right| = 0. \tag{1.46}$$

Notation: We write $I(\hbar) \sim \sum_{n=0}^{\infty} c_n \hbar^n$.

The series misses out transcendental terms like $e^{-\frac{1}{\hbar^2}} \sim 0$. However, these can evidently be important since obviously $e^{-\frac{1}{\hbar^2}} \neq 0$ for finite $\hbar$. These are called *non-perturbative contributions*. These become important in particular for non-Abelian gauge theories.

Take the ϕ-fourth action for a real scalar

$$S(\phi) = \underbrace{\frac{1}{2}m^2\phi^2}_{S_0(\phi)} + \underbrace{\frac{\lambda}{4!}\phi^4}_{S_1(\phi)} \qquad \begin{matrix} m^2 > 0 \\ \lambda > 0. \end{matrix} \tag{1.47}$$

Expand the exponential in the paritition function Z about the minimum of $S(\phi)$, which is $\phi = 0$.

$$Z = \int \mathrm{d}\phi \exp\left[-\frac{1}{\hbar}\left(\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right)\right] \tag{1.48}$$

$$= \int \mathrm{d}\phi\, e^{-S_0/\hbar} \overbrace{\sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\lambda}{4!\hbar}\right)^n \phi^{4n}}^{e^{-S_1/\hbar}}. \tag{1.49}$$

In order to make progress, we truncate the series to be able to swap the order of summation and integration. This misses out transcendental terms. In the end, we end up with a series that is asymptotic to Z:

$$Z \sim \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^{N} \frac{1}{n!}\left(-\frac{\hbar\lambda}{4!m^4}\right)^n 2^{2v} \int_0^{\infty} \mathrm{d}t\, e^{-t} t^{2n+\frac{1}{2}-1}, \tag{1.50}$$

where $t = \frac{1}{2\hbar}m^2\phi^2$. We recognise the integral to be the Gamma function

$$\int_0^\infty \mathrm{d}t\, e^{-t} t^{2n+\frac{1}{2}-1} = \Gamma(2n+\frac{1}{2}) = \frac{(4n)!\sqrt{\pi}}{4^{2n}(2n)!}. \tag{1.51}$$

The partition function is

$$Z \sim \frac{\sqrt{2\pi\hbar}}{m} \sum_{n=0}^{N} \left(-\frac{\hbar\lambda}{m^4}\right)^n \underbrace{\frac{1}{(4!)n!}}_{(a)} \underbrace{\frac{(4n)!}{2^{2n}(2n)!}}_{(b)} \tag{1.52}$$

The factor on the right comes in part from (a) the Taylor expansion of the term $S_1(\phi) = (\phi)\frac{\lambda}{4!}\phi^4$ in the exponential and from (b) the number of ways of pairing the $4n$ fields of the n copies of ϕ^4. Stirling's approximation allows us to write $n! \approx e^{n\ln n}$. The factor in the partition function then become

$$\frac{(4n)!}{(4!)^n n! 2^{2n}(2n)!} \approx n!. \tag{1.53}$$

We end up with factorial growth, signalling that the series is not convergent, but asymptotic!

1.1.3 Diagrammatic Method

Let us now introduce a current J to obtain the generating function

$$Z(J) = \int \mathrm{d}\phi \exp\left[-\frac{1}{\hbar}(S_0(\phi) + S_1(\phi) + J\phi)\right] \tag{1.54}$$

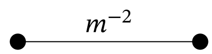
$$= \exp\left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right] \underbrace{\int \mathrm{d}\phi \exp\left[-\frac{1}{\hbar}(S_0 + J\phi)\right]}_{Z_0(J)} \tag{1.55}$$

$$\overset{(1.35)}{\propto} \exp\left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right] \exp\left(\frac{1}{2\hbar}J^T M^{-1} J\right), \qquad M = m^2 \tag{1.56}$$

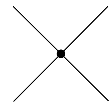
$$\sim \sum_{V=0}^{N} \frac{1}{V!}\left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right]^V \sum_{P=0}^{\infty} \frac{1}{P!}\left(\frac{1}{2\hbar}\frac{J^2}{2\hbar m^2}\right)^P. \tag{1.57}$$

This is called the *double expansion*. Diagrammatically, each of the P propagators, represented by a line as in Fig. 1.1a, give a factor of $M^{-1} = m^{-2}$. We use a large filled circle at the end of a line to represent a source factor J. Each of the V factors $\left(\frac{\partial}{\partial J}\right)^4$, originating from the interaction term $S_1(\phi)$, are associated with a vertex as in Fig. 1.1b. We use a small dot (or sometimes a small square) to mark a vertex.

Let us check that we reproduce the result (1.52) for $Z(0)$. For a term to be non-zero when $J = 0$, we need the number of derivations to be equal to the number of source terms coming from the end of propagators.



(a) Propagator with external sources J at both ends.

 $\sim \quad -\lambda\left(\frac{\partial}{\partial J}\right)^4$

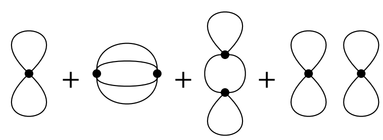
(b) Vertex

Figure 1.1: Components of the diagrammatic representation of the double series.

Notation (external sources)**:** We denote by E the number of external sources, which are left undifferentiated. For P propagtors and V vertices, the number of such sources is

$$E := 2P - 4V. \tag{1.58}$$

For $Z(0)$ we will require $E = 0$, whereas for n-point functions, we will want $E = n$. The first non-trivial terms are $(V, P) = (1, 2), (2, 4), \ldots$.

$Z(0) \propto 1 +$  $+ O(V = 3).$ (1.59)

1.1.4 Symmetry Factors

Definition 5 (symmetry factor)**:** The *symmetry factor S* is the number of ways of redrawing the unlabeled diagram, leaving it unchanged.

Definition 6 (pre-diagram)**:** A *pre-diagram* for a (V, P) term in the double expansion is a collection of V vertices and P propagators, where the ends of the vertex lines are labeled by numbers and the ends of the propagators labelled by letters.

We count the number of times each diagram appears in the double expansion by using such pre-diagrams.

Example 1.1.2 ($V = 1$)**:** Consider the first diagram with only a single vertex and two loops attached to it. There are $A = 4!$ ways of matching the sources a, a', b, b' to the derivatives at $1, 2, 3$, and 4. This is cancelled by a $4!$ in the denominator $F = (V!)(4!)^V (P!)2^P = 4! \cdot 2 \cdot 2^2$ of Eq. (1.52).

So the $(V, P) = (1, 2)$ diagram comes with a prefactor of $\frac{A}{F} = \frac{1}{8}$ (times $-\hbar\lambda m^{-4}$).

In general, S is given by the relation $\frac{A}{F} = \frac{1}{S}$, where A is the number of ways of assigning the sources to the derivatives and F the number of non-equivalent permutations of all vertices, each vertex's legs,

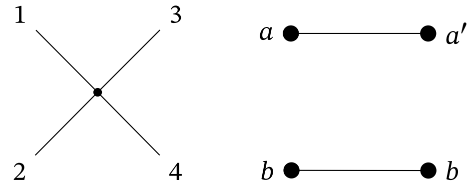
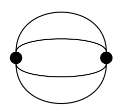


Figure 1.2: Pre-diagram of the $V = 1$ diagram with $P = 2$ loops.

all propagators, and their ends. However, the symmetry of each particular graph is important: If the diagram has a particular symmetry, then some permutations in F may be identical and have been double-counted. For the above diagram, consider the pairing $(1a, 2a', 3b, 4b')$. Swapping $a \leftrightarrow a'$ and $1 \leftrightarrow 2$ gives exactly the same graph, so it should not be counted twice.

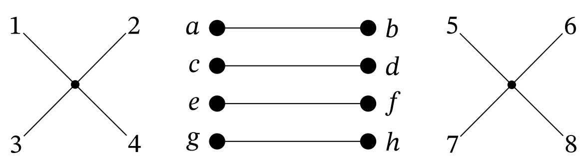
An alternative way to determine S is to consider the actions, which leave invariant the unlabelled diagram. These are called the automorphisms of the graph. For $(1, 2)$, we can swap the direction of upper and lower loops (2^2) and also swap upper and lower loops (2). Therefore, we obtain $S = 2 \cdot 2^2 = 8$.

Example 1.1.3 (basketball)**:** Let us look at a slightly more complicated example. The *basketball* diagram has the symmetry factor



$$S = 4! \cdot 2 = 48 \tag{1.60}$$

The pre-diagram associated to this is



(1.61)

We can simply calculate $F = 2 \cdot (4!)^2 \cdot 4! \cdot 2^4 = 4^3 \cdot 2^{14}$ and from the pre-diagram we determine the ways to pair currents and derivatives

$$A = \underbrace{(2 \cdot 4)}_{a} \underbrace{(4)}_{b} \underbrace{(2 \cdot 3)}_{c} \underbrace{(3)}_{d} \underbrace{(2 \cdot 2)}_{e} \underbrace{(2)}_{f} \underbrace{(2 \cdot 1)}_{g} \underbrace{(1)}_{h} = 3^2 \cdot 2^{10}. \tag{1.62}$$

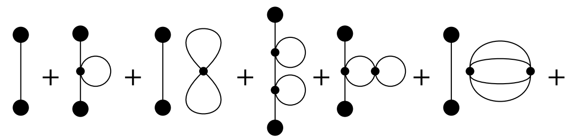
. There are probably multiple ways to obtain this factor, but the reasoning here was as follows: For the letter a, we have a choice (factor 2) whether to connect to the left or the right vertex. In each case, we have 4 numbers to connect to. Since the basketball shape has no loops, this means that b has no choice in which vertex to use; it always has to be the one that we did not choose for a. For b we only have a choice of 4 numbers to connect to. For c, we again have a choice of two vertices, but only three remaining numbers (since the others are filled by a or b). We proceed in the same way for the remaining letters. Thus $A/F = 1/48$.

For the other diagrams, we have

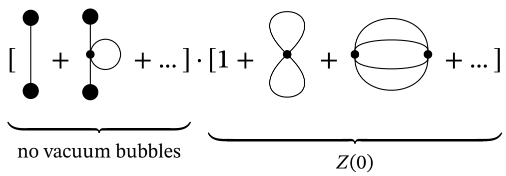
$$\frac{Z(0)}{Z_0(0)} = 1 - \frac{\lambda\hbar}{8m^4} + \frac{\hbar^2\lambda^2}{m^8}\left(\frac{1}{48} + \frac{1}{16} + \frac{1}{128}\right) \tag{1.63}$$

1.1.5 Diagrams with External Sources

As we have previously mentioned, diagrams with $E = n$ external sources need to be considered for the n-point correlation functions. Let us focus on those diagrams that have $E = 2$ external currents.

$$Z(J) \supset [$$

$$\dots] \qquad (1.64)$$

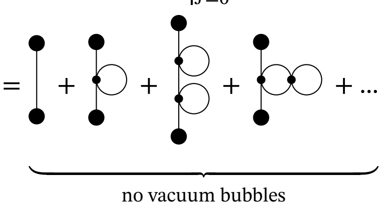
First we factor out the vacuum bubble diagrams

$$[\dots] = [$$


$$(1.65)$$

The 2-point expectation value, which is normalised by $Z(0)$, is then given by those diagrams with two external sources that do not have any vacuum bubbles

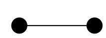
$$\langle \phi^2 \rangle = \frac{(-\hbar)^2}{Z(0)} \frac{\partial^2 Z(J)}{\partial J^2}\bigg|_{J=0} \qquad (1.66)$$

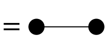
$$=$$


$$(1.67)$$

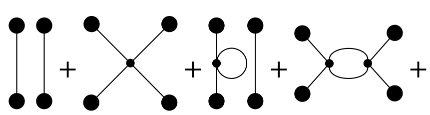
In this case, all diagrams are connected. This will not be true for higher moments.

