Advanced Quantum Field Theory

Part III Lent 2020

Lectures by Matthew B. Wingate

Report typos to: uco21@cam.ac.uk

More notes at: uco21.user.srcf.net

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1 Path Integrals in QM

The lecturer's notes can be found online at www.damtp.cam.ac.uk/user/wingate/AQFT.

Goal: Schrödinger's equation → path integral

Consider a Hamiltonian in one dimension $\hat{H} = H(\hat{x}, \hat{p})$, where position and momentum operators satisfy the common commutation relations $[\hat{x}, \hat{p}] = i\hbar$. Assume the it takes the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{1.1}$$

Schrödinger's equation then says that the time evolution of a state $|\psi(t)\rangle$ is given by

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle . \tag{1.2}$$

This has a formal solution, giving us the time-evolution operator

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle . \tag{1.3}$$

In the Schrödinger picture, the states are evolving in time whereas operators and their eigenstates are constant in time.

Definition 1 (wavefunction)**:** $\Psi(x,t) := \langle x|\psi(t)\rangle$

The Schrödinger equation then becomes

$$\langle x|\hat{H}|\psi(t)\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi(x,t). \tag{1.4}$$

We will turn this differential equation into an integral equation, where we will sum over particle paths—a path integral. We can introduce an integral by inserting a complete set of states 1 =

$\int dx_0 \left|x_0\right\rangle \left\langle x_0\right|$.

$$\Psi(x,t) = \langle x| \, e^{-i\hat{H}t/\hbar} \, |\psi(0)\rangle \tag{1.5}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}x_0 \, \langle x| \, e^{-i\hat{H}t/\hbar} \, |x_0\rangle \, \langle x_0|\psi(0)\rangle \tag{1.6}$$

$$:= \int_{-\infty}^{\infty} \mathrm{d}x_0 \, \underbrace{K(x,x_0;t)}_{\text{'kernel'}} \, \Psi(x_0,0) \tag{1.7}$$

We repeat this insertion for n intermediate times and positions.

Notation: Let $0 := t_0 < t_1 < \cdots < t_n < t_{n+1} := T$

And we also want to factor the exponential into n terms:

$$e^{i\hat{H}T/\hbar} = e^{-\frac{i}{\hbar}\hat{H}(t_{n+1}-t_n)} \cdots e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)}. \tag{1.8}$$

Then

$$K(x,x_0;T) = \int_{-\infty}^{\infty} \left[\prod_{r=1}^{n} \mathrm{d}x_r \, \langle x_{r+1}| \, e^{-\frac{i}{\hbar}\hat{H}(t_{r+1}-t_r)} \, |x_r\rangle\right] \langle x_1| \, e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)} \, |x_0\rangle \tag{1.9}$$

Integrals are over all possible position eigenstates at times $t_r, r = 1, \ldots, n$.

Free Theory

Consider the "free" theory, with $V(\hat{x}) = 0$. We will now play a similar but different trick to what we did before. Let us insert a complete set of momentum eigenstates $1 = \int_{-\infty}^{\infty} đp \, |p\rangle \langle p|$. We also note that these momentum eigenstates are plane waves $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$.

Definition 2 (barred differential)**:** We define the normalised differential in Fourier space to be

$$đp := \frac{\mathrm{d}p}{2\pi\hbar}. \tag{1.10}$$

In higher dimensional QFT, this generalises to

$$đ^n p := \frac{\mathrm{d}^n p}{(2\pi\hbar)^n} \tag{1.11}$$

The corresponding kernel is

$$K_0(x,x';t) = \langle x| \exp\left(\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t\right) |x'\rangle \, . \tag{1.12}$$

$$= \int_{-\infty}^{\infty} đp \, e^{-ip^2t/2m\hbar} e^{ip(x-x')/\hbar} \tag{1.13}$$

$$= \sqrt{\frac{m}{2\pi i\hbar t}} e^{\frac{im(x-x')^2}{2\hbar t}} \, . \tag{1.14}$$

Remark:

$$\lim_{t\to 0}\{K_0(x,x';t)\} = \delta(x-x'). \tag{1.15}$$

As expected from $\langle x|x'\rangle = \delta(x-x')$.

From the Baker-Campbell-Hausdorff formula, we know that

$$e^{\epsilon\hat{A}}e^{\epsilon\hat{B}} = \exp\left(\epsilon\hat{A}+\epsilon\hat{B}+\frac{\epsilon^2}{2}[\hat{A},\hat{B}]+\ldots\right) \neq e^{\epsilon(\hat{A}+\hat{B})} \tag{1.16}$$

$$\text{for small } \epsilon: \qquad e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}}e^{\epsilon\hat{B}}(1+O(\epsilon^2)) \tag{1.17}$$

Letting $\epsilon = 1/n$ and raising the above to the n^{th} power[1] gives

$$e^{\hat{A}+\hat{B}} = \lim_{n\to\infty}\left\{e^{\hat{A}/n}e^{\hat{B}/n}\right\}^n . \tag{1.18}$$

We will use this to separate kinetic and potential terms.

Take $t_{r+1}-t_r = \delta t$ with $\delta t \ll T$ and n large such that $n\delta t = T$.

$$e^{-\frac{i}{\hbar}\hat{H}\delta t} = \exp\left(-\frac{i\hat{p}^2\delta t}{2m\hbar}\right)\exp\left(-\frac{iV(\hat{x})\delta t}{\hbar}\right)[1+O(\delta t^2)] \tag{1.19}$$

Using the result (1.14),

$$\langle x_{r+1}|\exp\left(-\frac{i\hat{H}}{\hbar}\delta t\right)|x_r\rangle = e^{-iV(x_r)\delta t/\hbar}K_0(x_{r+1},x_r;\delta t) \tag{1.20}$$

$$= \sqrt{\frac{m}{2\pi i\hbar\delta t}}\exp\left[\frac{im}{2\hbar}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2\delta t - \frac{i}{\hbar}V(x_r)\delta t\right] \tag{1.21}$$

With $T = n\delta t$

$$K(x,x_0;T) = \int\left[\prod_{r=1}^{n}\mathrm{d}x_r\right]\left(\frac{m}{2\pi i\hbar\delta t}\right)^{\frac{n+1}{2}}\exp\left\{i\sum_{r=0}^{n}\left[\frac{m}{2\hbar}\left(\frac{x_{r+1}-x_r}{\delta t}\right)^2-\frac{1}{\hbar}V(x_r)\right]\delta t\right\} \tag{1.22}$$

In the limit $n\to\infty$, $\delta t\to 0$ with $n\delta t = T$ fixed, the exponent becomes

$$\frac{1}{\hbar}\int_0^T \mathrm{d}t\left[\frac{1}{2}m\dot{x}^2 - V(x)\right] = \int_0^T \mathrm{d}t\, L(x,\dot{x}), \tag{1.23}$$

where L is the classical Lagrangian, the Legendre transformation of the classical Hamiltonian. The classical action is $S = \int \mathrm{d}t\, L(x,\dot{x})$.

The main result therefore is that the path integral for the kernel is

$$\boxed{K(x,x_0;t) := \langle x|\, e^{-i\hat{H}t/\hbar}\,|x_0\rangle = \int \mathcal{D}x\, e^{\frac{i}{\hbar}S}} \tag{1.24}$$

[1] This step is sometimes called Suzuki–Trotter decomposition.

Definition 3 (functional integral)**:**

$$\mathcal{D}x = \lim_{\substack{\delta t \to 0 \\ n\delta T \text{ fixed}}} \left\{ (\sqrt{\ldots}) \prod_{r=1}^{n} (\sqrt{\ldots} dx_r) \right\} \tag{1.25}$$

We do not need to care about normalization factors.

Remark: In the limit $\hbar \to 0$, the interference of amplitudes is dominated by the ones close to the extremal path where δS. This leads to Hamilton's principle of least action.

We may analytically continue this to imaginary time. Let $\tau = it$. In terms of this imaginary time, we have

$$\langle x | \, e^{-\hat{H}\tau/\hbar} \, | x_0 \rangle = \int \mathcal{D}x \, e^{-S/\hbar}. \tag{1.26}$$

Mathematically, it makes these integrals much more well-defined, clearly convergent. Here the least-action principle is really evident from the $\hbar = 0$ argument. We also see the connection to statistical physics, interpreting $e^{-S/\hbar}$ as the Boltzmann factor $e^{-\beta H}$.

Quantum mechanics is quantum field theory in 0+1 dimensions. We are treating space differently from time: $\hat{x}(t)$ is a field, whereas t is a variable. However, Lorentz invariance forces us to put x and t on the same footing. In QFT, we solve this problem by demoting x from a field to another label. We then talk about fields $\phi(x, t)$ and want to know about the behaviour of these in all of spacetime.

> String theory gives another ansatz to this problem by promoting again, instead of demoting operators to labels.

1.1 Integrals and their Diagrammatic Expansions

In QFT, we are interested in correlation functions. The following discussion will be very similar to what we have seen in the *Statistical Field Theory* course in Michaelmas term, also no knowledge from that course will be assumed here.

For simplicity, consider a 0 -dimensional field $\varphi \in \mathbb{R}$. As if we are in imaginary time, let

$$Z = \int_{\mathbb{R}} \mathrm{d}\varphi \, e^{-\frac{S(\varphi)}{\hbar}}. \tag{1.27}$$

Assume that the action $S(\varphi)$ is an even polynomial and $S(\varphi) \to \infty$ as $\varphi \to \pm\infty$.

We will be interested in expectation values

$$\langle f \rangle = \frac{1}{Z} \int \mathrm{d}\varphi \, f(\varphi) e^{-S/\hbar}. \tag{1.28}$$

Again, assume f does not grow too fast as $\varphi \to \pm\infty$. Usually, f is polynomial in φ.

1.1.1 Free Theory

Say we have N scalar fields (in $0+1$ dimensions we should really just say 'variables') φ_a with $a = 1, \dots, N$, with action

$$S_0(\varphi) = \frac{1}{2} M_{ab}\varphi_a\varphi_b = \frac{1}{2}\varphi^T M\varphi, \tag{1.29}$$

where M is an $N \times N$ symmetric, positive definite ($\det M > 0$) matrix.

We can diagonalise this. There exists some orthogonal P such that $M = P\Lambda P^T$, where Λ is diagonal. Let $\chi = P^T\varphi$. Then the free partition function is

$$Z_0 = \int \mathrm{d}^N\varphi \exp\Big(-\frac{1}{2\hbar}\varphi^T M\varphi\Big) \tag{1.30}$$

$$= \int \mathrm{d}^N\chi \exp\Big(-\frac{1}{2\hbar}\chi^T\Lambda\chi\Big) \tag{1.31}$$

$$= \prod_{c=1}^{N} \int \mathrm{d}\chi_c\, e^{-\frac{\lambda_c}{2\hbar}\chi^2} = \sqrt{\frac{(2\pi\hbar)^N}{\det M}} \tag{1.32}$$

We want to get from the partition function to correlation functions. We can do this by introducing an N-component vector of external sources J to the action

$$S_0(\varphi) \to S_0 + J^T\varphi. \tag{1.33}$$

The partition function is then

$$Z_0(J) = \int \mathrm{d}^N\varphi \exp\Big(-\frac{1}{2\hbar}\varphi^T M\varphi - \frac{1}{\hbar}\Big)J^T\varphi. \tag{1.34}$$

Similar to solving an ordinary Gaussian integral, we complete the square by writing $\widetilde{\varphi} = \varphi + M^{-1}J$. One can then solve this integral to be

$$Z_0(J) = Z_0(0)\exp\Big(\frac{1}{2\hbar}J^T M^{-1}J\Big). \tag{1.35}$$

This is called the *generating function*[1]. Correlation functions are obtained from differentiating with respect to the auxiliary sources J and evaluating the whole expression at $J = 0$:

$$\langle\varphi_a\varphi_b\rangle = \frac{1}{Z_0(0)}\int \mathrm{d}^N\varphi\, \varphi_a\varphi_b \exp\Big(-\frac{1}{2\hbar}\varphi^T M\varphi - \frac{1}{\hbar}J^T\varphi\Big)\Big|_{J=0}. \tag{1.36}$$

$$= \frac{1}{Z_0(0)}\int \mathrm{d}^N\varphi \Big(-\hbar\frac{\partial}{\partial J_a}\Big)\Big(-\hbar\frac{\partial}{\partial J_b}\Big)\exp(\dots)\Big|_{J=0} \tag{1.37}$$

$$= \frac{1}{Z_0(0)}\Big(-\hbar\frac{\partial}{\partial J_a}\Big)\Big(-\hbar\frac{\partial}{\partial J_b}\Big)Z_0(J)|_{J=0} \tag{1.38}$$

$$= \hbar\,(M^{-1})_{ab} := a \;\text{———}\; b \tag{1.39}$$

[1]When we go to higher dimensions, where $J = J(x)$ this will be a generating functional $Z[J(x)]$.

Connecting this to the *Quantum Field Theory* course, we identify this as the *free propagator*.

More generally, let $l(\varphi)$ be a *linear* combination of φ_a.

$$l(\varphi) = \sum_{a=1}^{N} l_a \varphi_a, \qquad l_a \in \mathbb{R}. \tag{1.40}$$

Then the steps above are equivalent to swapping $l(\varphi)$ for $l(-\hbar\frac{\partial}{\partial J}) = -\hbar \sum_a l_a \frac{\partial}{\partial J_a}$.

The correlation function can again be evaluated explicitly by the introduction of an auxiliary current J:

$$\left\langle l^{(1)}(\varphi) \cdots l^{(p)}(\varphi) \right\rangle = \frac{1}{Z_0(0)} \int \mathrm{d}^N \varphi \prod_{i=1}^{p} l^{(i)}(\varphi) e^{-\frac{1}{2\hbar}\varphi^T M \varphi - \frac{1}{\hbar} J^T \varphi} \Bigg|_{J=0} . \tag{1.41}$$

$$= (-\hbar)^p \prod_{i=1}^{p} l^{(i)}(\frac{\partial}{\partial J})\, e^{\frac{1}{2\hbar} J^T M^{-1} J} \Bigg|_{J=0} \tag{1.42}$$

In other words, if p is odd, the integrand is odd in some φ_a and the integral over $\varphi_a \in (-\infty, +\infty)$ vanishes. For $p = 2k$, the terms which are non-zero as $J \to 0$ have the following form. We need half the derivatives to bring down components of $m^{-1}J$ and half to remove the J -dependence from those terms that earlier derivatives brought down. As such, we get exactly k factors of M^{-1} . This is Wick's theorem.

Example 1.1.1 (4-point function)**:** Consider the 4-point correlation function. One can check that the above result gives

$$\langle \varphi_a \varphi_b \varphi_c \varphi_d \rangle = \hbar^2 \left[(M^{-1})_{ab}(M^{-1})_{cd} + (M^{-1})_{ac}(M^{-1})_{bd} + (M^{-1})_{ad}(M^{-1})_{bc} \right] \tag{1.43}$$

$$= \begin{array}{cc} a & b \\ | & | \\ c & d \end{array} + \begin{array}{ccc} a & — & b \\ & & \\ c & — & d \end{array} + \begin{array}{ccc} a & & b \\ & \times & \\ c & & d \end{array} \tag{1.44}$$

We end up with three terms, one for each way of grouping the 4 fields into pairs.

In general, for $\langle \varphi_1 \ldots \varphi_{2k} \rangle$, the number of terms is the number of distinct ways of pairing the $2k$ fields. This is $(2k+1)!! = (2k)!/(2^k k!)$; the number of permutations of $2k$ fields is $(2k)!$, but we have to divide this by the 2^k permutations within the pairs and the $k!$ ways of rearranging the pairs.

Remark: For complex fields, M is Hermitian but not symmetric anymore. In that case, the order of indices of M^{-1} is important. We keep track of this by drawing the propagator with a directed line

$$\langle \phi_a \phi_b^* \rangle = \hbar\, (M^{-1})_{ab} \;:=\; a \longrightarrow b \tag{1.45}$$

1.1.2 Interacting Theory

We want to go beyond the free theory. The way we are going to achieve this is by an expansion about the classical result $\hbar$. The resulting integral will end up not being convergent.

Claim 1: Integrals like $\int \mathrm{d}\phi\, f(\phi)e^{-S/\hbar}$ do not have a Taylor expansion about $\hbar = 0$.

Proof (Dyson). If the expansion about $\hbar = 0$ existed for $\hbar > 0$, then in the complex plane, there must be some open neighbourhood of $\hbar$ in which the expansion converges. For $S(\phi)$ has a minimum, the integral is divergent if $\mathrm{Re}(\hbar) < 0$. Therefore, the radius of convergence cannot be greater than zero. □

So the $\hbar$-expansion is at best *asymptotic*.

Definition 4 (asymptotic)**:** A series $\sum_{n=0}^{\infty} c_n\hbar^n$ is asymptotic to a function $I(\hbar)$ if

$$\lim_{\hbar\to 0^+} \frac{1}{\hbar^N}\left| I(\hbar) - \sum_{n=0}^{N} c_n\hbar^n \right| = 0. \tag{1.46}$$

Notation: We write $I(\hbar) \sim \sum_{n=0}^{\infty} c_n\hbar^n$.

The series misses out transcendental terms like $e^{-\frac{1}{\hbar^2}} \sim 0$. However, these can evidently be important since obviously $e^{-\frac{1}{\hbar^2}} \neq 0$ for finite $\hbar$. These are called *non-perturbative contributions*. These become important in particular for non-Abelian gauge theories.

Take the ϕ-fourth action for a real scalar

$$S(\phi) = \underbrace{\frac{1}{2}m^2\phi^2}_{S_0(\phi)} + \underbrace{\frac{\lambda}{4!}\phi^4}_{S_1(\phi)} \qquad \begin{matrix} m^2 > 0 \\ \lambda > 0. \end{matrix} \tag{1.47}$$

Expand the exponential in the paritition function Z about the minimum of $S(\phi)$, which is $\phi = 0$.

$$Z = \int \mathrm{d}\phi \exp\left[-\frac{1}{\hbar}\left(\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right)\right] \tag{1.48}$$

$$= \int \mathrm{d}\phi\, e^{-S_0/\hbar} \overbrace{\sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\lambda}{4!\hbar}\right)^n \phi^{4n}}^{e^{-S_1/\hbar}}. \tag{1.49}$$

In order to make progress, we truncate the series to be able to swap the order of summation and integration. This misses out transcendental terms. In the end, we end up with a series that is asymptotic to Z:

$$Z \sim \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^{N} \frac{1}{n!}\left(-\frac{\hbar\lambda}{4!m^4}\right)^n 2^{2v} \int_0^{\infty} \mathrm{d}t\, e^{-t} t^{2n+\frac{1}{2}-1}, \tag{1.50}$$

where $t = \frac{1}{2\hbar}m^2\phi^2$. We recognise the integral to be the Gamma function

$$\int_0^\infty \mathrm{d}t\, e^{-t} t^{2n+\frac{1}{2}-1} = \Gamma(2n+\frac{1}{2}) = \frac{(4n)!\sqrt{\pi}}{4^{2n}(2n)!}. \tag{1.51}$$

The partition function is

$$Z \sim \frac{\sqrt{2\pi\hbar}}{m}\sum_{n=0}^{N}\left(-\frac{\hbar\lambda}{m^4}\right)^n \underbrace{\frac{1}{(4!)n!}}_{(a)}\underbrace{\frac{(4n)!}{2^{2n}(2n)!}}_{(b)} \tag{1.52}$$

The factor on the right comes in part from (a) the Taylor expansion of the term $S_1(\phi) = (\phi)\frac{\lambda}{4!}\phi^4$ in the exponential and from (b) the number of ways of pairing the $4n$ fields of the n copies of ϕ^4. Stirling's approximation allows us to write $n! \approx e^{n\ln n}$. The factor in the partition function then become

$$\frac{(4n)!}{(4!)^n n! 2^{2n}(2n)!} \approx n!. \tag{1.53}$$

We end up with factorial growth, signalling that the series is not convergent, but asymptotic!

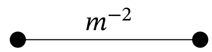
1.1.3 Diagrammatic Method

Let us now introduce a current J to obtain the generating function

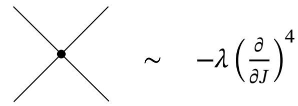
$$\begin{aligned}
Z(J) &= \int \mathrm{d}\phi \exp\left[-\frac{1}{\hbar}\left(S_0(\phi)+S_1(\phi)+J\phi\right)\right] && (1.54)\\
&= \exp\left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right]\underbrace{\int \mathrm{d}\phi \exp\left[-\frac{1}{\hbar}(S_0+J\phi)\right]}_{Z_0(J)} && (1.55)\\
&\overset{(1.35)}{\propto} \exp\left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right]\exp\left(\frac{1}{2\hbar}J^T M^{-1} J\right), \qquad M = m^2 && (1.56)\\
&\sim \sum_{V=0}^{N}\frac{1}{V!}\left[-\frac{\lambda}{4!\hbar}\left(\hbar\frac{\partial}{\partial J}\right)^4\right]^V \sum_{P=0}^{\infty}\frac{1}{P!}\left(\frac{1}{2\hbar}\frac{J^2}{2\hbar m^2}\right)^P. && (1.57)
\end{aligned}$$

This is called the *double expansion*. Diagrammatically, each of the P propagators, represented by a line as in Fig. 1.1a, give a factor of $M^{-1} = m^{-2}$. We use a large filled circle at the end of a line to represent a source factor J. Each of the V factors $\left(\frac{\partial}{\partial J}\right)^4$, originating from the interaction term $S_1(\phi)$, are associated with a vertex as in Fig. 1.1b. We use a small dot (or sometimes a small square) to mark a vertex.

Let us check that we reproduce the result (1.52) for $Z(0)$. For a term to be non-zero when $J = 0$, we need the number of derivations to be equal to the number of source terms coming from the end of propagators.



(a) Propagator with external sources J at both ends.



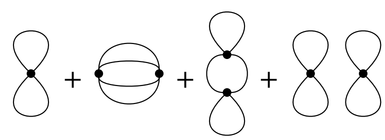
(b) Vertex

Figure 1.1: Components of the diagrammatic representation of the double series.

Notation (external sources)**:** We denote by E the number of external sources, which are left undifferentiated. For P propagtors and V vertices, the number of such sources is

$$E := 2P - 4V. \tag{1.58}$$

For $Z(0)$ we will require $E = 0$, whereas for n-point functions, we will want $E = n$. The first non-trivial terms are $(V, P) = (1, 2), (2, 4), \dots$.

$Z(0) \propto 1 +$  $+ O(V = 3).$ (1.59)

1.1.4 Symmetry Factors

Definition 5 (symmetry factor)**:** The *symmetry factor* S is the number of ways of redrawing the unlabeled diagram, leaving it unchanged.

Definition 6 (pre-diagram)**:** A *pre-diagram* for a (V, P) term in the double expansion is a collection of V vertices and P propagators, where the ends of the vertex lines are labeled by numbers and the ends of the propagators labelled by letters.

We count the number of times each diagram appears in the double expansion by using such pre-diagrams.

Example 1.1.2 ($V = 1$)**:** Consider the first diagram with only a single vertex and two loops attached to it. There are $A = 4!$ ways of matching the sources a, a', b, b' to the derivatives at $1, 2, 3$, and 4. This is cancelled by a 4! in the denominator $F = (V!)(4!)^V(P!)2^P = 4! \cdot 2 \cdot 2^2$ of Eq. (1.52).

So the $(V, P) = (1, 2)$ diagram comes with a prefactor of $\frac{A}{F} = \frac{1}{8}$ (times $-\hbar\lambda m^{-4}$).

In general, S is given by the relation $\frac{A}{F} = \frac{1}{S}$, where A is the number of ways of assigning the sources to the derivatives and F the number of non-equivalent permutations of all vertices, each vertex's legs,

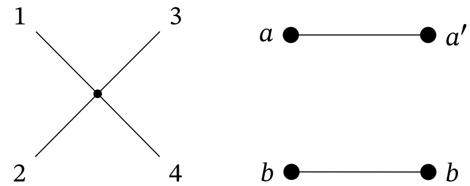
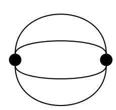


Figure 1.2: Pre-diagram of the $V = 1$ diagram with $P = 2$ loops.

all propagators, and their ends. However, the symmetry of each particular graph is important: If the diagram has a particular symmetry, then some permutations in F may be identical and have been double-counted. For the above diagram, consider the pairing $(1a, 2a', 3b, 4b')$. Swapping $a \leftrightarrow a'$ and $1 \leftrightarrow 2$ gives exactly the same graph, so it should not be counted twice.

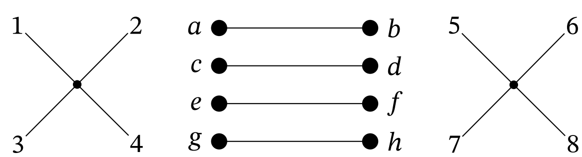
An alternative way to determine S is to consider the actions, which leave invariant the unlabelled diagram. These are called the automorphisms of the graph. For $(1, 2)$, we can swap the direction of upper and lower loops (2^2) and also swap upper and lower loops (2). Therefore, we obtain $S = 2 \cdot 2^2 = 8$.

Example 1.1.3 (basketball)**:** Let us look at a slightly more complicated example. The *basketball* diagram has the symmetry factor



$$S = 4! \cdot 2 = 48 \tag{1.60}$$

The pre-diagram associated to this is



(1.61)

We can simply calculate $F = 2 \cdot (4!)^2 \cdot 4! \cdot 2^4 = 4^3 \cdot 2^{14}$ and from the pre-diagram we determine the ways to pair currents and derivatives

$$A = \underbrace{(2 \cdot 4)}_{a} \underbrace{(4)}_{b} \underbrace{(2 \cdot 3)}_{c} \underbrace{(3)}_{d} \underbrace{(2 \cdot 2)}_{e} \underbrace{(2)}_{f} \underbrace{(2 \cdot 1)}_{g} \underbrace{(1)}_{h} = 3^2 \cdot 2^{10}. \tag{1.62}$$

. There are probably multiple ways to obtain this factor, but the reasoning here was as follows: For the letter a, we have a choice (factor 2) whether to connect to the left or the right vertex. In each case, we have 4 numbers to connect to. Since the basketball shape has no loops, this means that b has no choice in which vertex to use; it always has to be the one that we did not choose for a. For b we only have a choice of 4 numbers to connect to. For c, we again have a choice of two vertices, but only three remaining numbers (since the others are filled by a or b). We proceed in the same way for the remaining letters. Thus $A/F = 1/48$.

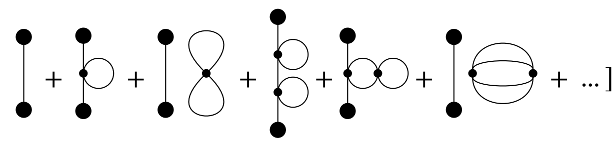
For the other diagrams, we have

$$\frac{Z(0)}{Z_0(0)} = 1 - \frac{\lambda\hbar}{8m^4} + \frac{\hbar^2\lambda^2}{m^8}\left(\frac{1}{48} + \frac{1}{16} + \frac{1}{128}\right) \tag{1.63}$$

1.1.5 Diagrams with External Sources

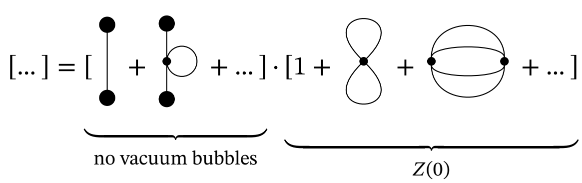
As we have previously mentioned, diagrams with $E = n$ external sources need to be considered for the n-point correlation functions. Let us focus on those diagrams that have $E = 2$ external currents.

$Z(J) \supset$



(1.64)

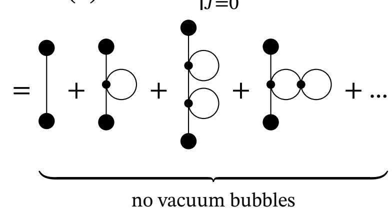
First we factor out the vacuum bubble diagrams



(1.65)

The 2-point expectation value, which is normalised by $Z(0)$, is then given by those diagrams with two external sources that do not have any vacuum bubbles

$$\langle \phi^2 \rangle = \frac{(-\hbar)^2}{Z(0)} \left.\frac{\partial^2 Z(J)}{\partial J^2}\right|_{J=0} \tag{1.66}$$

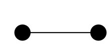


(1.67)

In this case, all diagrams are connected. This will not be true for higher moments.

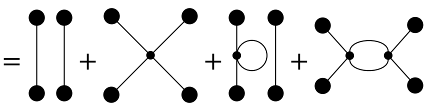
From $Z(J)$ (1.57), the $E = 2$, $V = 0$ ($P = 1$) term is

$$\frac{1}{2\hbar}\frac{J^2}{m^2} \tag{1.68}$$



We have $F = 2$ and $A = 1$, so $\frac{A}{F} = \frac{1}{2} = \frac{1}{S}$. The expectation value is $\langle \phi^2 \rangle = \frac{\hbar}{m^2} =$  as expected!

$\langle \phi^{2n} \rangle$ proceeds similarly, but note that there *are* disconnected diagrams. For example, for $n = 2$ we have

$\langle \phi^4 \rangle =$  (1.69)

1.2 Effective Actions

Definition 7 (Wilson effective action)**:** We define $W(\phi)$ such that $Z(J) = e^{-W(J)/\hbar}$.

