The Standard Model

Part III Lent 2020

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1 Introduction and History

Prerequisites

It is necessary to have attended the *Quantum Field Theory* and the *Symmetries, Fields, and Particles* courses, or to be familiar with the material covered in them. It would be advantageous to attend the *Advanced Quantum Field Theory* course during the same term as this course, or to study renormalisation and non-abelian gauge fixing.

1.1 Introduction

Definition 1 (standard model)**:** A theoretical physics construction (theory, model) that describes all known elementary particles and their interactions based on relativistic quantum field theory (QFT).

The Standard Model of particle physics is the most successful application of QFT we currently have. Based on the gauge group $SU(3)\times SU(2)\times U(1)$, it accurately describes, at the time of writing, all experimental measurements involving strong, weak, and electromagnetic interactions.

Ingredients

(i) spacetime: $3+1$ dimensional Minkowski space

symmetry: Poincaré group

(ii) particles:

spin $s=0$ Higgs

spin $s=1/2$ three families of quarks and leptons

(iii) interactions:

$s = 1$ three gauge interactions

$s = 1$ gravity[1]

Gauge (local) symmetry: $SU(3)_C \times SU(2)_L \times U(1)_Y \xrightarrow[Breaking]{Symmetry} SU(3)_C \times U(1)_{EM}$

C color: strong

L left: electroweak

Y hypercharge

These are related via $Q = T_3 + Y$.

Particle representations[2]:

- Quarks and Leptons: $\overbrace{3}^{\text{families (flavour)}}[\underbrace{(3,2;\frac{1}{6})}_{Q_L} + \underbrace{(\bar{3},1;-\frac{2}{3})}_{U_R} + \underbrace{(\bar{3},1;\frac{1}{3})}_{d_R} + \underbrace{(1,2;-\frac{1}{2})}_{L_L} + \underbrace{(1,1;1)}_{e_R} + \underbrace{(1,1;0)}_{\nu_R}]$
- Higgs: $(1,2;-\frac{1}{2})$
- Gauge: $\underbrace{(8,1;0)}_{gluons} + \underbrace{(1,3;0)}_{W^\pm,\mathbb{Z}} + \underbrace{(1,1;0)}_{\gamma}$

Comments

- interactions given by QFT
- main tool: symmetry
- total symmetry: spacetime $\otimes$ internal (gauge)[3]
- also accidental (global) symmetries $\sim$ baryon + lepton number
- plus approximate (flavour) symmetries:
- very rigid: $\sum Y = \sum Y^3 = 0$[4], $\#3 = \#\bar{3}$, $\#2$ even
- rich structure (3 phases: Coulomb, Higgs, confining)

[1] as important as it is, we will not be concerned with gravity for most of this course
[2] numbers tell us representations under $(C, L; Y)$
[3] Theorem: cannot mix these two symmetries. Supersymmetry provides a way around this.
[4] gravitational anomaly

Motivation

Why to learn about the SM?

- It is fundamental.
- It is based on elegant principles of symmetry.
- It is true!
  - outstanding predictions: ($\mathbb{Z}^0, W^{\pm}$, Higgs, ...)
  - precision tests:
    anomalous magnetic dipole moment of the electron:
    $$a = \frac{g-2}{2} = (1159.65218091 \pm 0.00000026) \times 10^{-6} \tag{1.1}$$
    fine structure constant (at $E \ll 10^3$GeV):
    $$\alpha^{-1} = \frac{\hbar c}{e^2} = 137.035999084(21) \tag{1.2}$$
- It is the best test of QFT.
- It is incomplete!

Take another look at the quarks and leptons. For $Q_L : (3, 2, \frac{1}{6})$ the second entry tells us that these are doublets under $SU(2)$. This means that we have $Q_L = \begin{pmatrix} v_L \\ d_L \end{pmatrix}$.

1.2 History

Weinberg has a good paper on advice he gives to young researchers. One of the advices is to study history and the history of physics in particular. This is because it gives us some sense of how physical theories developed and also that it makes us feel part of a bigger development in the pursuit of knowledge, no matter how small our contributions may seem.

t < 20$^{\text{th}}$ century: only two interactions (gravity and electromagnetism)

discreteness of matter not established

1896 Radioactivity (Becquerel, Pierre and Marie Curie, Rutherford) α, β, γ rays

This was a big discovery at the time; there is no inherent stability in nature! This was a manifestation that pointed to the existence of other interactions.

1897 J. J. Thomson: discovered the *electron* (e^-) and measured e/m a few hundred meters from where we are right now in Cambridge (close to The Eagle so you can enjoy that on your visit too). This was the first particle ever discovered, marking the beginning of particle physics.

1900's **1900-1930** Quantum mechanics developed. Probably the biggest revolution ever in science.

1905 Special relativity. These two still are the two basic theories to study in nature. The nature of quantum mechanics also implies that light behaves as a particle, which we now know as the photon.

In the same decade, Rutherford's group also discovered the atom.

1910's Francis Aston (1919) defined the 'whole number rule', for the ratio of different atomic nuclei to the hydrogen mass. This led to the discovery of the proton.

Cosmic rays were studied, in particular by using cloud chambers.

Einstein's theory of general relativity.

1920's Bose, Fermi statistics.

Beginning of QFT (Jordan, Heisenberg, Dirac, ...)

Dirac equation. This predicted a positive particle, which he thought could be the proton ...

1930's ...but then came to predict that this is the *positron* (e^+), which he just squeezed into the introduction of his paper on magnetic monopoles (1931).

1932 Anderson[1] discovered it.

1932 Chadwick discovered the *neutron*.

1930 Pauli predicted the *neutrino*: β-decay: $n \to p + e^- + \overline{\nu}$. Fermi described this in terms of a four-point field theory, illustrated in Fig. 1.1.

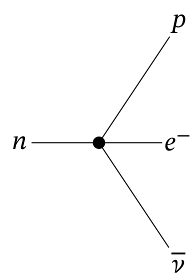


Figure 1.1: Four-point description of β-decay.

1934 Yukawa theory of strong interactions: scalar mediators of strong interactions (Pions). Potential $V(r) \sim e^{-mr}/r$, where $m \sim 100\text{MeV}$ is the pion mass. This explained the short range and kept field theory going, so people started to search for this new particle.

1936 Anderson discovered the *muon* ($m \sim 100\text{MeV}$).

1932 Heisenberg, 1936 (Condon et al.) introduced isospin $n \leftrightarrow p$. He thought that in the same way that the electron has spin-up and spin-down, the proton and neutron have such similar properties that they also have an internal symmetry.

1940's The history of physics is different to the history you learn at school; at much happens in physics during war-time.

1947 Lamb shift, QED (Schwinger, Feynman + Tomonaga + Dyson)

Pions π were discovered, explaining why the naive picture of Yukawa made sense.

1950's
- A time of great optimism. Particle accelerators and bubble chambers were built (E > MeV). People say that the 50's were a decade of wealth; the numbers of particles were also very rich. Dozens of new particles were discovered (kaons, hyperons, ...), mostly strongly interacting (*hadrons*), which are now classified into mesons (bosons) and baryons (fermions).
- Strangeness (Gell-Mann, Nishijima, Pais)
- Parity Violation (Lee and Yang) in 1936, Wu discovered it in 1957

[1] There were multiple people who arguably should get some more credit for this. Blackett discovered the positron but did not publish it fast enough. There were also a Russian scientist and a graduate student at CalTech who did similar discoveries.

- Discovery of (anti-)neutrino (Cowan- Reines, 1956)
- $V - A$ property of weak interactions (Marshak, Sudarshan, 1957)[1]
- Pontecorvo; neutrino oscillations proposed
- Yang–Mills 1954, non-Abelian gauge theory. In QED, there is a $U(1)$ gauge theory giving a massless photon. In Yang–Mills theory, with a greater group such as $SU(2)$, that this should give further massless / long-range particles. But these have never been seen so Pauli predicted correctly that this theory was not relevant to nature.

[1] Four experiements seemed to deny their theory. However, the theory was so strong that they were convinced that these experiments must have been wrong. All four actually turned out to be wrong.

1960's **1961** Eightfold way (Gell-Mann, Neemann)

Put order to zoo of discovered particles by considering representations of $SU(3)_{\text{flavour}}$. By con-

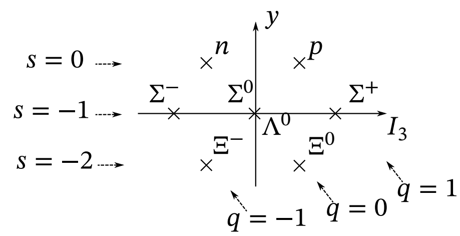


Figure 1.2: The eightfold way is the 8-dimensional representation of $SU(3)$.

sidering the 10-dimensional representation of $SU(3)$, depicted in Fig. 1.3 the Ω^- was predicted.

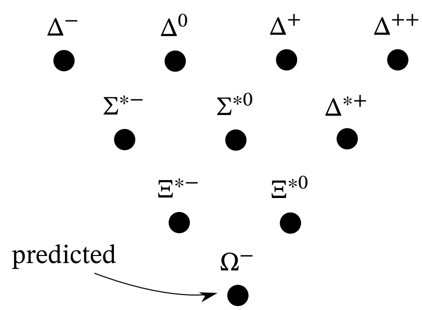


Figure 1.3: The 10-dimensional representation of $SU(3)$.

1964 Gell-Mann, Zweig came up with the theory of *quarks*. This theory was not accepted at the time since three quarks needed to be in the same state for some particles, violating Pauli's exclusion principle.

- $3 \oplus \overline{3} \to$ Mesons $s = 0$; $3 \otimes \overline{3} = 8 + 1$
- $3 \oplus 3 \otimes 3 \to$ baryons $s = \frac{1}{2}$; $3 \otimes 3 \otimes 3 = 10 + 8 + 8 + 1$

1964 Greenberg, 1965 (Nambu and Han) $\to$ colour

1967 Deep inelastic scattering. Evidence for substructure in the proton nucleus.

1961 Symmetry breaking (Nambu, Goldstone, Salam, Weinberg), Goldstone bosons (massless)

1964 Higgs Mechanism (Higgs, Brout, Englert, Kibble, Guralnik, Haden)

If the broken symmetry is local, then

- the gauge field is massive
- the Goldstone boson is is eaten and leaves behind a physical massive particle (Higgs)

The problem that Pauli pointed out to Yang and Mills is solved! Now you can have non-Abelian gauge symmetries, and broken symmetries.

1967-8 Weinberg, Salam, (Ward) tried non-Abelian gauge theory for the strong interaction, which failed. Trying it for the weak interaction gave Electroweak unification

$$\underbrace{SU(2)}_{L} \times \underbrace{U(1)}_{Y} \to \underbrace{U(1)}_{\text{EM}} \tag{1.3}$$

(Glashow 1962 identified $SU(2) \times U(1)$)

1964 experimental discovery of CP violation (Cronin, Fitsch) $\Rightarrow$ particle $\leftrightarrow$ antiparticle

1970's Glashow–Illoporlos–Maiani (GIM) mechanism. Explain no FCNC $\Rightarrow$ new quark: *charm* c. As such, the magic number of three, leading Gell-Mann to quarks, is not magic at all. The previous symmetry was only approximate, which was not noticed since c is very massive. In hindsight it was obvious that we needed a fourth quark:

1969 Jackiw–Bell–Adler; Anomalies. Need partner of s: $\begin{pmatrix} c \\ s \end{pmatrix}_L$.

1973 • weak neutral currents discovered

- Asymptotic Freedom (Gross, Wilczek, Politzer)

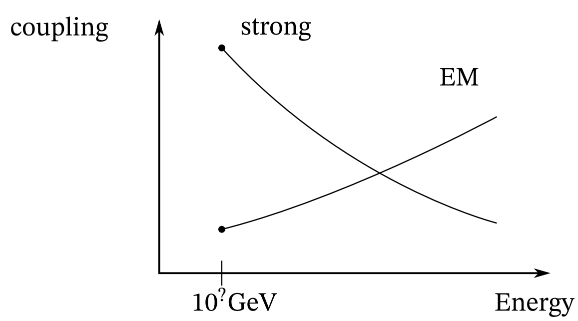


Figure 1.4: The running coupling gives hope for unification.

1974 J/ψ discovered $\to$ *charm*

1975-9 jets (quarks, gluons), for instance $e^+e^- \to qq$ gives 2 jets, but $e^+e^- \to qgq$ gives 3 jets.

$$R = \frac{e^+e^- \to \text{hadrons}}{e^+e^- \to \text{muons}} = \frac{33}{9} \tag{1.4}$$

depends on the number of colours. This gave evidence for 3 colours, confirming the idea of quarks.

1980's **1983** $Z^0, W^\pm$ discovered

1990's **1995** *top quark* discovered. This was not a surprise since people already knew about the bottom quark, which needed a partner. We end up with three families

$$\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix} \tag{1.5}$$

2000's *Tau neutrino*

2012 Higgs!

We are lucky to be taking this course in a time where the standard model is essentially solved. In this course we will see that this structure is essentially forced on us. The structure is very rigid.

2 Spacetime Symmetries

The symmetries we have are spacetime $\otimes$ internal (1967 Coleman–Mandula). In particular, the spacetime symmetry is the Poincaré symmetry.

2.1 Poincaré Symmetries and Spinors

A general transformation of the Poincaré group acts on spacetime x^μ as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \qquad \mu = 0,1,2,3 \tag{2.1}$$

where $\Lambda^\mu{}_\nu$ are the Lorentz transformations and a^μ translations. We write the Poincaré group therefore as $O(3,1) \rtimes \mathbb{R}^4$, where $\rtimes$ denotes the semi-direct product, which does not commute.

These transformations leave the Minkowski metric $ds^2 = dx^\mu \eta_{\mu\nu} dx^\nu$ invariant, where $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$: For $\Lambda \in O(3,1)$, we have

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \qquad \text{or} \qquad \Lambda^T \eta \Lambda = \eta \tag{2.2}$$

This means that we have a choice of either $\det \Lambda = \pm 1$.

$$(\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 - (\Lambda^2{}_0)^2 - (\Lambda^3{}_0)^2 = 1. \tag{2.3}$$

Then $\left|\Lambda^0{}_0\right| \geq 1$ implies that for each of the two choices of determinant, we can have either $\Lambda^0{}_0 \geq 1$ or $\Lambda^0{}_0 \leq -1$. Therefore, $O(3,1)$ has 4 disconnected components. The element continuously connected to the identity, $SO(3,1)^\uparrow$, is the proper orthochronous Lorentz group with $\det \Lambda = 1$ and $\Lambda^0{}_0 \geq 1$. Any other element in $O(3,1)$ can be obtained by combining elements of $SO(3,1)^\uparrow$ with

$$\{\mathbb{1}, \Lambda_P, \Lambda_T, \Lambda_{PT}\}, \qquad \text{Klein Group} \tag{2.4}$$

where $\Lambda_P = \mathrm{diag}(+1,-1,-1,-1)$ are the parity transformations and $\Lambda_T = \mathrm{diag}(-1,+1,+1,+1)$ time reversal.

From now on we work with $SO(3,1)^\uparrow \to SO(3,1)$.

2.1.1 Poincaré Algebra

As usual to derive the algebra, we consider the infinitesimal transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}; \qquad a^{\mu} = \epsilon^{\mu}; \qquad \omega^{\mu}{}_{\nu}, \epsilon^{\mu} \ll 1. \tag{2.5}$$

The invariance (2.2) of the metric then gives

$$(\delta^{\mu}_{\rho} + \omega^{\mu}{}_{\rho})\eta_{\mu\nu}(\delta^{\nu}_{\sigma} + \omega^{\nu}{}_{\sigma}) = \eta_{\rho\sigma}. \tag{2.6}$$

This implies that $\omega_{\sigma\rho} = -\omega_{\rho\sigma}$ is antisymmetric. As such, we have 6 parameters $\omega_{\mu\nu}$ for the Lorentz transformations. In addition to this, we have 4 parameters ϵ_{μ} for translations. In total, the Poincaré group has 10 dimensions.

To study the algebra of the Poincaré group, we will look at its representation on a Hilbert space, since we are interested in quantum theory. We are working with a state $|\psi\rangle$ and consider transformations enacted by unitary operators $U(\Lambda, a) = \exp\big(i[\omega_{\mu\nu}M^{\mu\nu} + \epsilon_{\mu}P^{\mu}]\big)$ as

$$|\psi\rangle \to U(\Lambda, a)\,|\psi\rangle\,, \tag{2.7}$$

where $U(\Lambda, a)$ form a representation of the Poincaré group and the generators $M^{\mu\nu}$ and P^{μ} of the Poincaré algebra are Hermitian. Near the identity, we can expand the exponential

$$U(1+\omega, \epsilon) = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + i\epsilon_{\mu}P^{\mu}, \tag{2.8}$$

Since $\omega_{\mu\nu}$ is antisymmetric, so is $M^{\mu\nu}$.

To determine the algebra, we also need to find the Lie brackets. Since the translations commute, $[P^{\mu}, P^{\nu}] = 0$. More complicated is the bracket $P^{\sigma}, M^{\mu\nu}$.

P^{μ} by itself has a double personality. It is a vector, which means that under infinitesimal Lorentz transformations it transforms as

$$\begin{aligned} P^{\sigma} \to \Lambda^{\sigma}{}_{\rho}P^{\rho} &\simeq (\delta^{\sigma}{}_{\rho} + \omega^{\sigma}{}_{\rho})P^{\rho} && (2.9)\\ &= P^{\sigma} + \frac{1}{2}(\omega_{\alpha\rho} - \omega_{\rho\alpha})\eta^{\sigma\alpha}P^{\rho} && (2.10)\\ &= P^{\sigma} + \frac{1}{2}\omega_{\alpha\rho}(\eta^{\rho\alpha}P^{\rho} - \eta^{\sigma\rho}P^{\alpha}). && (2.11) \end{aligned}$$

However, it is also an operator, which acts on the Hilbert space. Therefore, it transforms as

$$\begin{aligned} P^{\sigma} \to U^{\dagger}P^{\sigma}U &= (\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})P^{\sigma}(\mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}) && (2.12)\\ &= P^{\sigma} + \frac{i}{2}\omega_{\mu\nu}[M^{\mu\nu}, P^{\sigma}] + O(\omega^2). && (2.13) \end{aligned}$$

Comparing the above two expressions, we have

$$\boxed{[P^{\sigma}, M^{\mu\nu}] = -i(P^{\mu}\eta^{\nu\sigma} - P^{\nu}\eta^{\mu\sigma})} \tag{2.14}$$

As such, whenever we see an algebra $[X^{\mu_1\cdots}, M^{\rho\sigma}]$, then we should know that the right hand side actually tells us how $X^{\mu_1\cdots}$ transforms under Lorentz transformations!

Similarly,

$$\boxed{[M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho})}. \tag{2.15}$$

Again, this tells us that $M^{\mu\nu}$ transforms under Lorentz transformations as a tensor.

Example 2.1.1: A 4-dimensional matrix representation of $M^{\mu\nu}$ is given by

$$(M^{\rho\sigma})^{\mu}{}_{\nu} = -i(\eta^{\mu\sigma}\delta^{\rho}_{\nu} - \eta^{\rho\mu}\delta^{\sigma}_{\nu}) \tag{2.16}$$

Comment 1

Since $P^0 = H$ is the Hamiltonian, we find that the commutation relation have some physical meaning:

$$[P^0, P^\mu] = 0 \implies \text{Energy-Momentum conservation} \tag{2.17}$$

Comment 2

The algebra of $SO(3,1)$ is determined by the algebra of $SU(2)\times SU(2)$. Define Hermitian operators $J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$ and $K_i = M_{0i}$. Their algebra arises directly from the commutators (2.15) of the M_{ij}:

$$[J_i, J_j] = i\epsilon_{ijk}, \qquad [J_i, K_k] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{2.18}$$

We can also define $A_i = \frac{1}{2}(J_i + iK_i)$ and $B_i = \frac{1}{2}(J_i - iK_i)$. These are not Hermitian. However, this gives a nice separation between the algebras

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \qquad [B_i, B_j] = i\epsilon_{ijk}B_k, \qquad [A_i, B_j] = 0, \tag{2.19}$$

and in each case the A- and B-subalgebra look like the algebra of the J's: These are like $SU(2)$ algebras, except they are not Hermitian.