From $Z(J)$ (1.57), the $E = 2$, $V = 0$ ($P = 1$) term is

$$\frac{1}{2\hbar} \frac{J^2}{m^2} \qquad (1.68)$$


We have $F = 2$ and $A = 1$, so $\frac{A}{F} = \frac{1}{2} = \frac{1}{S}$. The expectation value is $\langle \phi^2 \rangle = \frac{\hbar}{m^2} =$  as expected!

$\langle \phi^{2n} \rangle$ proceeds similarly, but note that there *are* disconnected diagrams. For example, for $n = 2$ we have

$$\langle \phi^4 \rangle =$$

$$(1.69)$$

1.2 Effective Actions

Definition 7 (Wilson effective action)**:** We define $W(\phi)$ such that $Z(J) = e^{-W(J)/\hbar}$.

Claim 2: $W(0)$ is the sum of all connected vacuum diagrams and $W(J)$ is the sum of all connected diagrams.

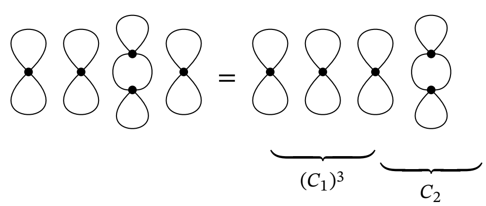
Proof. We denote the set of connected diagrams as $\{C_I\}$, which are taken to contain their respective symmetry factors. Any diagram D is a product of connected diagrams:

$$D = \frac{1}{S_D} \prod_I (C_I)^{n_I}, \tag{1.70}$$

where n_I is the number of times C_I appears in D, and S_D is the symmetry factor associated with rearranging diagrams C_I's, given by

$$S_D = \prod_I n_I!. \tag{1.71}$$

Example 1.2.1: Consider the diagram $D \propto (C_1)^3 \cdot C_2$:



(1.72)

The disconnected parts commute. We have $n_1 = 3$ parts of type C_1 and $n_2 = 1$ of C_2. The symmetry factor associated with the permutations of disconnected pieces is $S_D = 3! \cdot 1! = 6$.

Let $\{n_I\}$ be the set of integers specifying D, then

$$\frac{Z}{Z_D} = \sum_{\{n_I\}} D = \sum_{\{n_I\}} \prod_I \frac{1}{n_I!}(c_I)^{n_I} \tag{1.73}$$

$$= \prod_I \sum_{n_I} \frac{1}{n_I!}(c_I)^{n_I} \tag{1.74}$$

$$= \exp\left(\sum_I c_I\right) \tag{1.75}$$

$$= \exp(\text{sum of unique connected diagrams}) \tag{1.76}$$

$$:= e^{-(W-W_0)/\hbar}, \tag{1.77}$$

where $W = W_0 - \hbar \sum_I c_I$. □

Claim 3: $W(J)$ is the generating function for *connected correlation functions*.

Example 1.2.2 ($\langle \phi^2 \rangle$)**:** Taking logarithms, we have

$$-\frac{1}{\hbar}W(J) = \ln(Z(J)). \tag{1.78}$$

Differentiating with respect to J twice and evaluating at $J = 0$ gives

$$-\frac{1}{\hbar}\frac{\partial^2}{\partial J^2}W\bigg|_{J=0} = \frac{1}{Z(0)}\frac{\partial^2 Z}{\partial J^2}\bigg|_{J=0} - \frac{1}{(Z(0))^2}\left(\frac{\partial Z}{\partial J}\right)^2\bigg|_{J=0} \tag{1.79}$$

$$= -\frac{1}{\hbar^2}\left[\langle\phi^2\rangle - \cancel{\langle\phi\rangle}^2\right] = \frac{1}{\hbar^2}\langle\phi^2\rangle_{\text{connected}}. \tag{1.80}$$

There are also theories where $\langle\phi\rangle \neq 0$.

Example 1.2.3 ($\langle \phi^4 \rangle$)**:** Less trivially,

$$-\frac{1}{\hbar}\frac{\partial^4 W}{\partial J^4}\bigg|_{J=0} = \frac{1}{Z(0)}\frac{\partial^4 Z}{\partial J^4}\bigg|_{J=0} - \left(\frac{1}{Z(0)}\frac{\partial^2 Z}{\partial J^2}\right)^2\bigg|_{J=0} \tag{1.81}$$

$$= \langle\phi^4\rangle - \langle\phi^2\rangle^2 = \langle\phi^4\rangle_{\text{connected}} \tag{1.82}$$

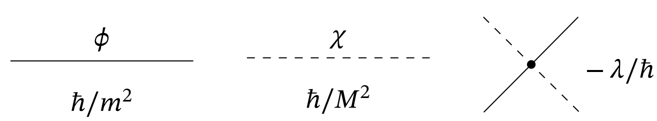
Remark: In statistics and statistical field theory, where ϕ is taken to be a random variable, the n-point correlation function $\langle\phi^n\rangle$ is often called the n^{th} *moment* of ϕ. The connected correlation functions $\langle\phi^n\rangle_c$ are then called *cumulants*. In this context $W(J)$ is the *cumulant generating function*[1], which is the natural logarithm of the *moment generating function* Z.

Interactions

Consider an action for two distinguishable fields

$$S(\phi,\chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2\chi^2. \tag{1.83}$$

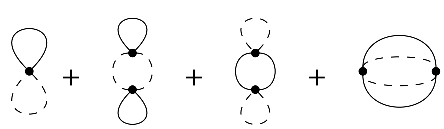
There is no factorial behind the factor 4. The Feynman rules are



(1.84)

The sum of connected vacuum diagrams is

$$-\frac{W}{\hbar} \sim$$

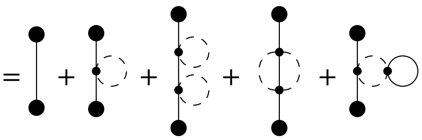


(1.85)

$$= -\frac{\hbar\lambda}{4m^2M^2} + \frac{\hbar^2\lambda^2}{m^4M^4}\left[\frac{1}{16} + \frac{1}{16} + \frac{1}{8}\right]. \tag{1.86}$$

As in (1.67), the two-point correlator $\langle \phi^2 \rangle$ is given by the sum of Feynman diagrams with two external source insertions, represented by a large dot:

$$\langle \phi^2 \rangle =$$



$$+ \dots \tag{1.87}$$

$$= \frac{\hbar}{m^2} - \frac{\hbar^2 \lambda}{2m^4 M^2} + \frac{\hbar^3 \lambda^2}{m^6 M^4}\left[\frac{1}{4} + \frac{1}{2} + \frac{1}{4}\right] + \dots \tag{1.88}$$

In the following section we will arrive at these results in a different way: Say we do not care about the field χ, maybe its mass is $M \gg m$ and it is never produced in our experiments. In such a case, we would want to 'integrate out' χ by defining $W(\phi)$ as

$$e^{-W(\phi)/\hbar} = \int \mathrm{d}\chi \, e^{-S(\phi,\chi)/\hbar}. \tag{1.89}$$

From this viewpoint, we treat $\phi^2\chi^2$ as a source term $J = \phi^2$ for χ^2. Once found, we can then use $W(\phi)$ to calculate expectation values $\langle f(\phi) \rangle$. It is in this sense that $W(\phi)$ indeed plays the role of an effective action for the field ϕ—one in which all the quantum corrections due to χ are taken into account.

1.2.1 Effective Field Theory: Integrating out χ

From the action (1.83), we obtained Feynman diagrams for both fields and calculated expressions for W and $\langle \phi^2 \rangle$. Let us now show that we can get the same thing by first removing χ from the theory and then calculating expectation values in the effective theory. In other words, we want to calculate correlation functions only involving ϕ fields as

$$\langle f(\phi) \rangle = \frac{1}{Z} \int \mathrm{d}\phi \, \mathrm{d}\chi \, f(\phi) e^{-S(\phi,\chi)/\hbar} = \frac{1}{Z} \, \mathrm{d}\phi \, f(\phi) e^{-W(\phi)/\hbar}. \tag{1.90}$$

In this simple example,

$$e^{-W(\phi)/\hbar} = \int \mathrm{d}\chi \, e^{-S(\phi,\chi)/\hbar} = e^{-m^2\phi^2/2\hbar} \sqrt{\frac{2\pi\hbar}{M^2 + \frac{1}{2}\lambda\phi^2}}. \tag{1.91}$$

Taking logarithms, we obtain

$$W(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\hbar}{2} \ln\left(1 + \frac{\lambda}{2M^2}\phi^2\right) + \frac{\hbar}{2} \ln\left(\frac{M^2}{2\pi\hbar}\right). \tag{1.92}$$

The final term is constant and in non-gravitational physics does not effect any expectation values.

In gravitational theories, this term is taken to be the origin of the cosmological constant.

We expand the logarithm

$$W(\phi) = \left(\frac{m^2}{2} + \frac{\hbar\lambda}{4M^2}\right)\phi^2 - \frac{\hbar\lambda^2}{16M^4}\phi^4 + \frac{\hbar\lambda^3}{48M^6}\phi^6 + \dots \tag{1.93}$$

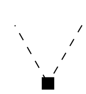
From a theory which did not have any self-interaction, integrating out the χ -field gave us infinitely many self-interaction terms for all the even powers. We can think of the first term as an effective mass and write the other couplings as

$$W(\phi) := \frac{m_{\text{eff}}^2}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_6}{6!}\phi^6 + \cdots + \frac{\lambda_{2k}}{(2k)!}\phi^{2k} + \dots, \tag{1.94}$$

where

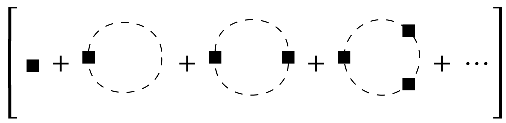
$$m_{\text{eff}}^2 = m^2 + \frac{\hbar\lambda}{2M^2} \qquad \lambda_{2k} = (-1)^{k+1}\hbar \frac{(2k)!}{2^{k+1}k} \frac{\lambda^k}{M^{2k}}. \tag{1.95}$$

In higher dimensions, we usually need to calculate $W(\phi)$ perturbatively. From $S(\phi,\chi)$ and the path integral over χ, we have the Feynman rules

$\hbar/M^2$ and  $\sim -\dfrac{\lambda\phi^2}{2\hbar}$. (1.96)

The propagator is the same as in the previous Feynman rules (1.84), but the factor associated with the vertex now accounts for the fact that we are treating the interaction term $\frac{\lambda}{4}\phi^2\chi^2$ as a source term for χ^2. The effective action is then given by the sum of connected diagrams

$$W(\phi) \sim -\hbar \quad \text{(1.97)}$$

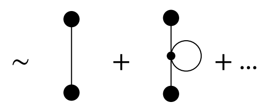


$$= S(\phi) + \frac{1}{2}\frac{\hbar\lambda}{2M}\phi^2 - \frac{1}{4}\frac{\hbar\lambda^2}{4M^4}\phi^4 + \frac{1}{3!}\frac{\hbar\lambda^3}{8M^6}\phi^6 + \cdots, \tag{1.98}$$

where in the first term $S(\phi) = S(\phi, 0)$ is the part of the action that is unaffected by the integral over χ.

We can now use $W(\phi)$ to calculate the expectation value

$$\langle\phi^2\rangle = \frac{1}{Z}\int \mathrm{d}\phi\, \phi^2 e^{-W(\phi)/\hbar} \sim \quad \text{(1.99)}$$



$$= \frac{\hbar}{m_{\text{eff}}^2} - \frac{\lambda_4\hbar^2}{2m_{\text{eff}}^6} + \ldots \tag{1.100}$$

The five diagrams of (1.87) have been reduced to just two diagrams with the effective action $W(\phi)$.

1.2.2 Quantum Effective Action Γ

Definition 8: We define the average field $\Phi = \langle\phi\rangle_J$ in the presence of an external source J as

$$\Phi := \frac{\partial W}{\partial J} = -\frac{\hbar}{Z(J)}\frac{\partial}{\partial J}\int \mathrm{d}\phi\, e^{-(S+J\phi)/\hbar} := \langle\phi\rangle_J, \tag{1.101}$$

where we have assumed that $S(\phi)$ as before is even in ϕ and has a minimum at $\phi = 0$.

