Precalculus

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Author Note

For most of you, this course will be unlike any mathematics course you have previously encountered. Why is this?

Learning a new language

Colleges and universities have been designed to help us discover, share and apply knowledge. As a student, the preparation required to carry out this three part mission varies widely, depending upon the chosen field of study. One fundamental prerequisite is fluency in a “basic language”; this provides a common framework in which to exchange ideas, carefully formulate problems and actively work toward their solutions. In modern science and engineering, college mathematics has become this “basic language”, beginning with *precalculus*, moving into *calculus* and progressing into more advanced courses. The difficulty is that college mathematics will involve genuinely new ideas and the mystery of this unknown can be sort of intimidating. However, everyone in this course has the intelligence to succeed!

Is this course the same as high school Precalculus?

There are key differences between the way teaching and learning takes place in high schools and universities. Our goal is much more than just getting you to reproduce what was done in the classroom. Here are some key points to keep in mind:

- The pace of this course will be faster than a high school class in precalculus. Above that, we aim for greater command of the material, especially the ability to extend what we have learned to new situations.

- This course aims to help you build the stamina required to solve challenging and lengthy multi-step problems.

- As a rule of thumb, this course should on average take 15 hours of effort per week. That means that in addition to the 5 classroom hours per week, you would spend 10 hours extra on the class. This is only an average and my experience has shown that 12–15 hours of study per week (outside class) is a more typical estimate. In other words, for many students, this course is the equivalent of a half-time job!

- Because the course material is developed in a highly cumulative manner, we recommend that your study time be spread out evenly over the week, rather than in huge isolated blocks. An analogy with athletics is useful: If you are preparing to run a marathon, you must train daily; if you want to improve your time, you must continually push your comfort zone.

Prerequisites

This course assumes prior exposure to the "mathematics" in Chapters 1-12; these chapters cover functions, their graphs and some basic examples. This material is fully developed, in case you need to brush up on a particular topic. If you have never encountered the concept of a function, graphs of functions, linear functions or quadratic functions, this course will probably seem too advanced. You are <u>not</u> assumed to have taken a course which focuses on mathematical problem solving or multi-step problem solving; that is the purpose of this course.

Internet

There is a great deal of archived information specific to this course that can be accessed via the World Wide Web at the URL address

`http://www.math.washington.edu/˜m120`

Why are we using this text?

Prior to 1990, the performance of a student in precalculus at the University of Washington was not a predictor of success in calculus. For this reason, the mathematics department set out to create a new course with a specific set of goals in mind:

- A review of the essential mathematics needed to succeed in calculus.

- An emphasis on problem solving, the idea being to gain both experience and confidence in working with a particular set of mathematical tools.

This text was created to achieve these goals and the 2004-05 academic year marks the eleventh year in which it has been used. Several thousand students have successfully passed through the course.

Notation, Answers, etc.

This book is full of worked out examples. We use the the notation "*Solution.*" to indicate where the reasoning for a problem begins; the symbol □ is used to indicate the end of the solution to a problem. There is a Table of Contents that is useful in helping you find a topic treated earlier in the course. It is also a good rough outline when it comes time to study for the final examination. The book also includes an index at the end. Finally, there is an appendix at the end of the text with ”answers” to most of the problems in the text. It should be emphasized these are ”answers” as opposed to ”solutions”. Any homework problems you may be asked to turn in will require you include all your work; in other words, a detailed solution. Simply writing down the answer from the back of the text would never be sufficient; the answers are intended to be a guide to help insure you are on the right track.

How to succeed in Math 120.

Most people learn mathematics by doing mathematics. That is, you learn it by active participation; it is very unusual for someone to learn the material by simply watching their instructor perform on Monday, Wednesday, and Friday. For this reason, the homework is THE heart of the course and more than anything else, study time is the key to success in Math 120. **We advise 15 hours of study per week, outside class.** Also, during the first week, the number of study hours will probably be even higher as you adjust to the viewpoint of the course and brush up on algebra skills.

Here are some suggestions: Prior to a given class, make sure you have looked over the reading assigned. If you can't finish it, at least look it over and get some idea of the topic to be discussed. Having looked over the material ahead of time, you will get FAR MORE out of the lecture. Then, after lecture, you will be ready to launch into the homework. If you follow this model, it will minimize the number of times you leave class in a daze. In addition, spread your study time out evenly over the week, rather than waiting until the day before an assignment is due.

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Comments

Send comments, corrections, and ideas to *colling@math.washington.edu* or *kdp@engr.washington.edu*.

Preface

Have you ever noticed this peculiar feature of mathematics: When you don't know what is going on, it is really hard, difficult, and frustrating. But, when you know what is going on, mathematics seems incredibly easy, and you wonder why you had trouble with it in the first place!

Here is another feature of learning mathematics: When you are struggling with a mathematical problem, there are times when the answer seems to *pop* out at you. At first, nothing is there, then very suddenly, in a flash, the answer is all there, and you sit wondering why you didn't "see" the solution sooner. We have a special name for this: It's the "A-Ha!" experience. Often the difficulty you have in studying mathematics is that the *rate* at which you are having an A-Ha! experience might be so low that you get discouraged or, even worse, you give up studying mathematics altogether. One purpose of this course is to introduce you to some strategies that can help you increase the rate of your mathematical A-Ha! experiences.

What is a story problem?

When we ask students if they like story problems, more often than not, we hear statements like: "I hate story problems!" So, what is it about these kinds of problems that causes such a negative reaction? Well, the first thing you can say about story problems is that they are mostly made up of *words*. This means you have to make a big effort to *read* and *understand* the words of the problem. If you don't like to read, story problems will be troublesome.

The second thing that stands out with story problems is that they force you to *think* about how things work. You have to give deep thought to how things in the problem *relate* to each other. This in turn means that story problems force you to connect many steps in the solution process. You are no longer given a list of *formulas* to work using memorized steps. So, in the end, the story problem is a multi-step process such that the "A-Ha!" comes only after lots of intense effort.

All of this means you have to spend *time* working on story problems. It is impossible to sit down and spend only a minute or two working each problem. With story problems, you have to spend much more time working toward a solution, and at the university, it is common to spend an hour or more working each problem. So another aspect of working these kinds of problems is that they demand a lot of *work* from you, the problem solver.

We can conclude this: *What works is work!* Unfortunately, there is no *easy* way to solve all story problems. There *are*, however, techniques that you can use to help you work efficiently. In this course, you will be presented with a wide range of mathematical tools, techniques, and strategies that will prepare you for university level problem solving.

What are the BIG errors?

Before we look at how to make your problem solving more efficient, let's look at some typical situations that make problem solving *inefficient*. If you want to be ready for university level mathematics, we are sure you have heard somewhere: "You must be prepared!" This means you need to have certain *well-developed mathematical skills* before you reach the university. We would like to share with you the three major sources of errors students make when working problems, especially when they are working exam problems. Every time we sit down and review solutions with a student who has just taken an exam, and who has lost a lot of points in that exam, we find errors falling pretty much into three categories, and these errors are the major cause of inefficient mathematical problem solving.

The first type of error that loses points is *algebra*. This is an error of not knowing all of the algebraic rules. This type of error also includes mistakes in the selection and use of mathematical symbols. Often, during the problem solving process, you are required to introduce mathematical *symbols*. But, without these symbols, you cannot make any further progress. Think of it this way: *Without symbols, you cannot do any mathematics involving equations!*

The second error we see in problem solving has to do with *visualization*. In this case, we're talking about more than the graphics you can get from a calculator. Graphing and curve sketching are very important skills. But, in doing story problems, you might find it almost impossible to create a solution without first drawing a *picture*[1] of your problem. Thus, by not drawing a good picture of the problem, students get stuck in their exams, often missing the solution to a problem entirely.

Finally, the third big source of error is not *knowing mathematical definitions.* Actually, this is a huge topic, so we will only touch on some of the main features of this kind of error. The key thing here is that by not knowing mathematical definitions, it becomes very hard to know what to *do* next in a multi-step solution to a story problem.

---

[1] Whenever we talk about a *picture* of your problem, we mean not just the drawing itself. In this case, the picture must include the drawing and the *labels* which clearly signify the quantities related to your problem.

Here is what it all boils down to: *Mathematical definitions, for the most part, provide little cookbook procedures for computing or measuring something.* For example, if you did not know the mathematical definition of "speed," you would not know that to measure speed, you first measure your *distance* and you simultaneously measure the *time* it takes to cover that distance. Notice this means you have *two* measuring instruments working at the same time. The second thing you must do, according to the definition of speed, is *divide* the distance you measured by the time you measured. The *result* of your division is a number that you will call *speed.* The definition is a step-by-step procedure that everyone agrees to when talking about "speed." So, it's easy to understand that if you are trying to solve a story problem requiring a speed computation and you did not know the definition or you could not remember the definition of speed, you are going to be "stuck" and no further progress will be possible!

What does all of this mean for you? As you study your mathematics, make sure you are the best you can be in these three areas: Algebra, Visualization, and Definitions. Do a little algebra every day. Always draw a picture to go with all your problems. And, know your mathematical definitions without hesitation. Do this and you will see a very large portion of your math errors disappear!

Problem Solving Strategies

This topic would require another book to fully develop. So, for now, we would like to present some problem solving ideas you can start using right away.

Let's look now at a common scenario: A student reads a story problem then exclaims, maybe with a little frustration: "If I only had the formula, I could solve this problem!" Does this sound familiar? What is going on here, and why is this student frustrated? Suppose you are this student. What are you actually trying to *do?* Let's break it down. First, you are reading some descriptive information in ***words*** and you need to *translate* this word information into ***symbols***. If you had the symbolic information, you would be in a position to mathematically solve your problem right away.

Unfortunately, you cannot solve anything without first translating your words into symbols. And, *going directly from words to symbols is usually very difficult!* So, here we are looking for some alternative approach for translating words into symbols. Figure 1 is the answer to this problem solving dilemma.

A lot is going on in Figure 1. Let's consider some of the main features of this diagram. First, it is suggesting that you are dealing with information in three different *forms:* Words, Pictures, and Symbols. The arrows in this diagram suggest that in *any* problem solving situation, you are

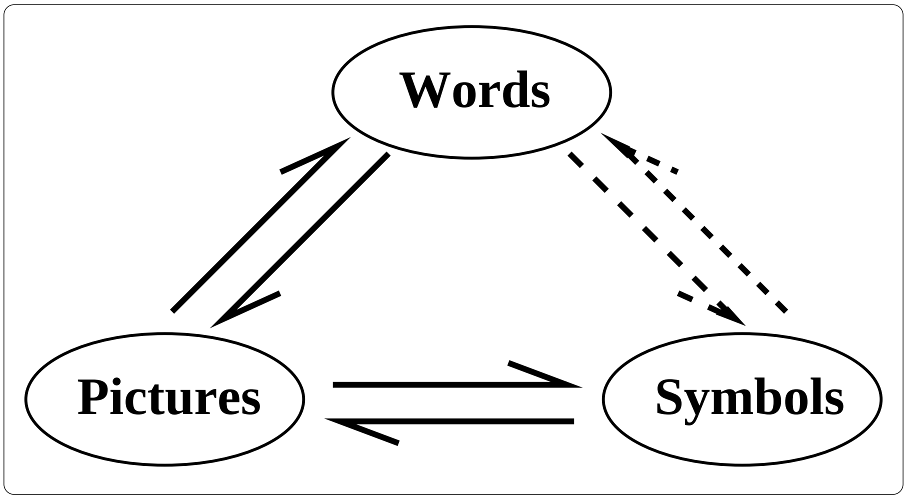


Figure 1: Problem solving as a transformation process.

actually translating information from one form to another. The arrows also suggest that there are alternative *paths* you can take to get from one form to another! This is a very, very important point: the idea that there is more than one way to get from words to symbols.

Let's rewind this discussion: You're reading a story problem. But, now, before giving any thought to what your formula is, that is, before worrying about your symbolic information, *you grab a blank sheet of paper and start drawing a* ***picture*** *of your problem.* And, to your picture you add symbols denoting the quantities you need in your problem. At this point in your problem solving, you are not trying to write any equations; you are only trying to *see* what your problem *looks* like. You are also concentrating on another extremely important step: Deciding what symbols to use in your problem!

Now you have a good picture of your problem. It shows not only what the problem looks like, but symbolically shows all the problem's variables and constants. You can start using this information to mathematically model your problem. The process of creating a mathematical model is actually nothing more than the arrow in the diagram going from ***pictures*** to ***symbols.*** Mathematical modeling is the jump you make from the visual information you have created to information contained in your formulas.

Let's summarize the problem solving process. You start with a description of a problem that is presented to you mainly in the form of words. Instead of trying to jump directly from words to symbols, you jump from words to pictures. Once you have a good picture, you jump from pictures to symbols. And, all the time, you are relying on mathematical definitions as you interpret the words of your problem; on visualization techniques as you draw pictures related to your problem; and, on your algebra skills as you are formulating the equations you need to solve your problem.

There is one final thing to notice about the diagram in this section. All of this discussion so far deals with the situation where your direction is from

$$\text{Words} \Longrightarrow \text{Pictures} \Longrightarrow \text{Symbols.}$$

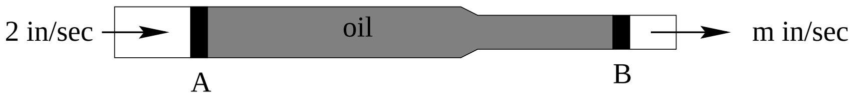
But when you study the diagram you see that the arrows go *both* ways! So, we will leave you with this to think about: What does it mean, within the context of problem solving, when you have

$$\text{Symbols} \Longrightarrow \text{Pictures} \Longrightarrow \text{Words}\,?$$

An Example.

Here is a worked example that is taken from a typical homework assignment for Section 1.1 of this book. See if you can recognize the multitude of steps needed to arrive at the equations that allow us to compute a solution. That is, try to identify the specific way in which information is being transformed during the problem solving process.

This problem illustrates the principle used to make a good “squirt gun”. A cylindrical tube has diameter 1 inch, then reduces to diameter d. The tube is filled with oil and piston A moves to the right 2 in/sec, as indicated. This will cause piston B to move to the right m in/sec. Assume the oil does not compress; that means the volume of the oil between the two pistons is always the same.



1. If the diameter of the narrow part of the tube is $\frac{1}{2}$ inch, what is the speed of piston B?
2. If B moves $11\,\frac{\text{in}}{\text{sec}}$, what is the diameter of the narrow part of the tube?

Solution.

The first thing to do with any story problem is to draw a picture of the problem. In this case, you might re-sketch the picture so that it looks 3-dimensional: See Figure 2. As you draw, add in *mathematical symbols* signifying quantities in the problem.

The next thing is to clearly define the variables in your problem:

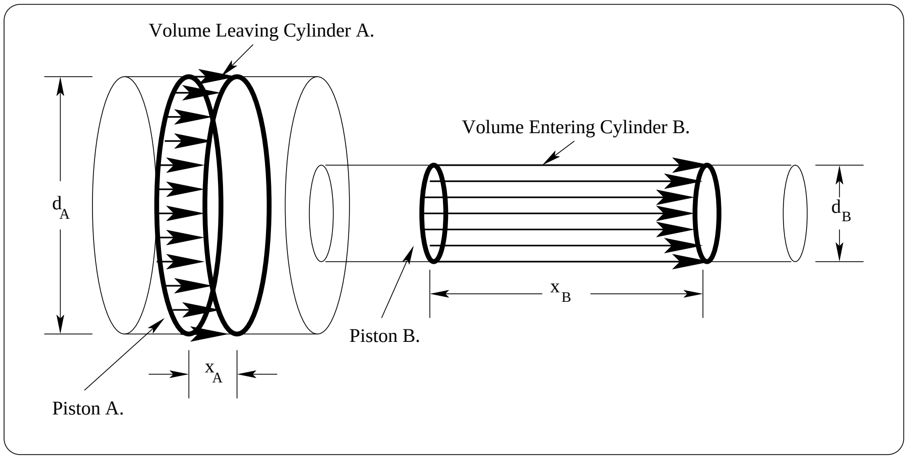


Figure 2: A re-sketch of the original given figure.

1. Let V_A and V_B stand for the change in volumes as piston A moves to the right.
2. Let d_A and d_B represent the diameters of each cylinder.
3. Let r_A and r_B represent the radii of each cylinder.
4. Let s_A and s_B stand for the speeds of each piston.
5. Let x_A and x_B stand for the distance traveled by each piston.

Now that you have some symbols to work with, you can write the given data down this way:

1. $s_A = 2\,\frac{\text{inches}}{\text{second}}$.
2. $d_A = 1.0\,\text{inch}$.

After you have studied this problem for a while, you would write down some useful relationships:

1. The volume of any cylinder is

   $$V = \pi r^2 h$$

   where r is the radius of the cylinder, and h is its height or length. From this, you can derive the volume of a cylinder in terms of its diameter, d:

   $$V = \frac{\pi}{4} d^2 h.$$

2. "Distance" = "Rate" × "Time". In terms of this problem, you would write

   $$x = st,$$

   where x is the distance your piston moves, and s is the speed of the piston's motion.

Now you are in a position to create a mathematical model that describes what is going on:

1. From the two relationships above, you can derive the volume equations for each cylinder so that the diameters and speed of the pistons are included:

   $$V_A = \frac{\pi d_A^2 x_A}{4} \Longrightarrow V_A = \frac{\pi d_A^2 (s_A t)}{4}$$

   and

   $$V_B = \frac{\pi d_B^2 x_B}{4} \Longrightarrow V_B = \frac{\pi d_B^2 (s_B t)}{4}$$

2. Since the oil does not compress, at each instant when piston A is moving, you must have $V_A = V_B$, thus:

   $$\frac{\pi d_A^2 (s_A t)}{4} = \frac{\pi d_B^2 (s_B t)}{4}.$$

   After canceling π, t, and 4, you end up with a *mathematical model* describing this problem that you can use to answer all sorts of interesting questions:

   $$d_A^2 s_A = d_B^2 s_B.$$

Using your model, you can compute the following solutions:

1. Given: $d_B = \frac{1}{2}$ in, $d_A = 1.0$ in, and $s_A = 2\,\frac{\text{in}}{\text{sec}}$, find s_B. From your model, you derive:

   $$d_A^2 s_A = d_B^2 s_B \Longrightarrow s_B = \frac{d_A^2 \cdot s_A}{d_B^2}$$

   from which you can compute

   $$m = s_B = 8\,\frac{\text{in}}{\text{sec}}.$$

2. Likewise, you can use your model to compute

   $$d_B = \sqrt{\frac{2}{11}}\,\text{in},$$

   exactly.

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Chapter 1

Warming Up

The basic theme of this book is to study *precalculus* within the context of problem solving. This presents a challenge, since skill in problem solving is as much an art or craft as it is a science. As a consequence, the process of learning involves an active apprenticeship rather than a passive reading of a text. We are going to start out by assembling a basic toolkit of examples and techniques that are essential in everything that follows. The main ideas discussed in the next couple of chapters will surely be familiar; our perspective on their use and importance may be new.

The process of going from equations to pictures involves the key concept of a *graph*, while the reverse process of going from pictures (or raw data) to equations is called *modeling*. Fortunately, the study of graphing and modeling need not take place in a theoretical vacuum. For example, imagine you have tossed a ball from the edge of a cliff. A number of natural questions arise: *Where and when does the ball reach its maximum height? Where and when does the ball hit the ground? Where is the ball located after* t *seconds?*

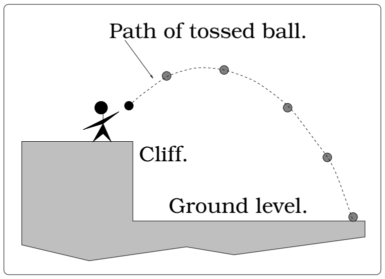


Figure 1.1: Ball toss.

We can attack these questions from two directions. If we knew some basic *physics*, then we would have equations for the motion of the ball. Going from these equations to the actual curved path of the ball becomes a *graphing* problem; answering the questions requires that we really understand the relationship between the symbolic equations and the curved path. Alternatively, we could approach these questions without knowing any *physics*. The idea would be to collect some data, keeping track of the height and horizontal location of the ball at various times, then find equations whose graphs will "best" reproduce the collected data points; this would be a *modeling* approach to the problem. Modeling is typically harder than graphing, since it requires good intuition and a lot of experience.

1.1 Units and Rates

A marathon runner passes the one-mile marker of the race with a clocked speed of 18 feet/second. If a marathon is 26.2 miles in length and this speed is maintained for the entire race, what will be the runner's total time?

This simple problem illustrates a key feature of modeling with mathematics: Numbers don't occur in isolation; a number typically comes with some type of *unit* attached. To answer the question, we'll need to recall a formula which precisely relates "total distance traveled" to "speed" and "elapsed time". But, we must be **VERY CAREFUL** to use consistent units. We are given *speed* units which involve distance in "feet" and the length of the race involves distance units of "miles". We need to make a judgment call and decide on a single type of distance unit to use throughout the problem; either choice is OK. Let's use "feet", then here is the fact we need to recall:

$$\begin{aligned} \text{(total distance traveled)} &= \text{(constant speed)} \times \text{(elapsed time)} \qquad (1.1) \\ \text{(ft)} &= \text{(ft/sec)} \times \text{(sec)} \end{aligned}$$

To apply the formula, let t represent the elapsed time in seconds and first carry out a "conversion of units" using the conversion factor "5,280 ft/mile". Recall, we can manipulate the units just like numbers, canceling common units on the top and bottom of a fraction:

$$\begin{aligned} 26.2\,\text{mile} \times (5{,}280\,\text{ft/mile}) &= (26.2)(5{,}280)\frac{\cancel{\text{mile}}\cdot \text{ft}}{\cancel{\text{mile}}} \\ &= (26.2)(5{,}280)\,\text{ft}. \end{aligned}$$

Formula 1.1 can now be applied:

$$\begin{aligned} (26.2)(5{,}280)\,\text{ft} &= 18\,\text{ft/sec} \times t \\ \left(\frac{(26.2)(5{,}280)}{18}\right)\frac{\cancel{\text{ft}}}{\cancel{\text{ft}}/\text{sec}} &= t \\ 7{,}685.33\,\frac{1}{1/\text{sec}} &= 7{,}685.33\,\text{sec} = t \end{aligned}$$

So, the runner would complete the race in $t = 7{,}685.33$ seconds. If we wanted this answer in more sensible units, we would go through yet another units conversion:

$$\begin{aligned} t &= 7{,}685.33\,\text{sec} \times (1\text{ min}/60\text{ sec}) \times (1\text{ hr}/60\text{ min}) \\ &= \left(\frac{7{,}685.33}{60^2}\right)\text{hr} \\ &= 2.1348\,\text{hr}. \end{aligned}$$

The finish clock will display elapsed time in units of "hours : minutes : seconds". Two further conversions (see Exercise 1.5) lead to our runner having a time of 2:08:05.33; this is a world class time!

Manipulation of units becomes especially important when we are working with the *density* of a substance, which is defined by

$$\text{density} \overset{\text{def}}{=} \frac{\text{mass}}{\text{volume}}.$$

For example, pure water has a density of $1\,\text{g/cm}^3$. Notice, given any two of the quantities "density, volume or mass," we can solve for the remaining unknown using the formula. For example, if $857\,\text{g}$ of an unknown substance has a volume of 2.1 liters, then the density would be

$$\begin{aligned} d &= \frac{\text{mass}}{\text{volume}} \\ &= \frac{857\,\text{g}}{2.1\,\text{L}} \\ &= \left(\frac{857}{2.1}\right) \times \left(\frac{\text{g}}{\cancel{\text{L}}}\right) \times \left(\frac{1\,\cancel{\text{L}}}{1{,}000\,\text{cm}^3}\right) \\ &= 0.408\,\text{g/cm}^3. \end{aligned}$$

Example 1.1.1. *A sphere of solid gold has a mass of* $100\,kg$ *and the density of gold is* $19.3\,g/cm^3$*. What is the radius of the sphere?*

Solution. This problem is more involved. To answer this, let r be the unknown radius of the sphere in units of cm. The volume of the sphere is $V = \frac{4}{3}\pi r^3$. Since the sphere is solid gold, the density of gold is the ratio

$$\text{density of gold} = \frac{\text{mass of sphere}}{\text{volume of sphere}}$$

Plugging in what we know, we get the equation

$$19.3\,\frac{\text{g}}{\text{cm}^3} = \left(\frac{100\text{kg}}{\frac{4}{3}\pi r^3}\right) = \left(\frac{100\text{kg}}{\frac{4}{3}\pi r^3}\right)\left(\frac{1000\,\text{g}}{1\,\text{kg}}\right) = \left(\frac{10^5\,\text{g}}{\frac{4}{3}\pi r^3}\right)$$

Solving for r^3 we find

$$r^3 = 1236.955516\,\text{cm}^3$$

from which we get

$$r = \left(1237\,\text{cm}^3\right)^{1/3} = 10.73457\,\text{cm}$$

$\square$

1.2 Total Change = Rate × Time

We live in a world where things are changing as time goes by: the temperature during the day, the cost of tuition, the distance you will travel after leaving this class, and so on. The ability to precisely describe how a quantity is changing becomes especially important when making any kind of experimental measurements. For this reason, let's start with a clear and careful definition. If a quantity is changing with respect to time (like temperature, distance or cost), we can keep track of this using what is called a *rate* (also sometimes called a *rate of change*); this is defined as follows:

$$\text{rate} \overset{\text{def}}{=} \frac{\text{change in the quantity}}{\text{change in time}}$$

This sort of thing comes up so frequently, there is special shorthand notation commonly used: We let the Greek letter Δ (pronounced "delta") be shorthand for the phrase "change in." With this agreement, we can rewrite our rate definition in this way:

$$\text{rate} \overset{\text{def}}{=} \frac{\Delta\,\text{quantity}}{\Delta\,\text{time}}$$

But, now the question becomes: How do we calculate a rate? If you think about it, to calculate "Δ quantity" in the rate definition requires that we compare two quantities at two different times and see how they differ (i.e., how they have changed). The two times of comparison are usually called the *final time* and the *initial time*. We really need to be precise about this, so here is what we mean:

$$\begin{aligned}\Delta\,\text{quantity} &= (\text{value of quantity at final time}) - \\ &\quad\ (\text{value of quantity at initial time}) \\ \Delta\,\text{time} &= (\text{final time}) - (\text{initial time}).\end{aligned}$$

For example, suppose that on June 4 we measure that the temperature at 8:00 am is 65° F and at 10:00 am it is 71° F. So, the final time is 10:00 am, the initial time is 8:00 am and the temperature is changing according to the

$$\begin{aligned}\text{rate} &= \frac{\Delta\,\text{quantity}}{\Delta\,\text{time}} \\ &= \frac{\text{final value of quantity} - \text{initial value of quantity}}{\text{final time} - \text{initial time}} \\ &= \frac{71 - 65\,\text{degrees}}{10{:}00 - 8{:}00\,\text{hours}} \\ &= 3\,\text{deg/hr}.\end{aligned}$$

As a second example, suppose on June 5 the temperature at 8:00 am is 71° and at 10:00 am it is 65°. So, the final time is 10:00 am, the initial time is 8:00 am and the temperature is changing according to the

$$\begin{aligned} \text{rate} &= \frac{\Delta\,\text{quantity}}{\Delta\,\text{time}} \\ &= \frac{\text{final value of quantity} - \text{initial value of quantity}}{\text{final time} - \text{initial time}} \\ &= \frac{65 - 71\,\text{degrees}}{10{:}00 - 8{:}00\,\text{hours}} \\ &= -3\,\text{deg/hr}. \end{aligned}$$

These two examples illustrate that a rate can be either a positive or a negative number. More importantly, it highlights that we really need to be careful when making a rate computation. In both examples, the initial and final times are the same and the two temperatures involved are the same, BUT whether they occur at the initial or final time is interchanged. If we accidently mix this up, we will end up being off by a minus sign.

There are many situations where the rate is the same for all time periods. In a case like this, we say we have a *constant rate*. For example, imagine you are driving down the freeway at a constant speed of $60\,\text{mi/hr}$. The fact that the speedometer needle indicates a steady speed of $60\,\text{mi/hr}$ means the rate your distance is changing is constant.

In cases when we have a constant rate, we often want to find the *total amount of change* in the quantity over a specific time period. The key principle in the background is this:

$$\text{Total Change in some Quantity} = \text{Rate} \times \text{Time} \tag{1.2}$$

It is important to mention that this formula only works when we have a constant rate, but that will be the only situation we encounter in this course. One of the main goals of calculus is to develop a version of (1.2) that works for non-constant rates. Here is another example; others will occur throughout the text.

Example 1.2.1. *A water pipe mounted to the ceiling has a leak. It is dripping onto the floor below and creates a circular puddle of water. The surface area of this puddle is increasing at a constant rate of 4 cm^2/hour. Find the surface area and dimensions of the puddle after 84 minutes.*

Solution. The quantity changing is "surface area" and we are given a "rate" and "time." Using (1.2) with time $t = 84$ minutes,

$$\begin{aligned} \text{Total Surface Area} &= \text{Rate} \times \text{Time} \\ &= \left(4\,\frac{\text{cm}^2}{\text{hr}}\right) \times \left(\frac{84}{60}\,\text{hr}\right) \\ &= 5.6\,\text{cm}^2. \end{aligned}$$

The formula for the area of a circular region of radius r is given at the back of this text. Using this, the puddle has radius $r = \sqrt{\frac{5.6}{\pi}} = 1.335\,\text{cm}$ at time $t = 84$ minutes. ☐

1.3 The Modeling Process

Modeling is a method used in disciplines ranging from architecture to zoology. This mathematical technique will crop up any time we are problem solving and consciously trying to both "describe" and "predict." Inevitably, mathematics is introduced to add structure to the model, but the clean equations and formulas only arise after some (or typically a lot) of preliminary work.

A model can be thought of as a caricature in that it will pick out certain features (like a nose or a face) and focus on those at the expense of others. It takes a lot of experience to know which models are "good" and "bad," in the sense of isolating the right features. In the beginning, modeling will lead to frustration and confusion, but by the end of this course our comfort level will dramatically increase. Let's look at an illustration of the problem solving process.

Example 1.3.1. *How much time do you anticipate studying precalculus each week?*

Solution. One possible response is simply to say "a little" or "way too much!" You might not think these answers are the result of modeling, but they are. They are a consequence of modeling the total amount of study time in terms of categories such as "a little," "some," "lots," "way too much," etc. By drawing on your past experiences with math classes and using this crude model you arrived at a preliminary answer to the question.

Let's put a little more effort into the problem and try to come up with a numerical estimate. If T is the number of hours spent on precalculus a given week, it is certainly the case that:

$$T = (\text{hours in class}) + (\text{hours reading text}) + (\text{hours doing homework})$$

Our time in class each week is known to be 5 hours. However, the other two terms require a little more thought. For example, if we can comfortably read and digest a page of text in (on average) 15 minutes and there are r pages of text to read during the week, then

$$(\text{hours reading text}) = \frac{15}{60}r \text{ hours.}$$

As for homework, if a typical homework problem takes (on average) 25 minutes and there are h homework problems for the week, then

$$(\text{hours doing homework}) = \frac{25}{60}h \text{ hours.}$$

We now have a mathematical model for the weekly time commitment to precalculus:

$$T = 5 + \frac{15}{60}r + \frac{25}{60}h \text{ hours.}$$

Is this a good model? Well, it is certainly more informative than our original crude model in terms of categories like “a little” or “lots.” But, the real plus of this model is that it clearly isolates the features being used to make our estimated time commitment and it can be easily modified as the amount of reading or homework changes. So, this is a pretty good model. However, it isn’t perfect; some homework problems will take a lot more than 25 minutes! □

1.4 Exercises

Problem 1.1. (a) Verify that 7685.33 seconds is 2 hours 8 minutes 5.33 seconds.

(b) Which is faster: 100 mph or 150 ft/s?

(c) Gina's salary is 1 cent/second for a 40 hour work week. Tiare's salary is $1400 for a 40 hour work week. Who has a higher salary?

(d) Suppose it takes 180 credits to get a baccalaureate degree. You accumulate credit at the rate of one credit per quarter for each hour that the class meets per week. For instance, a class that meets three hours each week of the quarter will count for three credits. In addition, suppose that you spend 2.5 hours of study outside of class for each hour in class. A quarter is 10 weeks long. How many total hours, including time spent in class and time spent studying out of class, must you invest to get a degree?

Problem 1.2. Sarah can bicycle a loop around the north part of Lake Washington in 2 hours and 40 minutes. If she could increase her average speed by 1 km/hr, it would reduce her time around the loop by 6 minutes. How many kilometers long is the loop?

Problem 1.3. The density of lead is 11.34 g/cm^3 and the density of aluminum is 2.69 g/cm^3. Find the radius of lead and aluminum spheres each having a mass of 50 kg.