Representations of $SU(2)\times SU(2)$

For representations of $SU(2)\times SU(2)$, recall that that the $SU(2)$ states are labelled by half-integers $j = 0, \frac{1}{2}, \ldots$. Then the A_i and B_i algebra states are labelled by $A, B = 0, \frac{1}{2}, \ldots$ respectively. Therefore, the representations of $SO(3,1)$ can be labelled by specifying (A, B).

Remark: Under parity

$$P: \qquad J_i \to J_i \tag{2.20}$$
$$K_i \to -K_i \tag{2.21}$$
$$A_i \leftrightarrow B_i \tag{2.22}$$
$$(A, B) \leftrightarrow (B, A) \tag{2.23}$$

Therefore, we can denote either one of these, say (A, B), as 'left'. Then (B, A) is 'right' and vice-versa.

Comment 3

We have the homomorphism

$$SO(3,1) \simeq SL(2,\mathbb{C}). \tag{2.24}$$

Consider first $SO(3,1)$. Let $X = X_\mu e^\mu = (X_0, X_1, X_2, X_3)$ denote a 4 -vector. Under Lorentz transformation $X \to \Lambda X$, where $\Lambda \in SO(3,1)$, the modulus squared $|X|^2 = X_0^2 - X_1^2 - X_2^2 - X_3^2$ remains invariant.

Now consider the space of 2×2 matrices with basis given by the Pauli matrices

$$\sigma^\mu := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{2.25}$$

We can then write any matrix $\widetilde{X}$ as a linear combination of these

$$\widetilde{X} = X_\mu \sigma^\mu = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & X_0 - X_3 \end{pmatrix}. \tag{2.26}$$

Taking the components (X_0, X_1, X_2, X_3) to be the same as the 4-vector above, this is just another way of representing the same information. Furthermore, the action of $SL(2,\mathbb{C})$ on $\widetilde{X}$ is

$$\widetilde{X} \to N\widetilde{X}N^\dagger, \qquad N \in SL(2,\mathbb{C}). \tag{2.27}$$

Since $N \in SL(2,\mathbb{C})$, we have $\det N = 1$. The determinant has the form $\det \widetilde{X} = X_0^2 - X_1^2 - X_2^2 - X_3^2$. Exactly as in the case of $SO(3,1)$, this quantity is kept invariant.

This is the defining feature. As such, the map from $SL(2,\mathbb{C}) \to SO(3,1)$ defined by (2.26) is homomorphic. This map is 2 to 1 since it maps $N = \pm\mathbb{1} \to \Lambda = \mathbb{1}$.

Claim 1: But $SL(2,\mathbb{C})$ is *simply connected.*

Proof. Polar decomposition: We can write $N = e^h U$, where $h = h^\dagger$ is hermitian and $U = (U^\dagger)^{-1}$ unitary. Since the eigenvalues of a hermitian matrix are positive, the trace of h is positive. Then, using that $\det e^h = e^{\operatorname{tr} h}$, we find that $\det N = 1$ implies that $\operatorname{tr} h = 0$ and $\det U = 1$.

$$h = \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix} \qquad U = \begin{pmatrix} x + iy & z + iw \\ -z + iw & x - iy \end{pmatrix}. \tag{2.28}$$

For h, the variables $a, b, c \in \mathbb{R}$ define the manifold $\mathbb{R}^3$. Similarly the components of U have the condition $x^2 + y^2 + z^2 + w^2 = 1$, defining the manifold S^3. Therefore, we have that $SL(2,\mathbb{C})$ has the manifold structure $\mathbb{R}^3 \times S^3$, which is simply connected. □

Corollary: Thus, the $SO(3,1)$ manifold, which is obtained from a 2 to 1 map from $SL(2,\mathbb{C})$, is $\mathbb{R}^3 \times S^3/\mathbb{Z}_2$.

Representations of $SL(2,\mathbb{C})$

Definition 2 (fundamental)**:** The *fundamental representation* ψ_α, $\alpha = 1, 2$ is given by

$$\psi'_\alpha = N_\alpha{}^\beta \psi_\beta. \tag{2.29}$$

The ψ_α transforming in this way are called *left-handed Weyl spinors*.

Definition 3 (conjugate)**:** The *antifundamental* or *conjugate representation* is given by *right-handed Weyl spinors* $\overline{\chi}_{\dot{\alpha}}$, $\dot{\alpha} = 1, 2$ transforming as

$$\overline{\chi}'_{\dot{\alpha}} = (N^*)_{\dot{\alpha}}{}^{\dot{\beta}} \overline{\chi}_{\dot{\beta}}. \tag{2.30}$$

Definition 4 (contravariant)**:** The *contravariant representation* is

$$\psi'^{\alpha} = \psi^{\beta} (N^{-1})_{\beta}{}^{\alpha}, \qquad \overline{\chi}'^{\dot{\alpha}} = \overline{\chi}^{\dot{\beta}} (N^{*-1})_{\dot{\beta}}{}^{\dot{\alpha}}. \tag{2.31}$$

To raise and lower indices, we need *invariant tensors*:

SO(3, 1) $\eta^{\mu\nu} = (\eta_{\mu\nu})^{-1}$

SL(2, $\mathbb{C}$) $\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot{\alpha}\dot{\beta}}$

Invariance means that $\epsilon^{\alpha\beta}$ transforms as

$$\epsilon'^{\alpha\beta} = \epsilon^{\rho\sigma} N_{\rho}{}^{\alpha} N_{\sigma}{}^{\beta} = \epsilon^{\alpha\beta} \det N = \epsilon^{\alpha\beta} \tag{2.32}$$

Mixed $SO(3,1)$ and $SL(2,\mathbb{C})$

$$\widetilde{X}_{\alpha\dot{\alpha}} = (X_\mu \sigma^\mu)_{\alpha\dot{\alpha}} \to N^\beta{}_\alpha (X_\nu \sigma^\nu)_{\beta\dot{\gamma}} (N^*)^{\dot{\gamma}}{}_\alpha = 1 \tag{2.33}$$

$$\Rightarrow \ \sigma^\mu{}_{\alpha\dot{\alpha}} = N_\alpha{}^\beta (\sigma^\nu)_{\beta\dot{\gamma}} (\Lambda^{-1})^\mu{}_\nu (N^*)^{\dot{\gamma}}{}_{\dot{\alpha}} \tag{2.34}$$

and similarly

$$(\overline{\sigma}^\mu)^{\alpha\dot{\alpha}} := \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^\mu)_{\beta\dot{\beta}} \tag{2.35}$$

Can choose

$$\sigma^\mu \overline{\sigma}^\nu + \sigma^\nu \overline{\sigma}^\mu = 2\eta^{\mu\nu}. \tag{2.36}$$

These are the *Dirac generators* of $SL(2,\mathbb{C})$.

Definition 5: From these, we can define new matrices

$$(\sigma^{\mu\nu})_\alpha{}^\beta := \frac{i}{4}\left(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu\right)_\alpha{}^\beta, \tag{2.37}$$

$$(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{4}\left(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu\right)^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{2.38}$$

Claim 2: The $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ are generators of the Lorentz group in spinor representations, meaning that they obey the commutation relations

$$[\sigma^{\mu\nu}, \sigma^{\lambda\rho}] = i\left(\eta^{\mu\rho}\sigma^{\nu\lambda} + \eta^{\nu\lambda}\sigma^{\mu\rho} - \eta^{\mu\lambda}\sigma^{\nu\rho} - \eta^{\nu\rho}\sigma^{\mu\lambda}\right) \tag{2.39}$$

and similar for $\overline{\sigma}$.

For $SL(2,\mathbb{C})$: Left handed (fundamental)

$$\psi_\alpha \to (e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}})_\alpha{}^\beta \psi. \tag{2.40}$$

Right handed (conjugate)

$$\overline{\chi}^{\dot{\alpha}} \to \left(e^{-\frac{i}{2}m_{\mu\nu}\overline{\sigma}^{\mu\nu}}\right)^{\dot{\alpha}}{}_{\dot{\beta}} \overline{\chi}^\beta. \tag{2.41}$$

Fundamental

$$J_i = \frac{1}{2}\epsilon_{ijk}\sigma_{jk} = \frac{1}{2}\sigma_i, \quad K_i = \sigma_{0i} = -\frac{i}{2}\sigma_i. \tag{2.42}$$

Exercise 2.1: Check $[\sigma_i, \sigma_j] = 2i\epsilon^{ijk}\sigma_k$.

$$A_i = \frac{1}{2}(J_i + iK_i) = \frac{\sigma_i}{2} \qquad B_i = \frac{1}{2}(J_i - iK_i) = 0 \tag{2.43}$$

$$(A,B) = (\frac{1}{2}, 0) \qquad \text{representation} \tag{2.44}$$

Conjugate:

$$(A,B) = (0, \frac{1}{2}) \tag{2.45}$$

Parity:

$$(A,B) \xrightarrow{P} (B,A) \tag{2.46}$$

Definition 6 (product)**:** Product of Weyl spinors:

$$\chi\psi := \chi^\alpha\psi_\alpha = -\chi_\alpha\psi^\alpha \tag{2.47}$$

$$\overline{\chi}\overline{\psi} := \overline{\chi}_{\dot{\alpha}}\overline{\psi}^{\dot{\alpha}} = -\overline{\chi}^{\dot{\alpha}}\overline{\psi}_{\dot{\alpha}} \tag{2.48}$$

In particular,

$$\psi\psi = \psi^\alpha\psi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\psi_\alpha = \psi_2\psi_1 - \psi_1\psi_2. \tag{2.49}$$

Choose ψ_α to be Grassmann numbers

$$\psi_1\psi_2 = -\psi_2\psi_1. \tag{2.50}$$

Then $\psi\psi = 2\psi_2\psi_1$.

All representations of Lorentz algebra can be obtained from products of $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$:

$$(\frac{1}{2}, 0)^m \otimes (0, \frac{1}{2})^n \quad \text{use } j_1 \otimes j_2 = |j_1 - j_2| \oplus \cdots \oplus j_1 + j_2. \tag{2.51}$$

Example 2.1.2:

$$(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2}) \tag{2.52}$$

$$\psi_\alpha\overline{\chi}_{\dot{\alpha}} = \frac{1}{2}(\psi\sigma_\mu\overline{\chi})\sigma^\mu{}_{\alpha\dot{\alpha}}. \tag{2.53}$$

Example 2.1.3:

$$(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) = (1, 0) \oplus (0, 0) \tag{2.54}$$

$$\psi_\alpha\psi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\underbrace{(\psi\chi)}_{\text{scalar}} + \frac{1}{2}(\sigma^{\mu\nu}\epsilon^T)_{\alpha\beta}\underbrace{(\psi\sigma_{\mu\nu}\chi)}_{(1,0)} \tag{2.55}$$

Connection to Dirac Matrices and Spinors

Definition 7 (Dirac spinor)**:** We define the *Dirac spinor* to be

$$\psi_D := \begin{pmatrix} \psi_\alpha \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{2.56}$$

Definition 8 (gamma matrices)**:** The Dirac γ-matrices are defined such that $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$, for example

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \tag{2.57}$$

Lorentz generators:

$$\Sigma^{\mu\nu} = \frac{i}{4}\gamma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{pmatrix} \tag{2.58}$$

Definition 9: We define a fifth gamma matrix

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{2.59}$$

Claim 3: This acts on Dirac spinors as

$$\gamma^5\psi_D = \begin{pmatrix} -\psi_\alpha \\ +\overline{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{2.60}$$

Definition 10 (projection operators)**:** We define *projection operators*

$$P_L := \frac{1}{2}(\mathbb{1} - \gamma^5), \qquad P_R := \frac{1}{2}(\mathbb{1} + \gamma^5) \tag{2.61}$$

Definition 11 (Dirac conjugation)**:** We define the *Dirac conjugate* of a Dirac spinor as

$$\overline{\psi}_D := (\chi^\alpha, \overline{\chi}_{\dot{\alpha}}). \tag{2.62}$$

Definition 12:

$$\psi_D^C := \begin{pmatrix} \chi_\alpha \\ \overline{\psi}^{\dot{\alpha}} \end{pmatrix} \tag{2.63}$$

Claim 4:

$$\psi_D^C = C\overline{\psi}_D^T, \qquad C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \tag{2.64}$$

Definition 13 (Majorano spinor)**:**

$$\psi_M := \begin{pmatrix} \psi_\alpha \\ \overline{\psi}^{\dot{\alpha}} \end{pmatrix} = \psi_M^C \tag{2.65}$$

We have

$$\psi_D = \psi_M^1 + i\psi_M^2. \tag{2.66}$$

No Majorana + Weyl spinor.

2.2 Unitary Representations of the Poincaré Group

This is due to Wigner in 1939.

2.2.1 Rotation Group

We already know how to find the representations of the rotation group in $3D$ $SO(3) \simeq SU(2)$. Its generators J_i obey $[J_i, J_j] = i\epsilon_{ijk}J_k$.

Definition 14 (Casimir operator)**:** The *Casimir operator* is

$$J^2 = J_1^2 + J_2^2 + J_3^2, \qquad [J^2, J_i] = 0. \tag{2.67}$$

The representations or multiplets of the rotation group, which are a collection of states, are labelled by the eigenvalues of the Casimir operators

$$J^2 |j\rangle = j(j+1)|j\rangle, \qquad j = 0, \frac{1}{2}, 1, ... \tag{2.68}$$

Within one representation, we can diagonalise one of the J_i generators, for instance J_3 . Its eigenvalues are j_3. Acting on any given state, labelled by j_3, in the representation that is labelled by j, we have

$$J_3 |j, j_3\rangle = j_3 |j, j_3\rangle, \qquad j_3 = -j, -j+1, \dots, +j. \tag{2.69}$$

Remark: For a compact group, the number of Casimir operators is the rank of the group, but for non-compact group we have to try by hand.

2.2.2 Poincaré Group

We follow exactly the same steps for the Poincaré group. The generators are P^μ and $M^{\mu\nu}$, obeying the commutation relations (2.14) and (2.15).

Definition 15 (Casimir operator)**:** The *Casimir operators* for the Poincaré group are

$$C_1 = P^\mu P_\mu \qquad C_2 = W^\mu W_\mu. \tag{2.70}$$

Remark: Note that we now have two Casimirs, so the representations will be indicated by two labels.

Claim 5: The Casimir operators commute with the Poincaré algebra's generators:

$$[C_{1,2}, P^\mu] = 0 = [C_{1,2}, M^{\mu\nu}]. \tag{2.71}$$

Exercise 2.2: Prove this!

Definition 16 (Pauli–Ljubanski vector)**:** The *Pauli–Ljubanski vector* is

$$W_M := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}. \tag{2.72}$$

Claim 6: The Pauli–Ljubanski vector and the Poincaré generators have the following commutation relations

$$[W_\mu, P_\nu] = 0, \qquad [W_\mu, M_{\rho\sigma}] = i(\eta_{\mu\rho}W_\sigma - \eta_{\mu\sigma}W_\rho). \tag{2.73}$$

Claim 7: Using Claim 6, one can show

$$[W_\mu, W_\nu] = -i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma, \tag{2.74}$$

so the W_μ do not form an algebra, except if the P^μ are fixed!

As before, we label representations of the Poincaré algebra by eigenvalues C_1 and C_2

$$|C_1, C_2\rangle. \tag{2.75}$$

Within an irreducible representation (irrep), we now need to pick a subset of generators that can be simultaneously diagonalised. Let us pick the P^μ, with eigenvalue p^μ, since they already commute with each other. Since $C_1 = P^\mu P_\mu$, the eigenvalues $p^\mu p_\mu$ can be bigger than, equal to, or less than zero.

$p^\mu p_\mu > 0$ In other words, p^μ is timelike. We can choose coordinates in which $p^\mu = (m, 0, 0, 0)$. It is immediately obvious that this vector is invariant under rotations J_i. We say that $SO(3)$ is the *little group*. Now $C_1 = P^\mu P_\mu = m^2$ and since $W_\mu = (0, -mJ_i)$, we have $C_2 = m^2 J^2$. The multiplet is then labelled by

$$|c_1, c_2; p^\mu, j_3\rangle = |m, j; p^\mu, j_3\rangle. \tag{2.76}$$

These are *massive one-particle states*!

Remark: We cannot overemphasise the importance of this result: elementary particles are irreducible representations of the Poincaré algebra.

$P^\mu P_\mu = 0$ We can find a frame in which $p^\mu = (E, 0, 0, E)$ and have $C_1 = 0$. The W_μ are given by

$$(W_0, W_1, W_2, W_3) = E(J_3, -J_1 + K^2, -J_2 - K_1, -J_3). \tag{2.77}$$

The commutation relations are

$$[W_1, W_2] = 0, \qquad [W_3, W_1] = -iEW_2, \qquad [W_3, W_2] = iEW_1. \tag{2.78}$$

The little group is the $2D$ Euclidean group E_2, which has infinite-dimensional representations. However, the corresponding particles have never been observed.

Remark: So far, we do not have a satisfactory explanation of why this particle should not be there. [Weinberg] gives a phenomenological explanation, whereas Wigner gave an explanation in terms of the heat capacity. The lecturer proposed an explanation in terms of string theory.

Set $W_1 = W_2 = 0$. Then W_3 is generating $SO(2)$, rotations around x_3. We have

$$W_\mu = EJ_3(1,0,0,-1) \propto p_\mu. \tag{2.79}$$

Finally, the second Casimir operator is $C_2 = W^\mu W_\mu = 0$. The irreducible representation is labelled

$$|0,0;p^\mu,\lambda\rangle := |p^\mu,\lambda\rangle\,, \tag{2.80}$$

where $\lambda = 0, \pm\frac{1}{2}, \pm 1, \dots$ is the eigenvalue of J_3, which is called *helicity*

$$e^{2\pi i\lambda}\,|p^\mu,\lambda\rangle = \pm\,|p^\mu,\lambda\rangle\,. \tag{2.81}$$

Remark: We will identify these with the Higgs ($\lambda = 0$), quarks and leptons ($\pm\frac{1}{2}$), gauge bosons (± 1) and the graviton (± 2). There are no higher spin particles! The $\lambda = \pm\frac{3}{2}$ particle, dubbed the gravitino, has not yet been found.

Remark: There is another state with $P^\mu P_\mu = 0$, which is simply $p^\mu = (0,0,0,0)$. This is what people call the vacuum; a state without any particles.

$P^\mu P_\mu < 0$ These are tachyonic states.

Remark: [Weinberg, Chapter 2] goes into further detail on this.

2.3 Discrete Spacetime Symmetries

We know of two discrete symmetries already. We have seen parity P, with $\Lambda_P = \text{diag}(+1, -1, -1, -1)$ and time reversal T with $\Lambda_T = \text{diag}(-1, +1, +1, +1)$. Previously we ignored them because these are not continuously connected to the identity. These transformations move us between the various disconnected pieces of the Lorentz group.

We can represent these as operators acting on a Hilbert space

$$P = U(\Lambda_P, 0) \qquad T = U(\Lambda_T, 0). \tag{2.82}$$

It will turn out that P is unitary while T is not.

Let us first consider how P and T act on operators of the Hilbert space. For any $U(\Lambda, a)$, these act as

$$PUP^{-1} = U(\Lambda_P, \Lambda, \Lambda_P^{-1}, \Lambda_P a), \qquad TUT^{-1} = U(\Lambda_T, \Lambda, \Lambda_T^{-1}, \Lambda_T a). \tag{2.83}$$

Infinitesimally, where we take the Lorentz transformations to be $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ and $\alpha^\mu = \epsilon^\mu$, with $\omega^\mu{}_\nu, \epsilon^\mu \ll 1$. We also write its unitary representation as $U(\Lambda, a) = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + i\epsilon_\mu P^\mu$. Then

$$PJ_iP^{-1} = J_i, \qquad PK_iP^{-1} = -K_i, \qquad PP_iP^{-1} = -P_i, \qquad PP_0P^{-1} = P_0. \tag{2.84}$$

Naively, we expect $TP_0T = -P_0$, but this would imply negative energy.

