Applications of Differential Geometry to Physics

Part III Lent 2019

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Contents

0.1 Kepler / Newton Orbits

$$\ddot{\mathbf{r}} = -\frac{GMv}{r^3}\mathbf{r} \quad \leftrightarrow \quad \text{conic sections} \tag{1}$$

General conic section is

$$ax^2 + by^2 + cxy + dx + ey + f = 0 \tag{2}$$

This is nowadays more generally studied in what we now call *algebraic geometry* rather than differential geometry.

Apolonius of Penge (?) asked 'what is the unique conic thorugh five points, no three of which are co-linear?'

The space of conics is $\mathbb{R}^6 - \{0\}/n = \mathbb{RP}^5$ (projective 5-space).

$$[a, b, c, d, e, f] \sim [\gamma a, \gamma b, \gamma c, \gamma d, \gamma c, \gamma f], \gamma \in \mathbb{R}^* \tag{3}$$

This is an application of geometry, rather than an application of differential geometry.

Remark: Apolonius proved this geometrically.

In this course however, we will look at the following.

1) Hamiltonian mechanics (mid 19^{th}). This is an elegant way of reformulating Newton's mechanics, turning second order differential equations into first order differential equations with the use of a function $H(p, q)$. The system of ODEs is

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{q} = -\frac{\partial H}{\partial q} \tag{4}$$

This led to the development of symplectic geometry (1960s). The connection is that the phase-space to which p and q belong has a 2 -form $dp \wedge dq$. Using the Hamiltonian function, one can find a vector field

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} \tag{5}$$

and looks for a one-parameter group of transformations, called symplectomorphisms, generated by this vector field. Under these symplectomorphisms, the 2 -form is unchanged meaning that the area illustrated in F2 is preserved. Details of this are going to come within the course.

2) General Relativity (1915) $\leftarrow$ Riemannian Geometry (1850)

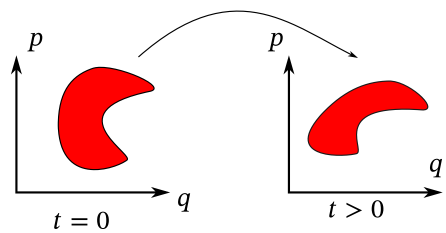

Figure 1

3) Gauge theory (Maxwell, Yang Mills) ↔ Connection on Principal Bundle (U(1) (Maxwell), SU(2), SU(3))

$$A_+ = A_- + dg \qquad g = \psi_+ - \psi_- \qquad \omega = \begin{cases} A_+ + d\psi_+ \\ A_- + d\psi_- \end{cases} \tag{6}$$

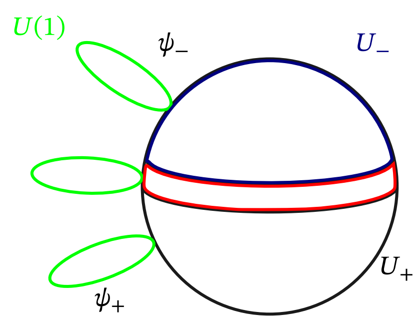

Figure 2

This course: cover 1, 2, 3 in some detail. Unifying feature: Lie groups.

- Prove some theorems, *lots of* examples (often instead of proofs)
- Want to be able to do calculations; compute characteristic classes etc.

We will assume that you took either Part III General Relativity, or Part III Differential Geometry, or some equivalent course.

1 Manifolds

Definition 1 (manifold)**:** An n -dimensional *smooth manifold* is a set M and a collection[2] of open sets U_α , labelled by $\alpha = 1, 2, 3, \dots$, called *charts* such that

- U_α cover M
- $\exists$ 1-1 maps $\phi_\alpha : U_\alpha \to V_\alpha \in \mathbb{R}^n$ such that

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{1.1}$$

  is a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

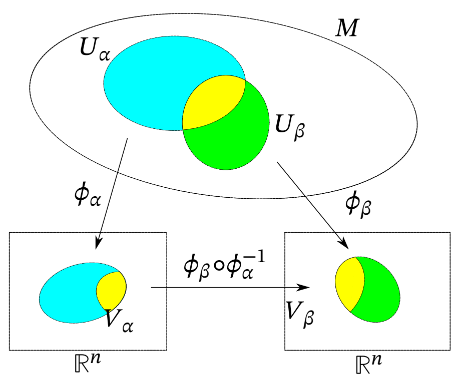

Figure 1.1: Manifold

As such, manifolds are topological spaces with additional structure, allowing us to do calculus.

Example ($M = \mathbb{R}^n$)**:** There is the *trivial manifold*, which can be covered by only one open set. There are other possibilities. In fact, there are infinitely many smooth structures on $\mathbb{R}^4$ (Proof by Donaldson in 1984 in his PhD. He used Gauge theory).

[2] In all examples that we will look at, there will be finitely α.

Example (sphere $S^n = \left\{\mathbf{r} \in \mathbb{R}^{n+1}, |\mathbf{r}| = 1\right\}$)**:** Have two open sets

$$U = S^n / \{0, 0, 0, , \dots, 0, 1\} \qquad \widetilde{U} = S^n / \{0, 0, 0, , \dots, 0, -1\} \tag{1.2}$$

We then define charts, where $\mathbb{R}^n = (x_1, \dots, x_n)$:

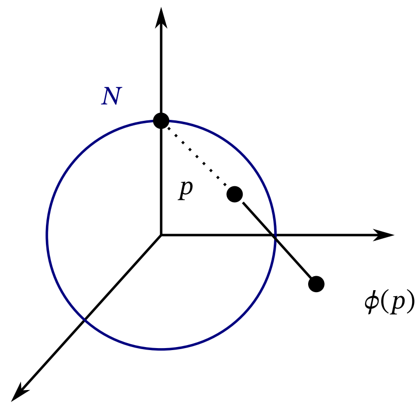

Figure 1.2

$$\begin{aligned} \phi(r_1, \dots, r_{n+1}) &= \left(\frac{r_1}{1 - r_{n+1}}, \dots, \frac{r_n}{1 - r_{n+1}}\right) \\ \text{on } \widetilde{U}, \qquad \widetilde{\phi}(r_1, \dots, r_{n+1}) &= \left(\frac{r_1}{1 - r_{n+1}}, \dots, \frac{r_n}{1 - r_{n+1}}\right) = (\widetilde{x}_1, \dots, \widetilde{x}_n). \end{aligned} \tag{1.3}$$

On $U \cap \widetilde{U}$,

$$\frac{r_k}{1 + r_{n+1}} = \frac{1 - r_{n+1}}{1 + r_{n+1}} \frac{r_k}{1 - r_{n+1}}, \qquad k = 1, \dots, n \tag{1.4}$$

$$\frac{1 - r_{n+1}}{1 + r_{n+1}} = \frac{(1 - r_{n+1})^2}{r_1^2 + r_2^2 + \cdots + r_n^2} = \frac{1}{x_1^2 + x_2^2 + \cdots + x_n^2} \tag{1.5}$$

So on $\phi(U \cap \widetilde{U})$,

$$(\widetilde{x}_1, \dots, \widetilde{x}_n) = \left(\frac{x_1}{x_1^2 + \cdots + x_n^2}, \dots, \frac{x_n}{x_1^2 + \cdots + x_n^2}\right) \tag{1.6}$$

are smooth maps from $\mathbb{R}^n \to \mathbb{R}^n$

Example: A Cartesian product of manifolds is a manifold, for example we have the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$.