Problem 1.4. The Eiffel Tower has a mass of 7.3 million kilograms and a height of 324 meters. Its base is square with a side length of 125 meters. The steel used to make the Tower occupies a volume of 930 cubic meters. Air has a density of 1.225 kg per cubic meter. Suppose the Tower was contained in a cylinder. Find the mass of the air in the cylinder. Is this more or less than the mass of the Tower?

Problem 1.5. Marathon runners keep track of their speed using units of *pace* = minutes/mile.

(a) Lee has a speed of 16 ft/sec; what is his pace?

(b) Allyson has a pace of 6 min/mile; what is her speed?

(c) Adrienne and Dave are both running a race. Adrienne has a pace of 5.7 min/mile and Dave is running 10.3 mph. Who is running faster?

Problem 1.6. Convert each of the following sentences into "pseudo-equations." For example, suppose you start with the sentence: "The cost of the book was more than $10 and the cost of the magazine was $4." A first step would be these "pseudo-equations":

(Book cost) > $10 and (Magazine cost) = $4.

(a) John's salary is $56,000 a year and he pays no taxes.

(b) John's salary is at most $56,000 a year and he pays 15% of his salary in taxes.

(c) John's salary is at least $56,000 a year and he pays more than 28% of his salary in taxes.

(d) The number of students taking Math 120 at the UW is somewhere between 1500 and 1800 each year.

(e) The cost of a new red Porsche is more than three times the cost of a new Ford F-150 pickup truck.

(f) Each week, students spend at least two but no more than three hours studying for each credit hour.

(g) Twice the number of happy math students exceeds five times the number of happy chemistry students. However, all of the happy math and chemistry students combined is less than half the total number of cheerful biology students.

(h) The difference between Cady's high and low midterm scores was 10%. Her final exam score was 97%.

(i) The vote tally for Gov. Tush was within one-hundredth of one percent of one-half the total number of votes cast.

Problem 1.7. Which is a better deal: A 10 inch diameter pizza for $8 or a 15 inch diameter pizza for $16?

Problem 1.8. The famous *theory of relativity* predicts that a lot of weird things will happen when you approach the speed of light $c = 3 \times 10^8$ m/sec. For example, here is a formula that relates the mass m_o (in kg) of an object at rest and its mass when it is moving at a speed v:

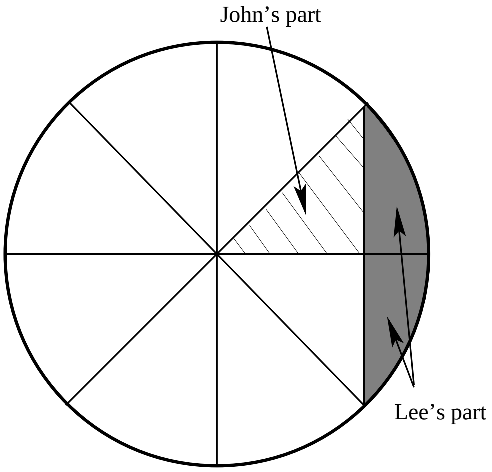
$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

(a) Suppose the object moving is Dave, who has a mass of $m_o = 66$ kg at rest. What is Dave's mass at 90% of the speed of light? At 99% of the speed of light? At 99.9% of the speed of light?

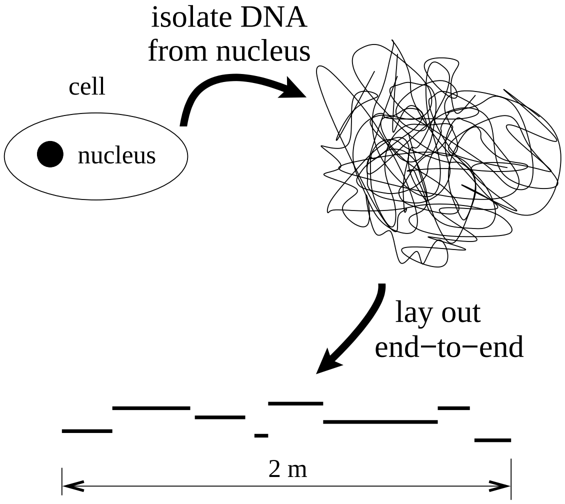
(b) How fast should Dave be moving to have a mass of 500 kg?

Problem 1.9. During a typical evening in Seattle, *Pagliacci* receives phone orders for pizza delivery at a constant rate: 18 orders in a typical 4 minute period. How many pies are sold in 4 hours? Assume *Pagliacci* starts taking orders at 5 : 00 pm and the profit is a constant rate of $11 on 10 orders. When will phone order profit exceed $1,000?

Problem 1.10. *Aleko's Pizza* has delivered a beautiful 16 inch diameter pie to Lee's dorm room. The pie is sliced into 8 equal sized pieces, but Lee is such a non-conformist he cuts off an edge as pictured. John then takes one of the remaining triangular slices. Who has more pizza and by how much?



Problem 1.11. A typical cell in the human body contains molecules of *deoxyribonucleic acid*, referred to as *DNA* for short. In the cell, this DNA is all twisted together in a tight little packet. But imagine unwinding (straightening out) all of the DNA from a single typical cell and laying it "end-to-end"; then the sum total length will be approximately 2 meters.



Assume the human body has 10^{14} cells containing DNA. How many times would the sum total length of DNA in your body wrap around the equator of the earth?

Problem 1.12. A water pipe mounted to the ceiling has a leak and is dripping onto the floor below, creating a circular puddle of water. The area of the circular puddle is increasing at a constant rate of 11 cm^2/hour.

(a) Find the area and radius of the puddle after 1 minute, 92 minutes, 5 hours, 1 day.

(b) Is the radius of the puddle increasing at a constant rate?

Problem 1.13. During the 1950s, Seattle was dumping an average of 20 million gallons of sewage into Lake Washington each day.

(a) How much sewage went into Lake Washington in a week? In a year?

(b) In order to illustrate the amounts involved, imagine a rectangular prism whose base is the size of a football field (100 yards × 50 yards) with height h yards. What are the dimensions of such a rectangular prism containing the sewage dumped into Lake Washington in a single day? (Note: There are 7.5 gallons in one cubic foot. Dumping into Lake Washington has stopped; now it goes into the Puget Sound.)

Problem 1.14. Dave has inherited an apple orchard on which 60 trees are planted. Under these conditions, each tree yields 12 bushels of apples. According to the local WSU extension agent, each time Dave removes a tree the yield per tree will go up 0.45 bushels. Let x be the number of trees in the orchard and N the yield per tree.

(a) Find a formula for N in terms of the unknown x. (Hint: Make a table of data with one column representing various values of x and the other column the corresponding values of N. After you complete the first few rows of the table, you need to discover the pattern.)

(b) What possible reason(s) might explain why the yield goes up when you remove trees?

Problem 1.15. Congress is debating a proposed law to reduce tax rates. If the current tax rate is $r\%$, then the proposed rate after x years is given by this formula:

$$\frac{r}{1+\frac{1}{1+\frac{1}{x}}}.$$

Rewrite this formula as a simple fraction. Use your formula to calculate the new tax rate after 1, 2, 5 and 20 years. Would tax rates increase or decrease over time? Congress claims that this law would ultimately cut peoples' tax rates by 75%. Do you believe this claim?

Problem 1.16. (a) The temperature at 7:00 am is 44°F and the temperature at 10:00 am is 50°F. What are the initial time, the final time, the initial temperature and the final temperature? What is the rate of change in the temperature between 7:00 am and 10:00 am?

(b) Assume it is 50°F at 10:00 am and the rate of change in the temperature between 10:00 am and 2:00 pm is the same as the rate in part (a). What is the temperature at 2:00 pm?

(c) The temperature at 4:30 pm is 54°F and the temperature at 6:15 pm is 26°F. What are the initial time, the final time, the initial temperature and the final temperature? What is the rate of change in the temperature between 4:30 pm and 6:15 pm?

Problem 1.17. (a) Solve for t: $3t-7=11+t$.

(b) Solve for a: $\sqrt{1+\frac{1}{a}}=3$.

(c) Solve for x: $\sqrt{x^2+a^2}=2a+x$.

(d) Solve for t: $1-t>4-2t$.

(e) Write as a single fraction:

$$\frac{2}{x}-\frac{1}{x+1}$$

Chapter 2

Imposing Coordinates

You find yourself visiting Spangle, WA and dinner time is approaching. A friend has recommended Tiff's Diner, an excellent restaurant; how will you find it?

Of course, the solution to this simple problem amounts to locating a "point" on a two-dimensional map. This idea will be important in many problem solving situations, so we will quickly review the key ideas.

2.1 The Coordinate System

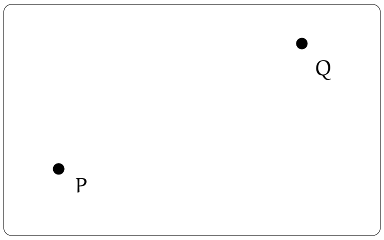
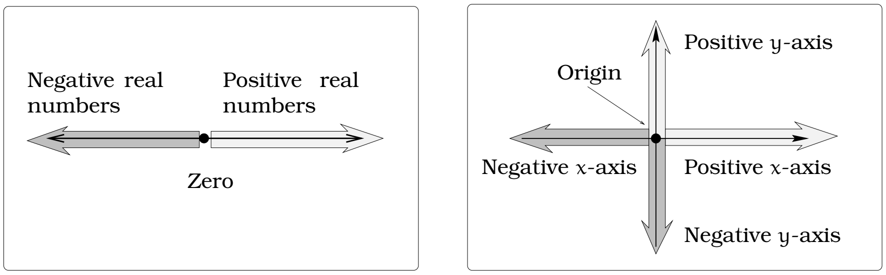


Figure 2.1: Two points in a plane.

If we are careful, we can develop the flow of ideas underlying two-dimensional coordinate systems in such a way that it easily generalizes to three-dimensions. Suppose we start with a blank piece of paper and mark two points; let's label these two points "P" and "Q." This presents the basic problem of finding a foolproof method to reconstruct the picture.

The basic idea is to introduce a *coordinate system* for the plane (analogous to the city map grid of streets), allowing us to catalog points in the plane using pairs of real numbers (analogous to the addresses of locations in the city).

Here are the details. Start by drawing two perpendicular lines, called the *horizontal axis* and the *vertical axis*, each of which looks like a copy of the real number line. We refer to the intersection point of these two lines as the *origin*. Given P in the plane, the plan is to use these two axes to obtain a pair of real numbers (x,y) that will give us the exact location of P. With this in mind, the horizontal axis is often called the x-axis and the vertical axis is often called the y-axis. Remember, a typical real number line (like the x-axis or the y-axis) is divided into three parts: the positive numbers, the negative numbers, and the number zero (see Figure 2.2(a)). This allows us to specify positive and negative portions of the x-axis and y-axis. Unless we say otherwise, we will always adopt the convention that the *positive x-axis* consists of those numbers to the right of the origin on the x-axis and the *positive* y*-axis* consists of those numbers above the origin on the y-axis. We have just described the xy*-coordinate system* for the plane:



(a) Number line. (b) xy-coordinate system.

Figure 2.2: Coordinates.

2.1.1 Going from P to a Pair of Real Numbers.

Imagine a coordinate system had been drawn on our piece of paper in Figure 2.1. Let's review the procedure of going from a point P to a pair of real numbers:

1. First, draw two new lines passing through P, one parallel to the x-axis and the other parallel to the y-axis; call these ℓ and ℓ^*, as pictured in Figure 2.3.
2. Notice that ℓ will cross the y-axis exactly once; the point on the y-axis where these two lines cross will be called "y." Likewise, the line ℓ^* will cross the x-axis exactly once; the point on the x-axis where these two lines cross will be called "x."
3. If you begin with two different points, like P and Q in Figure 2.1, you will see that the two *pairs* of points you obtain will be different; i.e., if Q gives you the pair (x^*,y^*), then either $x \neq x^*$ or $y \neq y^*$. This shows that two different points in the plane give two different pairs of real numbers and describes the process of assigning a pair of real numbers to the point P.

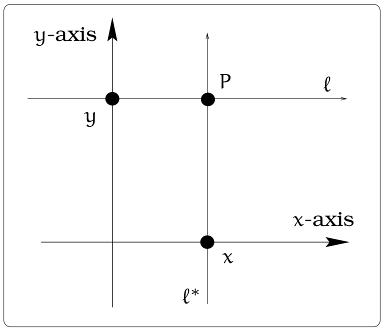


Figure 2.3: Coordinate pairs.

The great thing about the procedure we just described is that it is reversible! In other words, suppose you start with a pair of real numbers, say (x,y). Locate the number x on the x-axis and the number y on the y-axis. Now draw two lines: a line ℓ parallel to the x-axis passing through the number y on the y-axis and a line ℓ^* parallel to the y-axis passing through the number x on the x-axis. The two lines ℓ and ℓ^* will intersect in exactly one point in the plane, call it P. This procedure describes how to go from a given pair of real numbers to a point in the plane. In addition, if you start with two different pairs of real numbers, then the corresponding two points in the plane are going to be different. In the future, we will constantly be going back and forth between points in the plane and pairs of real numbers using these ideas.

Definition 2.1.1. ***Coordinate System:*** *Every point P in the xy-plane corresponds to a unique pair of real numbers (x, y), where x is a number on the horizontal x-axis and y is a number on the vertical y-axis; for this reason, we commonly use the notation "$P = (x,y)$."*

Having specified positive and negative directions on the horizontal and vertical axes, we can now divide our two dimensional plane into four *quadrants*. The first quadrant corresponds to all the points where both coordinates are positive, the second quadrant consists of points with the first coordinate negative and the second coordinate positive, etc. Every point in the plane will lie in one of these four quadrants or on one of the two axes. This quadrant terminology is useful to give a rough sense of location, just as we use the terminology "Northeast, Northwest, Southwest and Southeast" when discussing locations on a map.

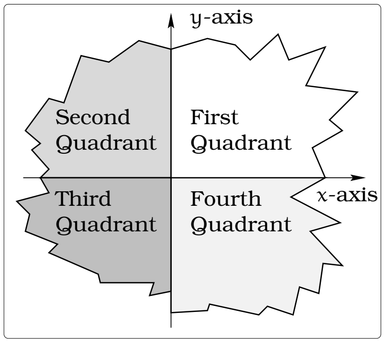


Figure 2.4: Quadrants in the xy-plane.

2.2 Three Features of a Coordinate System

A coordinate system involves *scaling, labeling* and *units* on each of the axes.

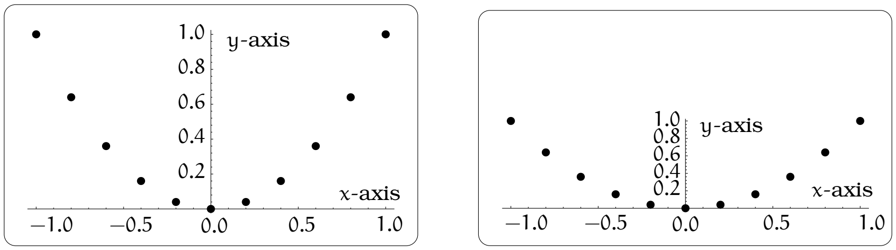
2.2.1 Scaling

Sketch two xy coordinate systems. In the first, make the scale on each axis the same. In the second, assume "one unit" on the x axis has the same length as "two units" on the y axis. Plot the points $(1,1)$, $(-1,1)$, $(-\frac{4}{5}, \frac{16}{25})$, $(-\frac{3}{5}, \frac{9}{25})$, $(-\frac{2}{5}, \frac{4}{25})$, $(-\frac{1}{5}, \frac{1}{25})$, $(0,0)$, $(\frac{1}{5}, \frac{1}{25})$, $(\frac{2}{5}, \frac{4}{25})$, $(\frac{3}{5}, \frac{9}{25})$, $(\frac{4}{5}, \frac{16}{25})$, $(1,1)$.

Both pictures illustrate how the points lie on a parabola in the xy-coordinate system, but the *aspect ratio* has changed. The aspect ratio is defined by this fraction:

$$\text{aspect ratio} \overset{\text{def}}{=} \frac{\text{length of one unit on the vertical axis}}{\text{length of one unit on the horizontal axis}}.$$

Figure 2.5(a) has aspect ratio 1, whereas Figure 2.5(b) has aspect ratio $\frac{1}{2}$. In problem solving, you will often need to make a rough assumption about the relative axis scaling. This scaling will depend entirely on the



(a) Aspect ratio = 1.

(b) Aspect ratio = $\frac{1}{2}$.

Figure 2.5: Coordinates.

information given in the problem. Most graphing devices will allow you to specify the aspect ratio.

2.2.2 Axes Units

Sometimes we are led to coordinate systems where each of the two axes involve different types of units (labels). Here is a sample, that illustrates the power of using pictures.

Example 2.2.1. *As the marketing director of Turboweb software, you have been asked to deliver a brief message at the annual stockholders meeting on the performance of your product. Your staff has assembled this tabular collection of data; how can you convey the content of this table most clearly?*

TURBOWEB SALES (in $1000's)									
week	*sales*	*week*	*sales*	*week*	*sales*	*week*	*sales*	*week*	*sales*
1	*11.0517*	*11*	*30.0417*	*21*	*81.6617*	*31*	*221.98*	*41*	*603.403*
2	*12.214*	*12*	*33.2012*	*22*	*90.2501*	*32*	*245.325*	*42*	*666.863*
3	*13.4986*	*13*	*36.693*	*23*	*99.7418*	*33*	*271.126*	*43*	*736.998*
4	*14.9182*	*14*	*40.552*	*24*	*110.232*	*34*	*299.641*	*44*	*814.509*
5	*16.4872*	*15*	*44.8169*	*25*	*121.825*	*35*	*331.155*	*45*	*900.171*
6	*18.2212*	*16*	*49.5303*	*26*	*134.637*	*36*	*365.982*	*46*	*994.843*
7	*20.1375*	*17*	*54.7395*	*27*	*148.797*	*37*	*404.473*	*47*	*1099.47*
8	*22.2554*	*18*	*60.4965*	*28*	*164.446*	*38*	*447.012*	*48*	*1215.1*
9	*24.596*	*19*	*66.8589*	*29*	*181.741*	*39*	*494.024*	*49*	*1342.9*
10	*27.1828*	*20*	*73.8906*	*30*	*200.855*	*40*	*545.982*	*50*	*1,484.131*

One idea is to simply flash an overhead slide of this data to the audience; this can be deadly! A better idea is to use a visual aid. Suppose we let the variable x represent the *week* and the variable y represent the *gross sales* (in thousands of dollars) in week x. We can then plot the points (x,y) in the xy-coordinate system; see Figure 2.6.

Notice, the units on the two axes are very different: y-axis units are "thousands of dollars" and x-axis units are "weeks." In addition, the aspect ratio of this coordinate system is not 1. The beauty of this picture is the visual impact it gives your audience. From the coordinate plot we can get a sense of how the sales figures are dramatically increasing. In fact, this plot is good evidence you deserve a big raise!

Mathematical modeling is all about relating concrete phenomena and symbolic equations, so we want to embrace the idea of visualization. Most typically, visualization will involve plotting a collection of points in the plane. This can be achieved by providing a “list” or a “prescription” for plotting the points. The material we review in the next couple of sections makes the transition from symbolic mathematics to visual pictures go more smoothly.

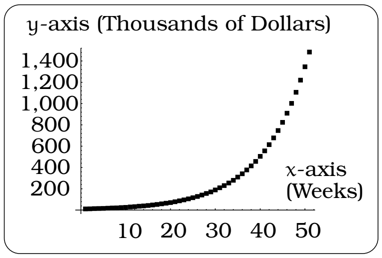


Figure 2.6: Turboweb sales.

2.3 A Key Step in all Modeling Problems

The initial problem solving or modeling step of deciding on a choice of xy-coordinate system is called *imposing a coordinate system*: There will often be many possible choices; it takes problem solving experience to develop intuition for a “natural” choice. This is a key step in all modeling problems.

Example 2.3.1. *Return to the tossed ball scenario on page 1. How do we decide where to draw a coordinate system in the picture?*

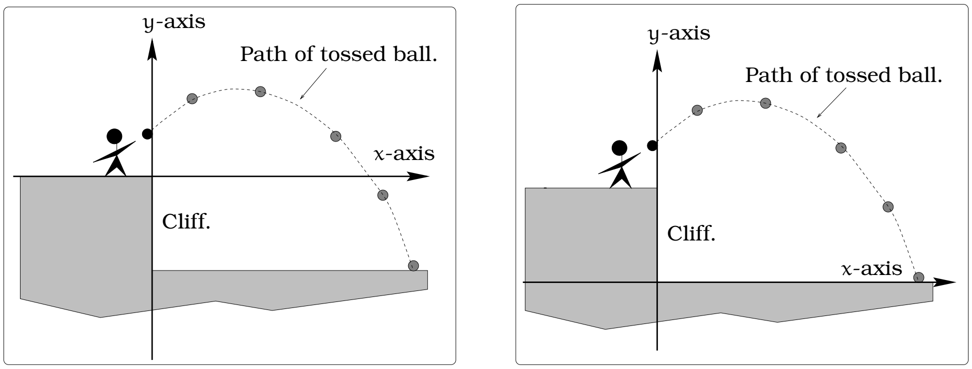
Figure 2.7 on page 16 shows four natural choices of xy-coordinate system. To choose a coordinate system we must specify the origin. The four logical choices for the origin are either the top of the cliff, the bottom of the cliff, the launch point of the ball or the landing point of the ball. So, which choice do we make? The answer is that any of these choices will work, but one choice may be more natural than another. For example, Figure 2.7(b) is probably the most natural choice: in this coordinate system, the motion of the ball takes place entirely in the first quadrant, so the x and y coordinates of any point on the path of the ball will be non-negative.

Example 2.3.2. *Michael and Aaron are running toward each other, beginning at opposite ends of a* $10{,}000$ *ft. airport runway, as pictured in Figure 2.8 on page 17. Where and when will these guys collide?*

Solution. This problem requires that we find the “time” and “location” of the collision. Our first step is to impose a coordinate system.

We choose the coordinate system so that Michael is initially located at the point $M = (0, 0)$ (the origin) and Aaron is initially located at the point $A = (10{,}000, 0)$. To find the coordinates of Michael after t seconds, we need to think about how distance and time are related.

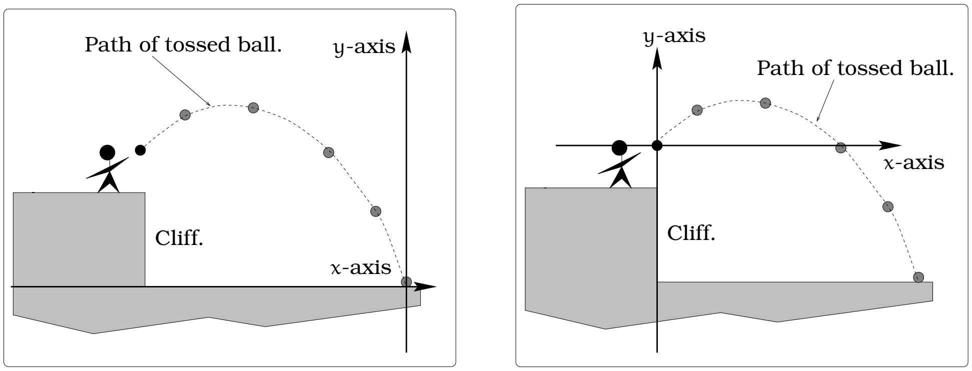
Since Michael is moving at the rate of 15 ft/second, then after one second he is located 15 feet right of the origin; i.e., at the point $(15, 0)$. After 2 seconds, Michael has moved an additional 15 feet, for a total of 30 feet; so he is located at the point $(30, 0)$, etc. Conclude Michael has traveled $15t$ ft. **to the right** after t seconds; i.e., his location is the point



y-axis
Path of tossed ball.
x-axis
Cliff.
y-axis
Path of tossed ball.
Cliff.
x-axis

(a) Origin at the top of the ledge.

(b) Origin at the bottom of the ledge.



Path of tossed ball.
y-axis
Cliff.
x-axis
y-axis
Path of tossed ball.
x-axis
Cliff.

(c) Origin at the landing point.

(d) Origin at the launch point.

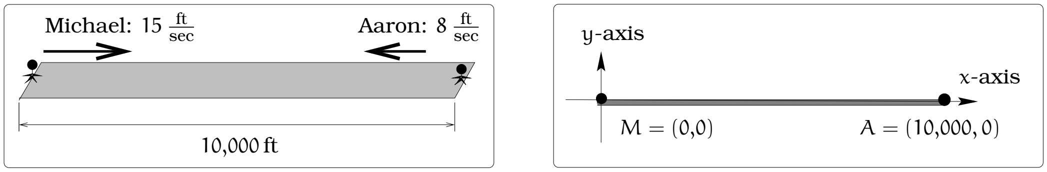
Figure 2.7: Choices when imposing an xy-coordinate system.

$M(t) = (15t, 0)$. Similarly, Aaron is located 8 ft. left of his starting location after 1 second (at the point $(9{,}992, 0)$), etc. Conclude Aaron has traveled $8t$ ft. **to the left** after t seconds; i.e., his location is the point $A(t) = (10{,}000 - 8t, 0)$.

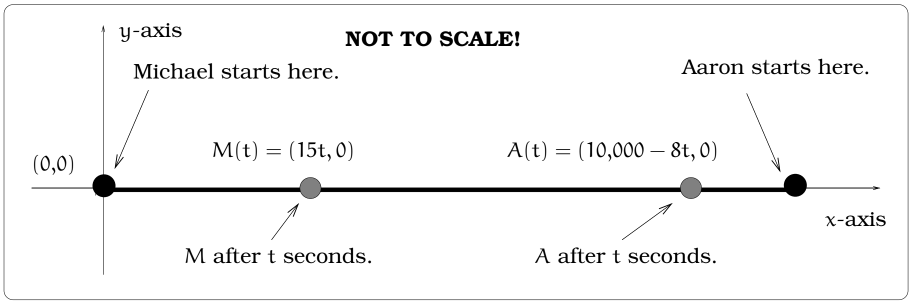
The key observation required to solve the problem is that the point of collision occurs when the coordinates of Michael and Aaron are equal. Because we are moving along the horizontal axis, this amounts to finding where and when the x-coordinates of $M(t)$ and $A(t)$ agree. This is a straight forward algebra problem:

$$\begin{aligned} 15t &= 10{,}000 - 8t \qquad (2.1)\\ 23t &= 10{,}000 \\ t &= 434.78 \end{aligned}$$

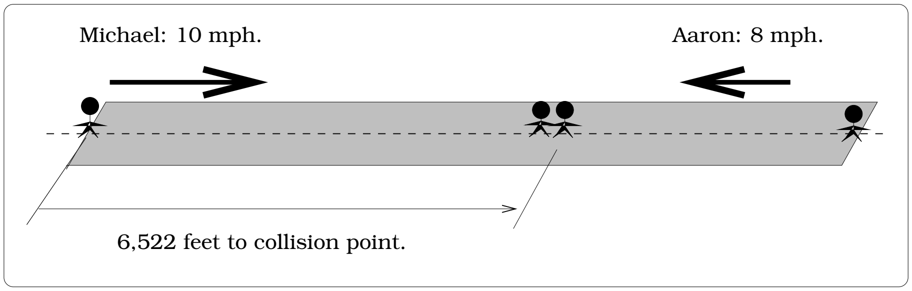
To the nearest tenth of a second, the runners collide after 434.8 seconds. Plugging $t = 434.78$ into either expression for the position: $M(434.8) = (15(434.8), 0) = (6{,}522, 0)$. □


Michael: 15 $\frac{\text{ft}}{\text{sec}}$
Aaron: 8 $\frac{\text{ft}}{\text{sec}}$
10,000 ft
y-axis
x-axis
M = (0,0)
A = (10,000, 0)

(a) The physical picture. (b) The xy-coordinate picture.


y-axis
NOT TO SCALE!
Michael starts here.
Aaron starts here.
(0,0)
M(t) = (15t, 0)
A(t) = (10,000 − 8t, 0)
x-axis
M after t seconds.
A after t seconds.

(c) Building a visual model


Michael: 10 mph.
Aaron: 8 mph.
6,522 feet to collision point.

(d) Michael and Aaron's collision point.

Figure 2.8: Michael and Aaron running head-on.

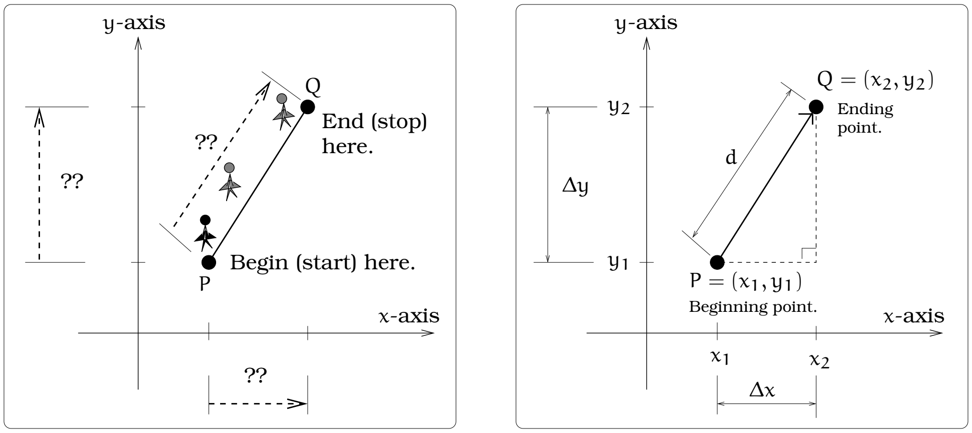
2.4 Distance

We end this Chapter with a discussion of direction and distance in the plane. To set the stage, think about the following analogy:

Example 2.4.1. *You are in an airplane flying from Denver to New York. How far will you fly? To what extent will you travel north? To what extent will you travel east?*

Consider two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the xy coordinate system, where *we assume that the units on each axis are the same*; for example, both in units of "feet." Imagine starting at the location P (Denver) and flying to the location Q (New York) along a straight line segment; see Figure 2.9(a). Now ask yourself this question: *To what overall extent have the* x *and* y *coordinates changed?*

To answer this, we introduce visual and notational aides into this figure. We have inserted an "arrow" pointing from the starting position P to the ending position Q; see Figure 2.9(b). To simplify things, introduce the notation Δx to keep track of the change in the x-coordinate and Δy



(a) Starting and stopping points. (b) Coordinates for P and Q.

Figure 2.9: The meaning of Δx and Δy.

to keep track of the change in the y-coordinate, as we move from P to Q. Each of these quantities can now be computed:

$$\begin{aligned} \Delta x &= \text{change in } x\text{-coordinate going from } P \text{ to } Q \\ &= (x\text{-coord of ending point}) - (x\text{-coord of beginning point}) \\ &= x_2 - x_1 \\ \Delta y &= \text{change in } y\text{-coordinate going from } P \text{ to } Q \\ &= (y\text{-coord of ending point}) - (y\text{-coord of beginning point}) \\ &= y_2 - y_1. \end{aligned} \tag{2.2}$$

We can interpret Δx and Δy using the right triangle in Figure 2.9(b). This means we can use the *Pythagorean Theorem* to write:

$$d^2 = (\Delta x)^2 + (\Delta y)^2;$$

that is,

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2},$$

which tells us the distance d from P to Q. In other words, d is the distance we would fly if we had flown along that line segment connecting the two points. As an example, if $P = (1, 1)$ and $Q = (5, 4)$, then $\Delta x = 5 - 1 = 4$, $\Delta y = 4 - 1 = 3$ and $d = 5$.

There is a subtle idea behind the way we defined Δx and Δy: You need to specify the "beginning" and "ending" points used to do the calculation in Equations 2.2. What happens if we had reversed the choices in Figure 2.9?

Then the quantities Δx and Δy will both be negative and the lengths of the sides of the right triangle are computed by taking the *absolute value* of Δx and Δy. As far as a distance calculation is concerned, the previous formula still works because of this algebra equality:

$$\begin{aligned} d &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(|\Delta x|)^2 + (|\Delta y|)^2}. \end{aligned}$$

We will sometimes refer to Δx and Δy as *directed distances* in the x and y directions. The notion of directed distance becomes important in our discussion of *lines* in Chapter 4 and later when you learn about *vectors*; it is also very important in *calculus*.

For example, if $P = (5,4)$ and $Q = (1,1)$, then $\Delta x = 1 - 5 = -4$, $\Delta y = 1 - 4 = -3$ and $d = 5$.

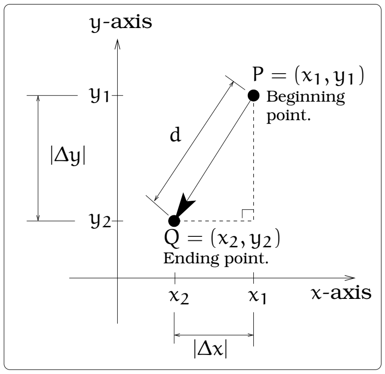


Figure 2.10: A different direction.