Theorem 1 (Wigner)**:** Transformations on a Hilbert space preserving probabilities are either unitary and linear or antiunitary and antilinear.

Proof. See [Weinberg, Vol. 1]. □

Unitary and Linear: We have two states $|\phi\rangle, |\psi\rangle$. Then

$$\langle U\phi|U\psi\rangle = \langle\phi|\psi\rangle \qquad U(\alpha\phi + \beta\psi) = \alpha U\phi + \beta U\psi. \tag{2.85}$$

Antiunitary and Antilinear: With similar states, we have instead

$$\langle U\phi|U\psi\rangle = \langle\phi|\psi\rangle^* \qquad U(\alpha\phi + \beta\psi) = \alpha^* U\phi + \beta^* U\psi. \tag{2.86}$$

In particular, if α is imaginary, then it changes sign, so 'U does not commute with i'.

Then pick T to be antiunitary and antilinear ($Ti = -iT$).

$$TJ_iT^{-1} = -J_i, \qquad TK_iT^{-1} = K_i, \qquad TP_iT^{-1} = -P_i, \qquad TP_0T^{-1} = P_0. \tag{2.87}$$

Now we want to know how these act on particle states.

Claim 8: For massive particles, we have

$$|m, i; p^\mu, j_3\rangle \xrightarrow{P} \eta_P \, |m, j; -p^\mu, j_3\rangle \,, \tag{2.88}$$

$$\xrightarrow{T} \eta_T (-1)^{j-j_3} \, |m, j; -p^\mu, -j_3\rangle \,, \tag{2.89}$$

where η_P and η_T represent some phase. For massless particles,

$$|p^\mu, \lambda\rangle \xrightarrow{P} \eta_P e^{\mp i\pi\lambda} \, |-p^\mu, -\lambda\rangle \,, \tag{2.90}$$

$$\xrightarrow{T} \eta_T e^{\pm i\pi\lambda} \, |-p^\mu, -\lambda\rangle \,. \tag{2.91}$$

Proof. See [Weinberg, Vol. 1]. □

Comments

- If $\lambda \neq 0$, then $|p^\mu, \lambda\rangle \rightarrow |-p^\mu, -\lambda\rangle$, so the states come in two polarisations $\pm\lambda$, so $\lambda = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$ This happens whenever parity is conserved. For most of the interactions, such as the photon or the graviton, this is true. However, for the weak interaction, parity is not conserved. We interpret the second helicity eigenvalue to belong to its antiparticle.
- For a massive particle of spin j, there are $2j + 1$ polarisation states, since $j_3 = -j, \ldots, +j$. For instance, a massive spin-1 particle has 3 polarisation states. However, the massless object of helicity λ has always only two states $\pm\lambda$. This difference between becomes even more pronounced for higher spins.
- We have not yet talked about any fields here. Special relativity and quantum mechanics is enough to imply the existence of particles. Fields will be introduced as a way to describe interactions.

2.4 From Particles to Fields

We do not only want to see that single particle states exist, but it is also extremely important to be able to describe *interactions* among many particles. Such interactions can be described by a Hamiltonian

$$H = H_0 + H_{\text{interaction}}, \tag{2.92}$$

where H_0 is the Hamiltonian of the free, non-interacting theory.

There are some conditions on the observables:

(i) Lorentz and translation invariance

(ii) Unitarity (guarantees that probabilities are conserved). Unitary evolution $U = e^{-iHt}$, where H is Hermitian.

(iii) Locality. This is where fields enter. We want to describe interactions at spacetime points, so we need some local structure there. The interactions are described as functions of x and t and those functions are the fields. The principle of *Cluster decomposition* means that things that are far away do not interact.

If all of these conditions are satisfied, then we are necessarily driven towards the introduction of fields.

We describe interactions between initial states at the infinite past $t \to -\infty$ to final states in the infinite future $t \to +\infty$, where both are assumed to act just like free states. This is described by the S-matrix:

$$S_{\beta\alpha} := \left\langle \beta^{\text{out}} \middle| \alpha^{\text{in}} \right\rangle = \delta_{\beta\alpha} + \delta(p^2 - m^2) M_{\beta\alpha}. \tag{2.93}$$

The $|\alpha\rangle$ and $|\beta\rangle$ are many particle states.

We can write the Hilbert space as

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots, \tag{2.94}$$

where H_n is the space of n -particle states. Notably, the vacuum $|0\rangle$ is the only state in $\mathcal{H}_0$, whereas $\mathcal{H}_1$ are the massless states $|p^\mu, \lambda\rangle = a^\dagger(p^\mu, \lambda)\,|0\rangle$. Furthermore, two particle states in $\mathcal{H}_2$ are then obtained from states in $\mathcal{H}_1$ as

$$\begin{aligned} \left| p_1^\mu, \lambda_1; p_2^\mu, \lambda_2 \right\rangle &= a^\dagger(p_2, \lambda_2) \left| p_1^\mu, \lambda \right\rangle & (2.95)\\ &= a^\dagger(p_2, \lambda_2) a^\dagger(p_1, \lambda_1)\,|0\rangle & (2.96)\\ &= \pm a^\dagger(p_1, \lambda) a^\dagger(p_2, \lambda_2)\,|0\rangle\,, & (2.97) \end{aligned}$$

where the sign depends on the type of particle. *Bosons* commute and have integer spin (helicity), whereas *fermions*, which have half-integer spin (helicity), anticommute.

2.4.1 General Conditions on Interactions

The conditions that we outlined earlier can now be written slightly differently:

(i) Unitarity (probabilities add up to 1) preserve by unitary time evolution $U = e^{-iHt}$. The S-matrix is unitary $S_{\beta\alpha} = \langle \beta | S\alpha \rangle$, $S^\dagger S = 1$.

(ii) Amplitudes (S-matrix) invariant under Poincaré transformations.

(iii) Locality (Cluster decomposition)

- The Hamiltonian has to be a local function $H = \int \mathrm{d}^3x\, \mathscr{H}(x,t)$; it is the sum of energy densities at each point. Similarly with the Lagrangian $L = \int \mathrm{d}^3x\, \mathscr{L}(x,t)$ and the Action $S = \int \mathrm{d}^4x\, \mathscr{L}(x^\mu)$.
- $\mathscr{H}$ and $\mathscr{L}$ are operators in position space x^μ, but particle states are defined in momentum space p^μ. Therefore, we need to Fourier transform to move to x-space. This leads us to defining a field. For instance, for a massless particle we can write down

$$A_\alpha(x) := \int đp\, e^{ipx} u_\alpha(p, \lambda) a(p, \lambda). \tag{2.98}$$

- Causality: operators at different spatial locations have to commute at the same time

$$[\phi_\alpha(x,t), \phi_\alpha^\dagger(y,t)] = 0. \tag{2.99}$$

The field A_α itself is local but not causal. To make it causal we introduce another field

$$B_\alpha(x) := \int đp\, e^{ipx} v_\alpha(p,\lambda) b(p,\lambda), \tag{2.100}$$

so that the total field, which satisfies (2.99) is

$$\phi_\alpha = A_\alpha + \xi B_\alpha^\dagger. \tag{2.101}$$

Therefore, field theory requires the existence of *antiparticles*!

(iv) Stability: Energy bounded from below ($|0\rangle$)

(v) Effective Field Theories ($\supset$ renormalisability and non-renormalisability)

- Physics is organised by scales. For instance, in QED we talk about electrons and photons. This is okay until we hit the energy threshold $E = 2m_\mu$ of the muon mass.
- The Lagrangian density can be written $\mathscr{L} = c_i \mathcal{O}_i(\phi_\alpha)$, where $\mathcal{O}_i$ are operators. We know $[\mathscr{L}] = 4$. The theory is said to be *renormalisable* if $[c_i] \geq 0$. This is very restrictive since $[\mathcal{O}_i] \geq 0$.

$$[c_i] + [\mathcal{O}_i] = 4. \tag{2.102}$$

This is very predictive since it allows only a few c_i.

The theory is non-renormalisable if there is a c_i with $[c_i] \leq 0$. For example, consider the scalar field with Lagrangian density

$$\mathscr{L} = \underbrace{\overbrace{\partial^\mu\phi\partial_\mu\phi - m^2\phi^2 - \lambda\phi^4}^{\text{renormalisable}} + \frac{\alpha_1}{M}\phi^5 + \frac{\alpha_2}{M^2}\phi^6 + \ldots}_{\text{non-renormalisable}}. \tag{2.103}$$

There is nothing wrong with this theory *as long as* we are working on energies much smaller than M, so that the expansion in (E/M) is under control. However, this fails for $E \sim M$. We require UV completion.

In a renormalisable theory like the Standard Model we cannot predict when the theory will break down, since we do not know of the scale of M. For gravity, which is non-renormalisable, we can be much more certain about the energy scales on which our predictions should be valid.

This is why in the standard model we will only tend to draw quadratic and quartic potentials; the requirement of renormalisability means that we cannot go beyond quartic and these are the only symmetric choices we have.

3 Internal Symmetries

The most general symmetries of the S-matrix are of the form

$$\text{spacetime} \otimes \text{internal}. \tag{3.1}$$

We have already seen the spacetime symmetries P^μ and $M^{\mu\nu}$. In general, the spacetime symmetries include supersymmetries Q^I_α , which anticommute $\{Q^I_\alpha, \overline{Q}^{-I}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu$. The corresponding representations lead to multiplets including fields of different spin and the fact that $N_{\text{boson}} = N_{\text{fermions}}$, which is not observed in nature so far. We will therefore not think about these further in this course, other than saying that the *gravitino* with $\lambda = \frac{3}{2}$ can only exist with supersymmetry.

Internal symmetries are transforming the operators and fields but leave them at the same spacetime position x

$$|\psi\rangle \to U\,|\psi\rangle\,, \qquad \mathcal{O}(x) \to \mathcal{O}'(x) = U^\dagger \mathcal{O} U. \tag{3.2}$$

These are symmetries if $[U, H] = 0$ and the action S is left invariant under these transformations.

3.0.1 Types of Symmetries

(i) Spacetime or internal

(ii) Continuous or discrete

For example, a continuous $U(1)$ symmetry is $\phi \to e^{i\alpha}\phi$, whereas an example of a discrete symmetry is $\phi \to -\phi$. Imposing this latter symmetry, odd terms ϕ^{2n+1} cannot exist in the Lagrangian.

(iii) Global or local

For example, the transformation $\phi \to e^{i\alpha\phi}$ is global if α is constant, and is local if $\alpha = \alpha(x)$ depends on position. In the latter case, we would need to adjust the Lagrangian to be invariant under this by replaying the partial with the covariant derivative $D_\mu\phi = \partial_\mu\phi - ieA_\mu\phi$. The symmetry not only transforms ϕ but also $A_\mu \to A_\mu + \partial_\mu\alpha$. This A_μ is called the *gauge field* and we add to the Lagrangian a kinetic term $F^{\mu\nu}F_{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The interactions between A_μ and ϕ are hidden in $D^\mu \phi D_\mu \phi^*$. This is a nice story: the interaction between electrons and photons comes from local symmetry. The beauty of this story will be questioned in Sec. 3.2.

Also for Dirac fields, the Lagrangian is

$$\mathscr{L} = \overline{\psi}\not{\partial}\psi = \overline{\psi}_L i\not{\partial}\psi_L + \overline{\psi}_R i\not{\partial}\psi_R - m(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R). \tag{3.3}$$

These fields transform as $\psi_{L,R} \to e^{i\alpha_{L,R}}\psi_{L,R}$. The kinetic terms have $U(1)_L \otimes U(1)_R$, which is a chiral symmetry. The mass terms $m \neq 0$ break this since $\alpha_R = \alpha_L$ and we only have $U(1)$. If $\alpha_L(x), \alpha_R(x)$ are local, then

$$\overline{\psi}\not{\partial}\psi \to \overline{\psi}\not{D}\psi; \qquad D_\mu = \partial_\mu - ieA_\mu. \tag{3.4}$$

We have made a separation between left- and right-handed since we already know that some interactions treat them differently.

(iv) Manifest or hidden

For example, in a spontaneous symmetry broken sombrero potential, the local vacuum does not see the overall symmetry, even though it is still there.

(v) Anomalous or non-anomalous

An anomalous symmetry is broken by quantum corrections, whereas non-anomalous symmetries are exact.

(vi) Real or accidental

(vii) Compact or non-compact

We will reserve the non-compact class only for the spacetime symmetries, since we know that the Poincaré group is non-compact. However, all the internal symmetries will be compact, since we want to have finite-dimensional representations.

(viii) Abelian or non-Abelian

An Abelian group is $U(1)$, whereas the non-Abelian ones are $SU(N), SO(N), Sp(2N), G_2, F_4$, E_6, and E_8, as we have seen from the Cartan classification in the *Symmetries, Particles, and Fields* course in Michaelmas term.

3.1 Noether's Theorem

If a Lagrangian $\mathscr{L}(\phi_\alpha)$ has a continuous symmetry for $\phi_a \to \phi'_\alpha$, then there exists a current j^μ that is conserved, $\partial_\mu j^\mu = 0$, when the field equations are satisfied. This implies that there exists a conserved charge

$$Q = \int \mathrm{d}^3x\, j^0, \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = \int \mathrm{d}^3x\, \partial_0 j^0 = -\int \mathrm{d}^3x\, \nabla \cdot j = 0. \tag{3.5}$$

Example 3.1.1 (Poincaré group)**:** We have already seen that translations $x^\mu \to x^\mu + a^\mu$ have an associated current ${T^\mu}_\nu$ with charges $P^0 = E = \int \mathrm{d}^3x\, T^{00}$ and $P^i = \int \mathrm{d}^3x\, T^{0i}$.

For rotations, $M^{ij} = \int \mathrm{d}^3x\, \left(x^i T^{0j} - x^j T^{0i}\right)$.

This is true for classical mechanics as well. What is different in QFT is that each of these conserved charges are also the generators of their corresponding group.

Example 3.1.2 (Internal symmetry)**:** If you have an internal symmetry generated by a non-Abelian group G, then

$$G:\ \psi^i \to \psi^i + i\alpha^a {(T_a)^i}_j \psi^j, \tag{3.6}$$

where the conserved charge is $T_a = \int \mathrm{d}^3x\, {J^0}_a$. Again, the conserved charge in the Noether theorem is the generator of the corresponding theory:

$$[T_a, \psi^i] = -{(T_a)^i}_j \psi_j. \tag{3.7}$$

For $U(1)$, the conserved charge Q is the electric charge.

3.2 Origin of Gauge (Local) Symmetries

Consider a massless helicity-1 field.

$$A_\mu(x) = \sum_{\lambda=\pm 1} \int \mathrm{d}p \left(\epsilon_\mu(p^\mu, \lambda) a_{p\lambda} e^{ipx} + \epsilon^*_\mu(p^\mu, \lambda) a^\dagger_{p\lambda} e^{-ipx} \right), \tag{3.8}$$

where ϵ_μ is the polarisation vector. As it is written, A_μ has four degrees of freedom $\mu = 0, 1, 2, 3$. However, we know that the massless helicity-1 field only has two degrees of freedom $\lambda = \pm 1$, so we need some (Lorentz invariant) constraints. We can impose

$$p^\mu \epsilon_\mu = 0, \tag{3.9}$$

which leads us from 4 to 3 degrees of freedom; this would be enough for a massive vector with $j_3 = -1, 0, 1$, but it is not satisfactory for the massless particle. In fact, there are no other Lorentz invariant constraints. Thus we know that the ϵ_μ have some extra degree of freedom. The constraint (3.9) leaves open a redundancy

$$\epsilon_\mu \equiv \epsilon_\mu + \alpha p_\mu, \tag{3.10}$$

which defines an equivalence class for the ϵ_μ . Transforming back to position space, the redundancy (3.10) becomes the gauge invariance condition

$$A_\mu \equiv A_\mu + \partial_\mu \alpha. \tag{3.11}$$

The origin of gauge invariance lies in the Lorentz invariant description of a massless helicity-1 field. One might say that Lorentz invariance implies gauge invariance!

Remark: The polarisation vector ϵ_μ is not a Lorentz covariant object despite carrying a vector index. This is because it transforms as $\epsilon_\mu \to \Lambda_\mu{}^\nu \epsilon_\nu + \alpha p_\mu$.

Similarly, for helicity $\lambda = 2$, we work with a field $h_{\mu\nu}$ and polarisations $\epsilon_{\mu\nu}$. We end up with a similar reundancy

$$\epsilon_{\mu\nu} \equiv \epsilon_{\mu\nu} + \alpha_\mu p_\nu + p_\mu \alpha_\nu \;\Rightarrow\; h_{\mu\nu} \equiv h_{\mu\nu} + \partial_\mu \alpha_\nu + \partial_\nu \alpha_\mu. \tag{3.12}$$

Again, the diffeomorphism invariance of (linearised) general relativity is implied by the Lorentz invariance of a massless helicity-2 particle.

All the beauty of the geometry of general relativity, or the gauge symmetry, is gone. It is all really Lorentz invariance.

Any amplitude (recall: $S_{\alpha\beta} = \delta_{\beta\alpha} + (2\pi)\delta(p_\alpha - p_\beta) M_{\alpha\beta}$) will be of the form

$$M_{\alpha\beta}(p_i^\mu, \lambda_i) = \epsilon^\mu (M_\mu)_{\alpha\beta}. \tag{3.13}$$

The redundancy (3.10) due to Lorentz invariance implies the *Ward identity*

$$\boxed{p^\mu M_\mu = 0}. \tag{3.14}$$

3.2.1 Charge Conservation

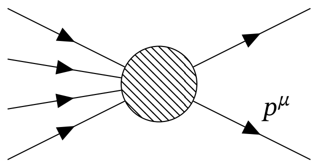


Figure 3.1

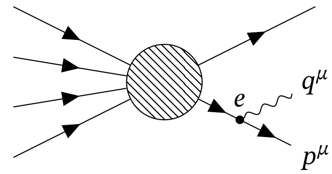


Figure 3.2

Consider the interaction illustrated in Fig. 3.1, which is described by some matrix element M_0. Fig. 3.2 depicts the same interaction, except that we added a 'soft photon' with momentum $q^\mu \ll p^\mu$ to the line with momentum p^μ. This interaction has matrix element $\mathcal{M}$. The most general vertex is $\Gamma^\mu = Fp^\mu + Qq^\mu$, where F, Q are functions of p^2, q^2, and $p \cdot q$. Computing $\epsilon^\mu \Gamma_\mu$ and $q^\mu q_\mu = 0$, we can forget the Q term. If $p^2 = m^2$ and $q^2 = 0$, then $F = F(\frac{p \cdot q}{m^2}) \approx F(0)$ without loss of generality. Thus we have

$$M = M_0 \times \left(\frac{\epsilon^\mu F p_\mu}{(p+q)^2 - m^2} \right) \simeq M_0 \times \left(\frac{F\epsilon \cdot p}{2p \cdot q} \right) \tag{3.15}$$

Adding soft photons to all external lines gives the Ward identity

$$M = M_0 \left(- \sum_{\text{incoming}} \frac{p_i \cdot \epsilon}{2p_i \cdot q} F_i(0) + \sum_{\text{outgoing}} \frac{p_i \cdot \epsilon}{2p_i \cdot q} F_i(0) \right). \tag{3.16}$$

This gives *charge conservation*

$$\sum_{\text{ingoing}} F_i(0) - \sum_{\text{outgoing}} F_i(0) = 0, \tag{3.17}$$

where $F_i(0) := Q_i$.

Let us do the same for $\lambda = 2$. This time we add a soft graviton instead of a soft photon. After going through the same considerations, one arrives at the Ward identity

$$\sum_{\text{in}} k_i p_i^\nu - \sum_{\text{out}} k_i p_i^\nu = 0. \tag{3.18}$$

The extra factors of the p_i^ν complicate this. How can you have an interaction where all the momenta are conserved and then this new combination of momenta is also conserved? This is only possible if all k_i are equal! This is the *principle of equivalence*: all particles interact gravitationally with the same strength.

