Supersymmetry

Part III Lent 2019

Lectures by David Skinner

Report typos to: uco21@cam.ac.uk
More notes at: uco21.user.srcf.net

January 28, 2020

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Lecture notes: www.damtp.cam.ac.uk/user/dbs26

Main book: Hori & Vafa "Mirror Symmetry" (chapters 8-16)

1 Introduction

1.1 Motivation: What is supersymmetry?

In a theory with bosons and fermions, the Hamiltonian can be written $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$, where $\mathcal{H}_{B(F)}$ has even (odd) number of fermionic excitations.

Such a theory is supersymmetric if there exists an operator Q mapping $\mathcal{H}_B \to \mathcal{H}_F$ and $\mathcal{H}_F \to \mathcal{H}_B$ such that

$$\left\{Q, Q^\dagger\right\} = 2H \qquad Q^2 = 0. \tag{1.1}$$

Here, $\{A, B\} = AB + BA$ is the *anti-commutator*, H is the Hamiltonian.

Consequences

i) The Hamiltonian and Q commute

$$\begin{aligned} [H, Q] &= \frac{1}{2}[\left\{Q, Q^\dagger\right\}, Q] && (1.2)\\ &= \frac{1}{2}[(QQ^\dagger + Q^\dagger Q)Q - Q(QQ^\dagger + Q^\dagger Q)] && (1.3)\\ &= 0. && (1.4) \end{aligned}$$

The two inner terms vanish since $Q^2 = 0$ and the two outer terms cancel identically. Therefore, the operator Q is *conserved*, and the transformations it generates will be *symmetries*. We call them *supersymmetries* because they mix bosons and fermions.

ii) All states ψ in our theory have non-negative energy

$$\begin{aligned} E = \langle\psi| H |\psi\rangle &= \frac{1}{2} \langle\psi| \left\{Q, Q^\dagger\right\} |\psi\rangle && (1.5)\\ &= \frac{1}{2}\|Q|\psi\rangle\|^2 + \frac{1}{2}\left\|Q^\dagger\psi\right\|^2 \geq 0, && (1.6) \end{aligned}$$

with equality if and only if $Q|\psi\rangle = 0 = Q^\dagger\psi$, meaning that the state is invariant under supersymmetry.

If we have a Lorentz invariant quantum field theory (QFT), then H is part of the momentum vector $P_m = (H, \mathbf{P})$. It is then natural to expect that there is a multiplet of Q's.

Indeed in $d = 4$ we have $\left\{Q_\alpha, Q^\dagger_{\dot{\alpha}}\right\} = 2\sigma^m_{\alpha\dot{\alpha}} P_m$, where $\sigma^m = (\mathbb{1}_{2\times 2}, \boldsymbol{\sigma})$.

For generic dimension d, this becomes $\left\{Q_A, Q^\dagger_B\right\} = 2\Gamma^m_{AB} P_m$.

Why study supersymmetry?

Traditionally, this question was answered by phenomenology. Supersymmetry was a promising approach to solve ongoing problems in dark matter, the unification of couplings in the standard model, as well as stabilizing the Higgs mass. The involvement of supersymmetry in the last issue was ruled out in experiments at CERN, and the above reasons will not be the motivation that drives us to study supersymmetry in this course.

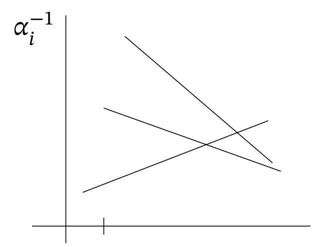

Figure 1.1: In many supersymmetric theories, the running couplings meet at the same point. This was taken to be indicative of a grand unified theory (GUT).

In this course, we will be driven by the following fact: QFT is hard! Usually, we have to study it via perturbation theory, as exemplified in 1.2. This is very different to quantum mechanics, where we

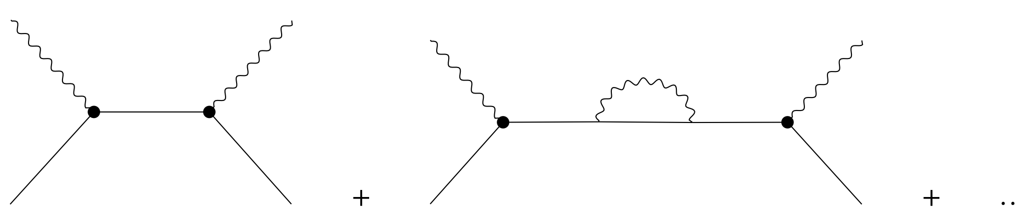

Figure 1.2: In the diagrammatic perturbation series in QFT, particles are almost always propagating freely, except at the interaction vertices.

first practice with exactly solvable systems. The reason for this is twofold. Firstly, they are usually better approximations to reality than a free particle; a spherically symmetric Coulomb potential is a better starting point to describe atoms than a free particle is, even though we still need to consider realistic models like (hyper)fine-structure perturbatively. Secondly, it helps us understand what quantum mechanics actually *is*.