Important Fact 2.4.2 (Distance formula)**.** *If* $P = (x_1, y_1)$ *and* $Q = (x_2, y_2)$ *are two points in the plane, then the straight line distance between the points (in the same units as the two axes) is given by the formula*

$$\begin{aligned} d &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \end{aligned} \tag{2.3}$$

If your algebra is a little rusty, a very common mistake may crop up when you are using the distance formula. For example,

$$\begin{aligned} \sqrt{3^2 + 4^2} &\overset{?}{=} \sqrt{3^2} + \sqrt{4^2} \\ \sqrt{9 + 16} &\overset{?}{=} 3 + 4 \\ 5 &\neq 7. \end{aligned}$$

Notice, you have an impossible situation: 5 **is never equal to** 7.

¡¡¡
CAUTION
!!!

Example 2.4.3. *Two cars depart from a four way intersection at the same time, one heading East and the other heading North. Both cars are traveling at the constant speed of* 30 *ft/sec. Find the distance (in miles) between the two cars after* 1 *hour* 12 *minutes. In addition, determine when the two cars would be exactly* 1 *mile apart.*

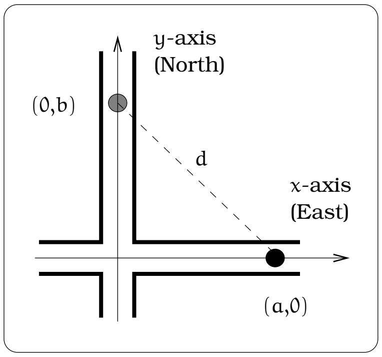


Figure 2.11: Two departing cars.

Solution. Begin with a picture of the situation. We have indicated the locations of the two vehicles after t seconds and the distance d between them at time t. By the distance formula, the distance between them is

$$\begin{aligned} d &= \sqrt{(a-0)^2+(0-b)^2} \\ &= \sqrt{a^2+b^2}. \end{aligned}$$

This formula is a first step; the difficulty is that we have traded the mystery distance d for two new unknown numbers a and b. To find the coordinate a for the Eastbound car, we know the car is moving at the rate of 30 ft/sec, so it will travel $30t$ feet after t seconds; i.e., $a = 30t$. Similarly, we find that $b = 30t$. Substituting into the formula for d we arrive at

$$\begin{aligned} d &= \sqrt{(30t)^2+(30t)^2} \\ &= \sqrt{2t^2(30)^2} \\ &= 30t\sqrt{2}. \end{aligned}$$

First, we need to convert 1 hour and 12 minutes into seconds so that our formula can be used:

$$\begin{aligned} \text{1 hr 12 min} &= 1 + 12/60 \text{ hr} \\ &= 1.2 \text{ hr} \\ &= (1.2\text{hr})\left(\frac{60 \text{ min}}{\text{hr}}\right)\left(\frac{60 \text{ sec}}{\text{min}}\right) \\ &= 4{,}320 \text{ sec}. \end{aligned}$$

Substituting $t = 4{,}320$ sec and recalling that 1 mile = 5,280 feet, we arrive at

$$\begin{aligned} d &= 129{,}600\sqrt{2}\,\text{feet} \\ &= 183{,}282 \text{ feet} \\ &= 34.71 \text{ miles}. \end{aligned}$$

For the second question, we specify the distance being 1 mile and want to find when this occurs. The idea is to set d equal to 1 mile and solve for

t. However, we need to be careful, since the units for d are feet:

$$\begin{aligned} 30t\sqrt{2} &= d \\ &= 5{,}280 \end{aligned}$$

Solving for t:

$$\begin{aligned} t &= \frac{5{,}280}{30\sqrt{2}} \\ &= \text{124.45 seconds} \\ &= \text{2 minutes 4 seconds.} \end{aligned}$$

The two cars will be 1 mile apart in 2 minutes, 4 seconds. □

2.5 Exercises

Problem 2.1. In the following four cases, let P be the initial (starting) point and Q the ending point; recall Equation 2.2 and Figure 2.10 on Page 19. Compute d = the distance from P to Q, Δx and Δy. Give your answer in **exact** form; eg. $\sqrt{2}$ is an exact answer, whereas 1.41 is an approximation of $\sqrt{2}$.

(a) $P = (0,0), Q = (1,1)$.

(b) $P = (2,1), Q = (1,-1)$.

(c) $P = (-1,2), Q = (4,-1)$.

(d) $P = (1,2), Q = (1+3t, 3+t)$, where t is a constant.

Problem 2.2. Start with two points $M = (a,b)$ and $N = (s,t)$ in the xy-coordinate system. Let d be the distance between these two points. Answer these questions and make sure you can justify your answers:

(a) TRUE or FALSE: $d = \sqrt{(a-s)^2 + (b-t)^2}$.

(b) TRUE or FALSE: $d = \sqrt{(a-s)^2 + (t-b)^2}$.

(c) TRUE or FALSE: $d = \sqrt{(s-a)^2 + (t-b)^2}$.

(d) Suppose M is the beginning point and N is the ending point; recall Equation 2.2 and Figure 2.10 on Page 19. What is Δx? What is Δy?

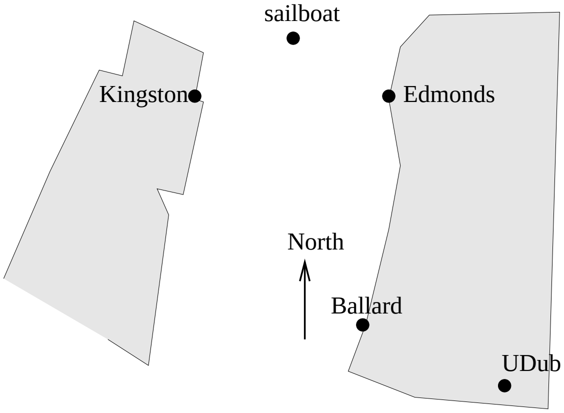
(e) Suppose N is the beginning point and M is the ending point; recall Equation 2.2 and Figure 2.10 on Page 19. What is Δx? What is Δy?

(f) If Δx=0, what can you say about the relationship between the positions of the two points M and N ? If Δy=0, what can you say about the relationship between the positions of the two points M and N? (Hint: Use some specific values for the coordinates and draw some pictures to see what is going on.)

Problem 2.3. Steve and Elsie are camping in the desert, but have decided to part ways. Steve heads North, at 6 AM, and walks steadily at 3 miles per hour. Elsie sleeps in, and starts walking West at 3.5 miles per hour starting at 8 AM.

When will the distance between them be 25 miles?

Problem 2.4. Erik's disabled sailboat is floating at a stationary location 3 miles East and 2 miles North of Kingston. A ferry leaves Kingston heading due East toward Edmonds at 12 mph. At the same time, Erik leaves the sailboat in a dinghy heading due South at 10 ft/sec (hoping to intercept the ferry). Edmonds is 6 miles due East of Kingston.



(a) Compute Erik's speed in mph and the Ferry speed in ft/sec.

(b) Impose a coordinate system and complete this table of data concerning locations (i.e., coordinates) of Erik and the ferry. Insert into the picture the locations of the ferry and Erik after 7 minutes.

Time	Ferry	Erik	Distance Between
0 sec			
30 sec			
7 min			
t hr			

(c) Explain why Erik misses the ferry.

(d) After 10 minutes, a Coast Guard boat leaves Kingston heading due East at a speed of 25 ft/sec. Will the Coast Guard boat catch the ferry before it reaches Edmonds? Explain.

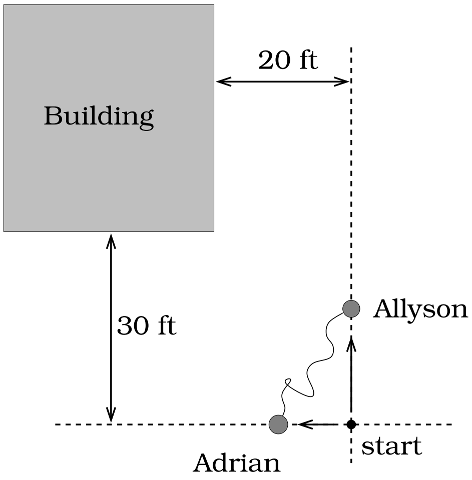
Problem 2.5. Suppose two cars depart from a four way intersection at the same time, one heading north and the other heading west. The car heading north travels at the steady speed of 30 ft/sec and the car heading west travels at the steady speed of 58 ft/sec.

(a) Find an expression for the distance between the two cars after t seconds.

(b) Find the distance in miles between the two cars after 3 hours 47 minutes.

(c) When are the two cars 1 mile apart?

Problem 2.6. Allyson and Adrian have decided to connect their ankles with a *bungee cord*; one end is tied to each person's ankle. The cord is 30 feet long, but can stretch up to 90 feet. They both start from the same location. Allyson moves 10 ft/sec and Adrian moves 8 ft/sec in the directions indicated.



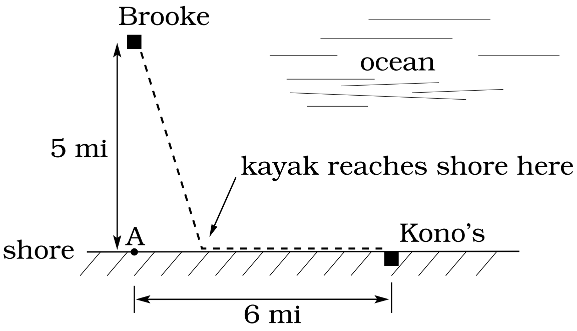
(a) Where are the two girls located after 2 seconds?

(b) After 2 seconds, will the slack in the bungee cord be used up?

(c) Determine when the bungee cord first becomes tight; i.e. there is no slack in the line. Where are the girls located when this occurs?

(d) When will the bungee cord first touch the corner of the building? (Hint: Use a fact about "similar triangles".)

Problem 2.7. Brooke is located 5 miles out from the nearest point A along a straight shoreline in her seakayak. Hunger strikes and she wants to make it to *Kono's* for lunch; see picture. Brooke can paddle 2 mph and walk 4 mph.



(a) If she paddles along a straight line course to the shore, find an expression that computes the total time to reach lunch in terms of the location where Brooke beaches the boat.

(b) Determine the total time to reach Kono's if she paddles directly to the point "A".

(c) Determine the total time to reach Kono's if she paddles directly to Kono's.

(d) Do you think your answer to (b) or (c) is the minimum time required for Brooke to reach lunch?

(e) Determine the total time to reach Kono's if she paddles directly to a point on the shore half way between point "A" and Kono's. How does this time compare to the times in parts (b) and (c)? Do you need to modify your answer to part (d)?

Problem 2.8. A spider is located at the position $(1,2)$ in a coordinate system, where the units on each axis are feet. An ant is located at the position $(15,0)$ in the same coordinate system. Assume the location of the spider after t minutes is $s(t) = (1+2t, 2+t)$ and the location of the ant after t minutes is $a(t) = (15-2t, 2t)$.

(a) Sketch a picture of the situation, indicating the locations of the spider and ant at times $t = 0,1,2,3,4,5$ minutes. Label the locations of the bugs in your picture, using the notation $s(0)$, $s(1)$,...,$s(5)$, $a(0)$, $a(1)$, ..., $a(5)$.

(b) When will the x-coordinate of the spider equal 5? When will the y-coordinate of the ant equal 5?

(c) Where is the spider located when its y-coordinate is 3?

(d) Where is each bug located when the y-coordinate of the spider is twice as large as the y-coordinate of the ant?

(e) How far apart are the bugs when their x-coordinates coincide? Draw a picture, indicating the locations of each bug when their x-coordinates coincide.

(f) A sugar cube is located at the position $(9{,}6)$. Explain why each bug will pass through the position of the sugar cube. Which bug reaches the sugar cube first?

(g) Find the speed of each bug along its line of motion; which bug is moving faster?

Problem 2.9. A Ferrari is heading south at a constant speed on Broadway (a north/south street) at the same time a Mercedes is heading west on Aloha Avenue (an east/west street). The Ferrari is 624 feet north of the intersection of Broadway and Aloha, at the same time that the Mercedes is 400 feet east of the intersection. Assume the Mercedes is traveling at the constant speed of 32 miles/hour. Find the speed of the Ferrari so that a collision occurs in the intersection of Broadway and Aloha.

Problem 2.10. Two planes flying opposite directions (North and South) pass each other 80 miles apart at the same altitude. The Northbound plane is flying 200 mph (miles per hour) and the Southbound plane is flying 150 mph. How far apart are the planes in 20 minutes? When are the planes 300 miles apart?

Problem 2.11. Here is a list of some algebra problems with "solutions." Some of the solutions are correct and some are wrong. For each problem, determine: (i) if the answer is correct, (ii) if the steps are correct, (iii) identify any incorrect steps in the solution (noting that the answer may be correct but some steps may not be correct).

(a) If $x \neq 1$,

$$\begin{aligned}\frac{x^2-1}{x+1} &= \frac{x^2+(-1)1}{x+1}\\ &= \frac{x^2}{x}+\frac{-1}{1}\\ &= x-1\end{aligned}$$

(b)

$$\begin{aligned}(x+y)^2-(x-y)^2 &= (x^2+y^2)-x^2-y^2\\ &= 0\end{aligned}$$

(c) If $x \neq 4$,

$$\begin{aligned}\frac{9(x-4)^2}{3x-12} &= \frac{3^2(x-4)^2}{3x-12}\\ &= \frac{(3x-12)^2}{3x-12}\\ &= 3x-12.\end{aligned}$$

Problem 2.12. Assume α, β are nonzero constants. Solve for x.

(a) $\alpha x+\beta=\frac{1}{\alpha x-\beta}$

(b) $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{1}{x}$

(c) $\alpha+\frac{1}{\beta}=\frac{1}{x}$

Problem 2.13. Simplify as far as possible.

(a) $(1-t)^2+(2+2t)^2$

(b) $(t+1)^2+(-t-1)^2-2$

(c) $\frac{1}{t-1}-\frac{1}{t+1}$ (write as a single fraction)

(d) $\sqrt{(2+t)^2+4t^2}$

Chapter 3

Three Simple Curves

Before we discuss graphing, we first want to become acquainted with the sorts of pictures that will arise. This is surprisingly easy to accomplish: Impose an xy-coordinate system on a blank sheet of paper. Take a sharp pencil and begin moving it around on the paper. The resulting picture is what we will call a *curve*. For example, here is a sample of the sort of "artwork" we are trying to visualize. A number of examples in the text will involve basic curves in the plane. When confronted with a curve in the plane, the fundamental question we always try to answer is this: *Can we give a condition (think of it as a "test") that will tell us precisely when a point in the plane lies on a curve?*

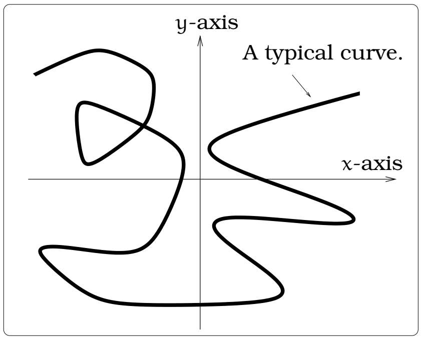


Figure 3.1: A typical curve.

Typically, the kind of condition we will give involves an equation in two variables (like x and y). We consider the three simplest situations in this chapter: horizontal lines, vertical lines and circles.

3.1 The Simplest Lines

Undoubtedly, the simplest curves in the plane are the horizontal and vertical lines. For example, sketch a line parallel to the x-axis passing through 2 on the y-axis; the result is a horizontal line ℓ, as pictured. This means the line ℓ passes through the point $(0, 2)$ in our coordinate system. A concise symbolic prescription for ALL of the points on ℓ can be given using "set notation":

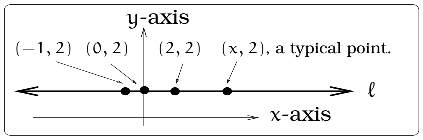


Figure 3.2: The points (x, y).

$$\ell = \{(x, 2) | x \text{ is any real number}\}.$$

We read the right-hand side of this expression as "*the set of all points* $(x, 2)$ *where* x = *any real number*." Notice, the points (x, y) on the line ℓ are EXACTLY the ones that lead to *solutions* of the equation $y = 2$; i.e., take any point on this line, plug the coordinates into the equation $y = 2$

and you get a true statement. Because the equation does not involve the variable x and only constrains y to equal 2, we see that x can take on any real value. In short, we see that plotting all of the solutions (x, y) to the equation $y = 2$ gives the line ℓ. We usually refer to the set of all solutions of the equation $y = 2$ as the *graph* of the equation $y = 2$.

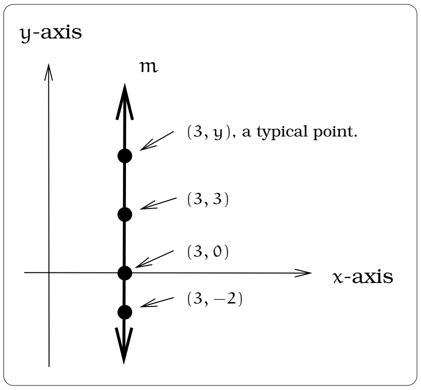


Figure 3.3: Points on $x = 3$.

As a second example, sketch the vertical line m passing through 3 on the x-axis; this means the line m passes through the point $(3, 0)$ in our coordinate system. A concise symbolic prescription for ALL of the points on m can be given using "set notation":

$$m = \{(3, y) : y \text{ is any real number}\}.$$

Notice, the points (x, y) on the line m are EXACTLY the ones that lead to *solutions* of the equation $x = 3$; i.e., take any point on this line, plug the coordinates into the equation $x = 3$ and you get a true statement. Because the equation does not involve the variable y and only specifies that $x = 3$, y can take on any real number value. Plotting all of the solutions (x, y) to the equation $x = 3$ gives the line m. We usually refer to the set of all solutions of the equation $x = 3$ as the *graph* of the equation $x = 3$.

These two simple examples highlight our first clear connection between a geometric figure and an equation; the link is achieved by plotting all of the solutions (x, y) of the equation in the xy-coordinate system. These observations work for any horizontal or vertical line.

Definition 3.1.1. ***Horizontal and Vertical Lines:*** *A horizontal line ℓ passing through k on the y-axis is precisely a plot of all solutions (x, y) of the equation $y = k$; i.e., ℓ is the graph of $y = k$. A vertical line m passing through h on the x-axis is precisely a plot of all solutions (x, y) of the equation $x = h$; i.e., m is the graph of $x = h$.*

3.2 Circles

Another common curve in the plane is a circle. Let's see how to relate a circle and an equation involving the variables x and y. As a special case of the distance formula (2.3), suppose $P = (0, 0)$ is the origin and $Q = (x, y)$ is any point in the plane; then

$$\begin{aligned} \text{distance from } P \text{ to } Q &= \sqrt{(x-0)^2 + (y-0)^2} \\ &= \sqrt{x^2 + y^2}. \end{aligned}$$

This calculation tells us that a point (x, y) is of distance r from the origin if and only if $r = \sqrt{x^2 + y^2}$ or, squaring each side, that $x^2 + y^2 = r^2$. This shows

$$\{(x, y) | \text{distance } (x, y) \text{ to origin is } r\} = \{(x, y) | x^2 + y^2 = r^2\}. \tag{3.1}$$

What is the left-hand side of Equation 3.1? To picture all points in the plane of distance r from the origin, fasten a pencil to one end of a non-elastic string (a string that will not stretch) of length r and tack the other end to the origin. Holding the string tight, the pencil point will locate a point of distance r from the origin. We could visualize all such points by simply moving the pencil around the origin, all the while keeping the string tight.

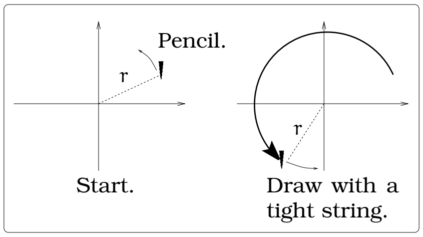


Figure 3.4: Drawing a circle.

What is the right-hand side of Equation 3.1? A point (x, y) in the right-hand set is a solution to the equation $x^2 + y^2 = r^2$; i.e., if we plug in the coordinates we get a true statement. For example, in Figure 3.5 we plot eight solutions $(r, 0)$, $(-r, 0)$, $(0, r)$, $(0, -r)$, $A = \left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right)$, $B = \left(\frac{r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}\right)$, $C = \left(\frac{-r}{\sqrt{2}}, \frac{-r}{\sqrt{2}}\right)$, $D = \left(\frac{-r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right)$, of the equation. To see that the last point is a solution, here is the sample calculation: $\left(\frac{-r}{\sqrt{2}}\right)^2 + \left(\frac{r}{\sqrt{2}}\right)^2 = \frac{r^2}{2} + \frac{r^2}{2} = r^2$.

Since the two sides of Equation 3.1 are equal, drawing the circle of radius r is the same as plotting all of the solutions of the equation $x^2 + y^2 = r^2$. The same reasoning can be used to show that drawing a circle of radius r centered at a point (h, k) is the same as plotting all of the solutions of the equation: $(x - h)^2 + (y - k)^2 = r^2$. We usually refer to the set of all solutions of the equation as the *graph* of the equation.

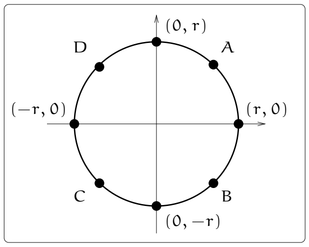


Figure 3.5: Computing points.

Definition 3.2.1 (Circles). *Let* (h, k) *be a given point in the* xy*-plane and* $r > 0$ *a given positive real number. The circle of radius* r *centered at* (h, k) *is precisely all of the solutions* (x,y) *of the equation*

$$(x - h)^2 + (y - k)^2 = r^2;$$

i.e., the circle is the graph of this equation.

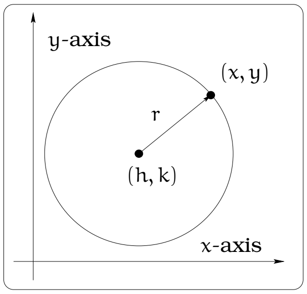


Figure 3.6: Defining a circle.

We refer to the equation in the box as the *standard form* of the equation of a circle. From this equation you know both the center and radius of the circle described.

¡¡¡
CAUTION
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Be very careful with the minus signs "−" in the standard form for a circle equation. For example, the equation

$$(x+3)^2 + (y-1)^2 = 7$$

is NOT in standard form. We can rewrite it in standard form:

$$(x-(-3))^2 + (y-1)^2 = (\sqrt{7})^2;$$

so, this equation describes a circle of radius $\sqrt{7}$ centered at $(-3, 1)$.

Examples 3.2.2. *Here are some of the ways we can discuss circles:*

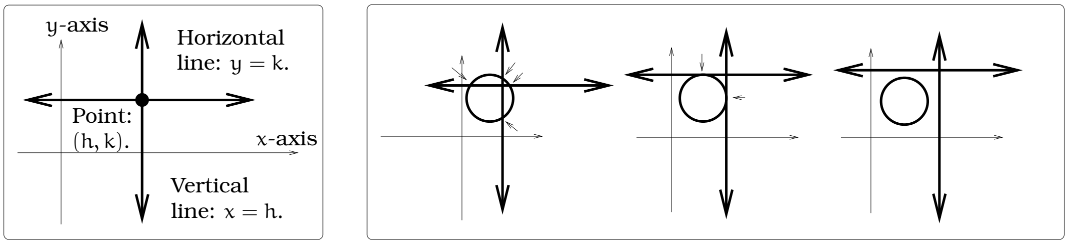
1. *The circle of radius 1 centered at the origin is the graph of the equation* $x^2 + y^2 = 1$. *This circle is called the unit circle and will be used extensively.*
2. *A circle of radius 3 centered at the point* $(h, k) = (1, -1)$ *is the graph of the equation* $(x-1)^2+(y-(-1))^2 = 3^2$*; or, equivalently* $(x-1)^2+(y+1)^2 = 3^2$; *or, equivalently* $x^2 + y^2 - 2x + 2y = 7$.
3. *The circle of radius* $\sqrt{5}$ *centered at* $(2, -3)$ *does not pass through the origin; this is because* $(0, 0)$ *is* ***not*** *a solution of the equation* $(x-2)^2 + (y+3)^2 = 5$.

3.3 Intersecting Curves I

In many problem solving situations, we will have two curves in the plane and need to determine where the curves intersect one another. Before we discuss a general procedure, let's make sure we really understand the meaning of the word "intersect." From Latin, the word "inter" means "within or in between" and the word "sectus" means "to cut." So, the intersection of two curves is the place where the curves "cut into" each other; in other words, where the two curves cross one another.

If the pictures of two curves are given to us up front, we can often visually decide whether or not they intersect. This is one good reason for drawing a picture of any physical problem we are trying to solve. We will need a small bag of tricks used for finding intersections of curves. We begin with intersections involving the curves studied in this section.

Two different horizontal lines (or two different vertical lines) will never intersect. However, a horizontal line always intersects a vertical line exactly once; Figure 3.7(a). Given a circle and a horizontal or vertical line, we may or may not have an intersection. Looking at Figure 3.7(b), you can convince yourself a given horizontal or vertical line will intersect a circle in either two points, one point or no points. This analysis is all pictorial; how do you find the explicit coordinates of an intersection point? Let's look at a sample problem to isolate the procedure used.



(a) Line equations. (b) Possible intersections.

Figure 3.7: Circles and lines.

Example 3.3.1. *Glo-Tek Industries has designed a new halogen street light fixture for the city of Seattle. According to the product literature, when placed on a* 50 *foot light pole, the resulting useful illuminated area is a circular disc* 120 *feet in diameter. Assume the light pole is located* 20 *feet east and* 40 *feet north of the intersection of Parkside Ave. (a north/south street) and Wilson St. (an east/west street). What portion of each street is illuminated?*

Solution. The illuminated area is a circular disc whose diameter and center are both known. Consequently, we really need to study the intersection of this circle with the two streets. Begin by imposing the pictured coordinate system; we will use units of feet for each axis. The illuminated region will be a circular disc centered at the point $(20, 40)$ in the coordinate system; the radius of the disc will be $r = 60$ feet.

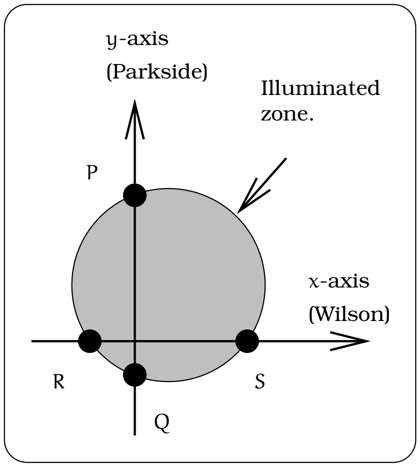


Figure 3.8: Illuminated street.

We need to find the points of intersection P, Q, R, and S of the circle with the x-axis and the y-axis. The equation for the circle with $r = 60$ and center $(h, k) = (20, 40)$ is

$$(x - 20)^2 + (y - 40)^2 = 3600.$$

To find the circular disc intersection with the y-axis, we have a system of two equations to work with:

$$\begin{aligned} (x - 20)^2 + (y - 40)^2 &= 3{,}600; \\ x &= 0. \end{aligned}$$

To find the intersection points we *simultaneously* solve both equations. To do this, we replace $x = 0$ in the first equation (i.e., we impose the conditions of the second equation on the first equation) and arrive at

$$\begin{aligned}
(0-20)^2+(y-40)^2 &= 3{,}600; \\
400+(y-40)^2 &= 3{,}600; \\
(y-40)^2 &= 3{,}200; \\
(y-40) &= \pm\sqrt{3{,}200} \\
y &= 40\pm\sqrt{3{,}200} \\
&= -16.57 \text{ or } 96.57.
\end{aligned}$$

Notice, we have two solutions. This means that the circle and y-axis intersect at the points $P=(0,96.57)$ and $Q=(0,-16.57)$. Similarly, to find the circular disc intersection with the x-axis, we have a system of two equations to work with:

$$\begin{aligned}
(x-20)^2+(y-40)^2 &= 3{,}600; \\
y &= 0.
\end{aligned}$$

Replace $y=0$ in the first equation (i.e., we impose the conditions of the second equation on the first equation) and arrive at

$$\begin{aligned}
(x-20)^2+(0-40)^2 &= 3{,}600; \\
(x-20)^2 &= 2{,}000; \\
(x-20) &= \pm\sqrt{2{,}000}; \\
x &= 20\pm\sqrt{2{,}000} \\
&= -24.72 \text{ or } 64.72.
\end{aligned}$$

Conclude the circle and x-axis intersect at the points $S=(64.72,0)$ and $R=(-24.72,0)$. □

The procedure we used in the solution of Example 3.3.1 gives us a general approach to finding the intersection points of circles with horizontal and vertical lines; this will be important in the exercises.

3.4 Summary

- Every horizontal line has equation of the form $y=c$.
- Every vertical line has equation of the form $x=c$.
- Every circle has equation of the form

  $$(x-h)^2+(y-k)^2=r^2$$

  where (h,k) is the center of the circle, and r is the circle's radius.

3.5 Exercises

Problem 3.1. This exercise emphasizes the "mechanical aspects" of circles and their equations.

(a) Find an equation whose graph is a circle of radius 3 centered at $(-3,4)$.

(b) Find an equation whose graph is a circle of diameter $\frac{1}{2}$ centered at the point $(3, -\frac{11}{3})$.

(c) Find four different equations whose graphs are circles of radius 2 through $(1,1)$.

(d) Consider the equation $(x-1)^2+(y+1)^2 = 4$. Which of the following points lie on the graph of this equation: $(1,1)$, $(1,-1)$, $(1,-3)$, $(1+\sqrt{3},0)$, $(0,-1-\sqrt{3})$, $(0,0)$.

Problem 3.2. Find the center and radius of each of the following circles.

(a) $x^2 - 6x + y^2 + 2y - 2 = 0$

(b) $x^2 + 4x + y^2 + 6y + 9 = 0$

(c) $x^2 + \frac{1}{3}x + y^2 - \frac{10}{3}y = \frac{127}{9}$

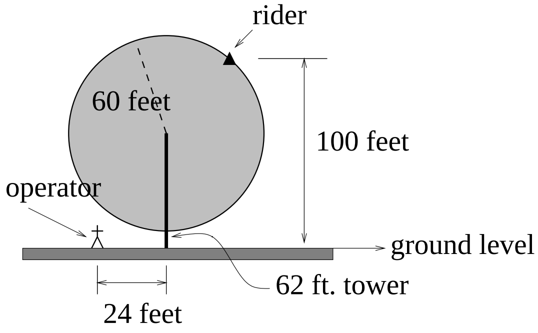
(d) $x^2 + y^2 = \frac{3}{2}x - y + \frac{35}{16}$

Problem 3.3. Water is flowing from a major broken water main at the intersection of two streets. The resulting puddle of water is circular and the radius r of the puddle is given by the equation $r = 5t$ feet, where t represents time in seconds elapsed since the the main broke.

(a) When the main broke, a runner was located 6 miles from the intersection. The runner continues toward the intersection at the constant speed of 17 feet per second. When will the runner's feet get wet?

(b) Suppose, instead, that when the main broke, the runner was 6 miles east, and 5000 feet north of the intersection. The runner runs due west at 17 feet per second. When will the runner's feet get wet?

Problem 3.4. An amusement park Ferris Wheel has a radius of 60 feet. The center of the wheel is mounted on a tower 62 feet above the ground (see picture). For these questions, the wheel is not turning.

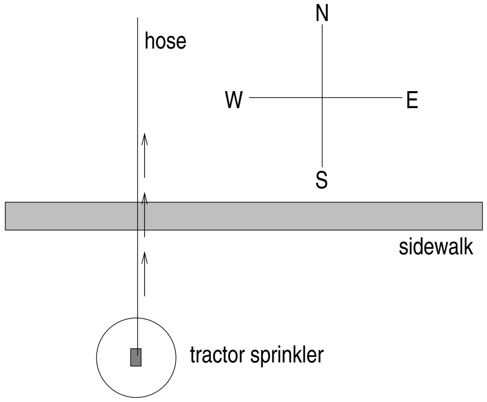


(a) Impose a coordinate system.

(b) Suppose a rider is located at the point in the picture, 100 feet above the ground. If the rider drops an ice cream cone straight down, where will it land on the ground?

(c) The ride operator is standing 24 feet to one side of the support tower on the level ground at the location in the picture. Determine the location(s) of a rider on the Ferris Wheel so that a dropped ice cream cone lands on the operator. (Note: There are two answers.)

