# String Theory

## Part III Lent 2020

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# Contents

# Admin Stuff

## Books and Lecture Notes

- String Theory Vol. 1 — Polchinski, CUP

  Probably fits this course most closely, although there will be some things that will be in this course that are not in the book and vice-versa.

- Superstrings Vol. 1 — Green et. al, CUP

- A String Theory Primer — Schomerus, CUP

## Online Resources

- Lecture notes by the professor are online.

- Probelem sheet 1 by the professor is online.

- David Tong's notes on arXiv:0908.0333.

- 'Why string theory?' Conlon (CRC) for some light reading and history

# 1 Introduction

## 1.1 What is String Theory?

We do not actually know. This question still needs fleshing out to be answered well. Most likely, the actual string theory that we are working towards will be very different to the content that we will cover in these lectures.

In some sense, string theory is an attempt at quantising the gravitational field.

Naive quantisation of the Einstein-Hilbert action presents a number of problems.

### Conceptual Problems

- The nature of time: Non-relativistic quantum mechanics is based on the Hamiltonian formulation. This is not necessarily a technical problem.
- How to quantise without a pre-existing causal structure?

  One of the first things we learn in QM, when going to QFT, is that it is important to know whether two operators are timelike or spacelike separated. We have a notion that all operators that are spacelike separated commute. However, if we are talking about general relativity, the metric contains information about and determines the causal structure. But this is what we want to quantise. So it is not immediately obvious what the algebra of operators should look like.
- The symmetry of general relativity is diffeomorphism invariance (coordinate reparametrisations).

  This is a gauge symmetry. One thing we will discuss (although not prove) when talking about scattering amplitudes in string theory, the fact is that there are no *local* diffeomorphism-invariant observables. As such, it is not even obvious what the quantum observables should be.

We will not be able to answer these deeper conceptual questions within the framework of string theory.

But perhaps more importantly, there are technical obstacles. You might ask, are there any assumptions that allow us to make progress and return to the difficult conceptual problems later?

## Technical obstacles

Let us look at perturbation theory.

In particle physics, we often expand $g_{\mu\nu}$ about some classical solution, e.g. Minkowski spacetime:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x). \tag{1.1}$$

This means that we can use the causal structure of the *background* classical metric $\eta_{\mu\nu}$ to talk about quantisation of the fluctuations $h_{\mu\nu}$. This is what we do in *Field Theory in Cosmology* for inflation. In spirit, this is very close to what we do in string theory.

However, in some sense this split between background and perturbation is artificial and arbitrary. Nonetheless, we can take the Einstein–Hilbert action in general dimensions $D$

$$S[g] = \frac{1}{k_D} \int \mathrm{d}^D x \sqrt{-g} R(g) \tag{1.2}$$

and expand it out by choosing a gauge. Choosing a gauge wisely, we get an action

$$S[h] \approx \frac{1}{k_D} \int \mathrm{d}^D x \left( h_{\mu\nu} \Box h^{\mu\nu} + \dots \right). \tag{1.3}$$

Since the Ricci scalar involves inverse powers of the metric, this expansion will not terminate. We say that this expression is *non-polynomial* in $h_{\mu\nu}$.

(a) Propagator (b) Vertices

Figure 1.1: Feynman rules

The quadratic term gives us a propagator of Fig. 1.1a and the interaction terms give us vertices illustrated in Fig. 1.1b These are the Feynman rules.

However, loops give divergences, which cannot be dealt with using standard techniques (renormalisation, c.f. *Advanced Quantum Field Theory*).

String theory provides a way to do quantum perturbation theory of the gravitational ‘field’ (and much more).

String theory gives us a framework to ask meaningful questions concerning quantum gravity (although we are not able to answer all of them at the moment).

There are also other approaches out there, although string theory is most thoroughly studied / understood (although this might be because most resources in this area flow into string theory).

One of the possibilities is trying to make classical gravity consistent with the standard model without needing to quantise it.

There will be a sense in which we will learn quite a lot about QFT from studying string theory.

To a certain extend, the motivation that we will take throughout this course is not whether string theory will give us a theory of the real world, but rather whether it might give us a hint on how to reconcile quantum mechanics and gravity. We have been trying to do this for the better part of a hundred years, so any hint would be appreciated.

## 1.2 Worldsheets and Embeddings

Popular science books and textbooks alike usually start off a discussion of string theory by starting with the assumption that particles are not point-like, but rather tiny little strings.

The starting point is to consider a worldsheet $\Sigma$ , a two-dimensional surface swept out by a string. This is analogous to a worldline swept out by a pointlike particle moving through spacetime, as illustrated in Fig. 1.2.

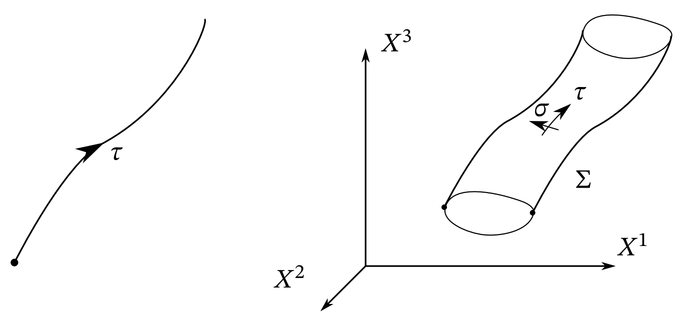


Figure 1.2: Particle worldline and string worldsheet

We put coordinates $(\sigma, \tau)$ on $\Sigma$ (at least locally) and we define an embedding of $\Sigma$ in the background spacetime $M$ by the functions $X^\mu(\sigma, \tau)$, where the $X^\mu$ are coordinates on $M$, i.e. $X : \Sigma \to M$.

There are rules (which we shall investigate) for glueing such worldsheets together, in a way that is consistent with the symmetries of the theory.

We shall see that diagrams such as Fig. 1.3 are in one-to-one correspondence with correlation functions in some quantum theory. It is natural to interpret such diagrams as Feynman diagrams in

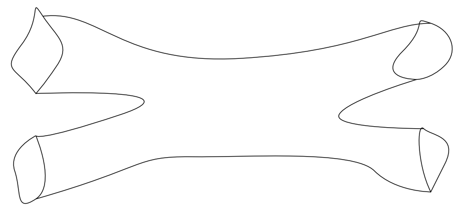

Figure 1.3: Not a particularly beautiful diagram…

a perturbative expansion of some theory about a vacuum.

# 2 The Classical Particle and String

In this section we will start systematically constructing a theory of quantum strings. As a precursor to this discussion, we will need to talk a bit about non-relativistic quantum mechanics.

In non-relativistic QM, we treat time ($t$) as a parameter and position ($\hat{x}$) as an operator. Obviously, this kind of distinction should not survive in a relativistic theory. This means there are some choices to be made.

**Second quantisation:** Both $x^i$ and $t$ are considered parameters. They are not the fundamental objects we quantise. Instead, we quantise quantum fields $\phi(\mathbf{x}, t)$, which are the basic objects of our theory. We require that the fields transform in appropriate ways under Lorentz transformations. This is historically overwhelmingly the most useful way to do it, seeing its success in quantum field theory and the standard model.

**First quantisation:** Here we make the other choice: we elevate $t$ to being an operator. This is the natural framework for describing the relativistic embedding of worldlines (-sheets, -volumes) into spacetime. Here, $X^\mu = (x^i, t)$ is an operator, the fundamental object we quantise, and there will be some other natural parameter entering the theory.

There is such a thing as string field theory, which employs the viewpoint of second quantisation. However, apart from a few exceptions, there is not much that you do not also obtain from first quantisation string theory. However, first quantisation string theory has made significant advances on problems that have not yet be solved with second quantisation.

### 2.0.1 Worldlines and Particles

We consider the embedding of a worldline $\mathcal{L}$ into spacetime $M$. The basic field is the embedding $X^\mu : \mathcal{L} \to M$. An action that describes this embedding might be

$$S[X] = -m \int_{x_2}^{x_1} \mathrm{d}s = -m \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \sqrt{-\eta^{\mu\nu} \dot{X}^\mu \dot{X}^\nu}, \tag{2.1}$$

where $\tau$ (a parameter) is the proper time and $X^\mu(\tau_2) = x_2^\mu, X^\mu(\tau_1) = x_1^\mu$ are end points of the world line.

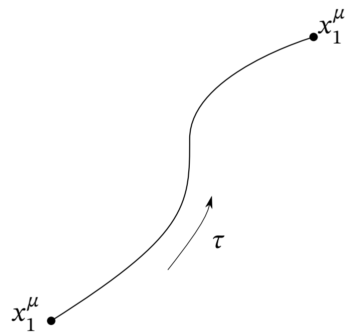


Figure 2.1

The momentum conjugate to $X^\mu(\tau)$ is $P_\mu(\tau) = -m\frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}}$. This satisfies $\dot{X}^2 = \eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu$. This satisfies $P^2 + m^2 = 0$ identically, an *on-shell condition*.

## Symmetries

**Rigid symmetry:** $X^\mu(\tau) \to \Lambda^\mu{}_\nu X^\nu(\tau) - a^\mu$, where $\Lambda^\mu{}_\nu$ is a Lorentz transformation and $a^\mu$ is a (constant) displacement.

**Reparameterisation invariance:** $\tau \to \tau + \xi(\tau)$. The embedding $X^\mu$ changes as

$$X^\mu(\tau) \to X^\mu(\tau + \xi) = X^\mu(\tau) + \xi\dot{X}^\mu(\tau) + \dots . \tag{2.2}$$

To first order, $\delta X^\mu(\tau) = \xi\dot{X}^\mu(\tau)$.

There is a rewriting of this action (2.1) that makes life a bit easier. We do this by introducing a new auxiliary field $e(\tau)$, which is a one-form, on the worldline $\mathcal{L}$.

$$S[X, e] = \frac{1}{2}\int_{\mathcal{L}} \mathrm{d}\tau \left(e^{-1}\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu - em^2\right). \tag{2.3}$$

> If you like, you can think of $e$ as something like a one-dimensional metric, which sets a scale for distances on the line.

> We will be able to do something analogous for strings, which will be very useful!

The equations of motion for $X^\mu(\tau)$ and $e(\tau)$ are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(e^{-1}\dot{X}^\mu\right) = 0. \tag{2.4}$$

However, the equation of motion for $e$ will not depend on $\dot{e}$, but be purely algebraic:

$$\dot{X}_2 + e^2 m^2 = 0. \tag{2.5}$$

As such, $e$ can be thought of as a Lagrange multiplier, enforcing a constraint. What is the constraint that it enforces?

The momentum conjugate to $X^\mu$ is

$$P_\mu = e^{-1}\dot{X}_\mu. \tag{2.6}$$

Combining (2.5) and (2.6), we can eliminate $e$ to find precisely the on-shell condition $P^2 + m^2 = 0$.

> The auxiliary field $e$ basically enforces energy-momentum conservation on the worldline.

**Exercise 2.1:** We can write $e^{-1} = m/\left|\dot{X}\right|$. Plug this into the action (2.3) to find that $S[X, e]$, subject to the equation of motion for $e(\tau)$ gives precisely the action (2.1).

There are two reasons why we should consider this new action instead. Firstly, in the $m \to 0$ limit, it will be easy to show that it describes null worldlines. Secondly, since there is no square root in this theory, it will be easier to quantise. This is at the cost of having to introduce a new non-dynamical field $e$.

The action $S[X, e]$ has the following symmetries:

- Poincaré invariance (where $e$ is invariant)
- Reparameterisation invariance:

$$\delta X^\mu = \xi \dot{X}^\mu \qquad \delta e = \frac{\mathrm{d}}{\mathrm{d}\tau}(\xi e) \tag{2.7}$$

  provided these variations vanish on the endpoints.

> Note that this transformation law under diffeomorphisms shows that $e$ needs to transform like a one-form.

> Locally, we can think of $e$ as a pure gauge.

**Remark:** We could generalise $\eta_{\mu\nu} \to g_{\mu\nu}\big(X(\tau)\big)$. The model becomes highly non-linear. We will look at this later in the context of strings.

## 2.1 Classical Strings: The Nambu–Goto Action

Our starting point will be the *Nambu-Goto Action*. We use units where $\hbar = c = 1$ throughout. The

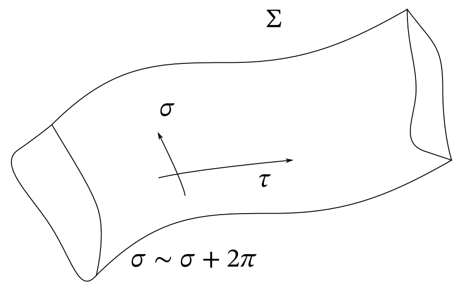


Figure 2.2

fundamental degree of freedom is

$$X : \sigma \to M, \tag{2.8}$$

where $M$ is called the *target space*.

**Definition 1** (Nambu–Goto Action)**:** The Nambu–Goto action is

$$S[X] = -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}\tau \, \mathrm{d}\sigma \sqrt{-\det\big(\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\big)}, \tag{2.9}$$

where $\alpha'$ is a constant with dimensions of (spacetime) area and $\sigma^a = (\tau, \sigma)$ and $\partial_a = \frac{\partial}{\partial\sigma^a}$ .

**Remark:** One often speaks of the *string length* $l_2 = 2\pi\sqrt{\alpha'}$.

**Definition 2** (string tension)**:** We introduce the *string tension* $T = \frac{1}{2\pi\alpha'}$.

> This turns out to be a good starting point; quantising this puts us on the right track. However, it is horrendously difficult to actually perform this quantisation. Instead, we will consider another action.

**Definition 3** (Polyakov Action)**:** The Polyakov action is

$$S[X, h] = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h} h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu. \tag{2.10}$$

The new feature as compared to the Nambu–Goto action is the new field $h_{ab}$, which is a metric on $\Sigma$. It is non-dynamical; there are no Einstein-Hilbert terms and similarly to $e$ previously it can be thought of as enforcing a constraint. We will find that $h_{ab}$ is extremely important.

> This form is quite suggestive. All this is, when quantised, is a two-dimensional massless Klein–Gordon field in three dimensions. It is difficult to find an easier theory than this.

Let us look at the equations of motion for the Polyakov action.

**Exercise 2.2:** If we vary the worldsheet metric $h_{ab} \to h_{ab} + \delta h_{ab}$, then the action changes as

$$\delta S = -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h} T_{ab} \delta h^{ab}, \tag{2.11}$$

with *stress tensor* $T_{ab} = \frac{1}{\alpha'} \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab} h^{cd} \partial_c X^\mu \partial_d X_\mu$.

> **Remark:** Indices are raised and lowered with the Minkowski metric.

So the $h_{ab}$ equation of motion is

$$\boxed{T_{ab} = 0} \tag{2.12}$$

**Remark:**
- The trace vanishes $h^{ab}T^{ab} = 0$, because $h_{ab}h^{ab} = 2$.
- Symmetric $T_{ab} = T_{ba}$ because $h_{ab} = h_{ba}$.
- The $X^\mu$ equation of motion is

$$\frac{1}{\sqrt{-h}} \partial_a \left( \sqrt{-h} h^{ab} \partial_b X^\mu \right) = 0. \tag{2.13}$$

  If the $h_{ab} = \mathrm{diag}(-1, 1)$, then this is the wave equation.

## 2.2 Classical Equivalence of Polyakov and Nambu–Goto

It is useful to define $G_{ab} = \eta_\mu \partial_a X^\mu \partial_b X^\nu$ . The Nambu–Goto action is then

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-\det(G_{ab})}. \tag{2.14}$$

By imposing the metric's equation of motion, the vanishing of $T_{ab}$ tells us that

$$G_{ab} - \frac{1}{2} h_{ab} \underbrace{G_{cd} h^{cd}}_{G := \mathrm{tr}(G_{ab})} = 0. \tag{2.15}$$

The determinant is

$$\det(G_{ab}) = \frac{1}{4} G^2 \det(h_{ab}) = \frac{1}{4} G^2 h. \tag{2.16}$$

So $\sqrt{-h}G = 2\sqrt{-\det G_{ab}}$ and

$$\frac{1}{2}\sqrt{-h} h^{cd} \partial_c X^\mu \partial_d X^\nu \eta_{\mu\nu} = \sqrt{-\det(G_{ab})}. \tag{2.17}$$

**Remark:** This is because $h_{ab}$ is not appearing dynamically in the action (no derivatives), so its equation of motion is just a constraint.

## 2.3 The Polyakov Action

$$S[X,h] = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}h^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu.$$

Can we generalise this action?

- We could replace $\eta_{\mu\nu}$ with a general $g_{\mu\nu}(x)$ (more later).
- What about a 2-$D$ Einstein–Hilbert term, to make $h_{ab}$ dynamical?

$$\frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}\,R(h) = \chi. \tag{2.18}$$

  In 2-$D$, this is a topological invariant: the Euler characteristic $\chi$ of the worldsheet.

- What about a cosmological constant on $\Sigma$?

$$\Lambda\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h} \tag{2.19}$$

  The equations of motion for $h_{ab}$ will be of the form

$$T_{ab} \propto \Lambda h_{ab}. \tag{2.20}$$

  However, we know that $T_{ab}$ is traceless. We therefore have $h^{ab}T_{ab} \propto 2\Lambda$, so we need $\Lambda = 0$.

- We could include background fields in the spacetime.

  For example, there could be a 2-form field $B(X) = \frac{1}{2}B_{\mu\nu}dX^\mu \wedge dX^\nu$. We could include the term

$$-\frac{1}{2\pi\alpha'}\int_\Sigma B = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}\,\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}. \tag{2.21}$$

  Or we might have a scalar field $\phi(X)$. The sort of term we might have is something like

$$\frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}\,\phi(X)R(h). \tag{2.22}$$

### 2.3.1 Symmetries

- Poincaré invariance: $X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu$, where $\Lambda^\mu{}_\nu$ and $a^\mu$ are independent of $(\sigma,\tau)$. (These are rigid / global symmetries, as opposed to gauge / local symmetries). The metric $h^{ab} \to h^{ab}$ does not change.

- Diffeomorphism invariance.

  Infinitesimally: $\sigma^a \to \sigma^a + \xi^a$. Then

$$\delta X^\mu = \xi^a \partial_a X^\mu \tag{2.23}$$
$$\delta h_{ab} = \xi^c \partial_c h_{ab} + (\partial_a \xi^c) h_{bc} + (\partial_b \xi^c) h_{ac}. \tag{2.24}$$

- Weyl invariance: $X^\mu \to X^\mu$ does not change. But the worldsheet is rescaled by some position dependent factor $h_{ab} \to e^{2\Lambda(\sigma,\tau)} h_{ab}$.

  Infinitesimally:

$$\delta X^\mu = 0, \qquad \delta h_{ab} = 2\Lambda h_{ab}. \tag{2.25}$$

### 2.3.2 Classical Solutions

The two-dimensional metric $h_{ab}$ has three degrees of freedom. We can use the diffeomorphism invariance to fix two of the degrees of freedom in $h_{ab}$ (at least locally) and write it as

$$h_{ab} = e^{2\Phi} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.26}$$

Moreover, Weyl invariance means that the factor $e^{2\phi}$ drops out, eliminating the final degree of freedom.

The action then becomes

$$S[X] = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \left( -\dot{X}^2 + (X')^2 \right), \qquad \dot{X}^\mu := \frac{\partial X^\mu}{\partial \tau}, \quad (X')^\mu := \frac{\partial X^\mu}{\partial \sigma}. \tag{2.27}$$

The equation of motion is

$$\Box X^\mu = 0 \qquad \Box = -\partial_\tau^2 + \partial_\sigma^2. \tag{2.28}$$

Solutions are split between left- and right-movers.

$$X^\mu(\sigma,\tau) = X_R^\mu(\tau+\sigma) + X_L^\mu(\tau+\sigma). \tag{2.29}$$

Introduce the Fourier modes $\alpha_n^\mu$ and $\overline{\alpha}_n^\mu$ (which are not complex conjugate to each other, but it will be useful to think of them as if they were). The solutions then become

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2} p^\mu (\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in(\tau-\sigma)} \tag{2.30}$$
$$X_R^\mu(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{\alpha'}{2} p^\mu (\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\overline{\alpha}_n{}^\mu}{n} e^{-in(\tau+\sigma)}. \tag{2.31}$$

The $x^\mu$ and $p^\mu$ are the centre of mass position and momentum of the string in spacetime.

**Notation:** We also introduce the notation

$$\alpha_0^\mu = \overline{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\,p^\mu \tag{2.32}$$

$X^\mu$ is real, so we also require that $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$.

## 2.4 Classical Hamiltonian Dynamics of the String

We continue to work in conformal gauge, where $h_{ab} = e^{\phi}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$.

**Definition 4** (canonical momentum)**:** We define the *canonical momentum field*, conjugate to $X^\mu$ as the functional derivative

$$P_\mu(\sigma,\tau) = \frac{\delta S(x)}{\delta X^\mu(\sigma,\tau)} = \frac{1}{2\pi\alpha'}\dot{X}_\mu. \tag{2.33}$$

**Definition 5:** Given the Lagrangian density $\mathscr{L}$ , the Hamiltonian density is

$$\mathscr{H} = P_\mu \dot{X}^\mu - \mathscr{L} = \frac{1}{4\pi\alpha'}(\dot{X}^2 + X'^2). \tag{2.34}$$

**Definition 6** (Poisson brackets)**:** We introduce Poisson brackets $\{\cdot,\cdot\}_{\text{PB}}$. For a particle theory, where our coordinates $x^\mu(\tau)$ and momenta $p_\mu(\tau)$ are our fundamental variables, it is useful to define

$$\{f,g\}_{\text{PB}} = \frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial x^\mu}. \tag{2.35}$$

**Example 2.4.1:** We have for example, $\{x^\mu, p_\nu\} = \delta^\mu_\nu$.

The Hamiltonian $\mathscr{H}$ is the generator of time translations

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \frac{\partial f}{\partial \tau} + \{f,\mathscr{H}\}. \tag{2.36}$$

Our field theoretic generalisation requires

$$\{X^\mu(\sigma,\tau), P_\mu(\sigma',\tau)\} = \delta^\mu_\nu \delta(\sigma-\sigma'). \tag{2.37}$$

Recall that the $X^\mu(\sigma,\tau)$ can be written in terms of Fourier modes $\alpha^\mu_n$ and $\overline{\alpha}^\mu_n$, where $\sigma \sim \sigma + 2\pi$ is periodic, which is the reason why $n$ takes on discrete values.

