Applications of Differential Geometry to Physics

Part III Lent 2019

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0.1 Kepler / Newton Orbits

$$\ddot{\mathbf{r}} = -\frac{GMv}{r^3}\mathbf{r} \quad \leftrightarrow \quad \text{conic sections} \tag{1}$$

General conic section is

$$ax^2 + by^2 + cxy + dx + ey + f = 0 \tag{2}$$

This is nowadays more generally studied in what we now call *algebraic geometry* rather than differential geometry.

Apolonius of Penge (?) asked 'what is the unique conic thorugh five points, no three of which are co-linear?'

The space of conics is $\mathbb{R}^6 - \{0\}/n = \mathbb{RP}^5$ (projective 5-space).

$$[a, b, c, d, e, f] \sim [\gamma a, \gamma b, \gamma c, \gamma d, \gamma c, \gamma f], \gamma \in \mathbb{R}^* \tag{3}$$

This is an application of geometry, rather than an application of differential geometry.

Remark: Apolonius proved this geometrically.

In this course however, we will look at the following.

1) Hamiltonian mechanics (mid 19^{th}). This is an elegant way of reformulating Newton's mechanics, turning second order differential equations into first order differential equations with the use of a function $H(p, q)$. The system of ODEs is

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{q} = -\frac{\partial H}{\partial q} \tag{4}$$

This led to the development of symplectic geometry (1960s). The connection is that the phase-space to which p and q belong has a 2 -form $dp \wedge dq$. Using the Hamiltonian function, one can find a vector field

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} \tag{5}$$

and looks for a one-parameter group of transformations, called symplectomorphisms, generated by this vector field. Under these symplectomorphisms, the 2 -form is unchanged meaning that the area illustrated in F2 is preserved. Details of this are going to come within the course.

2) General Relativity (1915) ← Riemannian Geometry (1850)

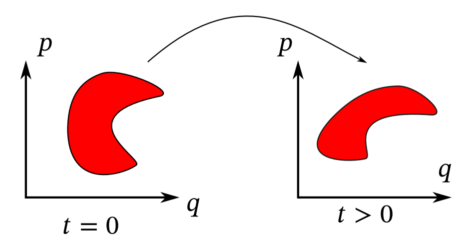

Figure 1

3) Gauge theory (Maxwell, Yang Mills) $\leftrightarrow$ Connection on Principal Bundle (U(1) (Maxwell), SU(2), SU(3))

$$A_+ = A_- + dg \qquad g = \psi_+ - \psi_- \qquad \omega = \begin{cases} A_+ + d\psi_+ \\ A_- + d\psi_- \end{cases} \tag{6}$$

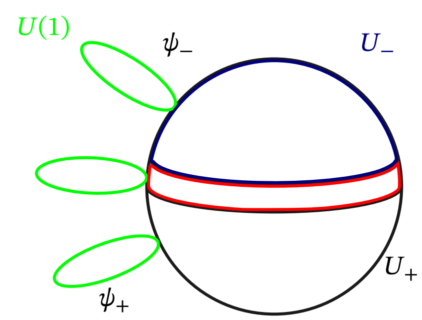

Figure 2

This course: cover 1, 2, 3 in some detail. Unifying feature: Lie groups.

- Prove some theorems, *lots of* examples (often instead of proofs)
- Want to be able to do calculations; compute characteristic classes etc.

We will assume that you took either Part III General Relativity, or Part III Differential Geometry, or some equivalent course.

1 Manifolds

Definition 1 (manifold)**:** An n -dimensional *smooth manifold* is a set M and a collection[2] of open sets U_α , labelled by $\alpha = 1, 2, 3, \ldots$, called *charts* such that

- U_α cover M
- $\exists$ 1-1 maps $\phi_\alpha : U_\alpha \to V_\alpha \in \mathbb{R}^n$ such that

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{1.1}$$

  is a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

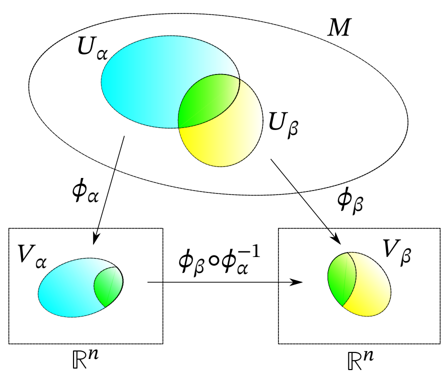

Figure 1.1: Manifold

As such, manifolds are topological spaces with additional structure, allowing us to do calculus.

