

# Applications of Differential Geometry to Physics

Part III Lent 2020

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## 0.1 Kepler / Newton Orbits

$$\ddot{\mathbf{r}} = -\frac{GMv}{r^3}\mathbf{r} \leftrightarrow \text{conic sections} \quad (1)$$

General conic section is

$$ax^2 + by^2 + cxy + dx + ey + f = 0 \quad (2)$$

This is nowadays more generally studied in what we now call *algebraic geometry* rather than differential geometry.

Apolonius of Penge (?) asked ‘what is the unique conic thorough five points, no three of which are co-linear?’

The space of conics is  $\mathbb{R}^6 - \{0\} / n = \mathbb{RP}^5$  (projective 5-space).

$$[a, b, c, d, e, f] \sim [\gamma a, \gamma b, \gamma c, \gamma d, \gamma e, \gamma f], \gamma \in \mathbb{R}^* \quad (3)$$

This is an application of geometry, rather than an application of differential geometry.

**Remark:** Apolonius proved this geometrically.

In this course however, we will look at the following.

- 1) Hamiltonian mechanics ( mid 19<sup>th</sup>). This is an elegant way of reformulating Newton’s mechanics, turning second order differential equations into first order differential equations with the use of a function  $H(p, q)$ . The system of ODEs is

$$\dot{q} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial q} \quad (4)$$

This led to the development of symplectic geometry ( 1960s). The connection is that the phase-space to which  $p$  and  $q$  belong has a 2 -form  $dp \wedge dq$  . Using the Hamiltonian function, one can find a vector field

$$X_H = \frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p} \quad (5)$$

and looks for a one-parameter group of transformations, called symplectomorphisms, generated by this vector field. Under these symplectomorphisms, the 2 -form is unchanged meaning that the area illustrated in F2 is preserved. Details of this are going to come within the course.

- 2) General Relativity (1915)  $\leftarrow$  Riemannian Geometry ( 1850)

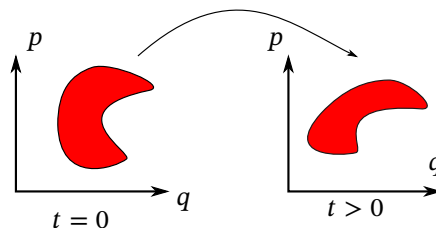


Figure 1

3) Gauge theory (Maxwell, Yang Mills)  $\leftrightarrow$  Connection on Principal Bundle (U(1) (Maxwell), SU(2), SU(3))

$$A_+ = A_- + dg \quad g = \psi_+ - \psi_- \quad \omega = \begin{cases} A_+ + d\psi_+ \\ A_- + d\psi_- \end{cases} \quad (6)$$

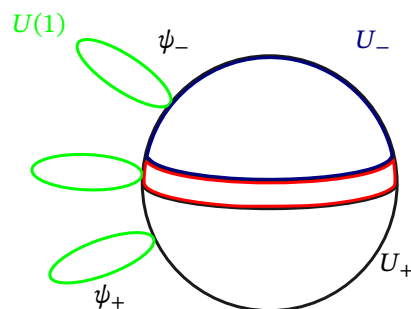


Figure 2

This course: cover 1, 2, 3 in some detail. Unifying feature: Lie groups.

- Prove some theorems, *lots of* examples (often instead of proofs)
- Want to be able to do calculations; compute characteristic classes etc.

We will assume that you took either Part III General Relativity, or Part III Differential Geometry, or some equivalent course.

# 1 Manifolds

**Definition 1** (manifold): An  $n$ -dimensional *smooth manifold* is a set  $M$  and a collection<sup>2</sup> of open sets  $U_\alpha$ , labelled by  $\alpha = 1, 2, 3, \dots$ , called *charts* such that

- $U_\alpha$  cover  $M$
- $\exists$  1-1 maps  $\phi_\alpha : U_\alpha \rightarrow V_\alpha \in \mathbb{R}^n$  such that

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta) \quad (1.1)$$

is a smooth map from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ .

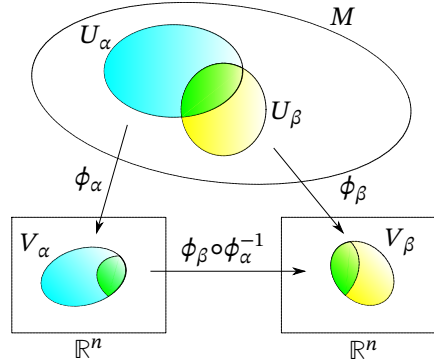


Figure 1.1: Manifold

As such, manifolds are topological spaces with additional structure, allowing us to do calculus.

**Example 1.0.1** ( $M = \mathbb{R}^n$ ): There is the *trivial manifold*, which can be covered by only one open set. There are other possibilities. In fact, there are infinitely many smooth structures on  $\mathbb{R}^4$  (Proof by Donaldson in 1984 in his PhD. He used Gauge theory).

<sup>2</sup>In all examples that we will look at, there will be finitely  $\alpha$ .

**Example 1.0.2** (sphere  $S^n = \{\mathbf{r} \in \mathbb{R}^{n+1}, |\mathbf{r}| = 1\}$ ): Have two open sets

$$U = S^n / \{0, 0, 0, \dots, 0, 1\} \quad \tilde{U} = S^n / \{0, 0, 0, \dots, 0, -1\} \quad (1.2)$$

We then define charts, where  $\mathbb{R}^n = (x_1, \dots, x_n)$ :

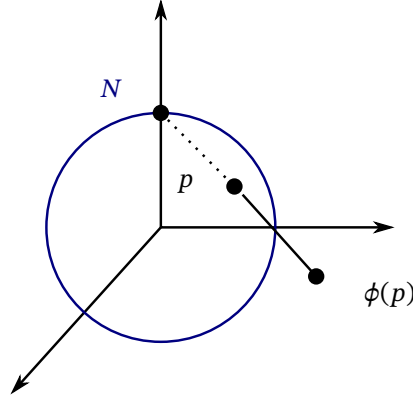


Figure 1.2

$$\begin{aligned} \phi(r_1, \dots, r_{n+1}) &= \left( \frac{r_1}{1 - r_{n+1}}, \dots, \frac{r_n}{1 - r_{n+1}} \right) \\ \text{on } \tilde{U}, \quad \tilde{\phi}(r_1, \dots, r_{n+1}) &= \left( \frac{r_1}{1 - r_{n+1}}, \dots, \frac{r_n}{1 - r_{n+1}} \right) = (\tilde{x}_1, \dots, \tilde{x}_n). \end{aligned} \quad (1.3)$$

On  $U \cap \tilde{U}$ ,

$$\frac{r_k}{1 + r_{n+1}} = \frac{1 - r_{n+1}}{1 + r_{n+1}} \frac{r_k}{1 - r_{n+1}}, \quad k = 1, \dots, n \quad (1.4)$$

$$\frac{1 - r_{n+1}}{1 + r_{n+1}} = \frac{(1 - r_{n+1})^2}{r_1^2 + r_2^2 + \dots + r_n^2} = \frac{1}{x_1^2 + x_2^2 + \dots + x_n^2} \quad (1.5)$$

So on  $\phi(U \cap \tilde{U})$ ,

$$(\tilde{x}_1, \dots, \tilde{x}_n) = \left( \frac{x_1}{x_1^2 + \dots + x_n^2}, \dots, \frac{x_n}{x_1^2 + \dots + x_n^2} \right) \quad (1.6)$$

are smooth maps from  $\mathbb{R}^n \rightarrow \mathbb{R}^n$

**Example 1.0.3:** A Cartesian product of manifolds is a manifold, for example we have the  $n$ -torus  $T^n = S^1 \times S^1 \times \dots \times S^1$ .