**Claim 1:** The Poisson bracket relationship between $X^\mu$ and $P_\mu$ requires

$$\boxed{\begin{aligned} \{\alpha^\mu_m, \alpha^\nu_n\}_{\text{PB}} &= -im\eta^{\mu\nu}\delta_{m+n,0} && (2.38\text{a})\\ \{\overline{\alpha}^\mu_m, \overline{\alpha}^\nu_n\}_{\text{PB}} &= -im\eta^{\mu\nu}\delta_{m+n,0} && (2.38\text{b})\\ \{\alpha^\mu_m, \overline{\alpha}^\nu_n\}_{\text{PB}} &= 0 && (2.38\text{c}) \end{aligned}}$$

*Proof.* Without loss of generality, let us check this at $\tau = 0$:

$$X^\mu(\sigma) = x^\mu + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha^\mu_n e^{+in\sigma} + \overline{\alpha}^\mu_n e^{-in\sigma}\right), \tag{2.39}$$

$$P^\mu(\sigma) = \frac{p^\mu}{2\pi} + \frac{1}{2\pi}\frac{1}{\sqrt{2\alpha'}}\sum_{n\neq 0}\left(\alpha^\mu_n e^{+in\sigma} + \overline{\alpha}^\mu_n e^{-in\sigma}\right). \tag{2.40}$$

$$\tag{2.41}$$

The Poisson bracket is

$$\left\{X^\mu(\sigma), P^\nu(\sigma')\right\}_{\text{PB}} = \frac{1}{2\pi}\left\{x^\mu, p^\nu\right\}_{\text{PB}} + \frac{i}{4\pi}\sum_{m,n\neq 0}\frac{1}{m}\left(\left\{\alpha_m^\mu, \alpha_n^\nu\right\}_{\text{PB}} e^{i(m\sigma+n\sigma')} + \left\{\overline{\alpha}_m^\mu, \overline{\alpha}_n^\nu\right\}_{\text{PB}} e^{-i(m\sigma+n\sigma')}\right) \tag{2.42}$$

$$= \frac{\eta^{\mu\nu}}{2\pi} + \frac{\eta^{\mu\nu}}{2\pi}\sum_{n\neq 0} e^{in(\sigma-\sigma')} = \frac{\eta^{\mu\nu}}{2\pi}\sum_n e^{in(\sigma-\sigma')}, \tag{2.43}$$

but $\frac{1}{2\pi}\sum_n e^{in(\sigma-\sigma')}$ is just the periodic version of the Dirac $\delta$-function, so

$$\left\{X^\mu(\sigma), P^\nu(\sigma')\right\}_{\text{PB}} = \eta^{\mu\nu}\delta(\sigma-\sigma'). \tag{2.44}$$

□

## 2.5 The Stress Tensor and Witt Algebra

Let us introduce (worldsheet) light-cone coordinates $\sigma^\pm = \tau \pm \sigma$. In these coordinates, the worldsheet metric looks like $e^\phi \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ and $\partial_\pm = \frac{\partial}{\partial\sigma^\pm}$. The action and equations of motion become

$$S = -\frac{1}{2\pi\alpha'}\int \mathrm{d}\sigma^+\,\mathrm{d}\sigma^-\,\partial_+ X\cdot\partial_- X, \qquad \partial_+\partial_- X^\mu = 0. \tag{2.45}$$

The stress tensor $T_{ab}$ is

$$T_{++} = -\frac{1}{\alpha'}\partial_+ X\cdot\partial_+ X, \qquad T_{--} = -\frac{1}{\alpha'}\partial_- X\cdot\partial_- X, \qquad \underbrace{T_{+-} = T_{-+} = 0}_{\text{effectively trace of } T_{ab}}. \tag{2.46}$$

The constraint is $T_{\pm\pm} = 0$. It is useful to introduce the Fourier modes of $T_{\pm\pm}$. We define (at $\tau = 0$) the charges

$$L_n = -\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma\, T_{--}(\sigma)e^{-in\sigma} \qquad \overline{L}_n = -\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\sigma\, T_{++}(\sigma)e^{+in\sigma}. \tag{2.47}$$

Recall that

$$\partial_- X^\mu(\sigma^-) = \sqrt{\frac{\alpha'}{2}}\sum_n \alpha_n e^{-in\sigma^-}, \tag{2.48}$$

where $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$. We find

$$L_n = \frac{1}{2\pi\alpha'}\int_0^{2\pi}\mathrm{d}\sigma\,\partial_- X^\mu(\sigma)\partial_- X_\mu(\sigma) \tag{2.49}$$

$$= \frac{1}{4\pi}\sum_{m,p}\alpha_m\cdot\alpha_p\int_0^{2\pi}\mathrm{d}\sigma\, e^{-i(m+p-n)\sigma} \tag{2.50}$$

$$= \frac{1}{4\pi}\sum_{m,p}\alpha_m\cdot\alpha_p 2\pi\delta_{p,n-m}, \tag{2.51}$$

and similarly for $\overline{L}_n$ . We have

$$\boxed{L_n = \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m, \qquad \overline{L}_n = \frac{1}{2}\sum_m \overline{\alpha}_{n-m}\cdot\overline{\alpha}_m,} \tag{2.52}$$

The constraint can be written as $L_n = 0 = \overline{L}_n$ . Using the algebra (2.38) for the $\alpha_n^\mu$ ($\overline{\alpha}_n^\mu$), we can compute the algebra for the $L_n$ ($\overline{L}_n$) to be

$$\{L_m, L_n\}_{\text{PB}} = -i(m-n)L_{m+n}, \tag{2.53a}$$

$$\{\overline{L}_m, \overline{L}_n\}_{\text{PB}} = -i(m-n)\overline{L}_{m+n}, \tag{2.53b}$$

$$\{L_m, \overline{L}_n\}_{\text{PB}} = 0. \tag{2.53c}$$

This is called the *Witt algebra*. We will see that if we set $L_n = 0 = \overline{L}_n$ at a given $\tau$, then the evolution of the system preserves $L_n = 0 = \overline{L}_n$.

## 2.6 A First Look at the Quantum Theory

If we choose the metric to be Minkowski up to Weyl transformation, we get a two-dimensional Klein–Gordon equation of motion. However, there is also the constraint $T_{ab} = 0$, which distinguishes the two-dimensional massless Klein–Gordon field from bosonic string theory. There are two approaches we can take. Either we impose the constraint first and then quantise, or we do it the other way around. *Lightcone quantisation* goes the first route, which has historically been very successful. However, in this course, we will not do this!

Instead, we will quantise the unconstrained theory and then impose the constraint $T_{ab} = 0$ as a physical condition on the Hilbert space of states.

### 2.6.1 Canonical Quantisation

We quantise by replacing our Poisson brackets with commutators:

$$\Big\{ \quad , \quad \Big\}_{\mathrm{PB}} \to -i[ \quad , \quad ]. \tag{2.54}$$

Giving equal time commutators:

$$[X^\mu(\sigma), X^\nu(\sigma')] = 0, \qquad [P_\mu(\sigma), P_\nu(\sigma')] = 0, \qquad [P_\mu(\sigma), X^\nu(\sigma')] = -i\delta_\mu{}^\nu \delta(\sigma - \sigma'). \tag{2.55}$$

Using the mode expansions for $X^\mu$ and $P_\mu$ , e.g.

$$X^\mu(\sigma,\tau) = x^\mu + p^\mu \alpha' \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \left( \alpha_n^\mu e^{-in(\tau-\sigma)} + \overline{\alpha}_n{}^\mu e^{-in(\tau+\sigma)} \right). \tag{2.56}$$

The commutation relations for the modes are

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}, \qquad [\alpha_m^\mu, \overline{\alpha}_n^\nu] = 0, \qquad [\overline{\alpha}_m^\mu, \overline{\alpha}_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}. \tag{2.57}$$

For each direction $\mu$, these are an infinite number of ladder operator relations $((\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu)$.

**Definition 7** (vacuum)**:** We introduce a *vacuum state* $|0\rangle$ on the worldsheet $\Sigma$ such that

$$\alpha_n^\mu |0\rangle = 0 \qquad n \geq 0. \tag{2.58}$$

We recall the Fourier modes of $T_{ab}$ are $L_n$ and $\overline{L}_n$, where

$$L_m = \frac{1}{2}\Sigma_n \alpha_{m-n} \cdot \alpha_n. \tag{2.59}$$

These are called the *Virasoro operators*.

The contraction is given by the spacetime Minkowski metric. Later we will briefly talk about how things change with a curved metric.

This expression Eq. (2.59) is ambiguous for $m = 0$ because $\alpha_n$ and $\alpha_{-n}$ do not commute. Going from the classical to the quantum theory, operator ordering starts to matter. We shall take

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0} \alpha_{-n} \cdot \alpha_n. \tag{2.60}$$

The ambiguity is still there as we will see later. We just shifted the ambiguity into the states.

Instead of thinking about $x^\mu$, $p_\mu$ and $T_{ab}$, it will be more useful to think about the mode operators $\alpha_m^\mu, \overline{\alpha}_m^\mu$ and $L_m$ and $\overline{L}_m$.

### 2.6.2 Physical State Conditions

We shall be imposing the conditions $T_{ab} = 0$ on the Hilbert space of the theory.

**Notation:** Let us define $N = \sum_{n>0} \alpha_{-n} \cdot \alpha_n$ and $\overline{N} = \sum_{n>0} \overline{\alpha}_{-n} \cdot \overline{\alpha}_n$.

We think of the $\alpha_n$ with positive (negative) $n$ to be annihilation (creation) operators. In that case $N$ is like a number operator. However, they are not quite like the ladder operators for the quantum harmonic operator, and $N$ is not quite a number operator; it is more like a weighted number operator.

Then $L_0 = \frac{\alpha'}{4}p^2 + N$, and $\overline{L}_0 = \frac{\alpha'}{4}p^2 + \overline{N}$, where $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$.

It would be too strong a condition for the operator to vanish $T_{ab} = 0$ identically as an operator expression; there would not be much of a theory left if we did that.

**Definition 8** (physical states)**:** Instead, we require that $T_{ab}\,|\phi\rangle = 0$ for $|\phi\rangle$ to be a *physical state*.

This means that $L_n\,|\phi\rangle = 0$ and $\langle\phi|\,L_{-n} = 0$ for $n > 0$ .

We shall also require

$$L_0\,|\phi\rangle = a\,|\phi\rangle \quad \text{and} \quad \overline{L}_0\,|\phi\rangle = a\,|\phi\rangle\,, \tag{2.61}$$

where the constant $a$ reflects the ambiguity in defining the ordering of operators constituting $L_0$ , Eq. (2.60), in the quantum theory.

For now we shall take $a = 1$.

We will find that we can interpret the canonical quantised states in a natural way when $a = 1$.

**Definition 9:** Let us define $L^{\pm} = L_0 \pm \overline{L}_0$.

Then our conditions for a state $|\phi\rangle$ to be physical translate to

$$\boxed{(L_0^+ - 2)\,|\phi\rangle = 0, \qquad L_0^-\,|\phi\rangle = 0, \qquad L_n\,|\phi\rangle = 0 = \overline{L}_n\,|\phi\rangle \quad \text{for } n > 0.} \tag{2.62}$$

## 2.7 The Spectrum

### 2.7.1 The Tachyon

**Definition 10:** We can construct a (spacetime) momentum eigenstate as

$$|k\rangle = e^{ik\cdot x}\,|0\rangle \tag{2.63}$$

In terms of a position basis in the target space, the momentum operator is $p_\mu = -i\frac{\partial}{\partial x^\mu}$, so $p_\mu\,|k\rangle = k_\mu\,|k\rangle$.

From the point of view of a massless Klein–Gordon theory, we would just stop here. However, we have got to check the physicality conditions, which we imposed. Straightforwardly, we get from (2.62) that

$$L_n\,|k\rangle = 0 = \overline{L}_n\,|k\rangle\,. \tag{2.64}$$

For the other two, we need to work a bit harder. We can write $L_0^- = N - \overline{N}$. This is sometimes called *level matching*.

This is the only constraint coupling the left- and right-moving sectors together.

Here, $N = \overline{N} = 0$. Finally, we only need to check that $(L_0^+ - 2)\,|k\rangle \overset{!}{=} 0$.

$$(L_0^+ - 2)\,|k\rangle = \left(\frac{\alpha'}{2}p^2 + N + \overline{N} - 2\right)|k\rangle = \left(\frac{\alpha'}{2}k^2 - 2\right)|k\rangle = 0. \tag{2.65}$$

This gives us a condition on $k^2$ : $k^2 - \frac{4}{\alpha'} = 0$. If we compare this with the energy-momentum condition $k^2 + M^2 = 0$, we see that the spacetime interpretation of this is that the state has negative mass squared:

$$\boxed{M^2 = -\frac{4}{\alpha'}} \tag{2.66}$$

The state $|k\rangle$ has a spacetime interpretation as a tachyon.

We found that $|k\rangle = e^{ik\cdot x}|0\rangle$ has a tachyon in the spectrum, so we are off to an inauspicious start. This problem can be cured by adding supersymmetry, but we will not do that in this course.

### 2.7.2 Massless States

Consider states of the form

$$|\epsilon\rangle = \epsilon_{\mu\nu}\alpha_{-1}^{\mu}\overline{\alpha}_{-1}^{\nu}|k\rangle, \tag{2.67}$$

where $\epsilon_{\mu\nu}$ is a constant tensor.

### Physical State Conditions

Clearly $N = \overline{N} = 1$, so $L_0^-|\phi\rangle = 0$. We can look at the energy-momentum condition $(L_0^+ - 2)|\epsilon\rangle = 0$. Since $N = \overline{N} = 1$, a calculation like (2.65) implies that $\frac{1}{2}\alpha' k^2 = 0$, so we require that $k^2 = 0$ is null.

Consider next $L_1|\epsilon\rangle = 0$:

$$0 = \frac{1}{2}\left(\sum_n \alpha_{1-n}\cdot\alpha_n\right)\epsilon_{\mu\nu}\alpha_{-1}^{\mu}\overline{\alpha}_{-1}^{\nu}|k\rangle = \epsilon_{\mu\nu}\overline{\alpha}_{-1}^{\nu}\alpha_0\cdot\alpha_1\alpha_{-1}^{\mu}|k\rangle. \tag{2.68}$$

We require then

$$0 = \epsilon_{\mu\nu}k_\rho\alpha_1^\rho\alpha_{-1}^\mu|k\rangle = \epsilon_{\mu\nu}k_\rho\left([\alpha_1^\rho,\alpha_{-1}^\mu] + \alpha_{-1}^\mu\alpha_1^\rho\right)|k\rangle \tag{2.69}$$

$$= \epsilon_{\mu\nu}\eta^{\mu\rho}k_\rho|k\rangle = \epsilon_{\rho\nu}k^\rho|k\rangle = 0. \tag{2.70}$$

The condition $L_1|\epsilon\rangle = 0$ requires us to impose $\epsilon_{\mu\nu}k^\mu = 0$. Interpreting $k^\mu$ as the centre of mass momentum, this means that there are no longitudinal polarisations. Similarly, $\overline{L_1}|\epsilon\rangle = 0$ requires that $\epsilon_{\mu\nu}k^\nu = 0$.

For all higher $n$, the $L$'s will commute with anything and annihilate the $|k\rangle$ in the definition (2.67) of $|\epsilon\rangle$; there is nothing else to check.

In summary, the physical state conditions on $|\epsilon\rangle$ are:

$$k^2 = 0, \qquad \epsilon_{\mu\nu}k^\mu = 0, \qquad \epsilon_{\mu\nu}k^\nu = 0. \tag{2.71}$$

We can decompose $\epsilon_{\mu\nu}$ into traceless symmetric ($h_{\mu\nu}$), traceless antisymmetric ($b_{\mu\nu}$), and trace part ($\phi$).

We call these parts by the following names:

$$\text{Dilaton}: \quad |\phi\rangle = \phi\alpha_{-1}^{\mu}\overline{\alpha}_{-1\mu}|k\rangle \tag{2.72}$$

$$\text{Graviton}: \quad |h\rangle = h_{\mu\nu}\alpha_{-1}^{\mu}\overline{\alpha}_{-1}^{\nu}|k\rangle \qquad (h_{\mu\nu} = h_{\nu\mu}) \tag{2.73}$$

$$\text{B-field}: \quad |b\rangle = b_{\mu\nu}\alpha_{-1}^{\mu}\overline{\alpha}_{-1}{}^{\nu}|k\rangle \qquad (b_{\mu\nu} = -b_{\nu\mu}) \tag{2.74}$$

### 2.7.3 Massive States

We could then look at the states with $N = \overline{N} = 2$ .

$$A_{\mu\nu}\alpha^{\mu}_{-2}\overline{\alpha}_{-2}{}^{\nu}\,|k\rangle + A_{\mu\nu\lambda}\alpha^{\mu}_{-2}\overline{\alpha}_{-1}{}^{\nu}\overline{\alpha}_{-1}{}^{\lambda}\,|k\rangle + \widetilde{A}_{\mu\nu\lambda}\overline{\alpha}_{-2}{}^{\mu}\alpha^{\nu}_{-1}\alpha^{\lambda}_{-1}\,|k\rangle + A_{\mu\nu\lambda\rho}\alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\overline{\alpha}_{-1}{}^{\lambda}\overline{\alpha}^{\rho}_{-1}\,|k\rangle\,. \quad (2.75)$$

The mass of such states is $m^2 = \frac{4}{\lambda'}$ .

## 2.8 The Big(-ish) Picture

We started with our Polyakov action

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\,\eta_{\mu\nu}\partial_a X^\mu \partial^a X^\nu.$$

We could deform this theory by adding a small (plane wave) deformation to the spacetime metric:

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}e^{ik\cdot X}. \quad (2.76)$$

The action changes by

$$\Delta S = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\, h_{\mu\nu}\partial_a X^\mu \partial^a X^\nu e^{ik\cdot X}. \quad (2.77)$$

The idea is that any deformation of the spacetime metric is in some sense associated with an operator

$$\mathcal{O} = h_{\mu\nu}\partial_a X^\mu \partial^a X^\nu e^{ik\cdot X}. \quad (2.78)$$

What makes string theory (or any 2-dimensional CFT) special is that there is a one-to-one correspondence between such operators $\mathcal{O}$, which correspond to a physical[1] deformation of the theory, and states in the Hilbert space. This *state-operator correspondence* will look something like

$$'' \lim_{\tau\to-\infty} \mathcal{O}\,|0\rangle = |h\rangle '' \quad (2.79)$$

We will make this more precise as we go on.

> The physical Hilbert space contains the information on how to deform the spacetime metric. This is in part why string theory allows us to talk about quantum gravity, where worldline field theory does not.

### Comment

What if we choose to start with a general metric?

$$S = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\, g_{\mu\nu}(X)\partial_a X^\mu \partial^a X^\nu. \quad (2.80)$$

[1]We will make this notion of 'physical' or 'reasonable' more precise in the language of CFT.

There are two things to note about this:

1. This is highly *nonlinear*. We might expand this metric as a power series in $X$ in Riemann normal coordinates. We get additional interaction terms in addition to the free kinetic term.

2. If we want to quantise this, we want all the classical symmetries to go through. Requiring the condition of Weyl-invariance ($h_{ab} \to e^{\omega(\sigma,\tau)} h_{ab}$) in the quantum theory constrains what $g_{\mu\nu}(X)$ we can have. One finds that $g_{\mu\nu}(X)$ has to satisfy

$$R_{\mu\nu}(g) + O(\alpha') = 0. \tag{2.81}$$

   To leading order in $\alpha'$, the metric must satisfy the vacuum Einstein equations. The higher order terms are *stringy corrections* to general relativity. More generally, if we have other background fields, we find that Weyl-invariance requires the full Einstein equations to be satisfied to leading order in $\alpha'$.

## 2.9 Spurious States and Gauge-invariance

We took $a = 1$ in the conditions (2.61), which are $(L_0 - a)\,|\phi\rangle = (\overline{L}_0 - a)\,|\phi\rangle = 0$. Why? Consider the state

$$|\chi\rangle = \sqrt{\frac{2}{\alpha'}}\left(\lambda_\lambda \alpha^\mu_{-1}\overline{L}_{-1} + \widetilde{\lambda}_\mu \overline{\alpha}_{-1}{}^\mu L_{-1}\right)|k\rangle \tag{2.82}$$

Clearly $|\chi\rangle$ is orthogonal to all physical states: If $|\phi\rangle \in \mathcal{H}$, then $\langle\phi|\chi\rangle = 0$ because $L_1\,|\phi\rangle = \overline{L}_1\,|\phi\rangle = 0$ . What conditions do $\lambda_\mu$ , $\widetilde{\lambda}_\mu$ , and $k$ have to satisfy for $|\chi\rangle$ to be physical? It is useful to write $|\chi\rangle$ as

$$|\chi\rangle = \left(\lambda_\mu k_\nu + \widetilde{\lambda}_\nu k_\mu\right)\alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}\,|k\rangle\,. \tag{2.83}$$

Keeping $a$ arbitrary, we find

$$(L_0^+ - 2a)\,|\chi\rangle = 0 \qquad \Rightarrow \qquad k^2 = \frac{4(a-1)}{\alpha'}. \tag{2.84}$$

We see that the physical state conditions are, unless $n = 1$, trivially satisfied since $N = \overline{N} = 1$. For these special cases with $n = 1$, we need to do a small calculation, which gives

$$L_1\,|\chi\rangle = 0 \qquad \text{if} \quad (\lambda\cdot k)k_\mu + \widetilde{\lambda}_\mu k^2 = 0 \tag{2.85}$$

$$\overline{L}_1\,|\chi\rangle = 0 \qquad \text{if} \quad (\widetilde{\lambda}\cdot k)k_\mu + \lambda_\mu k^2 = 0 \tag{2.86}$$

Is there any situation in which we can make sense of this? If $a = 1$, then $k^2 = 0$, and $|\chi\rangle$ is physical if in addition $\lambda\cdot k = 0 = \overline{\lambda}\cdot k$. Since $\langle\chi|\chi\rangle = \lambda^2 k^2 + 2(\lambda\cdot k)(\overline{\lambda}\cdot k) + \overline{\lambda}2k^2$, this guarantees that $\langle\chi|\chi\rangle = 0$.

They are orthogonal to every physical state, so in a correlation function they do not affect anything, even though they are physical.

Consider $\lambda_\mu = \widetilde{\lambda}_\mu$ and let us add $|\chi\rangle$ to the graviton state $|h\rangle = \epsilon_{\mu\nu}\alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}$, so

$$\left|h'\right\rangle = |h\rangle + |\chi\rangle . \tag{2.87}$$

Since $|\chi\rangle$ decouples entirely, its inner product with all physical states vanishing, this should give identical physics. Then

$$\left|h'\right\rangle = (\epsilon_{\mu\nu} + \lambda_\mu k_\nu + \lambda_\nu k_\mu)\alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}\,|k\rangle \tag{2.88}$$

$$= \epsilon'_{\mu\nu}\alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}\,|k\rangle . \tag{2.89}$$

All this does it that it changes the graviton to another graviton with a different polarisation. We recognise the transformation

$$\epsilon_{\mu\nu} \to \epsilon'_{\mu\nu} = \epsilon_{\mu\nu} + \lambda_\mu k_\nu + \lambda_\nu k_\mu \tag{2.90}$$

as a (linearised) diffeomorphism, the symmetry of general relativity. These states are interpreted as being gauge exact when we look at BRST quantisation later. This is a hint suggesting that $a = 1$ might be a sensible physical choice.

# 3 Path Integral Quantisation

This will be our favourite way to quantise the string, since the path integral formalism gives an easy way to computing correlation functions in terms of integrals over spaces of fields with given boundary conditions.

**Example 3.0.1** ($d = 1$)**:** In non-relativistic quantum mechanics, we may be interested in calculating the transition amplitude for a particle to go from $x_i(t_i)$ to $x_f(t_f)$ . The corresponding amplitude is given by a weighted sum of all possible ways of going from the start point to the end point

$$\langle x_i, t_i \,:\, x_f, t_f \rangle \propto \int_i^f \mathcal{D}X \, e^{iS[X]/\hbar}. \tag{3.1}$$

For most of the calculations we want to do we can treat this just as a generating function for correlation functions and we do not need to worry too much about the path integral measure $\mathcal{D}X$.