Claim 2: $W(0)$ is the sum of all connected vacuum diagrams and $W(J)$ is the sum of all connected diagrams.

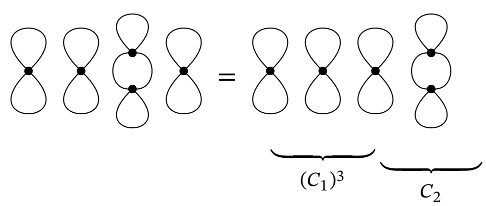
Proof. We denote the set of connected diagrams as $\{C_I\}$, which are taken to contain their respective symmetry factors. Any diagram D is a product of connected diagrams:

$$D = \frac{1}{S_D} \prod_I (C_I)^{n_I}, \tag{1.70}$$

where n_I is the number of times C_I appears in D, and S_D is the symmetry factor associated with rearranging diagrams C_I's, given by

$$S_D = \prod_I n_I!. \tag{1.71}$$

Example 1.2.1: Consider the diagram $D \propto (C_1)^3 \cdot C_2$:



(1.72)

The disconnected parts commute. We have $n_1 = 3$ parts of type C_1 and $n_2 = 1$ of C_2. The symmetry factor associated with the permutations of disconnected pieces is $S_D = 3! \cdot 1! = 6$.

Let $\{n_I\}$ be the set of integers specifying D, then

$$\frac{Z}{Z_D} = \sum_{\{n_I\}} D = \sum_{\{n_I\}} \prod_I \frac{1}{n_I!}(c_I)^{n_I} \tag{1.73}$$

$$= \prod_I \sum_{n_I} \frac{1}{n_I!}(c_I)^{n_I} \tag{1.74}$$

$$= \exp\left(\sum_I c_I\right) \tag{1.75}$$

$$= \exp(\text{sum of unique connected diagrams}) \tag{1.76}$$

$$:= e^{-(W-W_0)/\hbar}, \tag{1.77}$$

where $W = W_0 - \hbar \sum_I c_I$. □

Claim 3: $W(J)$ is the generating function for *connected correlation functions*.

Example 1.2.2 ($\langle\phi^2\rangle$)**:** Taking logarithms, we have

$$-\frac{1}{\hbar} W(J) = \ln(Z(J)). \tag{1.78}$$

Differentiating with respect to J twice and evaluating at $J = 0$ gives

$$-\frac{1}{\hbar}\left.\frac{\partial^2}{\partial J^2} W\right|_{J=0} = \frac{1}{Z(0)}\left.\frac{\partial^2 Z}{\partial J^2}\right|_{J=0} - \frac{1}{(Z(0))^2}\left.\left(\frac{\partial Z}{\partial J}\right)^2\right|_{J=0} \tag{1.79}$$

$$= -\frac{1}{\hbar^2}\left[\langle\phi^2\rangle - \cancel{\langle\phi\rangle^2}\right] = \frac{1}{\hbar^2}\langle\phi^2\rangle_{\text{connected}}. \tag{1.80}$$

There are also theories where $\langle\phi\rangle \neq 0$.

Example 1.2.3 ($\langle\phi^4\rangle$)**:** Less trivially,

$$-\frac{1}{\hbar}\left.\frac{\partial^4 W}{\partial J^4}\right|_{J=0} = \frac{1}{Z(0)}\left.\frac{\partial^4 Z}{\partial J^4}\right|_{J=0} - \left.\left(\frac{1}{Z(0)}\frac{\partial^2 Z}{\partial J^2}\right)^2\right|_{J=0} \tag{1.81}$$

$$= \langle\phi^4\rangle - \langle\phi^2\rangle^2 = \langle\phi^4\rangle_{\text{connected}} \tag{1.82}$$

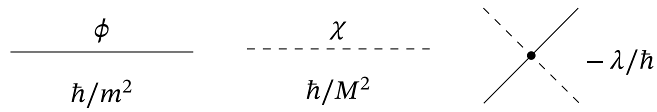
Remark: In statistics and statistical field theory, where ϕ is taken to be a random variable, the n-point correlation function $\langle\phi^n\rangle$ is often called the n^{th} *moment* of ϕ. The connected correlation functions $\langle\phi^n\rangle_c$ are then called *cumulants*. In this context $W(J)$ is the *cumulant generating function*[1], which is the natural logarithm of the *moment generating function* Z.

Interactions

Consider an action for two distinguishable fields

$$S(\phi, \chi) = \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2\chi^2. \tag{1.83}$$

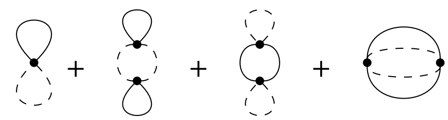
There is no factorial behind the factor 4. The Feynman rules are



(1.84)

The sum of connected vacuum diagrams is

$$-\frac{W}{\hbar} \sim$$

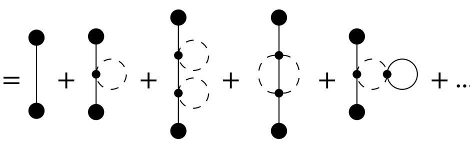


(1.85)

$$= -\frac{\hbar\lambda}{4m^2M^2} + \frac{\hbar^2\lambda^2}{m^4M^4}[\frac{1}{16} + \frac{1}{16} + \frac{1}{8}]. \tag{1.86}$$

As in (1.67), the two-point correlator $\langle\phi^2\rangle$ is given by the sum of Feynman diagrams with two external source insertions, represented by a large dot:

$$\langle\phi^2\rangle =$$



(1.87)

$$= \frac{\hbar}{m^2} - \frac{\hbar^2\lambda}{2m^4M^2} + \frac{\hbar^3\lambda^2}{m^6M^4}[\frac{1}{4} + \frac{1}{2} + \frac{1}{4}] + \dots \tag{1.88}$$

In the following section we will arrive at these results in a different way: Say we do not care about the field χ, maybe its mass is $M \gg m$ and it is never produced in our experiments. In such a case, we would want to 'integrate out' χ by defining $W(\phi)$ as

$$e^{-W(\phi)/\hbar} = \int \mathrm{d}\chi\, e^{-S(\phi,\chi)/\hbar}. \tag{1.89}$$

From this viewpoint, we treat $\phi^2\chi^2$ as a source term $J = \phi^2$ for χ^2. Once found, we can then use $W(\phi)$ to calculate expectation values $\langle f(\phi)\rangle$. It is in this sense that $W(\phi)$ indeed plays the role of an effective action for the field ϕ—one in which all the quantum corrections due to χ are taken into account.

1.2.1 Effective Field Theory: Integrating out χ

From the action (1.83), we obtained Feynman diagrams for both fields and calculated expressions for W and $\langle\phi^2\rangle$. Let us now show that we can get the same thing by first removing χ from the theory and then calculating expectation values in the effective theory. In other words, we want to calculate correlation functions only involving ϕ fields as

$$\langle f(\phi)\rangle = \frac{1}{Z}\int \mathrm{d}\phi\,\mathrm{d}\chi\, f(\phi)e^{-S(\phi,\chi)/\hbar} = \frac{1}{Z}\,\mathrm{d}\phi\, f(\phi)e^{-W(\phi)/\hbar}. \tag{1.90}$$

In this simple example,

$$e^{-W(\phi)/\hbar} = \int \mathrm{d}\chi\, e^{-S(\phi,\chi)/\hbar} = e^{-m^2\phi^2/2\hbar}\sqrt{\frac{2\pi\hbar}{M^2 + \frac{1}{2}\lambda\phi^2}}. \tag{1.91}$$

Taking logarithms, we obtain

$$W(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\hbar}{2}\ln\left(1 + \frac{\lambda}{2M^2}\phi^2\right) + \frac{\hbar}{2}\ln\left(\frac{M^2}{2\pi\hbar}\right). \tag{1.92}$$

The final term is constant and in non-gravitational physics does not effect any expectation values.

▌ In gravitational theories, this term is taken to be the origin of the cosmological constant.

We expand the logarithm

$$W(\phi) = \left(\frac{m^2}{2} + \frac{\hbar\lambda}{4M^2}\right)\phi^2 - \frac{\hbar\lambda^2}{16M^4}\phi^4 + \frac{\hbar\lambda^3}{48M^6}\phi^6 + \dots \tag{1.93}$$

From a theory which did not have any self-interaction, integrating out the χ -field gave us infinitely many self-interaction terms for all the even powers. We can think of the first term as an effective mass and write the other couplings as

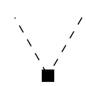
$$W(\phi) := \frac{m^2_{\text{eff}}}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_6}{6!}\phi^6 + \cdots + \frac{\lambda_{2k}}{(2k)!}\phi^{2k} + \dots, \tag{1.94}$$

where

$$m^2_{\text{eff}} = m^2 + \frac{\hbar\lambda}{2M^2} \qquad \lambda_{2k} = (-1)^{k+1}\hbar\frac{(2k)!}{2^{k+1}k}\frac{\lambda^k}{M^{2k}}. \tag{1.95}$$

In higher dimensions, we usually need to calculate $W(\phi)$ perturbatively. From $S(\phi,\chi)$ and the path integral over χ, we have the Feynman rules

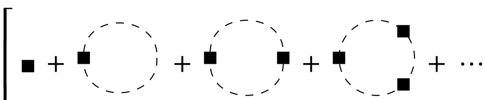
$\hbar/M^2$ and



$$\sim -\frac{\lambda\phi^2}{2\hbar}. \tag{1.96}$$

The propagator is the same as in the previous Feynman rules (1.84), but the factor associated with the vertex now accounts for the fact that we are treating the interaction term $\frac{\lambda}{4}\phi^2\chi^2$ as a source term for χ^2. The effective action is then given by the sum of connected diagrams

$W(\phi) \sim -\hbar\Big[$



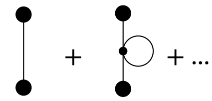
$\Big]$ (1.97)

$$= S(\phi) + \frac{1}{2}\frac{\hbar\lambda}{2M}\phi^2 - \frac{1}{4}\frac{\hbar\lambda^2}{4M^4}\phi^4 + \frac{1}{3!}\frac{\hbar\lambda^3}{8M^6}\phi^6 + \cdots, \tag{1.98}$$

where in the first term $S(\phi) = S(\phi, 0)$ is the part of the action that is unaffected by the integral over χ.

We can now use $W(\phi)$ to calculate the expectation value

$$\langle \phi^2 \rangle = \frac{1}{Z} \int \mathrm{d}\phi\, \phi^2 e^{-W(\phi)/\hbar} \sim \tag{1.99}$$



$$= \frac{\hbar}{m_{\text{eff}}^2} - \frac{\lambda_4 \hbar^2}{2m_{\text{eff}}^6} + \dots \tag{1.100}$$

The five diagrams of (1.87) have been reduced to just two diagrams with the effective action $W(\phi)$.

1.2.2 Quantum Effective Action Γ

Definition 8: We define the average field $\Phi = \langle \phi \rangle_J$ in the presence of an external source J as

$$\Phi := \frac{\partial W}{\partial J} = -\frac{\hbar}{Z(J)} \frac{\partial}{\partial J} \int \mathrm{d}\phi\, e^{-(S+J\phi)/\hbar} := \langle \phi \rangle_J, \tag{1.101}$$

where we have assumed that $S(\phi)$ as before is even in ϕ and has a minimum at $\phi = 0$.

Definition 9 (quantum effective action)**:** We have a Legendre transformation from the Wilsonian effective action $W(J)$ to the *quantum effective action* $\Gamma(\Phi)$:

$$\Gamma(\Phi) = W(J) - \Phi J. \tag{1.102}$$

Claim 4: As usual for Legendre transformations, we have

$$\boxed{\frac{\partial \Gamma}{\partial \Phi} = -J} \tag{1.103}$$

So $J \to 0$ corresponds to an extremum (in practice a minimum) of the effective action $\Gamma(\Phi)$.

Proof. Using the product rule, the chain rule, and the definition (1.101) of Φ,

$$\frac{\partial \Gamma}{\partial \Phi} = \frac{\partial W}{\partial \Phi} - J - \Phi \frac{\partial J}{\partial \Phi} = \frac{\partial W}{\cancel{\partial J}} \frac{\cancel{\partial J}}{\partial \Phi} - J - \cancel{\Phi \frac{\partial J}{\partial \Phi}} = -J \tag{1.104}$$

□

In higher dimensions, one performs a derivative expansion

$$\Gamma(\Phi) = \int \mathrm{d}^d x \left[-V(\Phi) - \frac{1}{2} \partial^\mu \Phi \partial_\mu \Phi + \dots \right] \tag{1.105}$$

where the first term in this expansion defines the effective potential $V(\Phi)$. The effective potential might shift the minimum of the action when including quantum effects. These quantum corrections can lead to spontaneous symmetry breaking.

1.2.3 Analogy with Statistical Mechanics

Cut bridge

The W is like the Helmholtz free energy F. In the presence of some external magnetic field h in some spin system, it is defined via

$$e^{-\beta F(h)} = \int \mathcal{D}s\, e^{-\beta H}. \tag{1.106}$$

We can define the magnetisation to be $M = -\frac{\partial F}{\partial h}$. One can then switch to the Gibbs free energy, analogous to Γ, by defining

$$G(M) = F(h) + Mh. \tag{1.107}$$

As $h \to 0$, the magnetisation of the system is the minimum of G.

1.2.4 Perturbative Calculation of $\Gamma(\Phi)$

We want to treat Φ as ϕ and write down a new Wilsonian effective action

$$e^{-W_\Gamma(J)/g} = \int \mathrm{d}\Phi\, e^{-(\Gamma(\Phi)+J\Phi)/g}, \tag{1.108}$$

where g is a new, fictitious Planck constant and J a source. This is in analogy to before, replacing the action S with the quantum effective action Γ. As before, $W_\Gamma(J)$ is the sum of connected vacuum diagrams and can be written as a power series in g

$$W_\Gamma(J) = \sum_{l=0}^{\infty} g^l W_\Gamma^{(l)}(J), \tag{1.109}$$

where l counts the loops. In particular, $W_\Gamma^{(0)}$ is composed of all the *tree* diagrams with Φ as legs.

In the limit of $g \to 0$, $W_\Gamma(J) \to W_\Gamma^{(0)}(J)$. Also as $g \to 0$, the integral in Φ is dominated by the minimum of the exponent, which is Φ such that

$$\frac{\partial\Gamma}{\partial\Phi} = -J \qquad \text{(steepest descent)} \tag{1.110}$$

Therefore, by analogy to the earlier definition with with action $S(\phi) + J\phi$, we have

$$W_\Gamma^{(0)}(J) = \Gamma(\Phi) + J\Phi = W(J). \tag{1.111}$$

The moral of the story is that the sum of connected diagrams in a theory with action $S(\phi) + J\phi$ (i.e. $W(J)$) can be constructed from a sum of tree diagrams with action $\Gamma(\Phi) + J\Phi$.

Definition 10 (bridge)**:** An edge in a connected graph is a *bridge* if removing it would leave the graph disconnected.

An example is shown in Fig. 1.3.

Figure 1.3: Cutting a bridge disconnects a graph.

Definition 11 (1PI)**:** A connected graph is called *one-particle irreducible* (1PI) if it has no bridges.

Example 1.2.4: Take the N-component scalar ϕ_a, $a = 1, \ldots, N$. Then the connected propagator is

$$\langle \phi_a \phi_b \rangle_J^c = -\hbar \frac{\partial^2 W}{\partial J_a \partial J_b} = -\hbar \frac{\partial \Phi_b}{\partial J_a}, \tag{1.112}$$

since $\Phi_b = \frac{\partial W}{\partial J_b}$. Using $J_a = -\frac{\partial \Gamma}{\partial \Phi_a}$ then gives

$$\langle \phi_a \phi_b \rangle_J^c = -\hbar \left(\frac{\partial J_a}{\partial \Phi_b} \right)^{-1} = \hbar \left(\frac{\partial^2 \Gamma}{\partial \Phi_b \Phi_a} \right)^{-1}. \tag{1.113}$$

The full connected propagator is $\hbar$ times the inverse of the quadratic term in $\Gamma(\Phi)$.

1.3 Fermions

Definition 12 (Grassmann numbers)**:** The *Grassmann numbers* are a set of n elements $\{\theta_a\}$ obeying $\theta_a \theta_b = -\theta_b \theta_a$. For any scalar $\phi \in \mathbb{C}$, we have $\theta_a \phi = \phi \theta_a$.

Remark: Anti-symmetry implies that $\theta^2 = 0$.

Remark: The product of an even number of θ's acts like a scalar and commutes with other θ's, for example $\theta_a(\theta_c \theta_d) = (\theta_c \theta_d)\theta_a$.

> Compare this with the behaviour of fermions (Grassmann variables, half integer spin) and bosons (scalars, integer spin) under exchange.

> The anticommuting Grassmann variables are frequently called *a-numbers*, in contrast to commuting *c-numbers*.

A general function of Grassmann variables can be written

$$f(\{\theta_a\}) = f(\theta) = f + \phi_a \theta_a + \frac{1}{2} g_{ab} \theta_a \theta_b + \cdots + \frac{1}{n!} h_{a_1,\ldots,a_n} \theta_{a_1} \ldots \theta_{a_n}. \tag{1.114}$$

The coefficients are anti-symmetric in their indices.

1.3.1 Grassmann Analysis

Definition 13 (differentiation)**:** On Grassmann variables, we define *differentiation* via

$$\left(\frac{\partial}{\partial\theta_a}\theta_b + \theta_b\frac{\partial}{\partial\theta_a}\right)\bullet = \delta_{ab}\bullet . \tag{1.115}$$

For a single Grassmann variable θ, where possible functions are of the form $F(\theta) = f + \phi\theta$, we need only specify $\int d\theta$ and $\int \theta d\theta$ to define *integration*. Requiring translational invariance gives

$$\int d\theta(\theta - \mu) = \int d\theta\theta, \tag{1.116}$$

when μ is a constant Grassmann variable. We can then choose a normalisation:

$$\int d\theta = 0 \qquad \int d\theta\theta = 1. \tag{1.117}$$

Remark: There is a similarity between differentiation and integration.

The *Berenzin rules* give

$$\int d\theta\frac{\partial}{\partial\theta}F(\theta) = 0. \tag{1.118}$$

Similarly, for n Grassmann variables, we define

$$\int \theta_1 \dots \theta_n \, \mathrm{d}^n\theta = 1. \tag{1.119}$$

Definition 14 (Berenzin integration)**:** In general, we define *Berenzin integration* over Grassmann variables as

$$\int \theta_{a_1} \dots \theta_{a_n} \, \mathrm{d}^n\theta = \epsilon_{a_1 \dots a_n}. \tag{1.120}$$

Let us consider how this integration measure changes under change of variables $\theta'_a = A_{ab}\theta_b$. Consider the integral

$$\begin{aligned}\int \mathrm{d}^n\theta\,\theta'_{a_1}\dots\theta'_{a_n} &= A_{a_1b_1}\dots A_{a_nb_n}\int \mathrm{d}^n\theta\,\theta_{b_1}\dots\theta_{b_n} && (1.121)\\ &= \det\{A\}\epsilon_{a_1\dots a_n} && (1.122)\\ &= \det\{A\}\int \mathrm{d}^n\theta'\,\theta'_{a_1}\dots\theta'_{a_n}. && (1.123)\end{aligned}$$

Therefore, the measure changes as $\mathrm{d}^n\theta' = [\det\{A\}]^{-1}\,\mathrm{d}^n\theta$, which is the inverse to bosonic integration!

1.3.2 Free Fermion Field Theory ($d = 0$)

Consider two fermionic fields θ_1, θ_2. The action must be bosonic, so up to an additive constant which can be absorbed into the normalisation of the partition function, we have $S = \frac{1}{2}A\theta_1\theta_2$, where $A \in \mathbb{R}$. The free partition function can be calculated explicitly by expanding the exponential

$$\mathcal{Z}_0 = \int \mathrm{d}^2\theta\, e^{-S(\theta)/\hbar} = \int \mathrm{d}^2\theta \left(1 - \frac{A}{2\hbar}\theta_1\theta_2\right) = -\frac{A}{2\hbar}. \tag{1.124}$$

Now, for $n = 2m$ fields θ_a have the action $S = \frac{1}{2}A_{ab}\theta_a\theta_b$, where A_{ab} is an anti-symmetric real matrix. The partition function is

$$\mathcal{Z}_0 = \int \mathrm{d}^{2m}\theta \sum_{j=0}^{m} \frac{(-1)^j}{(2\hbar)^j\, j!}(A_{ab}\theta_a\theta_b)^j \tag{1.125}$$

Since the sum terminates, we can exchange the order of differentiation and integration, giving

$$\begin{aligned}\mathcal{Z}_0 &= \frac{(-1)^m}{(2\hbar)^m m!}\int \mathrm{d}^{2m}\theta\, A_{a_1a_2}A_{a_3a_4}\dots A_{a_{2m-1}a_{2m}}\theta_{a_1}\dots\theta_{a_{2m}} &(1.126)\\ &= \frac{(-1)^m}{(2\hbar)^m m!}A_{a_1a_2}\dots A_{a_{2m-1}a_{2m}}\epsilon^{a_1a_2\dots a_{2m}} &(1.127)\\ &:= \frac{(-1)^m}{\hbar^m}\,\mathrm{Pfaff}\,A, &(1.128)\end{aligned}$$

which defines the *Pfaffian* $\mathrm{Pfaff}\,A$ of the matrix A .

Exercise 1.1: Show that $(\mathrm{Pfaff}\,A)^2 = \det A$.

Remark: Again, this result is, up to prefactors, the inverse of the bosonic free partition function

$$\mathcal{Z}_0 = \sqrt{\frac{(2\pi\hbar)^n}{\det M}}. \tag{1.129}$$

Introducing external, Grassmann-valued sources η_a to the action gives

$$S(\theta,\eta) = \frac{1}{2}A_{ab}\theta_a\theta_b + \eta_a\theta_a. \tag{1.130}$$

To compute the partition function, we complete the square as in the bosonic case:

$$S(\theta,\eta) = \frac{1}{2}\left(\theta_a + \eta_c(A^{-1})_{ca}\right)A_{ab}\left(\theta_b + \eta_d(A^{-1})_{db}\right) + \frac{1}{2}\eta_a(A^{-1})_{ab}\eta_b. \tag{1.131}$$

Using the translation invariance of Berezin integration, we have

$$\mathcal{Z}_0(\eta) = \mathcal{Z}_0(0)\exp\!\left(-\frac{1}{2\hbar}\eta^T A^{-1}\eta\right). \tag{1.132}$$

The propagator too can be evaluated explicitly to be

$$\langle\theta_a\theta_b\rangle = \frac{\hbar^2}{\mathcal{Z}_0(0)}\frac{\partial^2\mathcal{Z}_0(\eta)}{\partial\eta_a\partial\eta_b}\bigg|_{\eta=0} = \hbar(A^{-1})_{ab} \tag{1.133}$$

2 LSZ Reduction Formula

We will now start to move beyond $d = 0$ and transition to using path integrals. We will connect scattering amplitudes to correlation functions, specifically vacuum expectation values. This transitory topic will also allow us to have a glimpse at renormalisation. We will have in mind a weakly interacting theory, so that we can use the free theory as a good aproximation. Afterwards, we comment on how the interactions lead to deviations from the free theory. In doing so, we tell the story the way that Srednicki [Sred] does. However, the results that we will show can also be rigorously proved non-perturbatively, which usually happens in the final chapters of quantum field theory textbooks [P&S, Wb].

2.1 $2 \to 2$ Scattering

Consider a free scalar field in $3 + 1$ dimensions, built out of plane waves

$$\phi(x) = \int \frac{đ^3 k}{2E_{\mathbf{k}}} \left[a(\mathbf{k}) e^{-ik\cdot x} + a^\dagger(\mathbf{k}) e^{ik\cdot x} \right], \tag{2.1}$$

where $E_{\mathbf{k}}^2 = |\mathbf{k}|^2 + m^2$ and we are using the mostly-minus metric in Minkowski space so $k \cdot x = Et - \mathbf{k} \cdot \mathbf{x}$. The creation and annihilation operators are relativistically normalised, changing the factor from $\sqrt{2E} \to 2E$ as compared to the conventions from our *Quantum Field Theory* lectures. In a free theory, where we do not need to do any perturbative expansion, we set $\hbar = c = 1$. Let us find expressions for $a(\mathbf{k}) = a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ by looking a the following integrals:

$$\int \mathrm{d}^3 x \, e^{ik\cdot x} \phi(x) = \frac{1}{2E} a_{\mathbf{k}} + \frac{1}{2E} e^{2iEt} a_{-\mathbf{k}}^\dagger \tag{2.2}$$

$$\int \mathrm{d}^3 x \, e^{ik\cdot x} \partial_0 \phi(x) = -\frac{i}{2} a_{\mathbf{k}} + \frac{i}{2} e^{2iEt} a_{-\mathbf{k}}^\dagger. \tag{2.3}$$

We can solve these equations for $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{\dagger}$ to get

$$a(\mathbf{k}) = \int \mathrm{d}^3x\, e^{ik\cdot x}\left[i\partial_0\phi(x) + E\phi(x)\right] \tag{2.4}$$

$$a^{\dagger}(\mathbf{k}) = \int \mathrm{d}^3x\, e^{-ik\cdot x}\left[-i\partial_0\phi(x) + E\phi(x)\right]. \tag{2.5}$$

Let the initial state for the free theory be a one-particle state

$$|\mathbf{k}\rangle = a^{\dagger}(\mathbf{k})\,|\Omega\rangle\,, \tag{2.6}$$

where the vacuum $|\Omega\rangle$ satisfies $a(\mathbf{k})\,|\Omega\rangle = 0$ for all $\mathbf{k}$ and is normalised to $\langle\Omega|\Omega\rangle = 1$. The one-particle states are normalised to $\langle\mathbf{k}|\mathbf{k}'\rangle = (2E)\delta^3(\mathbf{k}-\mathbf{k}')$. We can take superpositions of these to introduce two Gaussian wavepackets with mean momentum $\mathbf{k}_1 \neq \mathbf{k}_2$ and width σ:

$$a_n^{\dagger} := \int \mathrm{d}^3k\, f_n(\mathbf{k})a^{\dagger}(\mathbf{k}), \qquad f_n(\mathbf{k}) \propto \exp\left[-\frac{|\mathbf{k}-\mathbf{k}_n|^2}{4\sigma^2}\right], \qquad n = 1,2. \tag{2.7}$$

Let us now evolve the Gaussians into the distant past and future, $t \to \pm\infty$, where the overlap in coordinate space is negligible. We will also assume that this works when including interactions. Interacting theories introduce the complication that $a^{\dagger}$ becomes time dependent, wherefore $a_1^{\dagger}$ and $a_2^{\dagger}$ depend on time. However, we will assume that the interacting $a_1^{\dagger}(t)$ and $a_2^{\dagger}(t)$ coincide with their free theory expressions (2.7) in the limit of $t \to \pm\infty$.

We want to consider the case where the initial and final states are two-particle states

$$|i\rangle = \lim_{t\to-\infty} a_1^{\dagger}(t)a_2^{\dagger}(t)\,|\Omega\rangle\,, \qquad |f\rangle = \lim_{t\to+\infty} a_{1'}^{\dagger}(t)a_{2'}^{\dagger}(t)\,|\Omega\rangle\,. \tag{2.8}$$

These are also normalised so that $\langle i|i\rangle = 1 = \langle f|f\rangle$ and we have $\mathbf{k}_1 \neq \mathbf{k}_2$ and $\mathbf{k}_1' \neq \mathbf{k}_2'$ respectively. Our goal is to find the *scattering amplitude* $\langle f|i\rangle$.

Let us find an expression that allows us to relate creation (annihilation) operators from the distant past (future) to the distant future (past). We employ the expressions (2.7) and (2.5) and the fundamental theorem of calculus to rewrite the difference of $a_1^{\dagger}$ at $t = \pm\infty$ as

$$a_1^{\dagger}(\infty) - a_1^{\dagger}(-\infty) = \int_{-\infty}^{\infty} \mathrm{d}t\, \partial_0 a_1^{\dagger}(t) \tag{2.9}$$

$$= \int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \mathrm{d}^4x\, \partial_0\left[e^{-ik\cdot x}(-i\partial_0\phi + E\phi)\right] \tag{2.10}$$

$$= -i\int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \mathrm{d}^3x\, e^{-ik\cdot x}(\partial_0^2 + E^2)\phi \tag{2.11}$$

$$= -i\int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \mathrm{d}^3x\, e^{-ik\cdot x}(\partial_0^2 - \overleftarrow{\nabla}^2 + m^2)\phi \tag{2.12}$$

$$= -i\int \mathrm{d}^3k\, f_1(\mathbf{k}) \int \mathrm{d}^3x\, e^{-ik\cdot x}(\partial^2 + m^2)\phi, \tag{2.13}$$

where in the fourth line the dispersion relation $E^2 = |\mathbf{k}|^2 + m^2$ is used and $|\mathbf{k}|^2$ turns into the laplacian ∇^2 acting leftward onto the exponential. In the last line we integrated by parts twice with $f_1(\mathbf{k})$ ensuring that surface terms vanish. In the free theory, the Klein–Gordon equation therefore implies that $a_1^\dagger(\infty) - a_1^\dagger(-\infty) = 0$. After a similar calculation for $a_2^\dagger$ and $a_{1',2'}$, we obtain

$$a_j^\dagger(-\infty) = a_j^\dagger(\infty) + i \int \mathrm{d}^3k\, f_j(\mathbf{k}) \int \mathrm{d}^4x\, e^{-ik\cdot x}(\partial^2 + m^2)\phi \tag{2.14a}$$

$$a_j'(\infty) = a_j'(-\infty) + i \int \mathrm{d}^3k\, f_j(\mathbf{k}) \int \mathrm{d}^4x\, e^{ik\cdot x}(\partial^2 + m^2)\phi. \tag{2.14b}$$

The $2 \to 2$ scattering amplitude is

$$\langle f|i\rangle = \langle\Omega|\, \mathcal{T} a_{1'}(\infty) a_{2'}(\infty) a_1^\dagger(-\infty) a_2^\dagger(-\infty)\, |\Omega\rangle\,, \tag{2.15}$$

where we were able to trivially insert the time-ordering operator $\mathcal{T}$, since the operators were already time-ordered.