What happens for $\lambda = 3$? We find that $\sum g_i p_i^\mu p_j^\nu$ is conserved. This implies that all $g_i = 0$. This is an extremely important result: there are no interacting theories for massless particles with helicity higher than $\lambda = 2$. Detailed proofs are given in [Weinberg, Vol. 1, Chapter 13] and in [Schwartz, Chapter 9].

3.2.2 Non Abelian

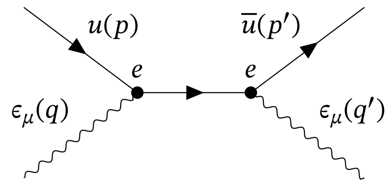


Figure 3.3: Compton scattering.

First, consider Compton scattering $e^+\gamma \to e^+\gamma$ in QED, illustrated by the Feynman diagram in Fig. 3.3. The matrix element can be written $M = M_{\mu\nu}\epsilon^\nu_{\text{in}}\epsilon^\mu_{\text{out}}$ where

$$M_{\mu\nu} = i(-ie)^2\overline{u}(p',\sigma')\left(\frac{\gamma_\mu(\not{p}+\not{q}+m)\gamma_\nu}{(p+q)^2-m^2} + \frac{\gamma_\nu(\not{p}-\not{q}'+m)\gamma_\mu}{(p-q')^2-m^2}\right)u(\mathbf{p},\sigma), \tag{3.19}$$

where $\sigma = g_s$. Consider the Ward identity $M_{\mu\nu}q^\nu\epsilon^\mu_{\text{out}} = 0$. Let us check whether this holds by considering the left hand side

$$M_{\mu\nu}q^\nu\epsilon^\mu_{\text{out}} = i(-ie)^2\overline{u}(p',\sigma')\left(\frac{\not{\epsilon}_{\text{out}}(\not{p}+\not{q}+m)\not{q}}{(p+q)^2-m^2} + \frac{\not{q}(\not{p}-\not{q}'+m)\not{\epsilon}_{\text{out}}}{(p'-q)^2-m^2}\right)u(\mathbf{p},\sigma), \tag{3.20}$$

where we used momentum conservation $p+q = p'+q'$. Using another trick: writing $\not{q} = \not{q}+\not{p}-m-(\not{p}-m)$ and using the Dirac equation

$$(\not{p}-m)u = 0 \qquad \overline{u}(\not{p}'-m) = 0, \tag{3.21}$$

we have

$$M_{\mu\nu}q^\nu\epsilon^\mu_{\text{out}} = i(-ie)^2\overline{u}(p',\sigma')\not{\epsilon}_{\text{out}}u(p,\sigma)\left(\frac{2p\cdot q}{\underbrace{(p+q)^2-m^2}_{2p\cdot q}} + \frac{2p'\cdot q}{\underbrace{(p'-q)^2-m^2}_{-2p'\cdot q}}\right) = 0. \tag{3.22}$$

Thus, the Ward identity holds as expected.

However, this is not the most general case we could have considered. Imagine that the couplings e_1 and e_2 at the two vertices would have been different. The terms in the big parenthesis in (3.22) would be $e_1e_2 - e_2e_1$ and, assuming these couplings do not commute, we could not have factored them out as we did. For the most general possible couplings T^a_{ij} between two fermions and a gauge boson, illustrated in Fig. 3.4, the ward identity implies

$$T^a_{ik}T^b_{kj} - T^b_{ik}T^a_{kj} = 0 \tag{3.23}$$

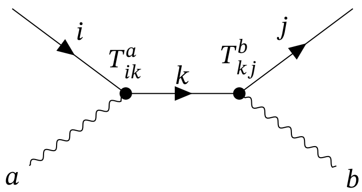


Figure 3.4: General couplings T^a_{ik} between two fermions i and k and a boson a.

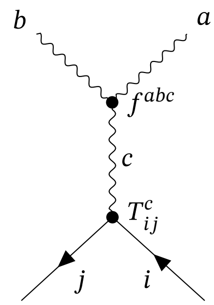


Figure 3.5: Allowing a general coupling f^{abc} between three bosons a, b, and c.

unless there is a coupling f^{abc} between the photons themselves. In that case, which is depicted in diagram 3.5, we have instead

$$T^a_{ik}T^b_{kj} - T^b_{ik}T^a_{kj} = f^{abc}T^c_{ij}. \tag{3.24}$$

$$[T^a, T^b] = if^{abc}T^c, \tag{3.25}$$

where the factor i in the second line is a matter of normalisation of the structure constants f^{abc}. This is the Lie algebra of a non-Abelian gauge (Yang–Mills) theory!

We find that a system with many massless helicity-1 fields comprises

- either photon-like particles, which do not interacting with themselves,
- or non-Abelian Yang–Mills gauge bosons with algebra $[T^a, T^b] = if^{abc}T^c$.

3.3 Yang–Mills Theory

Yang–Mills theory is the theory of a non-Abelian gauge group G.

(i) group elements $U = e^{i\theta^a T^a} \simeq \mathbb{1} + i\theta^a T^a$. The T^a are the generators and θ^a are the parameters or coordinates in the group manifold.

(ii) U acting on matter field $iU\psi$.

- We can then introduce the covariant derivative $D_\mu := \partial_\mu - igA_\mu$, which acts on ψ covariantly

$$D_\mu\psi \to UD_\mu\psi. \tag{3.26}$$

Requiring this, one can check that A_μ has to transform as

$$A_\mu \to A'_\mu = UA_\mu U^{-1} - \frac{i}{g}\partial_\mu U U^{-1}. \tag{3.27}$$

Infinitesimally, we may expand U in terms of the generators to give

$$A'^{\,a}_\mu = A^a_\mu + \frac{i}{g}\partial_\mu \alpha^a - f^{abc}\alpha^b A^c_\mu \tag{3.28}$$

- From the commutator

$$[D_\mu, D_\nu] = \left[-ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2[A_\mu, A_\nu]\right]\psi, \tag{3.29}$$

we obtain the gauge (Yang–Mills) field strength

$$F_{\mu\nu} := \frac{i}{g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{3.30}$$

It transforms covariantly as

$$F_{\mu\nu} \to F'_{\mu\nu} = UF_{\mu\nu}U^{-1} \tag{3.31}$$

$$\Rightarrow \; F'^{\,a}_{\mu\nu} = F^a_{\mu\nu} - f^{abc}\alpha^b F^c_{\mu\nu}. \tag{3.32}$$

We can build a gauge invariant (renormalisable) Lagrangian as

$$\mathscr{L} = -\frac{1}{4} g_{ab} F^a_{\mu\nu} F^{b\mu\nu} + \mathscr{L}_M(\psi, D_M\psi) + \theta F^a_{\mu\nu} \widetilde{F}^{a\mu\nu}, \tag{3.33}$$

where $\widetilde{F}^{a\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. Usually people do not write the (topological) θ-term since it is a total derivative and does not influence the physical classically. However, quantum mechanically it does! Non-perturbative features of Yang–Mills theories like instantons come from this term. Moreover, it is related to Chern–Simons theory and has some amazing mathematics behind it.

The second term $\mathscr{L}_M$ is the matter Lagrangian, coupling the gauge field to the matter fields ψ.

What about the first term? In order to behave like a real propagating field, we want positive kinetic energy. This implies that g_{ab} is constant, invariant $\mathscr{L}$, and positive definite. This implies that G cannot just be any group; it has to be compact, simple, or semi-simple. This is where it pays off that we studied the Cartan classification of these groups! We are restricted to work with the groups listed in Table 3.1.

G	rank	dimension
$SU(N)$	$N-1$	N^2-1
$SO(N)$	$\frac{N}{2}, \frac{N-1}{2}$	$\frac{N(N-1)}{2}$
$Sp(N)$	N	$N(2N+1)$
G_2	2	14
F_4	4	52
E_6	6	78
E_7	7	133
E_8	8	248

Table 3.1: Cartan Classification

Compactness implies $\text{Tr}(T^a T^a) >= 0$, which means that we have finite-dimensional representations with hermitian generators. For $SU(N)$, we have the representations:

fundamental: $\phi_i \to \phi_i + i\alpha^a (T^a_F)_{ij} \phi_j$, where T^a_F are hermitian.

antifundamental: Generators are $T^a_{AF} = -(T^a_F)^*$, so the corresponding elements of the representation $\phi^*_i \to \phi^*_i + i\alpha^a (T^a_{AF})_{ij} \phi^*_j = \phi^*_i - i\alpha^a (T^a_F)_{ji} \phi^*_j$.

adjoint: $(T^a_A)^{bc} := -if^{abc}$. This is an (N^2-1)-dimensional representation.

These are normalised as

$$\text{Tr}\big(T^a T^b\big) = \frac{1}{2}\delta^{ab} \tag{3.34}$$

$$T^a T^b = \frac{1}{2N}\delta^{ab} + \frac{1}{2} d^{abc} T^c + \frac{1}{2} i f^{abc} T^c, \tag{3.35}$$

where $d^{abc} = 2\,\mathrm{Tr}\big[T^a, \{T^b, T^c\}\big]$ is symmetric. The quadratic Casimir of the representation R is

$$C(R) = T_R^a T_R^a. \tag{3.36}$$

The index $T(R)$ is obtained as the trace of the product

$$\mathrm{Tr}\big[T_R^a T_R^b\big] = T(R)\delta^{ab}. \tag{3.37}$$

With the Lagrangian (3.33) we can look for the field equations. We take $g_{ab} = \delta_{ab}$ and ignore the θ-term since it is a total derivative, so that the Lagrangian is

$$\mathscr{L} = -\frac{1}{4}(F_{\mu\nu}F^{\mu\nu}) + \mathscr{L}_M(\psi, D\psi). \tag{3.38}$$

The Euler–Lagrange equations are

$$\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu A_\nu^a)}\right) = \frac{\partial\mathscr{L}}{\partial A_\nu^a}, \tag{3.39}$$

which give the field equations

$$-\partial_\mu F^{a\mu\nu} = -gF^{c\nu\mu}f^{abc}A_\mu^b - i\frac{\partial\mathscr{L}_\mu}{\partial(D_\nu\psi)}T^a\psi. \tag{3.40}$$

These field equations can also be written

$$\partial_\mu F_a^{\mu\nu} = -J_a^\nu, \tag{3.41}$$

with the current J_a^ν being defined as

$$J_a^\nu = -gf_{abc}F_c^{\nu\mu}A_{b\mu} - i\frac{\partial\mathscr{L}_\mu}{\partial(D_\nu\psi)}T_a\psi. \tag{3.42}$$

This current is conserved (by Noether's theorem), meaning that $\partial_\nu J_a^\nu = 0$. However, it is not gauge-covariant!

On the other hand, the field equations can be rewritten

$$\boxed{D_\mu F_a^{\mu\nu} = -j_a^\nu} \tag{3.43}$$

in terms of the covariant derivative and another current

$$j_a^\nu = -i\frac{\partial\mathscr{L}_\mu}{\partial(D_\nu\psi)}T_a\psi. \tag{3.44}$$

This current is not conserved, but it is covariantly conserved, meaning $D_\nu j_a^\nu = 0$!

This is very reminiscent of gravity, where the stress tensor is covariantly conserved.

We also have the Bianchi identity

$$D_\mu F^a_{\nu\lambda} + D_\nu F^a_{\lambda\mu} + D_\lambda F^a_{\mu\nu} = 0. \tag{3.45}$$

This gives

$$\boxed{D_\mu \widetilde{F}^{\mu\nu} = 0} \tag{3.46}$$

Equations (3.43) and (3.46) generalise Maxwell's equations.

This is very similar but not identical to electromagnetism. It generalises it in several ways. The term $F^a_{\mu\nu}F^{a\mu\nu}$ has terms ∂AAA and $AAAA$, which are self-interactions between the field that were absent for the classical photon! On the other hand, the similarity with gravity is compelling. There is a geometrical interpretation of definition (3.30) of the Maxwell tensor F as the curvature of the internal gauge space with A being the connection on the fibre bundle.

4 Broken Symmetries

4.1 Motivation

So far spin/helicity $0, \pm\frac{1}{2}$ OK

massless helicity $\pm 1 \ \Rightarrow$ QED, Yang–Mills

massless helicity $\pm 2 \ \Rightarrow$ gravity

No more interactions.

But massless Yang–Mills fields have not been observed.

What about massive $s = 1$? A massive spin-1 field $A_\mu(x,t)$ is associated with a polarisation vector $\epsilon_\mu(p)$. It has three polarisation states $j_z = -1, 0, 1$. Since $A_\mu(x,t)$ has 4 degrees of freedom, we impose the condition $p^\mu \epsilon_\mu = 0$ to get three polarisations. There is no gauge redundancy.

Pick $p^\mu = (E, 0, 0, p_z)$ with the condition that $E^2 - p_z^2 = m^2$. We have two transverse polarisations $\epsilon_1^\mu = (0,1,0,0)$ and $\epsilon_2^\mu = (0,0,1,0)$ and one longitudinal polarisation. With the normalisation condition $\epsilon_\mu^2 = -1$, there is only one possibility: $\epsilon_L^\mu = (\frac{p_z}{m}, 0, 0, \frac{E}{m})$.

Suppose we are at high energies, where $E \gg m$. We then have $p \approx E$. Then the longitudinal polarisation becomes $\epsilon_L^\mu \approx \frac{E}{m}(1,0,0,1)$. The amplitudes will have a term $M \sim g^2 \epsilon_L^0 \epsilon_L^z \sim g^2 \frac{E^2}{m^2}$. This blows up! However, the amplitudes are probabilities, which have to be less than one; so this blowing-up breaks unitarity. The theory of massless spin-1 particles must have a cutoff; it cannot be valid until high energies since it breaks perturbative unitarity.

Example 4.1.1: For a mass $m \simeq 100$GeV and coupling $g \sim 0.1$, the energy has to be less than $E \lesssim 1$Tev.

Massive spin-1 particles lead to interactions that can be valid only at small energies and have to be superseded by a consistent UV completion.

To search for a UV completion?

4.2 Spontaneously Broken Discrete Symmetries

Let us warm up to this concept slowly by considering discrete symmetries. Take the theory of a real scalar field with symmetry $\phi \to -\phi$. The most general renormalisable Lagrangian is

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V_\pm(\phi). \tag{4.1}$$

There are two possible scalar potentials

$$V_\pm(\phi) = \pm\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 + \kappa_\pm. \tag{4.2}$$

Here, $\lambda > 0$ for stability and $\kappa_\pm$ is introduced to be able to tune the value of V at the minimum.

Remark: Gauge invariance is imposed only for massless particles. For massive particles, the most general theory involves a mass-term, which breaks gauge invariance. This is totally okay, since gauge invariance is only secondary, coming from our requirement of Lorentz invariance.

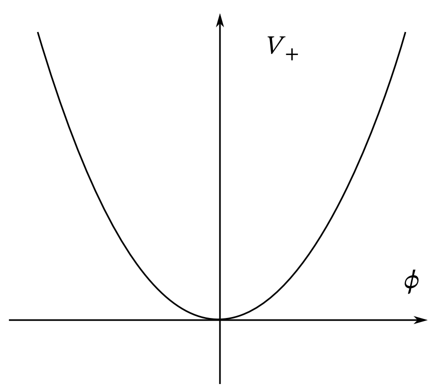


Figure 4.1

The potential V_+ has a minimum at $\phi = \phi_0 = 0$ and is symmetric under $\phi \to -\phi$, as illustrated in Fig. 4.1. We say the "symmetry is manifest".

In the quantum theory, we want to investigate vacuum expectation values (VEV)

$$\langle\phi\rangle := \langle 0|\,\phi\,|0\rangle = \int \mathcal{D}\phi\,\phi e^{\frac{i}{\hbar}\int\mathscr{L}\mathrm{d}^4x}. \tag{4.3}$$

Taking the classical limit $\hbar \to 0$, we recover the minimum of the potential $\langle\phi\rangle \to \phi_0$.

Perturbations around the vacuum

$$\phi = \phi_0 + \sigma(x) = \sigma(x). \tag{4.4}$$

The Lagrangian becomes

$$\mathscr{L} = \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma - \frac{1}{2}m^2\sigma^2 - \frac{\lambda}{4}\sigma^4, \tag{4.5}$$

so σ is a particle of mass m.

We can write the quartic V_- as

$$V_- = \frac{\lambda}{2}\left(\phi^2 - v^2\right)^2, \qquad v := \sqrt{\frac{m^2}{\lambda}}. \tag{4.6}$$

This potential is illustrated in Fig. 4.2. We have two vacua $\langle\phi\rangle = \pm v$ degenerate. Perturbing around

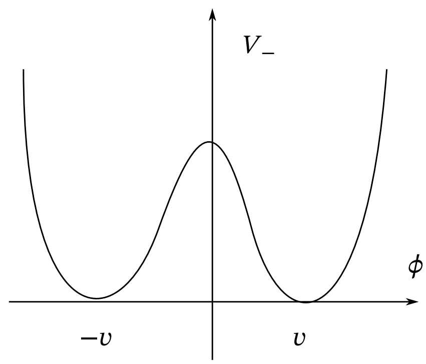


Figure 4.2: If you sit in the vacua $\phi = \pm v$, you do not see the symmetry of the potential.

the vacuum $\phi = \pm v + \sigma(x)$ gives

$$\mathscr{L} = \frac{1}{2}\partial_\mu\sigma\partial_\mu\sigma - \left(\lambda v^2\sigma^2 \pm \lambda v\sigma^3 + \frac{\lambda}{4}\sigma^4\right). \tag{4.7}$$

In this case $\lambda v^2\sigma^2 = m^2\sigma^2 + \dots$. So σ is a particle of squared mass $2m^2 \geq 0$. The second derivative of the potential gives the mass of the particle.

$$m^2 = \left.\frac{\partial^2 V_-}{\partial\phi^2}\right|_{\phi^2=v^2} = 2m^2. \tag{4.8}$$

We got a cubic term in the Lagrangian potential. The symmetry $\phi \to -\phi$ is *hidden* (spontaneously broken). However, the symmetry is of course still there since we have $\sigma \to -\sigma \mp 2v$. Some people say the symmetry is "non-linearly realised", when we have $\phi \to a\phi + X$, where X is something else.

If we would have expanded around $\phi = 0$, we have a tachyonic mass

$$\left.\frac{\partial^2 V}{\partial\phi^2}\right|_{\phi=0} = -m^2. \tag{4.9}$$

In principle, you can consider superpositions of the two vacua, which recovers the symmetry. This is discussed in [Weinberg, Vol. 2]; the upshot is that in large systems you still always have to choose one particular vacuum.

This system is very simple, just possessing a discrete symmetry, but is nonetheless very rich. Consider a thermodynamic system where the coefficient of ϕ^2 depends on the temperature. We have $m^2 \propto (T - T_c)$. For very high temperatures, $T \gg T_c$, we have a potential like V_+. Reducing the temperature to $T < T_c$, the symmetry is broken to the potential V_-. This model explains systems ranging from magnetic materials to cosmological systems. These models can have rich properties such as domain walls and other things we discussed in the *Statistical Field Theory* course.

4.3 Spontaneous Breaking of Continuous Global Symmetries

Let us generalise Sec. 4.2. Instead of a single field, consider an N-component scalar field $\phi = (\phi_1, \phi_2, \dots, \phi_N)^T$ with Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial^\mu \phi \cdot \partial_\mu \phi - V_\pm(\phi).. \tag{4.10}$$

In the case of V_+, the minima are symmetric. We will consider the more interesting case of V_-, which is given by

$$V_-(\phi) = \frac{\lambda}{4}(\phi\cdot\phi - v^2)^2, \qquad v^2 = \frac{m^2}{\lambda}, \quad \lambda > 0. \tag{4.11}$$

This Lagrangian is symmetric under the rotation group $O(N)$. The vacua lie at $\phi \cdot \phi = v^2$. The potential is shown in Fig. 4.3; we have a continuum of vacua.