Supersymmetry allows us to get exact results for (some observables) in QFT. This is especially true in $d < 4$, but also in $d = 4$.

These exact results are often closely related to deep maths, such as the Atiyah–Singer theorem (which we will meet in $d = 1$), mirror symmetry and enumerative geometry ($d = 2$), and Donaldson–Seiberg–Witten invariants ($d = 4$).

1.2 Fermions and Super Vector Spaces

Definition 1: A $\mathbb{Z}_2$-graded vector space is of the form $V = V_0 \oplus V_1$.

Definition 2 (parity)**:** We let the *parity* $|v|$ of $v \in V$ be

$$|v| = \begin{cases} 0, & \text{if } v \in V_0 \qquad \text{(even / bosonic)} \\ 1, & \text{if } v \in V_1 \qquad (odd/fermionic/Grassman) \end{cases} \tag{1.7}$$

Notation: If $\dim_{\mathbb{R}}(V_0) = p$ and $\dim_{\mathbb{R}}(V_1) = q$, then we write $V = \mathbb{R}^{p|q}$.

As usual, the dual V^* of a $\mathbb{Z}_2$-graded vector space (over $\mathbb{R}$) is the space of linear maps $\phi : V \to \mathbb{R}$ with $(V^*)_{0(1)}$ being those linear maps that vanish on $V_{1(0)}$ respectively.

Unsurprisingly, the direct sum of two $\mathbb{Z}_2$-graded vector spaces is

$$\begin{aligned} V \oplus W &= (V \oplus W)_0 \oplus (V \oplus W)_1 && (1.8) \\ &= (V_0 \oplus W_0) \oplus (V_1 \oplus W_1). && (1.9) \end{aligned}$$

Likewise, we can take the tensor product

$$(V \otimes W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1) \tag{1.10}$$

$$(V \otimes W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0) \tag{1.11}$$

Until now, we are just dealing with usual vector spaces, where we keep track of the fact that some elements have parity 1. To make V a super vector space, we define an unusual exchange operation.

$$\text{usually (bosonic):} \qquad \begin{aligned} s : \ U \otimes U' &\to U' \otimes U \\ u \otimes u' &\longmapsto u' \otimes u \end{aligned} \tag{1.12}$$

$$\text{super vector space:} \qquad \begin{aligned} s : \ V \otimes W &\to W \otimes V \\ v \otimes w &\longmapsto (-1)^{|v||w|} w \otimes v. \end{aligned} \tag{1.13}$$

Definition 3 (superalgebra)**:** Closely related is a superalgebra. This is a supervector space A with a multiplication map $\bullet : A \times A \to A$ with $|a \cdot b| = |a| + |b|$ (mod 2).

Definition 4 (commutative)**:** A is supercommutative (or just commutative) if $ab = (-1)^{|a||b|}ba$.

Example: To treat $\mathbb{R}^{p|q}$ as a superalgebra, we take

$$x^i x^j = x^j x^i, \qquad x^i \psi^a = \psi^a x^i, \qquad \text{but } \psi^a \psi^b = -\psi^b \psi^a, \tag{1.14}$$

where $x^i \in \mathbb{R}^{p|0}$ and $\psi^a \in \mathbb{R}^{0|q}$. In particular, $(\psi^a)^2 = 0$ for any fixed a.

Not all A are (super-)commutative.

Definition 5 (Lie superalgebra)**:** A *Lie superalgebra* is a supervector space $g = g_0 \oplus g_1$ with a bilinear Lie bracket operation $[,]: g \times g \to g$ that

- is 'graded anti-symmetric' $[X,Y] = -(-1)^{|x||y|}[Y,X]$
- obeys $[X,[Y,Z]] + (-1)^{|X|(|Y|+|Z|)}[Y,[Z,X]] + (-1)^{|Y|(|Z|+|X|)}[Z,[X,Y]] = 0$

Definition 6 (polynomials)**:** We can define polynomials on a super vector space $\mathbb{R}^{p|q} = v$ as $O(V) \simeq \mathrm{Sym}^*(V_0^*) \otimes \Lambda^*(V_1^*)$. They are of the form

$$c_{\underbrace{ijk \dots m}_{\text{symmetric}} \underbrace{abc \dots d}_{\text{antisymmetric}}} x^i x^j \dots x^m \psi^a \dots \psi^d. \tag{1.15}$$

Definition 7 (smooth functions)**:** We define smooth functions on a super vector space to be $C^\infty(V) = C^\infty(V_0) \otimes \Lambda^*(V_1^*)$, so a generic function has an expansion

$$F(x^i, \psi^a) = f(x^i) * \rho_a(x^i)\psi^a + g_{ab}(x^i)\psi^a\psi^b + \cdots + \frac{h(x)}{(\dim V_1)!}\epsilon_{ab\dots d}\psi^a\psi^b \dots \psi^d, \tag{1.16}$$

where the coefficients f, ρ_a, g_{ab}, …are smooth functions on V_0. We often call such functions $F(x^i, \psi^a)$ *superfields*, while the smooth functions $f, \rho_a, g_{ab}, \dots, h$ are the *component fields*. Note that if $F(x, \psi)$ is bosonic, then the component fields with even indices $f, g_{ab} = -g_{ba}, \dots$ are bosonic whilst the ones with odd indices $\rho_a, \dots$ are fermionic.