Example ($M = \mathbb{R}^n$)**:** There is the *trivial manifold*, which can be covered by only one open set. There are other possibilities. In fact, there are infinitely many smooth structures on $\mathbb{R}^4$ (Proof by Donaldson in 1984 in his PhD. He used Gauge theory).

[2] In all examples that we will look at, there will be finitely α.

Example (sphere $S^n = \{\mathbf{r} \in \mathbb{R}^{n+1}, |\mathbf{r}| = 1\}$): Have two open sets

$$U = S^n / \{0, 0, 0, , \dots, 0, 1\} \qquad \widetilde{U} = S^n / \{0, 0, 0, , \dots, 0, -1\} \tag{1.2}$$

We then define charts, where $\mathbb{R}^n = (x_1, \dots, x_n)$:

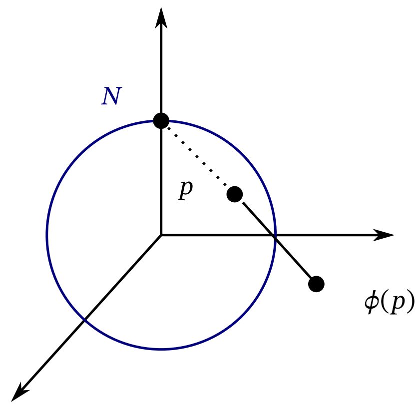

Figure 1.2

$$\begin{aligned} \phi(r_1, \dots, r_{n+1}) &= \left(\frac{r_1}{1 - r_{n+1}}, \dots, \frac{r_n}{1 - r_{n+1}} \right) \\ \text{on } \widetilde{U}, \qquad \widetilde{\phi}(r_1, \dots, r_{n+1}) &= \left(\frac{r_1}{1 - r_{n+1}}, \dots, \frac{r_n}{1 - r_{n+1}} \right) = (\widetilde{x}_1, \dots, \widetilde{x}_n). \end{aligned} \tag{1.3}$$

On $U \cap \widetilde{U}$,

$$\frac{r_k}{1 + r_{n+1}} = \frac{1 - r_{n+1}}{1 + r_{n+1}} \frac{r_k}{1 - r_{n+1}}, \qquad k = 1, \dots, n \tag{1.4}$$

$$\frac{1 - r_{n+1}}{1 + r_{n+1}} = \frac{(1 - r_{n+1})^2}{r_1^2 + r_2^2 + \cdots + r_n^2} = \frac{1}{x_1^2 + x_2^2 + \cdots + x_n^2} \tag{1.5}$$

So on $\phi(U \cap \widetilde{U})$,

$$(\widetilde{x}_1, \dots, \widetilde{x}_n) = \left(\frac{x_1}{x_1^2 + \cdots + x_n^2}, \dots, \frac{x_n}{x_1^2 + \cdots + x_n^2} \right) \tag{1.6}$$

are smooth maps from $\mathbb{R}^n \to \mathbb{R}^n$

Example: A Cartesian product of manifolds is a manifold, for example we have the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$.

Definition 2 (surface)**:** Let $f_1, \ldots, f_k : \mathbb{R}^N \to \mathbb{R}$ be smooth functions. A surface $f_1 = 0, \ldots, f_k = 0$ is a manifold of dimension $\dim n = N - k$ if the rank of the matrix $\frac{\partial f_\alpha}{\partial x^i}$, $\alpha = 1, \ldots, k$ and $i = 1, \ldots, N$ is maximal and equal to k at all points of $\mathbb{R}^N$..