We then use the integral expressions (2.14) to replace the creation (annihilation) operators of (2.15) with their counterparts at the distant future (past). The time-ordering operator then moves the resulting $a_i^\dagger(\infty)$ to the left and the $a_{j'}(-\infty)$ to the right, annihilating the vacuum. The only non-zero term is the one with the products of the integrals, yielding the unwieldy formula

$$\boxed{\begin{aligned}\langle f|i\rangle = i^4 \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\, \mathrm{d}^4x_1'\, \mathrm{d}^4x_2'\, & e^{-ik_1\cdot x_1} e^{-ik_2\cdot x_2} e^{ik_1'\cdot x_1'} e^{ik_2'\cdot x_2'} \\ & \times (\partial_1^2 + m^2)(\partial_2^2 + m^2)(\partial_{1'}^2 + m^2)(\partial_{2'}^2 + m^2) \\ & \times \langle\Omega|\, \mathcal{T}\phi(x_1)\phi(x_2)\phi(x_1')\phi(x_2')\, |\Omega\rangle\,, \quad (2.16)\end{aligned}}$$

where we have taken the narrow-width limit $\sigma \to 0$ of the Gaussian wave packets, turning $f(\mathbf{k}_j) \to \delta^3(\mathbf{k} - \mathbf{k}_j)$. This is a specific case of the *LSZ reduction* formula. In its more general form, it states that the information of $n \to n'$ scattering amplitudes can be recovered from the correlation functions $\langle\Omega|\, \mathcal{T}\phi(x_1) \dots \phi(x_n)\phi(x_1') \dots \phi(x_{n'}')\, |\Omega\rangle$.

This result can be proven without recourse to the free theory as we have done. However, the machinery needed to build this up is more involved. Our derivation assumed that the interactions do not alter the creation operators (2.7) at $t \to \pm\infty$. In fact, the general derivation requires only weaker assumptions:

1) We need a unique ground state $|\Omega\rangle$ and the first excited state has to be a single particle, which means that we need a single simple pole, rather than a continuum of particles for example.

2) We want the field $\phi\, |\Omega\rangle$ to be that single particle state, meaning that $\langle\Omega|\, \phi\, |\Omega\rangle = 0$. Usually, this is not a problem; if we have $\langle\Omega|\, \phi\, |\Omega\rangle = v \neq 0$, say when we have some spontaneous symmetry breaking, then we let $\widetilde{\phi} = \phi - v$ to obtain $\langle\Omega|\, \widetilde{\phi}\, |\Omega\rangle = 0$.

3) We want ϕ normalised such that $\langle k|\,\phi(x)\,|\Omega\rangle = e^{ik\cdot x}$ as in the free case. Usually, interactions spoil this and require us to rescale the field $\phi \to \mathcal{Z}_\phi^{1/2}\phi$.

This hints at the fact that we need to 'renormalise' the field and couplings when includng interactions. For example, in ϕ^4 theory we need to introduce renormalisation factors Z_ϕ, Z_m, and Z_λ:

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \to \frac{\mathcal{Z}_\phi}{2}\partial_\mu\phi\partial^\mu\phi - \frac{Z_m}{2}m^2\phi^2 - \frac{\lambda}{4!}Z_\lambda\phi^4. \tag{2.17}$$

We will meet this Lagrangian again in Sec. 3.4.

3 Scalar Field Theory

3.1 Wick Rotation

Let us make the connection between Euclidean and Minkowski spacetime a bit more concrete. It is convenient to start from the Minkowski metric with signature $(+---)$ and go to the Euclidean one with $(++++)$. The Lagrangian density in Minkowski space is

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V[\phi]. \tag{3.1}$$

We now use square brackets instead of parentheses to indicate that the function ϕ is itself a function of spacetime, so that $V[\phi]$ is a functional. In particular, the potential is

$$V[\phi] = \frac{1}{2}m^2\phi^2 + \sum_{n>2}\frac{1}{n!}V^{(n)}\phi^n. \tag{3.2}$$

The partition function is

$$Z = \int \mathcal{D}\phi\, e^{i\int \mathrm{d}x^0 L}, \tag{3.3}$$

where the Lagrangian L is the integral of the Lagrangian density $\mathscr{L}$ over space

$$L = \int \mathrm{d}^3x\, \mathscr{L}. \tag{3.4}$$

The free propagator is

$$\frac{i}{k^2 - m^2 + i\epsilon} = \frac{i}{(k^0)^2 - |\mathbf{k}|^2 - m^2 + i\epsilon}. \tag{3.5}$$

Let $ix^0 = x_4$ and use the Euclidean metric $(++++)$. Arrange the signs so that we have a Lagrangian density that has the same sign as (3.1) in the kinetic term

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V[\phi]. \tag{3.6}$$

The partition function is

$$Z = \int \mathcal{D}\phi\, e^{-\int \mathrm{d}x_4 L}. \tag{3.7}$$

One argument put forward for using the mostly plus metric in Minkowski space is that this transition to Euclidean space simply involves switching only one of the signs, rather than having to keep track of i's and minus signs as we do it here.

The propagator is

$$\widetilde{\Delta}_0(k) = \frac{1}{k^2 + m^2} = \frac{1}{(k_4)^2 + |\mathbf{k}|^2 + m^2}. \tag{3.8}$$

This means that we rotate the contour integral so that the poles lie on the imaginary axis, as illustrated in Fig. 3.1.

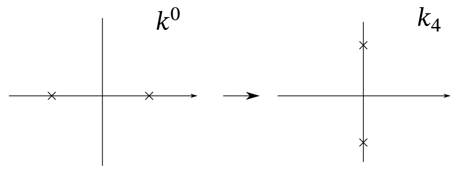


Figure 3.1

3.2 Feynman Rules

Take the free propagator as in Ch. ??.

$$S_0[\phi, J] = \int_{\mathbb{R}^4} \mathrm{đ}^4 x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + J(x)\phi(x)\right]. \tag{3.9}$$

Now write $\phi(x)$ and $J(\phi)$ as a Fourier integral

$$\phi(x) = \int \mathrm{đ}^4 k\, e^{ik\cdot x}\widetilde{\phi}(k). \tag{3.10}$$

The action then becomes

$$S_0[\widetilde{\phi}, \widetilde{J}] = \frac{1}{2}\int_{\mathbb{R}^4} \mathrm{đ}^4 k \left[\widetilde{\phi}(-k)(k^2 + m^2)\widetilde{\phi}(k) + \widetilde{J}(-k)\widetilde{\phi}(k) + \widetilde{J}(k)\widetilde{\phi}(-k)\right] \tag{3.11}$$

$$= \frac{1}{2}\int \mathrm{đ}^4 k \left[\widetilde{\chi}(-k)(k^2 + m^2)\widetilde{\chi}(k) - \frac{\widetilde{J}(-k)\widetilde{J}(k)}{k^2 + m^2}\right], \tag{3.12}$$

where $\widetilde{\chi} = \widetilde{\phi} + \widetilde{J}/(k^2 + m^2)$. Assume that the partition function is normalised such that $Z_0[0] = 1$. Then

$$Z_0[\widetilde{J}] = \exp\left[\frac{1}{2}\int \mathrm{đ}^4 k\, \frac{\widetilde{J}(-k)\widetilde{J}(k)}{k^2 + m^2}\right]. \tag{3.13}$$

The propagator is

$$\widetilde{\Delta}_0(q) = \frac{\delta^2 Z_0[\widetilde{J}]}{\delta\widetilde{J}(-q)\delta\widetilde{J}(q)}\Big|_{\widetilde{J}=0} = \frac{1}{q^2 + m^2}. \tag{3.14}$$

The functional derivative is defined in position and momentum space as

$$\frac{\delta}{\delta f(x_1)} f(x_2) = \delta^4(x_1 - x_2) \qquad \frac{\delta}{\delta \tilde{g}(k_1)} \tilde{g}(k_2) = \bar{\delta}^4(k_1 - k_2). \tag{3.15}$$

We can now Fourier transform back to obtain the propagator in position space

$$\Delta_0(x - x') = \int \text{đ}^4 k \, \frac{e^{ik\cdot(x-x')}}{k^2 + m^2}. \tag{3.16}$$

The partition functional in real space is then

$$Z_0[J] = \exp\left[\frac{1}{2} \int \mathrm{d}^4x \, \mathrm{d}^4x' \, J(x) \Delta(x - x') J(x')\right]. \tag{3.17}$$

3.2.1 Interactions

Let us now include interactions. The Lagrangian density becomes

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1, \tag{3.18}$$

where $\mathscr{L}_0 = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2$ is the free Lagrangian density. In presenting the following results we will skip a few steps since the derivations are very similar to what we have seen in $d = 1$ in the previous chapter. The partition functional is

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi \exp\left[-\int \mathrm{d}^4x \, (\mathscr{L}_0 + \mathscr{L}_1 + J\phi)\right] && (3.19) \\ &= \exp\left\{-\int \mathrm{d}^4y \, \mathscr{L}_1\left[-\frac{\delta}{\delta J(y)}\right]\right\} \exp\left[\frac{1}{2}\int \mathrm{d}^4x \, \mathrm{d}^4x' \, J(x) \Delta_0(x - x') J(x')\right] && (3.20) \\ &\sim \sum_{V=0}^{N} \frac{1}{V!} \left(-\int \mathrm{d}^4y \, \mathscr{L}_1\left[-\frac{\delta}{\delta J(y)}\right]\right)^V \sum_{P=0} \frac{1}{P!} \left[\frac{1}{2}\int \mathrm{d}^4x \, \mathrm{d}^4x' \, J(x) \Delta_0(x - x') J(x')\right]^P. && (3.21) \end{aligned}$$

For each term in $Z[J]$ there is a graph, as we have done before.

- Each of the P propagators $\Delta_0(x - x')$ is represented by

$$x \ \rule[0.5ex]{4em}{0.4pt} \ x' = \Delta_0(x - x'). \tag{3.22}$$

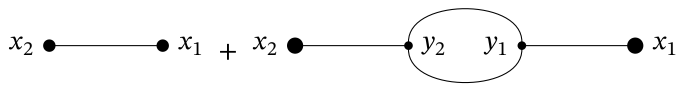
- We have V vertices with n lines from $\mathscr{L}_1\left[-\frac{\delta}{\delta J(y)}\right]$.
- We integrate over positions of all vertices.
- We have symmetry factors as before.
- Sources, represented by external large dots need to come in pairs; one derivative to bring down a J and the other to annihilate it.

$$J(x) \bullet\!\rule[0.5ex]{4em}{0.4pt}\!\bullet J(x') \tag{3.23}$$

Example 3.2.1 (**ϕ^3 theory**)**:** With the Wilsonian effective action $W[J] = -\ln Z[J]$, the two-point correlation (i.e. the propagator) is

$$\langle\phi(x_2)\phi(x_1)\rangle = -\left(-\frac{\delta}{\delta J(x_2)}\right)\left(-\frac{\delta}{\delta J(x_1)}W[J]\right). \tag{3.24}$$

For ϕ^3 theory, we have the interaction Lagrangian $\mathscr{L}_1 = \frac{\lambda}{3!}\phi^3$. Then we can expand the propagator in the diagrammatic series

$\langle\phi(x_2)\phi(x_1)\rangle =$  $+ \ldots$ (3.25)

The second diagram D represents the integral

$$D = \frac{\lambda^2}{2}\int \mathrm{d}^4y_1\,\mathrm{d}^4y_2\,\Delta_0(x_2-y_2)\Delta_0(y_1-x_1)(\Delta_0(y_2-y_1))^2 \tag{3.26}$$

with symmetry factor $S = 2$. The Fourier transform of the whole propagator is defined to be

$$\langle\widetilde{\phi}(p_2)\widetilde{\phi}(p_1)\rangle = \int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,e^{-i(p_1\cdot x_1+p_2\cdot x_2)}\langle\phi(x_2)\phi(x_1)\rangle, \tag{3.27}$$

where the sign in the exponential is a convention, which corresponds to choosing the momenta in the diagram to be direct outward [Br], which is illustrated in Fig. 3.2.

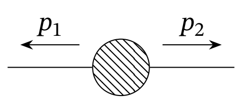


Figure 3.2: The Fourier sign convention in (3.27) corresponds to outward-directed external momenta in momentum-space Feynman diagrams.

Thus, the Fourier transform $\widetilde{D}$ of the second diagram D is

$$\begin{aligned}\widetilde{D} = \frac{\lambda^2}{2}&\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\,e^{-i(p_1\cdot x_1+p_2\cdot x_2)}\int \mathrm{d}^4y_1\,\mathrm{d}^4y_2\\ &\times\int\left[\prod_{j=1}^{4}\text{đ}^4k_j\right]e^{ik_2\cdot(x_2-y_2)}e^{ik_1\cdot(y_1-x_2)}e^{i(k_3+k_4)\cdot(y_2-y_1)}\widetilde{\Delta}_0(k_1)\widetilde{\Delta}_0(k_2)\widetilde{\Delta}_0(k_3)\widetilde{\Delta}_0(k_4),\end{aligned} \tag{3.28}$$

where we also expanded the position-space Δ_0 in terms of their momentum-space Fourier modes $\widetilde{\Delta}_0$. The only terms containing x_1 and x_2 are $e^{-i(p_1+k_1)\cdot x_1}$ and $e^{-i(p_2-k_2)\cdot x_2}$. The integral over x_1 and x_2 can therefore be performed to give the Fourier-normalised δ-functions $\text{đ}^4(p_1+k_1)$ and $\text{đ}^4(p_2-k_2)$. These allow us to perform the integrals over k_1 and k_2, giving

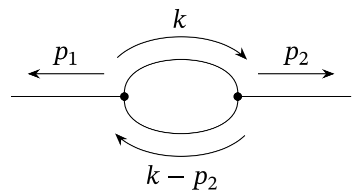
$$\begin{aligned}\widetilde{D} = \frac{\lambda^2}{2}\int \mathrm{d}^4y_1\,\mathrm{d}^4y_2\int \text{đ}^4k_3\,\text{đ}^4k_4\,e^{-i(p_1+k_3+k_4)\cdot y_1}e^{-i(p_2-k_3-k_4)\cdot y_2}&\\ \times\widetilde{\Delta}_0(-p_1)\widetilde{\Delta}_0(p_2)\widetilde{\Delta}_0(k_3)\widetilde{\Delta}_0(k_4).&\end{aligned} \tag{3.29}$$

In the same way, the y_1 and y_2 integrals give $\delta\!\!\!^{-4}(p_1+k_3+k_4)$ and $\delta\!\!\!^{-4}(p_2-k_3-k_4)$ respectively. Using these and performing the integral over k_3, we are left with the integral

$$\widetilde{D} = \frac{\lambda^2}{2}\int \text{đ}^4 k\, \delta\!\!\!^{-4}(p_1+p_2)\widetilde{\Delta}_0(-p_1)\widetilde{\Delta}_0(p_2)\widetilde{\Delta}_0(p_2-k)\widetilde{\Delta}_0(k), \tag{3.30}$$

where we relabeled $k_4 \to k$. We represent this diagrammatically as

$$\widetilde{D} \quad = \tag{3.31}$$



It is conventional to write loop diagrams with the momenta flowing in a consistent direction.

The momentum space Feynman rules can be summarised to be

- lines are propagators, which contribute factors $\widetilde{\Delta}_0(k) = (k^2+m^2)^{-1}$
- each n-point vertex contributes a factor of $-V^{(n)}$ in $\mathscr{L}_1$
- overall momentum conservation of outgoing momenta gives a factor $\delta\!\!\!^{-4}(\sum_j p_j)$
- each loop gives a momentum integral
  - take one of the propagators in the loop to be k (e.g. (3.31) it is the top one)
  - the momenta of the other propagators are given by momentum-conservation at the vertices (e.g. this determines the bottom momentum[1] one in (3.31))
- divide by the diagram's symmetry factor

3.3 Vertex Functions

Recall that $W[J]$ and $\Gamma[\Phi]$ are the sum of connected and 1PI diagrams respectively. Generalising the definition (1.102) from $d = 0$, the quantum effective action is the Legendre transform

$$\Gamma[\Phi] = W[J] - \int \mathrm{d}^4 x\, J(x)\Phi(x). \tag{3.32}$$

[1]Why did we not choose the bottom momentum to be $k + p_1$? The momentum conserving $\delta\!\!\!^{-4}(p_1+p_2)$ guarantees that these choices are equivalent!

Then we have the familiar relations, which are now functional derivatives

$$\frac{\delta W[J]}{\delta J(x)} = \Phi(x), \qquad \frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} = -J(x). \tag{3.33}$$

The connected n-point functions are

$$G^{(n)}(x_1, \dots, x_n) = (-1)^{n+1} \prod_{i=1}^{n} \frac{\delta}{\delta J(x_i)} W[J] = \langle \phi(x_1) \dots \phi(x_n) \rangle^{\text{connected}}. \tag{3.34}$$

Analogously, we define the n -point vertex functions

$$\Gamma^{(n)}(x_1, \dots, x_n) = (-1)^n \prod_{i=1}^{n} \frac{\delta}{\delta \Phi(x_i)} \Gamma[\Phi]. \tag{3.35}$$

Example 3.3.1 ($n = 2$)**:** The 2 -point functions are

$$G^{(2)}(x, y) = -\frac{\delta^2 W}{\delta J(x) \delta J(y)} = -\frac{\delta \Phi(y)}{\delta J(x)}, \tag{3.36}$$

$$\Gamma^{(2)}(x, y) = +\frac{\delta^2 \Gamma}{\delta \Phi(x) \delta \Phi(y)} = -\frac{\delta J(x)}{\delta \phi(y)}. \tag{3.37}$$

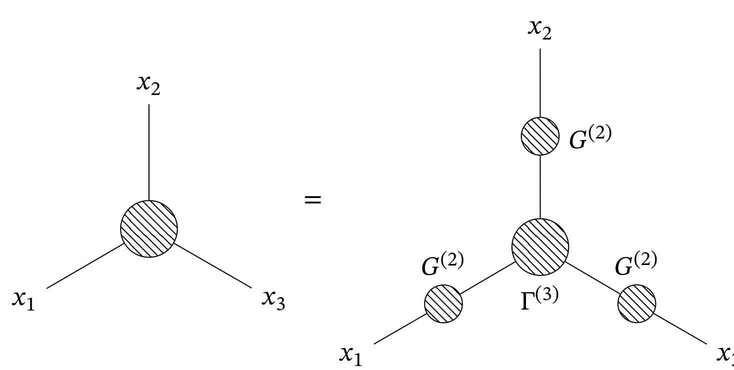
These are inverses of each other

$$\int \mathrm{d}^4 z \, G^{(2)}(x, z) \Gamma^{(2)}(z, y) = \delta^{(4)}(x - y). \tag{3.38}$$

Example 3.3.2 (Sheet I Question 4)**:** We can calculate

$$G^{(3)}(x_1, x_2, x_3) = \int \mathrm{d}^4 z_1 \, \mathrm{d}^4 z_2 \, \mathrm{d}^4 z_3 \, G^{(2)}(x_1, z_1) G^{(2)}(x_2, z_2) G^{(2)}(x_3, z_3) \underbrace{\left(-\frac{\delta^3 \Gamma}{\delta \Phi(z_1) \delta \Phi(z_2) \delta \Phi(z_3)} \right)}_{:=\Gamma^{(3)}(z_1, z_2, z_3)} \tag{3.39}$$

Diagrammatically, this can be represented



, (3.40)

where the central blob on the right hand side corresponds to the 1PI diagrams $\Gamma^{(3)}$ and the small blobs are the propagators $G^{(2)}$. The vertex functions $\Gamma^{(3)}$ come from *amputated* 3-point functions.

Equation (3.39) can be inverted

$$\Gamma^{(3)}(y_1, y_2, y_3) = \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\, \mathrm{d}^4x_3\, \Gamma^{(2)}(x_1, y_1)\Gamma^{(2)}(x_2, y_2)\Gamma^{(2)}(x_3, y_3)G^{(3)}(x_1, x_2, x_3). \tag{3.41}$$

It is useful to compare this to the LSZ reduction formula. We can generalise this to $n > 3$ and to momentum space. This is done in [Ryd, Sec. 7.3].

3.4 Renormalisation

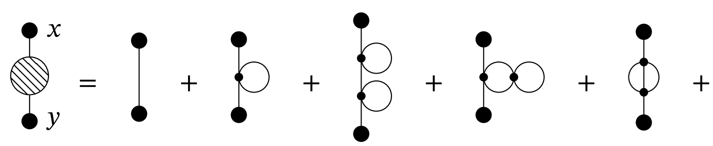
3.4.1 Self-Energy and Amputated Diagrams

Consider our favourite ϕ^4 scalar field theory in Euclidean spacetime

$$S[\phi] = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{3.42}$$

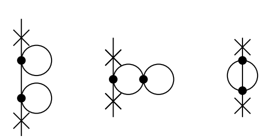
If we ignore numerical factors, the (connected) 2-point function, also called the *complete* or *dressed* propagator, is the sum of all connected diagrams

$$G^{(2)}(x, y) =$$



$$O(\lambda^3) \tag{3.43}$$

The first term is the 2-point vertex G_0 of the free theory, called the *bare propagator*. Including the interactions $O(\lambda)$ effects the propagator by changing the physical mass away from the *bare mass* m, and hence giving rise to *self-energy*. The first and last factors common to all diagrams are external propagators, so we define *amputated* or *truncated* graphs by multiplying the external legs by inverse propagators. Diagrammatically, an amputated external propagator is marked with a cross. For example the amputated diagrams of order λ^2 are shown in Fig. 3.3.



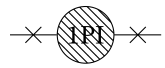
(3.44)

Figure 3.3: Amputated diagrams with two vertices in ϕ^4 theory.

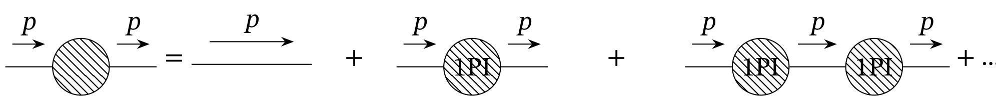
The first of these contains a propagator, along which it may be cut, and can therefore be viewed as a product of graphs of lower order; it is a 1-particle reducible graph. THe other two graphs are 1-particle irreducible (1PI).

Definition 15 (proper self-energy)**:** The (momentum space) *proper self-energy* function $\Pi(p^2)$ is the sum of 1PI graphs

$$\Sigma(p^2) \quad = \tag{3.45}$$



The complete propagator (in momentum space) $\widetilde{G}^{(2)}(p)$ may therefore be written in terms of the bare propagator $G_0(p) = (p^2 + m^2)^{-1}$ and the proper self-energy function $\Pi(p)$ as the following geometric series, which establishes the connection between connected and 1PI diagrams:



(3.46)

$$\widetilde{G}^{(2)}(p) = \frac{1}{p^2+m^2} + \overbrace{\frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2}} + \overbrace{\frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2}} + \ldots \tag{3.47}$$

$$= \frac{1}{p^2+m^2-\Pi(p^2)}. \tag{3.48}$$

Thus, we find that the 2-point vertex function is

$$\widetilde{\Gamma}^{(2)}(p) = [\widetilde{G}^{(2)}(p)]^{-1} = p^2 + m^2 - \Pi(p^2). \tag{3.49}$$

Moreover, defining the physical mass m_{phys} by the pole in the complete propagator

$$G^{(2)}(p) = \frac{1}{p^2 + m^2_{\text{phys}}}, \tag{3.50}$$

gives, on comparison with (3.48),

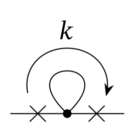
$$m^2_{\text{phys}} = m^2 - \Pi(p^2), \tag{3.51}$$

which justifies the appellation "self-energy" term to Π; it represents the mass from the 'bare' to the 'physical' value, calculated to all orders in perturbation theory.

3.4.2 Divergences

Perturbatively, we can write down the loop expansion in the same way as (3.43). In particular, the 1-loop contribution is

$$\Pi_1(p^2) =$$



$$= -\frac{\lambda}{2}\int đ^4k\, \frac{1}{k^2+m^2}. \tag{3.52}$$

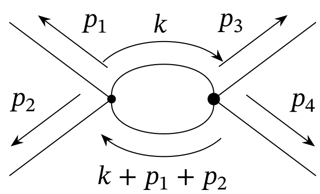
Now in spherical polar coordinates $\mathrm{d}^4k = k^3\,\mathrm{d}k\,\mathrm{d}\Omega$ and in general $\mathrm{d}^dk = S_d|k|^{d-1}\,\mathrm{d}|k|$ with $S_d = (2\pi)^{d/2}/\Gamma(d/2)$ is the surface area of the unit d-sphere, where Γ is the Euler-Gamma function. We see that this integral diverges. To tackle it, we introduce a cutoff Λ and change variables to $x = k^2/m^2$ so that

$$\Pi_1(p^2) = -\frac{\lambda}{2}\int^{\Lambda}\frac{\text{đ}^4k}{k^2+m^2} = -\frac{\lambda S_4}{4(2\pi)^4}\int^{\Lambda^2/m^2}\frac{x\,\mathrm{d}x}{1+x}, \tag{3.53}$$

$$= -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2\ln\left(1+\frac{\Lambda^2}{m^2}\right)\right]. \tag{3.54}$$

As expected, this is divergent as $\Lambda \to \infty$. We call this a *UV divergence*.

The (momentum space) 4-point function at 1-loop in ϕ^4-theory is given by the following sum of diagrams

$$\widetilde{\Gamma}^{(4)}(p_1,p_2,p_3,p_4) =$$ 

$+$ (permutations of momenta) (3.55)

$$= \frac{\lambda^2}{2}\int^{\Lambda}\text{đ}^4k\,\frac{1}{k^2+m^2}\sum_{p\in\{p_1+p_2,p_1+p_3,p_1+p_4\}}\frac{1}{(p+k)^2+m^2}. \tag{3.56}$$

As $k\to\infty$, we have $\mathrm{d}k\,/k^4$ in the integral and we expect a logarithmic $\ln(\Lambda/m)$ divergence. Since this integral is independent of the p's, it is convenient to evaluate the integral with external momenta set to zero.

$$\widetilde{\Gamma}^{(4)}(0,0,0,0) = \frac{3\lambda^2}{2}\int^{\Lambda}\text{đ}^4k\,\frac{1}{(k^2+m^2)^2} = \frac{3\lambda^2}{16\pi^2}\int^{\Lambda}\frac{k^3\,\mathrm{d}k}{(k^2+m^2)^2} \tag{3.57}$$

$$= \frac{3\lambda^2}{32\pi^2}\left[\ln\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{\Lambda^2+m^2}\right] \tag{3.58}$$

These diagrams must be dealt with.

On general grounds, we expect the full propagator to be of the form

$$\widetilde{G}^{(3)} = \sum_n \frac{\left|\langle\Omega|\,\phi(0)\,|n\rangle\right|^2}{p^2+m^2_{\text{phys}}} \tag{3.59}$$

where the sum (or the integral) over the states n comes about from inserting a complete set of states. The assumption is that the first excited state is a single particle $n=1$, which dominates the expression:

$$\widetilde{G}^{(3)} = \frac{\left|\langle\Omega|\,\phi(0)\,|1\rangle\right|^2}{p^2+m_1^2} + \dots, \tag{3.60}$$

where the additional terms are finite when $p^2 = -m_{\text{phys}}^2$. We expect m_{phys}^2 to be a physical mass with $\langle\Omega|\,\phi(0)\,|1\rangle = 1$. Loop diagrams spoil this; we have to renormalise the theory to control the divergences.