Problem 3.5. A *crawling tractor sprinkler* is located as pictured below, 100 feet South of a sidewalk. Once the water is turned on, the sprinkler waters a circular disc of radius 20 feet and moves North along the hose at the rate of $\frac{1}{2}$ inch/second. The hose is perpendicular to the 10 ft. wide sidewalk. Assume there is grass on both sides of the sidewalk.



(a) Impose a coordinate system. Describe the initial coordinates of the sprinkler and find equations of the lines forming the North and South boundaries of the sidewalk.

(b) When will the water first strike the sidewalk?

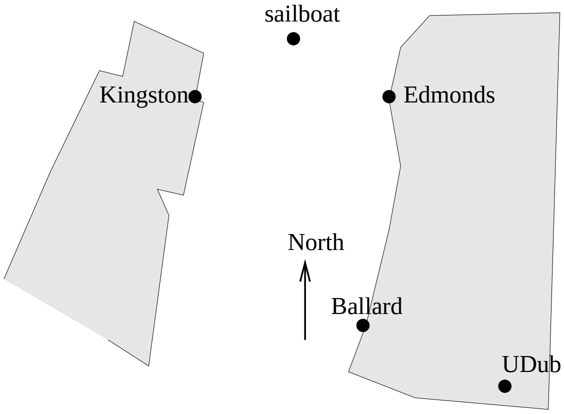
(c) When will the water from the sprinkler fall completely North of the sidewalk?

(d) Find the total amount of time water from the sprinkler falls on the sidewalk.

(e) Sketch a picture of the situation after 33 minutes. Draw an accurate picture of the watered portion of the sidewalk.

(f) Find the area of GRASS watered after one hour.

Problem 3.6. Erik's disabled sailboat is floating stationary 3 miles East and 2 miles North of Kingston. A ferry leaves Kingston heading toward Edmonds at 12 mph. Edmonds is 6 miles due east of Kingston. After 20 minutes the ferry turns heading due South. Ballard is 8 miles South and 1 mile West of Edmonds. Impose coordinates with Ballard as the origin.



(a) Find the equations for the lines along which the ferry is moving and draw in these lines.

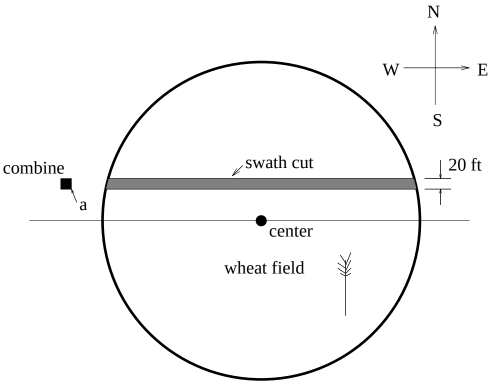
(b) The sailboat has a radar scope that will detect any object within 3 miles of the sailboat. Looking down from above, as in the picture, the radar region looks like a circular disk. The *boundary* is the "edge" or circle around this disc, the *interior* is the inside of the disk, and the *exterior* is everything outside of the disk (i.e. outside of the circle). Give a mathematical (equation) description of the boundary, interior and exterior of the radar zone. Sketch an accurate picture of the radar zone by determining where the line connecting Kingston and Edmonds would cross the radar zone.

(c) When does the ferry enter the radar zone?

(d) Where and when does the ferry exit the radar zone?

(e) How long does the ferry spend inside the radar zone?

Problem 3.7. Nora spends part of her summer driving a *combine* during the wheat harvest. Assume she starts at the indicated position heading east at 10 ft/sec toward a circular wheat field of radius 200 ft. The combine cuts a swath 20 feet wide and begins when the corner of the machine labeled "a" is 60 feet north and 60 feet west of the western-most edge of the field.



(a) When does Nora's rig first start cutting the wheat?

(b) When does Nora's rig first start cutting a swath 20 feet wide?

(c) Find the total amount of time wheat is being cut during this pass across the field.

(d) Estimate the area of the swath cut during this pass across the field.

Problem 3.8. (a) Solve for x:

$$\frac{x^2 - 2x + 1}{x + 5} = x - 2$$

(b) Solve for x:

$$\frac{x - 3}{x + 2} = 1$$

(c) If $x = -2$, find ALL solutions of the equation

$$(x + 1)^2 + (y - 1)^2 = 10$$

(d) If $y = 3$, find ALL solutions of the equation

$$5(x + 1)^2 + 2(y - 1)^2 = 10$$

Chapter 4

Linear Modeling

Sometimes, we will begin a section by looking at a specific problem which highlights the topic to be studied; this section offers the first such vista. View these problems as illustrations of precalculus in action, rather than confusing examples. Don't panic, the essential algebraic skills will be reviewed once the motivation is in place.

4.1 The Earning Power Problem

The government likes to gather all kinds of data. For example, Table 4.1 contains some data on the average annual income for full-time workers; these data were taken from the 1990 *Statistical Abstract of the U.S.* Given this information, a natural question would be: *How can we predict the future earning power of women and men?* One way to answer this question would be to use the data in the table to construct two different *mathematical models* that predict the future (or past) earning power for women or men. In order to do that, we would need to make some kind of initial assumption about the type of mathematical model expected. Let's begin by drawing two identical xy-coordinate systems, where the x-axis has units of "year" and the y-axis has units of "dollars;" see Figure 4.1. In each coordinate system, the data in our table gives us two points to plot: In the case of women, the data table gives us the points $P = (1970, 5{,}616)$ and $Q = (1987, 18{,}531)$. Likewise, for the men, the data table gives us the points $R = (1970, 9{,}521)$ and $S = (1987, 28{,}313)$.

(a) Women.

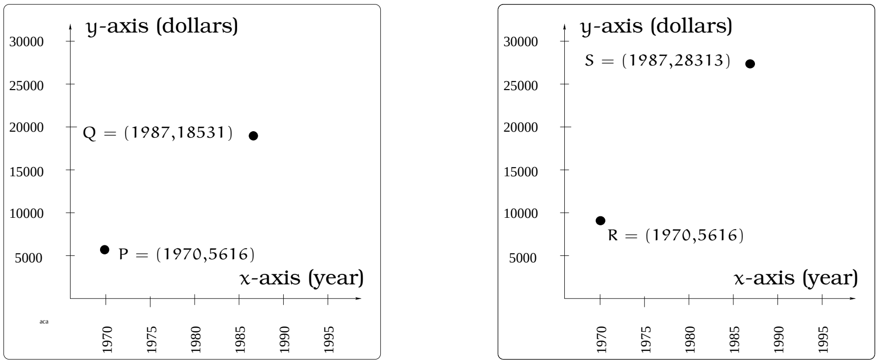
YEAR	WOMEN (dollars)
1970	$5,616
1987	$18,531

(b) Men.

YEAR	MEN (dollars)
1970	$9,521
1987	$28,313

Table 4.1: Earning power data.

To study the future earning power of men and women, we are going to make an assumption: *For women, if the earning power in year x is \$$y$,*



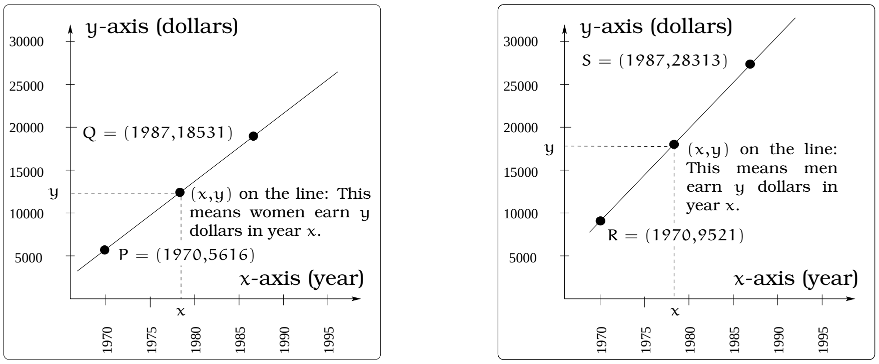
(a) Data points for women. (b) Data points for men.

Figure 4.1: Visualizing the data.

then the point (x, y) *lies on the line connecting* P *and* Q*. Likewise, for men, if the earning power in year* x *is* $\$y$*, then the point* (x, y) *lies on the line connecting* R *and* S*.*

In the real world, the validity of this kind of assumption would involve a lot of statistical analysis. This kind of assumption leads us to what is called a *linear model*, since we are demanding that the data points predicted by the model (i.e., the points (x, y)) lie on a straight line in a coordinate system. Now that we have made this assumption, our job is to find a way to mathematically describe when a point (x, y) lies on one of the two lines pictured in Figure 4.2.

Our goal in the next subsection is to review the mathematics necessary to show that the lines in Figure 4.2 are the so-called *graphs* of Equations 4.1 and 4.2.



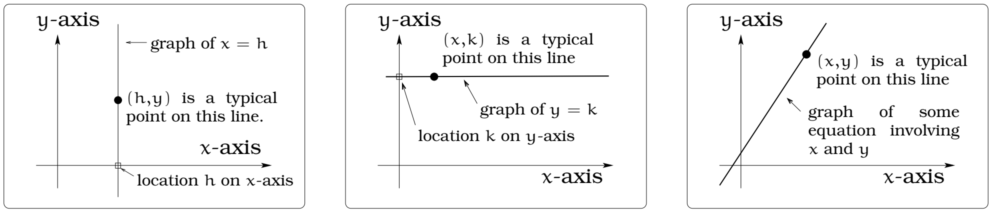
(a) Linear model for women. (b) Linear model for men.

Figure 4.2: Linear models of earning power.

$$
\begin{aligned}
y_{women} &= \left(\frac{18{,}531 - 5{,}616}{1987 - 1970}\right)(x - 1970) + 5{,}616 && (4.1)\\
&= \frac{12{,}915}{17}(x - 1970) + 5{,}616\\
y_{men} &= \left(\frac{28{,}313 - 9{,}521}{1987 - 1970}\right)(x - 1970) + 9{,}521 && (4.2)\\
&= \frac{18{,}792}{17}(x - 1970) + 9{,}521
\end{aligned}
$$

4.2 Relating Lines and Equations

A systematic approach to studying equations and their graphs would begin with the simple cases, gradually working toward the more complicated. Thinking visually, the simplest curves in the plane would be straight *lines*. As we discussed in Chapter 3, a point on the vertical line in Figure 4.3(a) will always have the same x-coordinate; we refer to this line as the graph of the equation $x = h$. Likewise, a point on the horizontal line in Figure 4.3(b) will always have the same y-coordinate; we refer to this line as the graph of the equation $y = k$. Figure 4.3(c) is different, in the sense that neither the x nor the y coordinate is constant; i.e., as you move a point along the line, both coordinates of the point are changing. It is reasonable to guess that this line is the graph of some equation involving both x and y. The question is: What is the equation?


y-axis
graph of x = h
(h,y) is a typical point on this line.
x-axis
location h on x-axis
y-axis
(x,k) is a typical point on this line
graph of y = k
location k on y-axis
x-axis
y-axis
(x,y) is a typical point on this line
graph of some equation involving x and y
x-axis

(a) Vertical line. (b) Horizontal line. (c) Sloped line.

Figure 4.3: Lines in a plane.

Here is the key geometric fact needed to model lines by mathematical equations:

Important Fact 4.2.1. *Two different points completely determine a straight line.*

This fact tells us that if you are given two different points on a line, you can reconstruct the line in a coordinate system by simply lining a ruler up with the two points. In our discussion, we will need to pay special attention to the difference between vertical and non-vertical lines.

4.3 Non-vertical Lines

Assume in this section that ℓ is a non-vertical line in the plane; for example, the line in Figure 4.3(c). If we are given two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ on a line ℓ, then Equation (2.2) on page 18 defined two quantities we can calculate:

$$\Delta x = \text{change in } x \text{ going from } P \text{ to } Q = x_2 - x_1.$$

$$\Delta y = \text{change in } y \text{ going from } P \text{ to } Q = y_2 - y_1.$$

We define the *slope* of the line ℓ to be the ratio of Δy by Δx, which is usually denoted by m:

$$\begin{aligned} m &\overset{\text{def}}{=} \frac{\Delta y}{\Delta x} \qquad (4.3) \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ \text{slope of } \ell &= \frac{\text{change in } y}{\text{change in } x} \end{aligned}$$

Notice, we are using the fact that the line is non-vertical to know that this ratio is always defined; i.e., we will never have $\Delta x = 0$ (which would lead to illegal division by zero). There is some additional terminology that goes along with the definition of the slope. The term Δy is sometimes called the *rise of* ℓ and Δx is called the *run of* ℓ. For this reason, people often refer to the slope of a line ℓ as "the rise over the run," meaning

$$\text{slope of } \ell \overset{\text{def}}{=} \frac{\text{rise of } \ell}{\text{run of } \ell} = \frac{\Delta y}{\Delta x} = m.$$

In addition, notice that the calculation of Δy involves taking the difference of two numbers; likewise, the calculation of Δx involves taking the difference of two numbers. For this reason, the slope of a line ℓ is sometimes called a *difference quotient*.

For example, suppose $P = (1,1)$ and $Q = (4,5)$ lie on a line ℓ. In this case, the rise $= \Delta y = 4$ and the run $= \Delta x = 3$, so $m = \frac{4}{3}$ is the slope of ℓ.

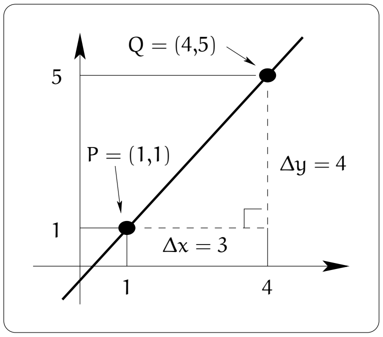


Figure 4.4: Computing the slope of a line.

When computing Δx, pay special attention that it is the x-coordinate of the destination point Q minus the x-coordinate of the starting point P; likewise, when computing Δy, it is the y-coordinate of the destination point Q minus the y-coordinate of the starting point P. We can reverse this order in **both** calculations and get the same slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_2 - y_1)}{-(x_2 - x_1)} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-\Delta y}{-\Delta x}.$$

We CANNOT reverse the order in just one of the calculations and get the same slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \neq \frac{y_2 - y_1}{x_1 - x_2}, \qquad \text{and} \qquad m = \frac{y_2 - y_1}{x_2 - x_1} \neq \frac{y_1 - y_2}{x_2 - x_1}.$$

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CAUTION
!!!

It is very important to notice that the calculation of the slope of a line does not depend on the choice of the two points P and Q. This is a real windfall, since we are then always at liberty to pick our favorite two points on the line to determine the slope. The reason for this freedom of choice is pretty easy to see by looking at a picture. If we were to choose two other points $P^* = (x_1^*, y_1^*)$ and $Q^* = (x_2^*, y_2^*)$ on ℓ, then we would get two similar right triangles: See Figure 4.5.

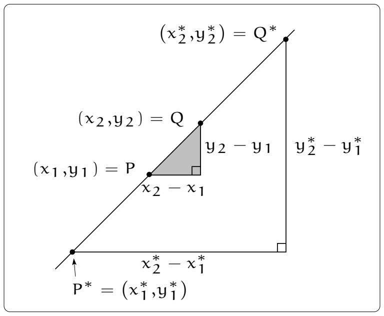


Figure 4.5: Using similar triangles.

Basic properties of similar triangles tell us ratios of lengths of common sides are equal, so that

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2^* - y_1^*}{x_2^* - x_1^*};$$

but this just says the calculation of the slope is the same for any pair of distinct points on ℓ. For example, lets redo the slope calculation when $P^* = P = (x_1, y_1)$ and $Q^* = (x, y)$ represents an arbitrary point on the line. Then the two ratios of lengths of common sides give us the equation

$$\begin{aligned} m &= \frac{y - y_1}{x - x_1}, \\ y - y_1 &= m(x - x_1). \end{aligned}$$

This can be rewritten as

$$
\begin{aligned}
y &= m(x - x_1) + y_1 \qquad (4.4)\\
&\text{or}\\
y &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1. \qquad (4.5)
\end{aligned}
$$

Equation 4.4 is usually called the *point slope formula* for the line ℓ (since the data required to write the equation amounts to a point (x_1,y_1) on the line and the slope m), whereas Equation 4.5 is called the *two point formula* for the line ℓ (since the data required amounts to the coordinates of the points P and Q). In any event, we now see that

Important Fact 4.3.1. (x,y) *lies on* ℓ *if and only if* (x,y) *is a solution to* $y = m(x - x_1) + y_1$.

We can plot the collection of ALL solutions to the equation in Fact 4.3.1, which we refer to as the *graph* of the equation. As a subset of the xy-coordinate system, the line

Important Fact 4.3.2. $\ell = \{(x, m(x - x_1) + y_1) \mid x$ *is any real number*$\}$.

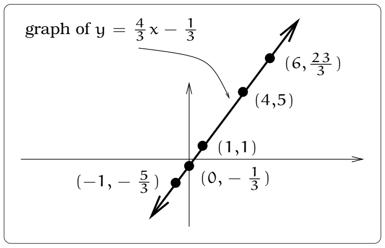


Figure 4.6: Verifying points on a line.

Example 4.3.3. *Consider the line* ℓ*, in Figure 4.6, through the two points* $P = (1,1)$ *and* $Q = (4,5)$*. Then the slope of* ℓ *is* $m = 4/3$ *and* ℓ *consists of all pairs of points* (x,y) *such that the coordinates* x *and* y *satisfy the equation* $y = \frac{4}{3}(x - 1) + 1$*. Letting* $x = 0, 1, 6$ *and* -1*, we conclude that the following four points lie on the line* ℓ*:* $(0,\frac{4}{3}(0 - 1) + 1) = (0,\frac{-1}{3})$, $(1,\frac{4}{3}(1 - 1) + 1) = (1,1)$, $(6,\frac{4}{3}(6 - 1) + 1) = (6,\frac{23}{3})$ *and* $(-1,\frac{4}{3}(-1 - 1) + 1) = (-1,\frac{-5}{3})$*. By the same reasoning, the point* $(0,0)$ *does not lie on the line* ℓ*. As a set of points in the plane, we have*

$$\ell = \left\{\left(x, \frac{4}{3}(x - 1) + 1\right) \mid x \textit{ is any real number}\right\}$$

Returning to the general situation, we can obtain a third general equation for a non-vertical line. To emphasize what is going on here, plug the specific value $x = 0$ into Equation 4.4 and obtain the point $R = (0,b)$ on the line, where $b = m(0 - x_1) + y_1 = -mx_1 + y_1$. But, Equation (4.4) can be written

$$
\begin{aligned}
y &= m(x - x_1) + y_1\\
&= mx - mx_1 + y_1\\
&= mx + b.
\end{aligned}
$$

The point R is important; it is precisely the point where the line ℓ crosses the y-axis, usually called the y-intercept. The *slope intercept equation of the line* is the form

$$y = mx + b,$$

where the slope of the line is m and b is the y-intercept of the line.

Summary 4.3.4. ***Non-vertical Lines:*** *Let ℓ be a non-vertical line in the xy-plane. There are three ways to obtain an equation whose graph is ℓ, depending on the data provided for ℓ:*

1. *If $P = (x_1,y_1), Q = (x_2,y_2)$ are two different points on the line, then the* ***two-point formula*** $y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1$ *gives an equation whose graph is ℓ.*
2. *If $P = (x_1,y_1)$ is a point on the line and m is the slope of ℓ, then the* ***point-slope formula*** $y = m(x - x_1) + y_1$ *gives an equation whose graph is ℓ.*
3. *If the line ℓ intersects the y-axis at the point $(0,b)$ and m is the slope of the line ℓ, then the* ***slope-intercept formula*** $y = mx + b$ *gives an equation whose graph is ℓ.*

4.4 General Lines

Summarizing, any line in the plane is the graph of an equation involving x and y and the equation always has the form

$$Ax + By + C = 0,$$

for some constants A, B, C. Equations like this are called *linear equations*. In general, non-vertical lines will be of the most interest to us, since these are the lines that can be viewed as the graphs of *functions*; we will discuss this in Chapter 5.

4.5 Lines and Rate of Change

If we draw a non-vertical line in the xy coordinate system, then its slope will be the *rate of change* of y with respect to x:

$$\begin{aligned} \text{slope} &= \frac{\Delta y}{\Delta x} \\ &= \frac{\text{change in } y}{\text{change in } x} \\ &\overset{\text{def}}{=} \text{rate of change of } y \text{ with respect to } x. \end{aligned}$$

We should emphasize that this rate of change is a constant; in other words, this rate is the same no matter where we compute the slope on the line. The point-slope formula for a line can now be interpreted as follows:

A line is determined by a point on the line and the rate of change of y *with respect to* x.

An interesting thing to notice is how the units for x and y figure into the rate of change calculation. For example, suppose that we have the equation $y = 10{,}000\,x + 200{,}000$, which relates the value y of a house (in dollars) to the number of years x you own it. For example, after 5 years, $x = 5$ and the value of the house would be $y = 10{,}000\,(5) + 200{,}000 = \$250{,}000$. In this case, the equation $y = 10{,}000\,x + 200{,}000$ is linear and already written in slope-intercept form, so the slope can be read as $m = 10{,}000$. If we carry along the units in the calculation of $\dfrac{\Delta y}{\Delta x}$, then the numerator involves "dollar" units and the denominator "years" units. That means that carrying along units, the slope is actually $m = 10{,}000$ dollars/year. In other words, the value of the house is changing at a rate of $10{,}000$ dollars/year.

At the other extreme, if the units for both x and y are the same, then the units cancel out in the rate of change calculation; in other words, the slope is a unit-less quantity, simply a number. This sort of thing will come up in the mathematics you see in chemistry and physics.

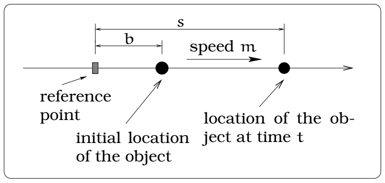

s
b
speed m
reference point
initial location of the object
location of the object at time t

Figure 4.7: Motion along a line.

One important type of rate encountered is the *speed* of a moving object. Suppose an object moves along a straight line at a constant speed m: See Figure 4.7.

If we specify a reference point, we can let b be the starting location of the moving object, which is usually called the *initial location* of the object. We can write down an equation relating the initial location b, the time t, the constant speed m and the location s at time t:

$$\begin{aligned} s &= \text{(location of object at time } t) \\ s &= \text{(initial location of object) + (distance object travels in time } t) \\ s &= b + mt. \end{aligned}$$

where t is in the same time units used to define the rate m. Notice, both b and m would be constants given to us, so this is a linear equation involving the variables s and t. We can graph the equation in the ts-coordinate system: See Figure 4.8.

It is important to distinguish between this picture (the graph of $s = mt + b$) and the path of the object in Figure 4.7. The graph of the equation should be thought of as a visual aid attached to the equation $s = mt + b$. The general idea is that using this visual aid can help answer various questions involving the equation, which in turn will tell us things about the motion of the object in Figure 4.7.

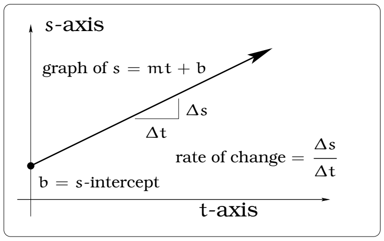
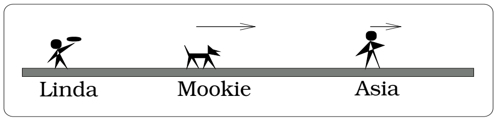


Figure 4.8: The graph of $s = mt + b$.

Two other comments related to this discussion are important. First, concerning notation, the speed m is often symbolized by v to denote constant *velocity* and b is written as $s_\circ$ (the subscript "0" meaning "time zero"). With these changes, the equation becomes $s = s_\circ + vt$, which is the form in which it would be written in a typical physics text. As a second note, if you return to Figure 4.8, you will notice we only drew in the positive t axis. This was because t represented time, which is always a non-negative quantity.

Example 4.5.1. *Linda, Asia and Mookie are all playing frisbee. Mookie is 10 meters in front of Linda and always runs 5 m/sec. Asia is 34 meters in front of Linda and always runs 4 m/sec. Linda yells "go!" and both Mookie and Asia start running directly away from Linda to catch a tossed frisbee. Find linear equations for the distances between Linda, Mookie and Asia after* t *seconds.*



Solution. Let s_M be the distance between Linda and Mookie and s_A the distance between Linda and Asia, after t seconds. An application of the above formula tells us

$$\begin{aligned} s_M &= \text{(initial distance between Linda and Mookie)} + \cdots \\ &\quad \cdots + \text{(distance Mookie runs in } t \text{ seconds)} \\ s_M &= 10 + 5t. \end{aligned}$$

Likewise,

$$\begin{aligned} s_A &= \text{(initial distance between Linda and Asia)} + \cdots \\ &\quad \cdots + \text{(distance Asia runs in } t \text{ seconds)} \\ s_A &= 34 + 4t. \end{aligned}$$

If s_{MA} is the distance between Mookie and Asia after t seconds, we compute

$$s_{MA} = s_A - s_M = (34 + 4t) - (10 + 5t) = (24 - t) \text{ meters.}$$

In all cases, the distances we computed are given by linear equations of the form $s = b + mt$, for appropriate b and rate m. $\square$

4.6 Back to the Earning Power Problem

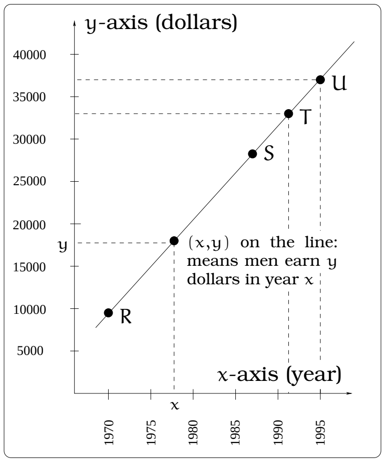


Figure 4.9: Linear model of Men's Earning Power.

We now return to the motivating problem at the start of this section. Recall the plot in Figure 4.1(b). We can model the men's earning power using the first and last data points, using the ideas we have discussed about linear equations. To do this, we should specify a "beginning point" and an "ending point" (recall Figure 2.9) and calculate the slope:

$$R_{begin} = (1970{,}9{,}521)$$

and

$$S_{end} = (1987{,}28{,}313).$$

We find that

Δy	Δx	m
$28{,}313 - 9{,}521$	$1987 - 1970$	$\frac{\Delta y}{\Delta x}$
$18{,}792$	17	$\frac{18{,}792}{17}$

If we apply the "point-slope formula" for the equation of a line, we arrive at the equation:

$$y = \frac{18{,}792}{17}(x - 1970) + 9{,}521. \tag{4.6}$$

The graph of this line will pass through the two points R and S in Figure 4.2. We sketch the graph in Figure 4.9, indicating two new points T and U.

We can use the model in (4.6) to make predictions of two different sorts: (i) predict earnings at some date, or (ii) predict when a desired value for earnings will occur. For example, let's graphically discuss the earnings in 1995:

- Draw a vertical line $x = 1995$ up to the graph and label the intersection point U.
- Draw a horizontal line ℓ through U. The line ℓ crosses the y axis at the point

  $$\frac{18{,}792}{17}(\underline{1995} - 1970) + 9521 = 37{,}156.$$

- The coordinates of the point $U = (1995, 37{,}156)$.

Conclude that \$37,156 is the Men's Earning Power in 1995. For another example, suppose we wanted to know when men's earning power will equal \$33,000? This means we seek a data point T on the men's earning curve whose y-coordinate is 33,000. By (4.6), T has the form

$$T = \left(x, \frac{18{,}792}{17}(x - 1970) + 9{,}521\right).$$

We want this to be a data point of the form $(x, 33{,}000)$. Setting these two points equal and equating the second coordinates leads to an algebra problem:

$$\begin{aligned} \frac{18{,}792}{17}(x - 1970) + 9{,}521 &= 33{,}000 \\ x &= 1991.24. \end{aligned}$$

This means men's earning power will be \$33,000 at the end of the first quarter of 1991. Graphically, we interpret this reasoning as follows:

- Draw a horizontal line $y = 33{,}000$ and label the intersection point T on the model.
- Draw a vertical line ℓ through T. The line ℓ crosses the x axis at the point 1991.24.
- The coordinates of the point $T = (1991.24, 33{,}000)$.

In the exercises, you will be asked to show that the women's earning power model is given by the equation

$$y = \frac{12{,}915}{17}(x - 1970) + 5{,}616.$$

Using the two linear models for the earning power of men and women, are women gaining on men? You will also be asked to think about this question in the exercises.

4.7 What's Needed to Build a Linear Model?

As we progress through this text, a number of different "types" of mathematical models will be discussed. We will want to think about the information needed to construct that particular kind of mathematical model. Why would we care? For example, in a laboratory context, if we knew a situation being studied was given by a linear model, this would effect the amount of data collected. In the case of linear models, we can now make this useful statement:

Important Fact 4.7.1. *A linear model is completely determined by:*

1. *One data point and a slope (a rate of change), or*
2. *Two data points, or*
3. *An intercept and a slope (a rate of change).*

4.8 Linear Application Problems

Example 4.8.1. *The yearly resident tuition at the University of Washington was* \$1827 *in* 1989 *and* \$2907 *in* 1995. *Assume that the tuition growth at the UW follows a linear model. What will be the tuition in the year 2000? When will yearly tuition at the University of Washington be \$10,000?*

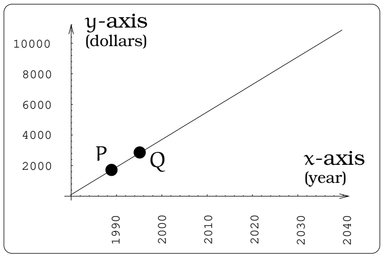


Figure 4.10: Linear tuition model.

Solution. If we consider a coordinate system where the x-axis represents the year and the y-axis represents dollars, we are given two data points: $P = (1989, 1{,}827)$ and $Q = (1995, 2{,}907)$. Using the two-point formula for the equation of line through P and Q, we obtain the equation

$$y = 180(x - 1989) + 1{,}827.$$

The graph of this equation gives a line through the given points as pictured in Figure 4.10.

If we let $x = 2000$, we get $y = \$3{,}807$, which tells us the tuition in the year 2000. On the other hand, if we set the equation equal to \$10,000, we can solve for x:

$$\begin{aligned} 10{,}000 &= 180(x - 1989) + 1{,}827 \\ 8{,}173 &= 180(x - 1989) \\ 2{,}034.4 &= x. \end{aligned}$$

Conclude the tuition is \$10,000 in the year 2035. □

4.9 Perpendicular and Parallel Lines

Here is a useful fact to keep in mind.

Important Facts 4.9.1. *Two non-vertical lines in the plane are parallel precisely when they both have the same slope. Two non-vertical lines are perpendicular precisely when their slopes are negative reciprocals of one another.*

Example 4.9.2. *Let ℓ be a line in the plane passing through the points $(1,1)$ and $(6,-1)$. Find a linear equation whose graph is a line parallel to ℓ passing through 5 on the y-axis. Find a linear equation whose graph is perpendicular to ℓ and passes through $(4,6)$.*

Solution. Letting $P = (1,1)$ and $Q = (6,-1)$, apply the "two point formula":

$$\begin{aligned} y &= \frac{-2}{5}(x-1)+1 \\ &= -\frac{2}{5}x + \frac{7}{5}. \end{aligned}$$

The graph of this equation will be ℓ. This equation is in slope intercept form and we can read off that the slope is $m = \frac{-2}{5}$. The desired line a is parallel to ℓ; it must have slope $m = -\frac{2}{5}$ and y-intercept 5. Plugging into the "slope intercept form":

$$y = \frac{-2}{5}x + 5.$$

The desired line b is a line perpendicular to ℓ (so its slope is $m' = \frac{-1}{\frac{-2}{5}} = \frac{5}{2}$) and passes through the point $(4,6)$, so we can use the "point slope formula":

$$y = \frac{5}{2}(x-4)+6.$$

□

4.10 Intersecting Curves II

We have already encountered problems that require us to investigate the intersection of two curves in the plane. Ultimately, this reduces to solving a system of two (or more) equations in the variables x and y. A useful tool when working with equations involving squared terms (i.e., x^2 or y^2), is the *quadratic formula*.

Important Fact 4.10.1. ***Quadratic Formula:*** *Consider the equation $az^2 + bz + c = 0$, where a,b,c are constants. The solutions for this equation are given by the formula*

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The solutions are real numbers if and only if $b^2 - 4ac \geq 0$.

The next example illustrates a typical application of the quadratic formula. In addition, we describe a very useful technique for finding the shortest distance between a "line" and a "point."