Definition 9 (quantum effective action)**:** We have a Legendre transformation from the Wilsonian effective action $W(J)$ to the *quantum effective action* $\Gamma(\Phi)$:

$$\Gamma(\Phi) = W(J) - \Phi J. \tag{1.102}$$

Claim 4: As usual for Legendre transformations, we have

$$\boxed{\frac{\partial\Gamma}{\partial\Phi} = -J} \tag{1.103}$$

So $J \to 0$ corresponds to an extremum (in practice a minimum) of the effective action $\Gamma(\Phi)$.

Proof. Using the product rule, the chain rule, and the definition (1.101) of Φ,

$$\frac{\partial\Gamma}{\partial\Phi} = \frac{\partial W}{\partial\Phi} - J - \Phi\frac{\partial J}{\partial\Phi} = \frac{\partial W}{\cancel{\partial J}}\frac{\cancel{\partial J}}{\partial\Phi} - J - \Phi\cancel{\frac{\partial J}{\partial\Phi}} = -J \tag{1.104}$$

□

In higher dimensions, one performs a derivative expansion

$$\Gamma(\Phi) = \int \mathrm{d}^d x \left[-V(\Phi) - \frac{1}{2}\partial^\mu \Phi \partial_\mu \Phi + \dots \right] \tag{1.105}$$

where the first term in this expansion defines the effective potential $V(\Phi)$. The effective potential might shift the minimum of the action when including quantum effects. These quantum corrections can lead to spontaneous symmetry breaking.

1.2.3 Analogy with Statistical Mechanics

The W is like the Helmholtz free energy F. In the presence of some external magnetic field h in some spin system, it is defined via

$$e^{-\beta F(h)} = \int \mathcal{D}s\, e^{-\beta H}. \tag{1.106}$$

We can define the magnetisation to be $M = -\frac{\partial F}{\partial h}$. One can then switch to the Gibbs free energy, analogous to Γ, by defining

$$G(M) = F(h) + Mh. \tag{1.107}$$

As $h \to 0$, the magnetisation of the system is the minimum of G.

1.2.4 Perturbative Calculation of $\Gamma(\Phi)$

We want to treat Φ as ϕ and write down a new Wilsonian effective action

$$e^{-W_\Gamma(J)/g} = \int \mathrm{d}\Phi\, e^{-(\Gamma(\Phi)+J\Phi)/g}, \tag{1.108}$$

where g is a new, fictitious Planck constant and J a source. This is in analogy to before, replacing the action S with the quantum effective action Γ. As before, $W_\Gamma(J)$ is the sum of connected vacuum diagrams and can be written as a power series in g

$$W_\Gamma(J) = \sum_{l=0}^{\infty} g^l W_\Gamma^{(l)}(J), \tag{1.109}$$

where l counts the loops. In particular, $W_\Gamma^{(0)}$ is composed of all the *tree* diagrams with Φ as legs.

In the limit of $g \to 0$, $W_\Gamma(J) \to W_\Gamma^{(0)}(J)$. Also as $g \to 0$, the integral in Φ is dominated by the minimum of the exponent, which is Φ such that

$$\frac{\partial \Gamma}{\partial \Phi} = -J \qquad \text{(steepest descent)} \tag{1.110}$$

Therefore, by analogy to the earlier definition with with action $S(\phi) + J\phi$, we have

$$W_\Gamma^{(0)}(J) = \Gamma(\Phi) + J\Phi = W(J). \tag{1.111}$$

The moral of the story is that the sum of connected diagrams in a theory with action $S(\phi) + J\phi$ (i.e. $W(J)$) can be constructed from a sum of tree diagrams with action $\Gamma(\Phi) + J\Phi$.

Definition 10 (bridge)**:** An edge in a connected graph is a *bridge* if removing it would leave the graph disconnected.

An example is shown in Fig. 1.3.

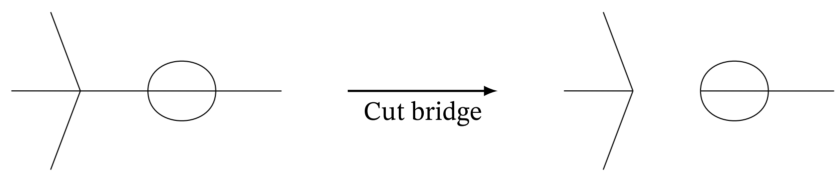


Figure 1.3: Cutting a bridge disconnects a graph.

Definition 11 (1PI)**:** A connected graph is called *one-particle irreducible* (1PI) if it has no bridges.

Example 1.2.4: Take the N-component scalar ϕ_a, $a = 1, \dots, N$. Then the connected propagator is

$$\langle \phi_a \phi_b \rangle_J^c = -\hbar \frac{\partial^2 W}{\partial J_a \partial J_b} = -\hbar \frac{\partial \Phi_b}{\partial J_a}, \tag{1.112}$$

since $\Phi_b = \frac{\partial W}{\partial J_b}$. Using $J_a = -\frac{\partial \Gamma}{\partial \Phi_a}$ then gives

$$\langle \phi_a \phi_b \rangle_J^c = -\hbar \left(\frac{\partial J_a}{\partial \Phi_b} \right)^{-1} = \hbar \left(\frac{\partial^2 \Gamma}{\partial \Phi_b \Phi_a} \right)^{-1}. \tag{1.113}$$

The full connected propagator is $\hbar$ times the inverse of the quadratic term in $\Gamma(\Phi)$.

1.3 Fermions

Definition 12 (Grassmann numbers)**:** The *Grassmann numbers* are a set of n elements $\{\theta_a\}$ obeying $\theta_a \theta_b = -\theta_b \theta_a$. For any scalar $\phi \in \mathbb{C}$, we have $\theta_a \phi = \phi \theta_a$.

Remark: Anti-symmetry implies that $\theta^2 = 0$.

Remark: The product of an even number of θ's acts like a scalar and commutes with other θ's, for example $\theta_a(\theta_c\theta_d) = (\theta_c\theta_d)\theta_a$.

> Compare this with the behaviour of fermions (Grassmann variables, half integer spin) and bosons (scalars, integer spin) under exchange.

> The anticommuting Grassmann variables are frequently called *a-numbers*, in contrast to commuting *c-numbers*.

A general function of Grassmann variables can be written

$$f(\{\theta_a\}) = f(\theta) = f + \phi_a \theta_a + \frac{1}{2} g_{ab} \theta_a \theta_b + \cdots + \frac{1}{n!} h_{a_1,\dots,a_n} \theta_{a_1} \dots \theta_{a_n}. \tag{1.114}$$

The coefficients are anti-symmetric in their indices.

1.3.1 Grassmann Analysis

Definition 13 (differentiation)**:** On Grassmann variables, we define *differentiation* via

$$\left(\frac{\partial}{\partial \theta_a} \theta_b + \theta_b \frac{\partial}{\partial \theta_a} \right) \bullet = \delta_{ab} \bullet . \tag{1.115}$$

For a single Grassmann variable θ, where possible functions are of the form $F(\theta) = f + \phi\theta$, we need only specify $\int d\theta$ and $\int \theta d\theta$ to define *integration*. Requiring translational invariance gives

$$\int d\theta(\theta - \mu) = \int d\theta\theta, \tag{1.116}$$

when μ is a constant Grassmann variable. We can then choose a normalisation:

$$\int d\theta = 0 \qquad \int d\theta\theta = 1. \tag{1.117}$$

Remark: There is a similarity between differentiation and integration.

The *Berenzin rules* give

$$\int d\theta \frac{\partial}{\partial\theta} F(\theta) = 0. \tag{1.118}$$

Similarly, for n Grassmann variables, we define

$$\int \theta_1 \dots \theta_n \, \mathrm{d}^n\theta = 1. \tag{1.119}$$

Definition 14 (Berenzin integration)**:** In general, we define *Berenzin integration* over Grassmann variables as

$$\int \theta_{a_1} \dots \theta_{a_n} \, \mathrm{d}^n\theta = \epsilon_{a_1 \dots a_n}. \tag{1.120}$$

Let us consider how this integration measure changes under change of variables $\theta'_a = A_{ab}\theta_b$. Consider the integral

$$\int \mathrm{d}^n\theta \, \theta'_{a_1} \dots \theta'_{a_n} = A_{a_1 b_1} \dots A_{a_n b_n} \int \mathrm{d}^n\theta \, \theta_{b_1} \dots \theta_{b_n} \tag{1.121}$$

$$= \det\{A\}\epsilon_{a_1 \dots a_n} \tag{1.122}$$

$$= \det\{A\} \int \mathrm{d}^n\theta' \, \theta'_{a_1} \dots \theta'_{a_n}. \tag{1.123}$$

Therefore, the measure changes as $\mathrm{d}^n\theta' = [\det\{A\}]^{-1} \, \mathrm{d}^n\theta$, which is the inverse to bosonic integration!

1.3.2 Free Fermion Field Theory ($d = 0$)

Consider two fermionic fields θ_1, θ_2. The action must be bosonic, so up to an additive constant which can be absorbed into the normalisation of the partition function, we have $S = \frac{1}{2}A\theta_1\theta_2$, where $A \in \mathbb{R}$. The free partition function can be calculated explicitly by expanding the exponential

$$\mathcal{Z}_0 = \int \mathrm{d}^2\theta \, e^{-S(\theta)/\hbar} = \int \mathrm{d}^2\theta \left(1 - \frac{A}{2\hbar}\theta_1\theta_2\right) = -\frac{A}{2\hbar}. \tag{1.124}$$

Now, for $n = 2m$ fields θ_a have the action $S = \frac{1}{2}A_{ab}\theta_a\theta_b$, where A_{ab} is an anti-symmetric real matrix. The partition function is

$$\mathcal{Z}_0 = \int \mathrm{d}^{2m}\theta \sum_{j=0}^{m} \frac{(-1)^j}{(2\hbar)^j j!}(A_{ab}\theta_a\theta_b)^j \tag{1.125}$$

Since the sum terminates, we can exchange the order of differentiation and integration, giving

$$\mathcal{Z}_0 = \frac{(-1)^m}{(2\hbar)^m m!}\int \mathrm{d}^{2m}\theta\, A_{a_1a_2}A_{a_3a_4}\dots A_{a_{2m-1}a_{2m}}\theta_{a_1}\dots\theta_{a_{2m}} \tag{1.126}$$

$$= \frac{(-1)^m}{(2\hbar)^m m!}A_{a_1a_2}\dots A_{a_{2m-1}a_{2m}}\epsilon^{a_1a_2\dots a_{2m}} \tag{1.127}$$

$$:= \frac{(-1)^m}{\hbar^m}\ \mathrm{Pfaff}\, A, \tag{1.128}$$

which defines the *Pfaffian* Pfaff A of the matrix A .

Exercise 1.1: Show that $(\mathrm{Pfaff}\, A)^2 = \det A$.

Remark: Again, this result is, up to prefactors, the inverse of the bosonic free partition function

$$\mathcal{Z}_0 = \sqrt{\frac{(2\pi\hbar)^n}{\det M}}. \tag{1.129}$$

Introducing external, Grassmann-valued sources η_a to the action gives

$$S(\theta,\eta) = \frac{1}{2}A_{ab}\theta_a\theta_b + \eta_a\theta_a. \tag{1.130}$$

To compute the partition function, we complete the square as in the bosonic case:

$$S(\theta,\eta) = \frac{1}{2}\left(\theta_a + \eta_c(A^{-1})_{ca}\right)A_{ab}\left(\theta_b + \eta_d(A^{-1})_{db}\right) + \frac{1}{2}\eta_a(A^{-1})_{ab}\eta_b. \tag{1.131}$$

Using the translation invariance of Berezin integration, we have

$$\mathcal{Z}_0(\eta) = \mathcal{Z}_0(0)\exp\left(-\frac{1}{2\hbar}\eta^T A^{-1}\eta\right). \tag{1.132}$$

The propagator too can be evaluated explicitly to be

$$\langle\theta_a\theta_b\rangle = \frac{\hbar^2}{\mathcal{Z}_0(0)}\frac{\partial^2\mathcal{Z}_0(\eta)}{\partial\eta_a\partial\eta_b}\bigg|_{\eta=0} = \hbar(A^{-1})_{ab} \tag{1.133}$$

2 LSZ Reduction Formula

We will now start to move beyond $d = 0$ and transition to using path integrals. This transitory topic will also allow us to have a glimpse at renormalisation. We will have in mind a weakly interacting theory, so that we can tell the story the way that Sredniki does. However, the results that we will show can also be rigorously proved non-perturbatively, which usually happens in the final chapters of quantum field theory textbooks.