Example: The n-sphere S^n is a surface in $\mathbb{R}^{n+1}$ with $f_1 = 1 - |\mathbf{r}|^2$.

Theorem 1 (Whitney)**:** Every smooth manifold of dimension n is an embedded surface in $\mathbb{R}^N$, where $N \leq 2n$.

> If you enjoy using geometrical intuition and looking at surfaces, this theorem ensures that you can always do that and not loose generality.

Definition 3 (real projective space)**:** The n-dimensional *real projective space* is defined as the quotient

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim, \tag{1.7}$$

where we quotient out the equivalence classes $[X_1, \ldots, X_n + 1] \sim [cX_1, \ldots, cX_{n+1}]$ for all $c \in \mathbb{R}^*$. The $[X_1, \ldots, X_{n+1}]$ are called *homogeneous coordinates.*

In other words, this is the space of all lines through the origin in $\mathbb{R}^{n+1}$.

Claim 1: $\mathbb{RP}^n$ is a smooth manifold of dimension n with $(n + 1)$ open sets.

Proof. Let us define our open sets with respect to the homogeneous coordinates. We define the set $U_\alpha : [X] \in \mathbb{RP}^n$ such that $X_\alpha \neq 0$ $\alpha = 1, \ldots, n + 1$. We can now find local coordinates on $\phi_\alpha : U_\alpha \to V_\alpha \in \mathbb{R}^n$

$$x_1 = \frac{X_1}{X_\alpha} \quad \ldots \quad x_{\alpha-1} = \frac{X_{\alpha-1}}{X_\alpha} \quad x_{\alpha+1} = \frac{X_{\alpha+1}}{X_\alpha} \quad \ldots \quad x_n = \frac{X_n}{X_\alpha}. \tag{1.8}$$

□

Exercise 1.1: Prove smoothness of $\phi_\beta \circ \phi_\alpha^{-1}$.

Now it turns out that this manifold is equivalent to $\mathbb{RP}^n = S^n / \mathbb{Z}^2$. From quantum mechanics, we know that this means in particular $\mathbb{RP}^3 = SO(3)$. This is illustrated in 1.3.

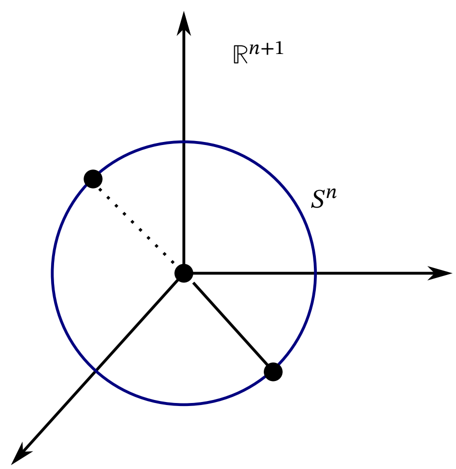

Figure 1.3: Real projective space $\mathbb{RP}^n$ is isomorphic to $S^n/\mathbb{Z}^n$, identifying antipodal points.

2 Vector Fields

Let $M, \widetilde{M}$ be smooth manifolds of dimension $n, \tilde{n}$.

Definition 4 (smooth map)**:** A map $f : M \to \widetilde{M}$ is *smooth* if $\widetilde{\phi}_\beta \circ f \circ \phi_\alpha^{-1}$ is a smooth map from $\mathbb{R}^n$ to $\widetilde{\mathbb{R}}^n$ for all α, β. We call $f : M \to \mathbb{R}$ a *function*, whereas we call $f : \mathbb{R} \to M$ a *curve*.

Let $\gamma : \mathbb{R} \to M$ be a curve. For some $U \in M$, $U \simeq \mathbb{R}^n$, we can define local coordinates $(x^1, \dots, x^n)$

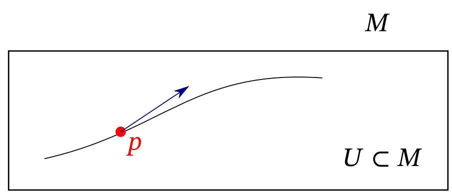

Figure 2.1

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Definition 5 (tangent vector)**:** A *tangent vector* V to γ at p is

$$V|_p = \left.\frac{\mathrm{d}\psi}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \in T_pM, \tag{2.1}$$

where T_pM is the *tangent space* to M at p.