Remark: This is reminiscent of a *polyform* $F \in \Omega^*(V)$

$$F(x^i, dx^i) = f(x) + \rho_i(x)dx^i + g_{ij}(x)dx^i \wedge dx^j + \cdots + h(x)dx^i \wedge \cdots \wedge dx^n. \tag{1.17}$$

There is a fundamental difference: the polyform indices run over i, while the superfield indices a need not be the same as i. However, if they have the same indexing set, then these are really similar.

1.3 Differentiation and Integration of Fermions

Definition 8 (derivation)**:** A *derivation* of a (super-)algebra A is a linear map $D : A \to A$ obeying

$$D(ab) = (Da)b + (-1)^{|a||D|}a(Db) \qquad \text{(graded Leibniz rule)}. \tag{1.18}$$

Example: On $\mathbb{R}^{p|q}$, we have even derivatives $\frac{\partial}{\partial x^i}$ and odd derivatives $\frac{\partial}{\partial \psi^a}$, which act in the way you would expect on single fields:

$$\frac{\partial}{\partial x^i}(x^j) = \delta_i^j \qquad \frac{\partial}{\partial x^i}\psi^a = 0 \qquad \frac{\partial x^j}{\partial \psi^a} = 0 \qquad \frac{\partial}{\partial \psi^a}(\psi^b) = \delta_a^b. \tag{1.19}$$

However,

$$\frac{\partial}{\partial \psi^a}(\psi^b\psi^c) = \delta_a^b\psi^c - \psi^b\delta_a^c. \tag{1.20}$$

More generally, a (smooth) vector field on $\mathbb{R}^{p|q}$ is

$$X(x, \psi) = X_0^i(x, \psi)\frac{\partial}{\partial x^i} + X_1^a(x, \psi)\frac{\partial}{\partial \psi^a}, \tag{1.21}$$

where $X_0^i, x_1^a \in C^\infty(\mathbb{R}^{p|q})$.

For integration, since $f(\psi) = \rho + a\psi$, we only need to define $\int 1\,\mathrm{d}\psi$ and $\int \psi\,\mathrm{d}\psi$. We require our measure $\mathrm{d}\psi$ to be translation invariant[1]: if $\psi' = \psi + \eta$ for some fixed fermionic $\eta \in \mathbb{R}^{0|1}$, then we want

$$\int \psi'\,\mathrm{d}\psi' = \int (\psi + \eta)\,\mathrm{d}\psi = \int \psi\,\mathrm{d}\psi + \eta \int \mathrm{d}\psi \;\Rightarrow\; \boxed{\int 1\,\mathrm{d}\psi = 0}. \tag{1.22}$$

We then normalise our measure by defining

$$\boxed{\int \psi\,\mathrm{d}\psi = 1} \tag{1.23}$$

These rules are called *Berezin integration*.

> **Remark:** Differentiation and integration is really the same thing. Not unlike complex variables.

Remark: These imply that

$$\int \frac{\partial}{\partial\psi}(F(\psi, \dots))\,\mathrm{d}\psi = 0. \tag{1.24}$$

In other words, when we perform integration by part for fermions, we never have to worry about boundary terms as long as we are careful about minus signs.

Suppose that we instead have a general case of n fermionic variables ψ^a . Then by iterated application of the previous rules, we define

$$\int \psi^1\psi^2 \dots \psi^n\,\mathrm{d}^n\psi = 1 \tag{1.25}$$

if they all appear, and zero otherwise. If they all appear, but not in the correct order, then we get extra minus signs

$$\int \underbrace{\psi^a\psi^b \dots \psi^c}_{n \text{ fermions}}\,\mathrm{d}^n\psi = \epsilon^{ab\dots c}. \tag{1.26}$$

> **Remark:** Note in particular that if any index appears twice, the square on the left-hand side vanishes, just like the Levi-Civita symbol on the right.

Suppose $\chi^a = N^a{}_b\psi^b$ for some $N \in GL(n, \mathbb{R})$. Then by linearity

$$\begin{aligned}
\int \chi^{a_1} \dots \chi^{a_n}\,\mathrm{d}^n\psi &= N^{a_1}{}_{b_1} \dots N^{a_n}{}_{b_n} \int \psi^{b_1} \dots \psi^{b_n}\,\mathrm{d}^n\psi && (1.27)\\
&= N^{a_1}{}_{b_1} \dots N^{a_n}{}_{b_n}\,\epsilon^{b_1 b_2 \dots b_n} && (1.28)\\
&= \det(N)\epsilon^{a_1 \dots a_n} && (1.29)\\
&= \det(N) \int \chi^{a_1}\chi^{a_2} \dots \chi^{a_n}\,\mathrm{d}^n\chi\,. && (1.30)
\end{aligned}$$

[1] In particular, this will be necessary to derive Ward identities in QFT.