**Definition 2** (surface): Let  $f_1, \dots, f_k : \mathbb{R}^N \rightarrow \mathbb{R}$  be smooth functions. A surface  $f_1 = 0, \dots, f_k = 0$  is a manifold of dimension  $\dim n = N - k$  if the rank of the matrix  $\frac{\partial f_\alpha}{\partial x^i}$ ,  $\alpha = 1, \dots, k$  and  $i = 1, \dots, N$  is maximal and equal to  $k$  at all points of  $\mathbb{R}^N$ .

**Example 1.0.4:** The  $n$ -sphere  $S^n$  is a surface in  $\mathbb{R}^{n+1}$  with  $f_1 = 1 - |\mathbf{r}|^2$ .

**Theorem 1** (Whitney): Every smooth manifold of dimension  $n$  is an embedded surface in  $\mathbb{R}^N$ , where  $N \leq 2n$ .

If you enjoy using geometrical intuition and looking at surfaces, this theorem ensures that you can always do that and not lose generality.

**Definition 3** (real projective space): The  $n$ -dimensional *real projective space* is defined as the quotient

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim, \quad (1.7)$$

where we quotient out the equivalence classes  $[X_1, \dots, X_{n+1}] \sim [cX_1, \dots, cX_{n+1}]$  for all  $c \in \mathbb{R}^*$ . The  $[X_1, \dots, X_{n+1}]$  are called *homogeneous coordinates*.

In other words, this is the space of all lines through the origin in  $\mathbb{R}^{n+1}$ .

**Claim 1:**  $\mathbb{RP}^n$  is a smooth manifold of dimension  $n$  with  $(n + 1)$  open sets.

*Proof.* Let us define our open sets with respect to the homogeneous coordinates. We define the set  $U_\alpha : [X] \in \mathbb{RP}^n$  such that  $X_\alpha \neq 0$   $\alpha = 1, \dots, n + 1$ . We can now find local coordinates on  $\phi_\alpha : U_\alpha \rightarrow V_\alpha \in \mathbb{R}^n$

$$x_1 = \frac{X_1}{X_\alpha} \quad \dots \quad x_{\alpha-1} = \frac{X_{\alpha-1}}{X_\alpha} \quad x_{\alpha+1} = \frac{X_{\alpha+1}}{X_\alpha} \quad \dots \quad x_n = \frac{X_n}{X_\alpha}. \quad (1.8)$$

□

**Exercise 1.1:** Prove smoothness of  $\phi_\beta \circ \phi_\alpha^{-1}$ .

Now it turns out that this manifold is equivalent to  $\mathbb{RP}^n = S^n / \mathbb{Z}_2$ . From quantum mechanics, we know that this means in particular  $\mathbb{RP}^3 = SO(3)$ . This is illustrated in 1.3.

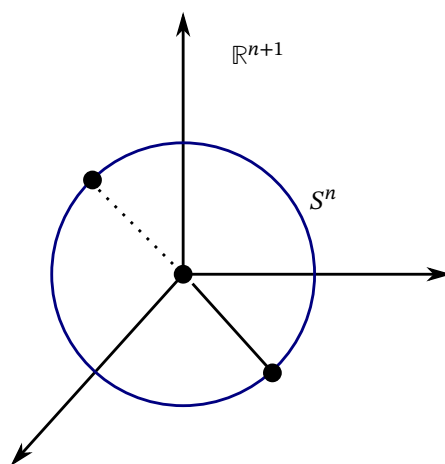


Figure 1.3: Real projective space  $\mathbb{RP}^n$  is isomorphic to  $S^n/\mathbb{Z}^n$ , identifying antipodal points.



## 2 Vector Fields

Let  $M, \tilde{M}$  be smooth manifolds of dimension  $n, \tilde{n}$ .

**Definition 4** (smooth map): A map  $f : M \rightarrow \tilde{M}$  is *smooth* if  $\tilde{\phi}_\beta \circ f \circ \phi_\alpha^{-1}$  is a smooth map from  $\mathbb{R}^n$  to  $\tilde{\mathbb{R}}^{\tilde{n}}$  for all  $\alpha, \beta$ . We call  $f : M \rightarrow \mathbb{R}$  a *function*, whereas we call  $f : \mathbb{R} \rightarrow M$  a *curve*.

Let  $\gamma : \mathbb{R} \rightarrow M$  be a curve. For some  $U \subset M$ ,  $U \simeq \mathbb{R}^n$ , we can define local coordinates  $(x^1, \dots, x^n)$

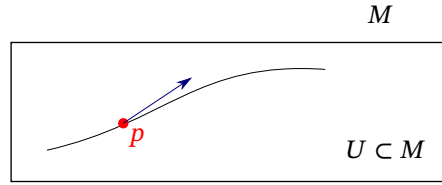


Figure 2.1

**Definition 5** (tangent vector): A *tangent vector*  $V$  to  $\gamma$  at  $p$  is

$$V|_p = \left. \frac{d\psi}{d\epsilon} \right|_{\epsilon=0} \in T_p M, \quad (2.1)$$

where  $T_p M$  is the *tangent space* to  $M$  at  $p$ .

**Definition 6** (tangent bundle): We define the *tangent bundle* as  $TM := \bigcup_{p \in M} T_p M$ .

**Definition 7** (vector field): A *vector field* assigns a tangent vector to all  $p \in M$ .

Let  $f : M \rightarrow \mathbb{R}$ . The rate of change of  $f$  along  $\gamma$  is

$$\left. \frac{d}{d\epsilon} f(x^a(\epsilon)) \right|_{\epsilon=0} = \sum_a \dot{x}^a \frac{\partial f}{\partial x^a} \quad (2.2)$$

$$= \sum_a V^a \left. \frac{\partial f}{\partial x^a} \right|_{\epsilon=0}, \quad (2.3)$$

where  $V^a := \dot{x}^a|_{\epsilon=0, \dots, x_n}$ .

Vector fields are first order differential operators

$$V = \sum_a V^a(\mathbf{x}) \frac{\partial}{\partial x^a}. \quad (2.4)$$

The derivatives  $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\} \Big|_p$  form a basis of  $T_p M$ .

## 2.1 Integral curves

**Definition 8** (integral curve): An *integral curve* (a *flow*) of a vector field is defined by

$$\dot{\gamma}(\epsilon) = V|_{\gamma(\epsilon)}, \quad (2.5)$$

where the dot denotes differentiation with respect to  $\epsilon$ .

On  $n$  first order ODEs:  $\dot{x}^a = V^a(x)$ .

There exists a unique solution given initial data  $X^a(0)$ . Given a solution  $X^a(\epsilon)$ , we can expand it in a Taylor series as

$$X^a(\epsilon) = X^a(0) + V^a \cdot \epsilon + O(\epsilon^2). \quad (2.6)$$

Up to first order in  $\epsilon$ , the vector field determines the flow. We call  $V$  a *generator* of its flow.