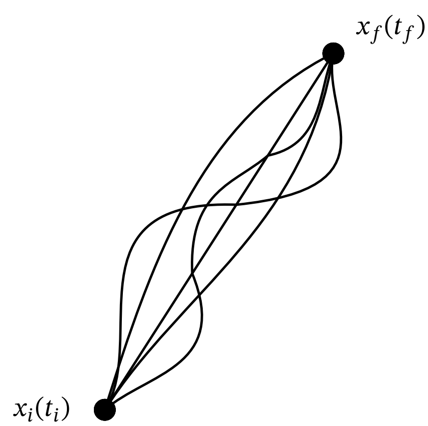


Figure 3.1: Path Integral

In string theory we take a very similar starting point. We take a weighted sum of all worldsheets with initial and final conditions specified. This is illustrated as shown in Fig.

We want to make sense of ($\hbar = 1$)

$$\langle \Psi_i | \Psi_f \rangle = \frac{1}{|\text{Diff} \times \text{Weyl}|} \int \mathcal{D}X\, \mathcal{D}h\, e^{iS[X,h]}, \tag{3.2}$$

where $S[X, h]$ is the Polyakov action (2.10) in flat spacetime. We want to separate the integral over $h_{ab}$ into those $h_{ab}$ related to each other by the action of diffeomorphisms and Weyl transformations, Diff $\times$ Weyl, and those that are 'physically distinct'. The sort of thing we want to do is to split the measure $\mathcal{D}h$ into

$$\mathcal{D}h = \mathcal{D}\nu \times \mathcal{D}h_{\text{phys}} \times \mathcal{J}, \tag{3.3}$$

where $\mathcal{D}\nu$ is the volume element of the infinite dimensional group Diff $\times$ Weyl, and $\mathcal{J}$ is some Jacobian coming from the separation, which we can see as a change of variable. Recall the general transformation of $h_{ab}$:

$$\delta h_{ab} = \delta_v h_{ab} + \delta_w h_{ab} + \delta_t h_{ab}, \tag{3.4}$$

where

- $\delta_v h_{ab} = \nabla_a v_b + \nabla_b v_,$ are diffeomorphisms (with parameter $v$), which take the form of a Lie derivative,
- $\delta_w h_{ab} = 2\omega h_{ab}$ are Weyl rescalings of $h_{ab}$,
- $\delta_t h_{ab}$ are physically distinct changes in $h_{ab}$.

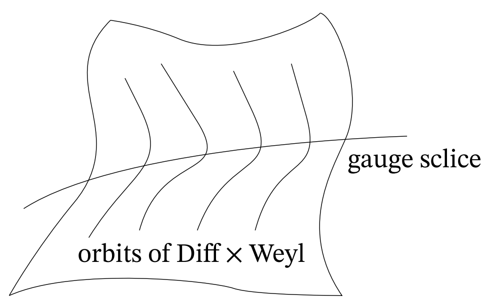


Figure 3.2: The 'space of $h_{ab}$'. Although we have not proved it, the gauge slice is chosen to cut through all orbits of Diff $\times$ Weyl exactly once.

We notice that $\delta_n h_{ab}$ has non-zero trace. It is useful to absorb this trace part into $\delta_w h_{ab}$. We introduce a traceless version of $\delta_v h_{ab}$:

$$\boxed{(\mathcal{P}v)_{ab} = \nabla_a v_b + \nabla_b v_a - h_{ab}(\nabla_c v^c)}. \tag{3.5}$$

And the Weyl transformation becomes

$$\delta_{\overline{\omega}} = 2\overline{\omega} h_{ab} \qquad \overline{\omega} = \omega + \frac{1}{2}\nabla \cdot v. \tag{3.6}$$

**Definition 11:** We call the space of physically distinct metrics $h_{ab}$ the *moduli space*.

And $\delta_t h_{ab}$ is a variation of the metric in the moduli space.

> So far, we are only talking about diffeomorphisms connected to the identitiv since we are working infinitesimally. We can extend this to 'large diffeomorphisms'.

We will work on worldsheets in which (at least infinitesimally) all metrics can be related to each other by diffeomorphisms and Weyl transformations.

## 3.1 A Crash Course on Riemann Surfaces

Potentially, we will work on all sorts of worldsheets with all sorts of topologies.

**Definition 12** (Riemann surface)**:** A *Riemann surface* is a 2 (real) dimensional Riemannian manifold, where we consider metrics related by a Weyl transformation $h_{ab} \to e^{2\omega} h_{ab}$ to be equivalent.

Schematically, we might say that the space of Riemann surfaces is the space of Riemannian manifolds modulo Weyl transformations.

### 3.1.1 Worldsheet Genus and Punctures

These things will distinguish different Riemann surfaces.

**Definition 13** (genus)**:** For $\Sigma$ without boundary (closed strings)[1] the topology of $\Sigma$ is encapsulated in the Euler characteristic

$$\chi = \frac{1}{4\pi} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h} R(h) = 2 - 2g, \tag{3.7}$$

where $g = 0, 1, \ldots$ is the *genus* of $\Sigma$.

Considering Fig., this starts to look a bit like a loop expansion, where $g$ will be related to the loop order in perturbation theory.

**Definition 14** (punctures)**:** *Punctures* on $\Sigma$ are marked points on $\Sigma$, i.e. points at a particular position.

**Example 3.1.1:** The Riemann surface illustrated in Fig has $g = 1$ and 3 marked points.

[1]For a cylinder, we might need to worry about the infinite past and future? We will discuss this later, but it turns out that it is not quite like a boundary. We will discuss this in terms of punctures.

Consider now the final piece $\delta_t h_{ab}$ of the transformation (3.4) of the metric. We can write these as

$$t^I \mu_{abI}, \qquad \mu_{abI} = \frac{\partial h_{ab}}{\partial m^I}, \tag{3.8}$$

where $m^I$ are coordinates of the moduli space of Riemann surfaces $M_g$ and $t^I \approx \delta m^I$ are tangent vectors to $M_g$.

### 3.1.2 Moduli Space of (Unpunctured) Riemann Surfaces

The moduli space $M_g$ of a genus $g$ Riemann surface $\Sigma_g$ is schematically

$$M_g = \frac{\{\text{metrics}\}}{\{\text{Diff} \times \text{Weyl}\}}. \tag{3.9}$$

The (real) dimension of $M_g$ is as follows

$$\dim(M_g) := s = \begin{cases} 0, & \text{if } g = 0 \\ 2, & \text{if } g = 1 \\ 6g - 6, & \text{if } g \geq 2. \end{cases} \tag{3.10}$$

**Example 3.1.2** ($g = 0$)**:** All metrics on $g = 0$ surfaces[1] may be brought to the form $e^{2\omega}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ by diffeomorphisms. In other words, they are trivial up to Weyl transformations.

**Example 3.1.3** ($g = 1$ (torus))**:** We can construct a torus by taking the complex plane $\mathbb{C}$ and imposing identifications such as

$$z \sim z + \lambda_1 m + \lambda_2 n, \tag{3.11}$$

where $\lambda_1, \lambda_2$ are lattice vectors and $m, n \in \mathbb{Z}$ . One can show that the ratio

$$\tau = \frac{\lambda_1}{\lambda_2} \in \mathbb{C} \tag{3.12}$$

is invariant under Diff $\times$ Weyl .

**Remark:** This $\tau$ is not worldsheet time! It is called the *complex structure*.

In other words, using diffeomorphisms and Weyl transformations, we can always bring the metric to the form

$$ds_2 = \left|dz + \tau d\bar{z}\right|^2. \tag{3.13}$$

[1] Everything we are saying pertains to two dimensional manifolds.

There are two degrees of freedom, which are encoded in this one complex degree of freedom. Without loss of generality, we can choose $\operatorname{Im}\tau > 0$ . We could write the identification $z \sim z + n_a\lambda^a$ , where $n_a = (m, n)$ and $\lambda^a = \begin{pmatrix}\lambda_1 \\ l_2\end{pmatrix}$. If we act with an $SL(2)$ transformation

$$n_a \to U_a{}^b n_b \tag{3.14}$$
$$\lambda^a \to (U^{-1})^a{}_b \lambda^b \tag{3.15}$$

this statement is preserved. Moreover, if $U \in SL(2;\mathbb{Z})$ , which has integer entries, then $(n, m)$ remain integer. So the moduli space $M_1$ at $g = 1$ can be identified with the upper half plane modulo $SL(2;\mathbb{Z})$.

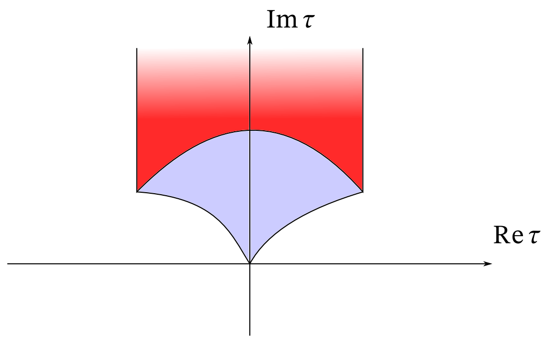


Figure 3.3

**Example 3.1.4** ($g \geq 2$)**:** It gets hard but there are analogues of the $SL(2;\mathbb{Z})$ (modular group) in all cases.

### 3.1.3 Conformal Killing Vectors

**Definition 15** (Conformal Killing group)**:** There is an overlap Diff∩Weyl called the *conformal Killing group* (CKG).

Is it possible to undo the effect of a diffeomorphism by a Weyl transformation? In other words, we want to find a pair $(v^a, \omega)$ such that

$$\delta h_{ab} = \nabla_a v_b - \nabla_b v_a + 2\omega h_{ab} = 0 \tag{3.16}$$

We take the trace to find that if

$$\omega = -\frac{1}{2}\nabla_a v^a, \tag{3.17}$$

then the diffeomorphism and Weyl transformation cancel out. We defined $Pv$ in Eq. (3.5).

**Definition 16** (Conformal Killing vectors)**:** Vectors that satisfy $(Pv)_{ab} = 0$ are what we shall call *conformal Killing vectors* (CKVs).

The CKG is generated by the CKVs. For a given genus, the (real) dimension of the conformal Killing group is

$$\dim_{\mathbb{R}}(\text{CKG}) := \varkappa = \begin{cases} 6, & \text{if } g = 0 \\ 2, & \text{if } g = 1 \\ 0, & \text{if } g \geq 2 \end{cases} \tag{3.18}$$

In particular, notice that these are finite-dimensional. We can figure out that for $g = 0$, the CKG is $SL(2, \mathbb{C})$ and on the torus $g = 1$ it is $U(1) \times U(1)$ . Moreover, for $g \geq 2$ , the CKG is empty; there is no overlap for higher $g$'s.

**Example 3.1.5** ($g = 2$)**:** The conformal Killing group is $SL(2; \mathbb{C})$[1]. This is called the *Möbius group*. If $z \in \mathbb{C}$, then this transformation acts as

$$z \to \frac{az + b}{cz + d}, \qquad ad - bc = 1, \qquad a, b, c, d \in \mathbb{Z}. \tag{3.19}$$

We can specify a particular element of the conformal Killing group at genus $g = 0$ by describing how three distinct points transform under the given map.

### 3.1.4 Moduli Space of Punctured Riemann Surfaces $M_{g,n}$

Suppose we have a Riemann surface of genus $g$ with $n$ punctures (or marked points). What is its moduli space? Naively, we might expect

$$M_{g,n} = M_g \otimes \Sigma^n. \tag{3.20}$$

But we also have to worry about the CKG. If we have a genus $g = 0$ surface with $n$ punctures, we can use the conformal Killing group to fix three of the locations of the punctures (i.e. 'three degrees of freedom'), leaving only $n - 3$ free punctures. Similarly, we can fix the CKG on the torus ($g = 1$) by fixing the location of one puncture.

> We will see very explicitly how this works when we look at the path integral.

So we should really take the moduli space (as relating to our path integral) to be

$$M_{g,n} = M_g \otimes \Sigma^{n - \frac{\varkappa}{2}}, \tag{3.21}$$

since each puncture has two bits $(\sigma, \tau)$ of information. Therefore, the dimension of the moduli space is

$$\boxed{\dim(M_{g,n}) = 6g - 6 + 2n} \tag{3.22}$$

[1]Strictly speaking this should be the projective group $PSL(2; \mathbb{C})$. However, this distinction will not matter much for our purposes.

## 3.2 Fadeev–Popov Determinant

Our goal will be to find some effective way to make sense of the path integral. When integrating over the worldsheet metrics, we want to count each physical metric only once, and avoid overcounting from to diffeomorphism and Weyl equivalence.

In other words, we heuristically want to mod out the equivalences of physical states

$$Z[0] = \frac{1}{|\mathrm{Diff} \times \mathrm{Weyl}|} \int \mathcal{D}X\, \mathcal{D}h\, e^{iS[h,X]} \tag{3.23}$$

**Definition 17** (Fadeev–Popov determinant)**:** To that end, we will introduce the *Fadeev–Popov* (FP) determinant $\Delta_{\mathrm{FP}}$, defined by

$$1 = \Delta_{\mathrm{FP}}[h] \int_{M_g} \mathrm{d}^3 t \int_{\mathrm{Diff}\times\mathrm{Weyl}} \mathcal{D}\omega\, \mathcal{D}v\, \delta[h - \hat{h}] \prod_{\substack{i=1 \\ a=1,2}}^{k} \delta(v^a(\hat{\sigma}_i)) \tag{3.24}$$

where $M_g$ is the integral over all Riemann surfaces of genus $g$ and the delta-functional

$$\delta[h - \hat{h}] = \prod_{\substack{a,b \\ \sigma,\tau}} \delta(h_{ab}(\sigma,\tau) - \hat{h}_{ab}(\sigma,\tau)), \tag{3.25}$$

fixes the metric to be a specifically chosen metric, where

$$h_{ab} - \hat{h}_{ab} = \delta h_{ab} = (Pv)_{ab} + 2\overline{\omega} h_{ab} + t^I \mu_{I\,ab}, \qquad \overline{\omega} = \omega + \dots \tag{3.26}$$

We need to fix the CKG (Diff ∩ Weyl).

> We will use the physical path integral methods, but in this particular case one can do the functional analysis rigorously, say with heat kernel methods.

## 3.3 Gauge-Fixing the Path Integral

Let us put (3.24) into (3.23):

$$\begin{aligned} Z[0] &= \frac{1}{\mathrm{Diff} \times \mathrm{Weyl}} \int_{\mathrm{Diff}\times\mathrm{Weyl}} \mathcal{D}\omega\, \mathcal{D}v \prod_{a,i} \delta(v^a(\sigma_i)) \int_{M_g} \mathrm{d}^3 t \int \mathcal{D}X\, \mathcal{D}h\, \delta[h - \hat{h}] \Delta_{\mathrm{FP}} e^{iS[h,X]} && (3.27) \\ &= \frac{1}{\mathrm{Diff} \times \mathrm{Weyl}} \int \mathcal{D}X \int_{M_g} \mathrm{d}^3 t \left( \int_{\mathrm{Diff}\times\mathrm{Weyl}} \mathcal{D}\omega\, \mathcal{D}v \prod_{a,i} \delta(v^a(\hat{\sigma}_i)) \right) e^{iS[\hat{h},X]} \Delta_{\mathrm{FP}}[\hat{h}]. && (3.28) \end{aligned}$$

In the parentheses, we have

$$\int_{\mathrm{Diff}\times\mathrm{Weyl}} \mathcal{D}\omega\, \mathcal{D}v \prod_{a,i} \delta(v^a(\hat{\sigma}_i)) = \frac{|\mathrm{Diff} \times \mathrm{Weyl}|}{|\mathrm{CKG}|}. \tag{3.29}$$

If this makes you uncomfortable, it should. We are trying to divide out infinite dimensional equivalence classes. There are also ways to make this rigorous using the theory of Riemann surfaces. However, they lack the physical intuition and would take much longer to derive.

We find a more respectable starting point

$$Z[0] = \frac{1}{|CKG|}\int_{M_g} \mathrm{d}^3 t \int \mathcal{D}X\, e^{iS[X,\hat{h}]}\Delta_{\mathrm{FP}}[\hat{h}]. \tag{3.30}$$

This looks much more reasonable. The conformal Killing group is finite-dimensional. For this to be a good starting point for the quantisation of the theory, we have to think a bit more about $\Delta_{\mathrm{FP}}[\hat{h}]$ .

## 3.4 A Field Theory Representation for $\Delta_{\mathrm{FP}}$

Inverting (3.24), we find that the inverse of the Fadeev–Popov determinant is

$$\Delta_{\mathrm{FP}}^{-1}[\hat{h}] = \int_{M_g} \mathrm{d}^3 t \int_{\mathrm{Diff}\times\mathrm{Weyl}} \mathcal{D}\overline{\omega}\,\mathcal{D}v\,\delta[\delta h]\prod_{a,i}\delta(v^a(\hat{\sigma}_i)). \tag{3.31}$$

Let us find an integral expression for the $\delta$-function(al)s by introducing auxiliary fields $\beta_{ab}(\sigma,\tau)$ and $2k$ elements $\xi_a^i$:

$$\Delta_{\mathrm{FP}}^{-1}[\hat{h}] = \int_{M_g} \mathrm{d}^3 t \int \mathcal{D}v\,\mathcal{D}\overline{\omega}\,\mathcal{D}\beta\,\mathrm{d}^{2k}\xi\,\exp\left(i(\beta \mid Pv + 2\overline{\omega}h + t^I\mu_I) + i\sum_{i=1}^{k}\xi_a^i\delta(v^a(\hat{\sigma}_i))\right), \tag{3.32}$$

where the inner product is defined as

$$(\beta \mid Pv + 2\overline{\omega}h + t^I\mu_I) := \int_{\xi} \mathrm{d}^2\sigma\,\sqrt{-h}\beta^{ab}((Pv)_{ab} + 2\overline{\omega}h_{ab} + t^I\mu_{I\,ab}). \tag{3.33}$$

We can do the $\overline{\omega}$ integral, which constrains $\beta_{ab}h^{ab} = 0$.

We have $\Delta_{\mathrm{FP}^{-1}}$ but we want $\Delta_{\mathrm{FP}}$. We introduce Grassmann variables. These are variables $\theta_i$ that anticommute $\theta_1\theta_2 = -\theta_2\theta_1$. They have lots of interesting properties, such as $\theta_i^2 = 0$. This means we can expand a function $f(x,\theta) = f(x) + f'(x)\theta$. Moreover, the delta function of a Grassmann variable is simply $\delta(\theta) = \theta$ and integral and derivative are equal $\int \mathrm{d}\theta\, f(\theta) = \frac{\partial f}{\partial\theta}$.

Let us consider some finite-dimensional Gaussian integral over a complex vector space $V$:

$$\frac{1}{\det M} = \int_V \mathrm{d}z\,\mathrm{d}\overline{z}\,e^{-(\overline{z},Mz)}, \tag{3.34}$$

where $(\cdot,\cdot)$ denotes some inner product on $V$. Writing the same type of integral with Grassmann variables,

$$\det M = \int \mathrm{d}\theta\,\mathrm{d}\overline{\theta}\,e^{-(\overline{\theta},M\theta)}. \tag{3.35}$$

(See Ryder's QFT book if curious). We can get from an integral expression for a determinant to the reciprocal by replacing the $c$-numbers with Grassmann numbers.

Similarly, we replace all the fields we integrate over in $\Delta_{\text{FP}}^{-1}$ with Grassmann-valued fields:

$$v^a(\hat{\sigma}_i) \to c^a(\sigma,\tau), \qquad \beta_{ab}(\sigma,\tau) \to b_{ab}(\sigma,\tau), \qquad t^I \to \zeta^I, \qquad \xi^i_a \to \eta^i{}_a. \tag{3.36}$$

Just like $\beta_{ab}$, we take $b_{ab}$ to be traceless also. With these Grassmann fields in place, we take $\Delta_{\text{FP}}[\hat{h}]$ to be given by

$$\Delta_{\text{FP}}[\hat{h}] = \int \mathrm{d}^3\zeta\, \mathcal{D}c\, \mathcal{D}b\, \mathrm{d}^{2k}\eta \exp\left[ i(b \mid Pc + \zeta^I \mu_I + i\sum_{i=1}^{\kappa} \eta^i{}_a c^a(\hat{\sigma}_i)) \right], \tag{3.37}$$

where $\kappa = |\text{CKG}|$. We can do these finite-dimensional integrals. The integral over $\zeta^I$ gives a $\delta$-functional.

$$\prod_{I=1}^{s} \delta(b \mid \mu_I) = \prod_{I=1}^{s} (b \mid \mu_I), \tag{3.38}$$

where $s = \dim M_g$.

The $\eta^i{}_a$ integral gives

$$\prod_{\substack{a=i,? \\ i=1,\dots,\kappa}} \delta(c^a(\hat{\sigma}_i)) = \prod_{i,a} c^a(\hat{\sigma}_i). \tag{3.39}$$

The expression for the Fadeev–Popov determinant becomes rather simple

$$\boxed{\Delta_{\text{FP}}[\hat{h}] = \int \mathcal{D}c\, \mathcal{D}b\, e^{iS[b,c]} \prod_{I=1}^{s} (b \mid \mu_I) \prod_{i,a} c^a(\hat{\sigma}_i)} \tag{3.40}$$

where the action for the $b, c$ (*ghost*) fields is

$$S[b,c] = (b \mid Pc) = \int_\Sigma \mathrm{d}^2\sigma \sqrt{-\hat{h}} b^{ab} (Pc)_{ab} = 2\int_\Sigma \mathrm{d}^2\sigma \sqrt{-\hat{h}} b^{ab} (\nabla_a c_b), \tag{3.41}$$

where we used that $b_{ab}$ is symmetric and traceless.

We shall refer to the (Grassmann) fields $b_{ab}$ and $c^a$ as Fadeev–Popov *ghosts*.

## 3.5 Calculating Observables

Our final form for $Z[0]$ for Riemann surfaces of genus $g$ is

$$Z_g[0] = \frac{1}{|\text{CKG}|}\int_{M_g} \mathrm{d}^3 t \int \mathcal{D}X\,\mathcal{D}b\,\mathcal{D}c\, e^{iS[X,b,c]} \prod_I (b \mid \mu_I) \prod_{i,a} c^a(\hat{\sigma}_i). \tag{3.42}$$

In general, we want to sum over all Riemann surfaces

$$Z[0] = \sum_{g=0}^{\infty} e^{\lambda\chi} Z_g[0], \tag{3.43}$$

where $\lambda$ is a constant and $\chi = lg - l$ is the Euler characteristic of the worldsheet. It will turn out that the constant weighting $\lambda$ will be determined by the theory rather than being a free parameter.

We can compute correlation functions of observables by including them in our path integral

$$\langle \phi_1 \dots \phi_n \rangle = \mathcal{N} \int \mathcal{D}\phi\, \phi_1 \dots \phi_n e^{iS[\phi]} \tag{3.44}$$

What might our observables look like? They will certainly need to be invariant under diffeomorphisms and Weyl transformations. We can build diffeomorphism-invariant observables by taking an operator $\mathcal{O}(\sigma,\tau)$ on a genus $g$ worldsheet $\Sigma_g$ and integrating it over $\Sigma_g$:

$$\mathcal{O} = \int_{\Sigma_g} \mathrm{d}^2\sigma\, \mathcal{O}(\sigma,\tau). \tag{3.45}$$

We will need to choose the $\mathcal{O}(\sigma,\tau)$ carefully in order for this to be Weyl invariant as well. We will talk about this when we come to CFTs.