We conclude that if $\chi^a = N^a{}_b \psi^b$, then $\mathrm{d}^n\chi = \frac{1}{\det(N)}\,\mathrm{d}^n\psi$.

> **Remark:** This is not the same as if you were doing bosonic integration, where you do not have the inverse of the determinant.

Example: If $\chi = a\psi$, then $\mathrm{d}\psi = d(a\psi) = \frac{1}{a}\,\mathrm{d}\psi$.

1.4 QFT in Zero Dimensions

1.4.1 Bosonic Theory

In $d = 0$, our whole universe is just a single point $M = \{\mathrm{pt}\}$. So a bosonic field is just a map $x : M \to \mathbb{R} \simeq \{\mathrm{pt}\} \to \mathbb{R}$, which is nothing else than a real variable. With n such real fields, the space of all field configurations is $\mathcal{C} = \mathbb{R}^n$. The path-integral measure $[\mathcal{D}X]$ is just the usual Lebesgue measure $\mathrm{d}^n x$. Then the partition function becomes $Z = \int_{\mathbb{R}^n} e^{-S(x)/\hbar}\,\mathrm{d}^n x$, where $S : \mathbb{R}^n \to \mathbb{R}$ is the action.

> Compare this with today's lecture on *Advanced Quantum Field Theory*.

There cannot be any kinetic terms since the universe is just a point and there cannot be anything we differentiate with respect to. However, we can have a mass term and maybe some sort of interaction such as

$$S(x^i) = \frac{m_2}{2}\delta_{ij}x^i x^j + \frac{\lambda^{ijkl}}{4}x^i x^j x^k x^l. \tag{1.31}$$

Now in $d = 0$ this is a finite-dimensional integral. But nonetheless, it is a difficult integral! Expanding the action to quadratic order around its stationary point, we can find that in the limit $\hbar \to 0^+$, the integral is asymptotic to

$$\int_{\mathbb{R}^n} e^{-S(x)/\hbar}\,\mathrm{d}^n x \sim (2\pi\hbar)^{n/2}\frac{e^{-S(x_*)}}{(\det\partial_i\partial_j S)^{1/2}|_{x=x_*}}\left(1 + \hbar A_1 + \hbar^2 A_2 + \dots\right), \qquad \text{(steepest descent)} \tag{1.32}$$

where x_* is a minimum of $S(x)$.

This is complicated! And approximate (zero radius of convergence)!

Remark: In the whole of the QFT course we basically just computed different numerators of this. In AQFT we will go on to loop diagrams and compute the denominator as well as the first order expansion. If you end up doing a PhD in the wrong area, you might compute higher and higher terms. But what is even the point? This series doesn't even converge!

1.4.2 Fermionic Theory

Let us now consider a purely fermionic theory. We need at least two fermions. Take $S = A\psi^1\psi^2$.

$$Z = \int e^{-S(\psi)/\hbar}\, \mathrm{d}^2\psi = \int \left(1 - \frac{A}{\hbar}\psi^1\psi^2\right) \mathrm{d}^2\psi = -\frac{A}{\hbar}. \tag{1.33}$$

More generally, for $2m$ fermions ψ^a and antisymmetric matrix A_{ab} ,

$$Z = \int e^{-\frac{A_{ab}}{2\hbar}\psi^a\psi^b}\, \mathrm{d}^{2m}\psi = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\hbar)^k k!} \int (A_{ab}\psi^a\psi^b)^k\, \mathrm{d}^{2m}\psi \tag{1.34}$$

$$= \frac{(-1)^m}{(2\hbar)^m m!} \epsilon^{a_1 b_1 \dots a_m b_m} A_{a_1 b_1} \dots A_{a_m b_m} \tag{1.35}$$

$$= \left(\frac{-1}{\hbar}\right)^m \mathrm{Pfaff}(A). \tag{1.36}$$

Definition 9 (Pfaffian)**:** In the preceding derivation, we stumbled across the *Pfaffian* of a $2m \times 2m$ antisymmetric matrix A, defined by

$$\mathrm{Pfaff}(A) = \frac{1}{2^m m!} e^{a_1 a_2 \dots a_{2m}} A_{a_1 a_2} \cdots A_{a_{2m-1} a_{2m}}. \tag{1.37}$$

Exercise 1.1: For antisymmetric A, show that $(\mathrm{Pfaff}\, A)^2 = \det A$.

This means that the Gaussian integral (1.36) can be written as $\pm\sqrt{\det A}$.