The following example illustrates how you get from a vector field to its flow.

**Example 2.1.1** ( $M = \mathbb{R}^2$ ,  $x^a = (x, y)$ ): Consider the vector field  $V = x \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ . The system of ODEs we solve is  $\dot{x} = x$  and  $\dot{y} = 1$ . This gives us the integral curve  $(x(\epsilon), y(\epsilon)) = (x(0)e^\epsilon, y(0) + \epsilon)$ . From this we can see that  $x(\epsilon) \cdot \exp(-y(\epsilon))$  is constant along  $\gamma$ . Using this we can draw the unparametrised integral curve in Fig. 2.2.

This example motivates the following definition.

**Definition 9** (invariant): An *invariant* of a vector field  $V$  is a function  $f$  constant along the flow of  $V$ .

$$f(x^a(0)) = f(x^a(\epsilon)) \quad \forall \epsilon. \quad (2.7)$$

Equivalently,  $V(f) = 0$ .

Let us now consider an example that goes the other way: from flow to vector field.

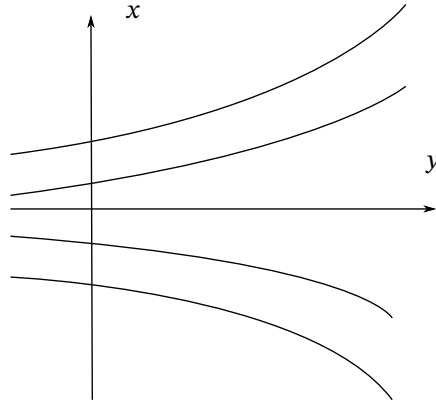


Figure 2.2

**Example 2.1.2:** Consider the 1-parameter group of rotations of a plane.

$$(x(\epsilon), y(\epsilon)) = (x_0 \cos \epsilon - y_0 \sin \epsilon, x_0 \sin \epsilon + y_0 \cos \epsilon). \quad (2.8)$$

The associated vector field is

$$V = \left( \frac{\partial y(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial y} + \frac{\partial x(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial x} \right) \Big|_{\epsilon=0} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}. \quad (2.9)$$

Now you can add vector fields, but there is also another operation.

**Definition 10** (Lie bracket): A *Lie bracket*  $[V, W]$  of two vector fields  $V, W$  is a vector field defined by

$$[V, W](f) = V(W(f)) - W(V(f)) \quad \forall f. \quad (2.10)$$

This is indeed another vector field since the commutator of two first order operators is another first order operator.

**Example 2.1.3:** Let  $V = x \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$  and  $W = \frac{\partial}{\partial x}$ . We then have  $[V, W] = -W$ .

This is not always the case but sometimes the Lie bracket reproduces some of the vector fields. There is an interesting algebraic structure to this.

**Definition 11** (Lie algebra): A *Lie algebra* is a vector space  $\mathfrak{g}$  with an anti-symmetric, bilinear operation  $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ , called a *Lie bracket*, which satisfies the *Jacobi identity*

$$[V, [U, W]] + [W, [V, U]] + [U, [W, V]] = 0 \quad \forall U, V, W \in \mathfrak{g}. \quad (2.11)$$

■ We will spend some time discussing this abstractly, but then focus on the Lie algebras of vector fields in the main part of this course.

Any two vector spaces of a given dimension are isomorphic; there is nothing special other than the dimension distinguishing vector spaces. For Lie algebras this is not so.

**Example 2.1.4:** Even in dimension 2, which is the lowest non-trivial dimension, there are two Lie algebras (up to isomorphism)

$$a) [V, W] = -W, \quad b) [V, W] = 0. \quad (2.12)$$

If the vector space underlying  $\mathfrak{g}$  is finite-dimensional, and  $V_\alpha, \alpha = 1, \dots, \dim \mathfrak{g}$  is a basis of  $\mathfrak{g}$ , we can define the Lie algebra by specifying the brackets

$$[V_\alpha, V_\beta] = \sum_\gamma f_{\alpha\beta}^\gamma V_\gamma, \quad (2.13)$$

where  $f_{\alpha\beta}^\gamma$  are the *structure constants*.

**Example 2.1.5** ( $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$ ): The vector space is given by  $n \times n$  real matrices, and the Lie bracket is the matrix commutator. The dimension of this Lie algebra is  $\dim \mathfrak{g} = n^2$ .

**Example 2.1.6** (Vector fields): The set of all vector fields on a manifold  $M$  form an infinite-dimensional Lie algebra.

**Example 2.1.7:** Consider  $\text{diff}(\mathbb{R})$  or  $\text{diff}(S^1)$ , vector fields on a line or on a circle respectively.

$$\text{diff}(\mathbb{R}), \quad x \in \mathbb{R}, \quad V_\alpha = -x^{\alpha+1} \frac{\partial}{\partial x} \quad (2.14)$$

$$\text{diff}(S^1), \quad \theta \in S^1, \quad V_\alpha = ie^{i\alpha\theta} \frac{\partial}{\partial \theta} \quad (2.15)$$

$$[V_\alpha, V_\beta] = (\alpha - \beta)V_{\alpha+\beta}. \quad (2.16)$$

**Example 2.1.8** (Virasoro algebra): The *Virasoro algebra*  $\text{Vir} = \text{diff}(S^1) \oplus \mathbb{R}$  is the central extension<sup>1</sup> of  $\text{diff}(S^1)$ , with *central charge*  $c = \mathbb{R}$ .

$$\begin{cases} [V_\alpha, c] = 0 \\ [V_\alpha, V_\beta]_{\text{vir}} = (\alpha - \beta)V_{\alpha+\beta} + \frac{c}{12}(\alpha^3 - \alpha)\delta_{\alpha+\beta, 0} \end{cases} \quad (2.17)$$

**Remark:**

$$[f(\theta) \frac{\partial}{\partial \theta}, g(\theta) \frac{\partial}{\partial \theta}] = \underbrace{(fg' - gf')}_{\text{Wronskian}} \frac{\partial}{\partial \theta} \quad (2.18)$$

‘After Witten’.

$$[f \frac{\partial}{\partial \theta}, g \frac{\partial}{\partial \theta}]_{\text{vir}} = [f \frac{\partial}{\partial \theta}, g \frac{\partial}{\partial \theta}] + \frac{ic}{48\pi} \int_0^{2\pi} (f'''g - g'''f) d\theta \quad (2.19)$$

**Theorem 2** (Ado): Every finite-dimensional Lie algebra is isomorphic to some matrix Lie algebra, a subalgebra of  $\mathfrak{gl}(n, \mathbb{R})$ .

<sup>1</sup>We will meet the concept of central extension and central charge in this term’s *String Theory* course.

**Remark:**  $n$  is not necessarily the dimension of the Lie algebra.

### 3 Lie Groups

**Definition 12** (Lie group): A *Lie group* is a smooth manifold  $G$ , which is also a group, such that the group operations

$$\text{multiplication} \quad G \times G \rightarrow G, \quad (g_1, g_2) \mapsto g_1 \cdot g_2 \quad (3.1)$$

$$\text{inverse} \quad G \rightarrow G, \quad g \mapsto g^{-1} \quad (3.2)$$

are smooth maps between manifolds.