So a correlation function of such observables would be:

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \sum_{g=0}^{\infty} \frac{e^{\lambda\chi}}{|\text{CKG}|} \int \prod_{i=0}^{n} \mathrm{d}^2\sigma_i \int_{M_g} \mathrm{d}^3 t \int \mathcal{D}X\,\mathcal{D}b\,\mathcal{D}c\, e^{iS[X,b,c]} \prod_I (b \mid \mu_I) \prod_{i,a} c^a(\hat{\sigma}_i) \mathcal{O}_1(\sigma_1) \dots \mathcal{O}_n(\sigma_n), \tag{3.46}$$

using the shorthand $\sigma_i = (\sigma_i, \tau_i)$.

**Remark:** We have the instruction to divide by the conformal Killing group, and to integrate over the moduli space of Riemann surfaces $M_g$ as well as all locations $\sigma_i$. We know we can fix the CKG on a sphere by keeping three locations fixed, and one on a torus. Instead of integrating over all points of the Riemann surface, we can thus write the instruction of dividing by the CKG as restricting what we integrate over:

$$\frac{1}{\text{CKG}} \int_{M_g} \mathrm{d}^3 t \int \prod_{i=1}^{n} \mathrm{d}^2\sigma_i = \int_{M_g} \mathrm{d}^3 t \int_{i=1}^{n-\kappa} \mathrm{d}^2\sigma_i = \int_{M_{g,n}} . \tag{3.47}$$

So when computing the correlation functions, we end up integrating only over $n - \varkappa$ locations

$$\boxed{\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \sum_{g=0}^{\infty} e^{\lambda\chi} \int_{M_{g,n}} \int \mathcal{D}X\, \mathcal{D}b\, \mathcal{D}c\, e^{iS[X,b,c]} \int_I (b \mid \mu_I) \prod_{i,a} c^a(\hat{\sigma}_i) \mathcal{O}_1 \dots \mathcal{O}_n} \tag{3.48}$$

This might seem like the final result depends on some locations, but as we will see the ghosts will take care of this.

# 4 Conformal Field Theory

This is a wonderful topic. There could be a whole Part III course in CFT alone. It is important in String Theory but also in Statistical and Condensed Matter Physics. Moreover, when trying to really understand QFT with mathematical rigour, your best bet is to start with low-dimensional CFT.

## 4.1 Conformal Field Theory

### 4.1.1 Conformal Invariance in General Dimension

Let us consider transformations that preserve angles. In particular, transformations $x^\mu \to x'^\mu(x)$ such that the metric on our space transforms as

$$\Lambda(x)\,\eta_{\mu\nu} = \eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}, \tag{4.1}$$

preserving the metric up to a local scale $\Lambda(x)$ .

Infinitesimally, we write a transformation as $x^\mu \to x^\mu(x) = x^\mu + \epsilon v^\mu(x) + \dots$ , where $\epsilon \ll 1$. The first order in $\epsilon$, the metric transforms as

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + \epsilon(\partial_\mu v_\nu + \partial_\nu v_\mu) = \Lambda(x)\,\eta_{\mu\nu}. \tag{4.2}$$

Let $\Lambda(x) = e^{\epsilon\omega(x)} \approx 1 + \epsilon\omega(x) + \dots$ . Thus, vector fields $v^\mu$ must satisfy

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \omega(x)\,\eta_{\mu\nu}. \tag{4.3}$$

Taking the trace of both sides,

$$\omega(x) = \frac{2}{d}\partial_\mu v^\mu, \tag{4.4}$$

where $d$ is the dimension few the spacetime. Therefore, vector fields that generate such conformal transformations satisfy

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{d}\partial_\lambda v^\lambda \eta_{\mu\nu}. \tag{4.5}$$

### 4.1.2 In Two Dimensions

Something special happens in $d = 2$ dimensions. Let us take the coordinates in two dimensions to be $\sigma$ and $\tau$. We take $\eta$ to be the Euclidean metric, diag$(1, 1)$. With this metric, Eq. (4.5) boils down to two independent equations

$$\boxed{\frac{\partial v_\tau}{\partial \tau} = \frac{\partial v_\sigma}{\partial \sigma}, \qquad \frac{\partial v_\sigma}{\partial \tau} = -\frac{\partial v_\tau}{\partial \sigma}} \tag{4.6}$$

where $v = \begin{pmatrix} v_\sigma \\ v_\tau \end{pmatrix}$. These are the Cauchy–Riemann equations!

**Remark:** We can now bring to bear the full power of complex analysis to understand quantum gravity.

If we introduce complex coordinates

$$w = \tau + i\sigma, \qquad \overline{w} = \tau - i\sigma, \qquad (\omega \in \mathbb{C}), \tag{4.7}$$

then the condition for the vector $v$ to generate a conformal transformation is that $v$ is holomorphic, meaning that $\overline{\partial} v = 0$.

If a theory is conformally invariant, then there is a conserved charge associated with every holomorphic $v$; there is an infinite number of conserved charges!

**Remark:** Given our current understanding, this is special in $d = 2$. This allows us to do quantum field theory with an easy that we just do not see in higher dimensions.

A particularly useful set of coordinates for our worldsheet is the exponential of $w$:

$$z = e^{\tau + i\sigma}, \qquad \overline{z} = e^{\tau - i\sigma}. \tag{4.8}$$

Under this map, the cylindrical worldsheet is mapped to the complex plane (if we choose a Euclidean metric, which we will do from now on).

Cylinder becomes radial evolution on the complex plane.

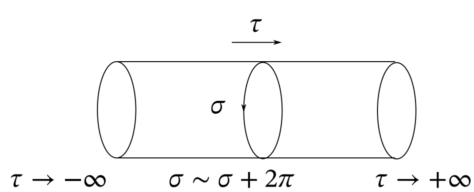


Figure 4.1

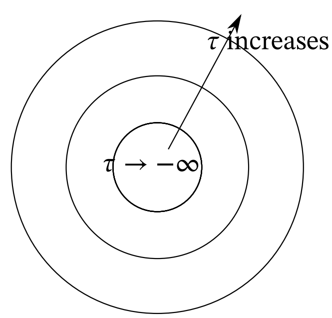


Figure 4.2

### 4.1.3 The Witt Algebra

We have seen that conformal transformations are generated by (anti-)holomorphic vectors $v(z)$ and $\overline{v}(z)$, i.e.

$$z \to z + v(z) + \ldots \qquad \overline{z} \to \overline{z} + \overline{v}(\overline{z}) + \ldots \tag{4.9}$$

We can write $v(z)$ as a power series expansion

$$v(z) = \sum_n v_n z^{n+1}. \tag{4.10}$$

We might allow for the possibility to have some isolated poles in this sum, but we will not worry too much about that for now. For now the exponent $z^{n+1}$ is just a convention. Infinitesimally, we have the transformation

$$z \to z + \sum_n v_n z^{n+1}, \tag{4.11}$$

which we can think of as being generated by the operators

$$l_n = -z^{n+1}\frac{\partial}{\partial z} \qquad \text{i.e.} \quad z \to z - \sum_n v_n l_n z. \tag{4.12}$$

The operators $l_n$ satisfy the Witt algebra

$$\boxed{[l_n, l_m] = (n-m)l_{n+m}} \tag{4.13}$$

The first time we saw this, the $l_n$ were the Fourier modes of the stress tensor. Here, another way to look at these is as the generators of conformal transformations. This hints at a connection between the stress tensor and conformal transformations.

### 4.1.4 Conformal Fields

Some definitions:

**Definition 18** (chiral field)**:** A *chiral field* is a field $\phi(z)$ that depends on $z$ alone.

**Definition 19** (conformal dimension)**:** The *conformal dimension* is a number $\Delta = h + \overline{h}$ that tells us how a field transforms under scaling.

Suppose we have a field $\phi(z, \overline{z})$ and rescale $(z, \overline{z}) \to (\lambda z, \overline{\lambda}\overline{z}) = (z', \overline{z}')$, where $\lambda$ and $\overline{\lambda}$ need not necessarily be complex conjugates of each other. Then

$$\phi(z, \overline{z}) \to \phi'(z', \overline{z}') = \lambda^h \overline{\lambda}^{\overline{h}} \phi(\lambda z, \overline{\lambda}\overline{z}). \tag{4.14}$$

**Remark:** A chiral field has $\overline{h} = 0$.

**Definition 20** (primary field)**:** Under the transformation $z \to f(z)$ , a *primary field* transforms as

$$\phi(z,\overline{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}} \phi\big(f(z), \overline{f}(\overline{z})\big), \tag{4.15}$$

where $f$ and $\overline{f}$ need not be complex conjugates of each other. What we are really saying is that primary fields are tensors under conformal transformations (not just under diffeomorphisms).

Similarly to how we focused on just the left-moving fields before, we will look only at chiral primary fields. (The correspondence between left (right) moving modes and (anti) holomorphic fields is given by the definition of our coordinates (4.8).)

For an infinitesimal transformation $f(z) = z + v(z) + \ldots$

$$\left(\frac{\partial f}{\partial z}\right)^h \approx (1+\partial v)^h \approx 1 + h\partial v \tag{4.16}$$

$$\phi(f(z)) = \phi(z + v + \ldots) = \phi(z) + v\partial\phi(z) + \ldots \tag{4.17}$$

and similarly for $\overline{z} \to \overline{z} + \overline{v}(\overline{z}) + \ldots$ . We find

$$\boxed{\delta_{v,\overline{v}}\phi(z,\overline{z}) = \left(h\partial v + \overline{h\partial v} + v\partial + \overline{v\partial}\right)\phi(z,\overline{z})} \tag{4.18}$$

### 4.1.5 Conformal Transformations and the Stress Tensor

We start with out favourite Polyakov action in flat space,

$$S[X] = -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\, \partial_a X^\mu \partial^a X^\nu \eta_{\mu\nu}. \tag{4.19}$$

Under a conformal transformation, the embedding field transforms as

$$\delta_v X^\mu = v^a \partial_a X^\mu. \tag{4.20}$$

We would like to find the Noether current associated with this. Under such a transformation, the action changes by

$$\delta_v S[X] = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\, (\partial^a v^b) T_{ab}. \tag{4.21}$$

We can integrate by parts to obtain

$$\delta_v S[X] = -\frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\, v^a(\partial^b T_{ab}) = 0. \tag{4.22}$$

For this to be a symmetry under all $v^a$, there has to be a conserved current

$$\boxed{\partial^a T_{ab} = 0 = \partial^b T_{ab}} \tag{4.23}$$

We could define conserved charges $Q_\pm$, which in lightcone coordinates ($\sigma^\pm = \tau \pm \sigma$) look like

$$Q_\pm = \frac{1}{2\pi}\oint \mathrm{d}\sigma\, v^\pm(\sigma) T_{\pm\pm}(\sigma), \tag{4.24}$$

where the normalisation is a convention. Infinitesimal conformal transformations are given by the Poisson bracket of of some field with $Q_\pm$ . In particular,

$$\delta_v X^\mu = \{Q_+ + Q_-, X^\mu\}_{\mathrm{PB}}, \tag{4.25}$$

where $\{\ ,\ \}_{\mathrm{PB}}$ is the classical Poisson bracket.

This classical statement gives us some hint that the stress tensor, thoguht of as a constraint, generates these conformal transformations and explains why the Witt algebra is appearing in these two apparently different circumstances. Put another way, the constraints coming from the vanishing of the stress tensor put physical conditions on the spacetime. We would like to recast this intuition into understanding how conformal symmetry gives rise to various properties of spacetime physics. We will soon look at a quantum version of (4.25).

## 4.2 Complex Coordinates

In this section, we use $z = e^{\tau + i\sigma}$, $\bar{z} = e^{\tau - i\sigma}$ and take $h_{ab}$ to be Euclidean on $\Sigma$. In these coordinates, the stress tensor has only two non-trivial coordinates

$$T := T_{zz} = -\frac{1}{\alpha'}\partial X^\mu \partial X^\nu \eta_{\mu\nu}, \qquad \bar{T} := T_{\bar{z}\bar{z}} = -\frac{1}{\alpha'}\bar{\partial} X^\mu \bar{\partial} X^\nu \eta_{\mu\nu}. \tag{4.26}$$

The off-diagonal entries $T_{z\bar{z}} = 0$ vanish as they are the trace of the stress tensor. Moreover,

$$\mathrm{d}t\,\mathrm{d}\sigma = -\frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2i|z|^2}. \tag{4.27}$$

The path integral becomes

$$\int \mathcal{D}X\, e^{iS[X]} \to \int \mathcal{D}X\, e^{-S[X]}, \tag{4.28}$$

where the action is

$$S[X] = -\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2 z\, \partial X^\mu \bar{\partial} X^\nu \eta_{\mu\nu}. \tag{4.29}$$

The equation of motion for the $X^\mu$ is now $\partial\bar{\partial} X^\mu = 0$. This means that we have the split

$$X^\mu(z,\bar{z}) = X^\mu_L(z) + X^\mu_R(\bar{z}). \tag{4.30}$$

Finally, our conservation equation (4.23) becomes

$$\bar{\partial} T(z) = 0 \qquad \partial \bar{T}(\bar{z}) = 0. \tag{4.31}$$

### 4.2.1 Ward Identities and Conformal Transformations

Given an infinitesimal conformal transformation, how do we compute it in terms of the operators in our theory. Ultimately, the fields that we are interested in are the embedding fields $X^\mu$, but also the ghosts $b$ and $c$. We will describe a general primary field as $\phi(z, \bar{z})$. Later we shall choose $\phi$ to be either $X^\mu$ or one of the $b, c$ ghost fields, which came from the quantum field theoretic description of the Fadeev–Popov determinant.

Let us consider the change in a correlation function $\langle \phi_1 \dots \phi_n \rangle = \int \mathcal{D}\phi\, e^{-S[\phi]} \phi_1 \dots \phi_n$ resulting from an infinitesimal transformation $\phi \to \phi' = \phi + \delta\phi$.

$$\langle \phi'(z_1) \dots \phi'(z_n) \rangle = \int \mathcal{D}\phi'\, e^{-S[\phi']} \phi'(z_1) \dots \phi'(z_n), \tag{4.32}$$

where we write $\phi'(z_i)$ as a shorthand for $\phi'(z_i, \bar{z}_i)$. We shall assume that the measure $\mathcal{D}\phi' = \mathcal{D}\phi$ does not change (meaning that there are no anomalies in the theory). However, in general the action might change. To leading order, we have

$$\langle \phi'(z_1) \dots \phi'(z_n) \rangle = \int \mathcal{D}\phi\, e^{-S[\phi] - \delta S[\phi]} (\phi(z_1) + \delta\phi(z_1)) \dots (\phi(z_n) + \delta\phi(z_n)) \tag{4.33}$$

$$= \langle \phi(z_1) \dots \phi(z_n) \rangle - \int \mathcal{D}\phi\, e^{-S[\phi]} \delta S[\phi] \phi(z_1) \dots \phi(z_n) \tag{4.34}$$

$$+ \int \mathcal{D}\phi\, e^{-S[\phi]} \sum_{k=1}^{n} \phi(z_1) \dots \phi(z_{k-1}) \delta\phi(z_k) \phi_{z_{k+1}} \dots \phi(z_n). \tag{4.35}$$

For our theory to be invariant under the change $\phi \to \phi'$ we require $\langle \phi' 1 \dots \phi_n' \rangle = \langle \phi_1 \dots \phi_n \rangle$ . Thus

$$\langle \delta S[\phi]\, \phi(z_1) \dots \phi(z_n) \rangle = \sum_{k=1}^{n} \langle \phi(z_1) \dots \delta\phi(z_k) \dots \phi(z_n) \rangle. \tag{4.36}$$

Let us now apply this to the bosonic string theory. Take $S$ to be the Polyakov action (with gauge fixed worldsheet metric) and $\delta\phi = v^a\partial_a\phi$ .

**Exercise 4.1:** Under such a transformation the action varies as

$$\delta S[\phi] = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\,(\partial_a v_b)T^{ab}, \tag{4.37}$$

where $T^{ab}$ is the stress tensor.

We have

$$\frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\,(\partial^a v^b)\langle T_{ab}\phi_1 \ldots \phi_n\rangle = \sum_{k=1}^{n}\langle\phi_1 \ldots \delta\phi_k \ldots \phi_n\rangle. \tag{4.38}$$

Our worldsheet looks locally like a cylinder with coordinates $(\sigma,\tau)$. We map each of these cylin-

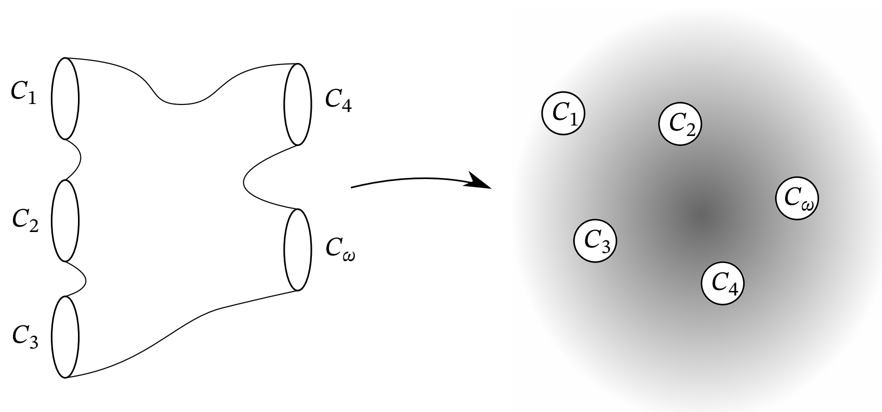


Figure 4.3

ders to the complex plane using $z = e^{\tau+i\sigma}$ and glue the regions together using (holomorphic) transition functions. This is illustrated in Fig. 4.3.

If we take the boundaries $C_i$ to points (the local $\tau$ coordinate goes to $-\infty$), we can associate the state on the boundary $C_i$ to a local operator $\phi(z_i,\bar{z}_i)$ .

We choose our parameter $v^a$ to be:

- zero on all $C_i$, except at $C_\omega$, which is the boundary associate to the operator at $z_i = \omega$, i.e. $\delta_v\phi_i = 0$ except for $\delta_v\phi(\omega,\overline{\omega})$,
- of the form $v^a = (v(z), \overline{v}(\overline{z}))$,
- arbitrary in the interior of $\Sigma$.

For this choice of $v^a$, we have

$$\frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\,(\partial_a v_b)T^{ab} = \langle \phi_1 \dots \delta\phi(\omega,\overline{\omega}) \dots \phi_n\rangle. \tag{4.39}$$

To put it another way, we look at a transformation where only one field changes for simplicity. We can rewrite the left-hand side of (4.39) as

$$\frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\,\partial^a(v^b\langle T_{ab}(\sigma)\phi_1 \dots \phi_n\rangle) - \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\, v^b\partial^a\langle T_{ab}(\sigma)\phi_1\dots\phi_n\rangle. \tag{4.40}$$

The first term is a boundary term ($\partial\Sigma = \bigcup_i C_i$). The only boundary contribution comes from $C_\omega$, where $v^a$ has holomorphic ($v(z)$) and antiholomorphic ($\overline{v}(\overline{z})$) components and is

$$\frac{1}{2\pi i}\oint_{C_\omega}\mathrm{d}z\, v(z)\langle T(z)\phi_1\dots\phi_n\rangle - \frac{1}{2\pi i}\oint_{C_\omega}\mathrm{d}\overline{z}\,\langle\overline{T}(\overline{z}))\phi_1\dots\phi_n\rangle, \tag{4.41}$$

where $T_{zz}(z) = T(z)$ and $T_{\overline{zz}}(\overline{z}) = \overline{T}(\overline{z})$. Thus

$$\begin{aligned}\langle\phi_1\dots\delta_v\phi(\omega,\overline{\omega})\dots\phi_n\rangle = &-\frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\, v^b\partial^a\langle T_{ab}\phi_1\dots\phi_n\rangle \\ &+\frac{1}{2\pi i}\oint_{C_\omega}\mathrm{d}z\, v(z)\langle T(z)\phi_1\dots\phi_n\rangle + \frac{1}{2\pi i}\oint \mathrm{d}\overline{z}\,\langle\overline{T}(\overline{z})\phi_1\dots\phi_n\rangle\end{aligned} \tag{4.42}$$

The $v$'s appear in very different ways here. In the term on the left hand side and the other two terms on the right, $v$ appears as a puncture. However, the first term on the right is different. But $v^a$ is arbitrary in the interior of $\Sigma$, so we must have

$$\partial^a\langle T_{ab}(\sigma)\phi_1\dots\phi_n\rangle = 0. \tag{4.43}$$

This is the analogue of the classical (Noether) statement $\partial^a T_{ab} = 0$.

All fields $\phi_i \neq \phi(\omega,\overline{\omega})$ did not contribute to the calculation and just came along for the ride. We conclude that we have the general operator statement

$$\boxed{\delta_v\phi(\omega,\overline{\omega}) = \oint_{C_\omega}\frac{\mathrm{d}z}{2\pi i}v(z)T(z)\phi(\omega,\overline{\omega}) - \oint_{C_\omega}\frac{\mathrm{d}\overline{z}}{2\pi i}\overline{v}(\overline{z})T(\overline{z})\phi(\omega,\overline{\omega})} \tag{4.44}$$

This is understood to hold as an equality once inserted into a correlation function.

We can start to see why it was so important to go to complex coordinates. By the residue theorem, the contributions on the right-hand side come from points where the functions on the right have poles. The singularity structure of operators as we bring them close together contains information about the physics. It will turn out that the local structure of the theory is contained within these singularities; not all singularities are bad!

## 4.3 Radial Ordering

Our task now is to understand better the singularity structure of these operators. Let us go back to the cylindrical worldsheet of Fig. 4.1. Mapping to the complex plane with $z = e^{\tau + i\sigma}$ in Fig. 4.2, we see that the notion of time ordering $\tau_1 > \tau_2$ translates into radial ordering $|z_1| > |z_2|$ .

**Definition 21** (radial ordering)**:** We introduce the notion of *radial ordering*:

$$\boxed{\mathcal{R}(A(z)B(w)) := \begin{cases} A(z)B(w), & \text{if } |z| > |w| \\ B(w)A(z), & \text{if } |z| < |w| \end{cases}} \tag{4.45}$$

We will eventually work our way towards Wick's theorem for radial ordered products.

Much of the discussion will be familiar from QFT although the setting will be different and the machinery of CFT will be much more potent.

A novel thing in two dimensions is that the fields transform in 2-dimensions as a contour integral. It will be quite common to see integrals around loops enclosing a pole, so we want to make sure those integrals are well-defined. In other words, we want to make sense of the expression

$$\oint_{C(w)} dz\mathcal{R}\big(A(z)B(w)\big), \tag{4.46}$$

where $C(w)$ is a contour around the point $z = w$.