We will connect scattering amplitudes to correlation functions.

2.1 $2 \to 2$ Scattering

Consider a free scalar field in $3 + 1$ dimensions, built out of plane waves

$$\phi(x) = \int \frac{đ^3 k}{2E_{\mathbf{k}}} \left[a(\mathbf{k}) e^{-ik\cdot x} + a^\dagger(\mathbf{k}) e^{ik\cdot x} \right], \tag{2.1}$$

where we are using the mostly minus metric in Minkowski space

$$k \cdot x = Et - \mathbf{k} \cdot \mathbf{x}, \tag{2.2}$$

and the relativistic normalisation for $a(\mathbf{k})$, changing the factor from $\sqrt{2E} \to 2E$ as compared to the conventions from our *Quantum Field Theory* lectures. In a free theory, where we do not need to do any perturbative expansion, we set $\hbar = c = 1$. Look at $\int \mathrm{d}^3 x \, e^{ik\cdot x} \phi(x)$ and $\int \mathrm{d}^3 x \, e^{ik\cdot x} \partial_0 \phi(x)$. These have

$$a(\mathbf{k}) = \int \mathrm{d}^3 x \, e^{ik\cdot x} \left[i\partial_0 \phi(x) + E\phi(x) \right] \tag{2.3}$$

$$a^\dagger(\mathbf{k}) = \int \mathrm{d}^3 x \, e^{-ik\cdot x} \left[-i\partial_0 \phi(x) + E\phi(x) \right]. \tag{2.4}$$

Let the initial state for the free theory be a one-particle state

$$|\mathbf{k}\rangle = a^\dagger(\mathbf{k}) \, |\Omega\rangle \, , \tag{2.5}$$

where $|\Omega\rangle$ satisfies $a(\mathbf{k})|\Omega\rangle = 0$ for all $\mathbf{k}$ and is normalised to $\langle\omega|\Omega\rangle = 1$. The one-particle states are normalised to

$$\left\langle \mathbf{k} \middle| \mathbf{k}' \right\rangle = (2E)\delta^3(\mathbf{k}-\mathbf{k}'), \tag{2.6}$$

where $E = \sqrt{\mathbf{k}^2 + m^2}$. We can take superpositions of these to introduce two Gaussian wavepackets

$$a_n^\dagger := \int \mathrm{d}^3k\, f_n(\mathbf{k}) a^\dagger(\mathbf{k}), \qquad f_n(\mathbf{k}) \propto \exp\left[-\frac{|\mathbf{k}-\mathbf{k}_n|^2}{4\sigma^2}\right], \qquad n = 1, 2. \tag{2.7}$$

Let us now evolve the Gaussians into the distant past and future, where the overlap in coordinate space is negligible. We will also assume that this works when including interactions. There is a complication since $a^\dagger(\mathbf{k})$ becomes time dependent, wherefore $a_1^\dagger(t)$ and $a_2^\dagger(t)$ depend on time. Assume that as $t \to \pm\infty$, the interacting $a_1^\dagger$ and $a_2^\dagger$ coincide with their free theory expressions. The initial and final states are

$$|i\rangle = \lim_{t\to-\infty} a_1^\dagger(t) a_2^\dagger(t)|\Omega\rangle \tag{2.8}$$

$$|f\rangle = \lim_{t\to+\infty} a_{1'}^\dagger(t) a_{2'}^\dagger(t)|\Omega\rangle. \tag{2.9}$$

These are also normalised $\langle i|i\rangle = 1 = \langle f|f\rangle$ and have $\mathbf{k}_1 \neq \mathbf{k}_2$ and $\mathbf{k}_1' \neq \mathbf{k}_2'$ respectively.

We want to find the *scattering amplitude* $\langle f|i\rangle$.

Remark: We have for example

$$\begin{aligned}
a_1^\dagger(\infty) - a_1^\dagger(-\infty) &= \int_{-\infty}^{\infty} \mathrm{d}t\, \partial_0 a_1^\dagger(t) && (2.10)\\
&= \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \mathrm{d}^4x\, \partial_0 \left[e^{-ik\cdot x}(-i\partial_0\phi + E\phi)\right] && (2.11)\\
&= -i\int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \mathrm{d}^3x\, e^{-ik\cdot x}(\partial_0^2 + E^2)\phi && (2.12)\\
&= -i\int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \mathrm{d}^3x\, e^{-ik\cdot x}(\partial_0^2 - \overleftarrow{\nabla}^2 + m^2)\phi && (2.13)\\
&= -i\int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \mathrm{d}^3x\, e^{-ik\cdot x}(\partial^2 + m^2)\phi, && (2.14)
\end{aligned}$$

where in the last line we integrated by parts twice. In the free theory, the Klein–Gordon equation therefore implies that $a_1^\dagger(\infty) - a_1^\dagger(-\infty) = 0$. Using a similar calculation for $a_j^\dagger$ and then a_j, we obtain

$$a_j^\dagger(-\infty) = a_j^\dagger(\infty) + i\int \mathrm{d}^3k\, f_j(\mathbf{k}) \int \mathrm{d}^4x\, e^{-ik\cdot x}(\partial^2 + m^2)\phi \tag{2.15a}$$

$$a_j(\infty) = a_j(-\infty) + i\int \mathrm{d}^3k\, f_j(\mathbf{k}) \int \mathrm{d}^4x\, e^{ik\cdot x}(\partial^2 + m^2)\phi. \tag{2.15b}$$

Using the time-ordering operator $\mathcal{T}$, we have

$$\langle f|i\rangle = \langle\Omega|\,\mathcal{T} a_{1'}(\infty) a_{2'}(\infty) a_1^\dagger(-\infty) a_2^\dagger(-\infty)\,|\Omega\rangle. \tag{2.16}$$

We can use the integral expressions (2.15), to commute these annihilation and creation operators, giving the rather unwieldy equation

$$\begin{aligned}\langle f|i\rangle = i^4 \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\, \mathrm{d}^4x_1'\, \mathrm{d}^4x_2'\, e^{-ik_1\cdot x_1}e^{-ik_2\cdot x_2}e^{ik_1'\cdot x_1'}e^{ik_2'\cdot x_2'} \\ \times (\partial_1^2+m^2)(\partial_2^2+m^2)(\partial_{1'}^2+m^2)(\partial_{2'}^2+m^2) \\ \times \langle\Omega|\, \mathcal{T}\phi(x_1)\phi(x_2)\phi(x_1')\phi(x_2')\,|\Omega\rangle\,,\end{aligned} \tag{2.17}$$

where we have taken $\sigma \to 0$ such that $f(\mathbf{k}_j) \to \delta^3(\mathbf{k}-\mathbf{k}_j)$. This is the *LSZ reduction* formula.

This result can be proven without recourse to the free theory as we have done. However, the machinery needed to build this up is more involved. This general derivation requires only weaker assumptions

1) We need a unique ground state $|\Omega\rangle$ and the first excited state has to be a single particle, which means that we need a single simple pole, rather than a continuum of particles for example.

2) We want the field $\phi\,|\Omega\rangle$ to be that single particle state, meaning that $\langle\Omega|\,\phi\,|\Omega\rangle = 0$. Usually, this is not a problem; if we have $\langle\Omega|\,\phi\,|\Omega\rangle = v \neq 0$, say when we have some spontaneous symmetry breaking, then we let $\widetilde{\phi} = \phi - v$ to obtain $\langle\Omega|\,\widetilde{\phi}\,|\Omega\rangle = 0$.

3) We want ϕ normalised such that $\langle k|\,\phi(x)\,|\Omega\rangle = e^{ik\cdot x}$ as in the free case. Usually, interactions spoil this and require us to rescale $\phi \to \mathcal{Z}_\phi^{1/2}\phi$.

   We see the need to 'renormalise', e.g.

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \to \mathscr{L} = \frac{\mathcal{Z}_\phi}{2}\partial_\mu\phi\partial^\mu\phi - \frac{Z_m}{2}m^2\phi^2 - \frac{\lambda}{4!}Z_\lambda\phi^4. \tag{2.18}$$

3 Scalar Field Theory

3.1 Wick Rotation

Let us make the connection between Euclidean and Minkowski spacetime a bit more concrete. It is convenient to start from the Minkowski metric with signature $(+---)$ and go to the Euclidean one with $(++++)$. The Lagrangian density in Minkowski space is

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V[\phi]. \tag{3.1}$$

We now use square brackets instead of parentheses to indicate that the function ϕ is itself a function of spacetime, so that $V[\phi]$ is a functional. In particular, the potential is

$$V[\phi] = \frac{1}{2}m^2\phi^2 + \sum_{n>2}\frac{1}{n!}V^{(n)}\phi^n. \tag{3.2}$$

The partition function is

$$Z = \int \mathcal{D}\phi\, e^{i\int \mathrm{d}x^0 L}, \tag{3.3}$$

where the Lagrangian L is the integral of the Lagrangian density $\mathscr{L}$ over space

$$L = \int \mathrm{d}^3x\, \mathscr{L}. \tag{3.4}$$

The free propagator is

$$\frac{i}{k^2 - m^2 + i\epsilon} = \frac{i}{(k^0)^2 - |\mathbf{k}|^2 - m^2 + i\epsilon}. \tag{3.5}$$

Let $ix^0 = x_4$ and use the Euclidean metric $(++++)$. Arrange the signs so that we have a Lagrangian density that has the same sign as (3.1) in the kinetic term

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V[\phi]. \tag{3.6}$$

The partition function is

$$Z = \int \mathcal{D}\phi\, e^{-\int \mathrm{d}x_4 L}. \tag{3.7}$$

One argument put forward for using the mostly plus metric in Minkowski space is that this transition to Euclidean space simply involves switching only one of the signs, rather than having to keep track of i's and minus signs as we do it here.

The propagator is

$$\widetilde{\Delta}_0(k) = \frac{1}{k^2 + m^2} = \frac{1}{(k_4)^2 + |\mathbf{k}|^2 + m^2}. \tag{3.8}$$

This means that we rotate the contour integral so that the poles lie on the imaginary axis, as illustrated in Fig. 3.1.

