We derive the likelihood ratio statistics for testing whether the proportion of a given bin B has changed over-time.

B can be, for example, the second bin, $(0, 1000]$.

There are 7 possible choices for B since 'min investment accepted' is discretized into 7 bins.

Notations:

- t denotes the t^{th} quarter, $1 \le t \le T$. e.g. $t = 2021Q4$.
- m_i is the 'minimum accepted investment' of the i^{th} observation
- t_i is the quarter of 'sale date' of the i^{th} observation
- p_{t_i} is the true probability of being in bin B in quarter t_i

Let $X_i = \begin{cases} 1, & \text{if } m_i \in B \\ 0, & \text{otherwise} \end{cases}$. In otherwords, X_i indicates whether or not the minimum accepted investment of the i^{th} observation, m_i, is in bin B.

We assume that:

1) Each observation is independent

2) $x_i \sim$ Bernouillre (p_{t_i})

So, $f(x_i | p_{t_i}) = p_{t_i}^{x_i} \cdot (1-p_{t_i})^{1-x_i}$

We form a hypothesis test:

$$H_0: p_1 = p_2 = \cdots = p_T$$

$$H_1: \exists\, i, j \text{ s.t. } p_i \neq p_j.$$

Then, $\theta = (\hat{p_1}, \hat{p_2}, \cdots\cdots, \hat{p_T})$, given that there are m quarters,

Then, the overall likelihood is given by:

$$L(\theta) = \prod_{i=1}^{n} f(x_i | p_{t_i})$$

$$= \prod_{i=1}^{n} p_{t_i}^{x_i} \cdot (1-p_{t_i})^{1-x_i}$$

$$= \prod_{t=1}^{m} \prod_{i \in S_t} p_t^{x_i} \cdot (1-p_t)^{1-x_i}$$, where S_t contain the observations in quarter t

Under the null hypothesis, $L(\theta) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$, so we are just maximizing one big joint-bernouillie likelihood, and we'll get $\hat{p_1} = \cdots = \hat{p_T} = \frac{\sum x_i}{n} = \bar{x}$

Otherwise, $L(\theta) = \prod_{t=1}^{T} \prod_{i \in S_t} p_t^{x_i} \cdot (1-p_t)^{1-x_i}$, so we will maximize the joint-bernouillie likelihood within each quarter, and will get $\hat{p_1} = \frac{\sum_{i \in S_1} x_i}{n_1} = \bar{x}_1$

$$\hat{p_2} = \frac{\sum_{i \in S_2} x_i}{n_2} = \bar{x}_2$$

$$\vdots$$

$$\hat{p}_T = \frac{\sum_{i \in S_T} x_i}{T} = \bar{x}_T$$

Thus, $\max_{\theta \in \Theta_0} L(\theta) = \prod_{i=1}^{n} (\bar{x})^{x_i} \cdot (1-\bar{x})^{1-x_i}$

$$\max_{\theta \notin \Theta_0} L(\theta) = \prod_{t=1}^{T} \prod_{i \in S_t} (\bar{x}_t)^{x_i} \cdot (1-\bar{x}_t)^{1-x_i}$$

and the test statistics is:

$$L = -2 \log \left(\frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \notin \Theta_0} L(\theta)} \right)$$

$$= -2 \left(\ln \bar{x} \cdot \sum_{i=1}^{n} x_i + \ln(1-\bar{x}) \cdot \left(n - \sum_{i=1}^{n} x_i\right) - \sum_{t=1}^{T} \ln(\bar{x}_t) \cdot \sum_{i \in S_t} x_i + \ln(1-\bar{x}_t) \cdot \left(n_t - \sum_{i \in S_t} x_i\right) \right)$$

where $L \sim \chi^2_{m-1}$.