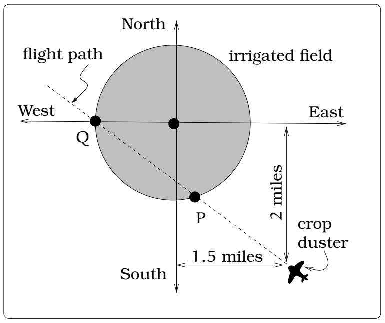


Figure 4.11: The flight path of a crop duster.

Example 4.10.2. *A crop dusting airplane flying a constant speed of* 120 *mph is spotted* 2 *miles South and* 1.5 *miles East of the center of a circular irrigated field. The irrigated field has a radius of* 1 *mile. Impose a coordinate system as pictured, with the center of the field the origin* $(0,0)$*. The flight path of the duster is a straight line passing over the labeled points* P *and* Q*. Assume that the point* Q *where the plane exits the airspace above the field is the Western-most location of the field. Answer these questions:*

1. *Find a linear equation whose graph is the line along which the crop duster travels.*
2. *Find the location* P *where the crop duster enters airspace above the irrigated field.*
3. *How much time does the duster spend flying over the irrigated field?*
4. *Find the shortest distance from the flight path to the center of the irrigated field.*

Solution.

1. Take $Q = (-1, 0)$ and $S = (1.5, -2)$ = duster spotting point. Construct a line through Q and S. The slope is $-0.8 = m$ and the line equation becomes:

   $$y = -0.8x - 0.8. \tag{4.7}$$

2. The equation of the boundary of the irrigated region is $x^2 + y^2 = 1$. We need to solve this equation AND the line equation $y = -0.8x - 0.8$ simultaneously. Plugging the line equation into the unit circle equation gives:

   $$\begin{aligned} x^2 + (-0.8x - 0.8)^2 &= 1 \\ x^2 + 0.64x^2 + 1.28x + 0.64 &= 1 \\ 1.64x^2 + 1.28x - 0.36 &= 0 \end{aligned}$$

   Apply the quadratic formula and find $x = -1, 0.2195$. Conclude that the x coordinate of P is 0.2195. To find the y coordinate, plug into the line equation and get $y = -0.9756$. Conclude that $P = (0.2195, -0.9756)$

3. Find the distance from P to Q by using the distance formula:

$$\begin{aligned} d &= \sqrt{(-1-0.2195)^2+(0-(-0.9756))^2} \\ &= 1.562\,\text{miles} \end{aligned}$$

Now, $\frac{1.562\,\text{miles}}{120\,\text{mph}} = 0.01302\,\text{hours} = 47\,\text{seconds}$.

4. The idea is to construct a line perpendicular to the flight path passing through the origin of the coordinate system. This line will have slope $m = -\frac{1}{-0.8} = 1.25$. So this perpendicular line has equation $y = 1.25x$. Intersecting this line with the flight path gives us the point closest to the center of the field. The x-coordinate of this point is found by setting the two line equations equal and solving:

$$\begin{aligned} -0.8x-0.8 &= 1.25x \\ x &= -0.3902 \end{aligned}$$

This means that the closest point on the flight path is $(-0.39, -0.49)$. Apply the distance formula and the shortest distance to the flight path is

$$\begin{aligned} d &= \sqrt{(-0.39)^2+(-0.49)^2} \\ &= 0.6263. \end{aligned}$$

□

4.11 Uniform Linear Motion

When an object moves along a line in the xy-plane at a constant speed, we say that the object exhibits **uniform linear motion**. Its location can be described using a pair of linear equations involving a variable which represents time. That is, we can find constants a, b, c, and d such that, at any time t, the object's location is given by (x,y), where

$$x = a + bt \text{ and } y = c + dt.$$

Such equations are called **parametric equations of motion**. The motion is defined in terms of the parameter t.

Since there are four constants to be determined, one needs four pieces of information to determine these equations. Knowing the object's location at two points in time is sufficient.

Example 4.11.1. *Bob is running in the xy-plane. He runs in a straight line from the point $(2,3)$ to the point $(5,-4)$, taking 6 seconds to do so. Find his equations of motion.*

Solution. We begin by setting a reference for our time parameter. Let's let $t = 0$ represent the instant when Bob is at the point $(2,3)$. In this way, t will represent the time since Bob left the point $(2,3)$. When $t = 6$, we know he will be at the point $(5,-4)$. This is enough information to determine his equations of motion.

We seek constants a, b, c, and d so that at time t, Bob's location is given by

$$x = a + bt \text{ and } y = c + dt.$$

When $t = 0$, we know Bob's location is $(2,3)$. That is, $x = 2$ and $y = 3$. Thus, with $t = 0$, we have the two equations

$$x = 2 = a + b(0) = a \text{ and } y = 3 = c + d(0) = c.$$

and so $a = 2$ and $c = 3$. We're half-way done.

When $t = 6$, we know Bob's location is $(5,-4)$. Thus, with $t = 6$, we have the two equations

$$x = 5 = a + b(6) = 2 + 6b \text{ and } y = -4 = c + d(6) = 3 + 6d$$

which we can solve to find

$$b = \frac{1}{2} \text{ and } d = -\frac{7}{6}.$$

So, we arrive at the equations of Bob's motion

$$x = 2 + \frac{1}{2}t \text{ and } y = 3 - \frac{7}{6}t.$$

Notice it is easy to check that these are correct. If we plug in $t = 0$, we find $x = 2, y = 3$ as required. If we plug in $t = 6$, we find $x = 5, y = -4$, as required. So we know we've done it right.

Now that we have these equations of motion, it is very easy to calculate Bob's location at any time. For instance, 30 seconds after leaving the point $(2,3)$, we can find that he is at the point $(17,-32)$ since

$$x = 2 + \frac{1}{2}(30) = 17, y = 3 - \frac{7}{6}(30) = -32.$$

□

Example 4.11.2. *Olga is running in the xy-plane, and the coordinate are given in meters (so, for example, the point $(1,0)$ is one meter from the origin $(0,0)$). She runs in a straight line, starting at the point $(3,5)$ and running along the line $y = -\frac{1}{3}x + 6$ at a speed of 7 meters per second, heading away from the y-axis. What are her parametric equations of motion?*

Solution. This example differs in some respects from the last example. In particular, instead of knowing where the runner is at two points in time, we only know one point, and have other information given to us about the speed and path of the runner. One approach is to use this new information to find where the runner is at some other point in time: this will then give us exactly the same sort of information as we used in the last example, and so we may solve it in an identical manner.

We know that Olga starts at the point $(3{,}5)$. Letting $t = 0$ represent the time when she starts, we then know that when $t = 0$, $x = 3$ and $y = 5$.

To get another point (and time), we can use the fact that we know what line she travels along, and which direction she runs. We may consider any point on the line in the correct direction: any will do. For instance, the point $(6{,}4)$ is on the line. We then need to find when Olga reaches this point. To do this, we find the distance from her starting point to the point $(6{,}4)$, and divide this by her speed. The time she takes to get to $(6{,}4)$ is thus

$$\frac{\sqrt{(6-3)^2+(4-5)^2}}{7} = 0.45175395 \text{ seconds .}$$

At this point, we are now in the same situation as in the last example. We know two facts: when $t = 0$, $x = 3$ and $y = 5$, and when $t = 0.45175395$, $x = 6$ and $y = 4$. As we saw in the last example, this is enough information to find the parametric equations of motion.

We seek a, b, c, and d such that Olga's location t seconds after she starts is $(x{,}y)$ where

$$x = a + bt \text{ and } y = c + dt.$$

When $t = 0$, $x = 3$, and $y = 5$, so

$$x = 3 = a + b(0) = a \text{ and } y = 5 = c + d(0) = c$$

and so $a = 3$ and $c = 5$. Also, when $t = 0.45175395$, $x = 6$ and $y = 4$, so

$$x = 6 = a+b(0.45175395) = 3+b(0.45175395) \text{ and } y = 4 = c+d(0.45175395) = 5+d(0.45175395).$$

Solving these equations for b and d, we find

$$b = \frac{3}{0.45175395} = 6.64078311 \text{ and } d = \frac{-1}{0.45175395} = -2.21359436.$$

Thus, Olga's equations of motion are

$$x = 3 + 6.64078311t, y = 5 - 2.21359436t.$$

□

4.12 Summary

- The equation of every non-vertical line can be expressed in the form

  $y = m(x - h) + k$ (the **point-slope** form)

  and

  $y = mx + b$ (the **slope-intercept** form)

- A vertical line has an equation of the form $x = c$.
- The shortest distance between a point, P, and a line, l, can be found by determining a line l_2 which passes through P and is perpendicular to l. Then the point at which l and l_2 intersect is the point on l which is closest to l. The distance from this point to P is the shortest distance between P and l.
- The location of an object moving at constant speed along a line can be described using a pair of equations (parametric equations)

  $x = a + bt,\ y = c + dt.$

4.13 Exercises

Problem 4.1. This exercise emphasizes the "mechanical aspects" of working with linear equations. Find the equation of a line:

(a) Passing through the points $(1,-1)$ and $(-2,4)$.

(b) Passing through the point $(-1,-2)$ with slope $m = 40$.

(c) With y-intercept $b = -2$ and slope $m = -2$.

(d) Passing through the point $(4,11)$ and having slope $m = 0$.

(e) Perpendicular to the line in (a) and passing through $(1,1)$.

(f) Parallel to the line in (b) and having y-intercept $b = -14$.

(g) Having the equation $3x + 4y = 7$.

(h) Crossing the x-axis at $x = 1$ and having slope $m = 1$.

Problem 4.2. Sketch an accurate picture of the line having equation $y = 2 - \frac{1}{2}x$. Let α be an unknown constant.

(a) Find the point of intersection between the line you have graphed and the line $y = 1 + \alpha x$; your answer will be a point in the xy plane whose coordinates involve the unknown α.

(b) Find α so that the intersection point in (a) has x-coordinate 10.

(c) Find α so that the intersection point in (a) lies on the x-axis.

Problem 4.3. (a) What is the area of the triangle determined by the lines $y = -\frac{1}{2}x + 5$, $y = 6x$ and the y-axis?

(b) If $b > 0$ and $m < 0$, then the line $y = mx + b$ cuts off a triangle from the first quadrant. Express the area of that triangle in terms of m and b.

(c) The lines $y = mx + 5$, $y = x$ and the y-axis form a triangle in the first quadrant. Suppose this triangle has an area of 10 square units. Find m.

Problem 4.4. Complete Table 4.2 on page 52. In many cases there may be several possible correct answers.

Problem 4.5. The (average) sale price for single family property in Seattle and Port Townsend is tabulated below:

YEAR	SEATTLE	PORT TOWNSEND
1970	\$38,000	\$8400
1990	\$175,000	\$168,400

(a) Find a linear model relating the year x and the sales price y for a single family property in Seattle.

(b) Find a linear model relating the year x and the sales price y for a single family property in Port Townsend.

(c) Sketch the graph of both modeling equations in a common coordinate system; restrict your attention to $x \geq 1970$.

(d) What is the sales price in Seattle and Port Townsend in 1983 and 1998?

(e) When will the average sales price in Seattle and Port Townsend be equal and what is this price?

(f) When will the average sales price in Port Townsend be \$15,000 less than the Seattle sales price? What are the two sales prices at this time?

(g) When will the Port Townsend sales price be \$15,000 more than the Seattle sales price? What are the two sales prices at this time?

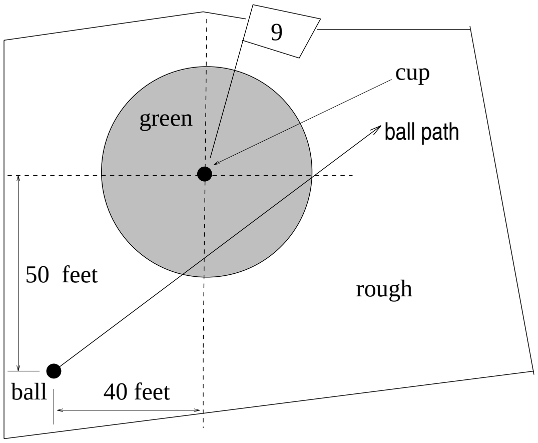
(h) When will the Seattle sales price be double the Port Townsend sales price?

(i) Is the Port Townsend sales price ever double the Seattle sales price?

Problem 4.6. The cup on the 9th hole of a golf course is located dead center in the middle of a circular green that is 70 feet in diameter. Your ball is located as in the picture below:

Equation	**Slope**	**y-intercept**	**Point on the line**	**Point on the line**
$y = 2x + 1$				
			$(3, -4)$	$(-1, 7)$
	-2	1		
	$\frac{1}{2}$		$(0, 1)$	
		$1{,}000$		
	0			
			$(3, 3)$	$(3, -2)$
			$(5, -9)$	

Table 4.2: Linear equations table for Problem 4.4.



The ball follows a straight line path and exits the green at the right-most edge. Assume the ball travels a constant rate of 10 ft/sec.

(a) Where does the ball enter the green?

(b) When does the ball enter the green?

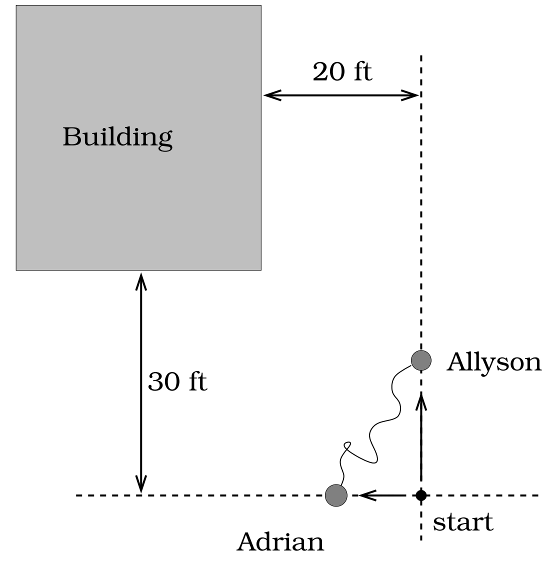
(c) How long does the ball spend inside the green?

(d) Where is the ball located when it is closest to the cup and when does this occur.

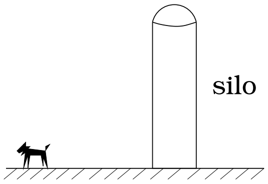
Problem 4.7. Allyson and Adrian have decided to connect their ankles with a *bungee cord*; one end is tied to each person's ankle. The cord is 30 feet long, but can stretch up to 90 feet. They both start from the same location. Allyson moves 10 ft/sec and Adrian moves 8 ft/sec in the directions indicated. Adrian stops moving at time $t = 5.5$ sec, but Allyson keeps on moving 10 ft/sec in the indicated direction.

(a) Sketch an accurate picture of the situation at time $t = 7$ seconds. Make sure to label the locations of Allyson and Adrian; also, compute the length of the bungee cord at $t = 7$ seconds.

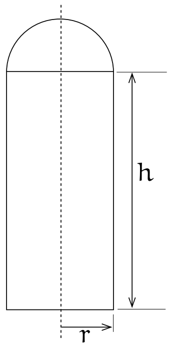
(b) Where is Allyson when the bungee reaches its maximum length?



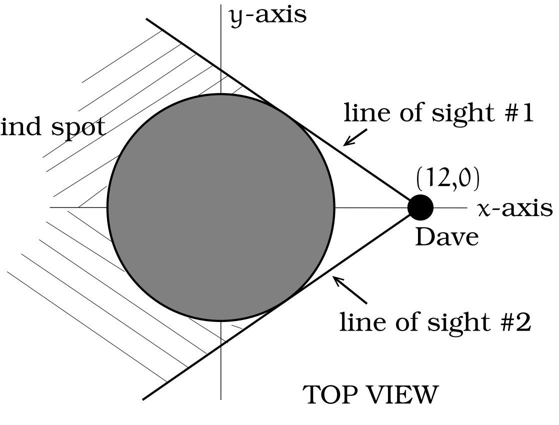
Problem 4.8. Dave is going to leave academia and go into business building *grain silos*. A grain silo is a cylinder with a hemispherical top, used to store grain for farm animals. Here is a 3D view, a cross-section, and the top view:



3D-view



cross-section



If Dave is standing next to a silo of cross-sectional radius $r = 8$ feet at the indicated position, his vision will be partially obstructed. Find the portion of the y-axis that Dave cannot see. (Hint: Let a be the x-coordinate of the point where line of sight #1 is tangent to the silo; compute the slope of the line using two points (the tangent point and (12,0)). On the other hand, compute the slope of line of sight #1 by noting it is perpendicular to a radial line through the tangency point. Set these two calculations of the slope equal and solve for a.)

Problem 4.9. While speaking on the phone to a friend in Oslo, Norway, you learned that the current temperature there was −23° Celsius (−23°C). After the phone conversation, you wanted to convert this temperature to Fahrenheit degrees °F, but you could not find a reference with the correct formulas. You then remembered that the relationship between °F and °C is linear.

(a) Using this and the knowledge that $32°F = 0°C$ and $212°F = 100°C$, find an equation that computes Celsius temperature in terms of Fahrenheit temperature; i.e., an equation of the form C= "an expression involving only the variable F."

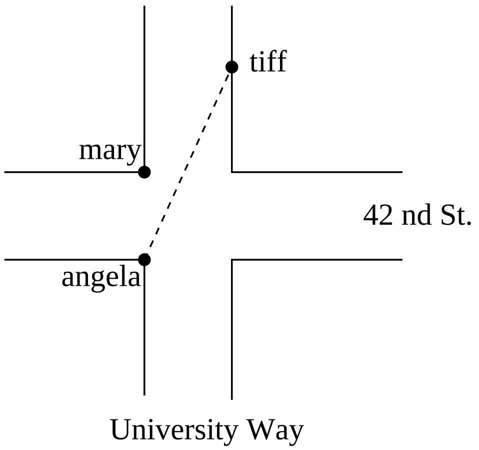
(b) Likewise, find an equation that computes Fahrenheit temperature in terms of Celsius temperature; i.e. an equation of the form F= "an expression involving only the variable C."

(c) How cold was it in Oslo in °F?

Problem 4.10. Pam is taking a train from the town of Rome to the town of Florence. Rome is located 30 miles due West of the town of Paris. Florence is 25 miles East, and 45 miles North of Rome.

On her trip, how close does Pam get to Paris?

Problem 4.11. Angela, Mary and Tiff are all standing near the intersection of University and 42nd streets. Mary and Tiff do not move, but Angela runs toward Tiff at 12 ft/sec along a straight line, as pictured. Assume the roads are 50 feet wide and Tiff is 60 feet north of the nearest corner. Where is Angela located when she is closest to Mary and when does she reach this spot?

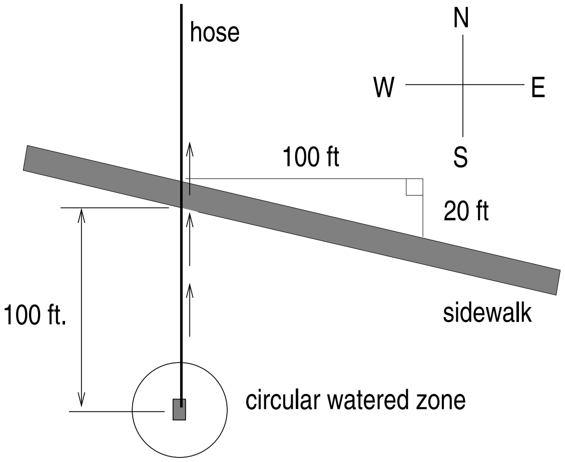


Problem 4.12. The infamous *crawling tractor sprinkler* is located as pictured below, 100 feet South of a 10 ft. wide sidewalk; notice the hose and sidewalk are not perpendicular. Once the water is turned on, the sprinkler waters a circular disc of radius 20 feet and moves North along the hose at the rate of $\frac{1}{2}$ inch/second.

(a) Impose a coordinate system. Describe the initial coordinates of the sprinkler and find the equation of the line forming the southern boundary of the sidewalk.

(b) After 33 minutes, sketch a picture of the wet portion of the sidewalk; find the length of the wet portion of the Southern edge of the sidewalk.

(c) Find the equation of the line forming the northern boundary of the sidewalk. (Hint: You can use the properties of right triangles.)



Problem 4.13. Margot is walking in a straight line from a point 30 feet due east of a statue in a park toward a point 24 feet due north of the statue. She walks at a constant speed of 4 feet per second.

(a) Write parametric equations for Margot's position t seconds after she starts walking.

(b) Write an expression for the distance from Margot's position to the statue at time t.

(c) Find the times when Margot is 28 feet from the statue.

Problem 4.14. Juliet and Mercutio are moving at constant speeds in the xy-plane. They start moving at the same time. Juliet starts at the point $(0,-6)$ and heads in a straight line toward the point $(10,5)$, reaching it in 10 seconds. Mercutio starts at $(9,-14)$ and moves in a straight line. Mercutio passes through the same point on the x axis as Juliet, but 2 seconds after she does.

How long does it take Mercutio to reach the y-axis?

Problem 4.15. (a) Solve for x:

$$\frac{1}{x} - \frac{1}{x+1} = 3.$$

(b) Solve for t: $2 = \sqrt{(1+t)^2 + (1-2t)^2}$.

(c) Solve for t: $\frac{3}{\sqrt{5}} = \sqrt{(1+t)^2 + (1-2t)^2}$.

(d) Solve for t: $0 = \sqrt{(1+t)^2 + (1-2t)^2}$.

Problem 4.16. (a) Solve for x:

$$x^4 - 4x^2 + 2 = 0$$

(b) Solve for y:

$$y - 2\sqrt{y} = 4$$

Chapter 5

Functions and Graphs

Pictures are certainly important in the work of an *architect*, but it is perhaps less evident that visual aids can be powerful tools for solving mathematical problems. If we start with an equation and attach a picture, then the mathematics can come to life. This adds a new dimension to both interpreting and solving problems. One of the real triumphs of modern mathematics is a theory connecting pictures and equations via the concept of a *graph*. This transition from "equation" to "picture" (called graphing) and its usefulness (called graphical analysis) are the theme of the next two sections. The importance of these ideas is HUGE and cannot be overstated. Every moment spent studying these ideas will pay back dividends in this course and in any future mathematics, science or engineering courses.

5.1 Relating Data, Plots and Equations

Imagine you are standing high atop an oceanside cliff and spot a seagull hovering in the air-current. Assuming the gull moves up and down along a vertical line of motion, how can we best describe its location at time t *seconds?*

There are three different (but closely linked) ways to describe the location of the gull:

- a table of **data** of the gull's height above cliff level at various times t;
- a **plot** of the data in a "time" (seconds) vs. "height" (feet) coordinate system;
- an **equation** relating time t (seconds) and height s (feet).

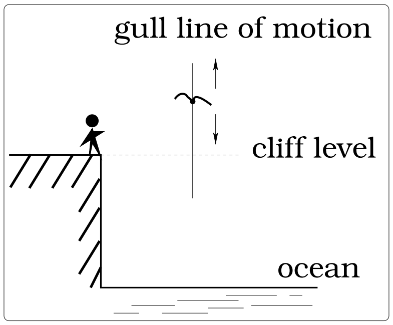
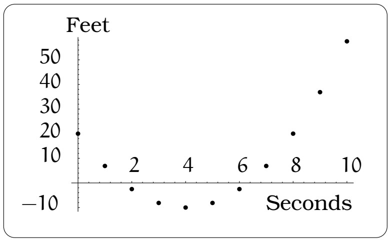


Figure 5.1: Seagull's height.

To make sure we really understand how to pass back and forth between these three descriptive modes, imagine we have tabulated (Figure 5.2) the height of the gull above cliff level at one-second time intervals for a 10 second time period. Here, a “negative height” means the gull is below cliff level. We can try to visualize the meaning of this data by plotting these 11 data points (t, s) in a time (sec.) vs. height (ft.) coordinate system.

Gull Height (feet above cliff level)					
t (sec)	**s (ft)**	**t (sec)**	**s (ft)**	**t (sec)**	**s (ft)**
0	20	4	-10	8	20
1	6.88	5	-8.12	9	36.88
2	-2.5	6	-2.5	10	57.5
3	-8.12	7	6.88		

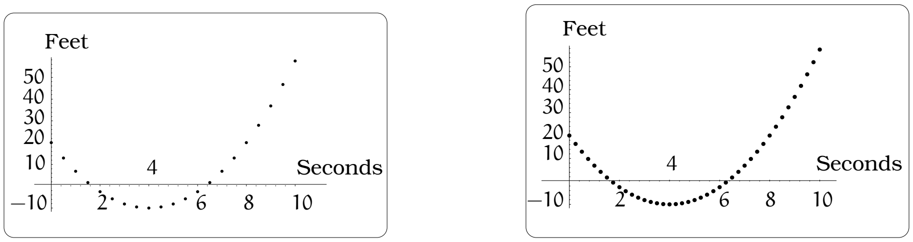
(a) Symbolic data.



(b) Visual data.

Figure 5.2: Symbolic versus visual view of data.

We can improve the quality of this description by increasing the number of data points. For example, if we tabulate the height of the gull above cliff level at 1/2 second or 1/4 second time intervals (over the same 10 second time period), we might get these two plots:



(a) $\frac{1}{2}$ second intervals.

(b) $\frac{1}{4}$ second intervals.

Figure 5.3: Shorter time intervals mean more data points.

We have focused on how to go from data to a plot, but the reverse process is just as easy: *A point* (t, s) *in any of these three plots is interpreted to mean that the gull is* s *feet above cliff level at time* t *seconds.*

Furthermore, increasing the amount of data, we see how the plotted points are “filling in” a portion of a parabola. Of course, it is way too tedious to create longer and longer tables of data. What we really want is a “formula” (think of it as a prescription) that tells us how to produce a data point for the gull’s height at any given time t. If we had such a formula, then we could completely dispense with the tables of data and just use the formula to crank out data points. For example, look at this equation involving the variables t and s:

$$s = \frac{15}{8}(t-4)^2 - 10.$$

If we plug in $t = 0, 1, 2, 9, 10$, then we get $s = 20$, 6.88, -2.5, 36.88, 57.5, respectively; this was some of our initial tabulated data. This same equation produces ALL of the data points for the other two plots, using 1/2 second and 1/4 second time intervals. (Granted, we have swept under the rug the issue of "*...where the heck the equation comes from...*"; that is a consequence of mathematically modeling the motion of this gull. Right now, we are focusing on how the equation relates to the data and the plot, assuming the equation is in front of us to start with.) In addition, it is very important to notice that having this equation produces an infinite number of data points for our gull's location, since we can plug in any t value between 0 and 10 and get out a corresponding height s. In other words, the equation is A LOT more powerful than a finite (usually called *discrete*) collection of tabulated data.

5.2 What is a Function?

Our lives are chock full of examples where two changing quantities are related to one another:

- The cost of *postage* is related to the *weight* of the item.
- The *value* of an investment will depend upon the *time* elapsed.
- The *population* of cells in a growth medium will be related to the amount of *time* elapsed.
- The *speed* of a chemical reaction will be related to the *temperature* of the reaction vessel.

In all such cases, it would be beneficial to have a "procedure" whereby we can assign a unique output value to any acceptable input value. For example, given the time elapsed (an input value), we would like to predict a unique future value of an investment (the output value). Informally, this leads to the broadest (and hence most applicable) definition of what we will call a *function*:

Definition 5.2.1. *A function is a procedure for assigning a unique output to any allowable input.*

The key word here is "procedure." Our discussion of the hovering seagull in 5.1 highlights three ways to produce such a "procedure" using data, plots of curves and equations.

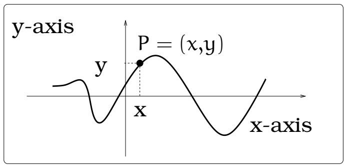


Figure 5.4: Graph of a procedure.

- A table of data, by its very nature, will relate two columns of data: The output and input values are listed as column entries of the table and reading across each row is the "procedure" which relates an input with a unique output.

- Given a curve in Figure 5.4, consider the "procedure" which associates to each x on the horizontal axis the y coordinate of the pictured point P on the curve.

- Given an equation relating two quantities x and y, plugging in a particular x value and going through the "procedure" of algebra often produces a unique output value y.

5.2.1 The definition of a function (equation viewpoint)

Now we focus on giving a precise definition of a function, in the situation when the "procedure" relating two quantities is actually given by an equation. Keep in mind, this is only one of three possible ways to describe a function; we could alternatively use tables of data or the plot of a curve. We focus on the equation viewpoint first, since it is no doubt the most familiar.

If we think of x and y as related physical quantities (e.g. *time* and *distance*), then it is sometimes possible (and often desirable) to express one of the variables in terms of the other. For example, by simple arithmetic, the equations

$$3x + 2y = 4 \qquad x^2 - x = \frac{1}{2}y - 4 \qquad y\sqrt{x^2+1} = 1,$$

can be rewritten as equivalent equations

$$y = \frac{1}{2}(4 - 3x) \qquad 2x^2 - 2x + 8 = y \qquad y = \frac{1}{\sqrt{x^2+1}}.$$

This leads to THE MOST IMPORTANT MATH DEFINITION IN THE WORLD:

Definition 5.2.2. *A function is a package, consisting of three parts:*

- *An equation of the form*

  $$y = \text{“a mathematical expression only involving the variable } x\text{,”}$$

  which we usually indicate via the shorthand notation $y = f(x)$. *This equation has the very special property that each time we plug in an* x *value, it produces exactly one (a unique)* y *value. We call the mathematical expression* $f(x)$ *"the rule".*

- *A set* D *of* x*-values we are allowed to plug into* $f(x)$, *called the "domain" of the function.*

- *The set* R *of output values* $f(x)$, *where* x *varies over the domain, called the "range" of the function.*

Any time we have a function $y = f(x)$, we refer to x as the *independent variable* (the "input data") and y as the *dependent variable* (the "output data"). This terminology emphasizes the fact that we have freedom in the values of x we plug in, but once we specify an x value, the y value is uniquely determined by the rule $f(x)$.

Examples 5.2.3. *(i) The equation* $y = -2x + 3$ *is in the form* $y = f(x)$, *where the rule is* $f(x) = -2x+3$. *Once we specify a domain of* x *values, we have a function. For example, we could let the domain be all real numbers.*

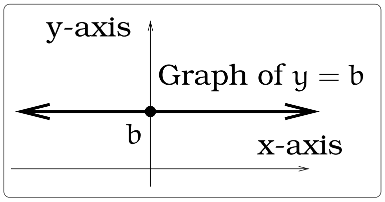


Figure 5.5: Constant function.

(ii) Take the same rule $f(x) = -2x + 3$ *from (i) and let the domain be all non-negative real numbers. This describes a function. However, the functions* $f(x) = -2x+3$ *on the domain of all non-negative real numbers and* $f(x) = -2x + 3$ *on the domain of all real numbers (from (i)) are different, even though they share the same rule; this is because their domains differ! This example illustrates the idea of what is called a restricted domain. In other words, we started with the function in (i) on the domain of all real numbers, then we "restricted" to the subset of non-negative real numbers.*

(iii) The equation $y = b$, *where* b *is a constant, defines a function on the domain of all real numbers, where the rule is* $f(x) = b$; *we call these the constant functions. Recall, in Chapter 3, we observed that the solutions of the equation* $y = b$, *plotted in the* xy *coordinate system, will give a horizontal line. For example, if* $b = 0$, *you get the horizontal axis.*

(iv) Consider the equation $y = \frac{1}{x}$, *then the rule* $f(x) = \frac{1}{x}$ *defines a function, as long as we do not plug in* $x = 0$. *For example, take the domain to be the non-zero real numbers.*

(v) Consider the equation $y = \sqrt{1 - x^2}$. *Before we start plugging in* x *values, we want to know the expression under the radical symbol (square root symbol) is non-negative; this insures the square root is a real number. This amounts to solving an inequality equation:* $0 \leq 1 - x^2$; *i.e.,* $-1 \leq x \leq 1$. *These remarks show that the rule* $f(x) = \sqrt{1 - x^2}$ *defines a function, where the domain of* x *values is* $-1 \leq x \leq 1$.

Typically, the domain of a function $y = f(x)$ will either be the entire number line, an interval on the number line, or a finite union of such intervals. We summarize the notation used to represent intervals in Table 5.1.