> **Remark:** Again, up to a normalisation of the measure, this is the inverse to what you expect from the bosonic counterpart of the Gaussian integral.

If our action contains finitely many fermions, it is always easy to compute the fermionic integral exactly (unlike the bosonic case).

Example: If we have a quartic action

$$S(\psi^1, \dots, \psi^4) = A(\psi^1\psi^2 + \psi^3\psi^4) + \lambda\psi^1\psi^2\psi^3\psi^4. \tag{1.38}$$

Then $S^2 \neq 0$, but $S^3 = 0$. So the exponential measure truncates just after the second term

$$e^{-S/\hbar} = 1 - \frac{S}{\hbar} + \frac{S^2}{2\hbar^2} \tag{1.39}$$

$$= 1 - \frac{1}{\hbar}[A(\psi^1\psi^2 + \psi^3\psi^4) + \lambda\psi^1\psi^2\psi^3\psi^4] \tag{1.40}$$

and hence, the integral extracts the piece

$$\int e^{-S(\psi)/\hbar}\, \mathrm{d}^4\psi = \frac{A^2}{\hbar^2} - \frac{\lambda}{\hbar}. \tag{1.41}$$

1.4.3 Supersymmetric Theory

A generic theory containing both fermions and bosons is intractable because of the bosonic integral. Even in $d = 0$, we get a complicated integral that is hard to solve. However, let us consider a theory containing one bosonic and two fermionic fields $(x, \psi, \overline{\psi})$.

> These fermionic fields can be considered $\psi = \psi^1 + i\psi^2$ and $\overline{\psi} = \psi^1 - i\psi^2$.

The most general action for these fields would be

$$S(x, \psi\overline{\psi}) = f(x) + g(x)\overline{\psi}\psi, \tag{1.42}$$

for some functions f and g. We can choose a very special relation between fermionic and bosonic fields by choosing the action

$$S(x, \psi\overline{\psi}) = \frac{1}{2}(\partial w)^2 - \overline{\psi}\psi\partial^2 w, \tag{1.43}$$

where $w = w(x)$ is a polynomial and $\partial w = \frac{\partial w}{\partial x}$.

Claim 1: This action is invariant under the flow generated by the fermionic vector fields

$$Q = \psi\frac{\partial}{\partial x} + (\partial w)\frac{\partial}{\partial\overline{\psi}} \qquad \text{and} \qquad Q^\dagger = \overline{\psi}\frac{\partial}{\partial x} - (\partial w)\frac{\partial}{\partial\psi}. \tag{1.44}$$

These are odd derivations of $\mathbb{R}^{1|2}$ with

$$\mathcal{Q}(x) = \psi \qquad \mathcal{Q}^\dagger(x) = \overline{\psi} \tag{1.45}$$

$$\mathcal{Q}(\psi) = 0 \qquad \mathcal{Q}^\dagger(\psi) = -\partial w(x) \tag{1.46}$$

$$\mathcal{Q}(\overline{\psi}) = \partial w(x) \qquad \mathcal{Q}^\dagger(\overline{\psi}) = 0 \tag{1.47}$$

Proof. We will only show this for $\mathcal{Q}^\dagger$.

$$\mathcal{Q}^\dagger(S) = \overline{\psi}\frac{\partial}{\partial x}\left(\frac{1}{2}(\partial w)^2 - \overline{\psi}\psi\partial^2 w\right) - (\partial w)\frac{\partial}{\partial \psi}\left(\frac{1}{2}(\partial w)^2 - \overline{\psi}\psi\partial^2 w\right) \tag{1.48}$$

$$= \overline{\psi}\partial w \partial^2 w - \overline{\psi}(\partial w)\partial^2 w = 0. \tag{1.49}$$

□

We say that $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ generate *supersymmetries* of this action.

We can also calculate the anti-commutation relations

$$\{\mathcal{Q},\mathcal{Q}\} = 2(\partial^2 w)\psi\frac{\partial}{\partial\overline{\psi}} \qquad \{\mathcal{Q}^\dagger,\mathcal{Q}^\dagger\} = -2(\partial^2 w)\overline{\psi}\frac{\partial}{\partial\psi}$$
$$\{\mathcal{Q},\mathcal{Q}^\dagger\} = -\partial w\left(\psi\frac{\partial}{\partial\psi} - \overline{\psi}\frac{\partial}{\partial\overline{\psi}}\right), \tag{1.50}$$

which also generate bosonic symmetries.

This supersymmetry obeys $\mathcal{Q}^2 = 0 = \mathcal{Q}^{\dagger 2}$ only up to the $\psi, \overline{\psi}$ 'equation of motion'[1]. To do better, we will need superfields.

