Applications of Differential Geometry to Physics

Part III Lent 2019

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Contents

0.1 Kepler / Newton Orbits

$$\ddot{\mathbf{r}} = -\frac{GMv}{r^3}\mathbf{r} \quad \leftrightarrow \quad \text{conic sections} \tag{1}$$

General conic section is

$$ax^2 + by^2 + cxy + dx + ey + f = 0 \tag{2}$$

This is nowadays more generally studied in what we now call *algebraic geometry* rather than differential geometry.

Apolonius of Penge (?) asked 'what is the unique conic thorugh five points, no three of which are co-linear?'

The space of conics is $\mathbb{R}^6 - \{0\}/n = \mathbb{RP}^5$ (projective 5-space).

$$[a, b, c, d, e, f] \sim [\gamma a, \gamma b, \gamma c, \gamma d, \gamma c, \gamma f], \gamma \in \mathbb{R}^* \tag{3}$$

This is an application of geometry, rather than an application of differential geometry.

Remark: Apolonius proved this geometrically.

In this course however, we will look at the following.

1) Hamiltonian mechanics (mid 19$^{\text{th}}$). This is an elegant way of reformulating Newton's mechanics, turning second order differential equations into first order differential equations with the use of a function $H(p, q)$. The system of ODEs is

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{q} = -\frac{\partial H}{\partial q} \tag{4}$$

This led to the development of symplectic geometry (1960s). The connection is that the phase-space to which p and q belong has a 2 -form $dp \wedge dq$. Using the Hamiltonian function, one can find a vector field

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} \tag{5}$$

and looks for a one-parameter group of transformations, called symplectomorphisms, generated by this vector field. Under these symplectomorphisms, the 2 -form is unchanged meaning that the area illustrated in F2 is preserved. Details of this are going to come within the course.

2) General Relativity (1915) ← Riemannian Geometry (1850)

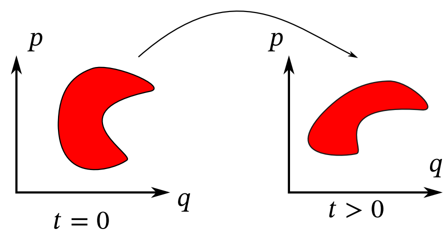

Figure 1

3) Gauge theory (Maxwell, Yang Mills) $\leftrightarrow$ Connection on Principal Bundle (U(1) (Maxwell), SU(2), SU(3))

$$A_+ = A_- + dg \qquad g = \psi_+ - \psi_- \qquad \omega = \begin{cases} A_+ + d\psi_+ \\ A_- + d\psi_- \end{cases} \tag{6}$$

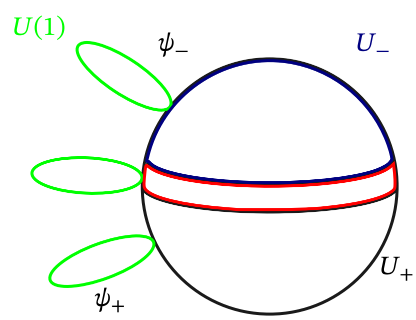

Figure 2

This course: cover 1, 2, 3 in some detail. Unifying feature: Lie groups.

- Prove some theorems, *lots of* examples (often instead of proofs)
- Want to be able to do calculations; compute characteristic classes etc.

We will assume that you took either Part III General Relativity, or Part III Differential Geometry, or some equivalent course.

1 Manifolds

Definition 1 (manifold)**:** An n -dimensional *smooth manifold* is a set M and a collection[2] of open sets U_α , labelled by $\alpha = 1, 2, 3, \ldots$, called *charts* such that

- U_α cover M
- $\exists$ 1-1 maps $\phi_\alpha : U_\alpha \to V_\alpha \in \mathbb{R}^n$ such that

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{1.1}$$

  is a smooth map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

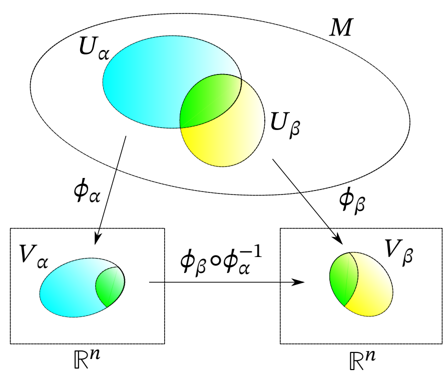

Figure 1.1: Manifold

As such, manifolds are topological spaces with additional structure, allowing us to do calculus.