Common Intervals on the Number Line

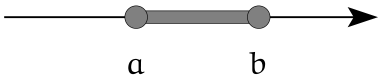
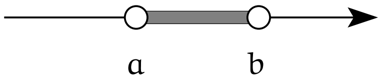
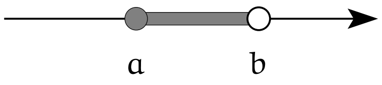
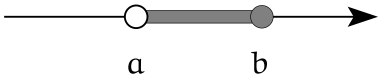
Description	**Symbolic Notation**	**Picture**
All numbers x between a and b, x possibly equal to either a or b	$a \leq x \leq b$	
All numbers x between a and b, $x \neq a$ and $x \neq b$	$a < x < b$	
All numbers x between a and b, $x \neq b$ and x possibly equal to a	$a \leq x < b$	
All numbers x between a and b, $x \neq a$ and x possibly equal to b	$a < x \leq b$	

Table 5.1: Interval Notations

We can interpret a function as a "prescription" that takes a given x value (in the domain) and produces a single unique y value (in the range). We need to be really careful and not fall into the trap of thinking that every equation in the world is a function. For example, if we look at this equation

$$x + y^2 = 1$$

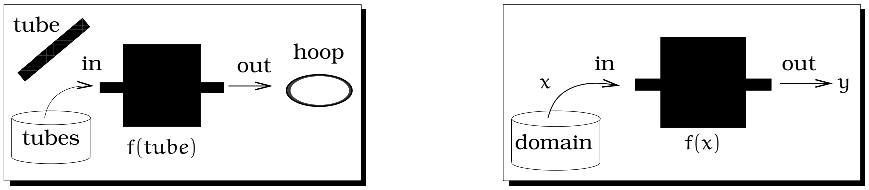
and plug in $x = 0$, the equation becomes

$$y^2 = 1.$$

This equation has two solutions, $y = \pm 1$, so the conclusion is that plugging in $x = 0$ does NOT produce a single output value. This violates one of the conditions of our function definition, so the equation $x + y^2 = 1$ is NOT a function in the independent variable x. Notice, if you were to try and solve this equation for y in terms of x, you'd first write $y^2 = 1 - x$ and then take a square root (to isolate y); but the square root introduces TWO roots, which is just another way of reflecting the fact there can be two y values attached to a single x value. Alternatively, you can solve the equation for x in terms of y, getting $x = 1 - y^2$; this shows the equation does define a function $x = g(y)$ in the independent variable y.

5.2.2 The definition of a function (conceptual viewpoint)

Conceptually, you can think of a function as a “process”: An allowable input goes into a “black box” and out pops a unique new value denoted by the symbol $f(x)$. Compare this with the machine making “hula-hoops” in Figure 5.6. While you are problem solving, you will find this to be a useful viewpoint when a function is described in words.



(a) A hoop machine as a “process” taking “tubes” to “hoops.”

(b) A function as a “process” which takes x to y.

Figure 5.6: Viewing a function as a “process.”

Examples 5.2.4. *Here are four examples of relationships that are functions:*

(i) ***The total amount of water used by a household since midnight on a particular day.*** *Let y be the total number of gallons of water used by a household between 12:00am and a particular time t; we will use time units of “hours.” Given a time t, the household will have used a specific (unique) amount of water, call it $S(t)$. Then $y = S(t)$ defines a function in the independent variable t with dependent variable y. The domain would be $0 \leq t \leq 24$ and the largest possible value of $S(t)$ on this domain is $S(24)$. This tells us that the range would be the set of values $0 \leq y \leq S(24)$.*

(ii) ***The height of the center of a basketball as you dribble, depending on time.*** *Let s be the height of the basketball center at time t seconds after you start dribbling. Given a time t, if we freeze the action, the center of the ball has a single unique height above the floor, call it $h(t)$. So, the height of the basketball center is given by a function $s = h(t)$. The domain would be a given interval of time you are dribbling the ball; for example, maybe $0 \leq t \leq 2$ (the first 2 seconds). In this case, the range would be all of the possible heights attained by the center of the basketball during this 2 seconds.*

(iii) ***The state sales tax due on a taxable item.*** *Let T be the state tax (in dollars) due on a taxable item that sells for z dollars. Given a taxable item that costs z dollars, the state tax due is a single unique*

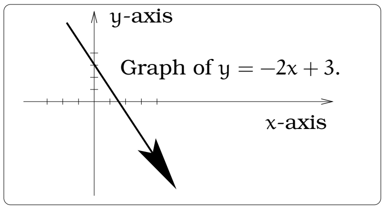
amount, call it $W(z)$. *So,* $T = W(z)$ *is a function, where the independent variable is* z. *The domain could be taken to be* $0 \leq z \leq 1{,}000{,}000$, *which would cover all items costing up to one-million dollars. The range of the function would be the set of all values* $W(z)$, *as* z *ranges over the domain.*

(iv) ***The speed of a chemical reaction depending on the temperature.*** *Let* v *be the speed of a particular chemical reaction and* T *the temperature in Celsius* °*C. Given a particular temperature* T, *one could experimentally measure the speed of the reaction; there will be a unique speed, call it* $r(T)$. *So,* $v = r(T)$ *is a function, where the independent variable is* T. *The domain could be taken to be* $0 \leq T \leq 100$, *which would cover the range of temperatures between the freezing and boiling points of water. The range of the function would be the set of all speeds* $r(T)$, *as* T *ranges over the domain.*

5.3 The Graph of a Function

Let's start with a concrete example; the function $f(x) = -2x + 3$ on the domain of all real numbers. We discussed this Example 5.2.3. Plug in the specific x values, where $x = -1, 0, 1, 2$ and tabulate the resulting y values of the function:

x	y	point (x,y)
-1	5	(-1,5)
0	3	(0,3)
1	1	(1,1)
2	-1	(2,-1)
$\vdots$	$\vdots$	$\vdots$
x	$-2x+3$	$(x,-2x+3)$



(a) Tabulated data. (b) Visual data.

Figure 5.7: Symbolic versus visual view of data.

This tells us that the points $(0,3)$, $(1,1)$, $(2,-1)$, $(-1,5)$ are solutions of the equation $y = -2x + 3$. For example, if $y = -2x + 3$, $x = 0$, $y = 3$, then $3 = -2 \cdot 0 + 3$ (which is true), or if $y = -2x + 3$, $x = 2$, $y = -1$, then $-1 = -2 \cdot 2 + 3$ (which is true), etc. In general, if we plug in x we get out $-2x + 3$, so the point $(x, -2x + 3)$ is a solution to the function equation $y = f(x)$. We can plot all of these solutions in the xy-coordinate system. The set of points we obtain, as we vary over all x in the domain, is called the set of *solutions* of the equation $y = -2x + 3$:

$$\text{Solutions} = \{\ (x, -2x + 3) \mid x \text{ any real number}\}.$$

Notice that plotting these points produces a line of slope $m = -2$ with y-intercept 3. In other words, the graph of the function $f(x) = -2x + 3$ is the same as the graph of the equation $y = -2x + 3$, as we discussed in Chapter 4.

In general, by definition, we say that a point (x,y) is a *solution* to the function equation $y = f(x)$ if plugging x and y into the equation gives a true statement.

How can we find ALL the solutions of the equation $y = f(x)$*?* In general, the definition of a function is “rigged” so it is easy to describe all solutions of the equation $y = f(x)$: Each time we specify an x value (in the domain), there is only one y value, namely $f(x)$. This means the point $P = (x, f(x))$ is the ONLY solution to the equation $y = f(x)$ with first coordinate x. We define the *graph* of the function $y = f(x)$ to be the plot of all solutions of this equation (in the xy coordinate system). It is common to refer to this as either the **“graph of $f(x)$”** or the **“graph of f.”**

$$\text{Graph} = \{(x,f(x)) \mid x \text{ in the domain}\} \tag{5.1}$$

Important Procedure 5.3.1. ***Points on a graph.*** *The description of the graph of a function gives us a procedure to produce points on the graph AND to test whether a given point is on the graph. On the one hand, if you are given* u *in the domain of a function* $y = f(x)$*, then you immediately can plot the point* $(u, f(u))$ *on the graph. On the other hand, if someone gives you a point* (u, v)*, it will be on the graph only if* $v = f(u)$ *is true. We illustrate this in Example 5.3.2.*

Example 5.3.2. *The function* $s = h(t) = \frac{15}{8}(t-4)^2 - 10$ *defines a function in the independent variable* t*. If we restrict to the domain* $0 \leq t \leq 10$*, then the discussion in Chapter 7 tells us that the graph is a portion of a parabola: See Figure 5.8. Using the above procedure, you can verify that the data points discussed in the seagull example (in §5.1) all lie on this parabola. On the other hand, the point* $(0,0)$ *is NOT on the graph, since* $h(0) = 20 \neq 0$*.*

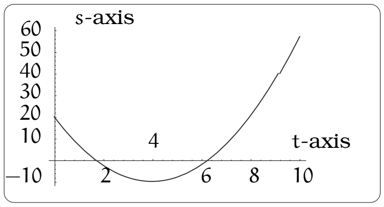


Figure 5.8: $s = h(t)$.

5.4 The Vertical Line Test

There is a pictorial aspect of the graph of a function that is very revealing: Since $(x, f(x))$ is the *only* point on the graph with first coordinate equal to x, a vertical line passing through x on the x-axis (with x in the domain) crosses the graph of $y = f(x)$ once and only once. This gives us a decisive way to test if a curve is the graph of a function.

Important Procedure 5.4.1. ***The vertical line test.*** *Draw a curve in the* xy*-plane and specify a set* D *of* x*-values. Suppose every vertical line through a value in* D *intersects the curve exactly once. Then the curve is* *the graph of some function on the domain* D. *If we can find a single vertical line through some value in* D *that intersects the curve more than once, then the curve is not the graph of a function on the domain* D.

For example, draw any straight line m in the plane. By the vertical line test, if the line m is not vertical, m is the graph of a function. On the other hand, if the line m is vertical, then m is not the graph of a function. These two situations are illustrated in Figure 5.9. As another example, consider the equation $x^2 + y^2 = 1$, whose graph is the unit circle and specify the domain D to be $-1 \leq x \leq 1$; recall Example 3.2.2. The vertical line passing through the point $\left(\frac{1}{2}, 0\right)$ will intersect the unit circle twice; by the vertical line test, the unit circle is **not** the graph of a function on the domain $-1 \leq x \leq 1$.

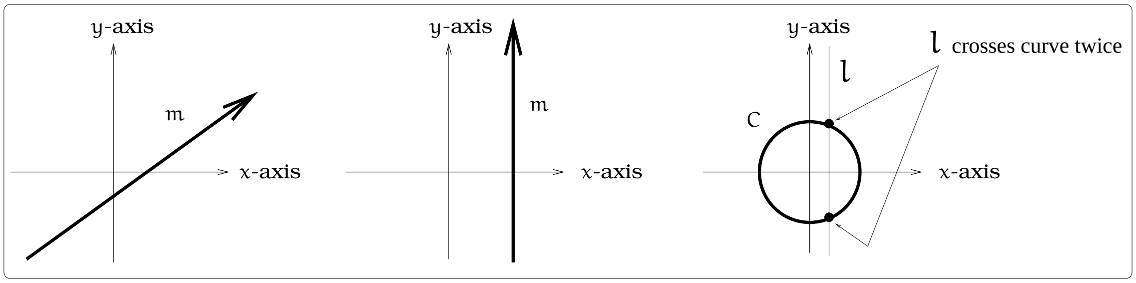


Figure 5.9: Applying the vertical line test.

5.4.1 Imposed Constraints

In physical problems, it might be natural to *constrain* (meaning to "limit" or "restrict") the domain. As an example, suppose the height s (in feet) of a ball above the ground after t seconds is given by the function

$$s = h(t) = -16t^2 + 4.$$

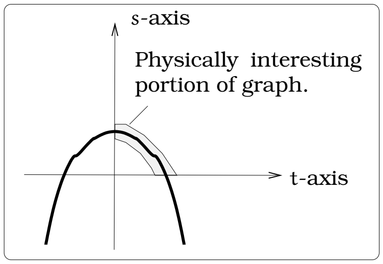


Figure 5.10: Restricting the domain.

We could look at the graph of the function in the ts-plane and we will review in Chapter 7 that the graph looks like a parabola. The physical context of this problem makes it natural to only consider the portion of the graph in the first quadrant; why? One way of specifying this quadrant would be to restrict the domain of possible t values to lie between 0 and $\frac{1}{2}$; notationally, we would write this constraint as $0 \leq t \leq \frac{1}{2}$.

5.5 Linear Functions

A major goal of this course is to discuss several different kinds of functions. The work we did in Chapter 4 actually sets us up to describe one very useful type of function called a *linear function.* Back in Chapter 4, we discussed how lines in the plane can be described using equations in the variables x and y. One of the key conclusions was:

Important Fact 5.5.1. *A non-vertical line in the plane will be the graph of an equation* $y = mx + b$, *where* m *is the slope of the line and* b *is the* y*-intercept.*

Notice that any non-vertical line will satisfy the conditions of the vertical line test, which means it must be the graph of a function. What is the function? The answer is to use the equation in x and y we already obtained in Chapter 4: The rule $f(x) = mx + b$ on some specified domain will have a line of slope m and y-intercept b as its graph. We call a function of this form a *linear function*.

Example 5.5.2. *You are driving 65 mph from the Kansas state line (mile marker 0) to Salina (mile marker 130) along I-35. Describe a linear function that calculates mile marker after* t *hours. Describe another linear function that will calculate your distance from Salina after* t *hours.*

Solution. Define a function $d(t)$ to be the mile marker after t hours. Using "distance=rate×time," we conclude that $65t$ will be the distance traveled after t hours. Since we started at mile marker 0, $d(t) = 65t$ is the rule for the first function. A reasonable domain would be to take $0 \leq t \leq 2$, since it takes 2 hours to reach Salina.

For the second situation, we need to describe a different function, call it $s(t)$, that calculates your distance from Salina after t hours. To describe the rule of $s(t)$ we can use the previous work:

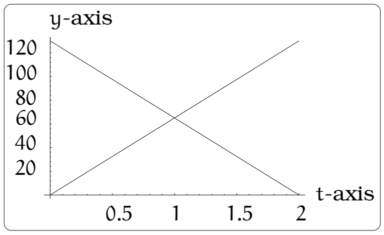


Figure 5.11: Distance functions.

$$\begin{aligned} s(t) &= (\text{mile marker Salina}) - \\ &\quad (\text{your mile marker at } t \text{ hrs.}) \\ &= 130 - d(t) \\ &= 130 - 65t. \end{aligned}$$

For the rule $s(t)$, the best domain would again be $0 \leq t \leq 2$. We have graphed these two functions in the same coordinate system: See Figure 5.11 (Which function goes with which graph?). □

5.6 Profit Analysis

Let's give a first example of how to interpret the graph of a function in the context of an application.

Example 5.6.1. *A software company plans to bring a new product to market. The sales price per unit is \$15 and the expense to produce and market* x *units is* $\$100(1+\sqrt{x})$. *What is the profit potential?*

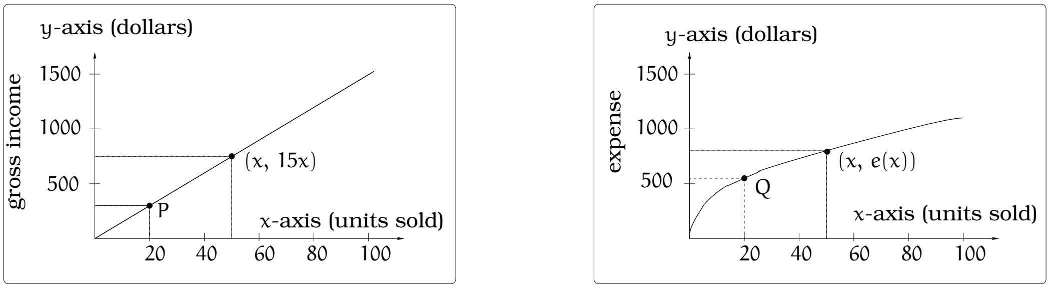
Two functions control the profit potential of the new software. The first tells us the *gross income*, in dollars, on the sale of x units. All of the costs involved in developing, supporting, distributing and marketing x units are controlled by the *expense* equation (again in dollars):

$$g(x) = 15x \qquad \text{(gross income function)}$$

$$e(x) = 100(1+\sqrt{x}) \qquad \text{(expense function)}$$

A *profit* will be realized on the sale of x units whenever the gross income exceeds expenses; i.e., this occurs when $g(x) > e(x)$. A *loss* occurs on the sale of x units when expenses exceed gross income; i.e., when $e(x) > g(x)$. Whenever the sale of x units yields zero profit (and zero loss), we call x a *break-even point*; i.e., when $e(x) = g(x)$.

The above approach is "symbolic." Let's see how to study profit and loss visually, by studying the graphs of the two functions $g(x)$ and $e(x)$. To begin with, plot the graphs of the two individual functions in the xy-coordinate system. We will focus on the situation when the sales figures are between 0 to 100 units; so the domain of x values is the interval $0 \le x \le 100$. Given any sales figure x, we can graphically relate three things:



(a) Gross income graph.

(b) Expenses graph.

Figure 5.12: Visualizing income and expenses.

- x on the horizontal axis;
- a point on the graph of the gross income or expense function;
- y on the vertical axis.

If $x = 20$ units sold, there is a unique point $P = (20, g(20)) = (20, 300)$ on the gross income graph and a unique point $Q = (20, e(20)) = (20, 547)$ on the expenses graph. Since the y-coordinates of P and Q are the function values at $x = 20$, the height of the point above the horizontal axis is controlled by the function.

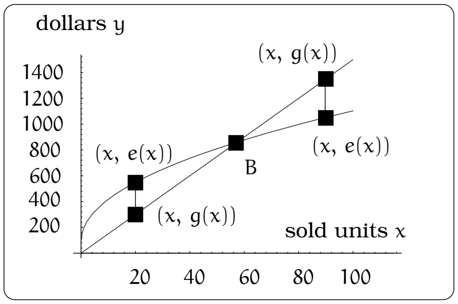


Figure 5.13: Modelling profit and loss.

If we plot both graphs in the same coordinate system, we can visually study the distance between points on each graph above x on the horizontal axis. In the first part of this plot, the expense graph is above the income graph, showing a loss is realized; the exact amount of the loss will be $e(x) - g(x)$, which is the length of the pictured line segment. Further to the right, the two graphs cross at the point labeled "B"; this is the break-even point; i.e., expense and income agree, so there is zero profit (and zero loss). Finally, to the right of B the income graph is above the expense graph, so there is a profit; the exact amount of the profit will be $g(x) - e(x)$, which is the length of the right-most line segment. Our analysis will be complete once we pin down the break-even point B. This amounts to solving the equation $g(x) = e(x)$.

$$\begin{aligned} 15x &= 100(1+\sqrt{x}) \\ 15x - 100 &= 100\sqrt{x} \\ 225x^2 - 3000x + 10000 &= 10000x \\ 225x^2 - 13000x + 10000 &= 0. \end{aligned}$$

Applying the quadratic formula, we get two answers: $x = 0.78$ or 57.

Now, we face a problem: Which of these two solutions is the answer to the original problem? We are going to argue that only the second solution $x = 57$ gives us the break even point. What about the other "solution" at $x = 0.78$? Try plugging $x = 0.78$ into the original equation: $15(0.78) \neq 100(1 + \sqrt{0.78})$. What has happened? Well, when going from the second to the third line, both sides of the equation were squared. Whenever we do this, we run the risk of adding extraneous solutions. What should you do? After solving any equation, look back at your steps and ask yourself whether or not you may have added (or lost) solutions. In particular, be wary when squaring or taking the square root of both sides of an equation. Always check your final answer in the original equation.

We can now compute the coordinates of the break-even point using either function:

$$B = (57, g(57)) = (57, 855) = (57, e(57)).$$

5.7 Exercises

Problem 5.1. For each of the following functions, find the expression for

$$\frac{f(x+h)-f(x)}{h}.$$

Simplify each of your expressions far enough so that plugging in $h=0$ would be allowed.

(a) $f(x)=x^2-2x$.

(b) $f(x)=2x+3$

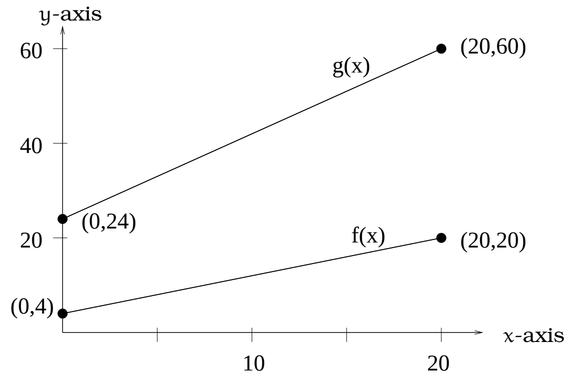
(c) $f(x)=x^2-3$

(d) $f(x)=4-x^2$

(e) $f(x)=-\pi x^2-\pi^2$

(f) $f(x)=\sqrt{x-1}$. (Hint: Rationalize the numerator)

Problem 5.2. Here are the graphs of two linear functions on the domain $0 \le x \le 20$. Find the formula for each of the rules $y=f(x)$ and $y=g(x)$. Find the formula for a NEW function $v(x)$ that calculates the vertical distance between the two lines at x. Explain in terms of the picture what $v(x)$ is calculating. What is $v(5)$? What is $v(20)$? What are the smallest and largest values of $v(x)$ on the domain $0 \le x \le 20$?



Problem 5.3. Dave leaves his office in Padelford Hall on his way to teach in Gould Hall. Below are several different scenarios. In each case, sketch a plausible (reasonable) graph of the function $s=d(t)$ which keeps track of Dave's distance s from Padelford Hall at time t. Take distance units to be "feet" and time units to be "minutes." Assume Dave's path to Gould Hall is along a straight line which is 2400 feet long.

gould
padelford

(a) Dave leaves Padelford Hall and walks at a constant speed until he reaches Gould Hall 10 minutes later.

(b) Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute. He then continues on to Gould Hall at the same constant speed he had when he originally left Padelford Hall.

(c) Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave then continues on to Gould Hall at twice the constant speed he had when he originally left Padelford Hall.

(d) Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Dave gets confused and stops for 1 minute to figure out where he is. Dave is totally lost, so he simply heads back to his office, walking the same constant speed he had when he originally left Padelford Hall.

(e) Dave leaves Padelford heading for Gould Hall at the same instant Angela leaves Gould Hall heading for Padelford Hall. Both walk at a constant speed, but Angela walks twice as fast as Dave. Indicate a plot of "distance from Padelford" vs. "time" for both Angela and Dave.

(f) Suppose you want to sketch the graph of a *new* function $s=g(t)$ that keeps track of Dave's distance s from Gould Hall at time t. How would your graphs change in (a)-(e)?

Problem 5.4. At 5 AM one day, a monk began a trek from his monastery by the sea to the monastery at the top of a mountain. He reached the mountain-top monastery at 11 AM, spent the rest of the day in meditation, and then slept the night there. In the morning, at 5 AM, he began walking back to the seaside monastery. Though walking downhill should have been faster, he dawdled in the beautiful sunshine, and ending up getting to the seaside monastery at exactly 11 AM.

(a) Was there necessarily a time during each trip when the monk was in exactly the same place on both days? Why or why not?

(b) Suppose the monk walked faster on the second day, and got back at 9 AM. What is your answer to part (a) in this case?

(c) Suppose the monk started later, at 10 AM, and reached the seaside monastery at 3 PM. What is your answer to part (a) in this case?

Problem 5.5. Sketch a reasonable graph for each of the following functions. Specify a reasonable domain and range and state any assumptions you are making. Finally, describe the largest and smallest values of your function.

(a) Height of a person depending on age.

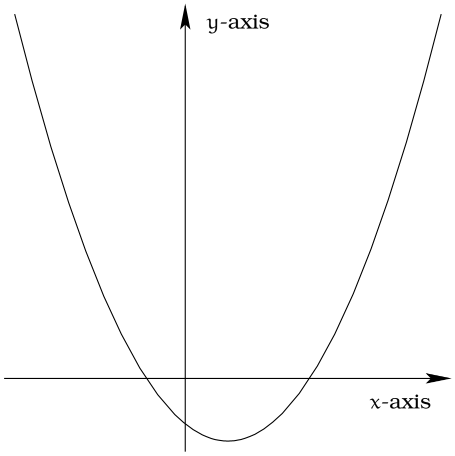
(b) Height of the top of your head as you jump on a pogo stick for 5 seconds.

(c) The amount of postage you must put on a first class letter, depending on the weight of the letter.

(d) Distance of your big toe from the ground as you ride your bike for 10 seconds.

(e) Your height above the water level in a swimming pool after you dive off the high board.

Problem 5.6. Here is a picture of the graph of the function $f(x) = 3x^2 - 3x - 2$.



Recall the procedure 5.3.1 on page 63.

(a) Find the x and y intercepts of the graph.

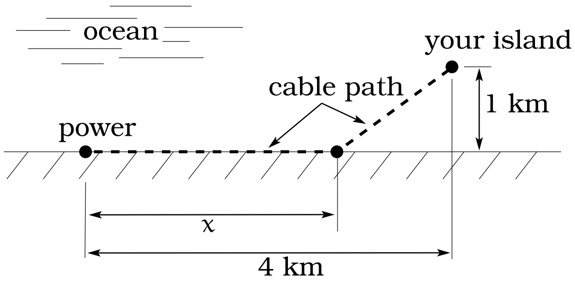
(b) Find the exact coordinates of all points (x,y) on the graph which have y-coordinate equal to 5.

(c) Find the coordinates of all points (x,y) on the graph which have y-coordinate equal to -3.

(d) Which of these points is on the graph: $(1,-2)$, $(-1,3)$, $(2.4,8)$, $(\sqrt{3},7-3\sqrt{3})$.

(e) Find the exact coordinates of the point (x,y) on the graph with $x = \sqrt{1+\sqrt{2}}$.

Problem 5.7. After winning the lottery, you decide to buy your own island. The island is located 1 km offshore from a straight portion of the mainland. There is currently no source of electricity on the island, so you want to run a cable from the mainland to the island. An electrical power sub-station is located 4 km from your island's nearest location to the shore. It costs \$50,000 per km to lay a cable in the water and \$30,000 per km to lay a cable over the land.



(a) Explain why we can assume the cable follows the path indicated in the picture;

i.e. explain why the path consists of two line segments, rather than a weird curved path AND why it is OK to assume the cable reaches shore to the right of the power station and the left of the island.

(b) Let x be the distance downshore from the power sub-station to where the cable reaches the land. Find a function $f(x)$ in the variable x that computes the cost to lay a cable out to your island.

(c) Make a table of values of $f(x)$, where $x = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \frac{7}{2}, 4$. Use these calculations to estimate the installation of minimal cost.

Problem 5.8. This problem deals with the “mechanical aspects” of working with the rule of a function. For each of the functions listed in (a)-(c), calculate: $f(0)$, $f(-2)$, $f(x+3)$, $f(\heartsuit)$, $f(\heartsuit + \triangle)$.

(a) The function $f(x) = \frac{1}{2}(x-3)$ on the domain of all real numbers.

(b) The function $f(x) = 2x^2 - 6x$ on the domain of all real numbers.

(c) The function $f(x) = 4\pi^2$.

Problem 5.9. Which of the curves in Figure 5.14 represent the graph of a function? If the curve is not the graph of a function, describe what goes wrong and how you might “fix it.” When you describe how to “fix” the graph, you are allowed to cut the curve into pieces and such that each *piece* is the graph of a function. Many of these problems have more than one correct answer.

Problem 5.10. Find an EXACT answer for each problem.

(a) Solve for x

$$\frac{x}{x+3} + \frac{5}{x-7} = \frac{30}{x^2 - 4x - 21}$$

(b) Solve for x

$$\sqrt{5x-4} = \frac{x}{2} + 2$$

(c) Solve for x

$$\sqrt{x} + \sqrt{x-20} = 10$$

(d) Solve for t

$$\sqrt{2t-1} + \sqrt{3t+3} = 5$$

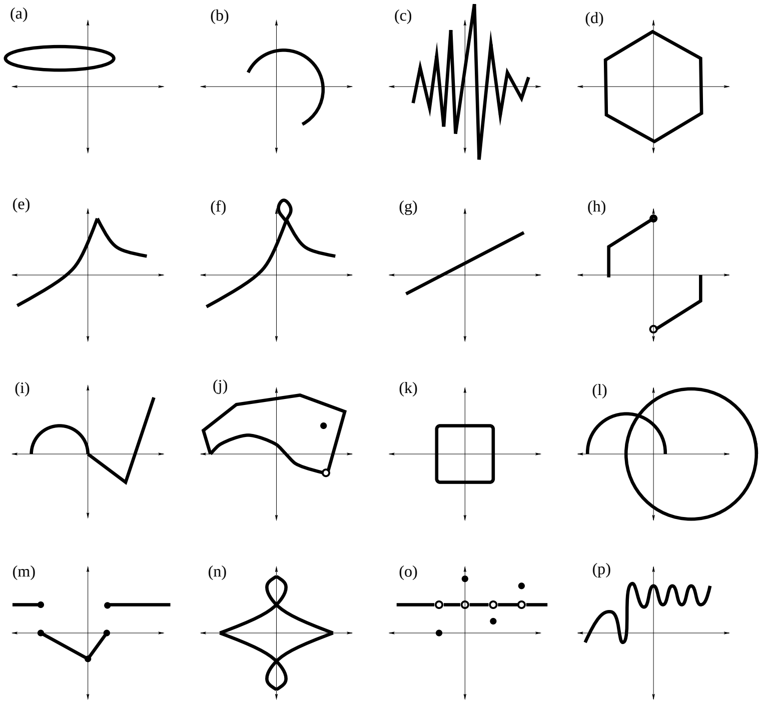

(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
(j)
(k)
(l)
(m)
(n)
(o)
(p)

Figure 5.14: Curves to consider for Problem 5.9

Chapter 6

Graphical Analysis

We ended the previous section with an in-depth look at a "Profit Analysis Problem." In that discussion, we looked at the graphs of the relevant functions and used these as visual aids to help us answer the questions posed. This was a concrete illustration of what is typically called "graphical analysis of a function." This is a fundamental technique we want to carry forward throughout the course. Let's highlight the key ideas for future reference.

6.1 Visual Analysis of a Graph

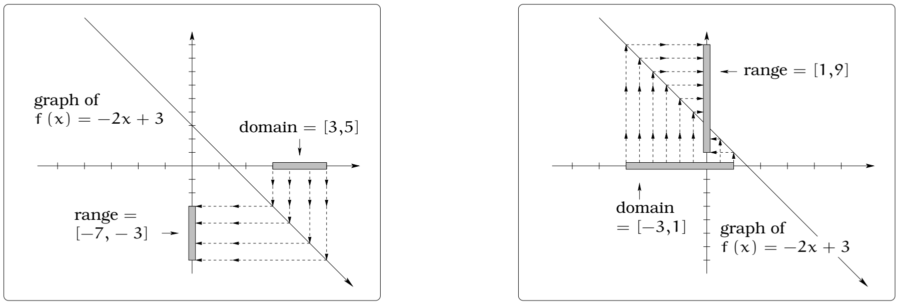
A variety of information can be visually read off of a function graph. To see this, we ask ourselves the following question: *What is the most basic qualitative feature of a graph?* To answer this, we need to return to the definition of the graph (see Equation (5.1) on page 63) and the surrounding discussion. The key thing about the graph of a function $f(x)$ is that it keeps track of a particular set of points in the plane whose coordinates are related by the function rule. To be precise, a point $P = (x,y)$ will be on the graph of the function $f(x)$ exactly when $y = f(x)$.

6.1.1 Visualizing the domain and range

A function is a package that consists of a rule $y = f(x)$, a domain of allowed x-values and a range of output y-values. The domain can be visualized as a subset of the x-axis and the range as a subset of the y-axis. If you are handed the domain, it is graphically easy to describe the range values obtained; here is the procedure:

Important Procedure 6.1.1. *Look at all points on the graph corresponding to domain values on the x-axis, then project these points to the y-axis. The collection of all values you obtain on the y-axis will be the range of the function. This idea of "projection" is illustrated in the two graphs be-*

low. We use arrows "→" to indicate going from a domain x*-value, up to the graph, then over to the* y*-axis:*


graph of
f (x) = −2x + 3
domain = [3,5]
range =
[−7, − 3]
range = [1,9]
domain
= [−3,1]
graph of
f (x) = −2x + 3

(a) Domain: $[3,5]$.

(b) Domain: $[-3,1]$.

Figure 6.1: Example projections.