In a supersymmetric theory (in $d = 0$) we can compute the partition function $Z = \int e^{-S/\hbar}\,\mathrm{d}x\,\mathrm{d}^2\psi$ exactly. To do this, let us rescale $w \to \lambda w$, for $\lambda \in \mathbb{R}_{\geq 0}$. Then the rescaled action $S_\lambda = \frac{\lambda^2}{2}(\partial w)^2 - \overline{\psi}\psi\lambda\partial^2 w$ is invariant under $Q_\lambda = \psi\frac{\partial}{\partial x} + \lambda(\partial w)\frac{\partial}{\partial\overline{\psi}}$ and $\mathcal{Q}_\lambda^\dagger$.

Claim 2: The key point is that $Z_\lambda = \int e^{-S_\lambda/\hbar}\,\mathrm{d}x\,\mathrm{d}^2\psi$ is independent of λ.

We will set $\hbar = 1$ from now on.[2]

Proof.

$$-\frac{\mathrm{d}}{\mathrm{d}\lambda}Z_\lambda = \int \frac{\mathrm{d}S_\lambda}{\mathrm{d}\lambda}e^{-S_\lambda}\,\mathrm{d}x\,\mathrm{d}^2\psi = \int\left(\lambda(\partial w)^2 - \overline{\psi}\psi\partial^2 w\right)e^{-S_\lambda}\,\mathrm{d}x\,\mathrm{d}^2\psi\,. \tag{1.51}$$

Observe that $\mathcal{Q}_\lambda^\dagger(\psi\partial w) = \overline{\psi}\psi\partial^2 w - \lambda(\partial w)^2 = -\frac{\mathrm{d}S_\lambda}{\mathrm{d}\lambda}$. Hence, we can write this as

$$\frac{\mathrm{d}Z_\lambda}{\mathrm{d}\lambda} = \int \mathcal{Q}_\lambda^\dagger(\psi\partial w)e^{-S^\lambda}\,\mathrm{d}x\,\mathrm{d}^2\psi = \int Q_\lambda^\dagger\left(\psi\partial w e^{-S_\lambda}\right)\mathrm{d}x\,\mathrm{d}^2\psi \tag{1.52}$$

where we used that the action itself is invariant under $Q_\lambda^\dagger$. Since $Q_\lambda^\dagger = \overline{\psi}\frac{\partial}{\partial x} - \lambda(\partial w)\frac{\partial}{\partial\psi}$, the second term does not survive the integral over $\mathrm{d}^2\psi$. The first term is a total derivative in x, so it dies under integration over $\mathrm{d}x$[3]. We conclude that $\frac{\mathrm{d}Z_\lambda}{\mathrm{d}\lambda} = 0$. □

[1] Of course we are in $d = 0$ and there is no time, so we do not have any time-evolution.

[3] We might need to worry about large x behaviour, but ∂w is some polynomial, so the exponential decay of e^{-S_λ} will always dominate at large x.

In particular, $Z(1) = \lim_{\lambda\to\infty} Z(\lambda)$. This is useful because it is easy to compute $Z(\lambda)$ at large λ: As $\lambda \to \infty$, the term $e^{\frac{\lambda^2}{2}(\partial w)^2}$ suppresses all contributions, except near critical points x_*, where $w'(x_*) = 0$. Suppose $w(x)$ is a generic polynomial of degree D with isolated[1], non-degenerate[2] critical points x_* . Then near any critical point,

$$w(x) = w(x_*) + \frac{c_*}{2}(x - x_*)^2 + \dots , \tag{1.53}$$

where $c_* = \partial^2 w(x_*)$. Hence, near x_*,

$$S(x, \psi, \overline{\psi}) = \frac{c_*^2}{2}(x - x_*)^2 - \overline{\psi}\psi c_* + \dots \tag{1.54}$$

The higher order terms in $\delta x = x - x_*$ will be negligible as $\lambda \to \infty$. Hence, near the critical point,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int e^{-S(x,\psi,\overline{\psi})} \, \mathrm{d}x \, \mathrm{d}^2\psi &= \frac{1}{\sqrt{2\pi}} \int e^{-\frac{c_*}{2}(x-x_*)^2} \left[-1 + c_*\overline{\psi}\psi\right] \mathrm{d}x \, \mathrm{d}^2\psi && (1.55)\\
&= \frac{c_*}{\sqrt{2\pi}} e^{-\frac{c_*}{2}(x-x_*)^2} \, \mathrm{d}x && (1.56)\\
&= \frac{c_*}{\sqrt{2\pi}} \sqrt{\frac{2\pi}{c_*^2}} = \frac{c_*}{|c_*|} && (1.57)\\
&= \mathrm{sgn}(\partial^2 w|_{x_*}). && (1.58)
\end{aligned}$$

Summing over all critical points,

$$Z = \sum_{x_* : \partial w|_{x_*} = 0} \mathrm{sgn}(\partial^2 w|_{x_*}). \tag{1.59}$$

As seen in Figure XXX, the partition function only really cares about the degree of the polynomials.

Let us think about what would have happened if we calculated this with the use of Feynman diagrams. If you did that, you would see that to all orders in the loop expansions, the diagrams cancel exactly. The reason for this is really this localisation.