Example ($M = \mathbb{R}^n$)**:** There is the *trivial manifold*, which can be covered by only one open set. There are other possibilities. In fact, there are infinitely many smooth structures on $\mathbb{R}^4$ (Proof by Donaldson in 1984 in his PhD. He used Gauge theory).

[2] In all examples that we will look at, there will be finitely α.

Example (sphere $S^n = \{\mathbf{r} \in \mathbb{R}^{n+1}, |\mathbf{r}| = 1\}$ **):** Have two open sets

$$U = S^n / \{0, 0, 0, , \dots, 0, 1\} \qquad \widetilde{U} = S^n / \{0, 0, 0, , \dots, 0, -1\} \tag{1.2}$$

We then define charts, where $\mathbb{R}^n = (x_1, \dots, x_n)$:

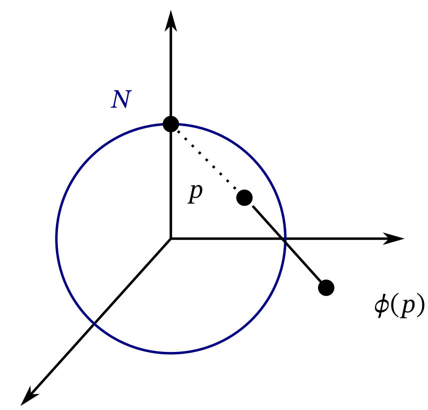

Figure 1.2

$$\begin{aligned} \phi(r_1, \dots, r_{n+1}) &= \left(\frac{r_1}{1 - r_{n+1}}, \dots, \frac{r_n}{1 - r_{n+1}} \right) \\ \text{on } \widetilde{U}, \qquad \widetilde{\phi}(r_1, \dots, r_{n+1}) &= \left(\frac{r_1}{1 - r_{n+1}}, \dots, \frac{r_n}{1 - r_{n+1}} \right) = (\widetilde{x}_1, \dots, \widetilde{x}_n). \end{aligned} \tag{1.3}$$

On $U \cap \widetilde{U}$,

$$\frac{r_k}{1 + r_{n+1}} = \frac{1 - r_{n+1}}{1 + r_{n+1}} \frac{r_k}{1 - r_{n+1}}, \qquad k = 1, \dots, n \tag{1.4}$$

$$\frac{1 - r_{n+1}}{1 + r_{n+1}} = \frac{(1 - r_{n+1})^2}{r_1^2 + r_2^2 + \cdots + r_n^2} = \frac{1}{x_1^2 + x_2^2 + \cdots + x_n^2} \tag{1.5}$$

So on $\phi(U \cap \widetilde{U})$,

$$(\widetilde{x}_1, \dots, \widetilde{x}_n) = \left(\frac{x_1}{x_1^2 + \cdots + x_n^2}, \dots, \frac{x_n}{x_1^2 + \cdots + x_n^2} \right) \tag{1.6}$$

are smooth maps from $\mathbb{R}^n \to \mathbb{R}^n$

Example: A Cartesian product of manifolds is a manifold, for example we have the n-torus $T^n = S^1 \times S^1 \times \cdots \times S^1$.

Definition 2 (surface)**:** Let $f_1, \dots, f_k : \mathbb{R}^N \to \mathbb{R}$ be smooth functions. A surface $f_1 = 0, \dots, f_k = 0$ is a manifold of dimension $\dim n = N - k$ if the rank of the matrix $\frac{\partial f_\alpha}{\partial x^i}$, $\alpha = 1, \dots, k$ and $i = 1, \dots, N$ is maximal and equal to k at all points of $\mathbb{R}^N$..

Example: The n-sphere S^n is a surface in $\mathbb{R}^{n+1}$ with $f_1 = 1 - |\mathbf{r}|^2$.

Theorem 1 (Whitney)**:** Every smooth manifold of dimension n is an embedded surface in $\mathbb{R}^N$, where $N \leq 2n$.

> If you enjoy using geometrical intuition and looking at surfaces, this theorem ensures that you can always do that and not loose generality.