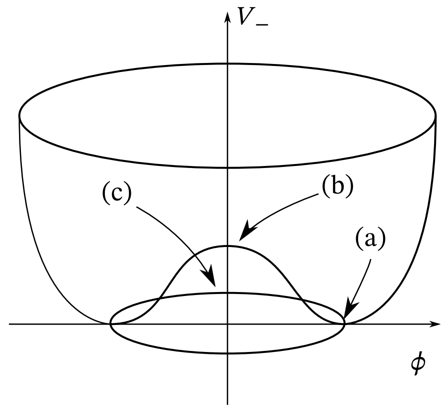


Figure 4.3: The curvature of the vacuum in the (a)-direction gives m_σ^2. The (c)-directions correspond to π's with 0 mass. The curvature at (b) gives the tachyon.

Pick $\langle \phi_0 \rangle = (0, \dots, 0, v)^T$. Fluctuations are then $\phi(x) = (\pi_1(x), \dots, \pi_{N-1}(x), V + \sigma(x))$. Making the substitution, the Lagrangian and potential are

$$\mathscr{L} = \frac{1}{2}\partial_\mu \pi \cdot \partial^\mu \pi + \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma - V_-(\pi_i, \sigma), \tag{4.12}$$

$$V_-(\pi, \sigma) = \frac{1}{2}(2\lambda v^2)\sigma^2 + \lambda(\sigma^2 + \pi^2)\sigma + \frac{\lambda}{4}(\sigma^2 + \pi^2)^2 \tag{4.13}$$

This is essentially the same thing we did for the discrete case. The only field that has a quadratic piece by itself in the potential is σ. There is no mass term for π, only for σ, where $m_\sigma^2 = 2\lambda v^2$. The mass matrix is

$$M_{ij} = \frac{\partial^2 V}{\partial \phi_i \partial \phi_j}|_{\phi = \langle\phi\rangle} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & m_\sigma^2 \end{pmatrix} \tag{4.14}$$

We have $N-1$ massless fields π, called *Goldstone bosons*. The symmetry is broken from $O(N) \to O(N-1)$.

4.3.1 Goldstone's Theorem

In general, if the Lagrangian $\mathscr{L}$ is invariant under a continuous (compact and semisimple) G such that the vacuum expectation value $\langle\phi\rangle \neq 0$ breaks symmetry $G \to H \subset G$. Define the *vacuum manifold* to be the space of all minima

$$\mathcal{M} = \{\phi_0 \mid V(\phi_0) = V_{\text{minimum}}\}, \qquad \phi_0 := \langle\phi\rangle. \tag{4.15}$$

Different vacua are related by group transformations $g \in G$ as $\phi_0' = g\phi_0$. We also define the invariant or stability group $H = \{h \in G \mid h\phi_0 = \phi_0\}$. Observing now that

$$\phi_0' = gh\phi_0 = (ghg^{-1})\phi'. \tag{4.16}$$

Therefore $ghg^{-1} \in H$. As such, $g \in G$, mapping one vacuum to another, defines equivalence classes: $g_1 \sim g_2$ if $\exists h \in H$ such that $g_1 = g_2 h$. We then have

$$\mathcal{M} \simeq \frac{G}{H}. \tag{4.17}$$

Consider now an infinitesimal transformation

$$g\phi = \phi + \delta\phi, \qquad \delta\phi = i\alpha^a T^a \phi, \quad a = 1, \dots, \dim G. \tag{4.18}$$

We can expand in a Taylor expansion and use the symmetry $V(\phi + \delta\phi) = V(\phi)$:

$$V(\phi + \delta\phi) - V(\phi) = i\alpha^a (T^a\phi)_r \frac{\partial V}{\partial \phi_r} = 0. \tag{4.19}$$

This is an important expression, which we will use later. If ϕ_0 is a minimum, then

$$V(\phi) - V(\phi_0) = \frac{1}{2}(\phi - \phi_0)_r \left.\frac{\partial^2 V}{\partial\phi_r \phi_s}\right|_{\phi=\phi_0} (\phi - \phi_0)_s. \tag{4.20}$$

We recognise the mass matrix appearing here. Differentiate (4.19) and evaluate at $\phi = \phi_0$:

$$(T^a\phi_0)_r \left.\frac{\partial^2 V}{\partial\phi_r \phi_r}\right|_{\phi_0} = 0. \tag{4.21}$$

The lessons from this are twofold.

- If the symmetry is unbroken and the vacuum unique, meaning that $g\phi_0 = \phi_0, \forall g \in G$, then $\delta\phi = 0$, then $(T^a\phi_0) = 0$ for all a.

- If $\exists g \in G$ such that there exists $T^a \phi_0 \neq 0$, then $T^a \phi_0$ is an eigenvector of the mass matrix with zero eigenvalue; these are the Goldstone bosons.

$$(T^a \phi_0)_r M^2_{rs} = 0. \tag{4.22}$$

How many massless states do we have? Let us split the generators T^a into two groups. We denote by $\widetilde{T}^i$ the elements of H and by $R^{\hat{a}}$ the remaining ones. The assumptions of compactness and semisimplicity mean that we have the orthogonality condition

$$\operatorname{Tr} \widetilde{T}^i R^{\hat{a}} = 0. \tag{4.23}$$

This means that each vector $R^{\hat{a}}_0$ is a unique eigenvector of M_{rs}. Since $i = 1, \dots, \dim H$ and $\hat{a} = \dim H, \dots, \dim G$, we find that the number of Goldstone bosons is

$$\dim \frac{G}{H} = \dim G - \dim H. \tag{4.24}$$

Example 4.3.1 ($O(N)$)**:** Take $G = O(N)$ and $H = O(N-1)$, then the number of Goldstone bosons is

$$\dim O(N) - \dim O(N-1) = \frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N - 1. \tag{4.25}$$

4.3.2 The Quantum Version

Claim 9: The Noether charges $Q^a = \int d^3x\, J_0^a$ are also generators of the symmetry.

$$[\phi_i, Q^a] = iT^a_{ij} Q_j, \tag{4.26}$$

where T^a_{ij} are the generators.

There are several points of observation we should make:

4.3.3 Order Parameter

The order parameter of spontaneous symmetry breaking is

$$\langle 0|\,\phi\,|0\rangle = \langle\phi\rangle \begin{cases} = 0, & \text{if unbroken} \\ \neq 0, & \text{if broken} \end{cases}. \tag{4.27}$$

If broken,

$$\langle\phi\rangle \neq 0 \;\Rightarrow\; \langle[\phi, Q]\rangle \neq 0 \;\Rightarrow\; \langle 0|\,(\phi Q - Q\phi)\,|0\rangle \neq 0 \;\Rightarrow\; Q\,|0\rangle \neq 0. \tag{4.28}$$

This is another way to state symmetry breaking: the symmetry is broken if the charges do not annihilate the vacuum. Equivalently, we can take an action U of the group on the minimum of the potential ϕ_0, so that for an unbroken symmetry we have

$$U\phi_0 = e^{i\alpha^a T^a}\phi_0 = (1 + i\alpha^a T^a)\phi_0 = 0 \;\Rightarrow\; i\alpha^a T^a \phi_0 = 0. \tag{4.29}$$

4.3.4 Degenerate Energies

Degenerate energies: Usually in QM, when we have states $|\psi\rangle = Q\,|\chi\rangle$ related by a symmetry Q, which obeys $[Q, H] = 0$, then

$$H\,|\psi\rangle = E_\psi\,|\psi\rangle = HQ\,|\chi\rangle = QH\,|\chi\rangle = E_\chi Q\,|\chi\rangle = E_\chi\,|\psi\rangle\,, \tag{4.30}$$

so the energies $E_\psi = E_\chi$ of the two states are degenerate.

However, in Field Theory, if ϕ_1, ϕ_2 are related by an action $i\phi_1 = [\phi_2, Q]$ of the charge Q, then

$$|1\rangle\, a_1^\dagger\,|0\rangle = i[a_2^\dagger, Q]\,|0\rangle = ia_2^\dagger Q\,|0\rangle - iQa_2^\dagger\,|0\rangle = -iQ\,|2\rangle + ia_2^\dagger Q\,|0\rangle\,. \tag{4.31}$$

So $|1\rangle \propto Q\,|2\rangle$ only if $Q\,|0\rangle = 0$, meaning that the symmetry is unbroken. If the vacuum does not preserve the energy, we have an extra term $ia_2^\dagger Q\,|0\rangle$, which prevents energy degeneracy.

Consider the case of a broken symmetry. Since $[H, Q^a] = 0$, we have

$$HQ^a |0\rangle = Q^a H |0\rangle = E_0 Q^a |0\rangle . \tag{4.32}$$

Therefore, if $Q^a |0\rangle \neq 0$, then the corresponding state $Q^a |0\rangle$ has the same energy as the vacuum $|0\rangle$ and we have degenerate energies.

Now consider the momentum states

$$|\pi^a(\mathbf{p})\rangle = K \int \mathrm{d}^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} J_0^a |0\rangle . \tag{4.33}$$

These have energy $E(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + m^2}$ due to the momentum and energy E_0 due to the vacuum. Since $|\pi^a(0)\rangle \propto Q^a |0\rangle$ has energy E_0, as shown in (4.32), this means that $E(\mathbf{p}) \to 0$ as $|\mathbf{p}| \to 0$. This in turn means that $m = 0$. These states are the Goldstone modes, labelled by broken generators Q^a. This is the quantum proof of the Goldstone theorem.

Effective Action

Recall that the Wilsonian effective action $W(J)$ is defined by

$$e^{iW(J)} := \int \mathcal{D}\phi\, e^{i\int (L+J\phi)\mathrm{d}^4x}. \tag{4.34}$$

It consists of the sum of connected diagrams. Taking the variational derivative, we define the classical field $\phi_c(x)$ as

$$\frac{\partial W}{\partial J} \frac{\int \mathcal{D}\phi\, \phi e^{i\int(\mathscr{L}+J\phi)\mathrm{d}^4x}}{\int \mathcal{D}\phi\, e^{i\int(\mathscr{L}+J\phi)\mathrm{d}^4x}} = \frac{\langle 0|\, \phi\, |0\rangle}{\langle 0|0\rangle} := \phi_c(x). \tag{4.35}$$

The quantum effective action Γ is then defined as the Legendre transform

$$\Gamma(\phi_c) = W(J) - \int \mathrm{d}^4x\, J\phi_c, \qquad \frac{\delta\Gamma}{\delta\phi_c} = -J. \tag{4.36}$$

As we know from *Advanced Quantum Field Theory*, Γ consists of the sum of 1PI diagrams and generates the inverse n-point functions. In particular, it generates the inverse propagator

$$\frac{\delta^2\Gamma}{\delta\phi_c\delta\phi_c} = \Delta^{-1}. \tag{4.37}$$

This is the mass. Spontaneous symmetry breaking is then the statement that

$$\frac{\delta\Gamma}{\delta\phi_c} = 0 \qquad \text{for} \qquad \phi_c \neq 0. \tag{4.38}$$

This means that

$$\Gamma = \int \mathrm{d}^4x \left(V_{\text{eff}} + \frac{1}{2}(\partial_\mu\phi_c)^2 + \dots \right). \tag{4.39}$$

At zero momentum, this means that we have

$$\frac{\delta\Phi}{\delta\phi_c} = 0 \;\Rightarrow\; \frac{\partial V_{\text{eff}}}{\partial\phi_c} = 0. \tag{4.40}$$

This is another proof of the Goldstone theorem.

4.4 Spontaneous Breaking of Gauge Symmetries

We have met three problems:

- Massive spin-1 field theory is not valid at high energies (unitary).
- Massless Yang–Mills fields are not seen.
- Massless Goldstone modes are not seen.

The first is physical but not consistent while the last two are consistent but not physical! We can address them all at once with the Higgs mechanism.

4.4.1 Abelian Higgs Model

Consider a theory governed by the Lagrangian

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^{\mu}\phi D_{\mu}\phi^{*} - V(|\phi|^2), \tag{4.41}$$

where the covariant derivative is $D_{\mu}\phi = (\partial_{\mu} + ieA_{\mu})\phi$ and the potential is

$$V = \frac{\lambda}{4}(|\phi|^2 - v^2)^2. \tag{4.42}$$

This exhibits a $U(1)$ symmetry

$$\phi \to e^{i\alpha(x)\phi}, \qquad A_{\mu} \to A_{\mu} - \frac{1}{e}\partial_{\mu}\alpha. \tag{4.43}$$

The vacuum lies at $\langle\phi\rangle = V\rho$, where $\rho = e^{i\theta}$ is a phase. Let us pick the real vacuum at $\theta = 0$.

Just as for the spontaneous symmetry breaking in the $O(N)$ case, we perturb around the vacuum

$$\phi = e^{i\xi(x)}(\eta(x) + v). \tag{4.44}$$

The kinetic and potential terms in the Lagrangian change as

$$D^{\mu}\phi D_{\mu}\phi^{*} \to \partial^{\mu}\eta\partial_{\mu}\eta + (\eta + v)^2(\partial^{\mu}\xi + eA^{\mu})^2 \tag{4.45}$$

$$V \to \frac{\lambda}{4}[(\eta + v)^2 - v^2]^2 = (v^2\eta)\eta^2 + (\lambda v)\eta^3 + \frac{1}{4}\eta^4. \tag{4.46}$$

Now we can see something magical happen. In the Lagrangian we had a kinetic term $F^{\mu\nu}F_{\mu\nu}$, which we did not touch. We also had a kinetic term $D^{\mu}\phi D_{\mu}\phi^{*}$. In the potential we have a mass term for η and self interactions, as well as interactions of η and A in the kinetic term of ϕ. However, curiously,

ξ does not appear in the potential. It only appears together with A. In particular, we can redefine the field A to include the Goldstone mode ξ.

$$A^\mu \to A^\mu + \frac{1}{e}\partial^\mu \xi. \tag{4.47}$$

This absorption, sometimes called the *unitary gauge*, is the *Higgs mechanism*. It results in A getting a mass term. The $F^{\mu\nu}F_{\mu\nu}$ term is unchanged under this gauge transformation and the total Lagrangian becomes

$$\mathscr{L} = \mathscr{L}^{\text{quadratic}} + \mathscr{L}^{\text{interaction}}, \tag{4.48}$$

$$\mathscr{L}^{\text{quadratic}} = -\frac{1}{4} + \frac{1}{2}\partial^\mu\eta\partial_\mu\eta - (V^2\lambda)\eta^2 + (\frac{1}{2}e^2v^2)A^\mu A_\mu \tag{4.49}$$

$$\mathscr{L}^{\text{interaction}} = (\lambda v)\eta^3 + \frac{\lambda}{4}\eta^4 + \frac{1}{2}(\eta^2 + 2v\eta)A^\mu A_\mu. \tag{4.50}$$

We can see that the *Higgs boson* η has a squared mass $m_\eta^2 = \frac{1}{2}v^2\lambda$ and the gauge field A_μ has squared mass $m_A^2 = e^2v^2$. This Higgs mechanism solves our problems: The gauge field acquires a mass, the Goldstone boson is “eaten”, and we have a consistent theory of a massive spin-1 field. We have 3 degrees of freedom; 2 from A_μ and 1 from the Goldstone boson.

The Abelian Higgs model describes the effective field theory of superconductivity. In this case, $\langle\phi\rangle \sim e^-e^-$ is the Cooper pair and $A^\mu A_\mu$ gives a magnetic field, which is energetically disfavoured. This is the Meissner effect. The penetration depth $R = \frac{1}{m_A}$ coincides with the mass of the gauge field. In the Yukawa case, it tells us how deeply the magnetic field enters into the superconductor. Moreover, we have vortex lines with correlation length $\xi = \frac{1}{m_\eta}$ and $\xi > R$ gives type I, whereas $\xi < R$ gives type II, where you can have vortices. Superconductivity is an example where you break to a discrete symmetry.

Since we map from a circular vacuum to three dimensional space, we have what is called *cosmic strings*:

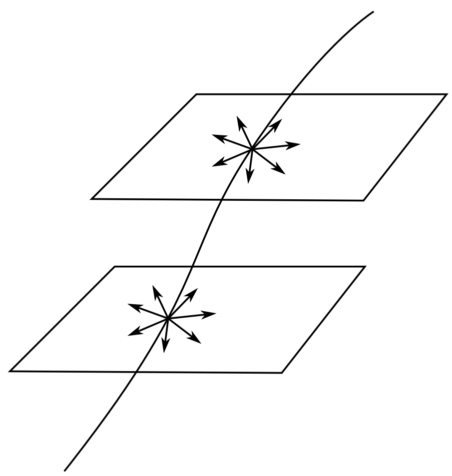


Figure 4.4: Cosmic Strings

4.5 Anomalies

▌ Non-examinable.

We have finished the spontaneous symmetry breaking part but for completeness we should mention another mechanism of symmetry breaking.

Definition 17 (anomalies)**:** *Anomalies* appear whenever you have classical symmetries, which are broken by quantisation.

In quantum theory, we have much for than a Lagrangian. It might be possible that the symmetries of the Lagrangian might disappear upon quantisation. Recall that in quantum field theory, we work with the path integral formulation of quantum mechanics

$$\int \mathcal{D}\phi \, e^{iS[\phi]}. \tag{4.51}$$

Consider a classical symmetry $\phi \to \phi'$, defined by the fact that the action transforms as $S \to S$. But the path integral measure also needs to be invariant for the physics to stay the same! The measure transforms as

$$\mathcal{D}\phi \to \mathcal{D}\phi' \, \mathcal{J}. \tag{4.52}$$

If the Jacobian $\mathcal{J} \neq 0$, then the symmetry is broken. The presence of anomalies has two different effects depending on the type of symmetry. In a global symmetry, the symmetry is simply broken and nothing bad happens. However, in the case of local (gauge) symmetries, anomalies are a killer: they imply that the theory is inconsistent. Every single gauge theory that we deal with has to be free of anomalies. It can be shown that chiral theories are automatically anomaly-free. This subject is very broad and deep, and we will not go into too much detail here, but it is encouraged to read up more on anomalies.

4.5.1 Anomalies in QED

Consider for instance QED:

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}_L(i\partial\!\!\!/ - eA\!\!\!/)\psi_L + \overline{\psi}_R(i\partial\!\!\!/ - eA\!\!\!/)\psi_R - m\overline{\psi}_L\psi_R - m\overline{\psi}_R\psi_L. \tag{4.53}$$

For the massless limit $m \to 0$, we have two symmetries:

$$\psi \to e^{i\alpha}\psi, \qquad \psi \to e^{i\beta\gamma_5}\psi \tag{4.54}$$

$$\psi_L \to e^{i(\alpha+\beta)}\psi_L, \qquad \psi_R \to e^{i(\alpha+\beta)}\psi_R. \tag{4.55}$$

We can then find conserved currents

$$J_V^\mu = \overline{\psi}\gamma^\mu\psi \qquad J_A^\mu = \overline{\psi}\gamma^\mu\gamma^5\psi, \tag{4.56}$$

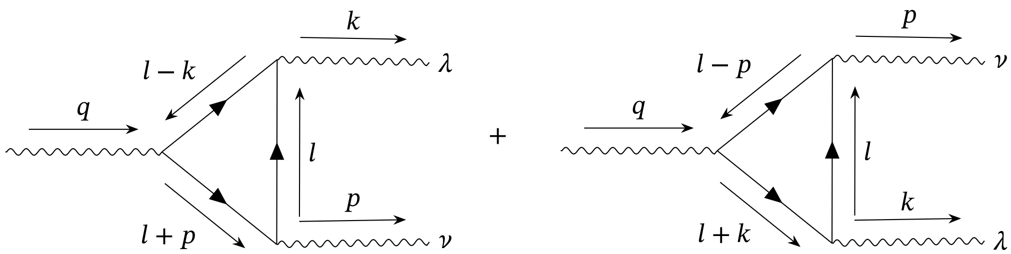
with $\partial^\mu J_\mu = \partial^\mu J_{\mu 5} = 0$. We call J_V^μ the *vector current* and J_A^μ the *axial current*. For the massive case $m \neq 0$,

$$\partial_\mu J_V^\mu = 0, \qquad \partial_\mu J_A^\mu = 2im\overline{\psi}\gamma^5\psi, \tag{4.57}$$

only the vector current is conserved.