Definition 6 (tangent bundle)**:** We define the *tangent bundle* as $TM := \bigcup_{p\in M} T_pM$.

Definition 7 (vector field)**:** A *vector field* assigns a tangent vector to all $p \in M$.

Let $f : M \to \mathbb{R}$. The rate of change of f along γ is

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(x^a(\epsilon))|_{\epsilon=0} = \sum_a \dot{x}^a \frac{\partial f}{\partial x^a} \tag{2.2}$$

$$= \sum_a V^a \left.\frac{\partial f}{\partial x^a}\right|_{\epsilon=0}, \tag{2.3}$$

whee $V^a := \dot{x}^a|_{\epsilon=0,\dots,x_n}$.

Vector fields are first order differential operators

$$V = \sum_a V^a(\mathbf{x}) \frac{\partial}{\partial x^a}. \tag{2.4}$$

The derivatives $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}\Big|_p$ form a basis of $T_p M$.

2.1 Integral curves

Definition 8 (integral curve)**:** An *integral curve* (a *flow*) of a vector field is defined by

$$\dot{\gamma}(\epsilon) = V|_{\gamma(\epsilon)}, \tag{2.5}$$

where the dot denotes differentiation with respect to ϵ.

On n first order ODEs: $\dot{x}^a = V^a(x)$.

There exists a unique solution given initial data $X^a(0)$. Given a solution $X^a(\epsilon)$, we can expand it in a Taylor series as

$$X^a(\epsilon) = X^a(0) + V^a \cdot \epsilon + O(\epsilon^2). \tag{2.6}$$

Up to first order in ϵ, the vector field determines the flow. We call V a *generator* of its flow.

The following example illustrates how you get from a vector field to its flow.

Example ($M = \mathbb{R}^2$, $x^a = (x, y)$)**:** Consider the vector field $V = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. The system of ODEs we solve is $\dot{x} = x$ and $\dot{y} = 1$. This gives us the integral curve $(x(\epsilon), y(\epsilon)) = (x(0)e^\epsilon, y(0) + \epsilon)$. From this we can see that $x(\epsilon) \cdot \exp(-y(\epsilon))$ is constant along γ. Using this we can draw the unparametrised integral curve in Fig. 2.2.

This example motivates the following definition.

Definition 9 (invariant)**:** An *invariant* of a vector field V is a function f constant along the flow of V.

$$f(x^a(0)) = f(x^a(\epsilon)) \qquad \forall \epsilon. \tag{2.7}$$

Equivalently, $V(f) = 0$.

Let us now consider an example that goes the other way: from flow to vector field.

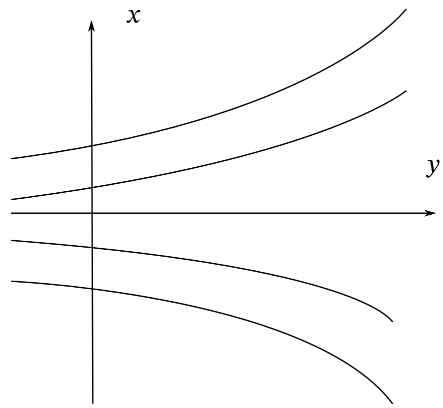

Figure 2.2

Example: Consider the 1 -parameter group of rotations of a plane.

$$(x(\epsilon), y(\epsilon)) = (x_0 \cos\epsilon - y_0 \sin\epsilon, x_0 \sin\epsilon + y_0 \cos\epsilon). \tag{2.8}$$

The associated vector field is

$$V = \left(\frac{\partial y(\epsilon)}{\partial \epsilon}\frac{\partial}{\partial y} + \frac{\partial x(\epsilon)}{\partial \epsilon}\frac{\partial}{\partial x}\right)\Bigg|_{\epsilon=0} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial y}. \tag{2.9}$$

Now you can add vector fields, but there is also another operation.