**Example 3.0.1** ( $G = GL(n, \mathbb{R}) \in \mathbb{R}^{n^2}$ ): The general linear group  $GL(n, \mathbb{R})$  is defined as the set of invertible matrices  $\{g \in G \mid \det g \neq 0\}$ . The dimension is  $\dim(G) = n^2$ .

**Example 3.0.2** ( $G = O(n, \mathbb{R})$ ): This is the group of orthogonal matrices, defined by  $\frac{1}{2}n(n+1)$  conditions  $g^T g = \mathbb{1}$ . The dimension is then  $\dim O(n, \mathbb{R}) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$ . We also have to check that these conditions define a manifold in the sense that the associated Jacobian has maximal rank.

**Definition 13** (group action): A *group action* on a manifold  $M$  is a map  $G \times M \rightarrow M$  mapping  $(g, p) \rightarrow g(p)$  such that

$$e(p) = p, \quad g_1(g_2(p)) = (g_1 \cdot g_2)(p) \quad (3.3)$$

for all  $p \in M$  and all  $g_1, g_2$  on  $G$ .

**Definition 14** (transformation group): If we have a group action, we refer to  $G$  as a group of *transformations*.

**Example 3.0.3:** Take  $M = \mathbb{R}^2$  and  $G = E(2)$ , the three-dimensional Euclidean group.

$$g \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \quad (3.4)$$

Take  $G_E \in G$  to be a one-parameter subgroup of  $G$ . There are three such subgroups

$$G_\theta : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos \theta \cdot x - \sin \theta \cdot y \\ \sin \theta \cdot x + \cos \theta \cdot y \end{pmatrix} \quad (3.5)$$

$$G_a : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x + a \\ y \end{pmatrix} \quad (3.6)$$

$$G_b : \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x \\ y + b \end{pmatrix}. \quad (3.7)$$

Each of these one-parameter subgroups generates a flow. We can think of this flow as being gener-

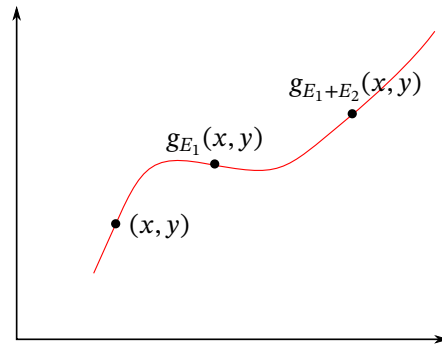


Figure 3.1

ated by a vector field  $V|_p = \left. \frac{d}{dE} g_E(p) \right|_{E=0}$ .

$$V_\theta = d \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \quad (3.8)$$

$$V_a = \left( \frac{d\tilde{x}}{da} \frac{\partial}{\partial \tilde{x}} + \frac{d\tilde{y}}{da} \frac{\partial}{\partial \tilde{y}} \right) \Big|_{a=0} = \frac{\partial}{\partial x} \quad (3.9)$$

$$V_b = \frac{\partial}{\partial y}. \quad (3.10)$$

We define a 3-dimensional Lie algebra of  $E(2)$  as

$$[V_a, V_\theta] = V_b \quad [V_b, V_\theta] = -V_a \quad [V_a, V_b] = 0, \quad (3.11)$$

represented by vector fields on  $M$ .



### 3.1 Geometry on Lie Groups

**Definition 15** (tangent map): Let  $f : M \rightarrow \tilde{M}$  be a smooth map between manifolds. We define the *tangent map* or *push forward* to be

$$\begin{aligned} f_* : T_p(M) &\rightarrow T_{f(p)}(\tilde{M}) \\ V &\mapsto f_*(V) = \left. \frac{d}{dE} f(\gamma(E)) \right|_{E=0}. \end{aligned} \quad (3.12)$$

This extends to the tangent bundle  $T(M)$ . If  $x^\alpha$  are coordinates of  $\mathcal{M} \supset M$ ,  $(y^{\alpha'})$  coordinates on  $\tilde{\mathcal{M}} \subset \tilde{M}$ , then

$$V = V^\alpha \frac{\partial}{\partial x^\alpha} \quad f_*(V) = V^\alpha \frac{\partial f^i}{\partial x^\alpha} \frac{\partial}{\partial y^i}. \quad (3.13)$$

**Definition 16** (Lie derivative): Let  $V, W$  be vector fields, where  $V$  generates a flow  $V = \dot{\gamma}$ . The *Lie derivative* is

$$\mathcal{L}_V W|_p := \lim_{\epsilon \rightarrow 0} \frac{W(p) - \gamma(\epsilon)_* W(p)}{\epsilon} \quad (3.14)$$

We can extend this definition over the whole manifold.

**Exercise 3.1:** Show that  $L_V W = [V, W]$ .

**Definition 17:** On functions  $f : M \rightarrow \mathbb{R}$ , we define the Lie derivative as  $\mathcal{L}_V(f) = V(f)$ .

**Claim 2** (Cartan's Magic Formula): On differential forms, we can use the Leibniz rule to show that

$$\mathcal{L}_V \Omega = d(\iota_V \Omega) + \iota_V(d\Omega) = d(V \lrcorner \Omega) + V \lrcorner d\Omega. \quad (3.15)$$

**Definition 18:** We define the cotangent space  $T_p^*M = \text{Span}\{dx^1, \dots, dx^n\}$  as the space of one-forms. The cotangent bundle is then

$$\bigcup_{p \in M} T_p^*M = T^*M. \quad (3.16)$$

**Definition 19** ( $r$ -form): Using the wedge product, which is anti-commutative on one-forms  $dx^i \wedge dx^j = -dx^j \wedge dx^i$ , we can define an  $r$ -form

$$\Omega = \frac{1}{r!} \Omega_{ij\dots k} dx^i \wedge dx^j \wedge \dots \wedge dx^k. \quad (3.17)$$

**Definition 20** (contraction): We write a *contraction* as

$$\frac{\partial}{\partial x^i} \lrcorner dx^j = \iota_{\frac{\partial}{\partial x^i}} dx^j = \delta_i^j. \quad (3.18)$$

For a general vector field  $V$  and one-form  $\Omega$ , we have

$$\iota_V \Omega = V \lrcorner \Omega = V^i \frac{\partial}{\partial x^i} \lrcorner \Omega_j dx^j = V^i \Omega_j \delta_i^j = V^i \Omega_i. \quad (3.19)$$

**Remark:** No metric is needed to define contraction.

**Definition 21:** A *Lie algebra*  $\mathfrak{g}$  of a Lie group  $G$  is the tangent space  $T_e G$  to  $G$  at the identity. The Lie bracket on  $\mathfrak{g}$  is the commutator of vector fields on  $G$ .