We will now see that, even in the massless limit, the axial current conservation will be broken.

Consider the following *triangle diagrams*



(4.58)

These kinds of diagrams created a big problem in the community, since people did not know how to deal with it. They are the source of the anomalies.

$$\int \mathrm{d}^4x\, e^{iq\cdot x} \langle p,k|\, J_A^\mu(x)\, |0\rangle = \delta^4(p+k-q)\epsilon_\nu(p)\epsilon_\lambda(k)\mathcal{M}^{\mu\nu\lambda}(p,k). \tag{4.59}$$

Computing $\partial_\mu J_A^\mu$ is equivalent to $q_\mu \mathcal{M}^{\mu\nu\lambda}$. We want to check whether the Ward identity is satisfied:

$$q_\mu M^{\mu\nu\lambda}\epsilon_\nu\epsilon_\lambda \stackrel{?}{=} 0. \tag{4.60}$$

We observe that both cannot vanish since

$$\langle p,k|\, \partial_\mu J^{\mu 5}\, |0\rangle = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}(-ip_\mu)\epsilon_\nu(p)(-ik_\alpha)\epsilon_\beta(k) \tag{4.61}$$

$$= -\frac{e^2}{2\pi^2}\langle p,k|\, e^{\alpha\lambda\beta\nu}F_{\alpha\beta}F_{\lambda\nu}\, |0\rangle\,. \tag{4.62}$$

$$\Rightarrow\ \partial_\mu J_A^\mu = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \tag{4.63}$$

This is caled the *Adler–Bell–Jackiw anomaly*. To obtain this, there is an integral

$$\int_{-\infty}^{\infty} (f(x+a) - f(x))\,\mathrm{d}x\,. \tag{4.64}$$

Naively, these seem to cancel each other, giving zero. However, if we expand the first term in a Taylor series, we obtain

$$\int \big(f'(x)a + \ldots\big)\,\mathrm{d}x = f(\infty) - f(-\infty), \tag{4.65}$$

where $f(\infty), f(-\infty)$ are constants that do not cancel each other.

For the path integral,

$$\int \mathcal{D}\psi\,\mathcal{D}\overline{\psi}\,\mathcal{D}A\, e^{i\int \mathrm{d}^4x\left(-\frac{1}{4}F_{\mu\nu}^2 + i\overline{\psi}\not{D}\psi\right)}. \tag{4.66}$$

Under the transformations (4.54), of the form

$$\psi \to \Delta\psi, \qquad \overline{\psi} \to \Delta_c\overline{\psi}, \tag{4.67}$$

we have $S \to S$, but the measure transforms as

$$\mathcal{D}\overline{\psi}\,\mathcal{D}\psi \to (\mathcal{J}_c \mathcal{J})^{-1}\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi, \tag{4.68}$$

$$= e^{\operatorname{Tr}\ln\Delta} = e^{\int \mathrm{d}^4x \langle x|\operatorname{Tr}\ln\Delta(x)|x\rangle} \tag{4.69}$$

where $\mathcal{J} = \det\Delta$, $\mathcal{J}_c = \det\Delta_c$. For $\Delta = e^i\beta\gamma_5$, we have

$$\mathcal{J} = e^{-i\int \mathrm{d}^4x\left(\beta\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right)}. \tag{4.70}$$

The result which we obtain from this:

$$\partial_\mu\langle J^{\mu 5}\mathcal{O}(x,\dots)\rangle = -\frac{e^2}{16\pi^2}\langle\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\mathcal{O}(x,\dots)\rangle \tag{4.71}$$

is in fact true to all loop orders.

4.5.2 Anomalies in Yang–Mills Theory

In the non-Abelian case, the corresponding diagrams contribute

$$\operatorname{Tr}\left(T^aT^bT^c\right) + \operatorname{Tr}\left(T^aT^cT^b\right) \tag{4.72}$$

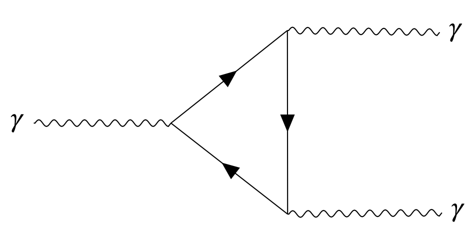
We define

$$A^{abc} = 2\operatorname{Tr}\left[T^a_R\{T^b_R, T^c_R\}\right] = A(R)d^{abc}, \tag{4.73}$$

where the subscript R denotes a particular fermion representation and d the fundamental. Then the axial current is

$$\partial_\mu(J^a_A)^\mu = \left(\sum_{\text{left}} A(R_l) - \sum_{\text{right}} A(R_r)\right)\frac{g^2}{128\pi^2}d^{abc}\epsilon^{\mu\nu\alpha\beta}F^b_{\mu\nu}F^c_{\alpha\beta} \tag{4.74}$$

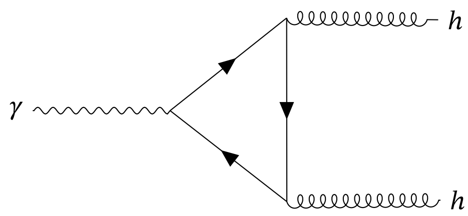
The key thing to compute is the quantity in parenthesis. If it is equal to zero, the theory is safe. If not, the theory is dead. This is a test of consistency for the theory. In general, the important thing is the difference between left- and right-handed spinors. In a chiral theory, in which left and right handed representations are the same, the anomalies will cancel automatically and the theory is consistent. For $U(1)^3$,



$$\sim \quad A \propto \operatorname{Tr}\left(Q^3\right)\begin{cases} = 0, & \text{consistent} \\ \neq 0, & \text{inconsistent}\end{cases} \tag{4.75}$$

Gravity

Finally, we can consider gravity.



$$A \propto \mathrm{Tr}(Q) \begin{cases} = 0, & \text{if consistent} \\ \neq 0, & \text{if inconsistent} \end{cases}, \tag{4.76}$$

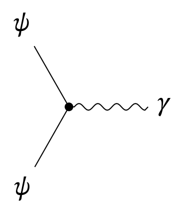
where h denotes a graviton.

5 Electroweak Interactions

5.1 Introduction

5.1.1 QED Interaction Processes

Recall in QED, we had the basic interaction vertex



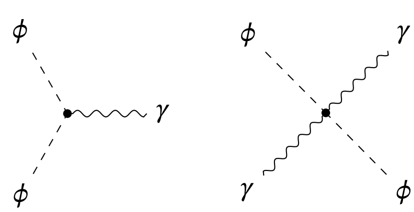
(5.1)

The current is $J^\mu = i\overline{\psi}\gamma^\mu\psi$. We can have various physical processes:

- $e^-e^- \to e^-e^-$
- $e^+e^- \to e^+e^-$
- $e^-\gamma \to e^-\gamma$
- $e^-e^- \to e^-e^-$
- $e^+e^- \to \mu^+e^-$
- $\ldots$

Remark: In the whole standard model, the last interaction is not possible. However, in QED itself, the lepton number does not need to be conserved.

Also, we have scalar QED with the diagrams



(5.2)

5.1.2 Weak Interaction Processes

(i) Leptonic:

- $\mu^- \to e^- \nu_\mu \overline{\nu}_e$
- $\nu_\mu e^- \to \nu_\mu e^-$
- $\nu_\mu e^- \to \nu_e \mu^-$

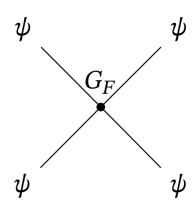
(ii) Semi-leptonic:

- $n \to p + e^- + \overline{\nu}_e$
  $(d \to u + e^- + \overline{\nu}_e)$

(iii) Non-leptonic:

- $\Lambda^0 \to p + \pi^-$
  $(s \to u + d + u)$

People found all of these interactions and needed to come up with a description to unify these. A nice proposal came from Fermi: he realised that all of these are four-body interactions, so we can have a *Fermi coupling*



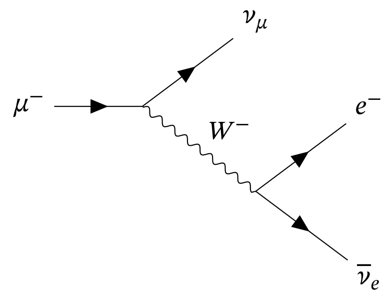
$$\mathscr{L} = G_F \psi^4 \tag{5.3}$$

Automatically, looking at the dimension, we have

$$G_F \sim \frac{1}{M^2} \tag{5.4}$$

and therefore the Lagrangian is non-renormalisable.

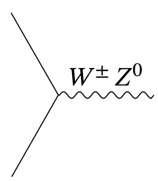
The natural thing to expect is that for energies smaller than a certain mass, this coupling is working, but needs to be replaced by something else at higher energies. This is akin to how electromagnetism sees a direct interaction between electrons; the photon does not appear at low energies. In such higher energies, we introduce the $W^\pm$ such that the first leptonic interaction becomes



(5.5)

Properties of Weak Interactions

The basic interaction vertex is



(5.6)

These interactions are short range (long lifetimes), which in analogy to the Yukawa potential, going as $e^{-M/r}/r$, means that the mediators must be massive $M_{W^\pm}, M_{Z^0} \neq 0$. For energies less than $M_{W^\pm}, M_{Z^0}$, we effectively have the Fermi interaction (5.3). Moreover, interactions are *chiral*, meaning that they break parity.

We know that the only way to describe these interactions in field theory valid at small and large energies is through a spontaneously broken gauge symmetry.

5.1.3 Identifying the Gauge theory

Start with electrons and neutrinos. We know that electrons e_L, e_R can be left- and right-handed. However, neutrinos ν_L only exist in their left-handed version, since we have not observed right-handed neutrinos in experiments. We cannot put particles with differing chiralities into the same multiplet, since it would not be Lorentz invariant. In general, we could have a doublet $\begin{pmatrix} \nu_l \\ e_L \end{pmatrix}$ and a singlet, which just contains e_R. This corresponds to the group

$$G = SU(2)_L \times U(1) \times U(1)_R \tag{5.7}$$

with generators T^a, Q_L, Q_R respectively. We know the fundamental representation for $SU(2)$ is given by the Pauli matrices

$$T^a = \frac{1}{2}\sigma^a, \tag{5.8}$$

acting on the doublet $\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$. Moreover, the Q_L and Q_R act as

$$Q_L \begin{cases} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\ e_R = 0 \end{cases}; \qquad Q_R \begin{cases} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = 0 \\ e_R = 1 \end{cases} \tag{5.9}$$

From this, we define the *hypercharge* Y and *electron lepton number* L_e as

$$Y := -Q_R - Q_L \qquad \Rightarrow Y\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \qquad Ye_R = -e_R, \tag{5.10}$$

$$L_e := 2Q_L + Q_R \qquad \Rightarrow L_e\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \qquad L_e e_R = e_R. \tag{5.11}$$

We then define the *electric charge*

$$\boxed{Q := T^3 + Y} \tag{5.12}$$

which acts on the mutliplets as

$$Q\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \begin{pmatrix} 0 \\ -e_L \end{pmatrix}, \qquad Qe_R = -e_R. \tag{5.13}$$

But remember that the gauge group (5.7) contains also a $U(1)_{L_e}$ factor. However, there is evidence that there is no gauge field corresponding to $U(1)_{L_e}$. Therefore, we work with

$$SU(2)_L \times U(1)_Y, \tag{5.14}$$

with charge $Q = T^3 + Y$.

5.2 Glashow–Weinberg–Salam Model

An element of this group is $U = e^{i\alpha_a T_a} e^{i\beta Y}$. The corresponding gauge fields are

$$SU(2)_L : W^a_\mu, \qquad W^a_{\mu\nu} := \partial_\mu W^a_\nu - \partial_\nu W_\mu - igf^{abc}[W^b_\mu, W^c_\nu] \tag{5.15}$$

$$U(1)_Y : B_\mu, \qquad B_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{5.16}$$

These change as

$$\delta W^a_\mu = \frac{1}{g}\partial_\mu \alpha^a - \epsilon^{abc}\alpha^b W^c_\mu \tag{5.17}$$

$$\delta B_\mu = \frac{1}{g'}\partial_\mu \beta, \tag{5.18}$$

where g, g' are gauge couplings. Once we impose renormalisability, the most general Lagrangian that gives us spontaneous symmetry breaking is

$$\mathscr{L} = -\frac{1}{4}(W_{\mu\nu})^2 - \frac{1}{4}(B_{\mu\nu})^2 + D_\mu H D^\mu H^+ + m^2 H^+ H - \lambda(H^+ H)^2 + \kappa, \tag{5.19}$$

where the SSB scalar H, the *Higgs*, is a complex $SU(2)_L$ doublet. It is by convention chosen to give the Higgs a hypercharge of $Y = \frac{1}{2}$:

$$H = \begin{pmatrix} H_+ \\ H_0 \end{pmatrix}_{Y=1/2}. \tag{5.20}$$

The covariant derivative is

$$D_\mu H = \partial_\mu H - igW_\mu^a T^a H - \frac{i}{2} g' B_\mu H. \tag{5.21}$$

For a broken symmetry, $\langle H \rangle \neq 0$, pick

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad v = \frac{m}{\sqrt{\lambda}}, \tag{5.22}$$

where v is the vacuum expectation value. Expand around the vacuum:

$$H = \frac{1}{\sqrt{2}} e^{-i\xi^a ??T_a} \begin{pmatrix} 0 \\ h(x) + v \end{pmatrix}, \tag{5.23}$$

where we identify the ξ_a as the Goldstones, and $h(x)$ as the real Higgs boson. Unitary:

$$\partial_\mu \xi^a T^a - gW_\mu^a T^a - g' B_\mu \to -gW_\mu^a T^a - g' B_\mu, \tag{5.24}$$

eating the Goldstones.

The Lagrangian is then a sum of quadratic and interaction terms

$$\mathscr{L} = \mathscr{L}^{\text{quadratic}} + \mathscr{L}^{\text{interaction}} \tag{5.25}$$

with

$$\begin{aligned}\mathscr{L}^{\text{quadratic}} &= -\frac{1}{4}(W^a_{\mu\nu})^2 - \frac{1}{4}(B_{\mu\nu})^2 + \frac{1}{2}\partial_\mu h \partial^\mu h - m_h^2 h^2 \\ &\quad + \frac{1}{2}\begin{pmatrix}0 & v\end{pmatrix}\left[(igW^a_\mu T^a + \frac{i}{2}g' B_\mu)(-ig[W^a]^\mu T^a - \frac{i}{2}g' B^\mu)\right]\begin{pmatrix}0 \\ v\end{pmatrix}.\end{aligned} \tag{5.26}$$

The term in square brackets is

$$\frac{g^2 v^2}{8??}\left[(W^1_\mu)^2 + (W^2_\mu)^2 + (\frac{g'}{g}B_\mu - W^3_\mu)^2\right]. \tag{5.27}$$

Let us diagonalise this mass matrix by introducing

$$Z^0_\mu := W^3_\mu \cos\theta_W - B_\mu \sin\theta_W, \qquad \cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}} \tag{5.28}$$

$$A_\mu := W^3_\mu \sin\theta_W + B_\mu \cos\theta_W, \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}, \tag{5.29}$$

where θ_W is called the *weak mixing angle*.

Also, we define

$$W^\pm := \frac{1}{\sqrt{2}}\left(W^1_\mu \mp i W^2_\mu\right). \tag{5.30}$$

From this we can read off the mass spectrum, which is shown in Table 5.1. We identify h as the

mass	experimentally
$m_h = \sqrt{2\lambda}v = \sqrt{2}m$	$m_h = 125.2\text{GeV}$
$m_{W^\pm} = \frac{1}{2}gv$	$m_W = 80.38\text{GeV}$
$m_{Z^0} = \frac{1}{2}v\sqrt{g^2 + g'^2} = m_W / \cos\theta_W > m_W$	$m_Z = 91.1876\text{GeV}$
$m_A = 0$	$m_\gamma < 10^{-18}\text{eV}$.

Table 5.1: Electroweak boson masses.

Higgs boson and A as the photon, which we know from QED.

Why is A_μ massless?

The reason why A is massless is that there is still some unbroken symmetry, meaning that $\delta\langle H\rangle = 0$. Recall

$$\delta\langle H\rangle = (i\alpha^a T^a + i\beta Y)\langle H\rangle \tag{5.31}$$

$$= \delta\begin{pmatrix} 0 \\ v/\sqrt{2}\end{pmatrix} = \frac{i}{2\sqrt{2}}\begin{pmatrix} \alpha_3+\beta & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -\alpha_2+\beta\end{pmatrix}\begin{pmatrix} 0 \\ v\end{pmatrix} = \frac{iv}{2\sqrt{2}}\begin{pmatrix} \alpha_1 - i\alpha_2 \\ \beta - \alpha_3\end{pmatrix} \tag{5.32}$$

If we want $\delta\langle H\rangle = 0$, then we need

$$\alpha_1 = \alpha_2 = 0, \qquad \beta = \alpha_3. \tag{5.33}$$

Thus

$$U = e^{i\alpha_3 T_3} e^{i\beta Y} = e^{i\beta(T_3+Y)} = e^{i\beta Q} \tag{5.34}$$

This unbroken symmetry is generated by Q, which we identified by physical reasoning as the electric charge. This justifies the identification of A_μ with the photon.

$$SU(2)_L \times U(1)_Y \xrightarrow{\langle H\rangle} U(1)_{EM}. \tag{5.35}$$

From this we can count the number of Goldstone modes

$$\text{number}(\xi^a) = 3 = \dim SU(2)\times U(1) - \dim U(1) = 4 - 1 = 3. \tag{5.36}$$

5.2.1 Charges of Physical Fields

Consider a global transformation $e^{i\beta Q}$ and let us see what happens to the Higgs.

$$\delta H = \left[\frac{i\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} + \frac{i\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\right] H = i\beta\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} = i\beta\begin{pmatrix} H_+ \\ 0\end{pmatrix}. \tag{5.37}$$

Therefore, H_+ has electric charge $+1$, whereas H_0 has electric charge 0.

Under the same transformation, the gauge fields change as

$$\delta W^a_\mu = -\epsilon^{abc}\alpha^b W^c_\mu. \tag{5.38}$$

Using $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = \beta$, we have

$$\delta\begin{pmatrix} W^1_\mu \\ W^2_\mu\end{pmatrix} = \beta\begin{pmatrix} W^2_\mu \\ -W^1_\mu\end{pmatrix} \quad\Rightarrow\quad \delta W^\pm_\mu = \pm i\beta W^\pm_\mu. \tag{5.39}$$

Therefore, $W^\pm_\mu$ have charges ± 1.

Finally,

$$\delta Z^0_\mu = \delta A_\mu = 0. \tag{5.40}$$

Therefore, both Z^0_μ and A_μ are uncharged.

Remark: In the covariant derivative,

$$D_\mu H = (\partial_\mu + igW^3_\mu T^3 + ig' B_\mu + \ldots)H, \tag{5.41}$$

where the second term contains $g\sin\theta_W A_\mu$, whereas the third term is $g'\cos\theta_W A_\mu$. We identify the electromagnetic coupling as

$$e = g\sin\theta_W = g'\cos\theta_W. \tag{5.42}$$

In summary, the parameters of the electroweak theory are listed in Table 5.2. In particular, we

m	λ	g	g'
$e = 0.303,$	$\theta_W = \arcsin 0.223$	$m_h = 125\text{GeV}$	$m_W = 80\text{GeV}$

Table 5.2: Parameters

obtain the fine-structure constant as

$$\alpha = \frac{e^2}{2\pi}. \tag{5.43}$$

Let us now expand the original Lagrangian, expanded with the Higgs. The quadratic part in the Lagrangian is

$$\mathscr{L}^{\text{quadratic}} = -\frac{1}{4}F^2_{\mu\nu} - \frac{1}{4}Z^2_{\mu\nu} - \frac{1}{2}(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)(\partial_\mu W^-_\mu - \partial_\nu W^-_\mu) \tag{5.44}$$

$$+\frac{1}{2}m_Z^2 Z^\mu Z_\mu + m_W^2 W^{+\mu}W^-_\mu + \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}m_h^2 h^2 \tag{5.45}$$

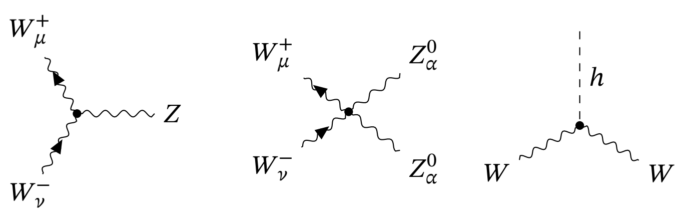
The interaction Lagrangian is quite unwieldy when fully expanded, and we split it into cubic and quartic pieces:

$$\mathscr{L}^{\text{interaction}} = \mathscr{L}^{\text{cubic}} + \mathscr{L}^{\text{quartic}}. \tag{5.46}$$

??