The difficulty is that different points of the contour, illustrated in Fig. 4.4, are different distance from $w$; in general, $|z| > |w|$ on some parts of $C(w)$ and $|z| < |w|$ on other parts. Since $|z| > |w|$

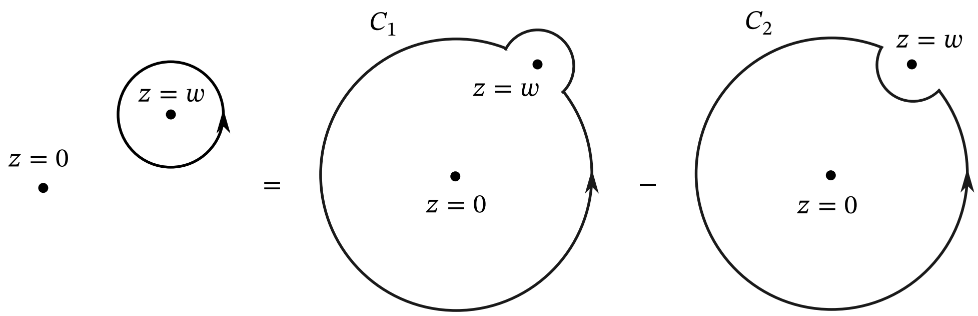


Figure 4.4

on $C_1$ and $|z| < |w|$ on $C_2$, we have

$$\oint_{C(w)} dz\mathcal{R}\big(A(z)B(w)\big) = \oint_{C_1} \mathrm{d}z\, A(z)B(w) - \oint_{C_2} \mathrm{d}z\, B(w)A(z). \tag{4.47}$$

In particular, consider

$$\mathcal{O} = \oint \mathrm{d}z\, A(z), \qquad B(w) \tag{4.48}$$

then

$$\oint_{C(w)} \mathrm{d}z\, \mathcal{R}\big(A(z)B(w)\big) := [\mathcal{O}, B(w)]. \tag{4.49}$$

This plays the role of a commutator in our radially ordered theory. If we consider the variation of a chiral primary field $\phi(w)$

$$\delta_v\phi(w) = \oint_{C(w)} \frac{\mathrm{d}z}{2\pi i}\mathcal{R}\big(v(z)T(z)\phi(w)\big) \tag{4.50}$$

$$= \oint_{|z|>|w|} \frac{\mathrm{d}z}{2\pi i} v(z)T(z)\phi(w) - \oint_{|z|<|w|} \frac{\mathrm{d}z}{2\pi i}\phi(w)v(z)T(z) \tag{4.51}$$

$$= [Q, \phi(w)], \tag{4.52}$$

where the charge $Q$ is given by

$$Q = \oint_{C(w)} \frac{\mathrm{d}z}{2\pi i} v(z) T(z). \tag{4.53}$$

This is similar to the Classical Poisson bracket expression for a transformation of a field $\phi(w)$.

The non-trivial contributions to $\delta_v \phi(w)$ will come from the poles of the contour integrals in (4.51).

## 4.4 Mode Expansions and Conformal Weights

This is a very brief aside that will give use the notation we need to evaluate these contour integrals. We are interested in the map (4.8) from the cylindrical worldsheet to the complex plane. On the cylinder, the mode expansion for a field $\phi(\tau, \sigma)$ might naturally be written as $\phi_{\text{cyl}}(w) = \sum_n \phi_n e^{-nw}$, where $w = \tau + i\sigma$ and similarly for a field with $\overline{w}$-dependence. Suppose $\phi_{\text{cyl}}$ has conformal weight $h$ (i.e. $\phi_{\text{cyl}}$ is a chiral primary). If we now transform to the complex plane with $z = e^w = e^{\tau + i\sigma}$,

$$\phi_{\text{cyl}}(w) \to \phi(z) = \left(\frac{\partial z}{\partial w}\right)^{-h} \phi_{\text{cyl}}(w) = z^{-h} \phi_{\text{cyl}}(w). \tag{4.54}$$

Therefore, on the complex plane with a Euclidean metric, the natural mode expansion of chiral primary fields is

$$\boxed{\phi(z) = \sum_n \phi_n z^{-n-h}}. \tag{4.55}$$

The only thing that has changed is the newly introduced exponent $-h$.

More generally, for a primary field of weight $(h, \overline{h})$, the natural mode expansion is

$$\boxed{\phi(z, \overline{z}) = \sum_{n,m} \phi_{nm} z^{-n-h} \overline{z}^{-m-\overline{h}}}. \tag{4.56}$$

**Example 4.4.1:** For example, we will see that the stress tensor has weight $(h, \overline{h}) = (2, 0)$ for $T(z)$ and $(0, 2)$ for $\overline{T}(\overline{z})$.

$$T(z) = \sum_n L_n z^{-n-2}, \qquad \overline{T}(\overline{z}) = \sum_n \overline{L}_n \overline{z}^{-n-2}. \tag{4.57}$$

**Example 4.4.2:** For reference,

$$X^\mu(z, \overline{z}) = x^\mu - i\frac{\alpha'}{2} p^\mu \ln |z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^\mu z^{-n} - \overline{\alpha}_n^\mu \overline{z}^{-n}\right). \tag{4.58}$$

The holomorphic derivative has a nice Laurent expansion

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu z^{n-1}, \tag{4.59}$$

where $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$. So $\partial X^\mu$ looks to be a conformal field of weight $(1, 0)$, but we will derive this later.

## 4.5 Radial Ordering and Normal Ordering

In QFT, we learned that Wick's theorem relates time-ordered and normal-ordered products of operators. The role of time-ordering is played here by radial ordering. For simplicity, let us consider the weight-$(1,0)$ chiral field $j^\mu(z) = \sum_n \alpha_n^\mu z^{-n-1}$. The reason we use the $\alpha_n^\mu$ as the modes is that we will eventually think of $j^\mu(z)$ as

$$j^\mu(z) = i\sqrt{\frac{2}{\alpha'}}\partial X^\mu(z). \tag{4.60}$$

Since $\alpha_n^\mu |0\rangle = 0$ for $n \geq 0$, we can split $j^\mu(z)$ into creation and annihilation modes:

$$j^\mu(z) = j_+^\mu(z) + j_-^\mu(z), \qquad j_+^\mu(z) = \sum_{n>=0} \alpha_n^\mu z^{-n-1} \quad \text{etc.} \tag{4.61}$$

**Definition 22** (normal ordering)**:** We define *normal ordering* $:\mathcal{O}:$ as moving all creation operators to the left in any string of operators $\mathcal{O}$.

In this case in particular, this means that

$$:j^\mu(z)j^\nu(w): = j_+^\mu(z)j_-^\nu(w) + j_-^\mu(z)j_-^\nu(w) + j_+^\mu(z)j_+^\nu(w) + j_-^\mu(z)j_-^\nu(w). \tag{4.62}$$

Note that the relationship between the normal ordered product and the general product

$$:j^\mu(z)j^\nu(w): = j^\mu(z)j^\nu(w) + [j_-^\nu(w), j_+^\mu(z)], \tag{4.63}$$

involves a commutator of our chiral primary. In QFT, we associated this commutator with a contraction or propagator of fields. We can evaluate this commutator by using the mode expansion: We know the algebra of the $\alpha_n^\mu$, so

$$[j_-^\mu(w), j_+^\nu(z)] = \sum_{\substack{m\geq 0\\ z>0}} [\alpha_{-m}^\mu, \alpha_n^\nu] w^{m-1} z^{-n-1} \tag{4.64}$$

$$= -\sum_{m,n} n\delta_{m,n}\eta^{\mu\nu} w^{m-1} z^{-n-1} \tag{4.65}$$

$$= -\frac{\eta^{\mu\nu}}{z^2}\sum_{n>0} n\left(\frac{w}{z}\right)^{n-1}. \tag{4.66}$$

This converges if $|z| > |w|$, giving

$$[j_-^\mu(w), j_+^\nu(z)] = \frac{\eta^{\mu\nu}}{(z-w)^2}. \tag{4.67}$$

As such, the normal ordered product for the $(h, \overline{h}) = (1, 0)$ chiral primary field $j(z)$ is

$$j^\mu(z)j^\nu(w) = :j^\mu(z)j^\nu(w): + \frac{\eta^{\mu\nu}}{(z-w)^2}, \qquad |z| > |w|. \tag{4.68}$$

This requirement that $|z| > |w|$ tells us that there is some connection between normal and radial ordering.

Similarly, swapping $z \leftrightarrow w$ and $\mu \leftrightarrow \nu$ in (4.68),

$$j^\nu(w) j^\mu(z) = \,:j^\mu(z) j^\nu(w):\, + \frac{\eta^{\mu\nu}}{(z-w)^2}, \qquad |z| < |w|. \tag{4.69}$$

Putting these together,

$$\mathcal{R}(j^\mu(z) j^\nu(w)) = \,:j^\mu(z) j^\nu(w):\, + \frac{\eta^{\mu\nu}}{(z-w)^2}. \tag{4.70}$$

We often write

$$\overbracket{j^\mu(z) j}^\nu(w) = \frac{\eta^{\mu\nu}}{(z-w)^2} \tag{4.71}$$

and refer to this as a *contraction* between $j^\mu(z)$ and $j^\nu(w)$.

For example, if we write $\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}}\, j^\mu(z)$, we have that

$$\partial_z \overbracket{X^\mu(z) \partial_w X}^\nu(w) = -\frac{\lambda'}{2} \frac{\eta^{\mu\nu}}{(z-w)^2}. \tag{4.72}$$

Integrating

$$\boxed{\overbracket{X^\mu(z) X}^\nu(w) = -\frac{\alpha'}{2} \eta^{\mu\nu} \ln(z-w)} \tag{4.73}$$

and there is of course an analogous statement for the antiholomorphic sector. Here we are splitting $X^\mu(z,\bar{z}) = X^\mu(z) + \bar{X}^\mu(\bar{z})$.

**Remark:** $\partial\bar{\partial} X^\mu(z,\bar{z}) = 0$

The generalisation of these statements are given by *Wick's theorem*:

$$\begin{aligned}
\mathcal{R}l(\phi_1(z_1) \dots \phi_n(z_n)) = \,&:\phi_1(z_1) \dots \phi_n(z_n): && (4.74)\\
&+ \sum_{(i,j)} :\phi_1(z_1) \dots \overbracket{\phi_i(z_i) \dots \phi_j}(z_j) \dots \phi_n(z_n): && (4.75)\\
&+ \sum_{(i,j)(k,l)} :\phi_1(z_1) \dots \overbracket{\phi_i(z_i) \dots \phi_j}(z_j) \dots \overbracket{\phi_k(z_k) \dots \phi_l}(z_l) \dots \phi_n(z_n): && (4.76)\\
&+ \dots\,. && (4.77)
\end{aligned}$$

## 4.6 Operator Product Expansions

Given a set of local operators $\mathcal{O}_i(w)$, the operator product expansion (OPE) characterises the behaviour of the theory at short distances. In particular, we are interested in how the operators behave when we bring them close together:

$$\lim_{w \to z} \mathcal{O}_i(w) \mathcal{O}_j(z). \tag{4.78}$$

The operator product expansion is a description of this composite operator in terms of the operators in the theory. In other words, we can think of it as

$$\lim_{z\to w} \mathcal{O}_i(w)\mathcal{O}_j(z) = \sum_k f_{ij}{}^k(z-w)\mathcal{O}_k(z). \tag{4.79}$$

We want to extract the singular terms of this limit.

For example, we could look at the following radial ordering operator:

$$\mathcal{R}\left(\partial X^\mu(z)\partial X^\nu(w)\right) = \,:\partial X^\mu(z)\partial X^\nu(w): -\frac{\lambda'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2}. \tag{4.80}$$

As $z \to w$, the first term is regular, while the second term is singular (and therefore contains the bits we are interested in!) We may write this as

$$\mathcal{R}\left(\partial X^\mu(z)\partial X^\nu(w)\right) = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2} + \dots, \tag{4.81}$$

where the dots denote the regular terms. More often than not, we will assume that radial ordering is implicitly applied, unless otherwise stated.

We could compute the correlation function (where radial ordering is understood)

$$\langle X^\mu(z)X^\nu(w)\rangle = \langle :X^\mu(z)X^\nu(w):\rangle - \frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-w), \tag{4.82}$$

$$= -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-w), \tag{4.83}$$

where we used that, by design, the vacuum expectation value of any normal ordered quantity will vanish. This is the Green's function for $\frac{1}{2\pi\alpha'}\partial\overline{\partial}$.

Some useful operator product expansions are

$$X^\mu(z)X^\nu(w) = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(z-w), \qquad \overline{X}^\mu(\overline{z})\overline{X}^\nu(\overline{w}) = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(\overline{z}-\overline{w}). \tag{4.84}$$

The operator product expansion of $X^\mu(z)\overline{X}^\nu(\overline{w})$ is regular.

We can use the knowledge of the $X^\mu$ operator product expansion to define composite operators such as the stress tensor, which, classically, is $T(z) = -\frac{1}{\alpha'}\partial X^\mu(z)\partial X_\mu(z)$. Using the operator product expansion (4.81) we define the composite operator as

$$T(z) = -\frac{1}{\alpha'}\lim_{m\to z}\left(\partial X^\mu(w)\partial X_\mu(z) + \frac{\alpha'}{2}\frac{D}{(z-w)^2}\right), \tag{4.85}$$

where the dimension $D = \eta_{\mu\nu}\eta^{\mu\nu}$.

## 4.7 $T(z)X^\mu(w)$ Operator Product Expansion and Conformal Transformations

Consider $T(z)X^\mu(w) = -\frac{1}{\alpha'} :\partial X^\nu \partial X_\nu(z): X^\mu(w)$. We use Wick's theorem and the operator product expansion

$$\partial X^\mu(z) X^\nu(w) = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{z-w} + \dots \tag{4.86}$$

to get

$$\begin{aligned} T(z)X^\mu(w) &= -\frac{2}{\alpha'} :\partial X^\nu \partial X_\nu(z): X^\mu(w) && (4.87)\\ &= -\frac{2}{\alpha'}\partial X_\nu(z)\left(-\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{z-w}\right) + \dots && (4.88)\\ &= \frac{\partial X^\mu(z)}{z-w} + \dots && (4.89) \end{aligned}$$

We could expand $\partial X^\mu(z)$ about $z = w$ as $\partial X^\mu(z) = \partial X^\mu(w) + (z-w)\partial^2 X^\mu(w) + \dots$. Then we have

$$T(z)X^\mu(w) = \frac{\partial X^\mu(w)}{z-w} + \dots . \tag{4.90}$$

We can compute the conformal transformation of $X^\mu(w)$:

$$\delta_v X^\mu(w) = \oint_{z=w} \frac{\mathrm{d}z}{2\pi i} v(z) T(z) X^\mu(w) = \oint_{z=w} \frac{\mathrm{d}z}{2\pi i} v(z) \frac{\partial X^\mu(w)}{z-w} = v(w)\partial X^\mu(w). \tag{4.91}$$

More generally, we expect a weight $(h, 0)$ primary field $\phi(z)$ to transform as

$$\delta_v \phi(z) = v(z)\partial\phi(z) + h\partial v(z)\phi(z), \tag{4.92}$$

at which point it is very tempting to call $X^\mu(z)$ a chiral primary of weight $h = 0$. However, we will see later that it is more subtle than that; there is something deeply non-local about the $X$'s. Here we used the residue theorem:

$$\frac{1}{(n-1)!}\partial_z^{n-1} f(z) = \oint_{w=z} \frac{\mathrm{d}w}{2\pi i}\frac{f(w)}{(w-z)^n}. \tag{4.93}$$

**Claim 2:** This requirement fixes the $T(w)\phi(z)$ operator product expansion to have the form

$$\boxed{T(w)\phi(z) = \frac{h}{(z-w)^2}\phi(z) + \frac{1}{z-w}\partial\phi(z) + \dots} \tag{4.94}$$

*Proof.* Exercise. □

From now on, we shall take this OPE (4.94) as the definition for a field $\phi(z)$ to be a chiral primary of weight $(h, 0)$.

We will be interested in OPEs for composite operators like $T(z)e^{ik\cdot X(w))}$ or what you might consider a consistency conditions $T(z)T(w) =?$. Given $T$ lies at the heart of what it means for a theory to be conformally invariant, we want it to transform in a certain way. This will have interesting repercussions for the spacetime in our theory.

Today we will look at how another pair of operators transforms.

### 4.7.1 $T(z) :e^{ik_\mu X^\mu(w)}:$ OPE

Our composite field is $:\exp\big(ik_\mu X^\mu(w)\big):$, where $k_\mu$ is some constant spacetime (i.e. target space) momentum vector. This will be a fundamental ingredient to any scattering amplitude, where $k_\mu$ is the centre-of-mass momentum of the string.

It is useful to write this

$$T(z) :e^{ik\cdot X(w)}: = T(z) \sum_{k\geq 0} \frac{i^n}{n!} k_{\mu_1} \dots k_{\mu_n} :X^{\mu_1}(w) \dots X^{\mu_n}(w):, \tag{4.95}$$

where $T(z) = -\frac{1}{\alpha'} :\partial X^\mu(z) \cdot \partial X_\mu(z):$. In the free theory, we can work this out exactly using Wick's theorem. The normal ordering reminds us that there are no internal contractions, only ones between the $\partial X$'s in $T$ and the $X$'s in $e^{ik\cdot X}$.

We shall make use of the OPE (4.86). There are two types of contribution to the OPE:

**Single Contractions:** These contribute

$$-\frac{2}{\alpha'} :\partial X(z) \cdot \partial X(z): \sum_{n\geq 0} \frac{i^n}{n!} k_{\mu_1} \dots k_{\mu_n} :X^{\mu_1}(w) \dots X^{\mu_n}(w): \tag{4.96}$$

where the factor 2 comes from the two ways to contract any given $X^{\mu_i}$ with any of the two $\partial X$. Of course, we get $n$ such terms, one for each $X^{\mu_i}$:

$$\begin{aligned} \dots &= \sum_{n>0} \frac{i^n}{n!} n (k\cdot X(w))^{n-1} k_\nu \left(\frac{\partial X^\nu(w)}{z-w}\right) && (4.97)\\ &= \sum_{n\geq 0} \frac{i^n}{n!} (k\cdot X(w))^n i k_\nu \left(\frac{\partial X^\nu(w)}{z-w}\right) && (4.98)\\ &= \frac{1}{z-w} \partial\left(e^{ik\cdot X(w)}\right). && (4.99)\end{aligned}$$

**Double Contractions:** These contribute

$$\begin{aligned} &-\frac{1}{\alpha'} :\partial X\cdot\partial X(z): \sum_{(i,j)}\sum_{n\geq 0} \frac{i^n}{n!} k_{\mu_1}\dots k_{\mu_i}\dots k_{\mu_j}\dots k_{\mu_n} :X^{\mu_1}(w)\dots X^{\mu_i}(w)\dots X^{\mu_j}(w)\dots X^{\mu_n}(w): \\ &= -\frac{1}{\alpha'} \sum_{n\geq 2} k_{\mu_2}\dots k_{\mu_{n-1}} \frac{i^n}{n!} n(n-1) X^{\mu_2}(w)\dots X^{\mu_n} \left(-\frac{\alpha'}{2}\right)^2 \frac{k^2}{(z-w)^2} && (4.100)\\ &= -\frac{\alpha'}{4}\frac{k^2}{(z-w)^2} :\sum_{n\geq 2} (k\cdot X(w))^{n-2}: i^2 i^{n-2} \frac{n!}{n!(n-2)!} && (4.101)\\ &= \frac{\alpha'}{4}\frac{k^2}{(z-w)^2} :e^{ik\cdot X(w)}: && (4.102)\end{aligned}$$

This is the second worst OPE you will have to do in this course.

This gives:

$$T(z) :e^{ik\cdot X(w)}: = \left(\frac{\alpha' k^2/4}{(z-w)^2} + \frac{\partial}{z-w}\right) e^{ik\cdot X(w)} + \dots . \tag{4.103}$$

From this we read off that $h = \frac{\alpha' k^2}{4}$. It is easy to see that a similar result holds for $\overline{T}(\overline{z}) e^{ik\cdot \overline{X}(\overline{w})}$ and more generally the operator $:\exp\bigl(ik_\mu X^\mu(w,\overline{w})\bigr):$ has conformal weight

$$\boxed{(h,\overline{h}) = \left(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4}\right)} \tag{4.104}$$

There is a sense in which this is a purely quantum mechanically result since our string coupling $\alpha'$ appears explicitly in exactly the same way $\hbar$ would appear. The other thing to note is that these numbers depend on $k^2$.

### 4.7.2 The $T(z)T(w)$ OPE and the Virasoro Algebra

This is basically the one field we cannot do without in a conformal field theory, so if it does not transform in the correct way we have a problem. We use

$$T(z) = -\frac{1}{\alpha'} :\partial X \cdot \partial X(z): \qquad \text{and} \qquad \partial X^\mu(z)\partial X^\nu(w) = -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2} + \dots \tag{4.105}$$

$$\begin{aligned} T(z)T(w) = {} & 4\left(-\frac{1}{\alpha'}\right) :\partial X^\mu \partial X_\mu(z): :\partial X^\nu \partial X_\nu(w): \\ & + 2\left(-\frac{1}{\alpha'}\right)^2 :\partial X_\mu \partial X^\mu(z): :\partial X^\nu \partial X_\nu(w): \end{aligned} \tag{4.106}$$

$$= -\frac{2}{\alpha'}\frac{\eta_{\mu\nu}}{(z-w)^2} :\partial X^\mu(z)\partial X^\nu(w): + \frac{1}{2}\frac{\delta^\mu{}_\nu}{(z-w)^2}\frac{\delta^\nu{}_\mu}{(z-w)^2}. \tag{4.107}$$

For the first term we can write

$$\partial X^\mu(z) = \partial X^\mu(w) + (z-w)\partial^2 X^\mu(w) + \dots \tag{4.108}$$

In the second term, we recognising the dimension of spacetime as $D = \operatorname{tr}(\eta_{\mu\nu})$. Then we have

$$T(z)T(w) = \frac{D/2}{(z-w)^4} - \frac{2}{\alpha'}\frac{1}{(z-w)^2} :\partial X_\mu \partial X^\mu(w): - \frac{2}{\alpha'}\frac{1}{(z-w)} :\partial X_\mu \partial^2 X^\mu(w): \tag{4.109}$$

$$\boxed{T(z)T(w) = \frac{D/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \dots} \tag{4.110}$$

This implicitly assumes radial ordering on the left-hand side.

The second term gives us the weight $h = 2$. However, the first term should not be there; we do not expect to see a pole of order 4. The stress tensor only transforms as a conformal field as we expect if $D = 0$. It will turn out that what we have been studying so far is only part of the stress tensor. We have forgotten about the ghosts, which will give a contribution of the stress tensor as well! Their contribution will allow for $D$ to be a finite integer, which is therefore determined by the consistency of the theory.

## 4.8 The Virasoro Algebra

We talked about the Witt algebra and saw that it had something to do with conformal transformations. The Witt algebra came from purely classical consideration. The TT OPE was obtained by including quantum mechanical effects, like double contractions.

The stress tensor is a field of weight 2 so we can expand it in terms of Laurent modes $L_n$:

$$T(z) = \sum_n L_n z^{-n-2}. \tag{4.111}$$

This may be inverted to give

$$L_n = \oint_{z=0} \frac{\mathrm{d}z}{2\pi i} z^{n+1} T(z) \tag{4.112}$$

and similarly for the tensor $\overline{T}(\overline{z})$ and its modes $\overline{L}_n$.