**Definition 22** (Left translations): For all  $g \in G$ , we define the *left translations*

$$\begin{aligned} L_g : G &\rightarrow G \\ h &\mapsto g \cdot h \end{aligned} \quad (3.20)$$

**Definition 23** (left invariant vector fields): Using the left translation maps, we define their push forward  $(L_g)_* : \mathfrak{g} \equiv T_e G \rightarrow T_g G$ , which maps  $V \in \mathfrak{g}$  to vector fields  $(L_g)_*(V)$  on  $G$ . This defines *left invariant vector fields* as sections  $V \in TG$  such that  $(L_g)_* V = V$  for all  $g \in G$ . Therefore,

$$[(L_g)_*V, (L_g)_*W] = (L_g)_*[V, W]_{\mathfrak{g}}. \quad (3.21)$$

**Remark:** It is important to understand the notation!

Left-invariant vector fields form a basis of  $\mathfrak{g}$ , meaning that  $\dim(G)$  is the number of global, non-vanishing vector fields on  $G$ .

**Definition 24** (parallelisable): A manifold  $M$  is *parallelisable* if there exists a set of vector fields  $\{V_i\}$ , where  $i = 1, \dots, \dim M$ , such that  $\forall p \in M$ , the tangent vectors  $\{V_i(p)\}$  form a basis for the tangent space  $T_p M$ .

**Claim 3:** Lie groups are parallelisable manifolds.

**Claim 4:** The converse is not true.

*Proof.*  $S^1, S^3, S^7$  are the only parallelisable spheres.

The first two are indeed manifolds:

$$S^1 = U(1), \quad S^3 = SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1 \right\}. \quad (3.22)$$

However,  $S^7$  is not a Lie group. □

■ This has introduced the field of  $K$ -theory.

**Claim 5:** Let  $\{L_\alpha\}$ ,  $\alpha = 1, \dots, \dim \mathfrak{g}$  be a basis of left invariant vector fields with  $[L_\alpha, L_\beta] = \sum_\gamma f_{\alpha\beta}^\gamma L_\gamma$ . Let  $\sigma^\alpha$  be a dual basis of left-invariant one-forms, meaning that  $L_\alpha \lrcorner \sigma^\beta = \delta_\alpha^\beta$ . Then

$$d\sigma^\alpha + \frac{1}{2} f_{\beta\gamma}^\alpha \sigma^\beta \wedge \sigma^\gamma = 0. \quad (3.23)$$

*Proof (Sheet 1).* Use the identity

$$d\Omega(V, W) = V(\Omega(W)) - W(\Omega(V)) - \Omega([V, W]). \quad (3.24)$$

□

■ Watch out for signs and factors in the upcoming derivations! Things can easily go wrong.

**Definition 25** (Maurer–Cartan 1-form): Assume that  $G$  is a matrix Lie group. The *Maurer–Cartan one-form* on  $G$  is then defined as

$$\rho := g^{-1} dg. \quad (3.25)$$

**Claim 6:** The Maurer–Cartan 1-form

1. is left invariant,
2. takes values in the Lie algebra,
3. obeys the *Maurer–Cartan equation*

$$d\rho + \rho \wedge \rho = 0. \quad (3.26)$$

*Proof.* 1. With  $g_0 \in G$  we have

$$(g_0 g)^{-1} d(g_0 g) = g^{-1} dg. \quad (3.27)$$

2. Take  $C$  a smooth curve  $g(s) \subset G$ .

$$g^{-1}(s)g(s + \epsilon) = \underbrace{\epsilon}_{\mathbb{1}} + \underbrace{\epsilon g^{-1} \frac{dg}{ds}}_{\in T_{g_0} G \simeq \mathfrak{g}}|_{\epsilon=0} + O(\epsilon^2). \quad (3.28)$$

So  $g^{-1}dg = \sum_{\alpha} \sigma^{\alpha} \otimes T_{\alpha}$ , where  $T_{\alpha}$  are matrices with  $[T_{\alpha}, T_{\beta}] = \sum_{\gamma} f_{\alpha\beta}^{\gamma} T_{\gamma}$ .

3. Consider first the exterior derivative term

$$d\rho = \sum_{\alpha} d\sigma^{\alpha} \cdot T_{\alpha} = -\frac{1}{2} f_{\beta\gamma}^{\alpha} \sigma^{\beta} \wedge \sigma^{\gamma} \cdot T_{\alpha}. \quad (3.29)$$

The wedge product term is

$$\rho \wedge \rho = \sigma^{\alpha} T_{\alpha} \wedge \sigma^{\beta} T_{\beta} = \frac{1}{2} \sigma^{\alpha} \wedge \sigma^{\beta} [T_{\alpha}, T_{\beta}] = \frac{1}{2} \sigma^{\alpha} \wedge \sigma^{\beta} f_{\alpha\beta}^{\gamma} T_{\gamma}. \quad (3.30)$$

□

**Example 3.1.1** (Heisenberg group): The *Heisenberg group* (sometimes just called *Nil*) is the group of upper-triangular matrices

$$g = \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} = \mathbb{1} + xT_1 + yT_2 + zT_3, \quad (3.31)$$

where  $\{T_i\}$  are also the generators of the Lie algebra. Explicitly, we have

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.32)$$

and so their commutation relations are

$$[T_1, T_2] = T_3, \quad [T_1, T_3] = 0 = [T_2, T_3]. \quad (3.33)$$

We can interpret  $T_1 = \hat{x}$  as position,  $T_2 = \hat{p}$  as momentum, and  $T_3 = i\hbar \hat{1}$  as the identity.

**Remark:** Examples such as the lecture today will be important for the exam! If someone gives you a matrix Lie group, you will proceed in this order.

Taking the inverse of (3.31), we construct the Maurer-Cartan 1-form

$$\rho = g^{-1}dg = \begin{pmatrix} 1 & -x & -z + xy \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx & dz \\ & dy \end{pmatrix} \quad (3.34)$$

$$= T_1 dx + T_2 dy + T_3(dz - xdy). \quad (3.35)$$

We define the dual basis of left-invariant one-forms

$$\sigma^1 = dx, \quad \sigma^2 = dy, \quad \sigma^3 = dz - xdy \quad (3.36)$$

$$d\sigma^1 = 0, \quad d\sigma^2 = 0, \quad d\sigma^3 = -dx \wedge dy = d\sigma^1 \wedge \sigma^2. \quad (3.37)$$

From these we find the left-invariant vector fields

$$L_1 = \frac{\partial}{\partial x}, \quad L_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad L_3 = \frac{\partial}{\partial z}. \quad (3.38)$$

We find the Lie algebra by computing the brackets

$$[L_1, L_2] = L_3, \quad [L_1, L_3] = 0, \quad [L_2, L_3] = 0. \quad (3.39)$$

This is the same as (3.33), the Lie algebra of the Heisenberg group. We have a choice of representing the Lie algebra either in terms of matrices,  $T_i$  or left-invariant vector fields  $L_i$ .

### 3.1.1 Right-Invariance

We could have defined the *right translations*  $R_g(h) = h \cdot g$ , and associated right-invariant vector fields and one-forms.

**Claim 7:** The commutation relations for right-invariant vector fields differs by a minus sign from left-invariant vector fields and they commute

$$[R_\alpha, R_\beta] = -f_{\alpha\beta}^\gamma R_\gamma, \quad [R_\alpha, L_\beta] = 0. \quad (3.40)$$

**Example 3.1.2** (Nil): For the Heisenberg group, using  $dg \cdot g^{-1}$

$$R_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad R_2 = \frac{\partial}{\partial y}, \quad R_3 = \frac{\partial}{\partial z}. \quad (3.41)$$

**Notation** (2): Right-invariant vector fields are said to generate left-translations and vice-versa.

## 3.2 Metrics on Lie Groups

We can do some local differential geometry by defining a left-invariant metric on  $G$ .