Definition 10 (Lie bracket)**:** A *Lie bracket* $[V, W]$ of two vector fields V, W is a vector field defined by

$$[V, W](f) = V(W(f)) - W(V(f)) \qquad \forall f. \tag{2.10}$$

> This is indeed another vector field since the commutator of two first order operators is another first order operator.

Example: Let $V = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $W = \frac{\partial}{\partial x}$. We then have $[V, W] = -W$.

> This is not always the case but sometimes the Lie bracket reproduces some of the vector fields. There is an interesting algebraic structure to this.

Definition 11 (Lie algebra)**:** A *Lie algebra* is a vector space $\mathfrak{g}$ with an anti-symmetric, bilinear operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called a *Lie bracket*, which satisfies the *Jacobi identity*

$$[V, [U, W]] + [W, [V, U]] + [U, [W, V]] = 0 \qquad \forall U, V, W \in \mathfrak{g}. \tag{2.11}$$

We will spend some time discussing this abstractly, but then focus on the Lie algebras of vector fields in the main part of this course.

Any two vector spaces of a given dimension are isomorphic; there is nothing special other than the dimension distinguishing vector spaces. For Lie algebras this is not so.

Example: Even in dimension 2, which is the lowest non-trivial dimension, there are two Lie algebras (up to isomorphism)

$$a)\quad [V,W] = -W, \qquad b)\quad [V,W] = 0. \tag{2.12}$$

If the vector space underlying $\mathfrak{g}$ is finite-dimensional, and V_α, $\alpha = 1, \dots, \dim \mathfrak{g}$ is a basis of $\mathfrak{g}$, we can define the Lie algebra by specifying the brackets

$$[V_\alpha, V_\beta] = \sum_\gamma f_{\alpha\beta}{}^{\gamma} V_\gamma, \tag{2.13}$$

where $f_{\alpha\beta}{}^{\gamma}$ are the *structure constants.*

Example ($\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$)**:** The vector space is given by $n \times n$ real matrices, and the Lie bracket is the matrix commutator. The dimension of this Lie algebra is $\dim \mathfrak{g} = n^2$.

Example (Vector fields)**:** The set of all vector fields on a manifold M form an infinite-dimensional Lie algebra.

Example: Consider diff($\mathbb{R}$) or diff(S^1), vector fields on a line or on a circle respectively.

$$\mathrm{diff}(\mathbb{R}), \quad x \in \mathbb{R}, \quad V_\alpha = -x^{\alpha+1}\frac{\partial}{\partial x} \tag{2.14}$$

$$\mathrm{diff}(S^1), \quad \theta \in S^1, \quad V_\alpha = ie^{i\alpha\theta}\frac{\partial}{\partial\theta} \tag{2.15}$$

$$[V_\alpha, V_\beta] = (\alpha - \beta)V_{\alpha+\beta}. \tag{2.16}$$

Example (Virasoro algebra)**:** The *Virasoro algebra* Vir $=$ diff(S^1) $\oplus\ \mathbb{R}$ is the central extension[1] of diff(S^1), with *central charge* $c = \mathbb{R}$.

$$\begin{cases} [V_\alpha, c] = 0 \\ [V_\alpha, V_\beta]_{\mathrm{vir}} = (\alpha-\beta)V_{\alpha+\beta} + \dfrac{c}{12}(\alpha^3 - \alpha)\delta_{\alpha+\beta,0} \end{cases} \tag{2.17}$$

Remark:

$$[f(\theta)\frac{\partial}{\partial\theta}, g(\theta)\frac{\partial}{\partial\theta}] = \underbrace{(fg' - gf')}_{\text{Wronskian}}\frac{\partial}{\partial\theta} \tag{2.18}$$

'After Witten'.

$$[f\frac{\partial}{\partial\theta}, g\frac{\partial}{\partial\theta}]_{\mathrm{vir}} = [f\frac{\partial}{\partial\theta}, g\frac{\partial}{\partial\theta}] + \frac{ic}{48\pi}\int_0^{2\pi}(f'''g - g'''f)\,\mathrm{d}\theta \tag{2.19}$$

Theorem 2 (Ado)**:** Every finite-dimensional Lie algebra is isomorphic to some matrix Lie algebra, a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.