We are just counting the number of things, this is the first sniff at some sort of index theorem.

[1] For each critical point, there is always some open neighbourhood around it that does not contain any others.

[2] This means that the second derivative of w does *not* vanish at such a point.

1.5 The Duistermaat–Heckmann Theorem

Definition 10 (symplectic manifold)**:** A *symplectic manifold* (M, ω) is a smooth manifold M of dimension $\dim_{\mathbb{R}}(M) = 2n$ on which we have a 2-form ω, which is

- closed: $d\omega = 0$,
- non-degenerate: $\omega(X, Y) = 0$ for all vector fields Y iff $X = 0$.

Equivalently, the non-degeneracy condition can be expressed as $\omega^n = \det(\omega_{ab}) dx^1 \wedge \cdots \wedge dx^{2n}$ is non-vanishing, and therefore provides a (Liouville) volume form.

Let (M, ω) be a symplectic manifold. Suppose X is a vector field on (M, ω) with ω invariant along the flow generated by X. This means that the Lie derivative vanishes

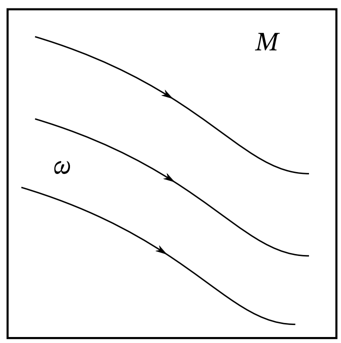

Figure 1.3

$$0 = \mathcal{L}_X \omega = (\iota_X d + d\iota_X)\,\omega = d(\iota_X \omega), \tag{1.60}$$

where in the last equation we used that $d\omega = 0$.

Definition 11 (Hamiltonian)**:** We say X is a *Hamiltonian vector field* if there exists a map $h : M \to \mathbb{R}$ such that $\iota_X \omega = -dh$.

Example: Let $M = \mathbb{R}^{2n}$ and $\omega = dp_i \wedge dq^i$. Take $X = \frac{\partial}{\partial q^i}$. Then $\iota_X \omega = -dp_i$. So translations are Hamiltonian with p_i as the Hamiltonian function.

We will be interested in compact (M, ω) with $\partial M = \emptyset$. We will also require that X generates a $U(1)$ action on M, meaning that generic orbits of X are circles.

Example: Consider $M = S^2$ with $\omega = \sin\theta d\theta \wedge d\phi$. Take $X = \frac{\partial}{\partial \phi}$. There are two fixed points at the north and south poles, as illustrated in 1.4.

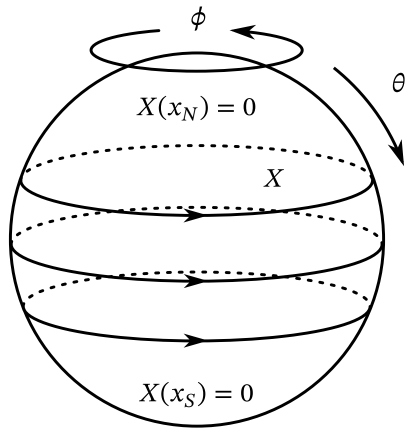

Figure 1.4

Then the Duistermaat–Heckmann theorem states that for any $\alpha \in \mathbb{R}$

$$\int_M e^{i\alpha h(x)} \frac{\omega^n}{n!} \tag{1.61}$$

localises to fixed points $x_* \in M$, where $X(x_*) = 0$.

Example $((M, \omega) = (S^2, \sin\theta d\theta \wedge d\phi))$**:** Have $X = -\frac{\partial}{\partial\phi}$ and $h = \cos\theta$. The integral can simply be done without using a fancy localisation theorem. Use the substitution $z = \cos\theta$:

$$\int_{S^2} e^{i\alpha\cos\theta} \sin\theta d\theta \wedge d\phi = 2\pi \int_{-1}^{+1} e^{i\alpha z}\, \mathrm{d}z = \frac{2\pi}{i\alpha}\left[e^{i\alpha} - e^{-i\alpha}\right], \tag{1.62}$$

which is simply the value of $e^{i\alpha h(x_*)}$ at the north and south poles.

Proof. We can derive this using supersymmetry. Our 'fields' are (x^a, ψ^b), where the ψ^a transform as vectors on M. Thus the space of fields is $\mathcal{C} = \Pi TM$. A generic superfield is

$$F(x,\psi) = f(x) + \rho_a(x)\psi^a + g_{ab}(x)\psi^a\psi^b + \cdots + h(x)\psi^1\psi^2\cdots\psi^{2n}. \tag{1.63}$$

The Π is just a notation that reminds us that the ψ^a are anti-commuting.

As before, we can identify the space of smooth functions $C^\infty(\Pi TM) = \Omega^*(M)$ to be the space of polyforms on the manifold.