3.4.3 Renormalisation Schemes: Separating the Divergences from the Finite Pieces

Clearly we have to do something about the divergences in the one-loop calculations of the previous section. A practical approach is to prescribe a way of separating the divergences from the finite pieces. In doing so, we will introduce some arbitrary choice in the theory and sacrifice some predictability; however, it turns out that we do not loose as much as one might at first expect. Let us decorate the original theory by adding 0-subscripts

$$\mathscr{L}_0 = \frac{1}{2}(\partial\phi_0)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4. \tag{3.61}$$

Generically, contribution from loops spoil this and we have to rescale

$$\phi_0 = Z_\phi^{\frac{1}{2}}\phi. \tag{3.62}$$

The rescaling Z_ϕ is determined by proper normalisation for LSZ, namely that

$$\langle\Omega|\,\widetilde{\phi}(0)\,|1\rangle = 1. \tag{3.63}$$

The Lagrangian becomes

$$\mathscr{L}_0 = \frac{Z_\phi}{2}(\partial\phi)^2 + \frac{Z_\phi}{2}m_0^2\phi^2 + \frac{Z_\phi^2\lambda}{4!}\phi^4. \tag{3.64}$$

What we want to do is separate out two sets of terms. We want to write the original Lagrangian in terms of the renormalised Lagrangian and any counter terms. In particular, the renormalised Lagrangian should be of the same form as the original action, giving

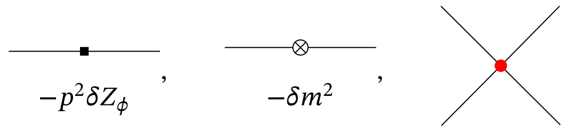
$$\mathscr{L}_0 = \mathscr{L}_{\text{ren}} + \mathscr{L}_{\text{ct}}, \tag{3.65}$$

$$= \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right] + \left[\frac{\delta Z_\phi}{2}(\partial\phi)^2 + \frac{\delta m^2}{2}\phi^2 + \frac{\delta\lambda}{4!}\phi^4\right]. \tag{3.66}$$

Equating coefficients, we have

$$\delta Z_\phi = Z_\phi - 1, \qquad \delta m^2 = Z_\phi m_0^2 - m^2, \qquad \delta\lambda = Z_\phi^2\lambda_0 - \lambda. \tag{3.67}$$

Of course the Feynman rules for the renormalised Lagrangian $\mathscr{L}_{\text{ren}}$ are the same as for the original Lagrangian $\mathscr{L}_0$ but with the coefficients, m^2 and λ, interpreted as the renormalised ones. We need to find the Feynman rules for the counter terms $\mathscr{L}_{\text{ct}}$: Diagrammatically, we treat these as vertices



$$-\,\delta\lambda. \tag{3.68}$$

Generally, $\delta Z_\phi, \delta m^2, \delta\lambda$ are $O(\hbar)$ at most. Therefore, tree diagrams containing $\mathscr{L}_{\text{ct}}$ vertices are the same order as 1-loop diagrams from $\mathscr{L}_{\text{ren}}$.

The 2-point vertex is

$$\widetilde{\Gamma}^{(2)}(p) = [\widetilde{G}^{(2)}(p)]^{-1} = p^2 + m^2 - \Pi(p^2). \tag{3.69}$$

From $\mathscr{L}_{\text{ren}}$, we get $\Pi_1(p^2)$ at one loop just as in (3.54), but with m^2, λ interpreted as renormalised quantities of (3.66). From $\mathscr{L}_{\text{ct}}$,

$$\Pi_{1,\text{ct}} = -\delta m^2 - p^2\delta Z_\phi. \tag{3.70}$$

Finite result for $\Pi_{1,\text{ ren}} = \Pi_1(p^2) + \Pi_{1,\text{ct}}$ is obtained by choosing

$$\delta Z_\phi = 0 \qquad \text{and} \qquad \delta m^2 = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2\ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right]. \tag{3.71}$$

With this choice $\Pi_{1,\text{ren}} = 0$. The freedom to choose where to put finite points is called the *renormalisation scheme*. This arbitrary choice obviously looses some predictability, but we do it to make progress with the divergences. The above scheme is called the *on-shell* scheme, based on the requirement that

$$\Pi_{\text{ren}}(-m^2_{\text{phys}}) \overset{!}{=} m^2 - m^2_{\text{phys}} \qquad \text{and} \qquad \left.\frac{\partial\Pi_{\text{ren}}}{\partial p^2}\right|_{p^2=-m^2_{\text{phys}}} = 0. \tag{3.72}$$

The first term usually cancels out $m^2 - m^2_{\text{phys}} = 0$.

With the on-shell scheme,

$$\widetilde{G}^{(2)}(p) = \frac{1}{p^2 + m^2 - \Pi_{\text{ren}}(p^2)} = \frac{1}{p^2 + m^2_{\text{phys}}}. \tag{3.73}$$

This has a pole at $p^2 = -m^2_{\text{phys}}$, which is the reason for the name *on-shell*. We are giving up on predicting the mass of the particle; we dial it in by hand after obtaining it from some experimental measurement. The residue from the LSZ is 1.

Next, choose $\delta\lambda$ to cancel the divergences in $\widetilde{\Gamma}^{(4)}_1(0,0,0,0)$. We have $\widetilde{\Gamma}^{(4)}_{1,\text{ct}} = -\delta\lambda$. Choosing

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\ln\left(\frac{\Lambda^2}{m^2} - 1\right). \tag{3.74}$$

After a bit of algebra, this gives

$$\lambda_{\text{eff}} := \widetilde{\Gamma}^{(4)}_{\text{ren}}(0,0,0,0) = \lambda + \widetilde{\Gamma}^{(4)}_{1}(0,0,0,0) + \widetilde{\Gamma}^{(4)}_{1,\text{ct}} \tag{3.75}$$

$$= \lambda - \frac{3\lambda^2}{32\pi^2}\left[\ln\left(1 + \frac{m^2}{\Lambda^2}\right) + \frac{m^2}{m^2+\Lambda^2}\right]. \tag{3.76}$$

This is finite as $\Lambda \to \infty$. (In fact, the infinite piece is chosen such that $\lambda_{\text{eff}} \to \lambda$.)

This term really acts like an effective coupling, which—at least to one-loop order—incorporates the quantum corrections. If we did this calculation to all orders in a well defined theory, it would be solved.

3.5 Dimensional Regularisation

Since physical predictions come from tree diagrams with effective couplings like λ_{eff}, physical quantities should be independent of the cutoff Λ in the end. We will do this not with a hard momentum cutoff, but with a method that works more generally for gauge theories. Hard momentum cutoff are not compatible with gauge invariance, so we want to come up with a different regularisation method.

A more mathematically elegant way to do this is given by *dimensional regularisation*. It is a trick we do order-by-order in perturbation theory.

In the context of perturbation theory, divergences can be regulated by working in $d = 4 - \epsilon$ dimensions. Usually, we think about $0 < \epsilon \ll 1$, but we have already seen that taking $\epsilon \to 1$ gave us useful results in *Statistical Field Theory*.

Let us start from the original Lagrangian (and drop the zero subscripts)

$$S = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(m\phi)^2 + \frac{\lambda}{4!}\phi^4\right]. \tag{3.77}$$

3.5.1 Dimensional Analysis

Denote by square brackets $[\bullet]$ the mass dimension and let $\hbar = c = 1$. Given that $[S] = 0$ and $[\partial] = [m] = -[x] = 1$, we find that the field has mass dimension

$$[m^2\phi^2] = 2[m] + 2[\phi] = d \implies [\phi] = \frac{d}{2} - 1. \tag{3.78}$$

From this we find that the coupling has dimension

$$[\lambda\phi^4] = d \implies [\lambda] = 4 - d = \epsilon. \tag{3.79}$$

Working with a dimensionful coupling is annoying, so we introduce an arbitrary renormalisation scale μ with mass dimension $[\mu] = 1$, which is not to be taken to ∞ like the cutoff was. Then we write

$$\lambda = \mu^\epsilon g(\mu) \tag{3.80}$$

such that g is dimensionless. Of course, g will depend on whichever choice we end up making for the renormalisation scale μ.

Thus our action is

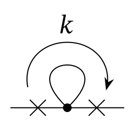
$$S = \int \mathrm{d}^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g\mu^\epsilon}{4!}\gamma^4 \right]. \tag{3.81}$$

Refer to Appendix B for the mathematical results used in this section.

3.5.2 One-Loop Renormalisation

Let us now return to the amputated diagram (3.52), writing $\lambda = g\mu^\epsilon$:

$$\Pi_1(p^2) =$$



$$= -\frac{g\mu^\epsilon}{2} \int_{\mathbb{R}^d} \frac{\bar{d}^d k}{k^2 + m^2} = -\frac{g\mu^\epsilon}{2} \frac{S_d}{(2\pi)^d} \int_0^\infty \frac{k^{d-1}\,\mathrm{d}k}{k^2 + m^2}, \tag{3.82}$$

where $k = |\mathbf{k}|$ is the norm of the d-dimensional vector $\mathbf{k}$. The integral can be performed by changing integration variables $u = m^2/(k^2 + m^2)$

$$\mu^\epsilon \int_0^\infty \frac{k^{d-1}\,\mathrm{d}k}{k^2 + m^2} = \frac{1}{2}\mu^\epsilon \int_0^\infty \frac{(k^2)^{d/2-1}\,\mathrm{d}k^2}{k^2 + m^2} = \frac{m^2}{2}\left(\frac{\mu}{m}\right)^\epsilon \int_0^1 \mathrm{d}u\, u^{-d/2}(1-u)^{d/2-1} \tag{3.83}$$

$$= \frac{m^2}{2}\left(\frac{\mu}{m}\right)^\epsilon \frac{\Gamma(\frac{d}{2})\Gamma(1-\frac{d}{2})}{\Gamma(1)}, \tag{3.84}$$

where we used the Euler beta function. Hence,

$$\Pi_1 = -\frac{gm^2}{2(4\pi)^{d/2}}\left(\frac{\mu}{m}\right)^\epsilon \Gamma(1 - \frac{d}{2}). \tag{3.85}$$

Using that

$$\Gamma(1-\frac{d}{2}) = \Gamma(\frac{\epsilon}{2} - 1) = -\frac{1}{1-\frac{\epsilon}{2}}\Gamma(\frac{\epsilon}{2}) = -\frac{2}{\epsilon} + \gamma - 1 + O(\epsilon). \tag{3.86}$$

We have thus exposed the divergence as $\epsilon \to 0$ that we want to tame.

Also

$$\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon/2} = 1 + \frac{\epsilon}{2}\ln\left(\frac{4\pi\mu^2}{m^2}\right) + O(\epsilon^2). \tag{3.87}$$

The result of putting the Lagrangian $\mathscr{L}_0$ into the diagram and finding the one-loop contribution to the vertex function is

$$\Pi_1(p^2) = -\frac{gm^2}{32\pi^2}\left[\frac{2}{\epsilon} - \gamma + 1 + \ln\left(\frac{4\pi\mu^2}{m^2}\right)\right] + O(\epsilon). \tag{3.88}$$

As in (3.65), we must add a counter term $\frac{1}{2}\delta m^2\phi^2$ to the Lagrangian with the intention to cancel the divergence. There are various possible renormalisation schemes we can choose. In practice, we often use one of

Minimal subtraction (MS): Just absorb the pole

$$\delta m^2 = -\frac{gm^2}{16\pi^2\epsilon}. \tag{3.89}$$

Modified minimal subtraction ($\overline{\mathbf{MS}}$) :

$$\delta m^2 = -\frac{gm^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) \tag{3.90}$$

Under this scheme, our result for the two-point function is

$$\Pi_1^{\overline{\mathrm{MS}}} = \frac{gm^2}{32\pi^2}\left(\ln\frac{\mu^2}{m^2} - 1\right). \tag{3.91}$$

Let us also calculate the four-point vertex function. The divergent piece is given by setting all external momenta to zero:

$$\widetilde{\Gamma}^{(4)}(0,0,0,0) = \frac{3g^2\mu^{2\epsilon}}{2}\int\frac{đ^d k}{(k^2+m^2)^2}. \tag{3.92}$$

We perform exactly the same tricks as before, performing angular integration and changing variables to make it dimensionless, as well as identifying the Euler-beta function. In the end, we get a very similar result

$$\widetilde{\Gamma}^{(4)}(0,0,0,0) = \frac{3g^2\mu^{\epsilon}}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln\frac{4\pi\mu^2}{m^2}\right) + O(\epsilon). \tag{3.93}$$

We have left one μ^ϵ outside on dimensional grounds in the next step. Again, this is divergent, so we introduce a counter term for g as well:

$$\mu^\epsilon\delta g = \frac{3g^2\mu^\epsilon}{32\pi^2}\underbrace{\Bigg(\overbrace{\frac{2}{\epsilon}}^{\mathrm{MS}} - \gamma + \ln 4\pi\Bigg)}_{\overline{\mathrm{MS}}}. \tag{3.94}$$

3.6 Calculating β-functions

3.6.1 The Old-Fashioned Approach to Investigating μ-dependence

We have the original Lagrangian

$$\mathscr{L}_0 = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4. \tag{3.95}$$

Adding counter terms we have

$$\mathscr{L}_{\text{ren}} + \mathscr{L}_{\text{ct}} = \frac{1+\delta Z_\phi}{2}(\partial\phi)^2 + \frac{m^2+\delta m^2}{2}\phi^2 + \frac{(g+\delta g)\mu^\epsilon}{4!}\phi^4. \tag{3.96}$$

This is just supposed to be a reshuffling of terms and divergences, so the two Lagrangians should be the same. Equating coefficients: the original parameters are μ -independent, since this splitting scale is arbitrarily chosen.

Look at dimensionless derivatives of the couplings (and masses)

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}g = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}g. \tag{3.97}$$

This is called the β -function, which tells us how the coupling constant 'runs' depending on the renormalisation scale.

We want

$$0 \overset{!}{=} \frac{\mathrm{d}}{\mathrm{d}\ln\mu}\lambda_0 = \frac{\mathrm{d}}{\mathrm{d}\ln\mu}\left[(g+\delta g)\mu^\epsilon\right] = \epsilon g(1+\frac{3g}{16\pi^2\epsilon}) + \beta(g)\left(1+\frac{3g}{8\pi\epsilon^2}\right), \tag{3.98}$$

where we employed the MS-scheme result of (3.94) (to save writing). The β-function is

$$\beta(g) = -\left(\frac{3g^2}{16\pi^2}+\epsilon g\right)\left(1+\frac{3g}{8\pi\epsilon^2}\right)^{-1}. \tag{3.99}$$

With a little bit of slight of hand, we forget that we want to take $\epsilon \to 0$ and hold ϵ fixed. We then expand the latter term in a binomial series

$$\beta(g) = \frac{3g^2}{16\pi^2} - \epsilon g + O(\frac{g^2}{\epsilon}, 2\text{ loop}). \tag{3.100}$$

Remark: In this whole calculation, we have ignored two-loop diagrams and higher.

As $\epsilon \to 0$, only the first term survives. Note that $\beta(\gamma) > 0$. As such, we obtained a differential equation, which tells us how g depends on μ:

$$\mu\frac{\mathrm{d}g}{\mathrm{d}\mu} = \frac{3g^2}{16\pi^2}. \tag{3.101}$$

We can solve this differential equation by separation of variables

$$\frac{\mathrm{d}g}{g^2} = \frac{3}{16\pi^2}\frac{\mathrm{d}\mu}{\mu}. \tag{3.102}$$

Integrating gives

$$\frac{1}{g(\mu')} = \frac{1}{g(\mu)} - \frac{3}{16\pi^2}\ln\frac{\mu'}{\mu}, \tag{3.103}$$

so

$$g(\mu') = \frac{g(\mu)}{1-\frac{3g}{16\pi^2}\ln\frac{\mu'}{\mu}} \overset{g\text{ small}}{\approx} g(\mu) + \frac{3(g(\mu))^2}{16\pi^2}\ln\frac{\mu'}{\mu}. \tag{3.104}$$

For $\mu' > \mu$, $g(\mu') > g(\mu)$. The coupling "runs" to larger values as μ increases.

Remark: If $\mu' \to \Lambda_{\phi^4}$, where Λ_{ϕ^4} is defined via

$$\frac{3g}{16\pi^2}\ln\frac{\Lambda_{\phi^4}}{\mu} = 1, \qquad \text{1-loop} \tag{3.105}$$

then $g(\mu') \to \infty$. This Λ_{ϕ^4} can be used as a scheme-dependent reference mass scale.

$$g(\mu) = \frac{16\pi^2}{3}\left[\ln\left(\frac{\Lambda_{\phi^4}}{\mu}\right)\right]^{-1}. \tag{3.106}$$

The appearance of the scale Λ_{ϕ^4} is "dimensional transmutation"; it seems like magic / alchemy that the regularisation of a dimensionless interaction introduces a scheme-dependent scale. It is an order-of-magnitude estimate of where the theory becomes non-perturbative.

Remark: Perturbation theory requires $\mu \ll \Lambda_{\phi^4}$.

3.6.2 The Modern Approach

Quantum effective action and the vertex functions $\Phi^{(n)}(\dots)$ should be physical. These go into the LSZ formula (2.16). Write $\phi_0 = Z_\phi^{\frac{1}{2}}\phi$, then

$$\Gamma_0^{(n)}(x_1,\dots,x_n) = (-1)^n\frac{\delta^n\Gamma[\phi_0]}{\delta\phi_0(x_1)\dots\delta\phi_0(x_n)} = (-1)^n Z_\phi^{1/2}\frac{\delta^n\Gamma[\phi]}{\delta\phi(x_1)\dots\delta\phi(x_n)}. \tag{3.107}$$

Definition 16 (anomalous dimension)**:** We define the analogue of the beta function, the *anomalous dimension* of ϕ , to be

$$\gamma_\phi = -\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_\phi. \tag{3.108}$$

With this definition, we have

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} Z_\phi^{1/2} = -\frac{n}{2} Z_\phi^{-n/2} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln Z_\phi = (n\gamma_\phi) Z_\phi^{-n/2}. \tag{3.109}$$

We require that terms in Γ should be independent of scale, which implies

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \Gamma^{(n)} = 0 = \left(\mu \frac{\partial}{\partial \mu} + \mu \frac{\mathrm{d}m^2}{\mathrm{d}\mu} \frac{\partial}{\partial m^2} + \beta(g) + n\gamma_\phi \right) \Gamma^{(n)}_{\text{ren}}(x_1, \dots, x_n). \tag{3.110}$$

Equations like this, which govern the running of the parameters of the theory by looking at n -point functions, are called *Callan–Symanzyk equations*.

All but the first term in the parentheses are (at least) order $\hbar$ at 1 -loop. In ϕ^4 theory $Z_\phi = 1$ to 1-loop order.

$$\Pi_1^{\overline{\text{MS}}} = \frac{gm^2}{32\pi^2} \left(\ln \frac{\mu^2}{\mu^2} - 1 \right) \tag{3.111}$$

$$\widetilde{\Gamma}^{(2)}(p^2 = 0) = \cancel{p^2} + m^2 - \frac{gm^2}{32\pi^2} \left(\ln \frac{\mu^2}{m^2} - 1 \right). \tag{3.112}$$

Using the Callan–Symanzyk equation, we have

$$0 = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \widetilde{\Gamma}^{(2)}(0) = \mu \frac{\mathrm{d}m^2}{\mathrm{d}\mu} - \frac{gm^2}{16\pi^2} + O(\hbar^2). \tag{3.113}$$

From this we see that the dimensionally regularised mass changed as

$$\mu \frac{\mathrm{d}m^2}{\mathrm{d}\mu} = \frac{gm^2}{16\pi^2}. \tag{3.114}$$

Including the leading order term and one-loop correction, we have

$$\widetilde{\Gamma}^{(4)}(0,0,0,0) = -g\mu^\epsilon + \frac{3g^2\mu^\epsilon}{32\pi^2} \ln \frac{\mu^2}{m^2}. \tag{3.115}$$

Differentiating this (and ignoring the μ^ϵ since the derivatives of those will vanish at $\epsilon \to 0$), we have

$$0 = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \widetilde{\Gamma}^4 = -\beta(g)\mu^\epsilon + \frac{3g^2\mu^\epsilon}{16\pi^2} + O(\hbar^2). \tag{3.116}$$

Once again, our beta function is

$$\beta(g) = \frac{3g^2}{16\pi^2}, \tag{3.117}$$

which is the same as (3.100). This argument is the more modern approach.

4 The Renormalisation (Semi-)Group

The idea, which comes from statistical field theory, is that we are studying quantum field theories that fall into universality classes.

We impose some cutoff, which has a degree of arbitrariness to it, in the UV. However, at lower energies, we want to see the same universal IR physics emerging from theories with different regularisation and renormalisation schemes and scales. We will have to tune these scales to get the correct low-energy physics.

Remark: When we say *UV*, we mean an unobtainable high-energy regime, such as the Planck scale, which we cannot probe with experiment. In contrast, the *IR* is all the interesting physics that is accessible to us.

The idea behind the renormalisation group (RG) is that we want to study how the microscopic features (i.e. the couplings) change along the "lines of constant IR physics".

Consider a real scalar field with momentum cutoff Λ_0 in $d \in \mathbb{N}$ dimensions. Generically, the action will be

$$S_{\Lambda_0}[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \sum_i \frac{1}{\Lambda_0^{d_i - d}} g_{i\mathcal{O}} \mathcal{O}_i(x) \right], \tag{4.1}$$

where the "operators" $\mathcal{O}_i[\phi(x)]$ are products of the field ϕ and its derivatives $\mathcal{O}_i = (\partial\phi)^{r_i}\phi^{s_i}$. In particular, they are *local* operators with mass dimension $d_i > 0$. The factors Λ_0 are introduced so that g_{i0} are dimensionless. Note that the mass term is included in the sum over all possible operators.

The partition function with cutoff Λ_0 is a function of all the couplings g_{iO}

$$\mathcal{Z}_{\Lambda_0}(g_{iO}) = \int^{\Lambda_0} \mathcal{D}\phi \, e^{-S_{\Lambda_0}[\phi]}. \tag{4.2}$$

The notation on the path integral indicates that the integral is over field modes that have $|p| \leq \Lambda_0$, meaning that the Fourier integral of ϕ in $\mathcal{Z}_{\Lambda_0}$ is

$$\phi(x) = \int_{|p| \leq \Lambda_0} \mathrm{đ}^d p \, e^{ip\cdot x} \widetilde{\phi}(p). \tag{4.3}$$

4.1 Effective Actions

The next step is to look at a slightly smaller cutoff Λ and split the fields into high and low momentum modes

$$\phi(x) = \phi^-(x) + \phi^+(x) \tag{4.4}$$
$$= \int_{|p|<\Lambda} đ^d p\, e^{ip\cdot x}\widetilde{\phi}(p) + \int_{\Lambda<|p|\leq\Lambda_0} \dots . \tag{4.5}$$

We then want to obtain a Wilsonian effective action $S_\Lambda^{\text{eff}}[\phi]$ (like a W) by integrating out the ϕ^+ as

$$S_\Lambda^{\text{eff}}[\phi] = -\ln \int_\Lambda^{\Lambda_0} \mathcal{D}\phi^+\, e^{-S_{\Lambda_0}[\phi+\phi^+]}. \tag{4.6}$$

The RG equations will tell us how S_Λ^{eff} and S_{Λ_0} are related. Separate out the terms in the action that couple UV and IR modes

$$S_{\Lambda_0}[\phi + \phi^+] = S^0[\phi^-] + S^0[\phi^+] + S_{\Lambda_0}^{\text{int}}[\phi^-, \phi^+], \tag{4.7}$$

with free action $S^0[\phi] = \int \mathrm{d}^d x \left[(\partial\phi)^2 + m^2\phi^2\right]$. There is no quadratic term $\phi^-\phi^+$ since in Fourier space we would get a delta function

$$\widetilde{\phi^-}(k)\phi^+(k')\delta^{(d)}(k + k'), \tag{4.8}$$

which vanishes for all k both above or below Λ. Another way of saying this is that ϕ^+ and ϕ^- have disjoint support in the momentum space. An example of a non-zero term is $\widetilde{\phi^-}(k)\widetilde{\phi^-}(k')\widetilde{\phi}^+(k'')\delta(k+k'+k'')$. We also have effective interactions

$$S_\Lambda^{\text{int}}[\phi] = -\ln \int \mathcal{D}\phi^+\, e^{-S^0[\phi^+]-S_{\Lambda_0}^{\text{int}}[\phi^-,\phi^+]}. \tag{4.9}$$

4.2 Running Couplings

If we want the physics to be independent of whether we use the cutoff Λ or Λ_0, then we need the associated partition functions to be equal

$$\mathcal{Z}(g_i(\Lambda)) = \mathcal{Z}_{\Lambda_0}(g_{i0}; \Lambda_0). \tag{4.10}$$

The right-hand side is independent of Λ , which means that the left-hand side must be, too. Thus the couplings $g_i(\Lambda)$ must "run" to compensate. We have the Callan–Symanzik (or RG) equation

$$\Lambda\frac{\mathrm{d}\mathcal{Z}_\Lambda(g)}{\mathrm{d}\Lambda} = \left(\Lambda \left.\frac{\partial}{\partial\Lambda}\right|_{g_i} + \Lambda\frac{\mathrm{d}g_i}{\mathrm{d}\Lambda} \left.\frac{\partial}{\partial g_i}\right|_{\Lambda}\right)\mathcal{Z}_\Lambda(g) = 0. \tag{4.11}$$

The effective action S_Λ^{eff} has the same form as S_{Λ_0} :

$$S_\Lambda^{\text{eff}}[\phi] = \int \mathrm{d}^d x \left[\frac{1}{2} Z_\Lambda (\partial\phi)^2 + \sum_i \frac{Z_\Lambda^{n_i/2}}{\Lambda^{d_i - d}} g_i(\Lambda_0) \mathcal{O}_i(x) \right], \tag{4.12}$$

where n_i is the number of fields ϕ in the operator $\mathcal{O}_i(x)$. We have also accounted for the fact that integrating out ϕ^+ modes may give us a normalisation $Z_\Lambda \neq 1$. We thus renormalise the field to restore the quadratic term in the action:

$$\phi^r = Z_\Lambda^{1/2} \phi, \tag{4.13}$$

where we use the field-renormalisation function Z (not the $\mathcal{Z}$).

Any remaining Λ -dependence must be described by $g_i(\Lambda)$. The classical β-function is $\beta_i^{\text{cl}} = d_i - d$ from the sum in (4.12). The quantum β -function is $\beta^{\text{qu}} = \Lambda \frac{\mathrm{d}g_i}{\mathrm{d}\Lambda}$, which gives the total $\beta = \beta^{\text{cl}} + \beta^{\text{qu}}$.

4.3 Vertex Functions

Recall the anomalous dimension $\gamma_\phi = -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\Lambda} \ln Z_\Lambda$. Look at n-point functions. Iterate this mode-thinning (the integrating-out of high-momentum modes). Let $0 < s < 1$.

$$Z_{s\Lambda}^{-n/2} \Gamma_{s\Lambda}^{(n)}(x_1, \dots, x_n; g_i(s\Lambda)) = Z_\Lambda^{-n/2} \Lambda_\Lambda^{(n)}(x_1, \dots, x_n; g(\Lambda)). \tag{4.14}$$

The infinitesimal version of this is the differential equation

$$\Lambda \frac{\mathrm{d}}{\mathrm{d}\Lambda} \Gamma_\Lambda^{(n)}(x_1, \dots, x_n; g(\Lambda)) = \left(\Lambda \frac{\partial}{\partial\Lambda} + \beta_i \frac{\partial}{\partial g_i} + n\gamma_\phi \right) \Gamma_\Lambda^{(n)}(x_1, \dots, x_n; g(\Lambda)). \tag{4.15}$$

This can be obtained by letting $s\Lambda = \Lambda'$ for fixed Λ. Differentiate with respect to s

$$s \frac{\mathrm{d}}{\mathrm{d}s} Z_s^{-n/2} = n\gamma_\phi \tag{4.16}$$

using $s\frac{\mathrm{d}}{\mathrm{d}s} = \Lambda' \frac{\mathrm{d}}{\mathrm{d}\Lambda'}$. Then relabel Λ' as Λ.

The RG transformation constitutes two steps:

1) Integrate out momentum modes over the annulus $(s\Lambda, \Lambda]$ in momentum-space.
2) Rescale coordinates $x' = sx$ so that the cutoff again sits at Λ.

Under rescaling, the kinetic term must be made properly renormalised

$$\phi^r(sx) = s^{1-\frac{d}{2}} \phi^r(x). \tag{4.17}$$

Then the rest of the action is invariant if we also rescale $\Lambda \to \Lambda/s$.

The n-point vertex functions should be the same

$$\Gamma_{\Lambda}^{(n)}(x_1, \dots, x_n; g(\Lambda)) = \left(\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2} \Gamma_{s\Lambda}^{(n)}(x_1, \dots, x_n; g(s\Lambda)). \tag{4.18}$$

However, these have different cutoffs, so they are difficult to compare. We want to compare the theory after performing both RG steps, not just the first one. We rescale coordinates / cutoff and the field, giving

$$\dots = \left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2} \Gamma_{\Lambda}^{(n)}(sx_1, \dots, sx_n; g(s\Lambda)). \tag{4.19}$$

But the numerical values of $Z_{s\Lambda}$ and $g(s\Lambda)$ do not get rescaled.

Reconsider points we look at. Instead of x_I argument, look at x_i/s.

$$\Gamma_{\Lambda}^{(n)}(\frac{x_1}{s}, \dots, \frac{x_n}{s}; g(\Lambda)) = \left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2} \Gamma_{\Lambda}^{(n)}\big(x_1, \dots, x_n; g(s\Lambda)\big). \tag{4.20}$$

As s gets smaller, the left-hand side $\left|x_i - x_j\right|$ gets bigger. On the right-hand side, the couplings are running to the IR. From this we see a connection between the running of the coupling and the physics of scale.

What about the pre-factor? This is the coefficient of the quantum action. For small $\delta s = 1 - s$,

$$\left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{1/2} = 1 + \left[\frac{d-2}{2} + \gamma_\phi\right]\delta s. \tag{4.21}$$

Therefore, the fields, n of which are in the n-point vertex function, behave as if their mass-dimensions were

$$\Delta_\phi := \frac{d-2}{2} + \gamma_\phi. \tag{4.22}$$

The first term $(d-2)/2$ is the "engineering dimension", which we obtained from looking at the dimension of the Lagrangian. The γ_ϕ is the "anomalous dimension". Both of these sum to give the "scaling dimension" Δ_ϕ.