[1] We will meet the concept of central extension and central charge in this term's *String Theory* course.

> **Remark:** n is not necessarily the dimension of the Lie algebra.

3 Lie Groups

Definition 12 (Lie group)**:** A *Lie group* is a smooth manifold G, which is also a group, such that the group operations

$$\text{multiplication} \qquad G \times G \to G, \qquad (g_1, g_2) \to g_1 \cdot g_2 \tag{3.1}$$

$$\text{inverse} \qquad G \to G \qquad g \to g^{-1} \tag{3.2}$$

are smooth maps between manifolds.

Example ($G = GL(n, \mathbb{R}) \in \mathbb{R}^{n^2}$)**:** The general linear group $GL(n, \mathbb{R})$ is defined as the set of invertible matrices $\{g \in G \mid \det g \neq 0\}$. The dimension is $\dim(G) = n^2$.

Example ($G = O(n, \mathbb{R})$)**:** This is the group of orthogonal matrices, defined by $\frac{1}{2}n(n+1)$ conditions $g^T g = \mathbb{1}$. The dimension is then $\dim O(n, \mathbb{R}) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$. We also have to check that these conditions define a manifold in the sense that the associated Jacobian has maximal rank.

Definition 13 (group action)**:** A *group action* on a manifold M is a map $G \times M \to M$ mapping $(g, p) \to g(p)$ such that

$$e(p) = p, \qquad g_1(g_2(p)) = (g_1 \cdot g_2)(p) \tag{3.3}$$

for all $p \in M$ and al g_1, g_2 on G.

Definition 14 (transformation group)**:** If we have a group action, we refer to G as a group of *transformations*.

Example: Take $M = \mathbb{R}^2$ and $G = E(2)$, the three-dimensional Euclidean group.

$$g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \tag{3.4}$$

Take $G_E \in G$ to be a one-parameter subgroup of G. There are three such subgroups

$$G_\theta : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos\theta \cdot x - \sin\theta \cdot y \\ \sin\theta \cdot x + \cos\theta \cdot y \end{pmatrix} \tag{3.5}$$

$$G_a : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x + a \\ y \end{pmatrix} \tag{3.6}$$

$$G_b : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x \\ y + b \end{pmatrix}. \tag{3.7}$$

Each of these one-parameter subgroups generates a flow. We can think of this flow as being gener-

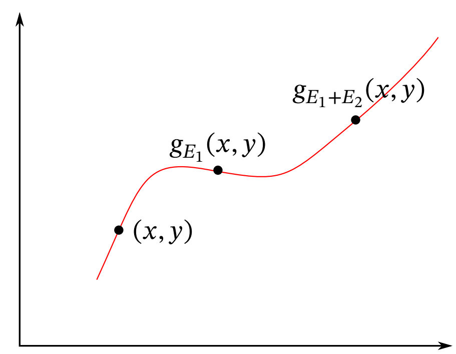

Figure 3.1

ated by a vector field $V|_p = \frac{\mathrm{d}}{\mathrm{d}E} g_E(p)\Big|_{E=0}$.

$$V_\theta = d\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{3.8}$$

$$V_a = \left(\frac{\mathrm{d}\tilde{x}}{\mathrm{d}a}\frac{\partial}{\partial \tilde{x}} + \frac{\mathrm{d}\tilde{y}}{\mathrm{d}a}\frac{\partial}{\partial \tilde{y}}\right)\Big|_{a=0} = \frac{\partial}{\partial x} \tag{3.9}$$

$$V_b = \frac{\partial}{\partial y}. \tag{3.10}$$

We define a 3-dimensional Lie algebra of $E(2)$ as

$$[V_a, V_\theta] = V_b \qquad [V_b, V_\theta] = -V_a \qquad [V_a, V_b] = 0, \tag{3.11}$$

represented by vector fields on M.