**Definition 26** (left-invariant metric): A *left-invariant metric*  $h$  on a Lie group  $G$  is of the form

$$h = g_{\alpha\beta} \sigma^\alpha \odot \sigma^\beta, \quad \alpha, \beta = 1, \dots, \dim G, \quad (3.42)$$

where  $g_{\alpha\beta}$  is a non-degenerate constant matrix.

Given the group and left-invariant metric, the right-invariant vector fields are generating isometries for this metric, which means that they are Killing vectors. Recall that the Maurer–Cartan 1-form is  $\rho = g^{-1}dg = \sigma^\alpha \otimes T_\alpha$ , where  $\sigma^\alpha$  are invariant under  $g \rightarrow g_0g$ . Therefore

$$\mathcal{L}_{R_\alpha} \sigma^\beta = 0 \quad \forall \alpha, \beta. \quad (3.43)$$

Therefore, right invariant vector fields are Killing vectors for  $h$ , meaning that

$$\mathcal{L}_{R_\alpha}(h) = 0. \quad (3.44)$$

**Example 3.2.1** (Nil): The metric is defined as

$$h = \delta_{\alpha\beta} \sigma^\alpha \cdot \sigma^\beta = dx^2 + dy^2 + (dz - xdy)^2. \quad (3.45)$$

How do we find the isometries?

- We can see that the metric components do not involve  $z$ , so it is invariant under  $z \rightarrow z + \omega$ , which is generated by  $\frac{\partial}{\partial z} = R_3$ .
- Similarly, we can see the same for  $y \rightarrow y + \epsilon$ , which is generated by  $\frac{\partial}{\partial y} = R_2$ .
- Finally, let us consider what happens for  $x \rightarrow x + \epsilon$ . As it stands, this is not an isometry. The parenthesis includes a term  $\delta dy$ , which we can get rid off by introducing another transformation  $z \rightarrow z + \epsilon y$ . This is generated by  $\frac{\partial}{\partial x} + y \frac{\partial}{\partial z} = R_1$ .

These agree with the right-invariant vector fields of Eq. (3.41).

### 3.2.1 Kaluza–Klein Interpretation

Consider the motion of a charged particle on the space of orbits of  $R_3 = \frac{\partial}{\partial z}$ .

This was introduced as a way to combine the two known forces at the time: gravity and electromagnetism. This is done by introducing extra dimensions, by looking at gravity in dimension  $d = 5$ , which reduces to gravity and electromagnetism in dimension  $d = 4$ . The higher dimension is taken to be compactified into a circle of very large radius, so it is not detected by experiment. This idea is still stuck with us today in *String Theory*.

**Example 3.2.2:** For the Heisenberg group, the metric is independent of  $z$ , so we can take our manifold to be periodic in  $z$ .

To find the equations of motion, it is useful to write down the geodesic Lagrangian

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + (\dot{z} - x\dot{y})^2), \quad (3.46)$$

where the dot denotes  $\bullet = \frac{d}{ds}$ . The Euler–Lagrange equations are

$$\frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = \frac{\partial \mathcal{L}}{\partial x^i} \quad (3.47)$$

$$\ddot{x} = -\dot{y}(\dot{z} - x\dot{y}) \quad (3.48)$$

$$\frac{d}{ds}(\dot{y} - x(\dot{z} - x\dot{y})) = 0 \quad (3.49)$$

$$\frac{d}{ds}(\dot{z} - x\dot{y}) = 0. \quad (3.50)$$

Using the final equation to introduce a constant  $c = \dot{z} - x\dot{y}$ , the first two equations reduce to

$$\ddot{x} = -c\dot{y} \quad \& \quad \ddot{y} = c\dot{x}. \quad (3.51)$$

Compare this with geodesic motion in a magnetic field. Let the spacetime be the Riemannian manifold  $(M, g = g_{ij}dx^i \odot dx^j)$  and the magnetic field be the closed 2-form  $F = \frac{1}{2}F_{ij}dx^i \wedge dx^j$ . The components of the Levi–Civita connection associated to  $g$  are

$$\Gamma_{jk}^i = \frac{1}{2}g^{il} \left( \frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right). \quad (3.52)$$

The geodesic equation of motion is then given by

$$\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = cF^i_j \dot{x}^j, \quad (3.53)$$

where we see  $F^i_j$  as an endomorphism rather than a 2-form. This is the general form of geodesic motion in a magnetic field. We want to compare this to (3.51), so we take  $M = \mathbb{R}^2$  and  $g_{ij} = \delta_{ij}$ , which gives  $\Gamma_{ij}^k = 0$ . Moreover, we take  $F = -dx \wedge dy$ , meaning that  $F_{ij} = -\epsilon_{ij}$  is the volume-form. So geodesics of the left-invariant metric  $h$  on  $G = \text{Nil}$  projects to the trajectories of a charged particle in a constant magnetic field. We think of this as in Fig. 3.2.

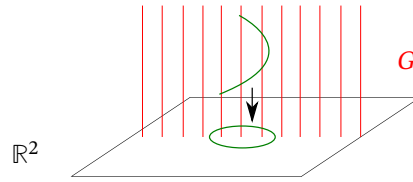


Figure 3.2: Kaluza–Klein reduction.

For the general case of Kaluza–Klein reduction, we have

$$h = (dz + A)^2 + g_{ij} dx^i \odot dx^j. \quad (3.54)$$

Take coordinates  $(z, x^i)$ , where  $z$  is the extra dimension. Moreover,  $A$  is a one-form on  $M$  and we define  $F = dA$ . The geodesic equation is

$$\dot{z} + A_i \dot{x}^i = c \quad (\text{conserved charge}) \quad (3.55)$$



Assume  $z$  is periodic. On  $G$ , the Laplacian is

$$\nabla^2 = L_1^2 + L_2^2 + L_3^2. \quad (3.56)$$

The Schrödinger equation is then  $-\nabla^2 \phi = E\phi$ .

$$\phi_{xx} + \phi_{zz} + (\partial_y + x d_z)^2 \phi = -E\phi. \quad (3.57)$$

Changing variables to  $\phi = \Psi(x, y)e^{iez}$ , we have

$$\Psi_{xx} - e^2 \Psi + (\partial_y + ixe)^2 \Psi = -E\Psi. \quad (3.58)$$

We can package some of the terms together to obtain a Maxwell potential. Recall that  $F = dy \wedge dx = dA$ . The equation for a charged particle moving in a magnetic field is

$$(\partial_x - ieA_x)^2 \Psi + (\partial_y - ieA_y)^2 \Psi = -(E - e^2)\Psi. \quad (3.59)$$

Comparing these equations, find that  $A_x = 0$  and  $A_y = -x$ , which means that we have

$$A = -x dy, \quad dA = -dx \wedge dy = F. \quad (3.60)$$

Landau problem;  $0 \leq z < 2\pi L$ . Charge quantisation  $e \cdot L \in \mathbb{Z}$ .