Let us consider the commutator of two of the Laurent modes

$$[L_m, L_n] = \oint_{z=0} \frac{\mathrm{d}z}{2\pi i} \oint_{w=0} \frac{\mathrm{d}w}{2\pi i} z^{m+1} w^{n+1} [T(z), T(w)]. \tag{4.113}$$

We have argued that we should think of

$$\oint_{z=0} \frac{\mathrm{d}z}{2\pi i} z^{m+1} [T(z), T(w)] = \oint_{z=w} \frac{\mathrm{d}z}{2\pi i} \mathcal{R}(T(w)T(w)) z^{m+1}. \tag{4.114}$$

Therefore, the commutator (4.113) becomes

$$[L_m, L_n] = \oint_{w=0} \frac{\mathrm{d}w}{2\pi i} w^{n+1} \oint_{z=w} \frac{\mathrm{d}z}{2\pi i} \mathcal{R}\big(T(z)T(w)\big). \tag{4.115}$$

We use the $TT$ operator product expansion to evaluate this

$$[L_m, L_n] = \oint_{w=0} \frac{\mathrm{d}w}{2\pi i} w^{n+1} \oint_{z=w} \frac{\mathrm{d}z}{2\pi i} z^{m+1} \left( \frac{D/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \right). \tag{4.116}$$

Noting that

$$\oint_{z=w} \frac{\mathrm{d}z}{2\pi i} \frac{z^{m+1}}{(z-w)^2} = (m+1) w^m \tag{4.117}$$

$$\oint_{z=w} \frac{\mathrm{d}z}{2\pi i} \frac{z^{m+1}}{(z-w)^4} = \frac{1}{3!} (m+1) m (m-1) w^{m-2}, \tag{4.118}$$

we have

$$[L_m, L_n] = \oint_{w=0} \frac{\mathrm{d}w}{2\pi i} w^{n+1} \left( \frac{D}{12} m(m^2-1) w^{m-2} + 2T(w)(m+1) w^m + \partial T(w) w^{m+1} \right) \tag{4.119}$$

$$\boxed{[L_m, L_n] = \frac{D}{12} (m^2-1) \delta_{m+n,0} + (m-n) L_{m+n}} \tag{4.120}$$

We obtain what we expect from the classical Witt algebra, plus a *central extension*, which is also often called an *anomaly*. This is called the *Virasoro algebra*. The violation of conformal invariance, which goes away for $D = 0$, is quite implicit in this Lie algebra. Similarly for $[\overline{L}_m, \overline{L}_n]$ and the $T(z)\overline{T}(w)$ OPE is regular, so

$$[L_m, \overline{L}_n] = 0. \tag{4.121}$$

There is an inconsistency in the theory for $D \neq 0$.

## 4.9 The $b, c$ Ghost System

We have not been considering the whole of the theory. The Fadeev–Popov procedure required the introduction of (anti-commuting) ghost fields: $b_{ab}(z)$ and $c^a(z)$:

$$c^a(z) c^b(w) = -c^b(w) c^a(z). \tag{4.122}$$

The ghost action is

$$S_{\text{gh}}[b,c] = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}h^{ac}b_{ab}(\nabla_c c^b) \tag{4.123}$$

so the total action is $S = S_{\text{Polyakov}}[X] + S_{\text{gh}}[b,c]$. There will be a contribution to the stress-energy tensor from the ghosts. We would like the theory to have conformal symmetry, so we expect $b$ and $c$ to transform in some non-trivial way under those. Since the stress tensor generates these transformations, we expect to have some ghost stress tensor.

**Claim 3:** The ghost stress tensor is

$$(T_{\text{gh}})_{ab} = -\frac{1}{2}c^c\nabla_{(c}b_{b)c} - (\nabla_{(a}c^c)b_{b)c} - \text{trace} \tag{4.124}$$

Since the field theory describing the $X$'s, and the one describing the $b$'s and $c$'s are independent, the total stress tensor is schematically

$$\mathcal{T} = T_X + T_{\text{gh}}, \tag{4.125}$$

where $T_X$ is the stress tensor from the Polyakov action. Working with a Euclidean metric on $\Sigma$ and complex coordinates as before, we can write the two degrees of freedom in $b_{ab}$, which is traceless and symmetric, as $b(z)$ and $\overline{b}(\overline{z})$. Similarly, we write the two degrees of freedom in $c^a$ as $c(z)$ and $\overline{c}(z)$. This latter splitting behaviour can be seen rather directly when considering that the action becomes

$$S_{\text{gh}}[b,c] = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2z\, b\overline{\partial}c + \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2z\,\overline{b}\partial\overline{c}. \tag{4.126}$$

**Remark:** This sector of the theory does not know about $\alpha'$. It also does not depend in any way on the spacetime metric. The ghosts came from dealing with the infinite overcounting in the path integral; nothing to do with the metric.

The $b$, and $\overline{b}$ equations of motion just give $\overline{\partial}c = 0$ and $\partial\overline{c} = 0$, which is why the $c^a$ also split into the two sectors claimed above. The total stress tensor is $\mathcal{T}(z) = T_x(z) + T_{\text{gh}}(z)$, where

$$T_X(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu(z): \tag{4.127}$$

$$T_{\text{gh}}(z) = :(\partial b)c(z): - 2\partial(:bc(z):) \tag{4.128}$$

with analogous expressions for $\overline{T}_X(\overline{z})$ and $\overline{T}_{\text{gh}}(\overline{z})$. We will need to compute the OPE of $T_{\text{gh}}$ with itself. The ghost sector does not know anything about the spacetime, so we do not expect a term $D$ in a pole of order 4 to pop up. Instead, there could be a fixed number, which we interpret as the critical dimension of the theory.

**Remark:** We will show that the OPE between $T_{\text{gh}}$ and $T_X$ will vanish, which makes sense since the two sectors are independent.

### 4.9.1 The $b, c$ OPE

The 2-point function is given by Wick's theorem

$$\langle b(z)c(w)\rangle = \langle :b(z)c(w):\rangle + \contraction{}{b}{(z)}{c}b(z)c(w) \tag{4.129}$$

$$= \contraction{}{b}{(z)}{c}b(z)c(w). \tag{4.130}$$

This 2-point function, which is a Greens function of the $b\bar\partial c$ part of the ghost action, gives the singular part of the $b(z)c(w)$ OPE.

$$b(z)c(w) = \contraction{}{b}{(z)}{c}b(z)c(w) + \ldots \tag{4.131}$$

The classical Green's function for the operator $\bar\partial$ on the (Riemann) sphere is $(z-w)^{-1}$, so our basic OPE is

$$\boxed{b(z)c(w) = \frac{1}{z-w} + \cdots = c(z)b(w)} \tag{4.132}$$

**Remark:** We could have done the same thing for the Laplacian to work out the $X$-OPE; the Laplacian Green's function in two dimensions goes as the natural logarithm of the separation, which is the first term of the $X$-OPE.

**Remark:** The $cc$ and $bb$ operator product expansions are trivial, since there are no propagators between them in the action.

### 4.9.2 Conformal weight of b(z)

The stress tensor is $T = (\partial b)c - 2\partial(bc)$ and so the operator product expansion we want is

$$T(z)\,b(w) = (\partial b(z))\,\contraction{}{c}{(z)}{b}c(z)b(w) - 2\partial_z(b(z)\,\contraction{}{c}{(z))\,}{b}c(z))\,b(w) + \ldots \tag{4.133}$$

$$= \frac{\partial b(z)}{z-w} - 2\partial_z\left(\frac{b(z)}{z-w}\right) + \ldots \tag{4.134}$$

$$= \frac{\partial b(z)}{z-w} - 2\frac{\partial b(z)}{z-w} + 2\frac{b(z)}{(z-w)^2} + \ldots \tag{4.135}$$

$$= -\frac{\partial b(z)}{z-w} + 2\frac{b(z)}{(z-w)^2} + \ldots \tag{4.136}$$

Using $b(z) = b(w) + 2(z-w)\partial b(w) + \ldots$,

$$T(z)b(w) = -\frac{\partial b(w)}{z-w} + 2\frac{b(w)}{(z-w)^2} + 2\frac{\partial b(w)}{(z-w)} + \ldots \tag{4.137}$$

and so

$$\boxed{T_{\text{gh}}(z)b(w) = \frac{2b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w} + \ldots} \tag{4.138}$$

We conclude $b(w)$ has weight $h = 2$.

**Exercise 4.2:** Similarly, we could compute

$$T_{\text{gh}}(z)c(w) = -\frac{1}{(z-w)^2} + \frac{1}{(z-w)}\partial c(w) + ... \tag{4.139}$$

to find that $c(w)$ has weight $(h, \overline{h}) = (-1, 0)$.

These results seem intuitive. We might think of $b_{ab}$ as a metric (but with the wrong statistics) and $c^a$ as a vector field (with the wrong statistics).

### 4.9.3 $T_{\mathbf{gh}}(z)T_{\mathbf{gh}}(w)$ OPE

We can now find $T_{\text{gh}}(z)T_{\text{gh}}(w)$. Proceeding in exactly the same way as the TT-OPE (4.110), we have

$$T_{\text{gh}}(z)T_{\text{gh}}(w) = -\frac{26/2}{(z-w)^4} + \frac{2}{(z-w)^2}T_{\text{gh}}(w) + \frac{1}{(z-w)}\partial T_{\text{gh}}(w) + \dots \tag{4.140}$$

So $T_{\text{gh}}(w)$ has weight $h = 2$. Combining this with (4.110), we find that the OPE of total stress tensor $\mathcal{T}(z) = T_X(z) + T_{\text{gh}}(z)$ is

$$\boxed{\mathcal{T}(z)\mathcal{T}(w) = \frac{(D-26)/2}{(z-w)^4} + \frac{2}{(z-w)}\mathcal{T}(w) + \frac{1}{(z-w)}\partial\mathcal{T}(w) + \dots} \tag{4.141}$$

We see that if $D = 26$, the anomaly term (unwanted pole of order 4) vanishes, and we have a consistent quantum conformal theory on our worldsheet $\Sigma$.

This would have been more complicated if the two sectors interacted. Moreover, if there were other fields in our theory, a different value of $D$ might make this vanish. In particular, if we made the theory supersymmetric (to cure the tachyon in our spectrum), adding in worldsheet fermions, we would obtain other contributions to the stress tensor. Doing the calculation, we would obtain $D = 10$.

There are three possible reactions we can have here. One approach is to say, "Well, nice try. This is obviously nonsense, so back to the drawing board it is."

Another approach is to consider the possibility of higher dimensions. Other theories do not really answer the question of why the universe has the dimensionality it has. (Or in the case of string theory, explaining why the universe has the dimensionality that it has not.) This makes fruitful contact with Kaluza–Klein theory.

Alternatively, in quantum theory, one may think about the dimensions of spacetime to be emergent. An example of this is the AdS/CFT correspondence.

### 4.9.4 Mode Expansions for Ghosts

Having found the weights of the ghosts, we have the mode expansions

$$b(z) = \sum_n b_n z^{-n-2}, \qquad c(z) = \sum_n c_n z^{-n+1}. \tag{4.142}$$

We can invert these to get expressions for the modes

$$b_n = \oint \frac{\mathrm{d}z}{2\pi i} z^{n+1} b(z), \qquad c_n = \oint \frac{\mathrm{d}z}{2\pi i} z^{n-2} c(z). \tag{4.143}$$

Ghosts are imposing the classical constraints that the stress tensor must vanish, and keep us on the correct gauge slice, which makes sure that we do not overcount in the Fadeev–Popov determinant. So they are physically relevant objects, although not measurable. However, despite having integer weights (which is the closest thing to spin we can have in two dimensions), ghosts violate the spin-statistics theorem and obey fermionic statistics. To quantise fields, we therefore want to explore anticommutation relations

$$\{b_m, c_n\} := b_m c_n + c_n b_m \tag{4.144}$$
$$= \oint_{z=0} \frac{\mathrm{d}z}{2\pi i} z^{m+1} \oint_{w=0} \frac{\mathrm{d}w}{2\pi i} w^{n-2} \{b(z), c(w)\} \tag{4.145}$$
$$= \oint_{z=0} \frac{\mathrm{d}z}{2\pi i} \oint_{w=z} \frac{\mathrm{d}w}{2\pi i} z^{m+1} w^{n-2} \mathcal{R}\big(b(z)c(w)\big), \tag{4.146}$$

where we write the radial ordering explicitly, although it is usually implicitly assumed. Using the $b(z)c(w)$ OPE, we can evaluate this anticommutation relation as

$$\boxed{\{b_m, c_n\} = \delta_{m+n,0}} \tag{4.147}$$

## 4.10 The State-Operator Correspondence

Our physical space of states contains, for example, the tachyon $|k\rangle = e^{ik\times X}|0\rangle$ or the graviton $\epsilon_{\mu\nu}\alpha^{\mu}_{-1}\overline{\alpha}^{\nu}_{-1}|k\rangle$, together with their physical state conditions. On the other hand, we have also been talking about operators, but have not yet explored their connection.

In $2d$ CFT, for each physical state there is a corresponding operator in the operator algebra of the theory. For example,

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_n \alpha^\mu_n z^{-n-1}. \tag{4.148}$$

We can construct a (in this case non-physical) state as follows: Consider the expression

$$\lim_{z\to 0} \partial X^\mu(z)|0\rangle = -i\sqrt{\frac{\alpha'}{2}} \lim_{z\to 0} \sum_n \frac{\alpha^\mu_n}{z^{n+1}}|0\rangle. \tag{4.149}$$

The terms with $n+1<0$ drop out. However, we have potential divergences for $n \geq -1$. However, recall that $\alpha^\mu_n|0\rangle = 0$ for $n \geq 0$. So all that we are left with is the case of $n+1=0$:

$$\lim_{z\to 0} \partial X^\mu(z)|0\rangle = -i\sqrt{\frac{\alpha'}{2}} \alpha^\mu_{-1}|0\rangle. \tag{4.150}$$

It is not too hard to see that we similarly have

$$\lim_{z,\bar{z}\to 0} \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X(z,\bar{z})} |0\rangle = \alpha^\mu_{-1}\bar{\alpha}^\nu_{-1} |k\rangle . \tag{4.151}$$

And it also goes the other way: given a state in the Hilbert space, there is an associated operator. In a quantum theory, this is not always true. However, in our case, we can think of physical states and operators interchangeably.

More generally, if we have a weight $h$ chiral field $\phi(z) = \sum_n \phi_n z^{-n-h}$, then for $\lim_{z\to 0} \phi(z) |0\rangle$ to exist, we require that

$$\boxed{\phi_n |0\rangle = 0 \quad \text{for} \quad n > -h} \tag{4.152}$$

Knowing the weight of a field tells us something about its relation to the vacuum. Conversely, we might say that the weight tells us about the definition of the vacuum for that field. Then

$$\lim_{z\to 0} \phi(z) |0\rangle = \phi_{-n} |0\rangle . \tag{4.153}$$

For example, knowing the weights for the ghosts means that we know the vacuum for the ghosts.

## 4.11 BRST Symmetry

After Fadeev–Popov, we had the action

$$S[X,b,c] = S[X] + S_{\text{gh}}[b,c], \tag{4.154}$$

a sum of the Polyakov action and the ghost action. Importantly, $h_{ab}$ was fixed to $\hat{h}_{ab}$, for example to the Euclidean metric. It is useful to do something a little strange and 'unfix it'. We would like to see how the choice $\hat{h}_{ab}$ does (not) influence the physics. (We hope that the choice does not affect the physics at all.) To that end, we introduce a Lagrange multiplier field $B_{ab}$ and the gauge-fixing term in the action

$$S_{\text{gf}}[h,B] = \frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h} B^{ab}(\hat{h}_{ab} - h_{ab}). \tag{4.155}$$

Since this term appears in the exponential of the partition function path integral, doing a functional integral over the $B$'s gives a delta functional that imposes the gauge-choice $\hat{h}_{ab} = h_{ab}$. As such, we are still making a choice in the spirit of Fadeev–Popov, but have the freedom to take any choice $\hat{h}_{ab}$ that we like, such as the Euclidean metric.

Our action will now be

$$S = S_{\text{Polyakov}}[X,h] + S_{\text{gh}}[b,c,h] + S_{\text{gf}}[B,h]. \tag{4.156}$$

There is a rigid symmetry of this theory. The fields transform under this rigid fermionic symmetry as:

$$\delta_Q X^\mu = i\epsilon c^a \partial_a X^\mu, \tag{4.157}$$

where $\epsilon$ is a Grassmann number (i.e. $\epsilon\widetilde{\epsilon} = -\widetilde{\epsilon}\epsilon$) and $c^a$ is the worldsheet ghost. The metric transforms as

$$\delta_Q h_{ab} = \epsilon(\mathcal{P}c)_{ab}, \tag{4.158}$$

where the operator $\mathcal{P}$, introduced in (3.5) gives the traceless part of the diffeomorphism:

$$(\mathcal{P}c)_{ab} = \nabla_a c_b + \nabla_b c_a - h_{ab}(\nabla_c c^c). \tag{4.159}$$

Moreover, the ghosts transform as

$$\delta_Q c^a = i\epsilon c^b \partial_b c^a \tag{4.160}$$

$$\delta_Q b_{ab} = i\epsilon B_{ab}, \tag{4.161}$$

$$\delta_Q B_{ab} = 0. \tag{4.162}$$

For $X^\mu$ and $h_{ab}$ we see that $\delta_Q$ acts like a diffeomorphism with parameter $v^a = \epsilon c^a$.

Let us first check that this is genuinely a symmetry of our action, before we discuss why we care at all about this symmetry.

## Checking the Symmetry

The Polyakov action $S_P[X,h]$ is clearly invariant under this transformation as it is invariant under Diff × Weyl. We notice that $\delta_Q^2$ on any field vanishes. To do the calculation with just one example:

$$\delta_Q^2 b_{ab} = \delta_Q(i\epsilon B_{ab}) = 0. \tag{4.163}$$

So (at least classically)

$$\delta_Q^2 = 0. \tag{4.164}$$

**Exercise 4.3:** Verify this.

**Definition 23** (gauge-fixing fermion)**:** Furthermore, if we introduce the *gauge-fixing fermion*

$$\Psi = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-h}\,b^{ab}(h_{ab} - \hat{h}_{ab}), \tag{4.165}$$

which is a Grassmann valued functional (due to the anticommuting $b^{ab}$).

**Remark:** Since this is Grassmann valued, we cannot just add this into the action.

One can show that the variation of the gauge-fixing fermion under the BRST transformation is

$$\delta_Q \Psi = i\epsilon\big(S_{\text{gh}}[h,b,c] + S_{\text{gf}}[B,h]\big). \tag{4.166}$$

Let us also introduce a charge $\mathcal{Q}_B$ that generates this transformation, e.g.

$$\delta_Q X^\mu = i\epsilon[\mathcal{Q}_B, X^\mu]. \tag{4.167}$$

Then our action is of the form

$$S = S_P[X,h] + \{\mathcal{Q}_B, \Psi\}. \tag{4.168}$$

The observation that $\delta_Q^2 = 0$ on all fields is equivalent to

$$\mathcal{Q}_B^2 = 0. \tag{4.169}$$

Since $\{\mathcal{Q}_B, \Psi\}$ is exact in $\mathcal{Q}_B$ (proportional to $\delta_Q\Psi$), acting with another transformation gives zero. Hence $S_{\text{gh}} + S_{\text{gf}}$ is also invariant. Therefore, this is a hidden rigid symmetry in our theory!

## Imposing the Equations of Motion

This should hold in principle even if we integrate out $B_{ab}$, since that does not change the theory. We can integrate out the Lagrange multiplier field $B_{ab}$. At the classical level, this is equivalent to finding and imposing the equations of motion for $B_{ab}$. It does not appear with any derivatives, so it only gives a constraint. The gauge fixing term (4.155) couples the $B_{ab}$ to the metric $h_{ab}$, whose equation of motion gives

$$B_{ab} = \mathcal{T}_{ab}, \tag{4.170}$$

the total stress tensor.

Integrating out $B^{ab}$ leaves $S_P[X] + S_{\text{gh}}[b,c]$, which has the rigid (i.e. global, nongauge) *BRST symmetry*

$$\delta_Q X^\mu = i\epsilon c^a \partial_a X^\mu \tag{4.171}$$

$$\delta_Q c^a = i\epsilon c^b \partial_b c^a \tag{4.172}$$

$$\delta_Q b_{ab} = i\epsilon \mathcal{T}_{ab}. \tag{4.173}$$

### 4.11.1 BRST Cohomology and the Physical Spectrum

We still have to worry about two things. Firstly, why do we care about this symmetry? Secondly, these are symmetries of classical actions and so we need to check that there are no anomalies upon quantisation.

We shall see that physical states of the theory live in the cohomology of $\mathcal{Q}_B$.

**Definition 24** (kernel)**:** We take a state $|\phi\rangle$ for which $\mathcal{Q}_B|\phi\rangle = 0$ to be in the kernel of $\mathcal{Q}_B$. Such states $|\phi\rangle \in \ker(\mathcal{Q}_B)$ is called *closed*.

**Definition 25** (image)**:** A state $|\phi\rangle$ of the form $|\phi\rangle = \mathcal{Q}_B|\chi\rangle$ are in the *image* of $\mathcal{Q}_B$. Such states $|\phi\rangle \in \mathrm{im}(\mathcal{Q}_B)$ are called *exact*.

**Definition 26** (cohomology)**:** When $\mathcal{Q}_B^2 = 0$, we can define the *cohomology* of $\mathcal{Q}_B$ as

$$\mathrm{Cohom}(\mathcal{Q}_B) = \frac{\ker(\mathcal{Q}_B)}{\mathrm{im}(\mathcal{Q}_B)}. \tag{4.174}$$

In other words, $|\phi\rangle \in \mathrm{Cohom}(\mathcal{Q}_B)$ are those states for which $\mathcal{Q}_B|\phi\rangle = 0$, but are not of the form $\mathcal{Q}_B|\chi\rangle = |\phi\rangle$.

**Definition 27** (physical states)**:** *Physical states* are those in the chomology of the BRST charge $\mathcal{Q}_B$.

This is a different starting point than the physically motivated one that we discussed in prior sections, but in some sense it feels a bit deeper; all the previous results will fall out from consideration of this cohomology.

## The Kernel $|\phi\rangle \in \ker(\mathcal{Q}_B)$

Let us first consider the physical meaning of the statement

$$|\phi\rangle \in \ker(\mathcal{Q}_B). \tag{4.175}$$

The transition amplitude between initial and final states in QFT is

$$\langle\phi_i|\phi_f\rangle = \int \mathcal{D}\phi\, \phi_i\phi_f e^{iS[\phi]+i\{\mathcal{Q}_B,\Psi\}}. \tag{4.176}$$

**Remark:** From our point of view, $S[\phi]$ is the Polyakov action and $\{\mathcal{Q}_B,\Psi\}$ gives the ghosts and gauge fixing action. However, this holds more generally, say for Yang–Mills theory.

Let us now introduce an infinitesimal change of gauge and let us see the effect on $\langle\phi_i|\phi_f\rangle$. The gauge-fixing action $S_{\mathrm{gf}}$ encodes this.

$$\begin{aligned}
\delta\langle\phi_i|\phi_f\rangle &= \int \mathcal{D}\phi\, \phi_i\phi_f e^{iS[\phi]+i\{\mathcal{Q}_B,\Psi+\delta\Psi\}} - \int \mathcal{D}\phi\, \phi_i\phi_f e^{iS[\phi]+i\{\mathcal{Q}_B,\Psi\}} && (4.177)\\
&= i\int \mathcal{D}\phi\, \phi_i\phi_f\{\mathcal{Q}_B,\delta\Psi\}e^{iS[\phi]+i\{\mathcal{Q}_B,\Psi\}} + \ldots && (4.178)\\
&= i\langle\phi_i|\{\mathcal{Q}_B,\delta\Psi\}|\phi_f\rangle + \ldots. && (4.179)
\end{aligned}$$

For $\delta\Psi$ generic, we require $\langle\phi_i|\mathcal{Q}_B = 0$, and $\mathcal{Q}_B|\phi_f\rangle = 0$ for $\delta\langle\phi_i|\phi_f\rangle = 0$. Taking $\mathcal{Q}_B = \mathcal{Q}_B^\dagger$ to be Hermitian, we find all physical states must satisfy $\mathcal{Q}_B|\phi\rangle = 0$, meaning that $|\phi\rangle \in \ker(\mathcal{Q}_B)$. This explains our fascination for the Kernel.