Definition 3 (real projective space)**:** The n-dimensional *real projective space* is defined as the quotient

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim, \tag{1.7}$$

where we quotient out the equivalence classes $[X_1, \dots, X_n + 1] \sim [cX_1, \dots, cX_{n+1}]$ for all $c \in \mathbb{R}^*$. The $[X_1, \dots, X_{n+1}]$ are called *homogeneous coordinates.*

In other words, this is the space of all lines through the origin in $\mathbb{R}^{n+1}$.

Claim 1: $\mathbb{RP}^n$ is a smooth manifold of dimension n with $(n+1)$ open sets.

Proof. Let us define our open sets with respect to the homogeneous coordinates. We define the set $U_\alpha : [X] \in \mathbb{RP}^n$ such that $X_\alpha \neq 0 \ \alpha = 1, \dots, n+1$. We can now find local coordinates on $\phi_\alpha : U_\alpha \to V_\alpha \in \mathbb{R}^n$

$$x_1 = \frac{X_1}{X_\alpha} \quad \dots \quad x_{\alpha-1} = \frac{X_{\alpha-1}}{X_\alpha} \quad x_{\alpha+1} = \frac{X_{\alpha+1}}{X_\alpha} \quad \dots \quad x_n = \frac{X_n}{X_\alpha}. \tag{1.8}$$

$\square$

Exercise 1.1: Prove smoothness of $\phi_\beta \circ \phi_\alpha^{-1}$.

Now it turns out that this manifold is equivalent to $\mathbb{RP}^n = S^n / \mathbb{Z}^2$. From quantum mechanics, we know that this means in particular $\mathbb{RP}^3 = SO(3)$. This is illustrated in 1.3.

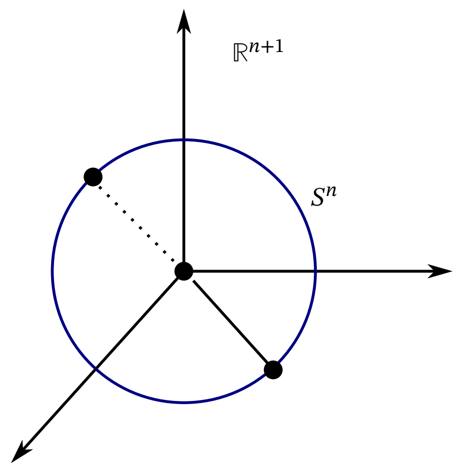

Figure 1.3: Real projective space $\mathbb{RP}^n$ is isomorphic to $S^n/\mathbb{Z}^n$, identifying antipodal points.

2 Vector Fields

Let $M, \widetilde{M}$ be smooth manifolds of dimension $n, \tilde{n}$.

Definition 4 (smooth map)**:** A map $f : M \to \widetilde{M}$ is *smooth* if $\widetilde{\phi}_\beta \circ f \circ \phi_\alpha^{-1}$ is a smooth map from $\mathbb{R}^n$ to $\widetilde{\mathbb{R}}^n$ for all α, β. We call $f : M \to \mathbb{R}$ a *function*, whereas we call $f : \mathbb{R} \to M$ a *curve*.

Let $\gamma : \mathbb{R} \to M$ be a curve. For some $U \in M, U \simeq \mathbb{R}^n$, we can define local coordinates $(x^1, \dots, x^n)$

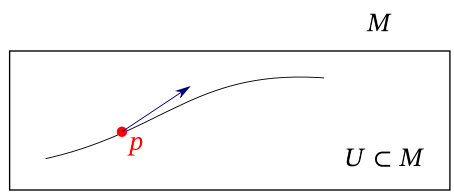

Figure 2.1

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Definition 5 (tangent vector)**:** A *tangent vector* V to γ at p is

$$V|_p = \left.\frac{\mathrm{d}\psi}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \in T_pM, \tag{2.1}$$

where T_pM is the *tangent space* to M at p.

Definition 6 (tangent bundle)**:** We define the *tangent bundle* as $TM := \bigcup_{p \in M} T_pM$.

Definition 7 (vector field)**:** A *vector field* assigns a tangent vector to all $p \in M$.