6.1.2 Interpreting Points on the Graph

We can visually detect where a function has positive or negative values:

Important Fact 6.1.2. *The function values* $f(x)$ *control the height of the point* $P(x) = (x,f(x))$ *on the graph above the* x*-axis; if the function value* $f(x)$ *is negative, the point* $P(x)$ *is below the* x*-axis.*

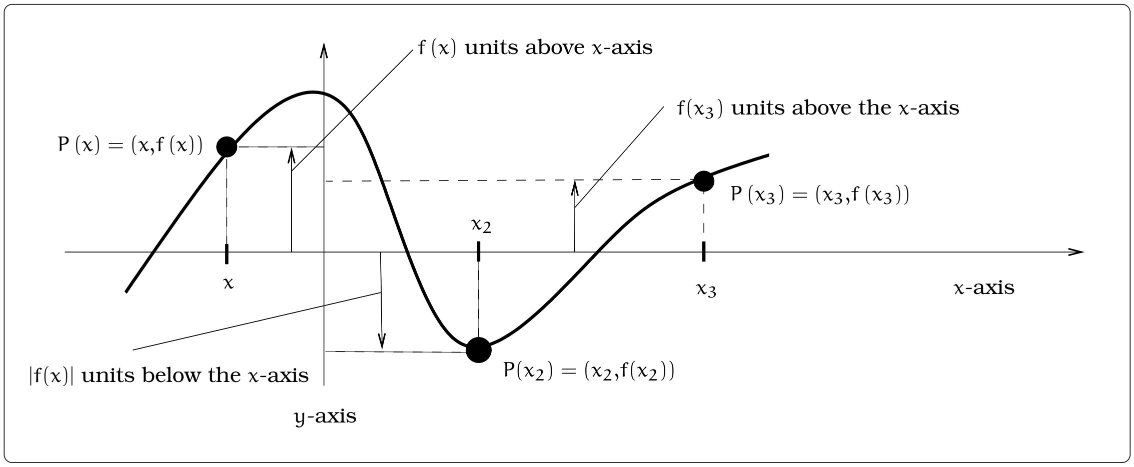

f (x) units above x-axis
f(x3) units above the x-axis
P (x) = (x,f (x))
P (x3) = (x3,f (x3))
x2
x
x3
x-axis
|f(x)| units below the x-axis
P(x2) = (x2,f(x2))
y-axis

Figure 6.2: Interpreting points on a graph.

In Figure 6.2 we can now divide the domain (in this case the whole number line) into segments where the function is above, below or crossing the axis. Keeping track of this information on a number line is called a *sign plot* for the function. We include a "shadow" of the graph in Figure 6.3

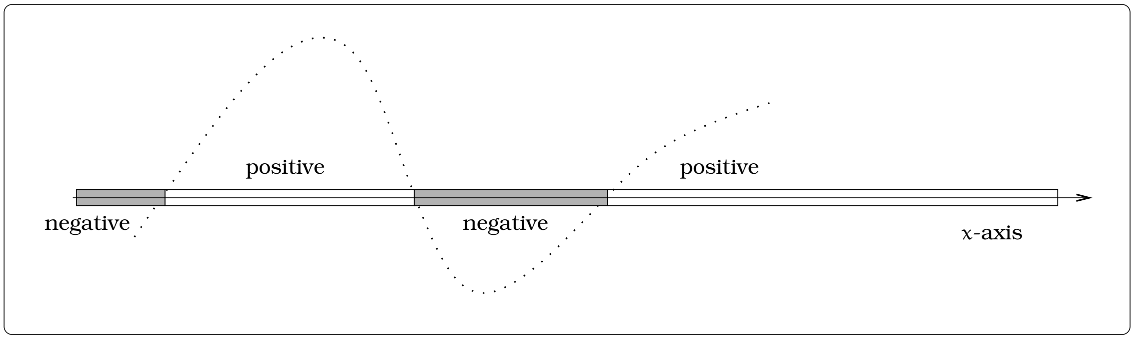

positive
positive
negative
negative
x-axis

Figure 6.3: Sign plot.

to emphasize how we arrived at our "positive" and "negative" labeling of the sign plot; in practice we would only provide a labeled number line.

By moving through a sequence of x values we can investigate how the corresponding points on the graph move "up and down"; this then gives us a dynamic visual sense of how the function values are changing. For example, in Figure 6.4, suppose we let x move from 1 to 5, left to right; we have indicated how the corresponding points on the curve will move and how the function values will change.

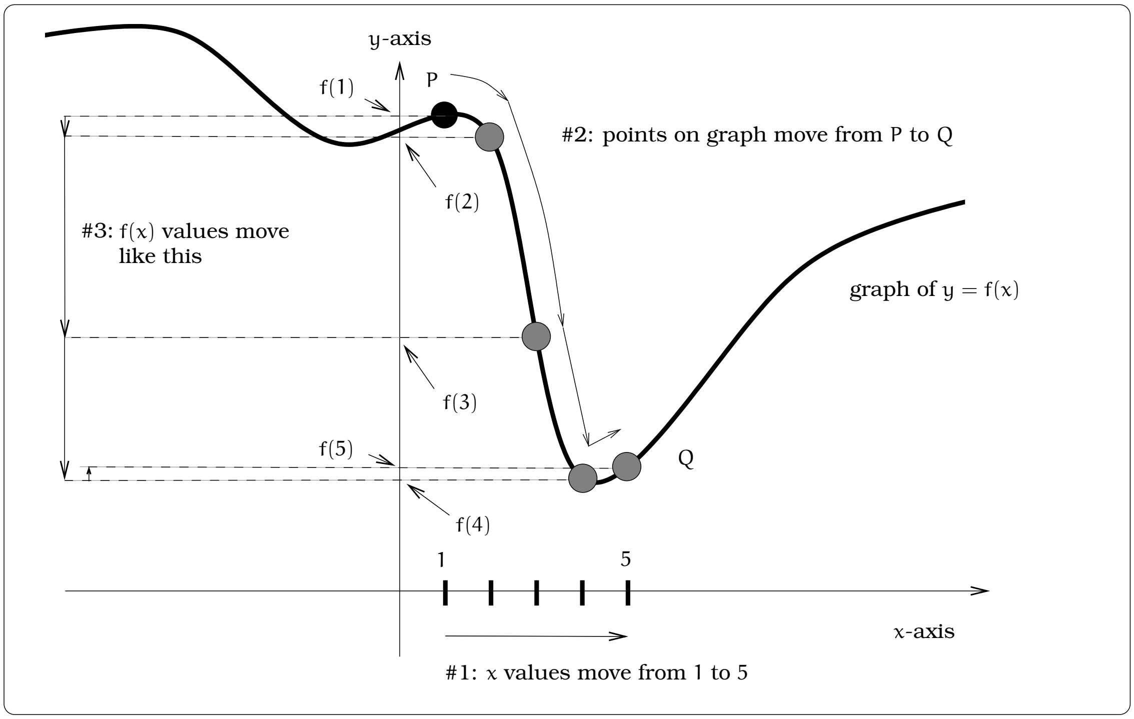

y-axis
f(1)
P
#2: points on graph move from P to Q
f(2)
#3: f(x) values move like this
graph of y = f(x)
f(3)
f(5)
Q
f(4)
1
5
x-axis
#1: x values move from 1 to 5

Figure 6.4: Dynamic interpretation of a graph.

6.1.3 Interpreting Intercepts of a Graph

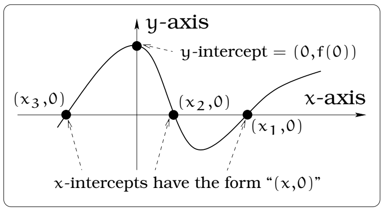
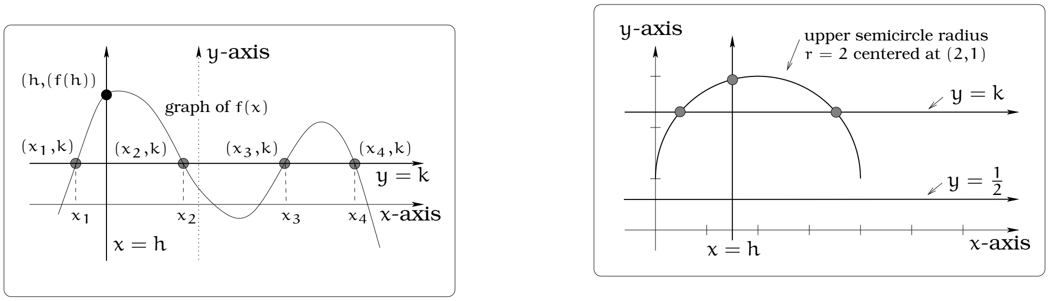

y-axis
y-intercept = (0,f(0))
(x₃,0)
(x₂,0)
x-axis
(x₁,0)
x-intercepts have the form "(x,0)"

Figure 6.5: Intercepts of a graph.

The places where a graph crosses the axes are often significant. We isolate each as an important feature to look for when doing graphical analysis. The graph of the function $y = f(x)$ crosses the y-axis at the point $(0,f(0))$; so, the y-intercept of the graph is just $f(0)$. The graph of the function $y = f(x)$ crosses the x-axis at points of the form $(x_0,f(x_0))$, where $f(x_0) = 0$. The values x_0 are called *roots* or *zeros* of the function $f(x)$. There can be at most one y-intercept, but there can be several x-intercepts or no x-intercept: See Figure 6.5

The graph of a function $y = f(x)$ crosses the vertical line $x = h$ at the point $(h,f(h))$. To find where the graph of a function $y = f(x)$ crosses the horizontal line $y = k$, first solve the equation $k = f(x)$ for x. If the equation $k = f(x)$ has solutions x_1, x_2, x_3, x_4, then the points of intersection would have the coordinates given.


(h,(f(h))
y-axis
graph of f(x)
(x₁,k)
(x₂,k)
(x₃,k)
(x₄,k)
y = k
x₁
x₂
x₃
x₄
x-axis
x = h
y-axis
upper semicircle radius r = 2 centered at (2,1)
y = k
y = 1/2
x = h
x-axis

(a) General curve.

(b) Semicircle.

Figure 6.6: Crossing horizontal and vertical lines.

As another example, the graph in Figure 6.6(b) above will cross the horizontal line $y = k$ twice if and only if $1 \leq k < 3$; the graph will cross the horizontal line $y = k$ once if and only if $k = 3$. The graph will not intersect the line $y = \frac{1}{2}$ and the graph will cross the vertical line $x = h$ if and only if $0 \leq h \leq 4$.

6.1.4 Interpreting Increasing and Decreasing

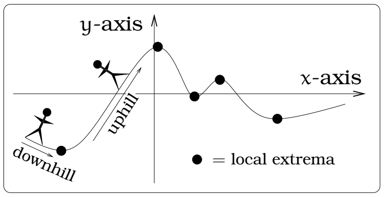

y-axis
x-axis
uphill
downhill
● = local extrema

Figure 6.7: Graphically interpreting increasing and decreasing.

We use certain terms to describe how the function values are changing over some domain of x values. Typically, we want to study what is happening to the values $f(x)$ as x moves from "left to right" in some interval. This can be linked graphically with the study of "uphill" and "downhill" portions of the function graph: If you were "walking to the right" along the graph, the function values are *increasing* if you are walking uphill. Likewise, if you were "walking to the right" along the graph, the function values are *decreasing* if you are walking downhill.

Once we understand where the graph is moving uphill and downhill, we can isolate the places where we change from moving uphill to downhill, or vice versa; these "peaks" and "valleys" are called *local maxima* and *local minima*. Some folks refer to either case as a *local extrema*. People have invested a lot of time (centuries!) and energy (lifetimes!) into the study of how to find local extrema for particular function graphs. We will see some basic examples in this course and others will surface in future courses once you have the tools of calculus at your disposal. Examples range from business applications that involve optimizing profit to understanding the three-dimensional shape a of biological molecule.

Example 6.1.3. *A hang glider launches from a gliderport in La Jolla. The launch point is located at the edge of a 500 ft. high cliff over the Pacific Ocean. The elevation of the pilot above the gliderport after* t *minutes is given by the graph in Figure 6.8:*

1. *When is the pilot climbing and descending?*
2. *When is the pilot at the glider port elevation?*
3. *How much time does the pilot spend flying level?*

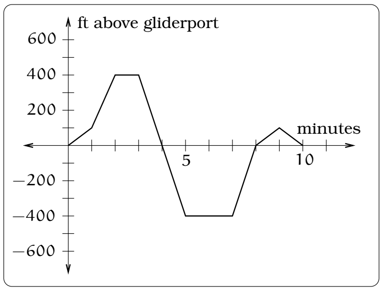


Figure 6.8: Hanglider elevations.

Solution.

1. Graphically, we need to determine the portions of the graph that are increasing or decreasing. In this example, it is increasing when $0 \leq t \leq 2$ and $7 \leq t \leq 9$. And, it is decreasing when $3 \leq t \leq 5$ and $9 \leq t \leq 10$.
2. Graphically, this question amounts to asking when the elevation is 0, which is the same as finding when the graph crosses the horizontal axis. We can read off there are four such times: $t = 0, 4, 8, 10$.
3. Graphically, we need to determine the portions of the graph that are made up of horizontal line segments. This happens when $2 \leq t \leq 3$ and $5 \leq t \leq 7$. So, our pilot flies level for a total of 3 minutes.

□

6.2 Circles and Semicircles

Back in Chapter 3, we discussed equations whose graphs were circles: We found that the graph of the equation

$$(x - h)^2 + (y - k)^2 = r^2 \tag{6.1}$$

is a circle of radius r centered at the point (h, k). It is possible to manipulate this equation and become confused. We could rewrite this as $(y - k)^2 = r^2 - (x - h)^2$, then take the square root of each side. However, the resulting equivalent equation would be

$$y = k \pm \sqrt{r^2 - (x-h)^2}$$

and the presence of that ± sign is tricky; it means we have two equations:

$$\begin{aligned} y &= k + \sqrt{r^2 - (x-h)^2} \quad \text{or} \\ y &= k - \sqrt{r^2 - (x-h)^2}. \end{aligned}$$

Each of these two equations defines a function:

$$f(x) = k + \sqrt{r^2 - (x-h)^2} \quad \text{or} \tag{6.2}$$

$$g(x) = k - \sqrt{r^2 - (x-h)^2}. \tag{6.3}$$

So, even though the Equation 6.1 is not a function, we were able to obtain *two* different functions $f(x)$ and $g(x)$ from the original equation. The relationship between the graph of the original equation and the graphs of the two functions in (6.2) and (6.3) is as follows: The upper semicircle is the graph of the function $f(x)$ and the lower semicircle is the graph of the function $g(x)$.

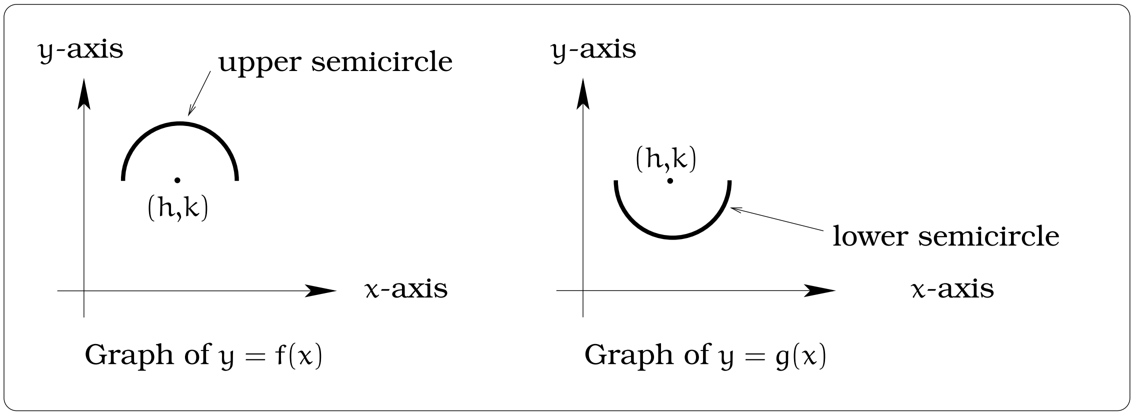
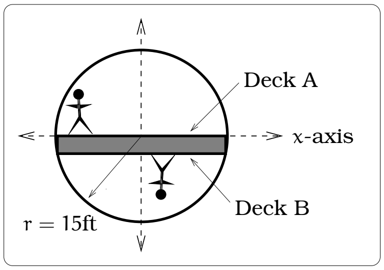


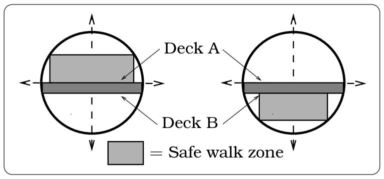
Figure 6.9: Upper and lower semicircles.

Example 6.2.1. *A tunnel connecting two portions of a space station has a circular cross-section of radius 15 feet. Two walkway decks are constructed in the tunnel. Deck A is along a horizontal diameter and another parallel Deck B is 2 feet below Deck A. Because the space station is in a weightless environment, you can walk vertically upright along Deck A, or vertically upside down along Deck B. You have been assigned to paint "safety stripes" on each deck level, so that a 6 foot person can safely walk upright along either deck. Determine the width of the "safe walk zone" on each deck.*

Solution. Impose a coordinate system so that the origin is at the center of the circular cross section of the tunnel; by symmetry the walkway is centered about the origin. With this coordinate system, the graph of the equation $x^2+y^2 = 15^2 = 225$ will be the circular cross-section of the tunnel. In the case of Deck A, we basically need to determine how close to each edge of the tunnel a 6 foot high person can stand without hitting his or her head on the tunnel; a similar remark applies to Deck B. This means we are really trying to fit two six-foot-high rectangular safe walk zones into the picture:



(a) Cross-section of tunnel.



(b) Walk zones.

Figure 6.10: Space station tunnels.

Our job is to find the coordinates of the four points P, Q, R, and S. Let's denote by x_1, x_2, x_3,and x_4 the x-coordinates of these four points, then $P = (x_1, 6)$, $Q = (x_2, 6)$, $R = (x_3, -8)$, and $S = (x_4, -8)$. To find x_1, x_2, x_3, and x_4, we need to find the intersection of the circle in Figure 6.10(b) with two horizontal lines:

- Intersecting the the upper semicircle with the horizontal line having equation $y = 6$ will determine x_1 and x_2; the upper semicircle is the graph of $f(x) = \sqrt{225 - x^2}$.
- Intersecting the lower semicircle with the horizontal line having equation $y = -8$ will determine x_3 and x_4; the lower semicircle is the graph of $g(x) = -\sqrt{225 - x^2}$.

For Deck A, we simultaneously solve the system of equations

$$\left\{ \begin{array}{ll} y & = \sqrt{225 - x^2} \\ y & = 6 \end{array} \right\}.$$

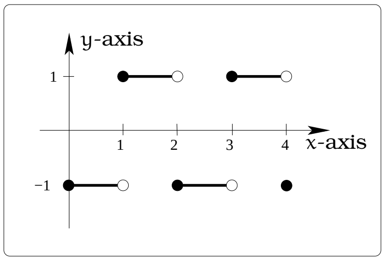
Plugging in $y = 6$ into the first equation of the system gives $x^2 = 225 - 6^2 = 189$; i.e., $x = \pm\sqrt{189} = \pm 13.75$. This tells us that $P = (-13.75, 6)$ and $Q = (13.75, 6)$. In a similar way, for Deck B, we find $R = (-12.69, -8)$ and $S = (12.69, -8)$.

In the case of Deck A, we would paint a safety stripe 13.75 feet to the right and left of the centerline. In the case of Deck B, we would paint a safety stripe 12.69 feet to the right and left of the centerline. ☐

6.3 Multipart Functions

So far, in all of our examples we have been able to write $f(x)$ as a nice compact expression in the variable x. Sometimes we have to work harder.

As an example of what we have in mind, consider the graph in Figure 6.11(a):



(a) Graphing a multipart function.

$$f(x) = \begin{cases} -1 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x < 2 \\ -1 & \text{if } 2 \leq x < 3 \\ 1 & \text{if } 3 \leq x < 4 \\ -1 & \text{if } x = 4 \end{cases}$$

(b) Writing a multipart function.

Figure 6.11: A multipart function.

The curve we are trying to describe in this picture is made up of five pieces; four little line segments and a single point. The first thing to notice is that on the domain $0 \leq x \leq 4$, this curve will define the graph of some function $f(x)$. To see why this is true, imagine a vertical line moving from left to right within the domain $0 \leq x \leq 4$ on the x-axis; any one of these vertical lines will intersect the curve exactly once, so by the *vertical line test*, the curve must be the graph of a function. Mathematicians use the shorthand notation above to describe this function. Notice how the rule for $f(x)$ involves five cases; each of these cases corresponds to one of the five pieces that make up the curve. Finally, notice the care with the "open" and "closed" circles is really needed if we want to make sure the curve defines a function; in terms of the rule, these open and closed circles translate into strict inequalities like $<$ or weak inequalities like $\leq$. This is an example of what we call a *multipart function*.

The symbolic appearance of multipart functions can be somewhat frightening. The key point is that the graph (and rule) of the function will be broken up into a number of separate cases. To study the graph or rule, we simply "home in" on the appropriate case. For example, in the above illustration, suppose we wanted to compute $f(3.56)$. First, we would find which of the five cases covers $x = 3.56$, then apply that part of the rule to compute $f(3.56) = 1$.

Our first multipart function example illustrated how to go from a graph in the plane to a rule for $f(x)$; we can reverse this process and go from the rule to the graph.

Example 6.3.1. *Sketch the graph of the multipart function*

$$g(x) = \begin{cases} 1 & \textit{if } x \leq -1 \\ 1 + \sqrt{1 - x^2} & \textit{if } -1 \leq x \leq 1 \\ 1 & \textit{if } x \geq 1 \end{cases}$$

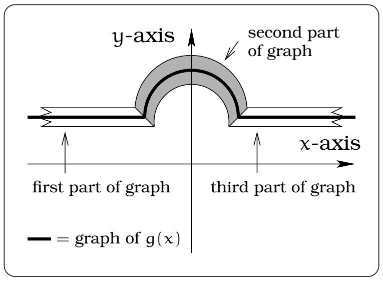


Figure 6.12: Multipart function $g(x)$.

Solution. The graph of $g(x)$ will consist of three pieces. The first case consists of the graph of the function $y = g(x) = 1$ on the domain $x \leq -1$, this consists of all points on the horizontal line $y = 1$ to the left of and including the point $(-1,1)$. We have "lassoed" this portion of the graph in Figure 6.12. Likewise, the third case in the definition yields the graph of the function $y = g(x) = 1$ on the domain $x \geq 1$; this is just all points on the horizontal line $y = 1$ to the right of and including the point $(1, 1)$. Finally, we need to analyze the middle case, which means we need to look at the graph of $1 + \sqrt{1 - x^2}$ on the domain $-1 \leq x \leq 1$. This is just the upper semicircle of the circle of radius 1 centered at $(0,1)$. If we paste these three pieces together, we arrive at the graph of $g(x)$. □

Example 6.3.2. *You are dribbling a basketball and the function* $s = h(t)$ *keeps track of the height of the ball's center above the floor after* t *seconds. Sketch a reasonable graph of* $s = h(t)$.

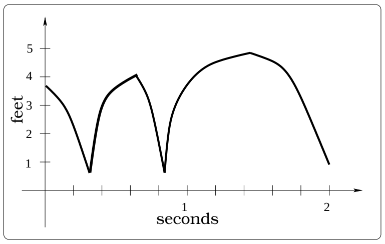


Figure 6.13: Dribbling.

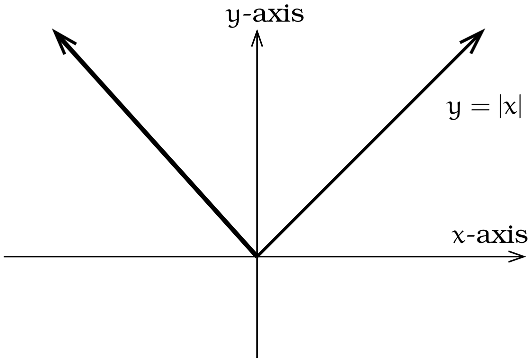
Solution. If we take the domain to be $0 \leq t \leq 2$ (the first 2 seconds), a reasonable graph might look like Figure 6.13. This is a multipart function. Three portions of the graph are decreasing and two portions are increasing. Why doesn't the graph touch the t axis? □

6.4 Exercises

Problem 6.1. The *absolute value* function is defined by the multipart rule:

$$|x| = \begin{cases} x & \text{if } 0 \leq x \\ -x & \text{if } x < 0 \end{cases}$$

The graph of the absolute value function is pictured below:



(a) Calculate: $|0|$, $|2|$, $|-3|$.

(b) Solve for x: $|x| = 4$; $|x| = 0$, $|x| = -1$.

(c) Sketch the graph of $y = \frac{1}{2}x+2$ and $y = |x|$ in the same coordinate system. Find where the two graphs intersect, label the coordinates of these point(s), then find the area of the region bounded by the two graphs.

Problem 6.2. For each of the following functions, graph $f(x)$ and $g(x) = |f(x)|$, and give the multipart rule for $g(x)$.

(a) $f(x) = -0.5x - 1$

(b) $f(x) = 2x - 5$

(c) $f(x) = x + 3$

Problem 6.3. Solve each of the following equations for x.

(a) $g(x) = 17$, where $g(x) = |3x + 5|$

(b) $f(x) = 1.5$ where

$$f(x) = \begin{cases} 2x & \text{if } x < 3, \\ 4 - x & \text{if } x \geq 3. \end{cases}$$

(c) $h(x) = -1$ where

$$h(x) = \begin{cases} -8 - 4x & \text{if } x \leq -2, \\ 1 + \frac{1}{3}x & \text{if } x > -2. \end{cases}$$

Problem 6.4. (a) Let $f(x) = x + |2x - 1|$. Find all solutions to the equation

$$f(x) = 8.$$

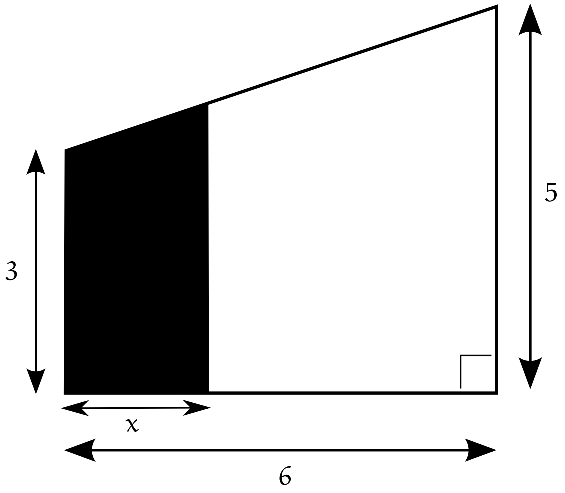
(b) Let $g(x) = 3x - 3 + |x + 5|$. Find all values of a which satisfy the equation

$$g(a) = 2a + 8.$$

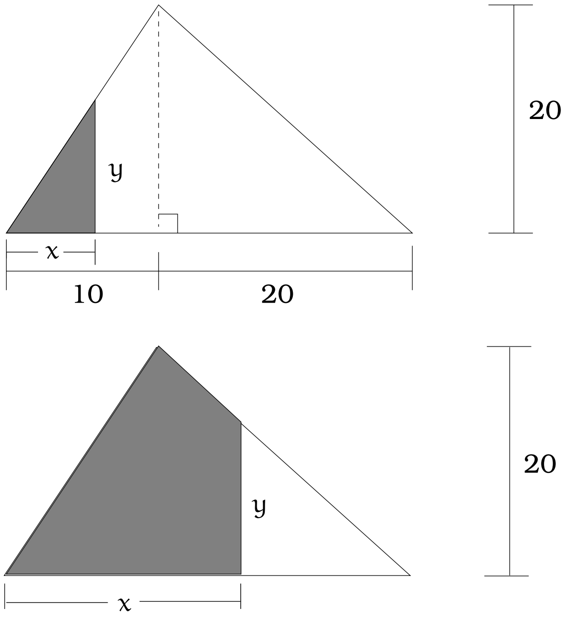
(c) Let $h(x) = |x| - 3x + 4$. Find all solutions to the equation

$$h(x - 1) = x - 2.$$

Problem 6.5. Express the area of the shaded region below as a function of x. The dimensions in the figure are centimeters.



Problem 6.6. Pizzeria Buonapetito makes a triangular-shaped pizza with base width of 30 inches and height 20 inches as shown. Alice wants only a portion of the pizza and does so by making a vertical cut through the pizza and taking the shaded portion. Letting x be the bottom length of Alice's portion and y be the length of the cut as shown, answer the following questions:



(a) Find a formula for y as a multipart function of x, for $0 \leq x \leq 30$. Sketch the graph of this function and calculate the range.

(b) Find a formula for the area of Alice's portion as a multipart function of x, for $0 \leq x \leq 30$.

(c) If Alice wants her portion to have half the area of the pizza, where should she make the cut?

Problem 6.7. This problem deals with cars traveling between Bellevue and Spokane, which are 280 miles apart. Let t be the time in hours, measured from 12:00 noon; for example, $t = -1$ is 11:00 am.

(a) Joan drives from Bellevue to Spokane at a constant speed, departing from Bellevue at 11:00 am and arriving in Spokane at 3:30 pm. Find a function $j(t)$ that computes her distance from Bellevue at time t. Sketch the graph, specify the domain and determine the range.

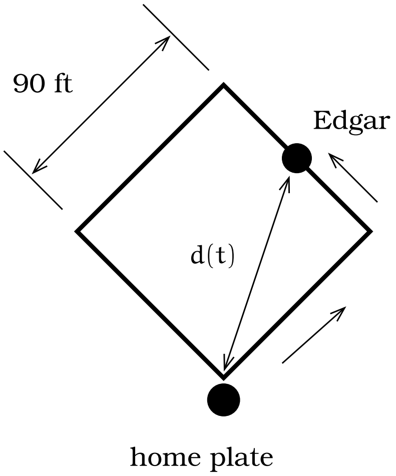
(b) Steve drives from Spokane to Bellevue at 70 mph, departing from Spokane at 12:00 noon. Find a function $s(t)$ for his distance from Bellevue at time t. Sketch the graph, specify the domain and determine the range.

(c) Find a function $d(t)$ that computes the distance between Joan and Steve at time t.

Problem 6.8. Arthur is going for a run. From his starting point, he runs due east at 10 feet per second for 250 feet. He then turns, and runs north at 12 feet per second for 400 feet. He then turns, and runs west at 9 feet per second for 90 feet.

Express the (straight-line) distance from Arthur to his starting point as a function of t, the number of seconds since he started.

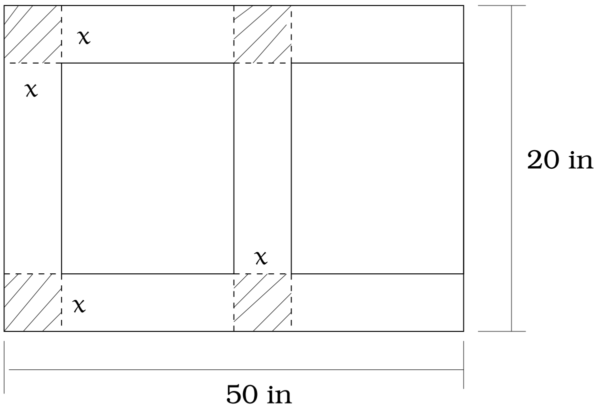
Problem 6.9. A baseball diamond is a square with sides of length 90 ft. Assume Edgar hits a home run and races around the bases (counterclockwise) at a speed of 18 ft/sec. Express the distance between Edgar and home plate as a function of time t. (Hint: This will be a multipart function.) Try to sketch a graph of this function.



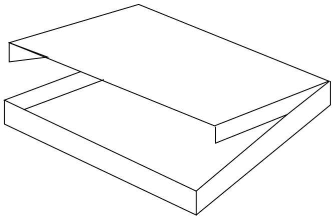
Problem 6.10. *Pagliacci Pizza* has designed a cardboard delivery box from a single piece of cardboard, as pictured.

(a) Find a polynomial function $v(x)$ that computes the volume of the box in terms of x. What is the degree of v?

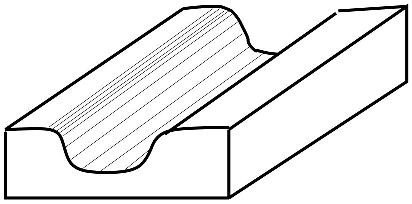
(b) Find a polynomial function $a(x)$ that computes the **exposed** surface area of the closed box in terms of x. What is the degree of a? What are the explicit dimensions if the **exposed** surface area of the closed box is 600 sq. inches?



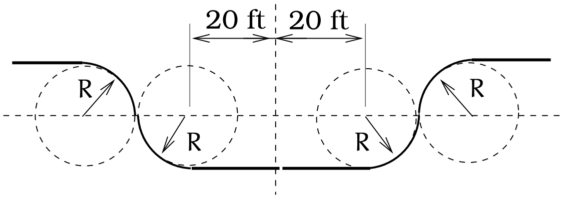
remove shaded squares and fold to get:



Problem 6.11. The vertical cross-section of a drainage ditch is pictured below:



3D−view of ditch



vertical cross-section

Here, R indicates a circle of radius 10 feet and all of the indicated circle centers lie along the common horizontal line 10 feet above and parallel to the ditch bottom. Assume that water is flowing into the ditch so that the level above the bottom is rising 2 inches per minute.