4.4 Renormalisation Group Flow

Definition 17 (RG Flows)**:** An *RG flow* is a line in coupling constant space corresponding to how the set of couplings $\{g_i\}$ change as we integrate modes.

These RG flows are governed by the β-functions $\{\beta_i(\{g_i\})\}$ and are illustrated in Fig. 4.1.

Theories lying along the same flow line describe the same IR physics.

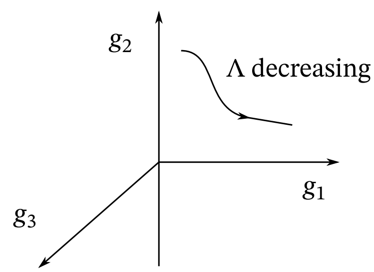


Figure 4.1: RG flow in coupling constant space.

4.4.1 Fixed Points of RG Equations

Definition 18 (fixed point)**:** A *fixed point* or *critical point* is a point $\{g_i^*\}$ in coupling constant space where the β-functions vanish:

$$\beta_i|_{\{g_j^*\}} = 0. \tag{4.23}$$

Recall that the β-functions are comprised of two parts:

$$\beta_i(\{g_i\}) = \underbrace{(d_i - d)g_i}_{\text{classical part}} + \overbrace{\Lambda\frac{\mathrm{d}g_i}{\mathrm{d}\Lambda}}^{\text{quantum part}}\,. \tag{4.24}$$

For the moment we will neglect dimensionful couplings like the mass, which contribute to the classical part.

Example 4.4.1 (Gaussian fixed point)**:** We can consider a free massless theory in which $g_j^* = 0$ for all j. This is called the *Gaussian fixed point*.

In general, there are also non-trivial fixed points. However, these require cancellation of the classical and quantum parts of the β-function; these are hard to find.

Let us think about what must happen at a fixed point.

Scale Invariance at Fixed Points

Part of the RG transformations was rescaling the fields and couplings after integrating out high-momentum modes. If all the couplings are invariant under RG transformations, there must be scale invariance. The fixed-point couplings g_i^* are independent of scale. Thus, other dimensionless functions of g_i are scale invariant as well. One example of this is the anomalous dimension $\gamma_\phi(g_i^*) := \gamma_\phi^*$.

The β-functions vanish. This means that the Callan–Symanzik equation becomes

$$\Lambda\frac{\partial}{\partial\Lambda}\Gamma^{(2)}_{\Lambda}(x,y) = -2\gamma^*_\phi\Gamma^{(2)}(x,y). \tag{4.25}$$

If we impose translational and rotational invariance, the two-point function $\Gamma^{(2)}(x,y) = \Gamma^{(2)}(|x-y|)$ should only depend on the relative separation of the points, rather than their absolute positions. Like $\langle\phi(x)\phi(y)\rangle$, the engineering dimension of $\Gamma^{(2)}$ is Λ^{d-2}. Putting these together, accounting for anomalous dimension, dimensional analysis gives us

$$\Gamma^{(2)}_{\Lambda}(x,y;g^*_i) = \frac{\Lambda^{d-2}}{\Lambda^{2\Delta_\phi}}\frac{c(g^*_i)}{|x-y|^{2\Delta_\phi}}, \tag{4.26}$$

where $\Delta_\phi = \frac{1}{2}(d-2)+\gamma^*_\phi$ is the scaling dimension of ϕ. This power law behaviour of 2-point functions or propagators is characteristic of scale invariant theories (think about classical gravity or electromagnetism, which obey power law force equations). You can contrast this to theories with a characteristic scale, $M = \frac{1}{\xi}$ (a mass or inverse correlation length). In such theories, the functions typically fall of exponentially like $\Gamma^{(2)}(x,y) \sim \frac{e^{-M|x-y|}}{|x-y|^{2\Delta_\phi}}$.

Behaviour Near a Fixed Point

In this course, we do not want to consider the conformal symmetry arising at a fixed point, but rather consider behaviour near a fixed point. Near a fixed point, we can linearise the RG equations. Let $\delta g_i = g_i - g^*_i$. Then we can write the RG equation as a matrix equation

$$\Lambda\frac{\mathrm{d}g_i}{\mathrm{d}\Lambda}\bigg|_{\{g^*_i+\delta g_i\}} = B_{ij}\delta g_j + O(\delta g^2). \tag{4.27}$$

Understanding the flow near a fixed point now amounts to a set of first order differential equations. Naturally, we want to look at the eigenvectors σ_i and eigenvalues $\Delta_i - d$ of the matrix B_{ij}. We call Δ_i the *scaling dimension* associated with the eigenvector σ_i.

Remark: Each axis in the coupling constant space corresponds to an operator in the action. Thus the eigenvectors σ_i, which represent a combination of directions in coupling constant space, generally consist of linear combinations of operators $\mathcal{O}_i$ in $S[\phi]$.

The linearised RG flow equations then become

$$\Lambda\frac{\mathrm{d}\sigma_i}{\mathrm{d}\Lambda} = (\Lambda_i - d)\sigma_I \tag{4.28}$$

$$\Rightarrow\ \sigma_i(\Lambda) = \left(\frac{\Lambda}{\Lambda_0}\right)^{\Delta_i - d}\sigma_i(\Lambda_0), \tag{4.29}$$

with initial scale $\Lambda_0 > \Lambda$. If the exponent is positive, when $\Lambda_i > d$, then we have $\sigma_i(\Lambda) < \sigma_i(\Lambda_0)$. In other words, we flow back to the fixed point as Λ decrease. These are accordingly called *irrelevant* directions. Conversely, if the signs are reversed, $\Lambda_i < d$, then we have a *relevant* direction, where $\sigma_i(\Lambda) > \sigma_i(\Lambda_0)$ and we flow away from the fixed point. Finally, we also have the *marginal* case $\Lambda_i = d$, where we cannot tell the direction of the flow.

Remark: Usually, including the second order in the calculation allows us to determine the direction of the flow when the first order approximation yields a marginal fixed point.

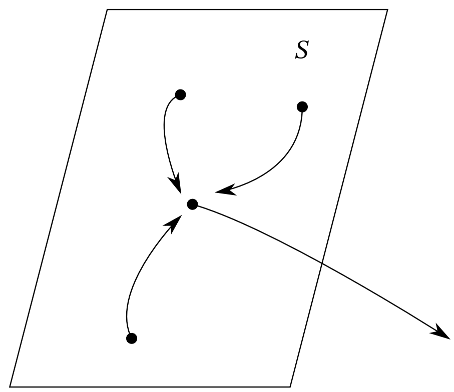


Figure 4.2: Coupling space with one relevant direction.

Consider the case of a theory in which we have one relevant direction at a given fixed point in coupling space. As illustrated in Fig. 4.2, we can then have a "surface" (submanifold) S of irrelevant couplings, called the *critical surface*.

In general, in infinite-dimensional coupling space, the critical surface is ∞-dimensional. Its co-dimension is finite, and is equal to the number of relevant operators. The trajectory leaving the fixed point is the *renormalised trajectory*.

4.5 Continuum Limit and Renormalisability

So far we have integrated out the high-momentum modes from Λ_0 down to Λ and obtained a family of effective actions

$$e^{-S_\Lambda^{\text{eff}}[\phi^-]} = \int_\Lambda^{\Lambda_0} \mathcal{D}\phi^+ \, e^{-S_{\Lambda_0}[\phi^- + \phi^+]}. \tag{4.30}$$

Requiring that we get the same physics out of this gave us the β-functions.

$$\beta_i(\{g_j\}) = (d_i - d)g_i + \Lambda\left(\frac{\mathrm{d}}{\mathrm{d}\Lambda} g_i\right)(\{g_j\}). \tag{4.31}$$

We want to take the *continuum limit*[1] $\Lambda_0 \to \infty$. Renormalisability is the sensitivity to initial couplings g_{0i}.

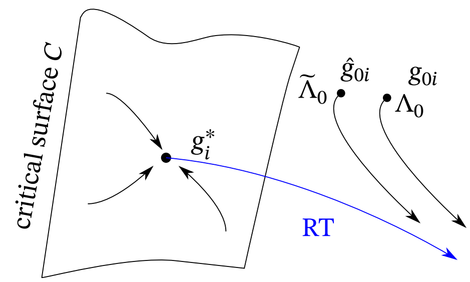


Figure 4.3

Cases:

(i) The theory has only irrelevant couplings. Then g_{0i} must lie in C. The $g_i(\Lambda)$ flow to the fixed point g_i^* irrespective of initial conditions g_{0i}. Therefore, the limit of S_Λ^{eff} as $\Lambda_0 \to \infty$ exists and gives a scale-invariant theory. We have $g_i(\Lambda) = g_i^*$. No renormalisation conditions are needed. An example of this is given by $\mathcal{N} = 4$ Super-Yang–Mills (SYM) in $d = 4$ dimensions.

(ii) There is at least 1 relevant (or marginal) direction. If $\Lambda_0 \to \infty$ for fixed g_{0i}, the flow leaves the vicinity of the fixed point (physics no longer universal). Therefore, we are sensitive to initial coupling (corresponds to relevant (or marginal) operators). Solution: Tune initial couplings to be closer to C as $\Lambda_0 \to \infty$. Then the flow is slower and Λ_0/Λ can be made larger. Renormalisation conditions are needed to fix g_{i0} for relevant directions (also marginal). Having a finite number of conditions gives a *renormalisable* theory. Examples of this include Yang–Mills, QCD, ..., which are asymptotically free ($\beta(g) < 0$).

(iii) Irrelevant operators need to be tuned for correct physics. An example of this is Fermi theory with an interaction term $(\overline{\psi}_i \Gamma \psi_j)(\overline{\psi}_k \Gamma \psi_l)$ and four-point coupling $G_F \sim g^2/M_W^2$.

Then we cannot allow couplings to flow into the fixed point. The $\Lambda_0 \to \infty$ limit cannot be taken. In principle, an infinite number of initial conditions are needed. We call this a *non-renormalisable* theory. However, this is not entirely useless, since we still have some universality. In practice, there is usually some principle by which we can order the importance of irrelevant operators.

As we will see, ϕ^4 and QED are *perturbatively renormalisable*. Beyond perturbation theory, one has to do more work to show that these couplings are marginally irrelevant. The couplings of ϕ^4

[1] This terminology is a bit more natural in the context of a lattice theory in condensed matter systems, where Λ^{-1} is the lattice spacing.

and, as we will see, QED are marginally irrelevant. We still have a finite number of renormalisation conditions. In dimensional regularisation, we have $g(\mu) \propto [\ln \frac{\Lambda_{\phi^4}}{\mu}]^{-1}$, which is called a *Landau pole* at high scales $\mu \simeq \Lambda_{\phi^4}$. Logarithmic running is so slow that in practice we never need to worry—we do not get close to Λ_{ϕ^4}.

5 Quantum Electrodynamics

We will work in Euclidean spacetime with action

$$S[\psi,\overline{\psi},A] = \int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(\not{D} + m)\psi\right], \tag{5.1}$$

where the covariant derivative[2] $\not{D} = \gamma^\mu(\partial_\mu + ieA_\mu)$ and $\psi, \overline{\psi}$ are 4-component Grassmann variables. Moreover, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the partition function is

$$\mathcal{Z} = \int \mathcal{D}\psi\, \mathcal{D}\overline{\psi}\, \mathcal{D}A\, e^{-S[\psi,\overline{\psi},A]}. \tag{5.2}$$

In Euclidean spacetime, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. We use the convention $\gamma_\mu^\dagger = \gamma_\mu$.

Example 5.0.1: One may use the representation

$$\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix}, \qquad \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.3}$$

5.1 QED Feynman Rules

Electron Propagator

Let us first find the Feynman propagator for the fermionic electrons ψ. We go to momentum space with Fourier convention

$$\psi(x) = \int đ^4p\, e^{ip\cdot x}\psi(p), \tag{5.4}$$

where, instead of using another symbol, we distinguish the field $\psi(x)$ from its modes $\psi(p)$ by keeping the argument explicit. The free part of the fermion action is

$$S_f[\psi,\overline{\psi}] = \int đ^4p\, \overline{\psi}(-p)(i\not{p} + m)\psi(p). \tag{5.5}$$

[2]Many textbooks use the convention $\not{D} = \gamma^\mu(\partial_\mu - ieA_\mu)$.

Introducing fermionic charges $\eta, \overline{\eta}$, the (momentum space) generating functional is

$$\mathcal{Z}[\eta,\overline{\eta}] = \int \mathcal{D}\psi\, \mathcal{D}\overline{\psi} \exp\Big\{-\int đ^4 p \left[\overline{\psi}(i\not{p} + m)\psi - \overline{\eta}\psi + \overline{\psi}\eta\right]\Big\} \tag{5.6}$$

$$= \mathcal{Z}[0,0] e^{-\int \overline{\eta}(i\not{p}+m)^{-1}\eta}. \tag{5.7}$$

The free propagator is therefore

$$G_F(p) = \left.\frac{\delta^2 \mathcal{Z}}{\delta\overline{\eta}\delta\eta}\right|_{\eta,\overline{\eta}=0} = \frac{1}{i\not{p} + m}. \tag{5.8}$$

We see that we could have just read off the propagator to be the inverse of the operator acting on the quadratic part of the Fourier-transformed action (5.5).

Photon Propagator

In the same way, the electromagnetic photon propagator is obtained from the Fourier-integral representation of the associated action

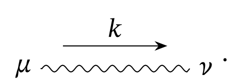
$$S_{EM}[A] = \frac{1}{4}\int \mathrm{d}^4 x\, F_{\mu\nu}F^{\mu\nu} \tag{5.9}$$

$$= -\frac{1}{2}\int \mathrm{d}^4 x\, A_\mu(x)[\delta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu]A_\nu(x) \tag{5.10}$$

$$= \frac{1}{2}\int đ^4 k\, A_\mu(-k)[k^2\delta^{\mu\nu} - k^\mu k^\nu]A_\nu(k), \tag{5.11}$$

where we integrated by parts in the second line after using $F_{\mu\nu}F^{\mu\nu} = 2[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu]$. In going to the third sign, we pick up another minus sign due to the two factors of i brought down by the derivatives. From this we can read off the EM propagator

$$D^{\mu\nu}(k) = \frac{1}{k^2}(\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}) = \tag{5.12}$$



Remark: It turns out that there is a subtlety here that was brushed under the rug in lectures: As explained above, we should really look at the *inverse* of the operator $k^2\delta^{\mu\nu} - k^\mu k^\nu$ acting on the quadratic part of the Fourier-transformed action (5.11). This means we want $D^{\mu\nu}(k)$ such that

$$(k^2\delta_{\mu\nu} - k_\mu k_\nu)D^{\nu\rho}(k) = \delta_\mu{}^\rho. \tag{5.13}$$

However, this equation has no solution since the 4×4 matrix $(k^2\delta_{\mu\nu} - k_\mu k_\nu)$ is singular [P&S, Sec. 9.4]. Equivalently, the expression (5.11) vanishes whenever $A_\mu(k) = k_\mu \alpha(k)$ for any scalar function $\alpha(k)$. For all these functions, the integrand of $\int \mathcal{D}A\, e^{-S[A]}$ is 1 and therefore the path integral is divergent (there is no Gaussian damping). In fact, this difficulty arises due to gauge invariance: the functional integral is divergent because we are redundantly integrating over a continuous infinity of physically equivalent field configurations, with the troublesome modes being those for which $A_\mu(x) = \frac{1}{e}\partial_\mu \alpha(x)$, i.e. those that are gauge-equivalent to $A_\mu(x) = 0$. This redundancy can be eliminated by *Faddeev–Popov gauge fixing*. To not be repetitive, this is explained for the more general non-Abelian case in Sec. 7.3. The upshot of this gauge-fixing procedure will be that we obtain a sensible photon propagator by including an extra term $-(\partial^\mu A_\mu)^2/2\xi$ (7.79) to the Lagrangian, so that (5.13) becomes

$$[k^2\delta_{\mu\nu} - (1 - \frac{1}{\xi}k_\mu k_\nu)]D^{\nu\rho}(k) = \delta_\mu{}^\rho, \tag{5.14}$$

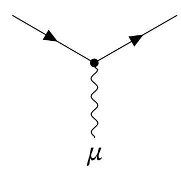
which is indeed invertible and has the solution

$$D^{\mu\nu}(k) = \frac{1}{k^2}\left(\delta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right). \tag{5.15}$$

When making computations, we may choose a specific value for ξ. In (5.12), we used $\xi = 0$, which is called *Landau gauge*. The *Feynman gauge* $\xi = 1$ is also commonly used, and sometimes the *Yennie gauge* $\xi = 3$ makes an appearance.

QED Vertex

The vertex between a fermion-antifermion pair $\overline{\psi}, \psi$ and a photon A_μ is



$$= -ie\gamma^\mu \tag{5.16}$$

which corresponds to the interaction Lagrangian $\mathscr{L} = \overline{\psi}(ie\gamma^\mu A_\mu)\psi$ term, which comes from expanding out the covariant derivative term $\overline{\psi}\not{D}\psi$.

5.2 Photon Generating Functional

Let us couple an external source to the photon following Maxwell's equation

$$\partial^\nu F^{\nu\mu} = J^\mu. \tag{5.17}$$

Then the generating functional whose Euler–Lagrange equations give these equations of motion is

$$\mathcal{Z}_0[J] = \int \mathcal{D}A \exp\left\{-\int d^4x\, [F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu]\right\} = \int \mathcal{D}A\, e^{-S_g[A,J]} \tag{5.18}$$

Claim 5: This action S_g is gauge invariant.

Proof. Define the gauge transformation to be $A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x)$[1] , we have

$$\int \mathrm{d}^4x\, J^\mu(A'_\mu - A_\mu) = \int \mathrm{d}^4x\, J^\mu(\partial_\mu \alpha) \tag{5.19}$$

$$= \int \mathrm{d}^4x\, \partial_\mu(J^\mu \alpha) - \int \mathrm{d}^4x\, (\partial_\mu J^\mu)\alpha = 0, \tag{5.20}$$

where the first term is a boundary term, which vanishes for suitable behaviour at $|x| \to \infty$, and the second term vanishes by conservation of the current (evident from Maxwell's equation (5.17)):

$$\partial_\mu J^\mu = \partial_\mu \partial_\nu F^{\mu\nu} = 0, \tag{5.21}$$

Since $F^{\mu\nu} = -F^{\nu\mu}$. □

The Fourier transform of the action is

$$S_g[A,J] = \frac{1}{2}\int đ^4k \left[A_\mu(-k)(k^2\delta^{\mu\nu} - k^\mu k^\nu)A_\nu(k) + J^\mu(-k)A_\mu(k) + J^\mu(k)A_\mu(-k)\right]. \tag{5.22}$$

However, this path integral formulation of QED has a problem: Consider the field configurations $A^{(0)}$ that are gauge-equivalent to $A_\mu(k) = 0$. The action vanishes and $e^{-S_g} = 1$. Therefore, we have an apparent divergence in the photon generating functional $\mathcal{Z}_0[J] \supset \int \mathcal{D}A^{(0)} \cdot 1$.

From the position-space gauge invariance, we get the momentum space condition

$$A_\mu(x) = \partial_\mu \alpha(x) \quad \to \quad A_\mu(k) = k_\mu \alpha(k). \tag{5.23}$$

Define the 4×4 matrix

$$P^{\mu\nu}(k) = \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}. \tag{5.24}$$

This is a *projection matrix*, which obeys ${P^\mu}_\nu(k)P^{\nu\rho}(k) = P^{\mu\rho}(k)$, so its eigenvalues are either 0 or 1. The k_ν are the zero-eigenvectors

$$P^{\mu\nu}(k)k_\nu = 0 \tag{5.25}$$

and $k_\mu J^\mu(k) = 0$ (from $\partial_\mu J^\mu(p) = 0$). The other three eigenvalues of $P^{\mu\nu}(k)$ are 1. The trace is the sum of these eigenvalues

$$\delta_{\mu\nu}P^{\mu\nu} = 3. \tag{5.26}$$

Solution: Interpret $\int \mathcal{D}A$ as integration over modes that are not gauge equivalent to $A_\mu(k) = 0$. In other words, we integrate over only those modes with $k^\mu A_\mu(k) = 0$. In position space, this is $\partial^\mu A_\mu(x) = 0$, which is called the *Lorenz gauge* or *Landau gauge*. As such, we are basically gauge fixing here. What we can do is introduce a delta function, which forces us to integrate along a

[1] Compared to the *Standard Model* course, we are absorbing the factor of $\frac{1}{e}$ into $\alpha(x)$.

particular path that cuts through all the gauge orbits exactly once. We will have to do this anyways in Chapter 7, where we talk about non-Abelian gauge theory, but it is a lot of work and not really necessary to right now, so we defer this discussion to then.

In this subspace, $P^{\mu\nu}$ is just the identity

$$\mathcal{Z}_0[J(k)] = \exp\left[\frac{1}{2}\int đ^4k\, J_\mu(-k)\frac{P^{\mu\nu}}{k^2}J_\nu(k)\right]. \tag{5.27}$$

The matrix can be identified as $\frac{1}{k^2}P^{\mu\nu}D^{\mu\nu}(k) = \frac{1}{k^2}(\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2})$, which again is the Lorenz Landau gauge condition.

Again, $k^\mu J_\nu(k) = 0$, so we can drop the second term in $P^{\mu\nu}(k)$ to find

$$\mathcal{Z}_0[J(k)] = \exp\left[\frac{1}{2}\, đ^4k\, J_\mu(-k)\frac{\delta^{\mu\nu}}{k^2}J_\nu(k)\right]. \tag{5.28}$$

This gives $D^{\mu\nu}(k) = \frac{1}{k^2}\delta^{\mu\nu}$, which is called *Feynman gauge*. Of course, we did not need to drop the whole of the second term. We could have introduced a parameter, which leads to a whole family of gauges. We did this in the *Quantum Field Theory* course and will see this again when we discuss the non-Abelian case.

Interactions

We can introduce an interaction term into the action

$$ieA_\mu(x)\overline{\psi}^\alpha(x)(\gamma^\mu)^{\alpha\beta}\psi^\beta(x), \tag{5.29}$$

where α, β are spinor indices. Introducing Grassmann-valued sources $\eta, \overline{\eta}$, the generating functional becomes

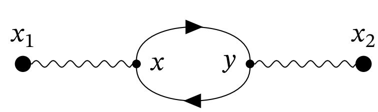
$$\mathcal{Z}[\eta,\overline{\eta},J] \propto \exp\left[-ie(\gamma^\mu)^{\alpha\beta}\int d^4x\left(-\frac{\delta}{\delta J(x)}\right)\left(-\frac{\delta}{\delta\eta^\alpha(x)}\right)\left(-\frac{\delta}{\delta\overline{\eta}^\beta(x)}\right)\mathcal{Z}_0[\eta,\overline{\eta}]\mathcal{Z}_0[J]\right] \tag{5.30}$$

It is a tedious exercise to see that for each fermion loop, we get an overall factor (-1) from anticommuting the Grassmann-valued sources.

Recall the Feynman propagator

$$S_F^{\alpha\beta}(x-y) = \langle \psi^\alpha(x)\overline{\psi}^\beta(y)\rangle. \tag{5.31}$$

The one-loop contribution is



(5.32)

This contributes

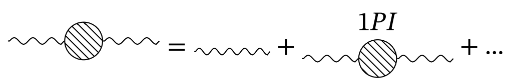
$$(-ie)^2 \int \mathrm{d}^4x\,\mathrm{d}^4y\,\langle A_\mu(x_1)\overline{\psi}^\alpha(x)\not{A}^{\alpha\beta}(x)\psi^\beta(x)\overline{\psi}^\gamma(y)\not{A}^{\gamma\delta}(y)\psi^\delta(y)A_\nu(x_2)\rangle \tag{5.33}$$

$$=(-ie)^2 \int \mathrm{d}^4x\,\mathrm{d}^4y\,\langle A\text{'s} \ldots (-1)^3\,\underbrace{\psi^\delta(y)\overline{\psi}^\alpha(x)}_{S_F^{\delta\alpha}(y-x)}\,\underbrace{\psi^\beta(x)\overline{\psi}^\gamma(y)}_{S_F^{\beta\gamma}(x-y)}\rangle, \tag{5.34}$$

and we can see that the $(-1)^3 = (-1)$ contributes the minus sign for the fermion loop.

5.3 Vacuum Polarisation

Quantum corrections to the photon propagator

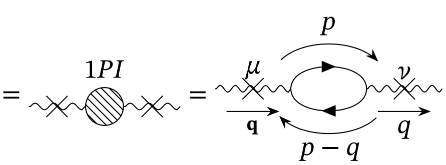


(5.35)

$$= \frac{1}{1-\Pi(q)}, \tag{5.36}$$

where we have the truncated diagrams

$\Pi^{\mu\nu}(q) =$



$+\, O(\text{2-loop}).$ (5.37)

Use dimensional regularisation. Let $e^2 = \mu^\epsilon g^2(\mu)$, where $\epsilon = 4 - d$. Then

$$\Pi^{\mu\nu}_{\text{1-loop}}(q^2) = -\mu^\epsilon(-ig)^2 \int đ^d p\,\mathrm{tr}\left(\frac{1}{i\not{p}+m}\gamma^\mu \frac{1}{i(\not{p}-\not{q})+m}\gamma^\nu\right) \tag{5.38}$$

$$= -\mu^\epsilon(ig)^2 \int đ^d p\,\frac{\mathrm{tr}[(-i\not{p}+m)\gamma^\mu(-i(\not{p}-\not{q})+m)\gamma^\nu]}{(p^2+m^2)((p-q)^2+m^2)}. \tag{5.39}$$

Let us now use Feynman parametrisation

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\,\frac{\delta(x+y-1)}{[Ay+Bx]^2}. \tag{5.40}$$

Then looking at just the denominator,

$$\int_0^1 \frac{\mathrm{d}x}{\{(p^2+m^2)(1-x)+[(p-q)^2+m^2]x\}^2} = \int_0^1 \frac{\mathrm{d}x}{[(p-qx)^2+m^2+q^2x(1-x)]^2}. \tag{5.41}$$

Shift integration variables to $p' = p - qx$. Then drop the prime. This gives

$$\Pi^{\mu\nu}_{\text{1-loop}}(q) = \mu^\epsilon g^2 \int đ^d p \int_0^1 \mathrm{d}x\,\frac{\mathrm{tr}\{[-i(\not{p}+\not{q}x)+m]\gamma^\mu[-i(\not{p}-\not{q}(1-x))+m]\gamma^\nu\}}{(p^2+\Delta)^2}, \tag{5.42}$$

where we have defined the shorthand $\Delta = m^2 + q^2 x(1-x)$.

We now make use of the following identities:

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4\delta^{\mu\nu}, \tag{5.43}$$

$$\mathrm{Tr}(\gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma) = 4\left(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}\right). \tag{5.44}$$

Employing these, the numerator can be brought into the form

$$\begin{aligned}\mathrm{Tr}\{\dots\} = 4\Big[&-(p+qx)^\mu[p-q(1-x)]^\nu + (p+qx)\cdot[p-q(1-x)]\delta^{\mu\nu} \\ &- (p+qx)^\nu[p-q(1-x)]^\mu + m^2\delta^{\mu\nu}\Big].\end{aligned} \tag{5.45}$$

As $d \to 4$, integrals over odd powers of p^μ vanish, so we can drop these terms. E.g. only the diagonal parts of $p^\mu p^\nu$ will integrate to something non-zero. Replace:

$$p^\mu p^\nu \to \frac{1}{d}\delta^{\mu\nu} p^2 \tag{5.46}$$

$$p^\mu p^\rho p^\nu p^\sigma \to \frac{(p^2)^2}{d(d+2)}(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}). \tag{5.47}$$

Now the integral depends only on p^2:

$$đ^d p \to S_{d-1} p^{d-1}\, đ^d p = \frac{(p^2)^{\frac{d}{2}-1}\,\mathrm{d}p^2}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}. \tag{5.48}$$

Putting things together, we find

$$\begin{aligned}\Pi_1^{\mu\nu}(q) = 4\mu^\epsilon \frac{g^2}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)} \int_0^1 \mathrm{d}x \int_0^\infty \mathrm{d}p^2\,(p^2)^{d/2-1}\frac{1}{(p^2+\Delta)^2} \\ \times\left[p^2\left(1-\frac{2}{d}\right)\delta^{\mu\nu} + \left(2q^\mu q^\nu - q^2\delta^{\mu\nu}\right)x(1-x) + m^2\delta^{\mu\nu}\right].\end{aligned} \tag{5.49}$$

These are Euler-B-functions:

$$\int_0^\infty \mathrm{d}p^2\,\frac{(p^2)^{d/2-1}}{(p^2+\Delta)^2} = \left(\frac{1}{\Delta}\right)^{2-d/2}\frac{\Gamma(2-d/2)\Gamma(d/2)}{\Gamma(2)} \tag{5.50}$$

$$\int_0^\infty \mathrm{d}p^2\,\frac{(p^2)^{d/2}}{(p^2+\Delta)^2} = \left(\frac{1}{\Delta}\right)^{1-d/2}\frac{\Gamma(1+d/2)\Gamma(1-d/2)}{\Gamma(2)}. \tag{5.51}$$

Hence,

$$\Pi_1^{\mu\nu}(q) = \frac{4q^2\mu^\epsilon}{(4\pi)^{d/2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 \frac{\mathrm{d}x}{\Delta^{\epsilon/2}}\Big(\delta^{\mu\nu}[m^2 - x(1-x)q^2] - \delta^{\mu\nu}[m^2 + x(1-x)q^2] + 2x(1-x)q^\mu q^\nu\Big) \tag{5.52}$$

$$= \frac{8g^2\mu^\epsilon}{(4\pi)^{d/2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 \frac{\mathrm{d}x}{\Delta^{\epsilon/2}}\left(-q^2\delta^{\mu\nu} + q^\mu q^\nu\right)x(1-x) \tag{5.53}$$

$$= (q^2\delta^{\mu\nu} - q^\mu q^\nu)\Pi_1 q(2), \tag{5.54}$$

where we denote the Lorentz invariant piece

$$\Pi_1(q^2) := -\frac{8g^2}{(4\pi)^{d/2}}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 \mathrm{d}x\, x(1-x)\left(\frac{\mu^2}{\Delta}\right)^{\epsilon/2}. \tag{5.55}$$

Nore that $\gamma_\mu \Pi_1^{\mu\nu} = 0$. In the limit $d \to 4$,

$$\Pi_1(q^2) = -\frac{g^2}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\left[\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi\gamma^2}{\Delta}\right)\right] + O(\epsilon). \tag{5.56}$$

Hence we need to renormalise this: Write

$$\mathscr{L}_0 = \mathscr{L} + \mathscr{L}_{\mathrm{ct}} \tag{5.57}$$

$$S_0 = S + S_{\mathrm{ct}}. \tag{5.58}$$

We thus split the coefficients

$$e_0 = Z_e e = (1+\delta Z_e)e, \qquad m_0 = Z_m m = (1+\delta Z_m)m \tag{5.59}$$

$$\psi_0 = \sqrt{Z_2}\psi, \qquad A_0 = \sqrt{Z_3}A, \tag{5.60}$$

where the zero subscript denotes the unrenormalised (“bare”) coefficient.