### 4.11.2 The BRST Charge

Let us talk more concretely now about the BRST charge $\mathcal{Q}_B$. We shall split the BRST charge into chiral and antichiral pieces

$$\mathcal{Q}_B = Q_B + \overline{Q}_B \tag{4.180}$$

and the condition $\mathcal{Q}_B^2 = 0$ is

$$Q_B^2 = \frac{1}{2}\{Q_B, Q_B\} = 0, \qquad \overline{Q}_B^2 = \frac{1}{2}\{\overline{Q}_B, \overline{Q}_B\} = 0, \qquad \{Q_B, \overline{Q}_B\} = 0. \tag{4.181}$$

Let us focus on $Q_B$; there will be an analogous statement for $\overline{Q}_B$ and the total BRST charge is the sum of the two.

We want the BRST transformation to act on the $c$ ghost as

$$[Q_B, X^\mu(w)] = c(w)\partial X^\mu(w). \tag{4.182}$$

A good guess for $Q_B$ is

$$Q_B = \oint \frac{\mathrm{d}z}{2\pi i} c(z) T_X(z). \tag{4.183}$$

Then, using the singuar part of the OPE

$$\oint \frac{\mathrm{d}z}{2\pi i} c(z) T_X(z) X^\mu(w) = \oint \frac{\mathrm{d}z}{2\pi i} c(z)\left(\frac{\partial X^\mu(w)}{z-w} + \dots\right) = c(w)\partial X^\mu(w). \tag{4.184}$$

However, there are two problems with this $Q_B$: for one, it does not square to zero. Moreover, it gives the correct transformation for $X^\mu$ but not for the ghosts, which should be

$$\{Q_B, c(w)\} = c(w)\partial c(w), \qquad \{Q_B, b(w)\} = \mathcal{T}(w) = T_X(w) + T_{\mathrm{gh}}(w). \tag{4.185}$$

The correct charge is

$$\boxed{Q_B = \oint \frac{\mathrm{d}z}{2\pi i} c(z)\left[T_X(z) + \frac{1}{2} T_{\mathrm{gh}}(z)\right]} \tag{4.186}$$

This charge satisfies (4.185). Let us explicitly verify the first relation

$$\{Q_B, c(w)\} = \oint \frac{\mathrm{d}z}{2\pi i}\left\{c(z)\left[T_X(z) + \frac{1}{2}T_{\mathrm{gh}}(z)\right], c(w)\right\} \tag{4.187}$$

$$= \frac{1}{2}\oint \frac{\mathrm{d}z}{2\pi i} c(z)\left(-\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{z+w} + \dots\right), \tag{4.188}$$

using the $T_{\mathrm{gh}}(z)c(w)$ OPE. Expanding $c(z) = c(w) + \partial c(w)(z-w) + \dots$, and noting that $c$ is Grassmann and therefore $c^2(w) = 0$, gives

$$\{Q_B, c(w)\} = \frac{1}{2}\oint \frac{\mathrm{d}z}{2\pi i}\left[-\frac{\partial c(w)c(w)}{z-w} - \frac{c(w)\partial c(w)}{z-w} + \dots\right] \tag{4.189}$$

$$= \oint \frac{\mathrm{d}z}{2\pi i}\frac{c(w)\partial c(w)}{z-w} \tag{4.190}$$

$$= c(w)\partial c(w). \tag{4.191}$$

**Exercise 4.4:** Verify $\{Q_B, b(w)\} = \mathcal{T}(w)$.

## BRST Current

We can write $Q_B$ in terms of a conserved current.

$$Q_B = \oint \frac{\mathrm{d}z}{2\pi i} j_B(z), \tag{4.192}$$

where the *BRST current*

$$j_B(z) = c(z)\left[T_X(z) + \frac{1}{2}T_{\mathrm{gh}}(z)\right] + \frac{3}{2}\partial^2 c(z) \tag{4.193}$$

is a $h = 1$ chiral field.

One may also calculate the OPE with two currents

$$j_B(z)j_B(w) = -\frac{D-18}{2(z-w)^3}c(w)\partial c(w) - \frac{D-18}{4(z-w)^2}c(w)\partial^2 c(w) - \frac{1}{12}\frac{D-26}{(z-w)}c(w)\partial^3 c(w) + \dots. \tag{4.194}$$

Integrating this up should give $Q_B^2$.

$$\{Q_B, Q_B\} = \oint_{z=0} \frac{\mathrm{d}z}{2\pi i} \oint_{w=z} \frac{\mathrm{d}w}{2\pi i} j_B(z)j_B(w). \tag{4.195}$$

On the face of it, it looks like $Q_B^2 = 0$ only if $D$ is both 26 and 18. We have seen the $D = 26$ case before, but the $D - 18$ terms look alarming! It turns out that the contributions from the $D - 18$ terms cancel.

$$2Q_B^2 = \{Q_B, Q_B\} = \oint_{z=0} \frac{\mathrm{d}z}{2\pi i} \frac{D-26}{12} c(z)\partial^3 c(z). \tag{4.196}$$

The $c(z)\partial^3 c(z)$ will not vanish on its own, so $Q_B^2 = 0$ if $D = 26$. Similarly for $\overline{Q}_B$, so $\mathcal{Q}_B^2 = 0$ if $D = 26$.

## Meaning of $\mathcal{Q}_B^2 = 0$

Classically, the charge $\mathcal{Q}_B$ should commute with the Hamiltonian

$$[\mathcal{Q}_B, H] = 0. \tag{4.197}$$

The choice of gauge is encoded in the gauge-fixing fermion $\Psi$ of (4.165), which we write

$$\Psi = \int_\Sigma \mathrm{d}^2\sigma\, F_{\mathrm{gf}}, \tag{4.198}$$

where $F_{\mathrm{gf}} = \sqrt{-h} b^{ab}(h_{ab} - \hat{h}_{ab})$. A change of gauge gives a change in the Hamiltonian

$$\delta H = \{\mathcal{Q}_B, \delta F_{\mathrm{gf}}\}. \tag{4.199}$$

We expect this new Hamiltonian to preserve $\mathcal{Q}_B$, which requires

$$\begin{aligned} 0 &= [\mathcal{Q}_B, \delta H] && (4.200)\\ &= [\mathcal{Q}_B, \{\mathcal{Q}_B, \delta F_{\mathrm{gf}}\}] && (4.201)\\ &= -[\delta F_{\mathrm{gf}}, \{\mathcal{Q}_B, \mathcal{Q}_B\}] - [\mathcal{Q}_B, \overbrace{\{\mathcal{Q}_B, \delta F_{\mathrm{gf}}\}}^{\delta H}] && (4.202)\\ &= -[\delta F_{\mathrm{gf}}, \{\mathcal{Q}_B, \mathcal{Q}_B\}], && (4.203) \end{aligned}$$

where we used the Jacobi identity. For this to be true for any $\delta F_{\mathrm{gf}}$, we require that

$$\{\mathcal{Q}_B, \mathcal{Q}_B\} = 0 \quad \Rightarrow \quad \boxed{\mathcal{Q}_B^2 = 0}. \tag{4.204}$$

Therefore, $\mathcal{Q}_B^2 = 0$ is resulting from saying that $\mathcal{Q}_B$ generates a symmetry.

## States $|\phi\rangle \notin \mathbf{im}(\mathcal{Q}_B)$

Suppose we had a state $|\chi\rangle = \mathcal{Q}_B |\Lambda\rangle$ for some $|\Lambda\rangle$. Clearly, since $\mathcal{Q}_B^2 = 0$, $|\chi\rangle$ is closed $\mathcal{Q}_B |\chi\rangle = 0$. But also, if $|\phi\rangle \in \ker(\mathcal{Q}_B)$, then

$$\langle\phi|\chi\rangle = \langle\phi| \mathcal{Q}_B |\Lambda\rangle = 0. \tag{4.205}$$

Also

$$\langle\chi|\chi\rangle = \langle\Lambda| \mathcal{Q}_B^2 |\Lambda\rangle = 0. \tag{4.206}$$

The state $|\chi\rangle$ is orthogonal to all physical states (and itself). These are the same properties as for spurious states, which we met in Sec. 2.9. Such states will decouple from any process that involves physical states (or even itself). Hence, physical states are in $\ker(\mathcal{Q}_B)$ but not in $\mathrm{im}(\mathcal{Q}_B)$. In other words, all physical states may be defined up to the addition of an $\mathcal{Q}_B$-exact term

$$|\phi\rangle \sim |\phi\rangle + \mathcal{Q}_B |\Lambda\rangle. \tag{4.207}$$

The physical states live in the cohomology of $\mathcal{Q}_B$.

# 5 Scattering Amplitudes (S-matrix)

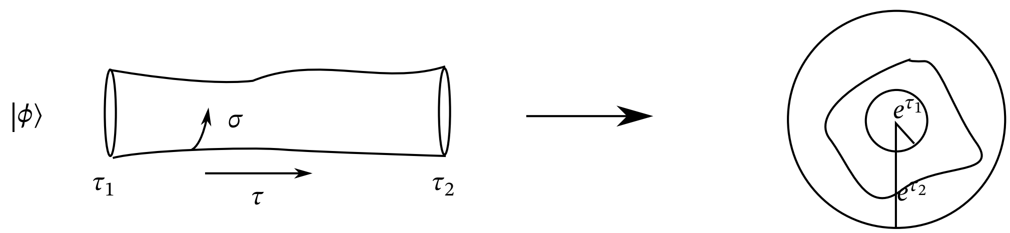


Figure 5.1: Propagator

As $\tau_1 \to -\infty$ and $\tau_2 \to \infty$, our picture Fig. 5.1 turns into Fig. 5.2.

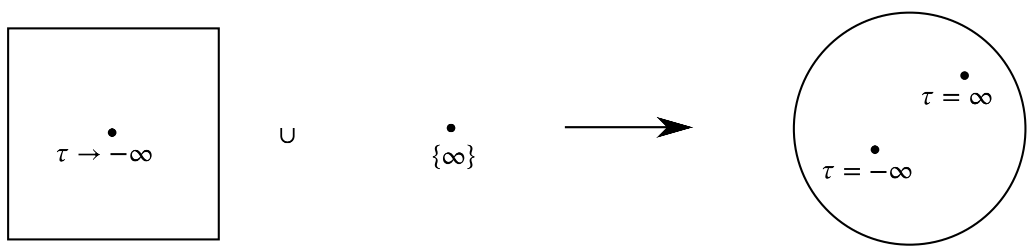


Figure 5.2

Interactions are depicted in Fig. 5.3. Let $w_2 = e^{\tau + i\sigma}$ be local coordinates on the free bits on the right-hand side. Then we have on the right a coordinate $z$, which compasses all of the punctures such that $z(w_2 = 0) = z_2$.

More generally, we can have any worldsheet topology, such as the one illustrated in Fig. 5.4. We use the state-operator correspondence to associate to each asymptotic state an operator at the given puncture. For example on the cylinder,

$$\lim_{t \to 0} \phi(z, \bar{z}) \left|0\right\rangle = \left|\phi\right\rangle . \tag{5.1}$$

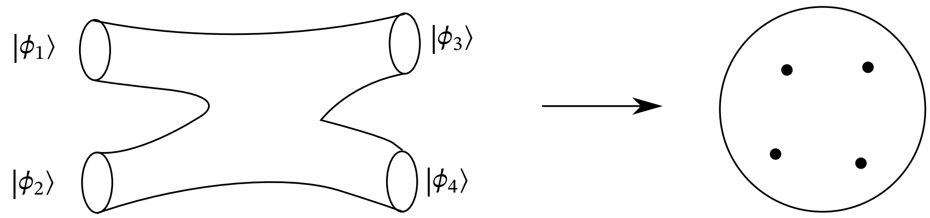


Figure 5.3: l20f3

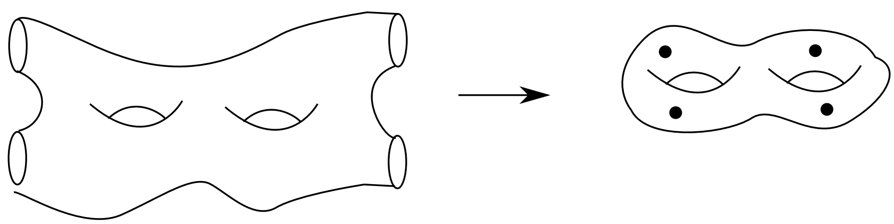

Figure 5.4: l20f4

## 5.1 The Dilaton and the String Coupling

The massless spectrum of the string includes the dilaton $\phi$, the graviton $g_{\mu\nu}$ and the (Kalb–Ramond) $B$-field $B_{\mu\nu}$.

What is a sensible starting point? We might try to add a curved metric to the Polyakov action and a similar background for the $B$-field and the dilaton:

$$\begin{aligned} S = &-\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}h^{ab}\partial_a X^\mu\partial_b X^\nu g_{\mu\nu}(X) && (5.2)\\ &-\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu B_{\mu\nu}(X) && (5.3)\\ &+\frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}R_\Sigma(h)\Phi(X). && (5.4)\end{aligned}$$

The first two terms are Weyl-invariant. However, the worldsheet Ricci scalar $R_\Sigma(h)$ does transform under Weyl transformations. However, these are cancelled out.

There are no dynamics here; recall that the Einstein–Hilbert like term just gives the Euler characteristic $\chi = 2g-2$ of the space, where $g$ is the genus. If $\Phi(X)$ is a constant (i.e. $\Phi_0 = \langle\Phi(X)\rangle$), then the dilaton term is a topological invariant

$$\frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{-h}R_\Sigma(h)\Phi_0 = \Phi_0\chi = \Phi_0(2g-2). \tag{5.5}$$

These were previously not interesting. However, for the scattering amplitudes, topological invariants give you some notion of loop order and are useful to keep track of. The dilaton contribution to the

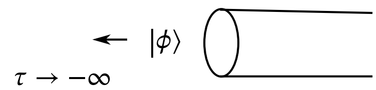


Figure 5.5

path integral is therefore

$$e^{\Phi_0(2g-2)}. \tag{5.6}$$

We see that $g_c := e^{\Phi_0}$ acts in the same way as a (spacetime) coupling constant. We add a factor of $g_c$ each time we add another loop into the worldsheet. The $c$ index refers to *closed*, since we are doing closed string theory. Most of our understanding of string theory comes from perturbation theory. There might be situations where $\Phi_0$ becomes large and we cannot say very much using perturbation theory. We are in a bizarre situation where we have the Feynman rules for our theory but have no deep understanding of where they are coming from. They are probably not coming from the same place as the QFT pendant.

From now on we will take $B_{\mu\nu}(X) = 0$, $g_{\mu\nu}(X) = \eta_{\mu\nu}$ and $\Phi(X) = \Phi_0$.

## 5.2 Tree Level

We shall focus on tree level amplitudes, which have $g = 0$.

### 5.2.1 Vertex Operators

We know that physical states are in the BRST cohomology:

$$\mathcal{Q}_B |\phi\rangle = 0, \qquad \text{but} \quad |\phi\rangle \neq \mathcal{Q}_B |\Lambda\rangle. \tag{5.7}$$

The requirement that an operator $\phi(z,\bar{z})$ gives a physical state (under the state operator correspondence) is that the operator is in the BRST cohomology:

$$[\mathcal{Q}_B, \phi(z,\bar{z})] = 0, \qquad \text{and} \quad \phi(z,\bar{z}) \neq \{\mathcal{Q}_B, \Lambda(z,\bar{z})\}. \tag{5.8}$$

Here we are assuming that $\phi(z,\bar{z})$ is bosonic. Since $\mathcal{Q}_B$ is fermionic the left-hand side uses the commutator, while the right-hand side uses the anti-commutator.

**Claim 4:** If we can find an operator $\phi(z,\bar{z})$ such that, if we split our BRST operator into chiral and anti-chiral pieces

$$\mathcal{Q}_B = Q_B + \overline{Q}_B, \tag{5.9}$$

we have commutators

$$[Q_B, \phi(z,\overline{z})] = \partial(c\phi) \qquad \text{and} \qquad [\overline{Q}_B, \phi(z,\overline{z})] = \overline{\partial}(\overline{c}\phi), \tag{5.10}$$

then we can easily write down a pair of BRST-invariant operators.

*Proof.* The operators

$$V_\phi = \int_\Sigma \mathrm{d}^2 z\, \phi(z,\overline{z}), \qquad U_\phi(z,\overline{z}) = c(z)\overline{c}(\overline{z})\phi(z,\overline{z}) \tag{5.11}$$

are BRST invariant. We have for example

$$\begin{aligned} [Q_B, U_\phi] &= (c\partial c)\overline{c}\phi + c\overline{c}\partial(c\phi) &(5.12)\\ &= (c\partial c)\overline{c}\phi + c\overline{c}(\partial c)\phi + c\overline{c}c\partial\phi = 0. &(5.13) \end{aligned}$$

□

**Remark:** Note that one of them is local and the other non-local. You cannot construct local diffeomorphism invariant operators, but you can construct operators like $V_\phi$, which are trivially diffeomorphism invariant. In the case of $U_\phi(z,\overline{z})$, we use the ghost fields.

Let us assume that $\phi(z,\overline{z})$ is a primary field of weight $(h,\overline{h})$. Let us further assume that $\phi$ does not depend on ghosts. This means that under a conformal transformation with vector field $v(z)$

$$\delta_v\phi = h(\partial v)\phi + v\partial\phi. \tag{5.14}$$

The BRST transformation is then given by replacing $v(z)$ with $c(z)$:

$$\begin{aligned} [Q_B, \phi] &= h(\partial c)\phi + c\partial\phi. &(5.15)\\ &= (h-1)\partial c\phi + \partial(c\phi). &(5.16) \end{aligned}$$

Thus $\phi(z,\overline{z})$ transforms as in (5.10) if $(h,\overline{h}) = (1,1)$.

**Remark:** It requires a little bit more thought to show that these are not also BRST-exact.

One might ask whether $U_\phi(z,\overline{z})$ and $V_\phi$ are two different observables or two different representations of the same observable? We will find that they describe the same observable, but we need both kinds to find the scattering amplitudes.

### 5.2.2 The Tachyon

In position space, if we wanted to calculate the scattering amplitude Fig. 5.6.

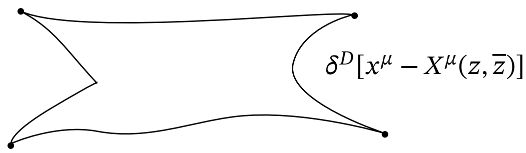


Figure 5.6: tachyon-scattering

Fourier transform to momentum space:

$$\int \mathrm{d}^D x\, \delta[x^\mu - X^\mu(z,\overline{z})] e^{ik\cdot x} = e^{ik_\mu X^\mu(z,\overline{z})}. \tag{5.17}$$

We compute the weight of $e^{ik\cdot X(z,\overline{z})}$

$$(h,\overline{h}) = \left(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4}\right). \tag{5.18}$$

So $e^{ik\cdot X(z,\overline{z})}$ has weight $(1,1)$ if

$$\boxed{\frac{\alpha' k^2}{4} = 1}. \tag{5.19}$$

This is the Tachyon mass condition. This suggests two vertex operators for the Tachyon

$$V_T = \frac{2g_c}{\alpha'} \int \mathrm{d}^2 z\, e^{ik\cdot X(z,\overline{z})}, \qquad U_T(z,\overline{z}) = \frac{2g_c}{\alpha'} c(z)\overline{c}(\overline{z}) e^{ik\cdot X(z,\overline{z})}, \tag{5.20}$$

where the normalisation constants are convention.

## 5.3 Massless Modes

Consider a correlation function

$$\langle \phi_1 \dots \phi_n \rangle_\eta = \int \mathcal{D}X \, e^{-S[X]} \phi_1 \dots \phi_n \tag{5.21}$$

in flat spacetime ($g_{\mu\nu} = \eta_{\mu\nu}$). How does this change if we let

$$g_{\mu\nu} = \eta_{\mu\nu} \longmapsto \eta_{\mu\nu} + \epsilon_{\mu\nu} e^{ik\cdot X}. \tag{5.22}$$

The Polyakov action changes by

$$S[X] \longmapsto S[X] - \frac{1}{4\pi\alpha'} \int_\sigma \mathrm{d}^2\sigma \sqrt{-h} h^{ab} \epsilon_{\mu\nu} e^{ik\cdot X} \partial_a X^\mu \partial_b X^\nu. \tag{5.23}$$

Assuming the change in $g_{\mu\nu}$ is small, then to leading order, the change in the correlation function is the insertion of the operator

$$\int_\Sigma \mathrm{d}^2\sigma \sqrt{-h} h^{ab} \epsilon_{\mu\nu} \partial_a X^\mu \partial_b X^\nu e^{ik\cdot X}. \tag{5.24}$$

This leads us to the graviton vertex operator:

$$V_g = g_c \int_\Sigma \mathrm{d}^2 z \, \epsilon_{\mu\nu} \partial X^\mu \overline{\partial} X^\nu e^{ik\cdot X(z,\overline{z})}, \qquad \epsilon_{\mu\nu} = \epsilon_{\nu\mu}. \tag{5.25}$$

We require all of our operators in the BRST cohomology. We found that a condition for this vertex operator to be BRST invariant if the integrand is of conformal weight $(1,1)$.

**Exercise 5.1:** Show that this happens when $k^2 = 0$, $\epsilon_{\mu\nu} k^\nu = 0$, and $\epsilon_{\mu\nu} k^\mu = 0$.

When first looking at the quantum string, we found exactly these conditions as physical state conditions for the graviton. Here they fall out naturally of the BRST considerations.

Once we have a $(1,1)$ primary field, we can construct two operators. In addition to $V_g$ also have the unintegrated vertex operator

$$U_g = g_c c(z) \overline{c}(z) \epsilon_{\mu\nu} \partial X^\mu \overline{\partial} X^\nu e^{ik\cdot X(z,\overline{z})}, \tag{5.26}$$

which is invariant under BRST.

The state/operator correspondence gives

$$\lim_{z\to 0} U_g = g_c c_1 \overline{c}_1 \epsilon_{\mu\nu} \alpha^\mu_{-1} \overline{\alpha}^\nu_{-1} \left| k \right\rangle. \tag{5.27}$$

Ignoring the ghost bits, this mode term is precisely what we interpreted as the graviton state.

Also

$B$**-field:** $(b_{\mu\nu} = -b_{\nu\mu})$

$$U_B = g_c b_{\mu\nu} c(z)\bar{c}(z)\partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X} \tag{5.28}$$

$$V_B = g_c \int_\Sigma \mathrm{d}^2 z\, b_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X}. \tag{5.29}$$

The similarity between the $B$-field and graviton is not accidental, but we will not talk much about the underlying similarities in this course.

**Dilaton:**

$$U_\phi = g_c c(z)\bar{c}(z)\phi\partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X} \tag{5.30}$$

$$V_\phi = g_c \int_\Sigma \mathrm{d}^2 z\, \phi \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X}. \tag{5.31}$$

### Massive States

Because the exponential $e^{ik\cdot X(z,\bar{z})}$ is not a conformal scalar, we can construct infinitely many other vertex operators. The massive states will be subject to renormalisation, which would take us beyond the scope of the course. Secondly, the unbroken gauge symmetry is found in the massless sector, which is really the interesting part of string theory. Finally, we do not really have much time left in this course, and need to be a bit selective in the choice of topics we are discussing.