Let $f : M \to \mathbb{R}$. The rate of change of f along γ is

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} f(x^a(\epsilon))|_{\epsilon=0} = \sum_a \dot{x}^a \frac{\partial f}{\partial x^a} \tag{2.2}$$

$$= \sum_a V^a \left.\frac{\partial f}{\partial x^a}\right|_{\epsilon=0}, \tag{2.3}$$

whee $V^a := \dot{x}^a|_{\epsilon=0,\dots,x_n}$.

Vector fields are first order differential operators

$$V = \sum_a V^a(\mathbf{x}) \frac{\partial}{\partial x^a}. \tag{2.4}$$

The derivatives $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}\Big|_p$ form a basis of T_pM.

2.1 Integral curves

Definition 8 (integral curve)**:** An *integral curve* (a *flow*) of a vector field is defined by

$$\dot{\gamma}(\epsilon) = V|_{\gamma(\epsilon)}, \tag{2.5}$$

where the dot denotes differentiation with respect to ϵ.

On n first order ODEs: $\dot{x}^a = V^a(x)$.

There exists a unique solution given initial data $X^a(0)$. Given a solution $X^a(\epsilon)$, we can expand it in a Taylor series as

$$X^a(\epsilon) = X^a(0) + V^a \cdot \epsilon + O(\epsilon^2). \tag{2.6}$$

Up to first order in ϵ, the vector field determines the flow. We call V a *generator* of its flow.

The following example illustrates how you get from a vector field to its flow.

Example ($M = \mathbb{R}^2$, $x^a = (x, y)$)**:** Consider the vector field $V = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. The system of ODEs we solve is $\dot{x} = x$ and $\dot{y} = 1$. This gives us the integral curve $(x(\epsilon), y(\epsilon)) = (x(0)e^\epsilon, y(0) + \epsilon)$. From this we can see that $x(\epsilon) \cdot \exp(-y(\epsilon))$ is constant along γ. Using this we can draw the unparametrised integral curve in Fig. 2.2.

This example motivates the following definition.

Definition 9 (invariant)**:** An *invariant* of a vector field V is a function f constant along the flow of V.

$$f(x^a(0)) = f(x^a(\epsilon)) \qquad \forall \epsilon. \tag{2.7}$$

Equivalently, $V(f) = 0$.

Let us now consider an example that goes the other way: from flow to vector field.

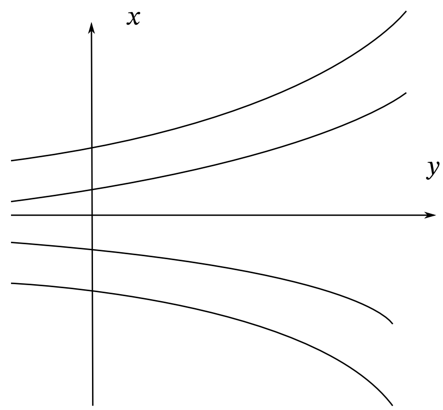

Figure 2.2

Example: Consider the 1 -parameter group of rotations of a plane.

$$(x(\epsilon), y(\epsilon)) = (x_0 \cos\epsilon - y_0 \sin\epsilon, x_0 \sin\epsilon + y_0 \cos\epsilon). \tag{2.8}$$

The associated vector field is

$$V = \left(\frac{\partial y(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial y} + \frac{\partial x(\epsilon)}{\partial \epsilon} \frac{\partial}{\partial x} \right)\bigg|_{\epsilon=0} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial y}. \tag{2.9}$$

Now you can add vector fields, but there is also another operation.

Definition 10 (Lie bracket)**:** A *Lie bracket* $[V, W]$ of two vector fields V, W is a vector field defined by

$$[V, W](f) = V(W(f)) - W(V(f)) \qquad \forall f. \tag{2.10}$$

This is indeed another vector field since the commutator of two first order operators is another first order operator.

Example: Let $V = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $W = \frac{\partial}{\partial x}$. We then have $[V, W] = -W$.

This is not always the case but sometimes the Lie bracket reproduces some of the vector fields. There is an interesting algebraic structure to this.

Definition 11 (Lie algebra)**:** A *Lie algebra* is a vector space $\mathfrak{g}$ with an anti-symmetric, bilinear operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called a *Lie bracket*, which satisfies the *Jacobi identity*

$$[V, [U, W]] + [W, [V, U]] + [U, [W, V]] = 0 \qquad \forall U, V, W \in \mathfrak{g}. \tag{2.11}$$