Write the right-hand side

$$S + S_{\mathrm{ct}} = \int \mathrm{d}^4x \left[\frac{1}{4}Z_3 F_{\mu\nu}F^{\mu\nu} + Z_2\overline{\psi}\partial\!\!\!/\psi + Z_m Z_2 m\overline{\psi}\psi + ieZ_1\overline{\psi}A\!\!\!/\psi\right], \tag{5.61}$$

where $Z_1 = Z_e Z_2\sqrt{Z_3}$. Let $Z_k := 1 + \delta Z_k$, with $k = e, m, 1, 2, 3$.

Remark: These have a matching Taylor expansion

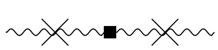
$$\delta Z_e = \delta Z_1 - \delta Z_2 - \frac{1}{2}\delta Z_3 + \dots \tag{5.62}$$

We will later show that gauge invariance implies that $Z_1 = Z_2$, so that

$$\delta Z_e = -\frac{1}{2}\delta Z_3. \tag{5.63}$$

Thus we look at the counterterm diagram, which gives a new 2-point interaction:

$$S_{\mathrm{ct}} \supset \int \mathrm{d}^4x\, \frac{\delta S_3}{4}F^2, \tag{5.64}$$



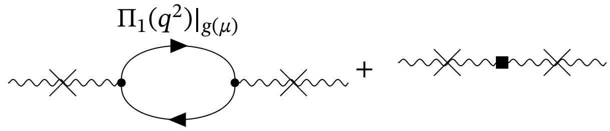
$$= -[k^2\delta^{\mu\nu} - (1-\xi)k^\mu k^\nu]\delta Z_3, \tag{5.65}$$

where $\xi = 0, 1$ in Landau / Feynman gauge respectively. Choose δZ_3 such that $\Pi_1^{\mathrm{ren}}(q^2)$ is finite. In the $\overline{\mathrm{MS}}$ scheme,

$$\delta Z_3 = -\frac{g^2(\mu)}{12\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi)\right). \tag{5.66}$$

Now $\int_0^1 \mathrm{d}x\,(1-x)x = \frac{1}{6}$, so $(\Pi_1^{\mu\nu}(q))_{\text{ren}}$, calculated in the "renormalised perturbation theory" yields:

$$[\Pi_1^{\mu\nu}(q)]_{\text{ren}} = \tag{5.67}$$



with

$$\Pi_1^{\text{ren}}(q^2) = \frac{g^2(\mu)}{2\pi^2}\int \mathrm{d}x\, x(1-x)\ln\left(\frac{m^2 + x(1-x)q^2}{\mu^2}\right). \tag{5.68}$$

Remark: There is a branch cut in the logarithm term when $m^2 + x(1-x)q^2 \le 0$. For $x \in [0,1]$, $0 \le x(1-x) \le \frac{1}{4}$. Going back to Minkowski space: $q_0 = iE$ such that the branch cut corresponds to

$$x(1-x)(E^2 - |q|^2) \ge m^2. \tag{5.69}$$

The smallest E on the cut is $E = 2m$, which is shown in Fig. 5.1. In other words, $E = 2m$ is the

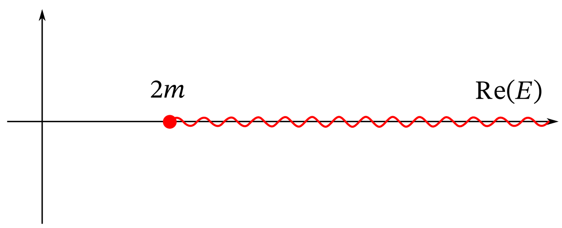


Figure 5.1

threshold energy for producing a real e^+e^- pair.

5.4 QED β-Function

$$e_0 = Z_e e = Z_1 Z_2^{-1} Z_3^{-1/2} e, \tag{5.70}$$

where we still need to show that $Z_1 = Z_2$. In the weak coupling limit,

$$e_0 = g_0 = Z_e g\mu^{\epsilon/2} = Z_3^{-1/2} g\mu^{\epsilon/2} = \left(1 - \frac{1}{2}\delta Z_3\right) g\mu^{\epsilon/2}. \tag{5.71}$$

The Callan–Symanzik equation tells us how the couplings run

$$\mu\frac{\mathrm{d}g_0}{\mathrm{d}\mu} = 0 = \left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)\left[\left(1 + \frac{g^2}{24\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi\right)\right)\right] g\mu^{\epsilon/2}. \tag{5.72}$$

We want to isolate the beta function by writing

$$\frac{\epsilon}{2} g\left(1 + \frac{g^2}{12\pi^2\epsilon}\right) + \beta(g)\left(1 + \frac{g^2}{4\pi^2\epsilon}\right) = 0, \tag{5.73}$$

where we dropped higher order terms in g. Collecting terms, we have

$$\beta(g) = -\left(\frac{\epsilon g}{2} + \frac{g^3}{24\pi^2}\right)\left(1 + \frac{g^2}{4\pi^2\epsilon}\right)^{-1} \tag{5.74}$$

$$= -\frac{\epsilon g}{2} + \frac{g^3}{12\pi^2} + O(2\text{ loop}) \tag{5.75}$$

As $d \to 4$, the first term vanishes and we obtain a positive beta-function

$$\beta(g) = \frac{g^3}{12\pi^2} > 0. \tag{5.76}$$

Just as we did for ϕ^4 theory, we integrate μ to μ', giving

$$\frac{1}{g^2(\mu')} = \frac{1}{g^2(\mu)} + \frac{1}{6\pi^2}\ln\frac{\mu}{\mu'}. \tag{5.77}$$

Just as for one-loop order, we introduce some scale Λ_{QED}, called the *Landau pole*, at which the coupling diverges.

$$g^2(\mu) = \frac{6\pi^2}{\ln\frac{\Lambda_{\text{QED}}}{\mu}}. \tag{5.78}$$

What scale does this happen at? Taking physical values for the electron mass $m_e = 0.511\text{MeV}$ and the fine-structure constant $\alpha = 1/137 = g^2(m_e)/(4\pi)$, this gives us a value

$$\Lambda_{\text{QED}} \simeq 10^{286}\text{GeV}. \tag{5.79}$$

Therefore, QED as an effective theory for electron-proton interactions is valid up to absurdly high energies.

Remark: We know that there is more to life than electrons and photons. In particular, the electroweak scale is $M_W = 100\text{GeV}$ and gravity should become important at 10^{19}GeV. Effectively, the cutoff scale for QED is much higher than any other physical scales at which corrections arise.

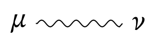
Alternate Derivation of β using Γ

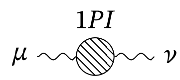
We can also use the quantum effective action Γ to derive the β-function. The free propagator is denoted

$$D_{\mu\nu}(q) = \mu \sim\sim\sim\sim\sim \nu\ . \tag{5.80}$$

The full propagator is

$$G^{(2)}_{\mu\nu}(p) = \int \mathrm{d}^d x\, e^{iq\cdot x}\langle A_\mu(x)A_\mu(0)\rangle \tag{5.81}$$

$=$ 

$+$ 

$+ \ldots$ (5.82)

$$= D_{\mu\nu} + D_{\mu\rho}\Pi^{\rho\sigma}D_{\sigma\nu} + \ldots \tag{5.83}$$

$$= D_{\mu\nu}(1 + \pi(q^2) + \pi^2(q^2) + \ldots) \tag{5.84}$$

$$= \frac{D_{\mu\nu}(q)}{1-\pi(q^2)}. \tag{5.85}$$

At 1-loop order, $\Pi^{\mu\nu} = \Pi^{\mu\nu}_1$ and $\pi = \pi_1$. Ar 1-loop, in Landau gauge, $G^{(2)}_{\mu\nu}(q^2)$ is obtained from

$$\Gamma[\psi,\overline{\psi},A] = \int \text{đ}^d p \left\{[1-\pi(p^2)][p^2\delta^{\mu\nu} - p^\mu p^\nu]\frac{1}{2}A_\mu(p)A_\nu(-p) + \ldots\right\}. \tag{5.86}$$

Rescale $A_\mu \to \frac{1}{e}A_\mu$, moving e from $\overline{\psi}\not{A}\psi$ term to kinetic term.

$$\Gamma[\psi,\overline{\psi},A] = \int \mathrm{d}^d x \left\{\frac{1-\pi(0)}{4e^2}F_{\mu\nu}F^{\mu\nu} + (\partial^2F^2 \text{ terms}) + (\text{more})\right\} \tag{5.87}$$

As such, the quantum effective action has more operators than we started with. The higher order operators do not lead to divergences. In ϕ^4 theory we did get corrections depending on momenta, but we were just interested in the divergent structure, so we looked at the $p = 0$ case. The finite momentum pieces do not contribute any divergences.

Instead of comparing to the bare Lagrangian, we compare the quantum effective actions $\Gamma[\psi,\overline{\psi},A]$ and demand their coefficients to be μ-independent. In particular, we may define

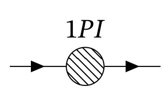
$$\frac{1}{e^2_{\text{phys}}} = \frac{1-\pi(0)}{e^2} = \frac{1}{\mu^\epsilon g^2}\left[1 - \frac{g^2}{2\pi}\int_0^1 \mathrm{d}x\, x(1-x)\ln\frac{\Delta}{\mu^2}\right]. \tag{5.88}$$

Taking the logarithmic derivative $\mu\frac{\mathrm{d}}{\mathrm{d}\mu}$ of both sides, we obtain the same β-function as before.

Full 1-loop Renormalisation

The full 1-loop renormalisation of QED requires taking into account fermion (electron) self-energy.

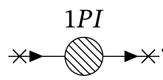
$$G(p) = \int \mathrm{d}^4 x\, e^{ip\cdot x}\langle\psi(x)\overline{\psi}(y)\rangle \tag{5.89}$$

$=$
$+$ 

$+ \ldots$ (5.90)

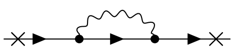
$$= \frac{1}{i\not{p} + m - \Sigma(p)}, \tag{5.91}$$

with self-energy $\Sigma(p)$ given by the truncated diagram

$$\Sigma(p) = $$ 

$$. \tag{5.92}$$

At 1-loop order,

$$\Sigma_1(p) = $$  $$\tag{5.93}$$

$$= -\frac{g^2}{16\pi^2}\int_0^1 \mathrm{d}x\,[(2-\epsilon)x(i\not{p}) + (4-\epsilon)m]\left[\frac{2}{\epsilon} + \gamma + \ln\left(\frac{4\pi\mu^2}{\Delta}\right) + O(\epsilon)\right]. \tag{5.94}$$

Remark: Note that the first terms is not proportional to $(i\not{p} + m)$.

Choose counter terms $\delta Z_2, \delta Z_m$ such that

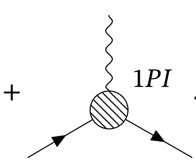
 $$= -\left[\delta Z_2(i\not{p}) + (\delta Z_2 + \delta Z_m)m\right]. \tag{5.95}$$

We may now choose renormalisation schemes from $\overline{\text{MS}}$, MS, or on-shell. In particular, on-shell ($\not{p} = im_{\text{phys}}$) gives

$$\Sigma(p)\big|_{\not{p}=im_{\text{phys}}} = 0 \quad \& \quad \frac{\mathrm{d}\Sigma}{\mathrm{d}\not{p}}\bigg|_{\not{p}=im_{\text{phys}}} = 0 \tag{5.96}$$

There is nothing wrong with the $\overline{\text{MS}}$ scheme, but the masses in the Lagrangian would not be equal to physical masses.

The correction to the vertex function is

$$\Gamma^{(3),\mu} = ie\gamma^\mu + $$ 

$$. \tag{5.97}$$

At 1-loop, we have the diagram in Fig. 5.2. One integral is divergent: it turns out that this requires

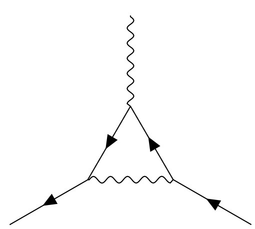


Figure 5.2

$\delta Z_1 = \delta Z_2$.

6 Symmetries and Path Integrals

In classical field theory, the Euler–Lagrange equations are derived by requiring the action be stationary under variations $\phi(x) \to \phi(x) + \epsilon(x)$, where $\epsilon(x)$ is an arbitrary function. Let us now investigate how the derivation is modified in the quantum theory.[2]

6.1 Schwinger–Dyson Equations for Scalars

Consider a free, massless scalar field $\phi = \phi(x)$ with action

$$S[\phi] = \frac{1}{2}\int \mathrm{d}^4 y\, \partial_\mu \phi \partial^\mu \phi = -\frac{1}{2}\int \mathrm{d}^4 y\, \phi \partial^2 \phi. \tag{6.1}$$

We first consider the 1-point function

$$\langle \phi(x) \rangle = -\frac{1}{Z[0]}\frac{\delta Z[J]}{\delta J(x)}\bigg|_{J=0} = \frac{1}{Z}\int \mathcal{D}\phi\, e^{\int \mathrm{d}^4 y \frac{1}{2}\phi \partial_y^2 \phi} \phi(x). \tag{6.2}$$

Consider a small variation $\phi(x) \to \phi(x) + \epsilon(x)$ inside the path integral. Since the path integral integrates over all field configurations, this redefinition of the field has to give the same result.

$$\langle \phi(x) \rangle = \frac{1}{Z}\int \mathcal{D}\phi [\phi(x) + \epsilon(x)] e^{\frac{1}{2}\int \mathrm{d}^4 y (\phi+\epsilon)\partial^2(\phi+\epsilon)}. \tag{6.3}$$

Remark: The path integral measure is unchanged by this transformation. Formally, the path integral measure changes as

$$\mathcal{D}\phi' = \det\left(\frac{\delta \phi'(x)}{\delta \phi(y)}\right)\mathcal{D}\phi\,. \tag{6.4}$$

As long as ϵ is linear in the fields ϕ, the Jacobian is field-independent and is cancelled out by the path integral in the normalisation Z.

[2] For reference, this discussion seems to follow [Schw, Cha. 14.7].

Expanding to first order in ϵ, the exponential factor becomes

$$e^{\int d^4y \frac{1}{2}(\phi+\epsilon)\partial^2(\phi+\epsilon)} \approx e^{\frac{1}{2}\int d^4y \phi\partial_y^2\phi}\left(1+\frac{1}{2}\int d^4z\,(\phi\partial_z^2\epsilon+\epsilon\partial_z^2\phi)\right) \tag{6.5}$$

$$= e^{\frac{1}{2}\int d^4y\phi\partial^2\phi}\left(1+\int d^4z\,\epsilon\partial^2\phi\right), \tag{6.6}$$

where we have integrated by parts twice to combine the $\epsilon\partial^2\phi$ and $\phi\partial^2\epsilon$ terms. Inserting the exponential back into (6.3), we have

$$\langle\phi(x)\rangle = \frac{1}{Z}\int \mathcal{D}\phi\, e^{\frac{1}{2}\int d^4y\phi\partial^2\phi}\left[\phi(x)+\epsilon(x)+\phi(x)\int d^4z\,\epsilon(z)\partial_z^2\phi(z)\right]. \tag{6.7}$$

The $\phi(x)$ term reproduces the original 1-point function (6.2), so the remaining two terms must add to zero. Writing now $\epsilon(x) = \int d^4z\,\epsilon(z)\delta^4(z-x)$, we can factor out the ϵ, giving

$$\int d^4z\,\epsilon(z)\int \mathcal{D}\phi\, e^{\frac{1}{2}\int d^4y\phi\partial^2\phi}\left[\phi(x)\partial_z^2\phi(z)+\delta^{(4)}(z-x)\right] = 0. \tag{6.8}$$

Since this should hold true for any variation $\epsilon(z)$, the term in square brackets must vanish. Moreover, as the path integral does not depend on z except through the field insertion $\phi(z)$, we can pull ∂_z^2 out of the integral in the first term, giving us the *Schwinger–Dyson equation*

$$\partial_z^2\langle\phi(z)\phi(x)\rangle = -\delta^4(z-x). \tag{6.9}$$

In other words, the Schwinger–Dyson equation for the 2-point function in a free scalar field theory simply reproduces the Green's function equation for the Feynman propagator. In general, these Schwinger–Dyson equations are related to classical equations of motion and additional contact terms. These contact terms indicate how the quantum field theory deviates from the corresponding classical field theory.

This derivation can be extended for n-point functions. For example,

$$\partial_z^2\langle\phi(z)\phi(x)\phi(y)\rangle = -\delta^{(4)}(z-x)\langle\phi(y)\rangle - \delta^{(4)}(z-y)\langle\phi(x)\rangle. \tag{6.10}$$

6.1.1 Adding Interactions

Let us now add an interaction Lagrangian $\mathscr{L}_{\text{int}}[\phi]$ to the action:

$$S[\phi] = \int d^4y\left(-\frac{1}{2}\phi\partial^2\phi + \mathscr{L}_{\text{int}}[\phi]\right). \tag{6.11}$$

The classical equations of motion become $\partial^2\phi = \mathscr{L}'_{\text{int}}[\phi]$. We Taylor-expand by employing functional derivation

$$\mathscr{L}_{\text{int}}[\phi+\epsilon] \approx \mathscr{L}_{\text{int}}[\phi] + \epsilon(x)\mathscr{L}'_{\text{int}}[\phi]. \tag{6.12}$$

Therefore, the addition of the potential $\mathscr{L}_{\text{int}}[\phi]$ contributes and additional term

$$-\int \mathrm{d}^4 z\, \epsilon(z) \mathscr{L}'_{\text{int}}[\phi(z)] \tag{6.13}$$

to the bracket in the path integral (6.7). Thus, (6.8) is modified to

$$\int \mathrm{d}^4 z\, \epsilon(z) \int \mathcal{D}\phi\, e^{-S} \left[\phi(x)\partial_z^2\phi(z) + \delta^{(4)}(z-x) - \phi(x)\mathscr{L}'_{\text{int}}[\phi(z)]\right] = 0. \tag{6.14}$$

Hence, the Schwinger–Dyson equation for the 2-point function for the interacting theory is

$$\partial_z^2 \langle \phi(z)\phi(x)\rangle = \langle \mathscr{L}'_{\text{int}}[\phi(z)]\,\phi(x)\rangle - \delta^{(4)}(z-x). \tag{6.15}$$

If we have more field insertions, the Schwinger–Dyson equations add contact interactions, contracting the field on which the operator acts with all the other fields in the correlator. For example, with three fields:

$$\partial_x^2 \langle \phi(x)\phi(y)\phi(z)\rangle = \langle \mathscr{L}'_{\text{int}}[\phi(x)]\,\phi(y)\phi(z)\rangle - \delta^4(x-z)\langle\phi(y)\rangle - \delta^4(x-y)\phi(z). \tag{6.16}$$

In this way, the complete set of Schwinger–Dyson equations can be derived. In general, for a massive scalar field [Schw, Sec. 14.7]

$$\begin{aligned}(\partial_x^2 + m^2)\,\langle \phi(x)\phi(x_1)\dots\phi(x_n)\rangle &= \left\langle \mathscr{L}'_{\text{int}}[\phi(x)]\phi(x_1)\dots\phi(x_n)\right\rangle \\ &\quad - \sum_i \delta^4(x-x_i)\left\langle \phi(x_1)\dots\phi(x_{i-1})\phi(x_{i+1})\dots\phi(x_n)\right\rangle,\end{aligned} \tag{6.17}$$

where $\mathscr{L}'_{\text{int}}[\phi] = \frac{\delta}{\delta\phi}\mathscr{L}_{\text{int}}[\phi]$ is the variational derivative of the interaction Lagrangian, and we are using $\langle \dots \rangle$ as an abbreviation for $\langle \Omega |\, T\{\dots\}\, |\Omega\rangle$ for time-ordered matrix elements in the interacting vacuum.

> **Remark:** These relations can be derived from the assumptions that the interacting quantum fields satisfy
>
> $$(\partial^2 + m^2)\phi = \mathscr{L}'_{\text{int}}[\phi], \qquad [\phi(x), \partial_0\phi(y)] = \delta^3(x-y). \tag{6.18}$$

The Schwinger–Dyson equations should be interpreted as saying that the difference between the quantum and classical equations of motion is given by contact terms, which specify the quantum theory.

6.1.2 Symmetries

As shown in Appendix A, the variational derivatives of the action S and the Lagrangian $\mathscr{L}$ are related via

$$\frac{\delta S}{\delta\phi} = \frac{\delta\mathscr{L}}{\delta\phi} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right). \tag{6.19}$$

For the action for a scalar Klein–Gordon field, this is

$$\frac{\delta S}{\delta \phi} = \mathscr{L}'_{\text{int}}[\phi] - \partial^2 \phi. \tag{6.20}$$

Therefore, the Schwinger–Dyson equation (6.15) becomes

$$\left\langle \frac{\delta S}{\delta \phi(z)} \phi(x) \right\rangle = \delta^{(4)}(z - x). \tag{6.21}$$

Suppose we have a symmetry, meaning that $\delta \mathscr{L} = 0$ under $\phi \to \phi + \epsilon$. We often have $\epsilon(x) = \alpha(x)\phi(x)$. Then, as shown in (A.11) of Appendix A, we have

$$\partial_\mu j^\mu = -\frac{\delta S}{\delta \phi} \epsilon. \tag{6.22}$$

This means that we have $\partial_\mu j^\mu = 0$ when the field equations are satisfied, which we expect from a Noether current. Together with (6.21), this yields the *Ward–Takahashi identity*

$$\frac{\partial}{\partial z^\mu} \langle j^\mu(z) \phi(x) \rangle = -\delta^{(4)}(z - x) \langle \epsilon(x) \rangle. \tag{6.23}$$

6.2 Ward–Takahashi Identity

In the derivation of Noether's theorem, we perform a variation of the field that is also a global symmetry of the Lagrangian, which leads to the existence of a classically conserved current. Performing a similar variation on the path integral and following the steps that led to the Schwinger–Dyson equations will produce a general and powerful relation among correlation functions known as the Ward–Takahashi identity. This not only implies the usual Ward identity and gauge invariance, but since it is non-perturbative it will also play an important role in the renormalisation of QED [Schw, Sec. 14.8].

6.2.1 Schwinger–Dyson for Fermions

Consider $\mathscr{L} = \overline{\psi} \not{\partial} \psi +$ (non-derivative terms). Under a transformation

$$\psi(x) \to e^{i\alpha(x)} \psi(x), \qquad \overline{\psi}(x) \to \overline{\psi} e^{-i\alpha(x)}, \tag{6.24}$$

we pick up an extra piece

$$\overline{\psi} \not{\partial} \psi \to \overline{\psi} \not{\partial} \psi + i \overline{\psi} \gamma^\mu \psi \partial_\mu \alpha. \tag{6.25}$$

We want to be able to compute the expectation value of our fields, so let us investigate the propagator $\langle \psi(x_1) \overline{\psi}(x_2) \rangle$. We expand about small $\alpha(x)$ and follow the same steps as in Sec. 6.1 to find that terms of order $O(\alpha)$ must vanish

$$0 = \int \mathcal{D}\psi\, \mathcal{D}\overline{\psi}\, e^{-S} \left[i \int \mathrm{d}^4 x\, \overline{\psi}(x)\gamma^\mu \psi(x) \partial_\mu \alpha(x) \right] \psi(x_1)\overline{\psi}(x_2) + \int \mathcal{D}\psi\, \mathcal{D}\overline{\psi}\, e^{-S} \left[i\alpha(x_1)\psi(x_1)\overline{\psi}(x_2) - i\alpha(x_2)\psi(x_1)\overline{\psi}(x_2) \right]. \tag{6.26}$$

Once again we integrate by parts and factor out the $\alpha(x)$. Requiring this to vanish for any $\alpha(x)$ gives the fermion Schwinger–Dyson equation associated with charge conservation

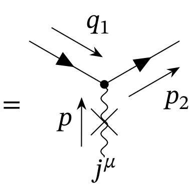
$$\partial_\mu \langle j^\mu(x)\psi(x_1)\overline{\psi}(x_2)\rangle = \left[-\delta^{(4)}(x - x_1) + \delta^{(4)}(x - x_2) \right] \langle \psi(x_1)\overline{\psi}(x_2)\rangle, \tag{6.27}$$

where $j^\mu(x) = i\overline{\psi}(x)\gamma^\mu\psi(x)$ is the Noether current for QED. This is a non-perturbative relation between the correlation functions. If has the same qualitative content as the other Schwinger–Dyson equations: the classical equations of motion, in this case $\partial_\mu j^\mu = 0$, hold within time-ordered correlation functions up to contact interactions.

6.2.2 Ward–Takahashi Identity in QED

The Schwinger–Dyson equation is going to be more useful for us in momentum space. Let us Fourier transform to obtain "off-shell" amplitudes, where we are not imposing momentum conservation. The Fourier transform of the matrix element of the current with fields is

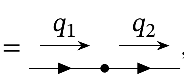
$$M^\mu(p, q_1, q_2) := \int \mathrm{d}^4 x\, \mathrm{d}^4 x_1\, \mathrm{d}^4 x_2\, e^{ip\cdot x} e^{iq_1\cdot x_1} e^{-iq_2\cdot x_2} \langle j^\mu(x)\psi(x_1)\overline{\psi}(x_2)\rangle =$$



$$, \tag{6.28}$$

where we have chosen signs so that the momenta represent $j(p) + e^-(q_1) \to e^-(q_2)$. Similarly, we define

$$M(q_1, q_2) := \int \mathrm{d}^4 x_1\, \mathrm{d}^4 x_2\, e^{iq_1\cdot x_1} e^{-iq_2\cdot x_2} \langle \psi(x_1)\overline{\psi}(x_2)\rangle =$$



$$, \tag{6.29}$$

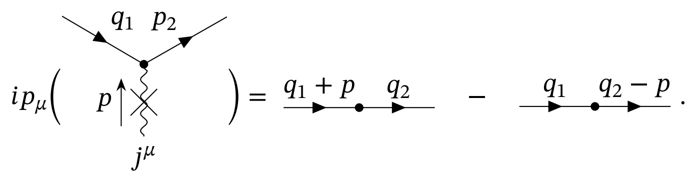
with signs to represent $e^-(q_1) \to e^-(q_2)$, so that

$$M(q_1 + p, q_2) := \int \mathrm{d}^4 x\, \mathrm{d}^4 x_1\, \mathrm{d}^4 x_2\, e^{iq_1\cdot x_1} e^{-iq_2\cdot x_2} \delta^4(x - x_1) \langle \psi(x_1)\overline{\psi}(x_2)\rangle \tag{6.30}$$

is the Fourier transform of the first term on the right of (6.27). The second term is similar and so the Fourier transform of the QED Schwinger–Dyson equation (6.27) is

$$ip_\mu M^\mu(p, q_1, q_2) = M(q_1 + p, q_2) - M(q_1, q_2 - p). \tag{6.31}$$

This is known as a *Ward–Takahashi* identity, and it can be represented diagrammatically as



(6.32)

These are not Feynman diagrams for S-matrix elements, since the momenta are not on-shell. Instead, they are Feynman diagrams for correlation functions, also sometime called *off-shell S-matrix* elements. The Feynman rules are the usual momentum space Feynman rules with the addition of propagators for external lines and without the external polarisation (i.e. without removing the stuff that the LSZ formula removes). Momentum is not necessarily conserved, which is why we can have $q_1 + p$ coming in and q_2 going out. Moreover, we have at no point used any perturbation theory, so these hold non-perturbatively.