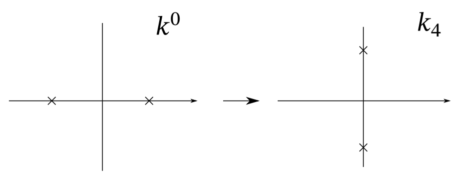


Figure 3.1

3.2 Feynman Rules

Take the free propagator as in Ch. ??.

$$S_0[\phi, J] = \int_{\mathbb{R}^4} \mathrm{d}^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + J(x)\phi(x)\right]. \tag{3.9}$$

Now write $\phi(x)$ and $J(\phi)$ as a Fourier integral

$$\phi(x) = \int \text{đ}^4k\, e^{ik\cdot x}\widetilde{\phi}(k). \tag{3.10}$$

The action then becomes

$$S_0[\widetilde{\phi}, \widetilde{J}] = \frac{1}{2}\int_{\mathbb{R}^4} \text{đ}^4k \left[\widetilde{\phi}(-k)(k^2 + m^2)\widetilde{\phi}(k) + \widetilde{J}(-k)\widetilde{\phi}(k) + \widetilde{J}(k)\widetilde{\phi}(-k)\right] \tag{3.11}$$

$$= \frac{1}{2}\int \text{đ}^4k \left[\widetilde{\chi}(-k)(k^2 + m^2)\widetilde{\chi}(k) - \frac{\widetilde{J}(-k)\widetilde{J}(k)}{k^2 + m^2}\right], \tag{3.12}$$

where $\widetilde{\chi} = \widetilde{\phi} + \widetilde{J}/(k^2 + m^2)$. Assume that the partition function is normalised such that $Z_0[0] = 1$. Then

$$Z_0[\widetilde{J}] = \exp\left[\frac{1}{2}\int \text{đ}^4k\, \frac{\widetilde{J}(-k)\widetilde{J}(k)}{k^2 + m^2}\right]. \tag{3.13}$$

The propagator is

$$\widetilde{\Delta}_0(q) = \frac{\delta^2 Z_0[\widetilde{J}]}{\delta\widetilde{J}(-q)\delta\widetilde{J}(q)}\Big|_{\widetilde{J}=0} = \frac{1}{q^2 + m^2}. \tag{3.14}$$

The functional derivative is defined in position and momentum space as

$$\frac{\delta}{\delta f(x_1)} f(x_2) = \delta^4(x_1 - x_2) \qquad \frac{\delta}{\delta \widetilde{g}(k_1)} \widetilde{g}(k_2) = \text{đ}^4(k_1 - k_2). \tag{3.15}$$

We can now Fourier transform back to obtain the propagator in position space

$$\Delta_0(x - x') = \int \text{đ}^4 k \, \frac{e^{ik\cdot(x-x')}}{k^2 + m^2}. \tag{3.16}$$

The partition functional in real space is then

$$Z_0[J] = \exp\left[\frac{1}{2} \int \mathrm{d}^4 x \, \mathrm{d}^4 x' \, J(x) \Delta(x - x') J(x')\right]. \tag{3.17}$$

3.2.1 Interactions

Let us now include interactions. The Lagrangian density becomes

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1, \tag{3.18}$$

where $\mathscr{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2$ is the free Lagrangian density. In presenting the following results we will skip a few steps since the derivations are very similar to what we have seen in $d = 1$ in the previous chapter. The partition functional is

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi \exp\left[-\int \mathrm{d}^4 x \, (\mathscr{L}_0 + \mathscr{L}_1 + J\phi)\right] && (3.19) \\ &= \exp\left\{-\int \mathrm{d}^4 y \, \mathscr{L}_1\left[-\frac{\delta}{\delta J(y)}\right]\right\} \exp\left[\frac{1}{2}\int \mathrm{d}^4 x \, \mathrm{d}^4 x' \, J(x) \Delta_0(x - x') J(x')\right] && (3.20) \\ &\sim \sum_{V=0}^{N} \frac{1}{V!} \left(-\int \mathrm{d}^4 y \, \mathscr{L}_1\left[-\frac{\delta}{\delta J(y)}\right]\right)^V \sum_{P=0} \frac{1}{P!} \left[\frac{1}{2}\int \mathrm{d}^4 x \, \mathrm{d}^4 x' \, J(x) \Delta_0(x - x') J(x')\right]^P. && (3.21) \end{aligned}$$

For each term in $Z[J]$ there is a graph, as we have done before.

- Each of the P propagators $\Delta_0(x - x')$ is represented by

$$x \;\frac{}{}\; x' = \Delta_0(x - x'). \tag{3.22}$$

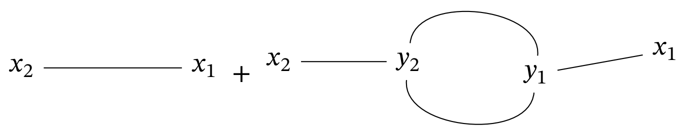
- We have V vertices with n lines from $\mathscr{L}_1\left[-\frac{\delta}{\delta J(y)}\right]$.
- We integrate over positions of all vertices.
- We have symmetry factors as before.
- Sources, represented by external large dots need to come in pairs; one derivative to bring down a J and the other to annihilate it.

$$J(x) \bullet\!\!\frac{}{}\!\!\bullet J(x') \tag{3.23}$$

Example 3.2.1 (ϕ^3 theory)**:** With the Wilsonian effective action $W[J] = -\ln Z[J]$, the two-point correlation is

$$\langle \phi(x_2)\phi(x_1)\rangle = -\left(-\frac{\delta}{\delta J(x_2)}\right)\left(-\frac{\delta}{\delta J(x_1)} W[J]\right). \tag{3.24}$$

Take $\mathscr{L}_1 = \frac{\lambda}{3!}\phi^3$. Then we have

$\langle \phi(x_2)\phi(x_1)\rangle =$  $+ \ldots$ (3.25)

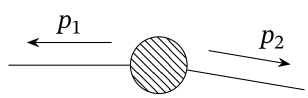
The second diagram is

$$D = \frac{\lambda^2}{2}\int \mathrm{d}^4 y_1\, \mathrm{d}^4 y_2\, \Delta_0(x_2 - y_2)\Delta_0(y_1 - x_1)(\Delta_0(y_2 - y_1))^2 \tag{3.26}$$

with symmetry factor $S = 2$. Fourier transforming this gives

$$\langle \widetilde{\phi}(p_2)\widetilde{\phi}(p_1)\rangle \int \mathrm{d}^4 x_1\, \mathrm{d}^4 x_2\, e^{-i(p_1\cdot x_1 + p_2\cdot x_2)}\langle \phi(x_2)\phi(x_1)\rangle, \tag{3.27}$$

where the exponential is a convention. This phase convention implies that all momenta are outward (see Brown Ch. 5.1). We then have the Fourier transform of the above diagram



(3.28)

$$\widetilde{D} = \int \mathrm{d}^4 x_1\, \mathrm{d}^4 x_2\, e^{-i(p_1\cdot x_1 + p_2\cdot x_2)} \int \mathrm{d}^4 y_1\, \mathrm{d}^4 y_2 \int \left[\prod_{j=1}^{4} \text{đ}^4 k_j\right]$$
$$\times\, e^{ik_2\cdot(x_2 - y_2)} e^{ik_1\cdot(y_1 - x_2)} e^{i(k_3+k_4)\cdot(y_2 - y_1)}$$
$$\times\, \widetilde{\Delta}\widetilde{\Delta}\widetilde{\Delta}?? \tag{3.29}$$

The only terms containing x_1 and x_2 are $e^{-i(p_1+k_1)\cdot x_1}$ and $e^{-i(p_2-k_2)\cdot x_2}$. These integrate to $\text{đ}^4(p_1 + k_1)$ and $\text{đ}^4(p_2 - k_2)$.

$$\widetilde{D} = \lambda^2 \int \mathrm{d}^4 y_1\, \mathrm{d}^4 y_2 \int \text{đ}^4 k_3\, \text{đ}^4 k_4\, e^{-i(p_1+k_3+k_4)\cdot y_1} e^{-i(p_2-k_3-k_4)\cdot y_2}$$
$$\times\, \widetilde{\Delta}_0(-p_1)\widetilde{\Delta}_0(p_2)\widetilde{\Delta}_0(k_3)\widetilde{\Delta}_0(k_4) \tag{3.30}$$

$$\cdots = \lambda^2 \int \text{đ}^4 k\, \text{đ}^4(p_1 + p_2)\widetilde{\Delta}_0(-p_1)\widetilde{\Delta}_0(p_2)\widetilde{\Delta}_0(p_2 - k)\widetilde{\Delta}_0(k). \tag{3.31}$$

It is conventional to write loop diagrams with the momenta flowing in a consistent direction

$$\widetilde{D} = NOTES \tag{3.32}$$

The momentum space Feynman rules can be summarised to be

- lines propagator $1/(k^2 + m^2)$
- vertices $-V^(n)$ in $\mathscr{L}_1$
- Momentum conserved at each vertex
- each loop gives a momentum integral
- overall momentum conservation comes from $\bar{\delta}^4(\sum_j p_j)$
- symmetry factors

3.3 Vertex Functions

Recall that $W[J]$ and $\Gamma[\Phi]$ are the sum of connected and 1PI diagrams respectively. Generalising the definition (1.102) from $d = 0$, the quantum effective action is the Legendre transform

$$\Gamma[\Phi] = W[J] - \int \mathrm{d}^4x\, J(x)\Phi(x). \tag{3.33}$$

Then we have the familiar relations, which are now functional derivatives

$$\frac{\delta W[J]}{\delta J(x)} = \Phi(x), \qquad \frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} = -J(x). \tag{3.34}$$

The connected n-point functions are

$$G^{(n)}(x_1, \dots, x_n) = (-1)^{n+1} \prod_{i=1}^{n} \frac{\delta}{\delta J(x_i)} W[J] = \langle \phi(x_1) \dots \phi(x_n) \rangle^{\text{connected}}. \tag{3.35}$$

Analogously, we define the n -point vertex functions

$$\Gamma^{(n)}(x_1, \dots, x_n) = (-1)^n \prod_{i=1}^{n} \frac{\delta}{\delta \Phi(x_i)} \Gamma[\Phi]. \tag{3.36}$$

Example 3.3.1 ($n = 2$)**:** The 2 -point functions are

$$G^{(2)}(x, y) = -\frac{\delta^2 W}{\delta J(x) \delta J(y)} = -\frac{\delta \Phi(y)}{\delta J(x)}, \tag{3.37}$$

$$\Gamma^{(2)}(x, y) = +\frac{\delta^2 \Gamma}{\delta \Phi(x) \delta \Phi(y)} = -\frac{\delta J(x)}{\delta \phi(y)}. \tag{3.38}$$

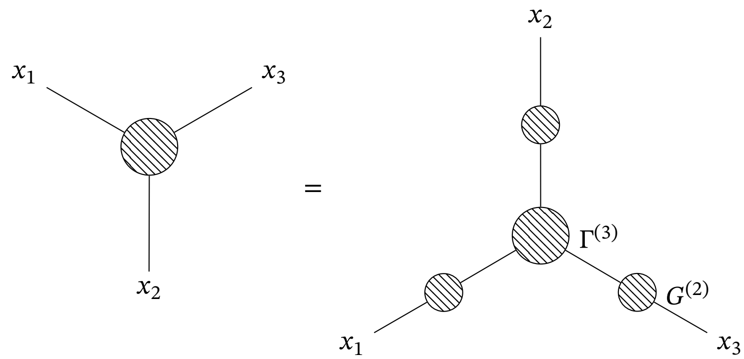
These are inverses of each other

$$\int \mathrm{d}^4z\, G^{(2)}(x, z) \Gamma^{(2)}(z, y) = \delta^{(4)}(x - y). \tag{3.39}$$

Example 3.3.2 (Sheet I Question 4)**:** We can calculate

$$G^{(3)}(x_1, x_2, x_3) = \int \mathrm{d}^4z_1\, \mathrm{d}^4z_2\, \mathrm{d}^4z_3 \cdot G^{(n)}(x_1, z_1) G^{(n)}(x_2, z_2) G^{(n)}(x_3, z_3) \underbrace{\left(-\frac{\delta^3 \Gamma}{\delta \Phi(z_1) \delta \Phi(z_2) \delta \Phi(z_3)} \right)}_{:=\Gamma^{(3)}(z_1, z_2, z_3)} \tag{3.40}$$

Diagrammatically, this can be represented



, (3.41)

where the central blob on the right hand side corresponds to the 1PI diagrams $\Gamma^{(3)}$ and the small blobs are the propagators $G^{(2)}$. The vertex functions $\Gamma^{(3)}$ come from *amputated* n-point functions. Equation (3.40) can be inverted

$$\Gamma^{(3)}(y_1, y_2, y_3) = \int d^4x_1\, d^4x_2\, d^4x_3\, \Gamma^{(2)}(x_1, y_1)\Gamma^{(2)}(x_2, y_2)\Gamma^{(2)}(x_3, y_3)G^{(3)}(x_1, x_2, x_3). \tag{3.42}$$

It is useful to compare this to the LSZ reduction formula. We can generalise this to n and in momentum space. This is done in Ryder Section 7.3.