On any coordinate patch we have a supervector space given by $\mathbb{R}^{2n|2n}$. For a general curved manifold we might need to worry about what happens on the overlaps of the coordinate patches, but we are not going to.

We choose our action to have 2 parts. Firstly,

$$S_0 = -i\alpha \left(h(x) + \omega_{ab}(x)\psi^a\psi^b \right). \tag{1.64}$$

Claim 3: This is invariant under supersymmetry transformations generated by the vector field

$$\mathcal{Q} = \psi^a \frac{\partial}{\partial x^a} + X^a(x)\frac{\partial}{\partial \psi^a}. \tag{1.65}$$

Proof.

$$\frac{i}{\alpha}Q(S_0) = \psi^a\partial_a h + \partial_a\omega_{bc}\psi^b\psi^b + 2X^a\omega_{ab}\psi^b \tag{1.66}$$

Now the second part vanishes since $d\omega = 0$.

$$\cdots = \psi^a(\partial_a h + 2X^b\omega_{ba}) = 0 \tag{1.67}$$

since $\iota_X\omega = -dh$ by Def. 11 □

We can write "$\mathcal{Q} = d + \iota_X$", giving

$$\frac{1}{2}\{\mathcal{Q}, \mathcal{Q}\} = (d + \iota_X)^2 = d\iota_X + \iota_X d = \mathcal{L}_X. \tag{1.68}$$

So $\mathcal{Q}^2 = 0$ on forms that are invariant along the flow of X. We now deform S by picking a positive definite metric g on M. Then for a constant $\lambda \in \mathbb{R}$ that measures the deformation, we have

$$S_\lambda = S_0 + \lambda\mathcal{Q}(g(X, \psi)). \tag{1.69}$$

Provided the metric is invariant under the flow, $\mathcal{Q}(S_\lambda) = \lambda Q^2(g_{ab}X^a\psi^b) = 0$. However, this corresponds to $\lambda\mathcal{L}_X(g_{ab}X^a dx^b)$.

The partition function of this is as always

$$Z = \int_{\Pi TM} e^{-S_\lambda}\, \mathrm{d}^{2n}x\, \mathrm{d}^{2n}\psi \tag{1.70}$$

Exercise 1.2: Check that the measure $\mathrm{d}^{2n}x\,\mathrm{d}^{2n}\psi$ is invariant under Diff(M).

Again, if we differentiate this with respect to the parameter λ, we have

$$-\frac{\mathrm{d}Z}{\mathrm{d}\lambda} = \int \mathcal{Q}(g(X,\psi))e^{-S_\lambda}\,\mathrm{d}^{2n}x\,\mathrm{d}^{2n}\psi = \int \mathcal{Q}(g(X,\psi)e^{-S_\lambda})\,\mathrm{d}^{2n}x\,\mathrm{d}^{2n}\psi = 0 \tag{1.71}$$

Because out whole action is supersymmetric, we were able to make $\mathcal{Q}$ act on everything in the first equality. Hence Z_λ is, as before, independent of λ.

In particular, this is useful since

$$Z(\lambda = 0) = \int_{\Pi TM} e^{-S_0}\, \mathrm{d}^{2n}x\, \mathrm{d}^{2n}\psi = (i\alpha)^n \int_M e^{i\alpha h(x)} \mathrm{Pfaff}(\omega_{ab})\, \mathrm{d}^{2n}x \tag{1.72}$$

$$= (i\alpha)^n \int_M e^{i\alpha h(x)} \frac{\omega^n}{n!}, \tag{1.73}$$

which is the Duistermaat–Heckmann integral.

▌ We have recast the problem into supersymmetric language.

Remark: The deformation term has two pieces $\mathcal{Q}(g(X,\psi)) = (\partial_c g_{ab} X^a)\psi^c\psi^b + \mathrm{g}(X,X)$. The second term is the important bit; it is purely bosonic and positive definite. Hence as we scale λ to a very large value $\lambda \to \infty$, we only get contributions from a neighbourhood of any critical points, where the vector field has zero length and thus vanishes $X(x_*) = 0$.

We know that Z_λ is independent of λ, so the original integral must localise. We can evaluate $\lim_{\lambda\to\infty} Z_\lambda$ by using steepest descent.

$$Z_\lambda = \lim_{\lambda\to\infty} \{Z_\lambda\} \sim \frac{(2\pi)^n}{(i\alpha)^n} \sum_{\substack{x_*\in M\\ X(x_*)=0}} e^{i\alpha h(x_*)} \left.\frac{\epsilon^{a_1 b_1 \ldots a_n b_n}(\partial_{a_1} X_{b_1}) \ldots (\partial_{a_n} X_{b_n})}{\sqrt{\det \partial_a \partial_b \mathrm{g}(X,X)}}\right|_{x=x_*} \tag{1.74}$$

where $X_b = g_{bc}X^c$. Localisation tells us that this result is exact. □