### 3.2.2 Killing Metric

Recall the Maurer–Cartan one-form  $\rho = g^{-1}dg = \sigma^\alpha \otimes T_\alpha$ . Define a metric to be

$$h := -\text{Tr}(g^{-1}dg \odot g^{-1}dg) \quad (3.61)$$

$$= -\text{Tr}(T_\alpha \cdot T_\beta) \sigma^\alpha \odot \sigma^\beta \quad (3.62)$$

$$= h_{\alpha\beta} \sigma^\alpha \odot \sigma^\beta \quad (3.63)$$

$$= -\text{Tr}(dg \cdot g^{-1} \odot dgg^{-1}). \quad (3.64)$$

This is both left-invariant and right-invariant. We say that the metric is *bi-invariant*.

**Example 3.2.3** ( $G = SU(2)$ ): As a manifold,  $SU(2)$  is the three-dimensional sphere  $S^3$ . The Killing metric will be the round metric on  $S^3$ . Its isometry group is  $SO(4)$ . It fits into  $SO(3) \rtimes SO(3)$ ; one of these generates the left-invariant vector fields and the other the right-invariant ones.

## 4 Hamiltonian Mechanics and Symplectic Geometry

Let  $M$  be a  $2n$ -dimensional manifold, which we refer to as the *phase space*. It does not come equipped with a metric, but there is another structure on it.

**Definition 27** (Poisson bracket): If  $f, g : M \rightarrow \mathbb{R}$  are functions on the phase space, then their *Poisson bracket* is

$$\{f, g\}_{\text{PB}} = \frac{\partial f}{\partial q^a} \frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a} \frac{\partial g}{\partial q^a}, \quad (4.1)$$

where  $(q^a, p_a)$ , with  $a = 1, \dots, n$  is a local coordinate system on  $M$ .

**Definition 28** (Hamiltonian): The *Hamiltonian* is a function  $H : M \rightarrow \mathbb{R}$  such that Hamilton's equations hold:

$$\dot{p}_a = -\frac{\partial H}{\partial q^a}, \quad \dot{q}^a = \frac{\partial H}{\partial p_a}. \quad (4.2)$$

**Definition 29** (Hamiltonian vector field): The *Hamiltonian vector field* is

$$X_H = \frac{\partial H}{\partial p_a} \frac{\partial}{\partial q^a} - \frac{\partial H}{\partial q^a} \frac{\partial}{\partial p_a}, \quad (4.3)$$

whose integral curves are  $t \rightarrow (\mathbf{p}(t), \mathbf{q}(t))$ .

A more general framework is kicked off by the following definition.

**Definition 30** (Poisson manifold): Let  $M$  be phase space and  $\omega^{ij} = \omega^{[ij]}$ , for  $i, j = 1, \dots, \dim M = m$ , be a tensor field on  $M$ . We call  $(M, \omega^{ij} = \omega)$  a *Poisson manifold* and  $\omega$  a *Poisson structure*, if the Poisson bracket

$$\{f, g\}_{\text{PB}} = \omega^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \quad (4.4)$$

is such that the Jacobi identity

$$\left\{f, \left\{g, h\right\}_{\text{PB}}\right\}_{\text{PB}} + \left\{h, \left\{f, g\right\}_{\text{PB}}\right\}_{\text{PB}} + \left\{g, \left\{h, f\right\}_{\text{PB}}\right\}_{\text{PB}} = 0, \quad (4.5)$$

holds for all functions  $f, g, h$  on  $M$ .

Not any anti-symmetric  $\omega^{ij}$  satisfies this; the Jacobi identity gives conditions on  $\omega^{ij}$ .

**Remark:** We do not distinguish between position and momenta  $x^i$ .

In the following, we will drop the subscript 'PB' to denote the Poisson bracket.

**Example 4.0.1:** Let  $M = \mathbb{R}^3$  and  $\omega^{ij} = \epsilon^{ijk} x^k$ . The Poisson brackets are

$$\{x^1, x^2\} = x^3, \quad \{x^3, x^1\} = x^2, \quad \{x^2, x^3\} = x^4. \quad (4.6)$$

We can then define the Casimir

$$f(r) = r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2. \quad (4.7)$$

This Poisson commutes with the  $x^i$ , meaning that  $\{f(r), x^i\} = 0$ . For example, we have

$$\{(x^1)^2 + (x^2)^2 + (x^3)^2, x^1\} = 2x^2 \overbrace{\{x^2, x^1\}}^{-x^3} + 2x^3 \overbrace{\{x^3, x^1\}}^{x^2} = 0. \quad (4.8)$$

Take the Hamiltonian to be

$$H = \frac{1}{2} \left( \frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + \frac{(x^3)^2}{a_3} \right). \quad (4.9)$$

Then the time-evolution is given by

$$\dot{g} = \{g, H\}, \quad \underbrace{\dot{x}^i = \omega^{ij} \frac{\partial H}{\partial x^j}}_{\text{Hamilton's equations}}. \quad (4.10)$$

Writing Hamilton's equations out explicitly gives Euler's equations for a rigid body

$$\dot{x}^1 = \frac{a_3 - a_2}{a_2 a_3} x^2 x^3, \quad \dot{x}^2 = \frac{a_1 - a_3}{a_1 a_3} x^1 x^3, \quad \dot{x}^3 = \frac{a_2 - a_1}{a_2 a_1} x^2 x^1. \quad (4.11)$$

**Example 4.0.2:** Let us restrict the Poisson structure from the first example to  $S^2 \subset \mathbb{R}^3$ .

$$x^1 = \sin \theta \cos \phi, \quad x^2 = \sin \theta \sin \phi, \quad x^3 = \cos \theta. \quad (4.12)$$

So  $\theta, \phi$  are functions of  $\mathbb{R}^3$ . Get

$$\{\theta, \phi\} = \frac{1}{\sin \theta} \quad (4.13)$$

(Exercise) and a Poisson structure on  $S^2$ , which is non-degenerate,

$$(\omega^{-1})_{ij} dx^i \wedge dx^j = \underbrace{\sin \theta d\theta \wedge d\phi}_{\text{symplectic structure}} \quad (4.14)$$

where  $x^1 = (\theta, \phi)$ .

Last time, we discussed Poisson structures, which we can now specialise.

**Definition 31** (symplectic manifold): A *symplectic manifold* is a smooth manifold  $M$  of dimension  $2n$  with a closed 2-form  $\omega \in \Lambda^2(\mathcal{M})$ , which is non-degenerate, meaning that

$$\underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_n \neq 0, \quad (4.15)$$

or  $\omega$  is a  $2 \times 2$  matrix of maximal rank.

The symplectic form  $\omega$  provides an isomorphism between  $TM$  and  $T^*M$  as

$$v \in TM \mapsto v \mapsto \omega \in T^*M. \quad (4.16)$$

However, we have to be more careful as this is now antisymmetric. If  $f : M \rightarrow \mathbb{R}$ , then  $df$  is a 1-form and is naturally associated to a Hamiltonian vector field  $X_f$ ,

$$X_f \mapsto \omega = -df, \quad (4.17)$$

where  $f$  is the Hamiltonian.