3.1 Geometry on Lie Groups

Definition 15 (tangent map)**:** Let $f: M \to \widetilde{M}$ be a smooth map between manifolds. We define the *tangent map* or *push forward* to be

$$\begin{aligned} f_*: T_p(M) &\to T_{f(p)}(\widetilde{M}) \\ V &\longmapsto f_*(V) = \frac{\mathrm{d}}{\mathrm{d}E} f(\gamma(E))\Big|_{E=0}. \end{aligned} \tag{3.12}$$

This extends to the tangent bundle $T(M)$. If x^α are coordinates of $\mathcal{M} \supset M$, $(y^{\alpha'})$ coordinates on $\widetilde{\mathcal{M}} \subset M$, then

$$V = V^\alpha \frac{\partial}{\partial x^a} \qquad f_*(V) = V^\alpha \frac{\partial f^i}{\partial x^a} \frac{\partial}{\partial y^i}. \tag{3.13}$$

Definition 16 (Lie derivative)**:** Let V, W be vector fields. Let V generate a flow $V = \dot{\gamma}$. The *Lie derivative* is

$$\mathcal{L}_V W|_p = \lim_{\epsilon \to 0} \frac{W(p) - \gamma(\epsilon)_* W(p)}{\epsilon} \tag{3.14}$$

$$\tag{3.15}$$

We can extend this definition over the whole manifold.

Exercise 3.1: Show that $L_V W = [V, W]$.

Definition 17: On functions $f : M \to \mathbb{R}$, we define the Lie derivative as $\mathcal{L}_V(f) = V(f)$.

On differential forms, we can use the Leibniz rule to show that

$$\mathcal{L}_V \Omega = d(\iota_V \Omega) + \iota_V(d\Omega). \tag{3.16}$$

Definition 18: We define the cotangent space $T_p^* M = \text{Span}\{dx^1, \ldots, dx^n\}$ as the space of one-forms. The cotangent bundle is then

$$\bigcup_{p \in M} T_p^* M = T^* M. \tag{3.17}$$

Using the wedge product, which is anti-commutative on one-forms $dx^i \wedge dx^j = -dx^j \wedge dx^i$, we can define an r -form

$$\Omega = \frac{1}{r!} \Omega_{ij\ldots k} dx^i \wedge dx^j \wedge \cdots \wedge dx^k. \tag{3.18}$$

Definition 19 (contraction)**:** We write a contraction as

$$\frac{\partial}{\partial x^i}(HOOK) dx^j = \iota_{\frac{\partial}{\partial x^i}} dx^j = \delta_i^{\ j}. \tag{3.19}$$

For a general vector field V and one-form Ω, we have

$$\iota_V \Omega = V^i \frac{\partial}{\partial x^i}(HOOK) \Omega_j dx^j = V^j \Omega_j \delta^j_{\ i} = V^i \Omega_i. \tag{3.20}$$

No metric is needed.

Definition 20: A *Lie algebra* $\mathfrak{g}$ of a Lie group G is the tangent space $T_e G$ to G at the identity. The Lie bracket on $\mathfrak{g}$ is the commutator of vector fields on G.

Definition 21 (Left translations)**:** For all $\mathfrak{g} \in G$, we define the *left translations*

$$L_g : G \to G \tag{3.21}$$

$$g \longmapsto g \cdot h. \tag{3.22}$$

Definition 22 (left invariant vector fields)**:** Using the left translation maps, we define their push forward $(L_g)_* : \mathfrak{g} \equiv T_e G \to T_g G$, which maps $v \in g$ to vector fields $(L_g)_*(V)$ on G. This defines *left invariant vector fields*.

$$[(L_g)_* V, (L_g)_* W] = (L_g)_* [V, W]_{\mathfrak{g}}. \tag{3.23}$$

Remark: It is important to understand the notation!

These form a basis of $\mathfrak{g}$, meaning that $\dim(G)$ is the number of global, non-vanishing vector fields on G.

Claim 2: Lie groups are parallelisable manifolds.