6.2.3 Relation to Renormalisation

In QED, we have

$$\mathscr{L} = \frac{1}{4}Z_3F^2 + Z_2\bar{\psi}\partial\!\!\!/\psi + Z_2Z_m m\bar{\psi}\psi + Z_1e\bar{\psi}A\!\!\!/\psi. \tag{6.33}$$

Our aim will be to prove that $Z_1 = Z_2$, as we assumed in the previous chapter. Let

$$\mathcal{M}(q_1, q_2) = \delta^4(q_1 - q_2)G(q\!\!\!/_1). \tag{6.34}$$

Also, consider for now a massless electron $m = 0$, since we are not interested in the renormalisation of the mass term for now. The renormalised propagator is then

$$G(p\!\!\!/) = \frac{1}{Z_2}\frac{1}{iq\!\!\!/}. \tag{6.35}$$

After one-loop renormalisation, we got the vertex function $\Gamma^{(2)\mu} \sim eZ_1\gamma^\mu$. We want to extend this vertex function to off-shell. This is done by taking the Green's function and amputating (see notes)

$$\Gamma^\mu(p, q_1, q_2)\delta^4(p+q_1-q_2) = \int \mathrm{d}^4x\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2\, e^{ip\cdot x}e^{iq_1\cdot x_1}e^{-iq_2\cdot x_2}G^{-1}(q_1)\langle j^\mu(x)\psi(x_1)\bar{\psi}(x_2)\rangle G^{-1}(p+q_1). \tag{6.36}$$

Relating the inverse Green's function to what we had before in momentum space, we have

$$G^{-1}(q_1)\mathcal{M}^\mu(p, q_1, q_2)?? \tag{6.37}$$

7 Nonabelian Gauge Theory

7.1 Lie Groups: Facts and Conventions

Let G be a continuous, connected group. For $U \in G$, we can write $U = \exp(i\theta^a T^a)\mathbb{1}$. The Hermitian generators are T^a and θ^a are numbers parametrising $U = U(\theta)$. The index a is running over the generators. The generators T^a form a Lie algebra under the Lie bracket

$$[T^a, T^b] = if^{abc}T^c, \tag{7.1}$$

with structure constants f^{abc}. We can and will choose a basis where f^{abc} is antisymmetric. We will think of matrix representations in which the Lie bracket is just the matrix commutator. This obeys a Jacobi identity

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0. \tag{7.2}$$

Which implies that the structure constants obey

$$f^{abd}f^{dce} + f^{bcd}f^{dae} + f^{cad}f^{dbe} = 0. \tag{7.3}$$

There is a normalisation for the structure constants, which we choose to be

$$f^{acd}f^{bcd} = N\delta^{ab}. \tag{7.4}$$

We classify Lie groups into unitary, orthogonal, symplectic, and the exceptional groups. In these lectures, we focus on unitary Lie groups. If $U \in U(N)$, then $U^\dagger U = \mathbb{1}$ and $U \in SU(N)$ satisfies $\det U = 1$. $SU(N)$ has $N^2 - 1$ generators, so $\dim SU(N) = N^2 - 1$.

7.1.1 Representations

Fundamental: Smallest, non-trivial representation of the Lie algebra. For $\mathfrak{su}(N)$, these are $N \times N$ traceless, Hermitian operators. For infinitesimal $SU(N)$ transformations, ϕ transforms in the fundamental representation as

$$\phi_i = \phi_i + i\alpha^a(T^a_{\text{fund}})_{ij}\phi_j, \tag{7.5}$$

where $1 \gg \alpha^a \in \mathbb{R}$. The indices $a = 1, \dots, N$ label the generators and $i, j = 1, \dots, N$ are particular to the representation we are in. The T^a_{fund} are Hermitian. We will often drop the subscript on the fundamental representation: $T^a = T^a_{\text{fund}}$.

Antifundamental: The generators are complex conjugate of the generators of the fundamental representation

$$(T^a_{\text{afund}}) = -(T^a_{\text{fund}})^*. \tag{7.6}$$

Infinitesimal transformations in the antifundamental representation are

$$\phi^*_i \to \phi^*_i i + \alpha^a (T^a_{\text{afund}})_{ij} \phi^*_j = \phi^*_i - i\alpha^a \phi^*_j (T^a_{\text{fund}})_{ji}. \tag{7.7}$$

Adjoint: Acts on vector space spanned by generator matrices:

$$(T^a_{\text{adj}})_{ij} = -i f^{aij}, \qquad a, i, j = 1, \dots, N^2 - 1. \tag{7.8}$$

Gauge fields transform in the adjoint representation.

7.1.2 Classifying Representations

Definition 19 (index)**:** The *index* $T(R)$ of a representation R is defined by an inner product

$$\operatorname{Tr}\big(T^a_R T^b_R\big) = T(R)\delta^{ab}, \tag{7.9}$$

where the trace is over the representation indices (rather than the generator indices).

fundamental:

$$T^a_{ij} T^b_{ji} = \frac{1}{2}\delta^{ab} \;\Rightarrow\; T(\text{fund}) = T_F = \frac{1}{2}. \tag{7.10}$$

adjoint:

$$f^{acd} f^{bcd} = N\delta^{ab} \;\Rightarrow\; T(\text{adj}) = T_A = N. \tag{7.11}$$

Definition 20 (quadratic Casimir)**:** The *quadratic Casimir* $C_2(R)$ is defined by

$$T^a_R T^a_R = C_2(R)\mathbb{1}. \tag{7.12}$$

Comparing with the definition of the index, setting $a = b$ and summing gives

$$T(R) \dim G = C_2(R) \dim R. \tag{7.13}$$

Therefore, we find that the Casimirs for the fundamental and adjoint representations for $SU(N)$ are

$$C_2(\text{fund}) = C_F = \frac{N^2 - 1}{2N}, \tag{7.14}$$

$$C_2(\text{adj}) = C_A = N. \tag{7.15}$$

7.2 Gauge Invariance and Wilson Lines

One approach would be to take QED and decorate the Lagrangian with indices that tell us under which representation these transform. We will get there. However, we will start from a slightly more geometric approach, involving Wilson lines.

7.2.1 Abelian Wilson Lines

Let us revisit QED, with fermions transforming under a local $U(1)$ transformation

$$\psi(x) \longmapsto e^{i\alpha(x)}\psi(y), \qquad \overline{\psi}(x) \longmapsto \overline{\psi}(x)e^{-i\alpha(x)}. \tag{7.16}$$

Then $\overline{\psi}\gamma\psi$ is not invariant. Consider the derivative in the direction of a unit vector n^μ

$$n^\mu \partial_\mu \psi = \lim_{a\to 0} \frac{1}{a}\left[\psi(x+an) - \psi(x)\right]. \tag{7.17}$$

Then fields at different points x and $x + an$ transform in a slightly different way

$$\psi(x+an) - \psi(x) \longmapsto e^{i\alpha(x+an)}\psi(x+an) - e^{i\alpha(x)}\psi(x). \tag{7.18}$$

A gauge covariant derivative fixes this. We want $D_\mu\psi$ to transform in the same ways as ψ:

$$D_\mu\psi(x) \longmapsto e^{i\alpha(x)}D_\mu\psi(x). \tag{7.19}$$

We define a *Wilson line* (a parallel transporter) $W(y, x)$ such that its gauge transformation is

$$W(y,x) \longmapsto e^{i\alpha(y)}W(y,x)e^{-i\alpha(x)}. \tag{7.20}$$

With the convention $W(x, x) = 1$, the Wilson line becomes a phase

$$W(y,x) = e^{i\phi(y,x)}, \qquad \phi \in \mathbb{R}. \tag{7.21}$$

We also assume the convention that $W(x, y) = [W(y, x)]^*$. We define D_μ such that

$$n^\mu D_\mu \psi = \lim_{a\to 0} \frac{1}{a}\left[\psi(x+an) - W(x+an,x)\psi(x)\right]. \tag{7.22}$$

Then $\overline{\psi}\not{D}\psi$ is gauge invariant.

For infinitesimal a, define A_μ in the middle of the Wilson line via

$$W(x+an,x) = \exp\left[iean^\mu A_\mu(x+\frac{1}{2}an)\right]. \tag{7.23}$$

Taking the limit in (7.22), we find the *gauge covariant derivative*

$$D_\mu\psi(x) = [\partial_\mu - ieA_\mu(x)]\psi(x) \tag{7.24}$$

which by construction transforms in the same way as the field

$$D_\mu\psi \longmapsto e^{i\alpha(x)}D_\mu\psi. \tag{7.25}$$

From this we can find the gauge transformation of A_μ as

$$A_\mu(x) \longmapsto A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x). \tag{7.26}$$

We have seen that if a field has the local transformation law (7.16), then its covariant derivative has the same transformation law. Thus the second covariant derivative of ψ, as well as its covariant derivative also transform according to (7.16):

$$D_\nu(D_\mu\psi) \longmapsto e^{i\alpha(x)}D_\nu(D_\mu\psi) \tag{7.27}$$

$$[D_\mu, D_\nu]\psi \longmapsto e^{i\alpha(x)}[D_\mu, D_\nu]\psi. \tag{7.28}$$

In particular, the commutator is not itself a derivative at all:

$$\begin{aligned}
[D_\mu, D_\nu]\psi &= [\partial_\mu, \partial_\nu]\psi - ie([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu])\psi - e^2[A_\mu, A_\nu]\psi && (7.29)\\
&= -ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi. && (7.30)
\end{aligned}$$

Since the factor ψ on the right-hand side accounts for the entire phase $e^{i\alpha(x)}$ in the transformation law (7.28), the field strength $F_{\mu\nu}$ defined as the curvature of the covariant derivative

$$[D_\mu, D_\nu] = -ie(\partial_\mu A_\nu - \partial_\nu A_\mu) = ieF_{\mu\nu} \tag{7.31}$$

must be gauge invariant in the case of Abelian gauge theory.

7.2.2 Abelian Wilson Loops

Wilson lines need not be infinitesimal and the field $A_\mu(x)$ not continuous. Let us reverse the logic of the previous section and start from a connection A_μ, assumed to have the transformation law (7.26). One can then show that the expression

$$W_C(z,y) = \exp\left[ie\int_C \mathrm{d}x^\mu\, A_\mu(x)\right] \tag{7.32}$$

transforms according to the defining relation (7.20) if the integral is taken along a path C that runs from y to z. Then $W_C(z,y)$ is called the *Wilson line*. A *Wilson loop* is a closed Wilson line $W_C(z,z)$, which starts and ends at the same point $y = z$. The Wilson loop C encloses a surface Σ as shown in 7.2. By Stokes' theorem, we can rewrite the Wilson loop as

$$W_C(z,z) = \exp\left[ie\oint_C A_\mu\,\mathrm{d}x^\mu\right] = \exp\left[\frac{ie}{2}\int_\Sigma F_{\mu\nu}\,\mathrm{d}\sigma^{\mu\nu}\right], \tag{7.33}$$

where $\mathrm{d}\sigma^{\mu\nu}$ is an area element on the surface Σ. We see that the field strength $F_{\mu\nu}$ appears from gauge invariant *Wilson loops*. Conversely, since (almost) all gauge-invariant functions of A_μ can be built up from $F_{\mu\nu}$, this suggests that the Wilson loop $W_C(y,y)$ is the most general gauge invariant.

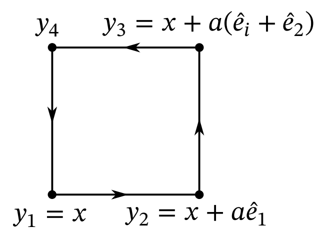


Figure 7.1: Plaquette

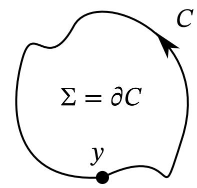


Figure 7.2: Wilson loop C

Example 7.2.1 (infinitesimal plaquette)**:** Take the Wilson loop to be a square, called a *plaquette*, as shown in Fig. 7.1. Taking the side length a to be infinitesimal, we can use the infinitesimal expression (7.23) for the Wilson line to expand the infinitesimal loop

$$P_{12}(x) = W(y_1, y_4)W(y_4, y_3)W(y_3, y_2)W(y_2, y_1) \tag{7.34}$$

$$\approx 1 - iea^2 F_{12}(x) + O(a^3). \tag{7.35}$$

Since the Wilson loop is gauge invariant, this is another way to view the gauge invariance of $F_{\mu\nu}$ geometrically.

7.2.3 Non-Abelian Case

Both the Wilson line and the Wilson loop can be generalised to the non-Abelian case. Take $V(x) \in G$ to be the fundamental representation of the gauge group G, such as $SU(N)$, acting on the n-plet of fields ψ as

$$\psi(x) \to V(x)\psi(x). \tag{7.36}$$

We want to construct the Wilson line that transforms in such a way to cancel out differences in transformations at different points

$$W(y, x) \to V(y)W(y, x)V^\dagger(x) \tag{7.37}$$

with normalisation $W(x, x) = \mathbb{1}$, the identity of G.

Consider an infinitesimal Wilson line (small a)

$$W(x + an, x) = 1 + igan^\mu A^a_\mu T^a. \tag{7.38}$$

Using the same methods as before, we find that the associated covariant derivative is

$$D_\mu = \partial_\mu - ig\mathcal{A}_\mu, \tag{7.39}$$

where we use the shorthand $\mathcal{A}_\mu := A^a_\mu T^a$ and T^a are the generators of the Lie algebra $\mathfrak{g}$. To find the form of the gauge transformation $\mathcal{A}_\mu \to \mathcal{A}'_\mu$, we require as in the Abelian case that the covariant derivative transforms in the same way as the field ψ:

$$D_\mu\psi \to D'_\mu\psi' = VD_\mu\psi \tag{7.40}$$
$$(\partial_\mu - ig\mathcal{A}'_\mu)V\psi = V(\partial_\mu - ig\mathcal{A}_\mu\psi). \tag{7.41}$$

Solving for $\mathcal{A}'_\mu$ gives

$$\mathcal{A}'_\mu = VA_\mu V^{-1} - \frac{i}{g}(\partial_\mu V)V^{-1} = V\left(\frac{i}{g}\overleftarrow{D}_\mu\right)V^{-1}. \tag{7.42}$$

Under an infinitesimal transformation

$$V(x) = \mathbb{1} + i\alpha^a(x)T^a + O(\alpha^2), \tag{7.43}$$

the fields ψ and A^a_μ transform as

$$\psi(x) \to (\mathbb{1} + i\alpha^a(x)T^a)\psi \tag{7.44}$$
$$A^a_\mu(x) \to A^a_\mu(x) + \frac{1}{g}\partial_\mu\alpha^a(x) + f^{abc}A^b_\mu(x)\alpha^c(x) + \frac{1}{g}\partial_\mu\alpha^a(x) \tag{7.45}$$
$$= A^a(x) + \frac{1}{g}[\partial_\mu\delta^{ac} - igA^b_\mu(x)\underbrace{(-if^{bac})}_{(T^b_{\text{adj}})^{ac}}]\alpha^c(x) \tag{7.46}$$
$$:= A^a_\mu(x) + \frac{1}{g}D^{ac}_\mu\alpha^c(x), \tag{7.47}$$

where we defined the gauge covariant derivative in the adjoint representation as

$$D^{ab}_\mu = \left[\delta^{ab}\partial_\mu - igA^c_\mu(T^c_{\text{adj}})^{ab}\right]. \tag{7.48}$$

This justifies our previous statement that the gauge fields transform in the adjoint representation.

7.2.4 Non-Abelian Wilson Loops

Not lectured. Taken from [P&S, Chapter 15.3].

We want to find the *finite* Wilson line that generalises (7.32) to the non-Abelian case. Since the matrices M do not in general commute at different points, and their exponentials combine non-trivially (according to the Baker–Campbell–Hausdorff formula), it would not be correct to simply replace $A_\mu \to A^a_\mu T^a$ in the exponent of the Abelian Wilson line (7.32). Instead, we must order the matrices in a particular way.

Let t be a parameter of the path C, running from 0 at $x = y$ to 1 at $x = z$. Then we define the Wilson line as the power-series expansion of the exponential, with the matrices in each term ordered so that the higher values of s stand to the left. This prescription is called *path-ordering* and is denoted by the symbol P. The non-Abelian Wilson line is

$$W_C(z,y) = P\left\{\exp\left[ig\int_0^1 \mathrm{d}t\,\frac{\mathrm{d}x^\mu}{\mathrm{d}t}A^a_\mu(x(t))T^a\right]\right\}. \tag{7.49}$$

The expression for the non-Abelian Wilson line is analogous to the time-ordered exponential in Dyson's formula for the interaction-picture propagator, which we have met in the *Quantum Field Theory* lectures last term.

Now what is the natural generalisation of the Abelian Wilson loop (7.33)? Since the matrices do not commute, the Wilson line $W_C(y,y)$ associated with a closed path returning to y is not gauge invariant, but transform as

$$W_C(y,y) \to V(y)W_C(y,y)V^{-1}(y). \tag{7.50}$$

We therefore define the Wilson loop for a non-Abelian gauge theory as the *trace* of the non-Abelian Wilson line around a closed path:

$$\operatorname{tr} W_C(y,y) = \operatorname{tr} P\left\{\exp\left[ig\int_0^1 \mathrm{d}t\,\frac{\mathrm{d}x^\mu}{\mathrm{d}t}A^a_\mu(x(t))T^a\right]\right\}. \tag{7.51}$$

Example 7.2.2 (infinitesimal plaquette (non-Abelian))**:** Let us again consider the infinitesimal square shown in Fig. 7.1. In this case, it turns out that one finds

$$P_{12}(x) = W_C(x,x) = 1 + iga^2F^a_{12}T^a + O(a^3), \tag{7.52}$$

where $F^a_{\mu\nu}$ is given by the full non-Abelian expression (7.55). Expanding the transformation law (7.50) to order a^2, one recovers the transformation law (7.56) for $F^a_{\mu\nu}$.

For $G = SU(2)$, where $T^a = \sigma^i$, we can evaluate the Wilson loop $\operatorname{tr} W_C(x,x)$ more explicitly, finding

$$\operatorname{tr} W_C(x,x) = 2 - \frac{1}{4}g^2a^2(F^a_{12})^2 + O(a^5). \tag{7.53}$$

The gauge invariance of $(F^a_{\mu\nu})^2$ can therefore be derived from this geometrical argument, just as in the Abelian case.

7.2.5 Non-Abelian Gauge Invariant Lagrangians

We define the field strength tensor through the commutator of the covariant derivatives

$$[D_\mu, D_\nu] = -igF^a_{\mu\nu}T^a. \tag{7.54}$$

Then the field strength tensor $F_{\mu\nu} = F^a_{\mu\nu}T^a$ has components

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu. \tag{7.55}$$

Under an infinitesimal transformation,

$$F^a_{\mu\nu} \to F^a_{\mu\nu} - f^{abc}\alpha^b F^c_{\mu\nu}, \tag{7.56}$$

so $F^a_{\mu\nu}$ is not gauge invariant! However,

$$F^a_{\mu\nu}(F^a)^{\mu\nu} = (F^a)^2 \tag{7.57}$$

is gauge-invariant. We have a Lagrangian

$$\mathscr{L} = \underbrace{\frac{1}{4}(F^a)^2}_{} + \underbrace{\overline{\psi}_i \left(\not\partial\delta_{ij} - ig\not A^a T^a_{ij} + m\delta_{ij}\right)\psi_j}_{} \tag{7.58}$$

$$= \mathscr{L}_{YM} + \overline{\psi}(\not D + m)\psi, \tag{7.59}$$

where the Yang–Mills Lagrangian is often called the *pure gauge*. Also, we can introduce another gauge invariant term

$$\mathscr{L}_\theta = \theta\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} = 2\theta\partial_\mu(\epsilon^{\mu\nu\rho\sigma}A^a_\nu F^a_{\rho\sigma}). \tag{7.60}$$

As this is a total derivative, it is a boundary term which does not contribute to any order in perturbation theory. However, it can contribute a nonperturbative topological term, which we miss out on in the perturbation expansion. It violates CP and therefore also T. Experimentally the strong interaction in QCD seems to conserve CP, constraining the θ to be vanishingly small. The *Strong CP Problem* is the task of finding a reason for why θ is so small.

7.3 Fadeev–Popov Gauge Fixing

Let us first think about an analogy. Consider an integral over two real variables

$$Z \propto \int \mathrm{d}x\,\mathrm{d}y\, e^{-S(x)}, \tag{7.61}$$

where the action $S(x)$ only depends on one of the variables. The integral over $y \in (-\infty, \infty)$ is divergent, but it is also redundant to any physics. In QED, i.e. in $U(1)$ gauge theory, we just set $y = 0$

$$Z = \int \mathrm{d}x\, e^{-S(x)} \tag{7.62}$$

Life is more complicated when the gauge fields interact with themselves and the previous method will not work in non-Abelian gauge theory. Instead, we may introduce a δ-function to fix y not to zero, but to an arbitrary function of x

$$Z = \int \mathrm{d}x\,\mathrm{d}y\,\delta(y - f(x))e^{-S(x)}. \tag{7.63}$$

Maybe we do not have y explicitly. All we need is that $y = f(x)$ is the unique solution to some equation $G(x, y) = 0$, which we call the *gauge-fixing condition*. To rewrite (7.63) in a more general way, we use the composition rule for δ-functions

$$\delta\big(G(x,y)\big) = \left|\frac{\partial G}{\partial y}\right|^{-1} \delta(y - f(x)), \tag{7.64}$$

where we use the assumption that $y - f(x)$ is the unique solution to $G(x, y) = 0$. Then, assuming further that $\frac{\partial G}{\partial y} > 0$,

$$Z = \int \mathrm{d}x\,\mathrm{d}y\,\frac{\partial G}{\partial y}\delta\big(G(x,y)\big)e^{-S(x)}. \tag{7.65}$$

These assumptions are fine in perturbation theory, but we have *Gribov copies* non-perturbatively.

This generalises to n variables as

$$Z = \int \mathrm{d}^n x\,\mathrm{d}^n y\,\det\!\left(\frac{\partial G}{\partial y}\right)\left[\prod_{i=1}^{n}\delta(G_i)\right]e^{-S(x)}. \tag{7.66}$$

In gauge theory, these are really path integrals. The above form means that we can somehow separate the path integral $\mathcal{D}A$ into the physical fields x and the gauge-equivalent degrees of freedom y.

7.3.1 Gauge Theory and the Faddeev–Popov Determinant

Let us now apply this to gauge field theory. The gauge-fixing condition, generalising $\partial_\mu A^\mu = 0$ from QED, is the generalised Lorentz gauge condition

$$G^a(x) = \partial^\mu A^a_\mu(x) - w^a(x) \tag{7.67}$$

where now the x represent spacetime labels and $w^a(x)$ are field-independent functions of x. Under gauge transformation,

$$G^a(x) \to G^a(x) + \frac{1}{g}\partial^\mu D^{ab}_\mu \alpha^b(x). \tag{7.68}$$

We can extract the differential operator with a functional derivative

$$\frac{\delta G^a(x)}{\delta\alpha^b(y)} = \frac{1}{g}\delta^{(4)}(x-y)(\partial^\mu D^{ab}_\mu). \tag{7.69}$$

This gives us the Fadeev–Popov determinant

$$\det\!\left(\frac{\delta G^a(x)}{\delta\alpha^b(y)}\right), \tag{7.70}$$

which are the components going into the path integral for the generating functional. The way to make sense of this formal object, a determinant over differential operators, is to rewrite it as a path integrals over fermionic ghost fields.

7.3.2 Faddeev–Popov Ghost Fields

We write the Faddeev–Popov determinant as a functional integral over a new set of *spinless* Grassmann fields $c, \overline{c}$

$$\det\left(\frac{\delta G^a(x)}{\delta \alpha^b(y)}\right) = \det\left(\frac{1}{g}\partial^\mu D_\mu\right) = \int \mathcal{D}c\,\mathcal{D}\overline{c}\exp\left[-\int \mathrm{d}^4x\,\overline{c}^a(-\partial^\mu D_\mu^{ab})c^b\right], \tag{7.71}$$

which adds to the Lagrangian the term

$$\mathscr{L}_{\text{gh}} = -\overline{c}^a\partial^\mu D_\mu^{ab}c^b. \tag{7.72}$$

Since this Lagrangian is the adjoint derivative, we see that $c, \overline{c}$ have to transform in the adjoint representation. To give the correct identity, c and $\overline{c}$ must be anticommuting fields that are scalars under Lorentz transformation, meaning that the quantum excitations of these fields have the wrong relation between spin and statistics to be physical particles. Nevertheless, we can treat these excitations as additional particles, called *Faddeev–Popov ghosts*, in the computation of Feynman diagrams. We will see that this is a completely benign trick since the unphysical c-fields, which violate the spin-statistics theorem, will end up not contributing to any scattering amplitudes.

We can integrate by parts to have an equivalent Lagrangian

$$\begin{aligned}\mathscr{L}_{\text{gh}} &= \partial^\mu\overline{c}^a D_\mu^{ab}c^b &(7.73)\\ &= \partial^\mu\overline{c}^a\partial_\mu c^a - ig(\partial^\mu\overline{c}^a)A_\mu^c(T^c_{\text{adj}})^{ab}c^b &(7.74)\\ &= \partial^\mu\overline{c}^a\partial_\mu c^a - gf^{abc}A_\mu^c(\partial^\mu\overline{c}^a)c^b &(7.75)\end{aligned}$$

which gives the same action S_{gh}. The first term gives the ghost propagator while the second term gives the interaction vertex between the c ghosts and the gauge field A. For $U(1)$, the structure constants $f^{abc} = 0$ vanish. So the ghosts decouple and are not needed.

7.3.3 Gauge Fixing

We want to rewrite the δ-function as a *gauge fixing* term in the action,

$$\delta[G^a(x)] = \delta[\partial^\mu A_\mu^a(x) - w^a(x)] \to e^{-S_{\text{gf}}}. \tag{7.76}$$

In order to do this, we want to multiply Z by a Gaussian in ω with width ξ,

$$Z \to \exp\left(-\int \mathrm{d}^4x\,\frac{\omega^2}{2\xi}\right)Z \tag{7.77}$$

and integrate over ω. As $\xi \to 0$, this becomes a δ-function.

The ω-integral is then easy since the δ-functional (7.76) simply replaces $\omega^a(x)$ with $\partial^\mu A_\mu^a(x)$. We then have the action

$$Z \propto \int \mathcal{D}A\, \mathcal{D}[c,\bar{c}] \exp\left(-S_{YM} - S_{\mathrm{gh}} - S_{\mathrm{gf}}\right), \tag{7.78}$$

with

$$S_{\mathrm{gf}} = \int \mathrm{d}^4x\, \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2. \tag{7.79}$$

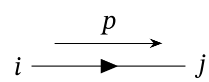
7.4 Feynman Rules for Fermions, Gauge Bosons, and Ghosts

Our Lagrangian has three different kinds of fields: gauge, fermion, and ghost

$$\mathscr{L} = \frac{1}{4}(F_{\mu\nu}^a)^2 + \bar{\psi}(\not{D} + m)\psi + \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 + \partial_\mu c^a \partial^\mu \bar{c}^a - g f^{abc} A_\mu^c \partial^\mu \bar{c}^a c^b. \tag{7.80}$$

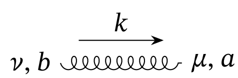
The propagators are:

Fermions



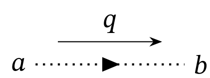
$$= \left(\frac{1}{i\not{p} + m}\right)^{ij} = \left(\frac{-i\not{p} + m}{p^2 + m^2}\right)^{ij} \tag{7.81}$$

Gauge bosons



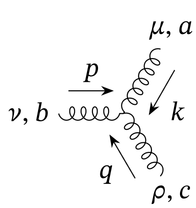
$$= \frac{1}{k^2}\left(\delta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)\delta^{ab} \tag{7.82}$$

Ghosts



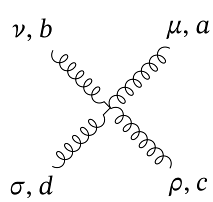
$$= \frac{\delta^{ab}}{q^2} \quad \text{(massless)} \tag{7.83}$$

The interactions come from writing out D_μ and $F_{\mu\nu}$:



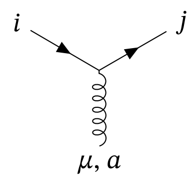
$$= -g f^{abc}\left[\delta^{\mu\nu}(k-p)^\rho + \delta^{\nu\rho}(p-q)^\mu + \delta^{\rho\mu}(q-k)^\nu\right], \tag{7.84}$$

from term $g f^{abc}(\partial_\mu A_\nu^a) A^{\mu,b} A^{\nu,c}$.

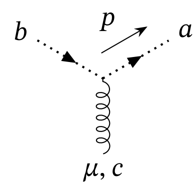


$$= \begin{aligned} -g^2\big[& f^{eab} f^{ecd}(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho}) \\ & f^{ace} f^{bde}(\delta^{\mu\nu}\delta^{\rho\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho}) \\ & f^{ade} f^{bce}(\delta^{\mu\nu}\delta^{\rho\sigma} - \delta^{\mu\rho}\delta^{\nu\sigma})\big], \end{aligned} \tag{7.85}$$

from the $\frac{1}{4}g^2(f^{abc}A^b_\mu A^c_\nu)(f^{ade}A^{\mu,d}A^{\nu,e})$ term. Moreover, taking T_{ij} to be in the fundamental representation, we have



$$= ig\gamma^\mu T^a_{ij},$$



$$= -gf^{abc}p^\mu, \tag{7.86}$$

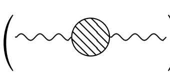
where the momentum p^μ is the one associated with the outgoing ghost $\bar{c}$.