Importantly, all of this depends only on the four parameters in Table 5.2.

Examples of Feynman rules include



(5.47)

Example 5.2.1: Consider the process ??

5.2.2 Coupling to Fermions

We will simultaneously treat the leptons L_L^i and quarks Q_L^i, which come in three families

$$L_L^i = \left\{ \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix} \right\}_{Y=-1/2} \tag{5.48}$$

$$Q_L^i = \left\{ \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right\}_{Y=1/6} \tag{5.49}$$

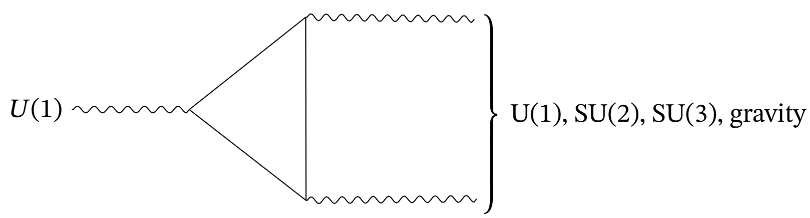
$$u_R^i = \{u_R, c_R, t_R\}_{Y=-2/5} \qquad d_R^i = \{d_R, s_R, b_R\}_{Y=-1/3} \tag{5.50}$$

$$e_R^i = \{e_R, \mu_R, \tau_R\}_{Y=-1} \qquad \nu_R^i = \{\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}\}_{Y=0} \tag{5.51}$$

Hypercharges specified by anomaly cancellation:

$$2\,\mathrm{Tr}\left[T^a \left\{T^b, T^c\right\}\right]_R = A(R) d^{abc}, \tag{5.52}$$

where the anomaly consistency condition is $A(R) = 0$.



(5.53)

- $U(1)^3 : \sum_{\text{left}} Y^3 - \sum_{\text{right}} Y^3 = 0$
- $SU(2)^2 \times U(1) : Y_L + 3Y_R = 0$
- $SU(2)^3 \times U(1) : 2Y_Q - Y_u - Y_d = 0$
- $\text{gravity}^2 \times U(1) : (2Y_L - Y_e - Y_u) + e(Y_Q - Y_u - Y_d) = 0$

With ν_R, also $U(1)_{B-L}$ satisfies the conditions.

5.2.3 Electroweak Interactions of Fermions

We split the Lagrangian

$$\mathscr{L}_{\text{fermions}} = \mathscr{L}_{\text{kinetic}}^{\text{fermions}} + \mathscr{L}_{\underbrace{\text{Higgs-fermions}}_{\text{Yukawa}}} \tag{5.54}$$

The general expression for the kinetic energy is

$$\mathscr{L}_{\text{kinetic}}^{\text{fermions}} = i\overline{L}_i \not{D} L_i + i\overline{Q}_{L_i} \not{D} Q_{L_i} + i\overline{e}_{R_i} \not{D} e_{R_i} + i\overline{\nu}_{R_i} \not{D} \nu_{R_i} + i\overline{u}_{R_i} \not{D} u_{R_i} + i\not{d}_{R_i} \not{D} d_{R_i}, \tag{5.55}$$

and the covariant derivative is

$$\begin{aligned} D_\mu &= \partial_\mu - igW_\mu^a T^a - ig' B_\mu Y & (5.56)\\ &= \partial_\mu - \frac{ig}{\sqrt{2}}\left(W_\mu^+ T^+ + W_\mu^- T^-\right) - \frac{igZ_\mu}{\cos\theta_W}\left(T^3 - \sin^2\theta_W Q\right) - ieA_\mu Q, & (5.57)\end{aligned}$$

where $T^\pm = T^1 \pm iT^2$.

The Yukawa potential is

$$\mathscr{L}_{\text{Yukawa}} = \mathscr{L}_{\text{Higgs-leptons}} + \mathscr{L}_{\text{Higgs-quarks}}, \tag{5.58}$$

with

$$\mathscr{L}_{\text{Higgs-quarks}} = -y_{ij}^d \overline{Q}_L^i H d_R^j - y_{ij}^u \overline{Q}_L^i \widetilde{H} u_R^j + \text{h.o.}, \tag{5.59}$$

where $\widetilde{H} := i\sigma_2 H^*$, and y_{ij}^u, y_{ij}^d are constants, called the Yukawas. These couplings are allowed by gauge invariance. Since the standard model has chiral symmetry, direct mass mass terms for quarks are not allowed by gauge invariance. Quarks get mass because $\langle H\rangle \neq 0$. Yukawa couplings to the Higgs give mass to fermions. This gives a mass term

$$\mathscr{L}_{\text{mass-quarks}} = -\frac{v}{\sqrt{2}}\left[\overline{d}_L^i y_{ij}^d d_R^j + \overline{u}_L^i y_{ij}^u u_R^j\right] + \text{h.o.} \tag{5.60}$$

Let us diagonalise the mass matrix

$$y^d = U_d M_d K_d^\dagger, \qquad y^u = U_u M_u K_u^\dagger, \tag{5.61}$$

where K, U are unitary and M_u, M_d are diagonal and real. Also

$$d_L \to U_d d_L, \qquad d_R \to K_d d_R \qquad u_L \to U_u u_L \qquad u_R \to K_u u_R. \tag{5.62}$$

The mass term is

$$\mathscr{L}_{\text{mass}} = -\frac{v}{\sqrt{2}}\left[\overline{d}_L^i (M_d)_{ii} d_R^i + \overline{u}_L^i (M_u)^{ii} u_R^i\right] + \text{h.o.}. \tag{5.63}$$

Thus, the masses are

$$m_d = \frac{v}{\sqrt{2}}(M_d)_{ii}, \qquad m_{u_i} = \frac{v}{\sqrt{2}}(M_u)_{ii}. \tag{5.64}$$

With these transformations, the masses are diagonal but the kinetic terms are not. The problem arises with the covariant derivatives.

$$\begin{aligned}\mathscr{L}_{\substack{\text{quarks}\\ \text{Higgs}\\ \text{gauge}}} = \mathscr{L}_{\text{kin}} &+ \frac{e}{\sin\theta_W} Z_m J_m^Z + eA_\mu J_{EM}^\mu - m_d^j(\overline{d}_L^j d_R^j + \overline{d}_R^j d_L^j) - m_u^j(\overline{u}_L^j u_R^j + \overline{u}_R^j u_L^j)\\ &+ \frac{e}{\sqrt{2}\sin\theta_W}\big[W_\mu^+ \underbrace{\overline{u}_L \gamma^\mu (V_{\text{CKM}})^{ii} d_L^j}_{J_\mu^+} + W_\mu^- \underbrace{\overline{d}_L^i \gamma^\mu (V_{\text{CKM}}^+)^{ij} u_L^j}_{J_\mu^-}\big], \end{aligned} \tag{5.65}$$

where the *Cabibbo–Kobayashi–Maskawa matrix* V_{CKM} arises from the diagonalisation and is unitary

$$V_{\text{CKM}} = U_u^\dagger U_d = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{5.66}$$

Note that this is a 3×3 unitary matrix, which generally has 9 free parameters. We can further reduce this number by 5 since we have $U(1)^6$ symmetries $d^i_{RL} \to e^{i\alpha_i} d_{R,L}$ and $u^i_{R,L} \to e^{i\beta_i} u^i_{R,L}$, where only the difference of the 6 rotations matters. We obtain $9-5=4$ degrees of freedom, with 3 real angles, $\theta_{12}, \theta_{13}, \theta_{23}$ and 1 complex phase δ. Writing $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$, one has

$$V_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{13} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{2?} - s_{12}s_{??}e^{i\delta} & s_{23}c_{13} \\ s_{11}s_{23} - c_{12}c_{2?}s_{??}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \tag{5.67}$$

This can be approximated as

$$V_{\text{CKM}} \approx \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(P-i\eta) \\ -\lambda & 1-\lambda^2 & A\lambda^2 \\ A\lambda^3(1-P-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4) \tag{5.68}$$

where $\lambda = s_{12} = \sin\theta_C \simeq 0.22$ and $\theta_{12} = \theta_C$ is the *Cabibbo angle*. The phase η implies that there is CP-violation.

Taking the modulus and forgetting about the phase we have

$$|V_{\text{CKM}}| \approx \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & \lambda^3 \\ -\lambda & 1-\frac{\lambda^2}{2} & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} + O(\lambda^4). \tag{5.69}$$

This is almost diagonal for small λ.

Historical Remarks

The mass eigenstates are not equal to the weak (flavour) eigenstates. We have charged currents $J_\mu^\pm$, which connect fermions of different flavours. However, the neutral currents J_μ^Z are flavour diagonal. In other words, interactions mediated by Z bosons do not mix between the different families. This is of historical significance. In 1974, people knew about three quarks, u, d, s. This means that they knew only about one family and a half. There would be some 'flavour changing neutral currents' (FCNC) allowed in that case. However, they were not seen experimentally. As a result, the Glashow–Iliopoulos–Maiani (GIM) mechanism was introduced, which suppressed the FCNCs in loop diagrams and predicted a fourth quark, partnering with the strange quark. This charm quark was discovered in 1974.

The V_{CKM} having one phase implies CP-violation. However, if we only had two families, this would not be there. Kobayashi and Maskawa predicted a new family to allow for the observed CP-violation. Later, the top and bottom quarks were discovered. The top, with $m_t \sim 176\text{GeV}$, was discovered in 1955.

Experimentally,

$$\theta_{12} = 13.02^\circ \pm 0.04^\circ \tag{5.70}$$
$$\theta_{13} = 0.2^\circ \pm 0.02^\circ \tag{5.71}$$
$$\theta_{23} = 2.56^\circ \pm 0.02^\circ \tag{5.72}$$
$$\delta = 69^\circ \pm 5^\circ. \tag{5.73}$$

Also, we have an accidental global symmetry

$$(d^i, u^i) \to e^{i\alpha}(d^i, u^i), \tag{5.74}$$

which leads to the conserved Baryon number

$$B(\phi_L) = B(u_R) = B(d_R) = \frac{1}{3} \tag{5.75}$$
$$B(\overline{Q}_L) = B(\overline{u}_R) = B(\overline{d}_R) = -\frac{1}{3}. \tag{5.76}$$

This accidental symmetry may be broken by introducing higher irrelevant couplings introduced in the process of renormalisation.

Leptons

We assume that there are no right-handed neutrinos ν_R^i.

$$\mathscr{L}_{\text{quadratic}}^{\text{leptons}} \supset -y_{ij}^e \overline{L}^i H e_R^j + \text{h. o.} \tag{5.77}$$

Then when $\langle H \rangle \neq 0$, only the electron gets a mass and the neutrino is massless. However, we now have compelling evidence that neutrinos do have a mass. It is therefore natural to include ν_R^i.

$$\mathscr{L}_{\text{quadratic}}^{\text{leptons}} \supset -y_{ij}^e \overline{L}^i H e_R^j - y_{ij}^\nu \overline{L}^i \widetilde{H} \nu_R^j - iM_{ij}^\nu (\nu_R^i)^c \nu_R^j + \text{h. o..} \tag{5.78}$$

The first two terms are the standard Yukawa couplings for the electrons and neutrinos. When $\langle H \rangle \neq 0$, this gives what is called the *Dirac mass* to e, ν. We were also allowed to introduce another mass term M_{ij}^ν, called the *Majorana mass*, which are the only mass terms allowed in the standard model. This is because the neutrinos are uncharged, often called *sterile*, which means that the introduction of this mass term therefore does not break gauge symmetry.

Mixing:

- mass eigenstate ν_L^i
- couplings to Z_μ diagonal
- Couplings to $W_\mu^\pm$: the *charged current* Lagrangian

$$\mathscr{L}_{cc} = -\frac{g}{\sqrt{2}} U_{\text{PMNS}}^{ij} (\overline{e}_{Li} \not{W} \nu_{Lj} + \text{h. o.}), \tag{5.79}$$

where the *Pontecorvo–Maki–Nakagawa–Sakata matrix* U_{PMNS} is again of the form (5.67)

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta'} \\ -s_{12}c_{13} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{2?} - s_{12}s_{??}e^{i\delta'} & s_{23}c_{13} \\ s_{11}s_{23} - c_{12}c_{2?}s_{??}e^{i\delta'} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta'} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & & \\ & e^{i\alpha_{12}/2} & \\ & & e^{i\alpha_{13}/2} \end{pmatrix} \tag{5.80}$$

The angles $\beta_{12}, \beta_{13}, \beta_{13}, \delta', \alpha_{12}$, and α_{13} are all new parameters, and $c_{ij} = \cos\beta_{ij}$ and $s_{ij} = \sin\beta_{ij}$. These parameters are not all fully measured yet.

Neutrino Oscillations

The masses of neutrinos are difficult to measure. Our best measurements come from the process of *neutrino oscillations*. The weak eigenstates are superpositions of mass eigenstates

$$|\nu_\alpha\rangle = \sum U^*_{\alpha_i} |\nu_i\rangle . \tag{5.81}$$

Not enough neutrinos produced in the sun were observed on the Earth. This was not explained by astrophysical phenomena but rather by the neutrino changing its family along the way of travel, which would require that there is a difference in mass between the families. This mass difference is of the order $m_2^2 - m_1^2 \leq 10^{-3}\text{eV}^2$.

6 Strong Interactions

6.1 Introduction

So far, we have seen electroweak interactions, which go through the phase transition

$$SU(2)_L \times U(1)_Y \xrightarrow{\langle H \rangle \neq 0} U(1)_{EM}. \tag{6.1}$$

The left-hand side is the *Higgs phase*, with massive $W_\mu^\pm, Z^0$, which mediate short range interactions. The right-hand side is the *Coulomb phase* with a massless γ, which mediates long range forces with potential $V \sim \frac{1}{r}$.

Can we also describe strong interactions with gauge theory? We will find that this is indeed possible, but it is not of a Higgs type. We have to ask whether there is another phase of gauge theories, which we have not yet explored.

6.2 Finding the Gauge Group

The natural gauge theory to consider is $SU_c(3)$, where the c is for *colour*. There are various arguments for this. One is the observation of the eightfold way, which predicted a new particle

$$\Omega^- = |s^r\rangle \otimes \left|s^b\right\rangle \otimes |s^g\rangle , \tag{6.2}$$

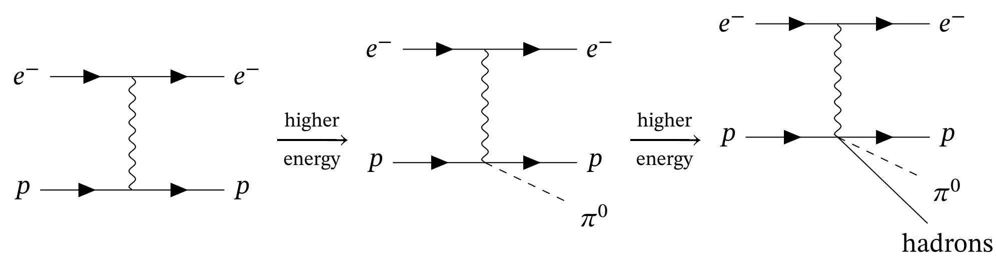
where people realised they needed to introduce a new quantum number, called colour, since otherwise the s quarks would be in the same quantum state, violating Pauli's exclusion principle.

Remark: There is nothing special about Ω^-. The same has to be true for particles with multiple $|d\rangle$ quarks for example.

Another argument is that leptons, which have no colour, do not interact with strong interactions. It is therefore natural to consider a gauge theory based on the three colours (Greenberg; Nambu and Han).

Since we have three colours, the options for the gauge group are $SO(3)$, $SU(3)$, and $U(3)$. In the case of $SO(3)$; we have $q\overline{q}$, and qq. These have non-integer charges, which we do not want. The qq would be a colour-charged object, which we do not observe. For $\mathfrak{u}(3) = \mathfrak{su}(3) \times \mathfrak{u}(1)$, we obtain long-range interactions, which we do not want. Therefore, we are left with $SU(3)_c$.

In experiments of *deep inelastic scattering* (DIS) $e^- p \to e^- p + \dots$, we have the following diagrams at progressively higher energies:



(6.3)

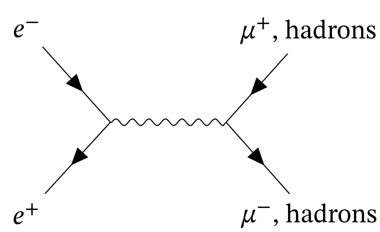
The first diagram is essentially elastic, whereas the higher energy diagram are inelastic. In these, the protons are behaving as if they had constituents, called *partons* (Feynman). The components seemed to behave more and more free at higher energies. It turns out that these partons were not just quarks, but quarks and gluons.

The parton cross sections σ were found to be independent of the total energy Q^2 at fixed x (Bjorken scaling), where

$$x = \frac{Q^2}{2m_p(E - E')}, \tag{6.4}$$

where m_p is the proton mass and E, E' are the electron energies.

An electron e^- and positron e^+ could interact to give $\mu^+ + \mu^-$ or hadrons



(6.5)

The ratio of cross-sections of these processes

$$\frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \propto N_c \sum Q_i^2. \tag{6.6}$$

The observations are consistent with $N_c = 3$ families.

6.3 Quantum Chromodynamics

For the theory of colour charge, quantum chromodynamics (QCD), we have the Lagrangian

$$\mathscr{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 + i\overline{q}_i \not{D}_{ij} q_j - m_i \overline{q}_i q_i. \tag{6.7}$$

Here the q_i are quarks, with i being colour indices. The covariant derivative is

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - ig_s G^a_\mu T^a_{ij}, \tag{6.8}$$

where G^a_μ are the *gluon* fields and $T^a = \frac{1}{2}\lambda^a$ are the generators with *Gell-Mann matrices*

$$\lambda^1 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix}, \qquad \lambda^2 = \begin{pmatrix} & -i & \\ i & & \\ & & \end{pmatrix}, \qquad \lambda^3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \end{pmatrix}, \tag{6.9}$$

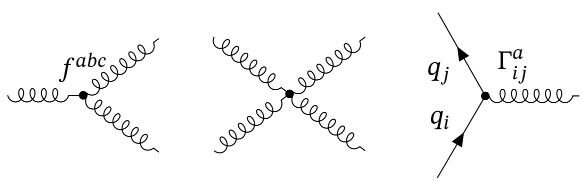
$$\lambda^4 = \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix}, \qquad \lambda^5 = \begin{pmatrix} & & -i \\ & & \\ i & & \end{pmatrix}, \qquad \lambda^6 = \begin{pmatrix} & & \\ & & 1 \\ & 1 & \end{pmatrix}, \tag{6.10}$$

$$\lambda^7 = \begin{pmatrix} & & \\ & & -i \\ & i & \end{pmatrix}, \qquad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}. \tag{6.11}$$

The field strength of the corresponding gluon is

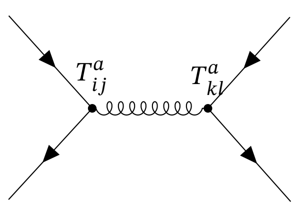
$$G^a_{\mu\nu} := \partial_\mu G^a_\nu - \partial_n G^a_\mu + g_s f^{abc} G^b_\mu G^c_\mu. \tag{6.12}$$

Typical vertices include



(6.13)

For example, we have gluon exchange

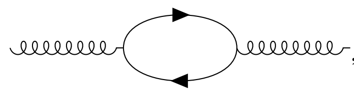


$$\propto T^a_{ji} T^a_{kl} \tag{6.14}$$

The representation decomposes as $3 \times \overline{3} = 8 + 1$. The 8 gives the repulsive baryon interactions $T^a T^a < 0$, whereas the 1 gives the attractive $T^a T^a > 0$. The limitation is that baryons and mesons carry no colour.