## 5.4 S-Matrix

We have weight $(h, \bar{h}) = (1, 1)$ operators $\phi_i = \phi(z_i, \bar{z}_i)$ and we compute the $n$-point amplitude

$$A_n = \sum_{g=0}^{\infty} g_c^{2g-2+n} \int_{M_g} \mathrm{d}^s t \int \prod_{i=\kappa+1}^{n} \mathrm{d}^2 z_i \int \mathcal{D}X\, \mathcal{D}[b,\bar{b}]\, \mathcal{D}[c,\bar{c}]\, e^{-S[X,b,c,\bar{b},\bar{c}]} \prod_{I=1}^{s} (b, \mu_I) \prod_{j=1}^{\kappa} c(z_j)\bar{c}(\bar{z}_j)\phi_1 \dots \phi_n \tag{5.32}$$

where $\kappa = \dim(\mathrm{CKG})$, $s = \dim(M_g)$.

Let us introduce some notation, which will make the BRST invariants (almost) obvious.

$$V_i = g_c \int \mathrm{d}^2 z_i\, \phi_i, \qquad U_i = g_c c(z_i)\bar{c}(\bar{z}_i)\phi_i. \tag{5.33}$$

Since the $\phi_i$ are weight $(1, 1)$ fields, we know that these are BRST invariant.

This gives us something, which looks a little bit more palatable.

$$A_n = \sum_{g=0}^{\infty} g_c^{2g-2} \int_{M_g} \mathrm{d}^s t\, \langle \prod_{I=1}^{s} (b, \mu_I) \prod_{i=1}^{\kappa} U_i \prod_{j=\kappa+1}^{n} V_i \rangle_{X,b,c}. \tag{5.34}$$

## Tree-level

We will focus on tree-level amplitudes, where $g = 0$: In this case, $M_g$ is a point; we can ignore the integral over the moduli space and $s = 0$. The CKG is $SL(2;\mathbb{C})$, so the complex dimension $\kappa = 3$; we fix three of our punctures. Then,

$$A_n^{\text{tree}} = g_c^{-2} \langle \prod_{i=1}^{3} U_i \prod_{j=4}^{n} V_j \rangle_{X,b,c}. \tag{5.35}$$

$$= g_c^{n-2} \int \prod_{i=4}^{n} \mathrm{d}^2 z_i \, \langle \prod_{j=1}^{3} c(z_j)\bar{c}(\bar{z}_j) \rangle_{b,c} \langle \phi_1 \dots \phi_n \rangle_X \tag{5.36}$$

To construct the scattering amplitudes for $n > 3$, we need both kinds of vertex operator, despite ultimately giving the same physical interpretation in target space, as we have seen for the graviton.

### 5.4.1 The Ghost Bit

The invariance under the CKG should guarantee that the expression does not depend on the location of the $n - 3$ punctures, although this is not at all obvious.

Consider

$$\langle c(z_1)c(z_2)c(z_3) \rangle = \langle 0 | \, c(z_1)c(z_2)c(z_3) \, | 0 \rangle \tag{5.37}$$

We can use our knowledge of the expansion of the $c$ ghosts, which have conformal weight $-1$:

$$c(z) = \sum_n c_n z^{-n+1}, \qquad c_n \, |0\rangle = 0 \quad n > 1, \tag{5.38}$$

we find

$$\langle c(z_1)c(z_2)c(z_3) \rangle = K(z_1 - z_2)(z_2 - z_3)(z_3 - z_1), \tag{5.39}$$

where $K = \langle 0 | \, c_{-1} c_0 c_1 \, | 0 \rangle = 1$ is our choice of normalisation.

We get a similar answer for the $\bar{c}$ ghosts. Therefore, the ghost bits gives

$$\boxed{\langle \prod_{j=1}^{3} c(z_j)\bar{c}(\bar{z}_j) \rangle = |z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2} \tag{5.40}$$

This is a universal, appearing in all tree-level scattering amplitudes. So

$$A_n^{\text{tree}} = g_c^{n-2} |z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2 \int \prod_{j=4}^{n} \mathrm{d}^2 z_j \, \langle \prod_{i=1}^{n} \phi(z_i, \bar{z}_i) \rangle_X. \tag{5.41}$$

## 5.5 Calculations Using the Path Integral

We want to compute the $n$-point function

$$\langle \phi_1 \dots \phi_n \rangle_X = \int \mathcal{D}X \, e^{-S_P[X]} \phi_1 \dots \phi_n, \tag{5.42}$$

where $S_P[X] = -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}^2 z \, \partial X^\mu \overline{\partial} X^\nu \eta_{\mu\nu}$. To do this, we introduce a source term

$$S_J[X] = \int_\Sigma \mathrm{d}^2 z \, X^\mu(z, \overline{z}) J_\mu(z, \overline{z}) \tag{5.43}$$

into the path integral. Consider the (normalised) generating functional

$$Z[J] = Z[0]^{-1} \int \mathcal{D}X \, e^{-S_P[X] - S_J[J,X]}. \tag{5.44}$$

From this generating functional, we obtain the $n$-point function (5.42) by differentiating $n$ times with respect to $J$ and then setting $J = 0$. We usually take this as a heuristic starting point motivating the development of our theory, but we will never actually take it that seriously, so we do not have to worry much about the proper definitions of, say, the path integral measure.

Let us write $X^\mu(z,\overline{z}) = x^\mu + \widetilde{X}(z,\overline{z})$ and similarly $\int \mathcal{D}X = \int \mathrm{d}^D x \int \mathcal{D}\widetilde{X}$. It will also be useful to identify the Green's function for the Polyakov action $S_P[X]$. The equation of motion for the $X^\mu(z,\overline{z})$ is just $\partial\overline{\partial}X^\mu(z,\overline{z}) = \Box X^\mu(z,\overline{z}) = 0$. The Green's function $G$ for the 2-dimensional Laplacian satisfies

$$-\frac{1}{\pi\alpha'} = \Box G(z,\overline{z};w,\overline{w}) = \delta^2(z-w) = \delta(z-w)\delta(\overline{z}-\overline{w}). \tag{5.45}$$

In the language of quantum field theory, $G$ is also the propagator for the theory. For brevity, we can write $G$ as

$$\boxed{G(z,w) = -\frac{\alpha'}{2}\ln|z-w|^2} \tag{5.46}$$

In the language of conformal field theory, this is the non-trivial part of the OPE between the $X$'s.

We can write $S_P[X] + S_J[J,X]$ in the following way by integrating by parts

$$\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2 z\, X^\mu \Box X^\nu \eta_{\mu\nu} + \int_\Sigma \mathrm{d}^2 z\, X^\mu J_\mu. \tag{5.47}$$

Splitting $X^\mu(z,\overline{z}) = x^\mu + \widetilde{X}(z,\overline{z})$, this becomes

$$\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2 z\, \widetilde{X}^\mu \Box \widetilde{X}^\nu \eta_{\mu\nu} + \int_\Sigma \mathrm{d}^2 z\, \widetilde{X}^\mu J_\mu + x^\mu \int_\Sigma \mathrm{d}^2 z\, J_\mu. \tag{5.48}$$

Let us complete the square in $\widetilde{X}^\mu$ by defining

$$Y^\mu(z) := \widetilde{X}^\mu(z) - \int_\Sigma \mathrm{d}^2 w\, G(z,w) J^\mu(w), \tag{5.49}$$

where we use $z$ everywhere to denote $z, \overline{z}$ and similarly for $w$. The action becomes

$$\frac{1}{2\pi\alpha'}\int_\Sigma \mathrm{d}^2 z\, Y^\mu(z)\Box Y_\mu(z) + \frac{1}{2}\int_{\Sigma\times\Sigma} \mathrm{d}^2 z\,\mathrm{d}^2 w\, J^\mu(z) G(z,w) J_\mu(w) + x^\mu \int_\Sigma \mathrm{d}^2 z\, J_\mu(z), \tag{5.50}$$

It is important to note that the second integral integrates over two copies of the worldsheet, with a Green's function tying together the two currents. Noting that $\mathcal{D}\widetilde{X} = \mathcal{D}Y$,

$$Z[J] = Z[0]^{-1}\int \mathrm{d}^D x\, \mathcal{D}Y \exp\!\Big(-\frac{1}{2\pi\alpha'}\Big)\int_\Sigma Y\Box Y - \frac{1}{2}\int \mathrm{d}^2 z\,\mathrm{d}^2 w\, J\cdot G\cdot J - x\cdot\int \mathrm{d}^2 x\, J. \tag{5.51}$$

We notice that

$$Z[0] = \int \mathcal{D}Y \exp\!\Big(-\frac{1}{2\pi\alpha'} Y\Box Y\Big). \tag{5.52}$$

We are left with

$$\boxed{Z[J] = \exp\!\Big(-\frac{1}{2}\int_{\Sigma\times\Sigma} \mathrm{d}^2 z\,\mathrm{d}^2 w\, J_\mu(z) G(z,w) J^\mu(w)\Big) \times \int \mathrm{d}^D x \exp\!\Big(x^\mu \int_\Sigma J_\mu(z)\,\mathrm{d}^2 z\Big)} \tag{5.53}$$

This is an exact statement, since we know $G$ exactly in the free theory. If we have interacting theories, we could only hope for something like this to leading order. In quantum field theory, we would now compute correlation functions of the $X^\mu$'s by taking functional integrals:

$$\langle X^{\mu_1}(z_1)\dots X^{\mu_n}(z_n)\rangle = \frac{\delta}{\delta J_{\mu_1}(z_1)}\cdots\frac{\delta}{\delta J_\mu(z_n)}Z[J]\bigg|_{J=0}. \tag{5.54}$$

But actually this is not something we want to do in string theory. In particular, the $X$'s are not BRST invariant. We are instead interested in the vertex operators.

### 5.5.1 Tachyon Scattering

Start with the correlation function relating to $n$-Tachyon scattering

$$\langle e^{ik_1\cdot X(z_1,\bar z_1)}\dots e^{ik_n\cdot X(z_n,\bar z_n)}\rangle_X, \tag{5.55}$$

where $e^{ik\cdot X}$ has weight $(h,\bar h) = (\frac{\alpha' k^2}{4},\frac{\alpha' k^2}{4}) \overset{!}{=} (1,1) \;\Rightarrow\; k^2 = \frac{4}{\alpha'}$. How might we evaluate this correlation function using the generating functional $Z[J]$? The correlation function may be written as

$$\langle e^{ik_1\cdot X(z_1,\bar z_1)}\dots e^{ik_n\cdot X(z_n,\bar z_n)}\rangle_X = \int \mathcal{D}X\, e^{-S_P[X]S_J[J,X]}, \tag{5.56}$$

where

$$S_J[J,X] = -i\sum_{j=0}^n k_{j\mu}X^\mu(z_j,\bar z_j) = -i\int_\Sigma \mathrm{d}^2 z\sum_{j=1}^n k_{j\mu}X^\mu(z,\bar z)\delta^2(z-z_j). \tag{5.57}$$

This is because all of the objects $e^{ik_j\cdot X(z_j)}$ inserted in the function integral are just functions, which commute and can be combined in the exponential in the usual way.

We can write this as

$$S_J[J,X] = \int_\Sigma \mathrm{d}^2 z\, X^\mu(z,\bar z)J_\mu(z,\bar z), \qquad \text{with} \quad J_\mu(z,\bar z) = -i\sum_{j=0}^n k_{j\mu}\delta^2(z-z_j). \tag{5.58}$$

In other words, the $n$-point tachyon correlation function is exactly $Z[J]$, where $J_\mu(z,\bar z)$ is given by (5.58).

Our expression (5.53) for $Z[J]$ included the term

$$x^\mu\int_\Sigma \mathrm{d}^2 z\, J_\mu(z,\bar z) = -ix^\mu\int_\Sigma \mathrm{d}^2 z\sum_{j=0}^n k_{\mu j}\delta^2(z-z_j) = -ix^\mu\left(\sum_{j=0}^n k_{j\mu}\right). \tag{5.59}$$

So the $x^\mu$-part of $Z[J]$ gives

$$\int \mathrm{d}^D x\exp\left(-ix^\mu\sum_{j=0}^n k_{\mu j}\right) = (2\pi)^D\delta\left(\sum_{j=0}^n k_{j\mu}\right), \tag{5.60}$$

which tells us that the total spacetime momentum is *conserved* in this process.

We also want to substitute our expression into the other term

$$\frac{1}{2}\int_{\Sigma\times\Sigma} \mathrm{d}^2z\,\mathrm{d}^2w\,J_\mu(z)G(z,w)J^\mu(w) = -\frac{1}{2}\sum_{i\neq j}\int \mathrm{d}^2z\,\mathrm{d}^2w\,k_{i\mu}\delta^2(z-z_i)k_j^\mu\delta^2(w-z_j)\times G(z,w). \quad (5.61)$$

We have to be a little careful here. Recall that

$$G(z,w) = -\frac{\alpha'}{2}\ln(z-w), \quad (5.62)$$

so $G(z,z)$ does not make sense. In particular, this is why we require that $i \neq j$ in (5.61). Physically, $z_i$ and $z_j$ label the punctures for the vertex operators, which we shall assume to not lie on the same point. In a more careful treatment, it is possible to regularise this expression to get something sensible when $z = w$, but we will be content with just assuming $i \neq j$. Then this integral is

$$\cdots = -\frac{1}{2}\sum_{i\neq j} k_i\cdot k_j G(z_i,z_j) = -\frac{1}{2}\sum_{i\neq j} k_i\cdot k_j\left(-\frac{\alpha'}{2}\ln\left|z_i-z_j\right|^2\right). \quad (5.63)$$

Putting this together, we have that our generating functional contains the interesting term

$$\exp\left(\frac{1}{2}\int_{\Sigma\times\Sigma} \mathrm{d}^2z\,\mathrm{d}^2w\,J_\mu(z)G(z,w)J^\mu(w)\right) = \exp\left(\frac{1}{2}\sum_{i\neq j}\alpha' k_i\cdot k_j\ln\left|z_i-z_j\right|\right) \quad (5.64)$$

$$= \prod_{i\neq j}\left|z_i-z_j\right|^{\alpha' k_i\cdot k_j/2} = \prod_{i>j}\left|z_i-z_j\right|^{\alpha' k_i\cdot k_j}. \quad (5.65)$$

Finally, we need to think about how to evaluate $k_i\cdot k_j$.

We have that the $X$-dependent part of the scattering amplitude (i.e. without the ghosts) is

$$\langle e^{ik_1\cdot X(z_1,\bar{z}_1)}\ldots e^{ik_n\cdot X(z_n,\bar{z}_n)}\rangle_X = (2\pi)^D\prod_{i>j}\left|z_i-z_j\right|^{\alpha' k_i\cdot k_j}\times\delta^D\left(\sum_{j=0}^n k_{\mu j}\right). \quad (5.66)$$

### 5.5.2 Examples

**n = 3:** For tachyons, $k_i^2 = 4/\alpha'$. Noting that momentum conservation gives $k_1^\mu + k_2^\mu + k_3^\mu = 0$, we use

$$(-k_3)^2 = (k_1 + k_2)^2 = k_1^2 + k_2^2 + 2k_1 \cdot k_2 \tag{5.67}$$

to rewrite the dot product

$$\alpha' k_1 \cdot k_2 = \frac{\alpha'}{2}(k_3^2 - k_1^2 - k_2^2) = \frac{\alpha'}{2}\left(-\frac{4}{\alpha'}\right) = -2. \tag{5.68}$$

Since there is nothing special about particles 1 and 2, we generally find that

$$\alpha' k_i \cdot k_j = (-1)^{\delta_{ij}} 2. \tag{5.69}$$

Then

$$\langle e^{ik_1 \cdot X(z_1,\bar{z}_1)} \dots e^{ik_3 \cdot X(z_3,\bar{z}_3)} \rangle_X = (2\pi)^D \delta^D \left( \sum_{i=1}^n k_{i\mu} \right) |z_1 - z_2|^{-2} |z_2 - z_3|^{-2} |z_3 - z_1|^{-2}. \tag{5.70}$$

The ghost contribution was $|z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2$, so we have

$$A_3 = g_c (2\pi)^D \delta^D \left( \sum_{i=1}^3 k_{\mu i} \right). \tag{5.71}$$

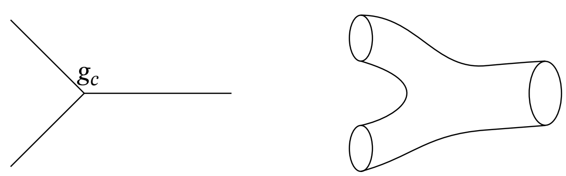


Figure 5.7

**n=4:** This is called the *Virasoro–Shapiro Amplitude*. For simplicity, we shall set $z_1 = 0, z_2 = 1$, and $z_3 = \lambda \to \infty$. We also have $z_4 = z$, which we cannot fix.

$$\left\langle \prod_{i=1}^3 c(z_i)\bar{c}(\bar{z}_i) \right\rangle_{b,c} \prod_{i<j} \left|z_i - z_j\right|^{\alpha' k_i \cdot k_j} = |\lambda|^{\alpha(k_1+k_2+k_4)\cdot k_3 + 4} |z|^{\alpha' k_1 \cdot k_4} |1 - z|^{\alpha' k_2 \cdot k_4}, \tag{5.72}$$

where we have used momentum conservation. As before, the $k_i \cdot k_j$ terms cancel out for $i, j = 1, 2, 3$ and only the dot products with $k_4$ remain. Note further that

$$\alpha'(k_1 + k_2 + k_4) \cdot k_3 + 4 = -\alpha' k_4^2 + 4 = 0. \tag{5.73}$$

The $\lambda$-dependence drops out, leaving

$$A_4 = g_c^2 \delta^{26}(\sum_{i=1}^4 k_{i\mu}) \int d^2z\, |z|^{\alpha' k_1 \cdot k_4} |1 - z|^{\alpha' k_2 \cdot k_4}. \tag{5.74}$$

You should be happy reproducing something like this in the exam, but the following should not be committed to memory.

From a field theory point of view, we might imagine that we are looking at a scattering amplitude like on the eft-hand side of Fig. 5.9. Unlike the 3-point amplitude however, this field theory point of view does not capture the physics on the string worldsheet on the right. We introduce

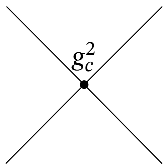


Figure 5.8

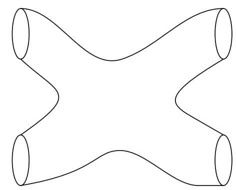

Figure 5.9

Mandelstram variables

$$s = -(k_1 + k_2)^2, \qquad t = -(k_1 + k_3)^2, \qquad u = -(k_1 + k_4)^2, \tag{5.75}$$

and

$$\alpha(s) = -1 - \frac{\alpha'}{4} s, \qquad \text{etc.} \tag{5.76}$$

We also introduce the $\Gamma$-function

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} \, \mathrm{d}x \,. \tag{5.77}$$

Then

$$\boxed{A_n = g_c^2 \delta^{26}(\sum_{i=1}^{4} k_{\mu i}) \frac{\Gamma(\alpha(s))\Gamma(\alpha(??))\Gamma(\alpha(u))}{\Gamma(\alpha(t)+\alpha(u))\Gamma(\alpha(s)+\alpha(u))\Gamma(\alpha(s)+\alpha(t))}} \tag{5.78}$$

## Comments

$s, t, u$-permutation invariant: historically, this lead to *dual models* in the description of strong interactions. There is no fundamental difference in string theory between $s, t$, and $u$ amplitudes. We can deform them into each other. The problem was that in intermediate states you always found gravitons inevitably creeping into the theory.

### 5.5.3 Scattering of Massless States

Vertex operators of the form

$$V = g_c \int_\Sigma \mathrm{d}^2 z \, \epsilon_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X}. \tag{5.79}$$

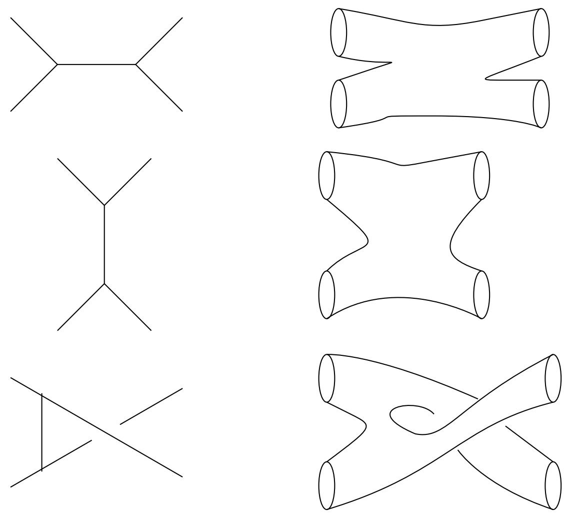

Figure 5.10

We can write

$$i\partial X^{\mu}(z_j)e^{ik_j\cdot X(z_j,\bar{z}_j)} = \left[\frac{\partial}{\partial\rho_{\mu j}}\exp\left(i\int_{\Sigma}\mathrm{d}^2z\,\delta^2(z-z_j)(k_{j\mu}+\rho_{j\nu}\frac{\partial}{\partial z})X^{\nu}(z,\bar{z})\right)\right]_{\rho_j=0} \tag{5.80}$$

or

$$\partial X^{\mu}_j\bar{\partial}X^{\nu}_j e^{ik\cdot X_j} = \left[-\frac{\partial^2}{\partial\partial\rho_{\mu}\partial\bar{\rho}_{\nu}}\exp\left(i\int_{\Sigma}\mathrm{d}^2z\,\delta^2(z-z_j)\left(k_{\lambda j}+\rho_{\lambda j}\frac{\partial}{\partial z}+\bar{\rho}_{\lambda j}\frac{\partial}{\partial\bar{z}}\right)X^{\lambda}(z,\bar{z})\right)\right]\Bigg|_{\rho_i=\bar{\rho}_i=0}. \tag{5.81}$$

If we define a current

$$J_{\mu}(z,\bar{z}) = -i\sum_{j=i}^{n}\delta^2(z-z_j)\left(k_{\mu j}+\rho_{\mu j}\frac{\partial}{\partial z}+\bar{\rho}_{\mu j}\frac{\partial}{\partial\bar{z}}\right) \tag{5.82}$$

then the scattering amplitude for $n$ such massless states is

$$A_n \sim g_c^{n-2}\left[\prod_{i=1}^{n}\epsilon^{(i)}_{\mu\nu}\frac{\partial^2}{\partial\rho_j\partial\bar{\rho}_j}\exp\left(\frac{1}{2}\int_{\Sigma\times\Sigma}\mathrm{d}^2z\,\mathrm{d}^2w\right)J_{\lambda}(z,\bar{z})J^{\lambda}(w,\bar{w})G(z,w)\right]_{\rho=\bar{\rho}=0}, \tag{5.83}$$

where

$$G(z,w) = -\frac{\alpha'}{2}\ln(z-w)-\frac{\alpha'}{2}\ln(\bar{z}-\bar{w}). \tag{5.84}$$

The current (5.82) involves derivatives of $z$ and also derivatives of $\bar{z}$. However, since $G$ splits these, it is useful to define a chiral current $j_{\mu}(z)$, which only takes derivatives with respect to $z$

$$j_{\mu}(z) = -i\sum_{i=1}^{n}\delta^2(z-z_i)\left(\frac{1}{2}k_{\mu i}+\rho_{\mu i}\frac{\partial}{\partial z}\right) \tag{5.85}$$

and $\bar{j}_{\mu}(\bar{z})$ defined similarly. The factor of $\frac{1}{2}$ is coming in since we get equal contributions of $j$ and $\bar{j}$ in that sector so that $J_{\mu}(z,\bar{z}) = j_{\mu}(z)+\bar{j}_{\mu}(\bar{z})$.