3.4 Renormalisation

Consider our favourite ϕ^4 scalar field theory in Euclidean spacetime

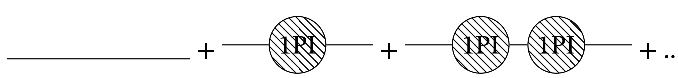
$$S[\phi] = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{3.43}$$

The full propagator

$$\widetilde{G}^{(2)}(p) = \int d^4x\, e^{-ip\cdot x}\langle\phi(x)\phi(0)\rangle^{\text{conn}} \tag{3.44}$$

can be written as a geometric series to make the connection between connected and 1PI diagrams. Diagrammatically,

$$\widetilde{G}^{(n)}(p) =$$



(3.45)

$$\begin{aligned} &= \frac{1}{p^2+m^2} + \frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2} + \ldots && (3.46)\\ &= \frac{1}{p^2+m^2-\Pi(p^2)}, && (3.47)\end{aligned}$$

where $\Pi(p^2) = \widetilde{\Gamma}^{(2)}(p)$. Perturbatively

$$\Pi(p^2) = DIAGRAMS \tag{3.48}$$

The first term in particular is

$$DIAGRAM = -\frac{\lambda}{2}\int đ^4k\,\frac{1}{k^2+m^2}. \tag{3.49}$$

Now in spherical polar coordinates $d^4k = k^3\,dk\,d\Omega$ and in general $d^dk = S_d|k|^{d-1}\,d|k|$ with $S_d = (2\pi)^{d/2}/\Gamma(d/2)$ is the surface area of the unit d-sphere, where Γ is the Euler-Gamma function. We see that this integral diverges. To tackle it, we introduce a cutoff Λ and change variables to $x = k^2/m^2$ so that

$$-\frac{\lambda}{2}\int^{\Lambda}\frac{đ^4k}{k^2+m^2} = -\frac{\lambda S_4}{4(2\pi)^4}\int^{\Lambda^2/m^2}\frac{x\,dx}{1+x}, \tag{3.50}$$

$$\Pi_i(p^2) = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right]. \tag{3.51}$$

As expected, this is divergent as $\Lambda\to\infty$. We call this a *UV divergence*.

The 4-point function at 1-loop is given by the following sum of diagrams

$$DIAGRAMS = \frac{\lambda^2}{2}\int^{\Lambda}đ^4k\,\frac{1}{k^2+m^2}\sum_{p\in\{p_1+p_2,p_1+p_3,p_1+p_4\}}\frac{1}{(p+k)^2+m^2}. \tag{3.52}$$

$$= \widetilde{\Gamma}^{(4)}(p_1,p_2,p_3,p_4) \tag{3.53}$$

As $k\to\infty$, we have $dk\,/k^4$ in the integral and we expect a logarithmic $\ln(\Lambda/m)$ divergence. Since this integral is independent of the p's, it is convenient to evaluate the integral with external momenta set to zero.

$$\widetilde{\Gamma}^{(4)}(0,0,0,0) = \frac{3\lambda^2}{2}\int^{\Lambda}đ^4k\,\frac{1}{(k^2+m^2)^2} = \frac{3\lambda^2}{16\pi^2}\int^{\Lambda}\frac{k^3\,dk}{(k^2+m^2)^2} \tag{3.54}$$

$$= \frac{3\lambda^2}{32\pi^2}\left[\ln\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2+m^2}\right] \tag{3.55}$$

These diagrams must be dealt with.

On general grounds, we expect the full propagator to be of the form

$$\widetilde{G}^{(3)} = \sum_n \frac{\left|\langle\Omega|\,\phi(0)\,|n\rangle\right|^2}{p^2+m^2} \tag{3.56}$$

where the sum (or the integral) over the states n comes about from inserting a complete set of states. The assumption is that the first excited state is a single particle $n=1$, which dominates the expression:

$$\widetilde{G}^{(3)} = \frac{\left|\langle\Omega|\,\phi(0)\,|1\rangle\right|^2}{p^2+m_1^2} + \dots, \tag{3.57}$$

where the additional terms are finite when $p^2 = -m_1^2$. We expect m_1^2 to be a physical mass with $\langle\Omega|\,\phi(0)\,|1\rangle = 1$. Loop diagrams spoil this; we have to renormalise the theory to control the divergences.

Let us decorate the original theory by adding 0 subscripts

$$\mathscr{L}_0 = \frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4. \tag{3.58}$$

Generically, contribution from loops spoil this and we have to rescale

$$\phi_0 = Z_\phi^{\frac{1}{2}}\phi. \tag{3.59}$$

The rescaling Z_ϕ is determined by proper normalisation for LSZ, namely that

$$\langle\Omega|\,\widetilde{\phi}(0)\,|1\rangle = 1. \tag{3.60}$$

The Lagrangian becomes

$$\mathscr{L}_0 = \frac{Z_\phi}{2}(\partial\phi)^2 + \frac{Z_\phi}{2}m_0^2\phi^2 + \frac{Z_\phi^2\lambda}{4!}\phi^4. \tag{3.61}$$

What we want to do is separate out two sets of terms. We want to write the original Lagrangian in terms of the renormalised Lagrangian and any counter terms. In particular, the renormalised Lagrangian should be of the same form as the original action, giving

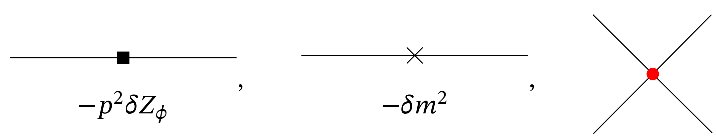
$$\mathscr{L}_0 = \mathscr{L}_{\text{ren}} + \mathscr{L}_{\text{ct}}, \tag{3.62}$$

$$= \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right] + \left[\frac{\delta Z_\phi}{2}(\partial\phi)^2 + \frac{\delta m^2}{2}\phi^2 + \frac{\delta\lambda}{4!}\phi^4\right]. \tag{3.63}$$

Equating coefficients, we have

$$\delta Z_\phi = Z_\phi - 1, \qquad \delta m^2 = Z_\phi m_0^2 - m^2, \qquad \delta\lambda = Z_\phi^2\lambda_0 - \lambda. \tag{3.64}$$

Of course the Feynman rules for the renormalised Lagrangian $\mathscr{L}_{\text{ren}}$ are the same as for the original Lagrangian $\mathscr{L}_0$ but with the coefficients, $m^2\&\lambda$, interpreted as the renormalised ones. We need to find the Feynman rules for the counter terms $\mathscr{L}_{\text{ct}}$:



$-p^2\delta Z_\phi$, $-\delta m^2$, $-\,\delta\lambda.$ (3.65)

Generally, $\delta Z_\phi, \delta m^2, \delta\lambda$ are $O(\hbar)$ at most. Therefore, tree diagrams containing $\mathscr{L}_{\text{ct}}$ vertices are the same order as 1-loop diagrams from $\mathscr{L}_{\text{ren}}$.

The 2-point vertex is

$$\widetilde{\Gamma}^{(2)}(p) = [\widetilde{G}^{(2)}(p)]^{-1} = p^2 + m^2 - \Pi(p^2). \tag{3.66}$$

From $\mathscr{L}_{\text{ren}}$, we get $\Pi_1(p^2)$ at one loop just as in (3.51), but with m^2, λ interpreted as renormalised quantities of (3.63). From $\mathscr{L}_{\text{ct}}$,

$$\Pi_{1,\text{ct}} = AMPUTATEDDIAGRAMS = -\delta m^2 - p^2\delta Z_\phi. \tag{3.67}$$

Finite result for $\Pi_{i\text{ren}} = \Pi_1(p^2) + \Pi_{1,\text{ct}}$ is obtained by choosing

$$\delta Z_\phi = 0 \qquad \text{and} \qquad \delta m^2 = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right]. \tag{3.68}$$

With this choice $\Pi_{1,\text{ren}} = 0$. The freedom to choose where to put finite points is called the *renormalisation scheme*. This arbitrary choice obviously looses some predictability, but we do it to make progress with the divergences. The above scheme is called the *on-shell* scheme, based on the requirement that

$$\Pi_{\text{ren}}(-m^2_{\text{phys}}) \overset{!}{=} m^2 - m^2_{\text{phys}} \qquad \text{and} \qquad \left.\frac{\partial \Pi_{\text{ren}}}{\partial p^2}\right|_{p^2=-m^2_{\text{phys}}} = 0. \tag{3.69}$$

The first term usually cancels out $m^2 - m^2_{\text{phys}} = 0$.

With the on-shell scheme,

$$\widetilde{G}^{(2)}(p) = \frac{1}{p^2 + m^2 - \Pi_{\text{ren}}(p^2)} = \frac{1}{p^2 + m^2_{\text{phys}}}. \tag{3.70}$$

This has a pole at $p^2 = -m^2_{\text{phys}}$, which is the reason for the name *on-shell*. We are giving up on predicting the mass of the particle; we dial it in by hand after obtaining it from some experimental measurement. The residue from the LSZ is 1.

Next, choose $\delta\lambda$ to cancel the divergences in $\Gamma^{(4)}_1(0,0,0,0)$. We have $\widetilde{\Gamma}^{(4)}_{1,\text{ct}} = -\delta\lambda$. Choosing

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2} \ln\left(\frac{\Lambda^2}{m^2} - 1\right). \tag{3.71}$$

After a bit of algebra, this gives

$$\lambda_{\text{eff}} := \widetilde{\Gamma}^{(4)}_{\text{ren}}(0,0,0,0) = \lambda + \widetilde{\Gamma}^{(4)}_1(0,0,0,0) + \widetilde{\Gamma}^{(4)}_{1,\text{ct}} \tag{3.72}$$

$$= \lambda - \frac{3\lambda^2}{32\pi^2}\left[\ln\left(1 + \frac{m^2}{\Lambda^2}\right) + \frac{m^2}{m^2 + \Lambda^2}\right]. \tag{3.73}$$

This is finite as $\Lambda \to \infty$. (In fact, the infinite piece is chosen such that $\lambda_{\text{eff}} \to \lambda$.)

This term really acts like an effective coupling, which—at least to one-loop order—incorporates the quantum corrections. If we did this calculation to all orders in a well defined theory, it would be solved.

3.5 Dimensional Regularisation

Since physical predictions come from tree diagrams with effective couplings like λ_{eff}, physical quantities should be independent of the cutoff Λ in the end. We will do this not with a hard momentum cutoff, but with a method that works more generally for gauge theories. Hard momentum cutoff are not compatible with gauge invariance, so we want to come up with a different regularisation method.

A more mathematically elegant way to do this is given by *dimensional regularisation*. It is a trick we do order-by-order in perturbation theory.

In the context of perturbation theory, divergences can be regulated by working in $d = 4 - \epsilon$ dimensions. Usually, we think about $0 < \epsilon \ll 1$, but we have already seen that taking $\epsilon \to 1$ gave us useful results in *Statistical Field Theory*.

Let us start from the original Lagrangian (and drop the zero subscripts)

$$S = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(m\phi)^2 + \frac{\lambda}{4!}\phi^4\right]. \tag{3.74}$$

3.5.1 Dimensional Analysis

Denote by square brackets $[\bullet]$ the mass dimension and let $\hbar = c = 1$. Given that $[S] = 0$ and $[\partial] = [m] = -[x] = 1$, we find that the field has mass dimension

$$[m^2\phi^2] = 2[m] + 2[\phi] = d \implies [\phi] = \frac{d}{2} - 1. \tag{3.75}$$

From this we find that the coupling has dimension

$$[\lambda\phi^4] = d \implies [\lambda] = 4 - d = \epsilon. \tag{3.76}$$

Working with a dimensionful coupling is annoying, so we introduce an arbitrary renormalisation scale μ with mass dimension $[\mu] = 1$, which is not to be taken to ∞ like the cutoff was. Then we write

$$\lambda = \mu^\epsilon g(\mu) \tag{3.77}$$

such that g is dimensionless. Of course, g will depend on whichever choice we end up making for the renormalisation scale μ.