Remark: All of these vertices involve the same coupling constant g. In fact, the coupling constants of all three nonlinear terms in the Yang–Mills Lagrangian must be equal in order to preserve the Ward identity and avoid the production of bosons with unphysical polarisation states. Conversely, the non-Abelian gauge symmetry guarantees that these couplings are equal, giving a consistent theory of physical vector particle interactions [P&S, pp. 508].

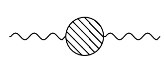
7.5 One-Loop Renormalisation

The Lagrangian (7.80) of non-Abelian gauge theory contains no interactions of dimension higher than 4 and is therefore renormalisable, meaning that the divergences can be removed by a finite number of counterterms.

However, as in QED, the gauge symmetry implies restrictions on the divergences. In QED, the Ward identity implies the relation

$$q^\mu \Bigg($$  $$\Bigg) = 0, \tag{7.87}$$

which in turn implies that the photon self-energy diagrams have the structure



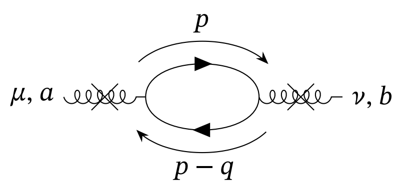
$$= (q^2\delta^{\mu\nu} - q^\mu q^\nu)\Pi(q^2). \tag{7.88}$$

The only divergence possible is a logarithmically divergent contribution to $\Pi(q^2)$. In non-Abelian gauge theories, (7.87) still holds and therefore the self-energy again has the Lorentz structure (7.88), although the contributions and cancellations ins this vacuum polarisation diagram are much more complex.

7.5.1 Vacuum Polarisation

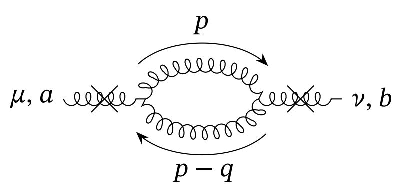
Our aim will be to show (7.88) and calculate the gauge boson self-energy $\Pi(q^2)$. At one-loop order, we have 5 diagrams contributing, all of which are truncated:

Fermion loop: $\mathfrak{M}_F^{ab,\mu\nu} =$



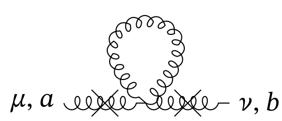
(7.89)

Gauge loops: $\mathfrak{M}_3^{ab,\mu\nu} =$



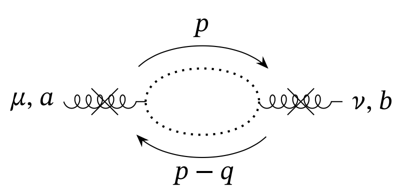
(7.90)

$\mathfrak{M}_4^{ab,\mu\nu} =$



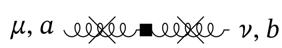
(7.91)

Ghost loop: $\mathfrak{M}_{\text{gh}}^{ab,\mu\nu} =$



(7.92)

Counterterm: $\mathfrak{M}_{ct}^{ab,\mu\nu} =$



(7.93)

The fermion loop (7.89) is as in QED: Using dimensional regularisation, we have

$$\mathfrak{M}_F^{ab\mu\nu} = -\operatorname{Tr}\big(T^a T^b\big)(ig)^2\mu^\epsilon \int đ^d p \, \frac{\operatorname{Tr}[(-i\not{p} + m)\gamma^\mu(-i(\not{p} - \not{q})\gamma^\nu)]}{[(p-q)^2 + m^2][p^2 + m^2]}. \tag{7.94}$$

Now the generators of the fundamental representation of $SU(N)$ obey the trace relation (7.10)

$$\operatorname{Tr}\big(T^a T^b\big) = T_F \delta^{ab} = \frac{1}{2}\delta^{ab}. \tag{7.95}$$

The rest is as in QED.

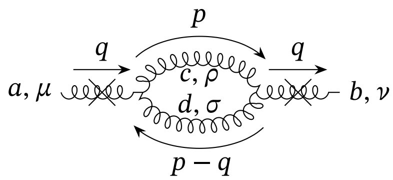
$$\mathfrak{M}_F^{ab\mu\nu} = -\frac{1}{2}\delta^{ab}\left(q^2\delta^{\mu\nu} - q^\mu q^\nu\right)\frac{g^2}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\left[\frac{2}{\epsilon} - \gamma + \ln\!\left(\frac{4\pi\mu^2}{\Delta}\right)\right] \tag{7.96}$$

where the momentum term in parentheses is expected for massless gauge boson and

$$\Delta = m^2 + q^2 x(1-x). \tag{7.97}$$

Let us now consider the pure gauge sector. We first look at the gauge loop (7.90), which is built using the three-gauge-boson vertex 7.84. Working in Feynman gauge $\xi = 1$, this diagram evaluates to

$$\mathfrak{M}_3^{ab\mu\nu} = \tag{7.98}$$



$$= \frac{g^2\mu^\epsilon}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{p^2(p-q)^2} f^{adc}f^{cdb}N^{\mu\nu}, \tag{7.99}$$

where we divided by the symmetry factor 2, and defined

$$\begin{aligned} N^{\mu\nu} = {}& [\delta^{\mu\sigma}(2q+p)^\rho + \delta^{\sigma\rho}(2p-q)^\mu + \delta^{\rho\mu}(-p-q)^\sigma] \\ & \times [\delta_{\rho\sigma}(2p-q)^\nu + \delta_\sigma{}^\nu(2q-p)_\rho + \delta^\nu{}_\rho(-q-p)_\sigma] \end{aligned} \tag{7.100}$$

First we want to isolate the loop momentum p^2 in the denominator using Feynman's trick

$$\frac{1}{p^2(p-q)^2} \propto \int_0^1 \mathrm{d}x\,\frac{1}{(l^2-\Delta)^2}, \tag{7.101}$$

where $l = p + qx$ and $\Delta = -x(1-x)q^2$. We then turn the crank on the numerator.

Remark: A calculation like this would be too long to ask in an exam.

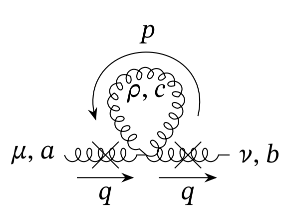
We can also evaluate $f^{acd}f^{bcd} = C_A\delta^{ab}$, where $C_A = C_2(G)$ is the *quadratic Casimir operator* of the adjoint representation. The end result is of the form

$$\mathfrak{M}_3^{ab\mu\nu} = \frac{g^2 C_A}{(4\pi)^{d/2}}\delta^{ab}\int_0^1 \frac{1}{\Delta^{2-d/2}}\left\{\left[\Gamma(1-\frac{d}{2})f(x) + \Gamma(2-\frac{d}{2})g(x)\right]\delta^{\mu\nu}q^2 - \Gamma(2-\frac{d}{2})h(x)q^\mu q^\nu\right\}. \tag{7.102}$$

It is not obvious since the x-dependence is in the factor, but it turns out that we cannot factor out $(q^2\delta^{\mu\nu} - q^\mu q^\nu)$, which is what we would like to see for a massless gauge boson. This is inconsistent with the Ward identity $q_\mu\mathfrak{M}_3^{ab\mu\nu} = 0$. It looks like we have developed a massive mode for the boson. However, we know of course that this is not the case.

It turns out that adding the four-gauge-vertex diagram

$$\mathfrak{M}_4^{ab,\mu\nu} = \tag{7.103}$$

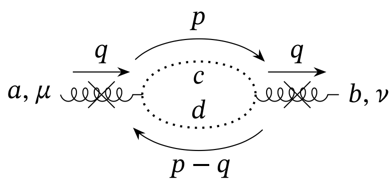


$$= -g^2 C_A\delta^{ab}\delta^{\mu\nu}(d-1)\int\frac{\mathrm{d}^d p}{p^2}$$

also does not help with the desired transverse Lorentz structure. In dimensional regularisation, the integral over p yields zero as $d \to 4$.

The problem is fixed by realising that we simply have not yet added the ghost loop amplitude (7.92):

$$\mathfrak{M}_{\text{gh}}^{ab\mu\nu} = \tag{7.104}$$



$$= -\int đ^d p(-g f^{cad} p^\mu)\frac{1}{p^2}(-g f^{dbc}(p-q)^\nu)\frac{1}{(p-q)^2} \tag{7.105}$$

$$= g^2 C_A \delta^{ab} \int đ^d p\, \frac{p^\mu (p-q)^\nu}{p^2(p-q)^2}. \tag{7.106}$$

There is no symmetry factor in this case, but just as for fermions we obtain a minus sign for the Grassmann loop. Again we use Feynman's trick on the denominator

$$\int \frac{đ^d p}{p^2(p-q)^2} = \int_0^1 \mathrm{d}x\, \frac{đ^d l}{[l^2+\Delta]^2}, \tag{7.107}$$

with $l^\mu = (p - xq)^\mu$ and $\Delta = q^2 x(1-x)$. Then we substitute $p^\mu = (l + xq)^\mu$ in the numerator, discarding any terms linear in l^μ and using $l^\mu l^\nu = \delta^{\mu\nu} l^2/d$, giving

$$p^\mu(p-q)^\nu = \delta^{\mu\nu} l^2/d - x(1-x)q^\mu q^\nu + \text{odd terms}. \tag{7.108}$$

Finally, using the integral identities (B.14), we find

$$\mathfrak{M}_{\text{gh}}^{ab\mu\nu} = \frac{g^2 C_a \delta^{ab}}{(4\pi)^{d/2}} \int_0^1 \frac{\mathrm{d}x}{\Delta^{2-d/2}} \left[\Gamma(1-\frac{d}{2})\frac{1}{2}x(1-x)\delta^{\mu\nu}q^2 - \Gamma(2-\frac{d}{2})x(1-x)q^\mu q^\nu\right]. \tag{7.109}$$

Now we can combine this with the result of (7.102). After some tricks, we have the sum

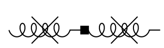
$$\mathfrak{M}_3^{ab\mu\nu} + \mathfrak{M}_{\text{gh}}^{ab\mu\nu} = \frac{g^2 C_a \delta^{ab}}{16\pi^2}(\delta^{\mu\nu}q^2 - q^\mu q^\nu)\left[\frac{5}{3}\left(\frac{2}{\epsilon} + (\ln\frac{\mu^2}{\Delta})\text{-term} + (\text{finite})\right)\right]. \tag{7.110}$$

Remark: This long calculation is not examinable.

Also, rescaling

$$\mathfrak{M}_F^{ab\mu\nu} = \frac{g^2 T_F \delta^{ab}}{16\pi^2}(\delta^{\mu\nu}q^2 - q^\mu q^\nu)\left[-\frac{4}{3}\frac{2}{\epsilon} + \log + \text{finite}\right] \tag{7.111}$$

We need a counterterm to renormalise the divergence (7.110)



$$= \delta_3 = \frac{g^2}{16\pi^2}\frac{2}{\epsilon}\left[\frac{5}{3}C_A - \frac{4}{3}T_F n_f\right]. \tag{7.112}$$

For every different type of fermion in the theory we get essentially the same diagram, which is why we just added the multiplicative factor n_f, which is the number of fermions (flavour).

Renormalised Lagrangian

If we unpack the whole Lagrangian into individual terms, all of which in principle can be renormalised separately, we obtain

$$\begin{aligned}\mathscr{L} = \frac{1}{4}Z_3(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + Z_{3g} g f^{abc}(\partial_\mu A^a_\nu)A^b_\mu A^c_\nu + \frac{1}{4}Z_{4g}g^2 f^{abe}f^{cde}A^a_\mu A^b_\nu A^{\mu,c}A^{\nu,d} \\ + Z_{2'}\bar{c}\partial^2 c - Z_{1'} g f^{abc}(\partial_\mu \bar{c}^a)A^b_\mu c^c \\ + Z_2\bar{\psi}\not{\partial}\psi + Z_m m\bar{\psi}\psi + Z_1 g\bar{\psi}\not{A}^a T^a\psi. \quad (7.113)\end{aligned}$$

The change to the first term is what we have been calculating in this section. The second term is not renormalised. The first line is the pure-gauge part of the action, whereas the second line is the ghost and the third the fermion part.

Claim 6: We can define some sort of effective coupling q^2_{eff} that is equal to the combination

$$q^2_{\text{eff}} = \frac{Z_1^2}{Z_2^2 Z_3}g^2\mu^\epsilon = \frac{Z_{1'}^2}{Z_{2'}^2 Z_3}g^2\mu^\epsilon = \frac{Z_{3g}^2}{Z_3^3}g^2\mu^\epsilon = \frac{Z_{4g}}{Z_3^2}g^2\mu^\epsilon. \tag{7.114}$$

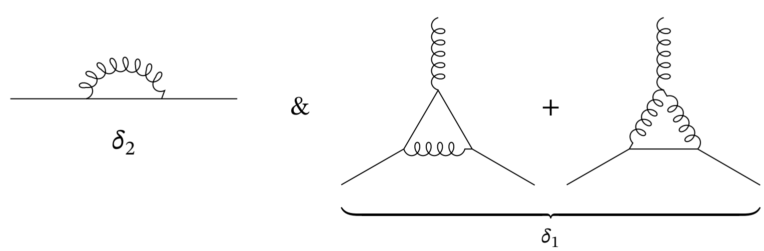
In QED, we used gauge invariance to write down the Ward–Takahashi identity. Here we do not have gauge invariance, but we have another kind of symmetry, called *BRST symmetry*, which will help us.

7.5.2 β-Functions and Asymptotic Freedom

We obtain the β-functions from requiring

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g_{\text{eff}} = 0. \tag{7.115}$$

We have $Z_3 = (1 + \delta_3)$. Need δ_2 and δ_1 from the diagrams



(7.116)

To write

$$\beta(g) = -\frac{\epsilon}{2}g - g\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\frac{Z_1}{Z_2 Z_3^{1/2}}\right) \tag{7.117}$$

$$= g\left[-\frac{\epsilon}{2} - \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\delta_1 - \delta_2 - \frac{1}{2}\delta_3\right)\right]. \tag{7.118}$$

where we use L.O. β-function to write

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g = -\frac{\epsilon}{2}d. \tag{7.119}$$

To get

$$\beta(g) = -\frac{\epsilon}{2} - \frac{g^3}{16\pi^2}\left(\frac{11}{3}c_A - \frac{4}{3}n_f T_F\right). \tag{7.120}$$

For $SU(3)$ (and fundamental fermions)

$$C_A = N = 3, \qquad T_F = \frac{1}{2}, \qquad d = 4. \tag{7.121}$$

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(11 - \frac{2}{3}n_f\right) := -\frac{g^3}{16\pi^2}\beta_0. \tag{7.122}$$

Now g is marginally relevant (for $n_f < 16$). So $g \to \infty$ in the IR. Conversely, $g \to 0$ at high energies. This phenomenon is called *asymptotic freedom*. Let Λ_{QCD} be the *low energy* scale where $g \to \infty$ (at this order)

$$g^2(\mu) = \frac{1}{\beta_0 \ln \frac{\mu^2}{\Lambda_{QCD}^2}}. \tag{7.123}$$

This asymptotic freedom is why we use non-Abelian gauge theories to describe the strong interactions.

7.6 BRST Symmetry [Becchi–Rouet–Stora–Tyutin]

The idea is that we have got the gauge-fixing term $\frac{1}{2\xi}(\partial_\mu A^\mu)^2$. Since it fixes the gauge, it breaks gauge invariance. But there is still a remnant global symmetry.

7.6.1 Abelian Case

It is easiest if we strip away the non-Abelian details for a moment and look at QED ($U(1)$ gauge theory). The gauge-fixed Lagrangian is

$$\mathscr{L} = \frac{1}{4}(F^a)^2 + \overline{\psi}(\not{D} + m)\psi + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \overline{c}\partial^2 c. \tag{7.124}$$

Under the infinitesimal gauge transformation

$$\psi \to \psi + i\alpha\psi \quad \text{or} \quad \delta\psi = i\alpha\psi \tag{7.125a}$$

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha \quad \text{or} \quad \delta A_\mu = \frac{1}{e}\partial_\mu\alpha, \tag{7.125b}$$

where $\alpha = \alpha(x)$. The term

$$(\partial_\mu A^\mu)^2 \to (\partial^\mu A_\mu + \frac{1}{e}\partial^2\alpha)^2 \tag{7.126}$$

is only invariant if $\partial^2\alpha(x) = 0$.

Remark: The $\bar{c}\partial^2 c$ term in $\mathscr{L}$ leads to equations of motion $\partial^2 c = 0 = \partial^2 \bar{c}$.

Consider restricting our gauge transformations to $\alpha(x) = \theta c(x)$, where θ is some x-independent Grassmann number. So if we then also transform

$$\bar{c}(x) \to \bar{c}(x) - \frac{\theta}{e\xi}\partial^\mu A_\mu(x), \quad \text{or} \quad \delta\bar{c} = -\frac{\theta}{e\xi}\partial^\mu A_\mu(x) \tag{7.127}$$

along with the previous ones (7.125), then $\mathscr{L}$ is invariant. These are the *BRST transformations of QED*.

7.6.2 Non-Abelian Case

$$\mathscr{L} = \frac{1}{4}(F^a)^2 + \bar{\psi}(\not{D} + m)\psi + \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 - \bar{c}\partial^\mu D_\mu c. \tag{7.128}$$

We now have to account for the fact that the derivative $D_\mu c^a = \partial_\mu c^a + g f^{abc} A^b_\mu c^c$. Let $\alpha^a(x) = \theta c^a(x)$. The BRST transformations are

$$\delta\psi_i = i\theta c^a T^a_{ij}\psi_j \tag{7.129}$$

$$\delta A^a_\mu = \frac{\theta}{g} D^{ab}_\mu c^b \tag{7.130}$$

$$\delta\bar{c} = -\frac{\theta}{g\xi}\partial^\mu A^a_\mu. \tag{7.131}$$

The A_μ in the covariant derivative contributes a term

$$D_\mu c^a \to D_\mu c^a - \theta f^{abc}(D_\mu c^b)c^c. \tag{7.132}$$

To cancel this, we need a transformation

$$\delta c^c = -\frac{\theta}{2} f^{abc} c^b c^c \tag{7.133}$$

to have $\mathscr{L}$ invariant. Thus we have a remnant global symmetry (BRST).

Sometimes it is convenient to introduce an auxiliary (non-dynamical) scalar field $B^a(x)$ (Nakanishi–Lautrup).

$$\mathscr{L} = \frac{1}{4}(F^a)^2 + \bar{\psi}(\not{D} + m)\psi - \frac{\xi}{2}(B^a)^2 + B^a\partial^\mu A^a_\mu - \bar{c}\partial^\mu D_\mu c. \tag{7.134}$$

We recover the original $\mathscr{L}$ by completing the square and integrating over $\mathcal{D}B$. Now

$$\delta\bar{c}^a = \theta B^a. \tag{7.135}$$

$$\delta B^a = 0. \tag{7.136}$$

7.6.3 BRST Cohomology

You can show that two successive transformations leave the fields invariant. That is, let $Q = Q_{\text{BRST}}$ be an operator which gives the transformed fields:

$$\psi \to \psi + \theta Q\psi \tag{7.137}$$

$$A_\mu \to A_\mu + \theta QA, \tag{7.138}$$

with $Q^2 = 0$ for all fields. In other words, the BRST operator is *nilpotent*.

> The following is non-examinable, proofs are out of scope for this course.

A nilpotent operator Q divides the Hilbert space $\mathcal{H}$ into subspaces:

closed states: those states annihilated by Q

$$Q\left|\Psi\right\rangle = 0 \;\Rightarrow\; \left|\Psi\right\rangle \in \mathcal{H}_{\text{closed}} \tag{7.139}$$

exact states: states in the image of Q

$$\exists \left|\Phi\right\rangle \in \mathcal{H} \text{ s.t. } \left|\Psi\right\rangle = Q\left|\Phi\right\rangle \;\Rightarrow\; \left|\Psi\right\rangle \in \mathcal{H}_{\text{exact}} \tag{7.140}$$

Note that $\mathcal{H}_{\text{exact}} \subset \mathcal{H}_{\text{closed}}$: If $\left|\Psi\right\rangle \in \mathcal{H}_{\text{exact}}$, then $\exists \left|\Phi\right\rangle \in \mathcal{H}$ such that

$$Q\left|\Psi\right\rangle = Q^2\left|\Phi\right\rangle \overset{Q^2=0}{=} 0 \;\Rightarrow\; \left|\Psi\right\rangle \in \mathcal{H}_{\text{closed}}. \tag{7.141}$$

Physical states are those in the quotient (or moduli) space

$$\mathcal{H}_{\text{phys}} := \frac{\mathcal{H}_{\text{closed}}}{\mathcal{H}_{\text{exact}}}. \tag{7.142}$$

This is called *BRST cohomology*.

It turns out that only the 2 transverse polarisations belong to $\mathcal{H}_{\text{phys}}$.

Appendices

A Variational Calculus

A.1 Variations of the Lagrangian

Let us make a bit more precise what we mean when we talk about the variations of the functional $\mathscr{L}$ with respect to the function ϕ. Say we have a small variation $\phi \to \phi + \delta\phi$. Then, for a Lagrangian $\mathscr{L}(x) = \mathscr{L}[\phi(x), \partial\phi(x)]$, we have the variation

$$\delta\mathscr{L}(x) = \left.\frac{\partial\mathscr{L}}{\partial\phi}\right|_x \delta\phi(x) + \left.\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right|_x \partial_\mu(\delta\phi(x)). \tag{A.1}$$

From this we write

$$\frac{\delta\mathscr{L}(x)}{\delta\phi(y)} = \left.\frac{\delta\mathscr{L}}{\delta\phi}\right|_x \delta^{(4)}(y-x) + \left.\frac{\delta\mathscr{L}}{\delta(\partial_\mu\phi)}\right|_x \partial_\mu\delta^{(4)}(y-x). \tag{A.2}$$

A.2 Variations of the Action

The action is given by the integral $S[\phi] = \int \mathrm{d}^4x\, \mathscr{L}[\phi(x), \partial_\mu\phi(x)]$. We say that $S[\phi]$ is a functional, since it depends only on the function $\phi(x)$, not on the position x itself. The functional derivative of S with respect to $\phi(y)$ is defined by the associated variation (A.1) of the function $\mathscr{L}$ under the integral:

$$\begin{aligned}
\frac{\delta S[\phi]}{\delta\phi(y)} &= \int \mathrm{d}^4x\, \frac{\delta\mathscr{L}(x)}{\delta\phi(y)} && \text{(A.3)}\\
&= \int \mathrm{d}^4x \left[\left.\frac{\partial\mathscr{L}}{\partial\phi}\right|_x \delta^{(4)}(x-y) + \left.\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right|_x \partial_\mu\delta^{(4)}(x-y) \right] && \text{(A.4)}\\
&= \left.\frac{\partial\mathscr{L}}{\delta\phi}\right|_y - \partial_\mu \left.\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right)\right|_y, && \text{(A.5)}
\end{aligned}$$

where we integrated by parts in the second line to deal with the derivative of the delta function.

Remark: The functional derivative is again a function of the coordinate y.

This is familiar from classical field theory, where the equations of motion are such as to minimise the action S. This requires the left-hand side to vanish, giving the Euler–Lagrange equations. In quantum field theory the action is not exactly minimised, and the Schwinger–Dyson equations allow us to quantify this.

A.3 Relation to Schwinger–Dyson

Let us now insert (A.5) back into (A.1). We obtain

$$\delta\mathscr{L} = \left[\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right) + \frac{\delta S}{\delta\phi}\right]\delta\phi + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) \tag{A.6}$$

$$= \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) + \frac{\delta S}{\delta\phi}\delta\phi \tag{A.7}$$

$$= \partial_\mu j^\mu + \frac{\delta S}{\delta\phi}\delta\phi, \tag{A.8}$$

where we identified the Noether current

$$j^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\phi. \tag{A.9}$$

We can then rearrange (A.7) to obtain the functional derivative of S in terms of the functional derivative of $\mathscr{L}$:

$$\frac{\delta S}{\delta\phi} = \frac{\delta\mathscr{L}}{\delta\phi} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right). \tag{A.10}$$

Moreover, from (A.8) we can see that if $\delta\mathscr{L} = 0$ under a transformation $\phi \to \phi + \delta\phi$, then

$$\frac{\delta S}{\delta\phi}\delta\phi = -\partial_\mu j^\mu. \tag{A.11}$$

B Identities for Loop Integrals and Dimensional Regularisation

To evaluate loop integrals, we often deal with exactly the same tricks. We list them here for convenience. We have added useful identities from [Schw, Apx. B] to the ones given in class.

B.1 Euler's Gamma and Beta Functions

Euler-Γ function: Use, for $\alpha > 0$, the analytic continuation of the factorial

$$\Gamma(\alpha) = \int_0^\infty \mathrm{d}x\, x^{\alpha-1} e^{-x}, \qquad \alpha\Gamma(\alpha) = \Gamma(\alpha+1), \qquad \Gamma(1) = 1, \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{B.1}$$

This has the series expansion

$$\ln\Gamma(\alpha+1) = -\gamma\alpha - \sum_{k=2}^{\infty}(-1)^k \frac{1}{k}\zeta(k), \tag{B.2}$$

where $\gamma = \gamma_E \approx 0.577216$ is the Euler–Mascheroni constant and $\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$ is the Riemann ζ-function. Usually we exponentiate this

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon). \tag{B.3}$$

In loop calculations, the following term often appears

$$\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \ln\Delta - \gamma + \ln(4\pi) + O(\epsilon)\right), \qquad \text{[P\&S, Eq. (A.52)]} \tag{B.4}$$

with $\epsilon = 4 - d$.

Euler-beta function:

$$B(s,t) = \int_0^1 \mathrm{d}x\, u^{s-1}(1-u)^{t-1} = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}. \tag{B.5}$$

Surface area S_d of a unit d-sphere: For integer dimension $d \in \mathbb{N}$, we can do d Gaussian integrals and convert to polar coordinates

$$(\sqrt{\pi})^d = \int_{\mathbb{R}^d} \prod_{i=1}^{d} \mathrm{d}x_i \, e^{-x_i^2} = S_d \int_0^\infty \mathrm{d}r \, r^{d-1} e^{-r^2} = \frac{1}{2} S_d \, \Gamma(\frac{d}{2}). \tag{B.6}$$

For non-integer $d \in \mathbb{C}$, we define S_d via analytic continuation as

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{B.7}$$

Remark: You will not be asked to prove these or even have these at hand in the exam.

B.2 Denominator: Feynman Parameters

Feynman parameters are a set of mathematical identities:

$$\frac{1}{AB} = \int_0^1 \frac{\mathrm{d}x}{[A + (B - A)x]^2} = \int_0^1 \mathrm{d}x \, \mathrm{d}y \, \frac{\delta(x + y - 1)}{[xA + yB]^2} \tag{B.8}$$

$$\frac{1}{ABC} = \int_0^1 \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, \delta(x + y + z - 1) \frac{2}{[xA + yB + zC]^3}. \tag{B.9}$$

We often use these to complete the square in the denominator of the loop integrals. For example

$$\int \frac{đ^4 k}{k^2 (k - p)^2} = \int đ^4 k \int_0^1 \frac{\mathrm{d}x}{[k^2 + [(k - p)^2 - k^2]x]^2} \tag{B.10}$$

$$= \int_0^1 \mathrm{d}x \, \frac{đ^4 k}{[(k - xp)^2 + \Delta]^2}, \tag{B.11}$$

where $\Delta = p^2 x(1 - x)$. We can then shift $k \to k + xp$, leaving an integral that only depends on k^2.

B.3 Scalar Integrals

Once we have manipulated the loop integrals so that they are only functions of the magnitude of the momentum, we use

$$\int \mathrm{d}^d k = S_d \int k^{d-1} \, \mathrm{d}k \,, \tag{B.12}$$

where S_d is given in (B.7). The resulting integrals over the (as usual Euclidean) momenta are evaluated as

$$\int \mathrm{d}k \, \frac{k^a}{(k^2 + \Delta)^b} = \Delta^{\frac{a+1}{2} - b} \frac{\Gamma\left(\frac{a+1}{2}\right) \Gamma\left(b - \frac{a+1}{2}\right)}{2\Gamma(b)}. \tag{B.13}$$

In particular, the most useful loop integrals are, to order $\epsilon = 4 - d$:

$$\int đ^d l \, \frac{1}{(l^2 + \Delta)^2} \sim \frac{1}{(4\pi)^2} \left[\frac{2}{\epsilon} - \ln \Delta + ... \right] \tag{B.14a}$$

$$\int đ^d l \, \frac{l^2}{(l^2 + \Delta)^2} \sim \frac{1}{(4\pi)^2} \left[\frac{2}{\epsilon} - \ln \Delta + ... \right] \times (-2\Delta). \tag{B.14b}$$

B.4 Clifford Algebra

In $d = 4 - \epsilon$, the contraction identities for the γ-matrices are [P&S, Eq. (7.89)]

$$\gamma^\mu \gamma^\nu \gamma_\mu = -(2 - \epsilon)\gamma^\nu \tag{B.15}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\delta^{\nu\rho} - \epsilon \gamma^\nu \gamma^\rho \tag{B.16}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu + \epsilon \gamma^\nu \gamma^\rho \gamma^\sigma. \tag{B.17}$$

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