**Claim 8:** We can define a Poisson bracket by

$$\{f, g\}_{\text{PB}} := X_g(f) = \omega(X_g, X_f) \quad (4.18)$$

**Remark:** Note that this is antisymmetric since  $\omega(X_g, X_f) = -\omega(X_f, X_g)$ .

In local coordinates, the Poisson bracket is

$$\{f, g\} = \sum_{i,j=1}^{2n} \omega^{ij} \frac{\partial f}{\partial x^j} \frac{\partial g}{\partial x^i}. \quad (4.19)$$

**Exercise 4.1:** The Jacobi identity follows from the closure  $d\omega = 0$ .

If we compute the Lie bracket of two vector fields  $X_f, X_g$ , we find the *anti-homomorphism*

$$[X_f, X_g] = -X_{\{f, g\}}. \quad (4.20)$$

Hamiltonian vector fields preserve the symplectic form. The finite way of saying this is that they generate a flow under which the symplectic form is invariant. Infinitesimally this is expressed with the Lie derivative as

$$\mathcal{L}_{X_f} \omega = d(X_f \lrcorner \omega) + X_f \lrcorner d\omega = -d(df) = 0. \quad (4.21)$$

**Theorem 3** (Darboux): Let  $(M, \omega)$  be a  $2n$ -dimensional symplectic manifold. There exist *local* coordinates  $x^1 = q^1, \dots, x^n = q^n, x^{n+1} = p_1, \dots, x^{2n} = p_n$  around any point in  $M$ , such that

$$\omega = \sum_{a=1}^n dp_a \wedge dq^a, \quad (4.22)$$

and the Poisson bracket takes the standard form.

*Proof.* The proof proceeds by induction with respect to half of the dimension of the symplectic manifold. We first choose an arbitrary function  $p_1 : M \rightarrow \mathbb{R}$  (which will later become the first momentum coordinate). Given this function, we search for another function  $q^1 : M \rightarrow \mathbb{R}$  such that

$$X_{p_1}(q^1) = 1. \quad (4.23)$$

We denote this as  $\dot{q}^1 = 1$ . This is an ordinary differential equation, which will have a solution<sup>1</sup> subject to some initial condition.

The second step is as follows. Consider the surface  $M_1 = \{x \in M, p_1 = \text{const.}, q^1 = \text{const.}\}$ . This is a submanifold  $M_1 \subset M$ , which we can prove from the embedding theorem; the maximal rank condition of the Jacobian is encoded in (4.23). This  $M_1$  is locally symplectic with symplectic form  $\omega_1 \equiv \omega|_{p_1, q^1 \text{ const.}}$ .

Now look for  $p_2, q^2$  and so on. A full proof is given in the book by Arnold. □

**Claim 9:** Let  $Q$ , which will play the role of configuration space, be an  $n$ -dimensional manifold. The cotangent bundle  $T^*Q$  admits a global symplectic structure.

*Proof.* Our goal is to describe what this symplectic structure is. We have a projection  $\pi : T^*Q \rightarrow Q$  acting as  $\pi(q, p) = q$ .

**Definition 32** (pull-back): Given a map  $f : M \rightarrow N$ , the *pull-back*  $f^* : T_{f(p)}^*N \rightarrow T_p^*M$  defined as

$$f^*(p)(V) = p(f_*V). \quad (4.24)$$

In local coordinates, taking  $x^i$ , with  $i = 1, \dots, \dim M$  coordinates on  $M$  and  $y^a$  with  $a = 1, \dots, \dim N$  coordinates on  $N$ , we can use the chain rule to write explicitly

$$f^*(dy^a) = \sum_i \frac{\partial f^a}{\partial x^i} dx^i. \quad (4.25)$$

The pull-back of  $\pi$  is a map

$$\pi^* : T^*(Q) \rightarrow T^*(T^*Q). \quad (4.26)$$

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<sup>1</sup>The assumption for the existence theorem would be that we work in a Lipschitz-class of functions.

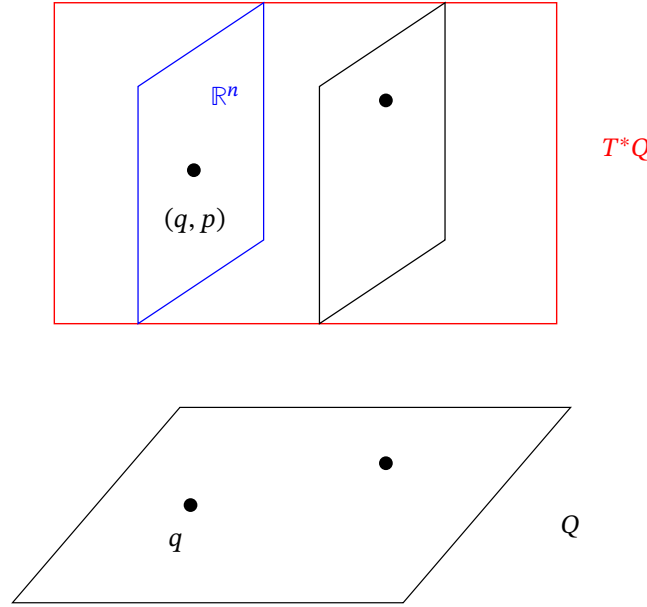


Figure 4.1

Say that  $p \in T^*(Q)$  is a one-form on  $Q$ . We can define  $\theta = \pi^*(p)$  and from this we get our canonical symplectic form

$$\omega = d\theta. \quad (4.27)$$

This is manifestly closed. This construction does not depend on coordinates, but usually in  $(p_a, q^a)$  coordinates we have

$$\theta = p_a dq^a, \quad \omega = d\theta = dp_a \wedge dq^a. \quad (4.28)$$

□

We can define yet another structure that symplectic geometry gives us.

**Definition 33** (Canonical transformations): Let  $(M, \omega)$  be a  $2n$ -dimensional symplectic manifold. An endomorphism  $f : M \rightarrow M$  is called *canonical* when  $f^*(\omega) = \omega$ .

The one-parameter groups of canonical transformations are generated by Hamiltonian vector fields. For dimension  $n = 1$ , these are area preserving maps.

Consider the canonical transformation  $(p_a, q^a) \xrightarrow{f} (P_a, Q^a)$ .

$$d(\mathbf{p} \cdot d\mathbf{q}) = -d(\mathbf{Q} \cdot d\mathbf{P}), \quad (4.29)$$

where  $\mathbf{P} = P(p, q)$  and  $\mathbf{Q} = Q(p, q)$ . Then

$$d(\mathbf{p} \cdot d\mathbf{q} + \mathbf{Q}d\mathbf{P}) = 0. \quad (4.30)$$

So the one-form  $\mathbf{p} \cdot d\mathbf{q} + \mathbf{Q} \cdot d\mathbf{P}$  is a closed-one form. Locally, this implies that it is exact, meaning that there is some *generating function*  $S = S(\mathbf{q}, \mathbf{P})$  such that

$$\mathbf{p} \cdot d\mathbf{q} + \mathbf{Q} d\mathbf{P} = dS. \quad (4.31)$$

From this function we can define  $Q$  and  $p$  as

$$Q^a = \frac{\partial S}{\partial p_a}, \quad p_a = \frac{\partial S}{\partial q^a}. \quad (4.32)$$

This gives  $\mathbf{P}(q, p)$  and  $Q(q, p)$ .