Thus our action is

$$S = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g\mu^\epsilon}{4!}\gamma^4\right]. \tag{3.78}$$

3.5.2 Mathematical Notes

Let us briefly collect a few mathematical results.

1. Surface area S_d of a unit sphere in d dimensions: For integer d, we can do d Gaussian integrals and convert to polar coordinates

$$(\sqrt{\pi})^d = \int_{\mathbb{R}^d} \prod_{i=1}^{d} \mathrm{d}x_i\, e^{-x_i^2} = S_d \int_0^\infty \mathrm{d}r\, r^{d-1} e^{-r^2} = \frac{1}{2} S_d\, \Gamma(\frac{d}{2}). \tag{3.79}$$

   For non-integer $d \in \mathbb{C}$, we define S_d via analytic continuation as

$$S_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}. \tag{3.80}$$

2. Analytic continuation of the Γ-function: Use, for $\alpha > 0$

$$\Gamma(\alpha) = \int_0^\infty \mathrm{d}x\, x^{\alpha-1} e^{-x} \qquad \alpha\Gamma(\alpha) = \Gamma(\alpha+1), \qquad \Gamma(1), \qquad \Gamma(\frac{1}{2}) = \sqrt{-\pi}. \tag{3.81}$$

3. Expansion of the Γ-function

$$\ln \Gamma(\alpha+1) = -\gamma\alpha - \sum_{k=2}^{\infty} (-1)^k \frac{1}{k} \zeta(k), \tag{3.82}$$

   where $\gamma = \gamma_E \approx 0.577216$ is the Euler–Mascheroni constant and $\zeta(k) = \sum_{n=1}^\infty \frac{1}{n^k}$ is the Riemann ζ-function. Usually we exponentiate this

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon). \tag{3.83}$$

4. Euler beta function:

$$B(s,t) = \int_\gamma^1 \mathrm{d}x\, u^{s-1}(1-u)^{t-1} = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}. \tag{3.84}$$

Remark: You will not be asked to prove these or even have these at hand in the exam.

Let us now return to the amputated diagram DIAGRAM

$$\Pi_1(p^2) = -\frac{g\mu^\epsilon}{2} \int \frac{đ^d k}{k^2 + m^2} = -\frac{g\mu^\epsilon}{2} \frac{S_d}{2(2\pi)^d} \int_0^\infty \frac{k^{d-1}\,\mathrm{d}k}{k^2 + m^2}. \tag{3.85}$$

The integral can be performed by changing integration variables $u = m^2/(k^2 + m^2)$

$$\mu^\epsilon \int_0^\infty \frac{k^{d-1}\,\mathrm{d}k}{k^2+m^2} = \frac{1}{2}\mu^\epsilon \int_0^\infty \frac{(k^2)^{d/2-1}\,\mathrm{d}k^2}{k^2+m^2} = \frac{m^2}{2}(\frac{\mu}{m})^\epsilon \int_0^1 \mathrm{d}u\, u^{-d/2}(1-u)^{d/2-1} \tag{3.86}$$

$$= \frac{m^2}{2}(\frac{\mu}{m})^\epsilon \frac{\Gamma(\frac{d}{2})\Gamma(1-\frac{d}{2})}{\Gamma(1)}, \tag{3.87}$$

where we used the Euler beta function. Hence,

$$\Pi_1 = -\frac{gm^2}{2(4\pi)^{d/2}}(\frac{\mu}{m})^\epsilon \Gamma(1-\frac{d}{2}). \tag{3.88}$$

Using that

$$\Gamma(1-\frac{d}{2}) = \Gamma(\frac{\epsilon}{2}-1) = -\frac{1}{1-\frac{\epsilon}{2}}\Gamma(\frac{\epsilon}{2}) = -\frac{2}{\epsilon} + \gamma - 1 + O(\epsilon). \tag{3.89}$$

We have thus exposed the divergence as $\epsilon \to 0$ that we want to tame.

Also

$$\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon/2} = 1 + \frac{\epsilon}{2}\ln\left(\frac{4\pi\mu^2}{m^2}\right) + O(\epsilon^2). \tag{3.90}$$

The result of putting the Lagrangian $\mathscr{L}_0$ into the diagram and finding the one-loop contribution to the vertex function is

$$\Pi_1(p^2) = -\frac{gm^2}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma + 1 + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right] + O(\epsilon). \tag{3.91}$$

As in (3.62), we must add a counter term $\frac{1}{2}\delta m^2\phi^2$ to the Lagrangian with the intention to cancel the divergence. There are various possible renormalisation schemes we can choose. In practice, we often use one of

Minimal subtraction (MS): Just absorb the pole

$$\delta m^2 = -\frac{gm^2}{16\pi^2\epsilon}. \tag{3.92}$$

Modified minimal subtraction ($\overline{\mathrm{MS}}$) :

$$\delta m^2 = -\frac{gm^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) \tag{3.93}$$

Under this scheme, our result for the two-point function is

$$\Pi_1^{\overline{\mathrm{MS}}} = \frac{gm^2}{32\pi^2}\left(\ln\frac{\mu^2}{m^2} - 1\right). \tag{3.94}$$

Let us also calculate the four-point vertex function. The divergent piece is given by setting all external momenta to zero:

$$\widetilde{\Gamma}^{(4)}(0,0,0,0) = \frac{3g^2\mu^{2\epsilon}}{2}\int \frac{\bar{d}^d k}{(k^2+m^2)^2}. \tag{3.95}$$

We perform exactly the same tricks as before, performing angular integration and changing variables to make it dimensionless, as well as identifying the Euler-beta function. In the end, we get a very similar result

$$\widetilde{\Gamma}^{(4)}(0,0,0,0) = \frac{3g^2\mu^{\epsilon}}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln\frac{4\pi\mu^2}{m^2}\right) + O(\epsilon). \tag{3.96}$$

We have left one μ^ϵ outside on dimensional grounds in the next step. Again, this is divergent, so we introduce a counter term for g as well:

$$\mu^\epsilon \delta g = \frac{3g^2\mu^\epsilon}{32\pi^2}\underbrace{\Bigg(\overbrace{\frac{2}{\epsilon}}^{\text{MS}} - \gamma + \ln 4\pi\Bigg)}_{\overline{\text{MS}}}. \tag{3.97}$$

3.6 Calculating β-functions

3.6.1 The Old-Fashioned Approach to Investigating μ-dependence

We have the original Lagrangian

$$\mathscr{L}_0 = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4. \tag{3.98}$$

Adding counter terms we have

$$\mathscr{L}_{\text{ren}} + \mathscr{L}_{\text{ct}} = \frac{1+\delta Z_\phi}{2}(\partial\phi)^2 + \frac{m^2+\delta m^2}{2}\phi^2 + \frac{(g+\delta g)\mu^\epsilon}{4!}\phi^4. \tag{3.99}$$

This is just supposed to be a reshuffling of terms and divergences, so the two Lagrangians should be the same. Equating coefficients: the original parameters are μ -independent, since this splitting scale is arbitrarily chosen.

Look at dimensionless derivatives of the couplings (and masses)

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}g = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}g. \tag{3.100}$$

This is called the β -function, which tells us how the coupling constant ‘runs’ depending on the renormalisation scale.

We want

$$0 \stackrel{!}{=} \frac{\mathrm{d}}{\mathrm{d}\ln\mu}\lambda_0 = \frac{\mathrm{d}}{\mathrm{d}\ln\mu}\left[(g+\delta g)\mu^{\epsilon}\right] = \epsilon g(1+\frac{3g}{16\pi^2\epsilon}) + \beta(g)\left(1+\frac{3g}{8\pi\epsilon^2}\right), \tag{3.101}$$

where we employed the MS-scheme result of (3.97) (to save writing). The β-function is

$$\beta(g) = -\left(\frac{3g^2}{16\pi^2}+\epsilon g\right)\left(1+\frac{3g}{8\pi\epsilon^2}\right)^{-1}. \tag{3.102}$$

With a little bit of slight of hand, we forget that we want to take $\epsilon \to 0$ and hold ϵ fixed. We then expand the latter term in a binomial series

$$\beta(g) = \frac{3g^2}{16\pi^2} - \epsilon g + O(\frac{g^2}{\epsilon}, 2\text{ loop}). \tag{3.103}$$

Remark: In this whole calculation, we have ignored two-loop diagrams and higher.

As $\epsilon \to 0$, only the first term survives. Note that $\beta(\gamma) > 0$. As such, we obtained a differential equation, which tells us how g depends on μ:

$$\mu\frac{\mathrm{d}g}{\mathrm{d}\mu} = \frac{3g^2}{16\pi^2}. \tag{3.104}$$

We can solve this differential equation by separation of variables

$$\frac{\mathrm{d}g}{g^2} = \frac{3}{16\pi^2}\frac{\mathrm{d}\mu}{\mu}. \tag{3.105}$$

Integrating gives

$$\frac{1}{g(\mu')} = \frac{1}{g(\mu)} - \frac{3}{16\pi^2}\ln\frac{\mu'}{\mu}, \tag{3.106}$$

so

$$g(\mu') = \frac{g(\mu)}{1 - \frac{3g}{16\pi^2}\ln\frac{\mu'}{\mu}} \overset{g\text{ small}}{\approx} g(\mu) + \frac{3(g(\mu))^2}{16\pi^2}\ln\frac{\mu'}{\mu}. \tag{3.107}$$

For $\mu' > \mu$, $g(\mu') > g(\mu)$. The coupling "runs" to larger values as μ increases.

Remark: If $\mu' \to \Lambda_{\phi^4}$, where Λ_{ϕ^4} is defined via

$$\frac{3g}{16\pi^2}\ln\frac{\Lambda_{\phi^4}}{\mu} = 1, \qquad \text{1-loop} \tag{3.108}$$

then $g(\mu') \to \infty$. This Λ_{ϕ^4} can be used as a scheme-dependent reference mass scale.

$$g(\mu) = \frac{16\pi^2}{3}\left[\ln\left(\frac{\Lambda_{\phi^4}}{\mu}\right)\right]^{-1}. \tag{3.109}$$

The appearance of the scale Λ_{ϕ^4} is "dimensional transmutation"; it seems like magic / alchemy that the regularisation of a dimensionless interaction introduces a scheme-dependent scale. It is an order-of-magnitude estimate of where the theory becomes non-perturbative.

Remark: Perturbation theory requires $\mu \ll \Lambda_{\phi^4}$.

3.6.2 The Modern Approach

Quantum effective action and the vertex functions $\Phi^{(n)}(\dots)$ should be physical. These go into the LSZ formula (2.17). Write $\phi_0 = Z_\phi^{\frac{1}{2}}\phi$, then

$$\Gamma_0^{(?)}(x_1, \dots, x_n) = (-1)^n \frac{\delta^n\Gamma[\phi_0]}{\delta\phi_0(x_1)\dots\delta\phi_0(x_n)} = (-1)^n Z_\phi^{1/2}\frac{\delta^n\Gamma[\phi]}{\delta\phi(x_1)\dots\delta\phi(x_n)}. \tag{3.110}$$

Definition 15 (anomalous dimension)**:** We define the analogue of the beta function, the *anomalous dimension* of ϕ , to be

$$\gamma_\phi = -\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_\phi. \tag{3.111}$$

With this definition, we have

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}Z_\phi^{1/2} = -\frac{n}{2}Z_\phi^{-n/2}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_\phi = (n\gamma_\phi)Z_\phi^{-n/2}. \tag{3.112}$$

We require that terms in Γ should be independent of scale, which implies

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu}\Gamma^{(n)} = 0 = \left(\mu\frac{\partial}{\partial \mu} + \mu \frac{\mathrm{d}m^2}{\mathrm{d}\mu}\frac{\partial}{\partial m^2} + \beta(g) + n\gamma_\phi\right)\Gamma^{(n)}_{\text{ren}}(x_1, \dots, x_n). \tag{3.113}$$

Equations like this, which govern the running of the parameters of the theory by looking at n -point functions, are called *Callan–Symanzyk equations*.