Claim 3: The converse is not true.

Proof. S^1, S^3, S^7 are the only parallelisable spheres.

The first two are indeed manifolds: $S^1 = U(1)$, $S^3 = SU(2) = \left\{ \begin{pmatrix} a & b \\ -b^* & a \end{pmatrix} \middle| \, |a|^2 + |b|^2 = 1 \right\}$, but S^7 is not a Lie group. □

▌ This has introduced the field of k-theory.

Claim 4: Let L_α, $\alpha = 1, \dots, \dim \mathfrak{g}$ be a basis of left invariant vector fields with structure constants

$$[L_\alpha, L_\beta] = \sum_\gamma f_{\alpha\beta}{}^\gamma L_\gamma. \tag{3.24}$$

Let σ^α be a dual basis of left-invariant one-forms

$$L_\alpha (hook) \sigma^\beta = \delta_\alpha{}^\beta. \tag{3.25}$$

Then

$$d\sigma^\alpha + \frac{1}{2} f^\alpha{}_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma = 0. \tag{3.26}$$

Proof (Sheet 1). Use the identity

$$d\Omega(V, W) = V(\Omega(W)) - W(\Omega(V)) - \Omega([V, W]). \tag{3.27}$$

□

▌ Watch out for signs and factors in the upcoming derivations! Things can easily go wrong.

Definition 23 (Maurer–Cartan)**:** Assume that G is a matrix Lie group. The *Maurer–Cartan 1-form* on G is

$$\rho := g^{-1} dg. \tag{3.28}$$

Claim 5: This one-form

- is left invariant.
- takes values in the Lie algebra.

Proof.
- $(g_0 g)^{-1} d(g_0 g) = g^{-1} dg$, g_0 =??
- Take C a smooth curve $g(s) \subset G$.

$$g^{-1}(s)g(s+\epsilon) = \underbrace{\epsilon}_{\mathbb{1}} + \epsilon \underbrace{g^{-1}\frac{\mathrm{d}g}{\mathrm{d}\epsilon}}_{\in T_e G \simeq \mathfrak{g}}|_{\epsilon=0} + O(\epsilon^2). \tag{3.29}$$

So $g^{-1}dg = \sum_\alpha \sigma^\alpha \otimes T_\alpha$, where T_α are matrices with $[T_\alpha, T_\beta] = \sum_\gamma f_{\alpha\beta}{}^\gamma T_\gamma$.

□

Claim 6: It obeys the Maurer–Cartan equation:

$$d\rho + \rho \wedge \rho = 0. \tag{3.30}$$

Proof. Consider first the exterior derivative term

$$d\rho = \sum_\alpha d\sigma^\alpha \cdot T_\alpha = -\frac{1}{2} f^\alpha{}_{\beta\gamma} \sigma^\beta \wedge \sigma^\gamma \cdot T_\alpha. \tag{3.31}$$

The wedge product term is

$$\rho \wedge \rho = \sigma^\alpha T_\alpha \wedge \sigma^\beta T_\beta = \frac{1}{2}\sigma^\alpha \wedge \sigma^\beta [T_\alpha, T_\beta] = \frac{1}{2}\sigma^\alpha \wedge \sigma^\beta f_{\alpha\beta}{}^\gamma T_\gamma. \tag{3.32}$$

□

Example (Heisenberg group)**:** The *Heisenberg group* (sometimes just called *Nil*) is the group of upper triangular matrices

$$g = \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} = \mathbb{1} + xT_1 + yT_2 + zT_3, \tag{3.33}$$

with

$$T_1 = \begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix} \qquad T_2 = \begin{pmatrix} & & \\ & & 1 \\ & & \end{pmatrix} \qquad T_3 = \begin{pmatrix} & & 3 \\ & & \\ & & \end{pmatrix} \tag{3.34}$$

The commutation relations are

$$[T_1, T_2] = T_3 \quad [T_1, T_3] = 0 = [T_2, T_3]. \tag{3.35}$$

We can interpret T_1 as position, T_2 as momentum and T_3 as the identity $i\hbar$.