6.4 Asymptotic Freedom

We have the vacuum polarisation diagram



(6.15)

where the loop is given by quarks. The gauge coupling g_s is energy-dependent, described by the β-function

$$\beta(\alpha_s) = \mu \frac{\mathrm{d}\alpha_s}{\mathrm{d}\mu}, \qquad \alpha_s = \frac{g_s^2}{4\pi}, \tag{6.16}$$

where μ is the energy scale that we are working on.

For $SU(N_c)$ with N_f flavours and cutoff Λ_{UV}^2,

$$\frac{1}{g_s^2(\mu)} = \frac{1}{g_{s0}^2} - \frac{1}{(4\pi)^2}\left[\frac{11}{3}N_c - \frac{2}{3}N_f\right]\ln\frac{\Lambda_{UV}^2}{\mu^2}. \tag{6.17}$$

For the standard model, where $N_c = 3$ and $N_f = 6$, the quantity in square brackets is positive.

This minus sign in the β-function makes the theory so different from QED. Recall that in QED the β-function was

$$\frac{1}{e^2(\mu)} = \frac{1}{e_0^2} + \frac{1}{12\pi^2}\ln\frac{\Lambda^2}{\mu^2}. \tag{6.18}$$

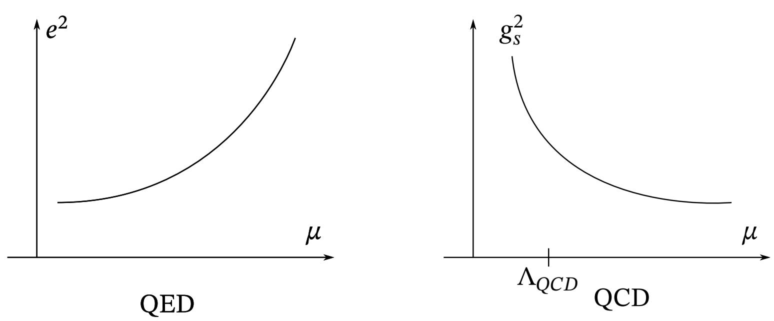


Figure 6.1

As shown in Fig. 6.1, the *Landau pole* in QED occurs at high energies, whereas QCD has the Landau pole $(1/g^2(\Lambda_{QCD}) \to 0)$ at low energies. In particular,

$$\boxed{\Lambda_{QCD} = \Lambda_{UV} e^{-4\pi^2/g_s^2}} \tag{6.19}$$

Inserting measured parameters, we find $\Lambda_{QCD} \simeq 200\text{MeV}$. This is called *dimensional transmutation*, since we are essentially exchanging a dimensionless parameter g_s for a dimensionful parameter Λ_{QCD}.

Remark: Expanding $e^{-4\pi^2/g_s^2}$ around $g_s = 0$ does not give a valid Taylor expansion: all derivatives are zero.

We see that quarks and gluons are confined into hadrons. This is called the *confinement phase*.

6.5 Effective Field Theory: Chiral Perturbation Theory

quarks	u	d	s	c	b	t
masses	1.7-3.3MeV	4.1-5.3MeV	104MeV	1270MeV	46GeV	173GeV

Table 6.1: Quark masses. Λ_{QCD} lies between the s and c quarks.

Consider QCD with only the lightest two quarks

$$\mathscr{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 + i\overline{u}_L \not{D} u_L + i\overline{u}_R \not{D} u_R + i\overline{d}\not{D} d_L + i\overline{d}\not{D} d_R - m_u \overline{u}_L u_R - m_d \overline{d}_L d_R. \tag{6.20}$$

We have kinetic and mass terms for the left-handed and right-handed up and down quarks. In the massless limit $m_u, m_d \to 0$, the theory has an approximate global *chiral symmetry*:

$$SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A, \tag{6.21}$$

where $SU(2)_{L,R}$ act on the left-handed and right-handed respectively, and the $U(1)_{V,A}$ are the phases acting on the up and down quarks. The $U(1)_A$ is anomalous.

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \to g_L \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad \begin{pmatrix} u_R \\ d_R \end{pmatrix} \to g_R \begin{pmatrix} u_R \\ d_R \end{pmatrix}. \tag{6.22}$$

$U(1)_V$ gives the baryon number. We will here focus on the chiral part $SU(2)_L \times SU(2)_R$.

We can define generators

$$T^a_V := T^a_L + T^a_R \to SU(2)_V, \qquad T^a_A := T^a_L - T^a_R \to SU(2)_A. \tag{6.23}$$

This will take

$$\begin{pmatrix} u \\ d \end{pmatrix} \to e^{i(\theta^a_V T^a + \gamma_5 \theta^a_A T^a)} \begin{pmatrix} u \\ d \end{pmatrix}. \tag{6.24}$$

Here the parameters $\theta^a_{V,A}$ generate $SU(2)_{V,A}$ respectively.

6.5.1 Chiral Symmetry Breaking

Under $SU(2)_A$, a hadron is sent to a hadron with opposite parity but other quantum numbers (charge, ...) unchanged. In particular, this would make these two hadrons degenerate, which has not been observed. As we have seen in Chapter 4, this degeneracy can be dealt with if $SU(2)_A$ is spontaneously broken. Then, we have chiral symmetry breaking:

$$SU(2)_L \times SU(2)_R \to SU(2)_V, \tag{6.25}$$

where we have *isospin* g such that

$$\begin{pmatrix} u \\ d \end{pmatrix} \to g \begin{pmatrix} u \\ d \end{pmatrix}. \tag{6.26}$$

Note further that since the proton consists of $p = uud$ and the neutron as $n = udd$, we have an approximate symmetry in nuclear physics

$$\begin{pmatrix} p \\ n \end{pmatrix} \to g \begin{pmatrix} p \\ n \end{pmatrix}. \tag{6.27}$$

This approximate symmetry, which is observed in nature, fundamentally arises since the masses of the up and down quarks are very small.

The order parameter of chiral symmetry breaking is

$$\langle \overline{u}_L u_R \rangle = \langle \overline{d}_L d_R \rangle \neq 0. \tag{6.28}$$

Remark: We cannot have a vacuum expectation value of quarks that is non-zero. In fact, only Lorentz scalars (like the Higgs) can have non-zero vacuum expectation value without violating Lorentz invariance.

These are the expectation values of *quark condensates* similar to Cooper pairs $\phi \sim e^- e^-$ in superconductivity. In order to describe the spontaneous symmetry breaking, we introduce scalar fields Σ_{ij}, where i, j are $SU(2)_L \times SU(2)_R$ indices, transforming under $SU(2)_L \times SU(2)_R$ as

$$\Sigma \to g_L \Sigma g_R^\dagger, \qquad \Sigma^\dagger \to g_R \Sigma^\dagger g_L^\dagger. \tag{6.29}$$

Effective Lagrangian:

$$\mathscr{L} = \operatorname{Tr}\big(\partial_\mu \Sigma\big)(\partial^\mu \Sigma)^\dagger + m^2 \operatorname{Tr} \Sigma\Sigma^\dagger - \frac{\lambda}{4} \operatorname{Tr}\big[\Sigma\Sigma^\dagger\Sigma\Sigma^\dagger\big] \tag{6.30}$$

$$\langle \Sigma_{ij} \rangle = \frac{v}{\sqrt{2}} \mathbb{1}_2, \qquad v = \frac{2m}{\sqrt{\lambda}}. \tag{6.31}$$

$$SU(2)_L \times SU(2)_R \to SU(2)_v. \tag{6.32}$$

Contact with quarks: $v \sim \Lambda_{QCD} \sim \langle \overline{u} u \rangle^{1/3}$ since $[u] = 3/2$.

Expand around the vacuum

$$\Sigma(x) = \frac{v + \sigma(x)}{\sqrt{2}} e^{2iT^a \pi^a(x)/F_\pi}, \tag{6.33}$$

where F_π was included for dimensional reasons. It turns out that $F_\pi = v$ is the vacuum expectation value. Here, $\sigma(x)$ is the massive Higgs field, which is invariant under $SU(2)$. The $\pi^a(x)$ are the massless Goldstone modes, which transform under the adjoint representation

$$\delta \pi^a = -\epsilon^{abc} \theta^b \pi^c. \tag{6.34}$$

Now we integrate out the massive $\sigma(x)$, since we are interested in the massless Goldstone modes.

Remark: Although the maths is the same, the focus here is very different to Chapter 4, where the Goldstone bosons were eaten by the Higgs, whereas here we want to trace them and find out their low energy theory.

At low energies,

$$U(x) = e^{2i\pi^a T^a/F_\pi} = \exp\left[\frac{i}{F_\pi}\begin{pmatrix} \pi^0 & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi^0 \end{pmatrix}\right], \tag{6.35}$$

where for notational convenience we defined $\pi^0 := \pi^3$ and $\pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm i\pi^2)$.

The chiral Lagrangian is

$$\mathscr{L}_\chi = \frac{F_\pi^2}{4}\operatorname{Tr}\left[D^\mu U (D_\mu U)^\dagger\right] + \lambda_1 \operatorname{Tr}\left[(D^\mu U)(D_\mu U^\dagger)\right]^2 + \dots \tag{6.36}$$

Expanding the exponentials in U:

$$\mathscr{L}_\chi = \frac{1}{2}(\partial_\mu \pi^0)(\partial^\mu \pi^0) + (\partial_\mu \pi^+)(\partial^\mu \pi^-)^\dagger + \frac{1}{F_\pi^2}\left[-\frac{1}{3}\pi^0\pi^0 D_\mu \pi^+ D^\mu \pi^- + \dots\right] + \dots \tag{6.37}$$

This is an expansion in powers of

$$\frac{E}{F_\pi} \sim \frac{E}{\Lambda_{QCD}} \tag{6.38}$$

valid for energies $E \ll \Lambda_{QCD}$. This is called chiral (χ-)perturbation theory. You can check that $\pi^0, \pi^\pm$ have the same quantum numbers (charge, etc.) as the pion fields. Essentially, pions, which are hadronic composites of quarks, can also be understood as (pseudo-)Goldstone bosons of chiral symmetry.

Looking at the quark mass table 6.1, we may wonder what happens with s, whose mass is still much lower than c. It turns out that we recover the eightfold way of Gell-Mann. In other words, starting from colour symmetry of QCD, we can use the hierarchy of masses of quarks to explain the origin of the approximate symmetries observed historically.

7 The Standard Model and its Limitations

7.1 The SM Lagrangian

Let us put it all together.

Gauge Fields ($\lambda = \pm 1$): the gauge group is

$$\begin{array}{ccc} SU(3)_c \times & \frac{SU}{2}_L & \times U(1)_Y \\ G^a_\mu & \underbrace{W^{\hat{a}}_\mu, B_\mu}_{W^\pm_\mu, Z^0_\mu, A_\mu} & \end{array} \tag{7.1}$$

$$\mathscr{L}_{\text{gauge}} = -\frac{1}{4}(G^a_{\mu\nu})^2 - \frac{1}{4}(W^a_{\mu\nu})^2 - \frac{1}{4}(B_{\mu\nu})^2 + \theta_3 G^a_{\mu\nu}\widetilde{G}^{a\mu\nu} + \theta_W W^{\hat{a}}_{\mu\nu}\widetilde{W}^{\hat{a}\mu\nu} - \theta_B B_{\mu\nu}\widetilde{B}^{\mu\nu}, \tag{7.2}$$

Kinetic and self interactions for $G^a_{\mu\nu}, W^{\hat{a}}_{\mu\nu}$. Here, $\widetilde{G}^{a\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}G^a_{\rho\sigma}$.

Matter Fields ($\lambda = \pm\frac{1}{2}$):

$$Q^i_L = \left\{\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ d_L \end{pmatrix}, \right\} \dots \tag{7.3}$$

$$\mathscr{L}^{\text{fermions}} = \mathscr{L}^{\text{fermions}}_{\text{kinetic}} + \mathscr{L}_{\text{Yukawa}} \tag{7.4}$$

$$\mathscr{L}^{\text{fermions}}_{\text{kinetic}} = i\overline{Q}^i_L \not{D} Q^i_L + i\overline{u}^i_R \not{D} u^i_R + i\overline{d}_R \not{D} d_{Ri} + i\overline{L}^i \not{D} L^i_L + i\overline{e}^i_R \not{D} e_R + (i\overline{\nu}^i_R \not{D} \nu^i_R)^* \tag{7.5}$$

where the gauge covariant derivative is

$$D_\mu = \partial_\mu - ig_s G^a_\mu T^a - igW^{\hat{a}}_\mu T^{\hat{a}} - ig' B_\mu Y. \tag{7.6}$$

The g, g' are relate to e, θ_w and $W^{\hat{a}}_\mu, B_\mu$ are related to $W^\pm_\mu, Z^0_\mu, A_\mu$.

$$L_{\text{Yukawa}} = -y^d_{ij}\overline{Q}^i_L H d^j_R - y^u_{ij}\overline{Q}^i_L \widetilde{H} u^j_R - y^e_{ij}\overline{L}^i_L H e^j_R - (y^\nu_{ij}\overline{L}^i \widetilde{H}\nu_R)^* + \text{h.c.} \tag{7.7}$$

Higgs: $H = (1, 2, \frac{1}{2}) = \begin{pmatrix} H_+ \\ H_0 \end{pmatrix}$

$$\mathscr{L}_{\text{Higgs}} = D_\mu H D^\mu H^\dagger + m^2 |H|^2 - \lambda |H|^4, \tag{7.8}$$

where m^2, λ are related to the VEV $\langle H \rangle = v, m_h$.

The total standard model Lagrangian is the sum of all these

$$\mathscr{L}_{\text{SM}} = \mathscr{L}_{\text{gauge}} + \mathscr{L}^{\text{fermions}}_{\text{kinetic}} + \mathscr{L}_{\text{Yukawa}} + \mathscr{L}_{\text{Higgs}}, \tag{7.9}$$

which is renormalisable.

Gravity: We can also couple this to gravity

$$\mathscr{L}_{\text{SM}} \to \sqrt{g} \left(\mathscr{L}_{\text{SM}} + \Lambda + R + O(R^2) \right), \tag{7.10}$$

where in $\mathscr{L}_{\text{SM}}$, we also change $D_\mu \to \mathcal{D}_\mu$ the gravity covariant derivatives. This is an effective field theory, valid at $E \ll M_{\text{pl}} = \sqrt{\frac{\hbar c}{G}} \sim 10^{1?}$GeV.

The arbitrary (free) parameters of the Standard Model Lagrangian are listed in Tab. 7.1.

		Physical	Number
Gauge	g_s, g, g' and $\theta_3, \theta_w, \theta_B$	e, θ_W, M_W	$3+3$
Higgs	m^2, λ	m_h, v	2
Yukawa Couplings	$y^u_{ij}, y^d_{ij}, y^e_{ij}, (y^\nu_{ij})^*$	m^u_i, m^d_i, m^e_i	9
		V_{CKM}	4
		U_{PMNS}	6
		$(m^\nu_i, M^\nu_i)^*$	

Table 7.1: Free parameters in the Standard Model

7.2 Open Questions

(i) Fundamental: UV completion for couplings to gravity.

(ii) Strong coupling regimes:

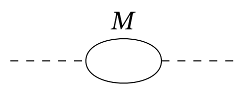
- Perturbative expansions (Feynman diagrams) are in $O(g^2)$, valid for $g^2 \ll 1$.
- Mathematically rigorous proof of confinement.

(iii) Naturalness. Writing $\mathscr{L}_{\mathrm{SM}} = \sum c_i \mathcal{O}_i$, where $[c_i] + [\mathcal{O}_i] = 4$.

- An operator of dimension $[\mathcal{O}] = 0$ is the cosmological constant Λ, which represents the vacuum energy: $R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = \langle T_{\mu\nu} \rangle + g_{\mu\nu}\Lambda$. The observed $\Lambda \sim (10^{-3}\mathrm{ev})^4$. But quantum corrections suggest an infinite vacuum energy. (Worst case is M_{pl}^4. This is a factor 10^{123} different!)
- An operator $[\mathcal{O}] = 2$ is the Higgs $m^2|H|^2$. We have

$$H \;\text{- - - -}\overset{m^2}{\times}\text{- - - -}\; H \tag{7.11}$$

This gets quantum corrections



$$+ \ldots \qquad M^2 \sim \int^{??} \frac{\text{đ}^4 k}{k^2} \tag{7.12}$$

We need to keep $m \ll M$. How? (Tuning of 15 orders of magnitude.)

- There are no operators of mass dimension $[\mathcal{O}] = 3$.
- For $[\mathcal{O}] = 4$, we have terms θ_W, θ_B that can be rotated away, but

$$\theta_3 G^a_{\mu\nu} \widetilde{G}^{a\mu\nu} \tag{7.13}$$

is in fact generated. It gives an electric dipole moment of the neutron, which is not observed. Thus puts an upper bound on it $\theta_3 < 10^{-10}$. Explaining this is called the *Strong CP problem*.

- Neutrino masses

$$y^\nu \overline{L} H \nu_R + M \nu_R \nu_R. \tag{7.14}$$

We find that $M \gg 100\mathrm{GeV}$ and $[\mathcal{O}] = 3$. The question of how neutrinos obtain masses is still an open problem.

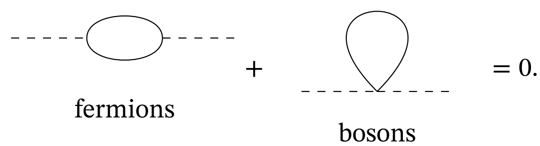
(iv) Flavour: Why are there so many free parameters in $\mathscr{L}_{\mathrm{Yukawa}}$ and huge hierarchy of masses.

7.3 Beyond the Standard Model

The whole of the standard model was built up from the assumptions of Poincaré invariance and quantum mechanics. How to we go beyond the standard model? There are multiple approaches that people are taking:

(i) Top down (Quantum gravity, etc)

Also, supersymmetry: We have seen all of the helicities, except $\lambda = \pm 3/2$, which does show up in supersymmetric theories. It also addresses the hierarchy problem, since fermions and bosons cancel



(7.15)

(ii) Model Building (add more gauge symmetries GUT, more particles, …)

(iii) Bottom-up Approaches:

- SM EFT

$$\mathscr{L}_{\text{SM}} = c_i \mathcal{O}_i = \mathscr{L}_{\text{SM}}^{\text{renormalisable}} + \frac{\kappa}{M}\mathcal{O}_5 + \left(\frac{\kappa'}{M}\right)^2 \mathcal{O}_6 + \dots \tag{7.16}$$

We can add non-renormalisable terms as long as we work within energies $E \ll M$. In particular, in dimension 5 there is only one operator $\mathcal{O}_5 : \frac{LLHH}{M}$. This is nice because if $\langle H \rangle \neq 0$, we obtain $\left(\frac{v^2}{M}\right) LL$, giving a mass term for the ν's for $M \sim 10^{14}$GeV.

In dimension 6, we have $\mathcal{O}_3 : 63$ operators. We have 4 terms $\frac{qqql}{M^2}$, which violate baryon number B. This gives proton decay $p \to \pi^0 + e^+$, and $M \geq 10^{15}$GeV.

(iv) Amplitudes: Forget about a Lagrangian, and only work with amplitudes that are consistent with our assumptions of unitarity, locality, etc. From these amplitudes we can reproduce all of the amplitudes in the standard model, and even go beyond.

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