All but the first term in the parentheses are (at least) order $\hbar$ at 1 -loop. In ϕ^4 theory $Z_\phi = 1$ to 1-loop order.

$$\Pi_1^{\overline{\text{MS}}} = \frac{gm^2}{32\pi^2}\left(\ln\frac{\mu^2}{\mu^2} - 1\right) \tag{3.114}$$

$$\widetilde{\Gamma}^{(2)}(p^2 = 0) = \cancel{p^2} + m^2 - \frac{gm^2}{32\pi^2}\left(\ln\frac{\mu^2}{m^2} - 1\right). \tag{3.115}$$

Using the Callan–Symanzyk equation, we have

$$0 = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\widetilde{\Gamma}^{(2)}(0) = \mu\frac{\mathrm{d}m^2}{\mathrm{d}\mu} - \frac{gm^2}{16\pi^2} + O(\hbar^2). \tag{3.116}$$

From this we see that the dimensionally regularised mass changed as

$$\mu\frac{\mathrm{d}m^2}{\mathrm{d}\mu} = \frac{gm^2}{16\pi^2}. \tag{3.117}$$

Including the leading order term and one-loop correction, we have

$$\widetilde{\Gamma}^{(4)}(0,0,0,0) = -g\mu^\epsilon + \frac{3g^2\mu^\epsilon}{32\pi^2}\ln\frac{\mu^2}{m^2}. \tag{3.118}$$

Differentiating this (and ignoring the μ^ϵ since the derivatives of those will vanish at $\epsilon \to 0$), we have

$$0 = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\widetilde{\Gamma}^4 = -\beta(g)\mu^\epsilon + \frac{3g^2\mu^\epsilon}{16\pi^2} + O(\hbar^2). \tag{3.119}$$

Once again, our beta function is

$$\beta(g) = \frac{3g^2}{16\pi^2}, \tag{3.120}$$

which is the same as (3.103). This argument is the more modern approach.

4 The Renormalisation (Semi-)Group

The idea, which comes from statistical field theory, is that we are studying quantum field theories that fall into universality classes.

We impose some cutoff, which has a degree of arbitrariness to it, in the UV. However, at lower energies, we want to see the same universal IR physics emerging from theories with different regularisation and renormalisation schemes and scales. We will have to tune these scales to get the correct low-energy physics.

Remark: When we say *UV*, we mean an unobtainable high-energy regime, such as the Planck scale, which we cannot probe with experiment. In contrast, the *IR* is all the interesting physics that is accessible to us.

The idea behind the renormalisation group (RG) is that we want to study how the microscopic features (i.e. the couplings) change along the "lines of constant IR physics".

Consider a real scalar field with momentum cutoff Λ_0 in $d \in \mathbb{N}$ dimensions. Generically, the action will be

$$S_{\Lambda_0}[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \sum_{\text{terms } i} \frac{1}{\Lambda_0^{d_i - d}} g_{iO}\mathcal{O}_i(x) \right], \tag{4.1}$$

where the "operators" $\mathcal{O}_i[\phi(x)]$ are simply some functions of the fields ϕ and their derivatives $\partial\phi$. We could have $\mathcal{O}_i = (\partial\phi)^{r_i}\phi^{s_i}$ for example. In particular, they are *local* operators with mass dimension $d_i > 0$.

The partition function with cutoff Λ_0 is a function of all the couplings g_{iO}

$$\mathcal{Z}_{\Lambda_0}(g_{iO}) = \int^{\Lambda_0} \mathcal{D}\phi\, e^{-S_{\Lambda_0}[\phi]}. \tag{4.2}$$

The notation on the path integral indicates that the integral is over field modes that have $|p| \leq \Lambda_0$, meaning that the Fourier integral of ϕ in $\mathcal{Z}_{\Lambda_0}$ is

$$\phi(x) = \int_{|p|\leq\Lambda_0} \text{đ}^d p\, e^{ip\cdot x}\widetilde{\phi}(p). \tag{4.3}$$

4.1 Effective Actions

The next step is to look at a slightly smaller cutoff Λ and split the fields into high and low momentum modes

$$\phi(x) = \phi^-(x) + \phi^+(x) \tag{4.4}$$

$$= \int_{|p|<\Lambda} \text{đ}^d p \, e^{ip\cdot x}\widetilde{\phi}(p) + \int_{\Lambda<|p|\leq\Lambda_0} \dots . \tag{4.5}$$

We then want to obtain a Wilsonian effective action $S_\Lambda^{\text{eff}}[\phi]$ (like a W) by integrating out the ϕ^+ as

$$S_\Lambda^{\text{eff}}[\phi] = -\ln \int_\Lambda^{\Lambda_0} \mathcal{D}\phi^+ \, e^{-S_{\Lambda_0}[\phi+\phi^+]}. \tag{4.6}$$

The RG equations will tell us how S_Λ^{eff} and S_{Λ_0} are related. Separate out the terms in the action that couple UV and IR modes

$$S_{\Lambda_0}[\phi + \phi^+] = S^0[\phi^-] + S^0[\phi^+] + S_{\Lambda_0}^{\text{int}}[\phi^-, \phi^+], \tag{4.7}$$

with free action $S^0[\phi] = \int \mathrm{d}^d x \left[(\partial\phi)^2 + m^2\phi^2\right]$. There is no quadratic term $\phi^-\phi^+$ since in Fourier space we would get a delta function

$$\widetilde{\phi^-}(k)\phi^+(k')\delta^{(d)}(k + k'), \tag{4.8}$$

which vanishes for all k both above or below Λ. Another way of saying this is that ϕ^+ and ϕ^- have disjoint support in the momentum space. An example of a non-zero term is $\widetilde{\phi^-}(k)\widetilde{\phi^-}(k')\widetilde{\phi}^+(k'')\delta(k+k' + k'')$. We also have effective interactions

$$S_\Lambda^{\text{int}}[\phi] = -\ln \int \mathcal{D}\phi^+ \, e^{-S^0[\phi^+] - S_{\Lambda_0}^{\text{int}}[\phi^-,\phi^+]}. \tag{4.9}$$

4.2 Running Couplings

If we want the physics to be independent of Λ, Λ_0, then we need the partition functions to be equal

$$\mathcal{Z}_\Lambda(g_i)(g_i(\Lambda)) = \mathcal{Z}_{\Lambda_0}(g_{i0}; \Lambda_0). \tag{4.10}$$

The right-hand side is independent of Λ , which means that the left-hand side must be, too. Thus the couplings $g_i(\Lambda)$ must "run" to compensate. We have the Callan–Symanzik (or RG) equation

$$\Lambda\frac{\mathrm{d}\mathcal{Z}_\Lambda(g)}{\mathrm{d}\Lambda} = \left(\Lambda \left.\frac{\partial}{\partial\Lambda}\right|_{g_i} + \Lambda\frac{\mathrm{d}g_i}{\mathrm{d}\Lambda}\frac{\partial}{\partial g_i}\right)\mathcal{Z}_\Lambda(g) = 0. \tag{4.11}$$

The effective action S_Λ^{eff} has the same form as S_{Λ_0} :

$$S_\Lambda^{\text{eff}}[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2} Z_\Lambda (\partial\phi)^2 + \sum_i \frac{Z_\Lambda^{n_i/2}}{\Lambda^{d_i - d}} g_i(\Lambda) \mathcal{O}_i(x) \right], \tag{4.12}$$

where n_i is the number of fields ϕ in the operator $\mathcal{O}_i(x)$. Integrating out ϕ^+ modes may imply that $Z_\Lambda \neq 1$. We thus renormalise the field to restore the quadratic term in the action:

$$\phi^r = Z_\Lambda^{1/2} \phi, \tag{4.13}$$

where we use the field-renormalisation function Z (not the $\mathcal{Z}$).

Any remaining Λ -dependence must be described by $g_i(\Lambda)$. The classical β-function is $\beta_i^{\text{cl}} = d_i - d$ from the sum in (4.12). The quantum β -function is $\beta^{\text{qu}} = \Lambda \frac{\mathrm{d}g_i}{\mathrm{d}\Lambda}$, which gives the total $\beta = \beta^{\text{cl}} + \beta^{\text{qu}}$.

4.3 Vector Functions

Recall the anomalous dimension $\gamma_\phi = -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\Lambda} \ln Z_\Lambda$. Look at n-point functions. Iterate this mode-thinning (the integrating-out of high-momentum modes). Let $0 < s < 1$.

$$Z_{s\Lambda}^{-n/2} \Gamma_{s\Lambda}^{(n)}(x_1, \dots, x_n; g_i(s\Lambda)) = Z_\Lambda^{-n/2} \Lambda_\Lambda^{(n)}(x_1, \dots, x_n; g(\Lambda)). \tag{4.14}$$

The infinitesimal version of this is the differential equation

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \Gamma_\Lambda^{(n)}(x_1, \dots, x_n; g(\Lambda)) = \left(\Lambda \frac{\partial}{\partial\Lambda} + \beta_i \frac{\partial}{\partial g_i} + n\gamma_\phi \right) \Gamma_\Lambda^{(n)}(x_1, \dots, x_n; g(\Lambda)). \tag{4.15}$$

This can be obtained by letting $s\Lambda = \Lambda'$ for fixed Λ. Differentiate with respect to s

$$s \frac{\mathrm{d}}{\mathrm{d}s} Z_s^{-n/2} = n\gamma_\phi \tag{4.16}$$

using $s\frac{\mathrm{d}}{\mathrm{d}s} = \Lambda' \frac{\mathrm{d}}{\mathrm{d}\Lambda'}$. Then relabel Λ' as Λ.

The RG transformation constitutes two steps:

1) Integrate out momentum modes over the annulus $(s\Lambda, \Lambda]$ in momentum-space.

2) Rescale coordinates $x' = sx$ so that the cutoff again sits at Λ.

Under rescaling, the kinetic term must be made properly renormalised

$$\phi^r(sx) = s^{1-\frac{d}{2}} \phi^r(x). \tag{4.17}$$

Then the rest of the action is invariant if we also rescale $\Lambda \to \Lambda/s$.

The n-point vertex functions should be the same

$$\Gamma^{(n)}_{\Lambda}(x_1, \dots, x_n; g(\Lambda)) = \left(\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2} \Gamma^{(n)}_{s\Lambda}(x_1, \dots, x_n; g(s\Lambda)). \tag{4.18}$$

However, these have different cutoffs, so they are difficult to compare. We want to compare the theory after performing both RG steps, not just the first one. We rescale coordinates / cutoff and the field, giving

$$\cdots = \left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2} \Gamma^{(n)}_{\Lambda}(sx_1, \dots, sx_n; g(s\Lambda)). \tag{4.19}$$

But the numerical values of $Z_{s\Lambda}$ and $g(s\Lambda)$ do not get rescaled.

Reconsider points we look at. Instead of x_I argument, look at x_i/s.

$$\Gamma^{(n)}_{\Lambda}(\frac{x_1}{s}, \dots, \frac{x_n}{s}; g(\Lambda)) = \left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2} \Gamma^{(n)}_{\Lambda}\big(x_1, \dots, x_n; g(s\Lambda)\big). \tag{4.20}$$

As s gets smaller, the left-hand side $\left|x_i - x_j\right|$ gets bigger. On the right-hand side, the couplings are running to the IR. From this we see a connection between the running of the coupling and the physics of scale.

What about the pre-factor? This is the coefficient of the quantum action. For small $\delta s = 1 - s$,

$$\left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{1/2} = 1 + \left[\frac{d-2}{2} + \gamma_\phi\right]\delta s. \tag{4.21}$$

Therefore, the fields, n of which are in the n-point vertex function, behave as if their mass-dimensions were

$$\Delta_\phi := \frac{d-2}{2} + \gamma_\phi. \tag{4.22}$$

The first term $(d-2)/2$ is the "engineering dimension", which we obtained from looking at the dimension of the Lagrangian. The γ_ϕ is the "anomalous dimension". Both of these sum to give the "scaling dimension" Δ_ϕ.