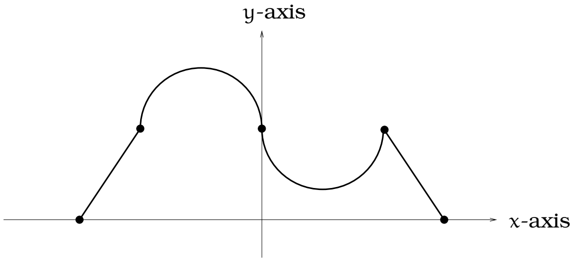
(a) When will the ditch be completely full?

(b) Find a multipart function that models the vertical cross-section of the ditch.

(c) What is the width of the filled portion of the ditch after 1 hour and 18 minutes?

(d) When will the filled portion of the ditch be 42 feet wide? 50 feet wide? 73 feet wide?

Problem 6.12. The graph of a function $y = g(x)$ on the domain $-6 \leq x \leq 6$ consistes of line segments and semicircles of radius 2 connecting the points $(-6,0)$,$(-4,4)$, $(0,4)$, $(4,4)$, $(6,0)$.



(a) What is the range of g?

(b) Where is the function increasing? Where is the function decreasing?

(c) Find the multipart formula for $y = g(x)$.

(d) If we restrict the function to the smaller domain $-5 \leq x \leq 0$, what is the range?

(e) If we restrict the function to the smaller domain $0 \leq x \leq 4$, what is the range?

Problem 6.13. (a) Simply as far as possible

$$\frac{1}{1+\frac{1}{a}} - \frac{a}{a+1}.$$

(b) Find a, b, c that simultaneously satisfy these three equations:

$$\begin{aligned} a+b-c &= 5 \\ 2a-3b+c &= 4 \\ a+b+c &= -1 \end{aligned}$$

Chapter 7

Quadratic Modeling

If you kick a ball through the air enough times, you will find its path tends to be parabolic. Before we can answer any detailed questions about this situation, we need to get our hands on a precise mathematical model for a parabolic shaped curve. This means we seek a function $y = f(x)$ whose graph reproduces the path of the ball.

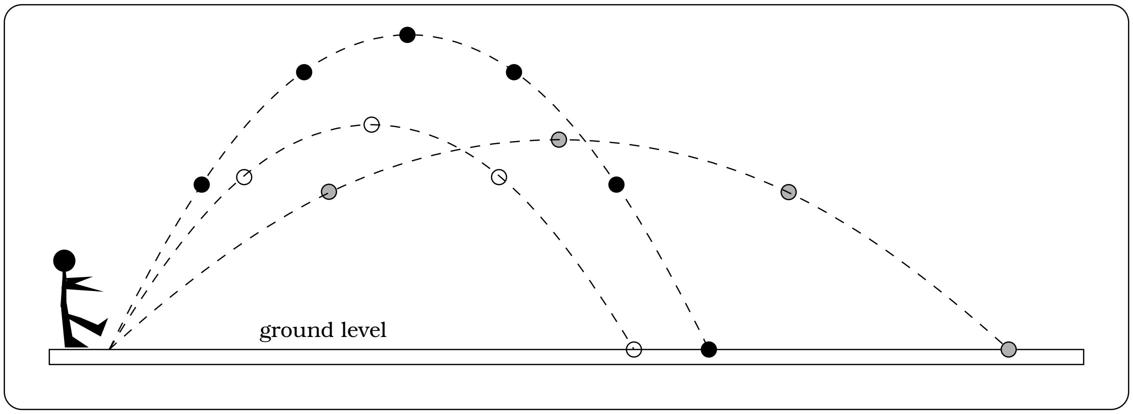


Figure 7.1: Possible paths for a kicked ball are parabolic.

7.1 Parabolas and Vertex Form

OK, suppose we sit down with an xy-coordinate system and draw four random parabolas; let's label them I, II, III, and IV: See Figure 7.2. The relationship between these parabolas and the fixed coordinate system can vary quite a bit: The key distinction between these four curves is that only I and IV are the graphs of functions; this follows from the *vertical line test*. A parabola that is the graph of a function is called a *standard parabola*. We can see that any standard parabola has three basic features:

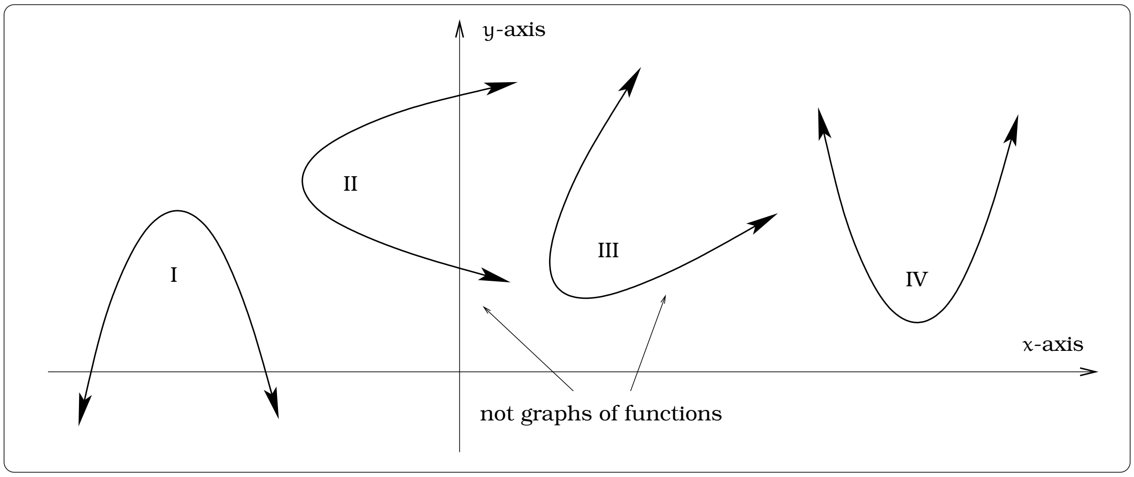


Figure 7.2: Relationship between a fixed coordinate system and various parabolas.

- the parabola will either open "upward" or "downward";
- the graph will have either a "highest point" or "lowest point," called the *vertex*;
- the parabola will be symmetric about some vertical line called the *axis* of symmetry.

Our first task is to describe the mathematical model for any standard parabola. In other words, what kind of function equations $y = f(x)$ give us standard parabolas as their graphs? Our approach is geometric and visual:

- Begin with one specific example, then show every other standard parabola can be obtained from it via some specific geometric maneuvers.
- As we perform these geometric maneuvers, we keep track of how the function equation for the curve is changing.

This discussion will amount to a concrete application of a more general set of tools developed in the following section of this chapter.

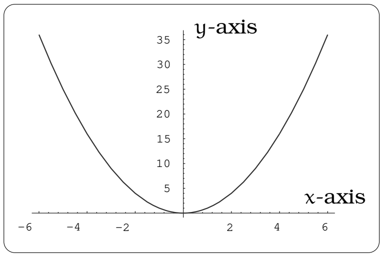


Figure 7.3: Graph of $y = x^2$.

Using a graphing device, it is an easy matter to plot the graph of $y = x^2$ and see we are getting the parabola pictured in Figure 7.3. The basic idea is to describe how we can manipulate this graph and obtain any standard parabola. In the end, we will see that standard parabolas are obtained as the graphs of functions having the form

$$y = ax^2 + bx + c,$$

for various constants a, b, and c, with $a \neq 0$. A function of this type is called a *quadratic function* and these play a central role throughout the course. We will divide our task into two steps.

First we show every standard parabola arises as the graph of a function having the form

$$y = a(x-h)^2 + k,$$

for some constants a, h, and k, with $a \neq 0$. This is called the *vertex form* of a quadratic function. Notice, if we were to algebraically expand out this equation, we could rewrite it in the $y = ax^2 + bx + c$ form. For example, suppose we start with the vertex form $y = 2(x-1)^2 + 3$, so that $a = 2, h = 1, k = 3$. Then we can rewrite the equation in the form $y = ax^2 + bx + c$ as follows:

$$2(x-1)^2 + 3 = 2(x^2 - 2x + 1) + 3 = 2x^2 - 4x + 5,$$

so $a = 2, b = -4, c = 5$. The second step is to show any quadratic function can be written in vertex form; the underlying algebraic technique used here is called *completing the square*. This is a bit more involved. For example, if you are simply handed the quadratic function $y = -3x^2+6x-1$, it not at all obvious why the vertex form is obtained by this equality:

$$-3x^2 + 6x - 1 = -3(x-1)^2 + 2.$$

The reason behind this equality is the technique of completing the square. In the end, we will almost always be interested in the vertex form of a quadratic. This is because a great deal of qualitative information about the parabolic graph can simply be "read off" from this form.

7.1.1 First Maneuver: Shifting

Suppose we start with the graph in Figure 7.3 and horizontally shift it h units to the right. To be specific, consider the two cases $h = 2$ and $h = 4$. To visualize this, imagine making a wire model of the graph, set in on top of the curve, then slide the wire model h units to the right. What you will obtain are the two "dashed curves" in Figure 7.4. We will call the process just described a *horizontal shift*. Since the "dashed curves" are no longer the original parabola in Figure 7.3, the corresponding function equations must have changed.

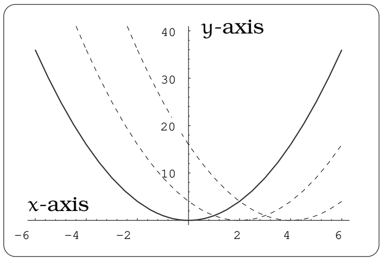


Figure 7.4: Shift to the right.

Using a graphing device, you can check that the corresponding equations for the dashed graphs would be

$$\begin{aligned} y &= (x-2)^2 \\ &= x^2 - 4x + 4, \end{aligned}$$

which is the plot with lowest point $(2,0)$ and

$$\begin{aligned} y &= (x-4)^2 \\ &= x^2-8x+16, \end{aligned}$$

which is the plot with lowest point $(4,0)$. In general, if h is positive, the graph of the function $y=(x-h)^2$ is the parabola obtained by shifting the graph of $y=x^2$ by h units to the right.

Next, if h is negative, shifting h units to the right is the same as shifting $|h|$ units left! On the domain $-6 \leq x \leq 6$, Figure 7.5 indicates this for the cases $h=-2,-4$, using "dashed curves" for the shifted graphs and a solid line for the graph of $y=x^2$. Using a graphing device, we can check that the corresponding equations for the dashed graphs would be

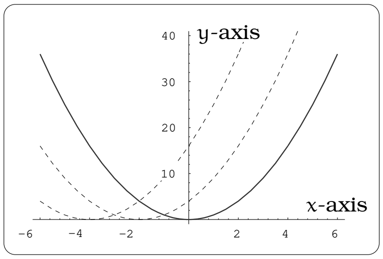


Figure 7.5: Shift to the left.

$$\begin{aligned} y &= (x-(-2))^2 \\ &= (x+2)^2 \\ &= x^2+4x+4, \end{aligned}$$

which is the plot with lowest point $(-2,0)$ and

$$\begin{aligned} y &= (x-(-4))^2 \\ &= (x+4)^2 \\ &= x^2+8x+16, \end{aligned}$$

which is the plot with lowest point $(-4,0)$. In general, if h is negative, the graph of the function $y=(x-h)^2$ gives the parabola obtained by shifting the graph of $y=x^2$ by $|h|$ units to the LEFT.

The conclusion thus far is this: Begin with the graph of $y=x^2$ in Figure 7.3. Horizontally shifting this graph h units to the right gives a new (standard) parabola whose equation is $y=(x-h)^2$.

We can also imagine *vertically shifting* the graph in Figure 7.3. This amounts to moving the graph k units vertically upward. It turns out that this vertically shifted graph corresponds to the graph of the function $y=x^2+k$. We can work out a few special cases and use a graphing device to illustrate what all this really means.

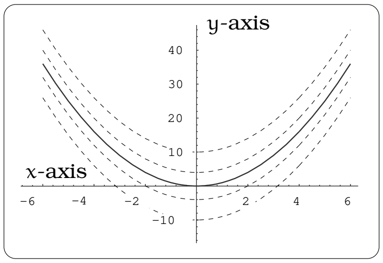


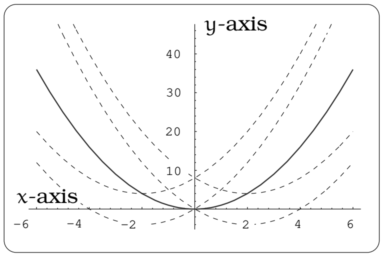
Figure 7.6: Vertical shifts.

Figure 7.6 illustrates the graphs of $y=x^2+k$ in the cases when $k=4$, 10 and $k=-4,-10$, leading to vertically shifted graphs. Positive values of k lead to the upper two "dashed curves" and negative values of k lead to the lower two "dashed curves"; the plot of $y=x^2$ is again the solid line. The equations giving these graphs would be $y=x^2-10$, $y=x^2-4$, $y=x^2+4$ and $y=x^2+10$, from bottom to top dashed plot.

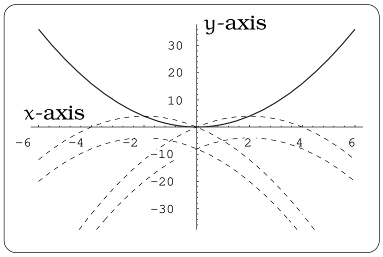
If we combine horizontal and vertical shifting, we end up with the graphs of functions of the form $y=(x-h)^2+k$. Figure 7.7(a) illustrates the four cases with corresponding equations $y = (x \pm 2)^2 \pm 4$; as an exercise, identify which equation goes with each curve.

7.1.2 Second Maneuver: Reflection

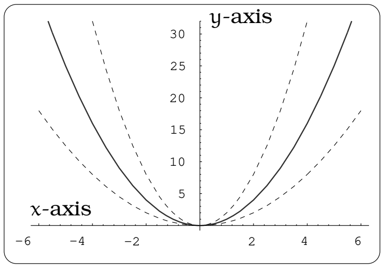
Next, we can reflect any of the curves $y = p(x)$ obtained by horizontal or vertical shifting across the x-axis. This procedure will produce a new curve which is the graph of the new function $y = -p(x)$. For example, begin with the four dashed curves in the previous figure. Here are the reflected parabolas and their equations are $y = -(x \pm 2)^2 \pm 4$: See Figure 7.7(b).



(a) Combined shifts.



(b) Reflections.



(c) Vertical dilations.

Figure 7.7: Shifts, reflections, and dilations.

7.1.3 Third Maneuver: Vertical Dilation

If a is a positive number, the graph of $y = ax^2$ is usually called a *vertical dilation* of the graph of $y = x^2$. There are two cases to distinguish here:

- If $a > 1$, we have a vertically expanded graph.
- If $0 < a < 1$, we have a vertically compressed graph.

This is illustrated for $a = 2$ (upper dashed plot) and $a = 1/2$ (lower dashed plot): See Figure 7.7(c).

7.1.4 Conclusion

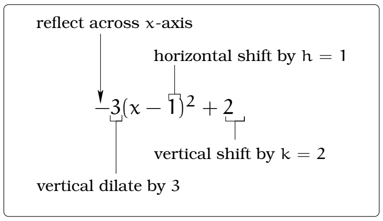
Starting with $y = x^2$ in Figure 7.3, we can combine together all three of the operations: shifting, reflection and dilation. This will lead to the graphs of functions that have the form:

$$y = a(x - h)^2 + k,$$

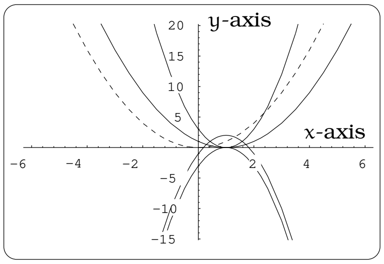
for some a, h and k, $a \neq 0$. If you think about it for awhile, it seems pretty easy to believe that any standard parabola arises from the one in Figure 7.3 using our three geometric maneuvers. In other words, what we have shown is that any standard parabola is the graph of a quadratic equation in vertex form. Let's summarize.

Important Fact 7.1.1. *A standard parabola is the graph of a function* $y = f(x) = a(x - h)^2 + k$, *for some constants* a, h, *and* k *and* $a \neq 0$. *The vertex of the parabola is* (h, k) *and the axis of symmetry is the line* $x = h$. *If* $a > 0$, *then the parabola opens upward; if* $a < 0$, *then the parabola opens downward.*

Example 7.1.2. *Describe a sequence of geometric operations leading from the graph of* $y = x^2$ *to the graph of* $y = f(x) = -3(x-1)^2 + 2$.



(a) What do the symbols of an equation mean?



(b) What does the equation look like?

Figure 7.8: Interpreting an equation.

Solution. To begin with, we can make some initial conclusions about the specific shifts, reflections and dilations involved, based on looking at the vertex form of the equation. In addition, by Fact 7.1.1, we know that the vertex of the graph of $y = f(x)$ is $(1, 2)$, the line $x = 1$ is a vertical axis of symmetry and the parabola opens downward.

We need to be a little careful about the order in which we apply the four operations highlighted. We will illustrate a procedure that works. The full explanation for the success of our procedure involves *function compositions* and we will return to that at the end of Chapter 8. The order in which we will apply our geometric maneuvers is as follows:

$$\begin{aligned} \textbf{horizontal shift} &\Rightarrow \textbf{vertical dilate} \\ &\Rightarrow \textbf{reflect} \\ &\Rightarrow \textbf{vertical shift} \end{aligned}$$

Figure 7.8(b) illustrates the four curves obtained by applying these successive steps, in this order. As a reference, we include the graph of $y = x^2$ as a "dashed curve":

- A horizontal shift by $h = 1$ yields the graph of $y = (x-1)^2$; this is the fat parabola opening upward with vertex $(1, 0)$.
- A dilation by $a = 3$ yields the graph of $y = 3(x-1)^2$; this is the skinny parabola opening upward with vertex $(1, 0)$.
- A reflection yields the graph of $y = -3(x-1)^2$; this is the downward opening parabola with vertex $(1, 0)$.
- A vertical shift by $k = 2$ yields the graph of $y = -3(x-1)^2 + 2$; this is the downward opening parabola with vertex $(1, 2)$.

□

7.2 Completing the Square

By now it is pretty clear we can say a lot about the graph of a quadratic function which is in vertex form. We need a procedure for rewriting a given quadratic function in vertex form. Let's first look at an example.

Example 7.2.1. *Find the vertex form of the quadratic function* $y = -3x^2 + 6x - 1$.

Solution. Since our goal is to put the function in vertex form, we can write down what this means, then try to solve for the unknown constants. Our first step would be to write

$$-3x^2 + 6x - 1 = a(x - h)^2 + k,$$

for some constants a, h, k. Now, expand the right hand side of this equation and factor out coefficients of x and x^2:

$$\begin{aligned} -3x^2 + 6x - 1 &= a(x - h)^2 + k \\ -3x^2 + 6x - 1 &= a(x^2 - 2xh + h^2) + k \\ -3x^2 + 6x - 1 &= ax^2 - 2xah + ah^2 + k \\ (-3)x^2 + (6)x + (-1) &= (a)x^2 + (-2ah)x + (ah^2 + k). \end{aligned}$$

If this is an equation, then it must be the case that the coefficients of like powers of x match up on the two sides of the equation in Figure 7.9. Now we have three equations and three unknowns (the a, h, k) and we can proceed to solve for these:

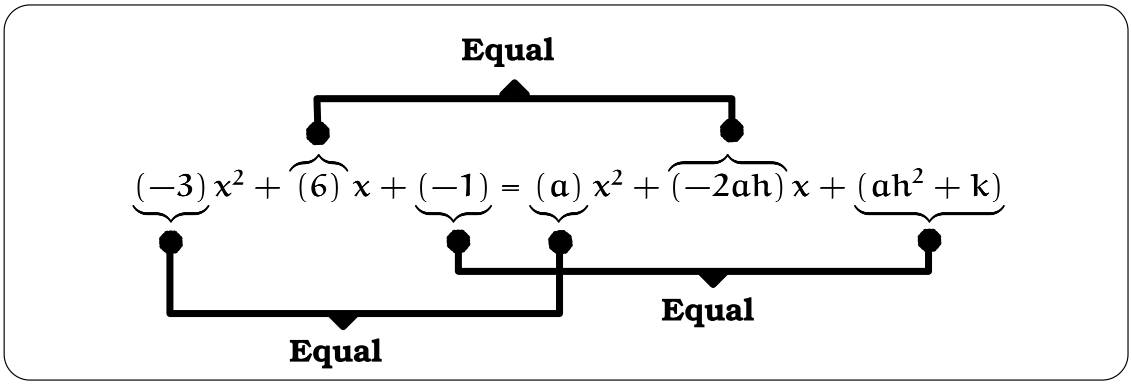

Equal
$\underbrace{(-3)}x^2 + \overbrace{(6)}x + \underbrace{(-1)} = \underbrace{(a)}x^2 + \overbrace{(-2ah)}x + \underbrace{(ah^2 + k)}$
Equal
Equal

Figure 7.9: Balancing the coefficients.

$$\begin{aligned} -3 &= a \\ 6 &= -2ah \\ -1 &= ah^2 + k \end{aligned}$$

The first equation just hands us the value of $a = -3$. Next, we can plug this value of a into the second equation, giving us

$$\begin{aligned} 6 &= -2ah \\ &= -2(-3)h \\ &= 6h, \end{aligned}$$

so $h = 1$. Finally, plug the now known values of a and h into the third equation:

$$\begin{aligned} -1 &= ah^2 + k \\ &= -3(1^2) + k \\ &= -3 + k, \end{aligned}$$

so $k = 2$. Our conclusion is then

$$-3x^2 + 6x - 1 = -3(x-1)^2 + 2.$$

Notice, this is the quadratic we studied in Example 7.1.2 on page 90. □

The procedure used in the preceding example will always work to rewrite a quadratic function in vertex form. We refer to this as *completing the square.*

Example 7.2.2. *Describe the relationship between the graphs of* $y = x^2$ *and* $y = f(x) = -4x^2 + 5x + 2$.

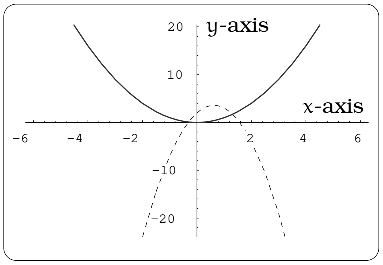


Figure 7.10: Maneuvering $y = x^2$.

Solution. We will go through the algebra to complete the square, then interpret what this all means in terms of graphical maneuvers. We have

$$\begin{aligned} -4x^2 + 5x + 2 &= a(x-h)^2 + k \\ (-4)x^2 + 5x + 2 &= ax^2 + (-2ah)x + (ah^2 + k). \end{aligned}$$

This gives us three equations:

$$\begin{aligned} -4 &= a \\ 5 &= -2ah \\ 2 &= ah^2 + k. \end{aligned}$$

We conclude that $a = -4, h = \frac{5}{8} = 0.625$ and $k = \frac{57}{16} = 3.562$. So, this tells us that we can obtain the graph of $y = f(x)$ from that of $y = x^2$ by these steps:

- Horizontally shifting by $h = 0.625$ units gives $y = (x - 0.625)^2$.
- Vertically dilate by the factor $a = 4$ gives $y = 4(x - 0.625)^2$.
- Reflecting across the x-axis gives $y = -4(x - 0.625)^2$.
- Vertically shifting by $k = 3.562$ gives $y = f(x) = -4(x - 0.625)^2 + 3.562$.

□

Example 7.2.3. *A drainage canal has a cross-section in the shape of a parabola. Suppose that the canal is 10 feet deep and 20 feet wide at the top. If the water depth in the ditch is 5 feet, how wide is the surface of the water in the ditch?*

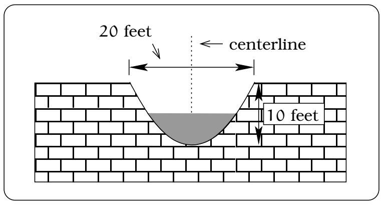


Figure 7.11: A drainage canal.

Solution. Impose an xy-coordinate system so that the parabolic cross-section of the canal is symmetric about the y-axis and its vertex is the origin. The vertex form of any such parabola is $y = f(x) = ax^2$, for some $a > 0$; this is because $(h,k) = (0,0)$ is the vertex and the parabola opens upward! The dimension information given tells us that the points $(10,10)$ and $(-10,10)$ are on the graph of $f(x)$. Plugging into the expression for f, we conclude that $10 = 100a$, so $a = 0.1$ and $f(x) = (0.1)x^2$. Finally, if the water is 5 feet deep, we must solve the equation: $5 = (0.1)x^2$, leading to $x = \pm\sqrt{50} = \pm 7.07$. Conclude the surface of the water is 14.14 feet wide when the water is 5 feet deep. □

7.3 Interpreting the Vertex

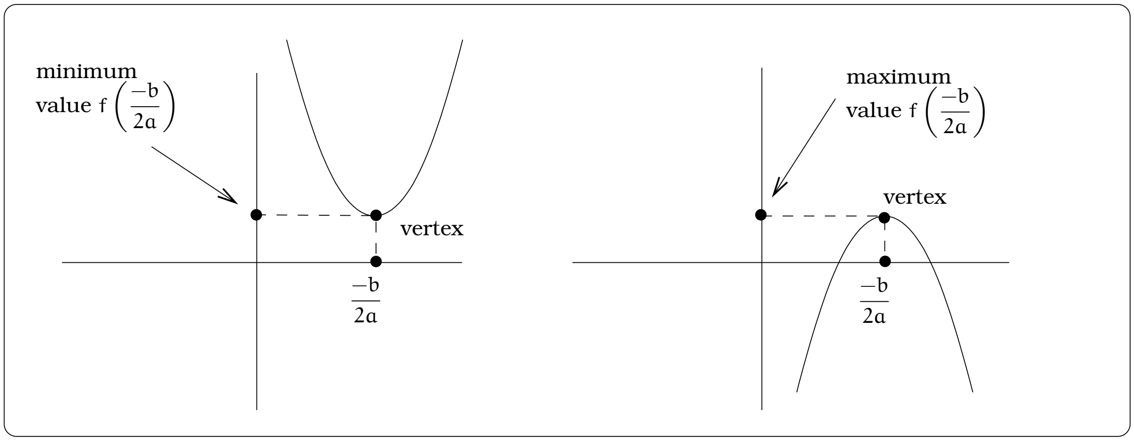

minimum value $f\left(\frac{-b}{2a}\right)$
vertex
$\frac{-b}{2a}$
maximum value $f\left(\frac{-b}{2a}\right)$
vertex
$\frac{-b}{2a}$

Figure 7.12: The vertex as the extremum of the quadratic function.

If we begin with a quadratic function $y = f(x) = ax^2+bx+c$, we know the graph will be a parabola. Graphically, the vertex will correspond to either the "highest point" or "lowest point" on the graph. If $a > 0$, the vertex is the lowest point on the graph; if $a < 0$, the vertex is the highest point on the graph. The maximum or minimum value of the function is the second coordinate of the vertex and the value of the variable x for which this extreme value is achieved is the first coordinate of the vertex. As we know, it is easy to read off the vertex coordinates when a quadratic function is written in vertex form. If instead we are given a quadratic function $y = ax^2 + bx + c$, we can use the technique of completing the square and arrive at a formula for the coordinates of the vertex in terms of a, b, and c. We summarize this below and label the two situations (upward or downward opening parabola) in the Figure 7.12. Keep in mind, it is always possible to obtain this formula by simply completing the square.

Important Fact 7.3.1. *In applications involving a quadratic function*

$$f(x) = ax^2 + bx + c,$$

the vertex has coordinates $P = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$. *The second coordinate of the vertex will detect the* ***maximum*** *or* ***minimum*** *value of* $f(x)$*; this is often a key step in problem solving.*

Example 7.3.2. *Discuss the graph of the quadratic function* $y = f(x) = -2x^2 + 11x - 4$.

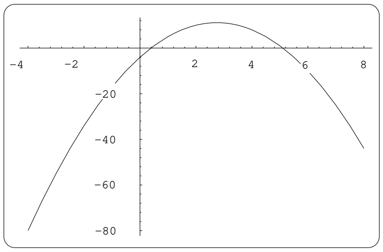


Figure 7.13: Sketching $y = f(x)$.

Solution. We need to place the equation $y = f(x)$ in vertex form. We can simply compute $a = -2$, $h = \frac{-b}{2a} = \frac{11}{4}$ and $k = f(\frac{11}{4}) = \frac{89}{8}$, using Fact 7.3.1:

$$\begin{aligned} f(x) &= -2x^2 + 11x - 4 \\ &= -2\left(x - \left(\frac{11}{4}\right)\right)^2 + \frac{89}{8}. \end{aligned}$$

This means that the graph of $f(x)$ is a parabola opening downward with vertex $\left(\frac{11}{4}, \frac{89}{8}\right)$ and axis $x = \frac{11}{4}$; see Figure 7.13. □

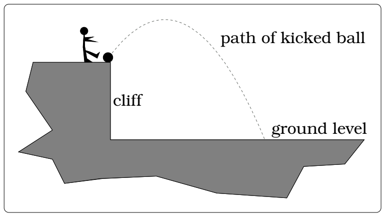
7.4 Quadratic Modeling Problems

The real importance of quadratic functions stems from the connection with motion problems. Imagine one of the three kicked ball scenarios in Figure 7.1 and impose a coordinate system with the kicker located at the origin. We can study the motion of the ball in two ways:

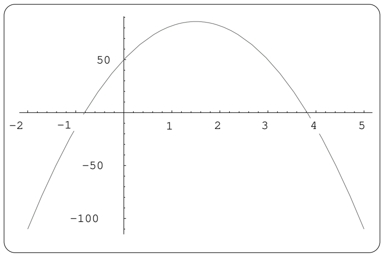
- Regard time t as the important variable and try to find a function $y(t)$ which describes the height of the ball t seconds after the ball is kicked; this would just be the y-coordinate of the ball at time t. The function $y(t)$ is a quadratic function. If we had this function in hand, we could determine **when** the ball hits the ground by solving the equation $0 = y(t)$, but we would not be able to determine **where** the ball hits the ground.

- A second approach is to forget about the time variable and simply try to find a function $y = f(x)$ whose graph models the exact path of the ball. In particular, we could find **where** the ball hits the ground by solving $0 = f(x)$, but we would not be able to determine **when** the ball hits the ground.

Example 7.4.1. *Figure 7.14(a) shows a ball is located on the edge of a cliff. The ball is kicked and its height (in feet) above the level ground is given by the function* $s = y(t) = -16t^2 + 48t + 50$, *where* t *represents seconds elapsed after kicking the ball. What is the maximum height of the ball and when is this height achieved? When does the ball hit the ground? How high is the cliff?*



(a) What it looks like physically.



(b) What it looks like graphically.

Figure 7.14: Different views of the ball's trajectory.

Solution. The function $y(t)$ is a quadratic function with a negative leading coefficient, so its graph in the ts-coordinate system will be a downward opening parabola. We use a graphing device to get the picture in Figure 7.14(b).

The vertex is the highest point on the graph, which can be found by writing $y(t)$ in standard form using Fact 7.3.1:

$$\begin{aligned} y(t) &= -16t^2 + 48t + 50 \\ &= -16\left(t - \frac{3}{2}\right)^2 + 86. \end{aligned}$$

The vertex of the graph of $y(t)$ is $\left(\frac{3}{2}, 86\right)$, so the maximum height of the ball above the level ground is 86 feet, occuring at time $t = \frac{3}{2}$.

The ball hits the ground when its height above the ground is zero; using the quadratic formula:

$$\begin{aligned} y(t) &= -16t^2 + 48t + 50 \\ &= 0 \\ t &= \frac{-48 \pm \sqrt{(48)^2 - 4(-16)(50)}}{2 \cdot 16} \\ &= 3.818\,\text{sec or } -0.818\,\text{sec} \end{aligned}$$

Conclude the ball hits the ground after 3.818 seconds. Finally, the height of the cliff is the height of the ball zero seconds after release; i.e., $y(0) = 50$ feet is the height of the cliff. □

Here are two items to consider carefully:

1. **The graph of $y(t)$ is NOT the path followed by the ball! Finding the actual path of the ball is not possible unless additional information is given. Can you see why?**
2. **The function $y(t)$ is defined for all t; however, in the context of the problem, there is no physical meaning when $t < 0$.**

iii CAUTION !!!

The next example illustrates how we must be *very* careful to link the question being asked with an appropriate function.

Example 7.4.2. *A hot air balloon takes off from the edge of a mountain lake. Impose a coordinate system as pictured in Figure 7.15 and assume that the path of the balloon follows the graph of* $y = f(x) = -\frac{2}{2500}x^2 + \frac{4}{5}x$. *The land rises at a constant incline from the lake at the rate of 2 vertical feet for each 20 horizontal feet. What is the maximum height of the balloon above lake level? What is the maximum height of the balloon above ground level? Where does the balloon land on the ground? Where is the balloon 50 feet above the ground?*

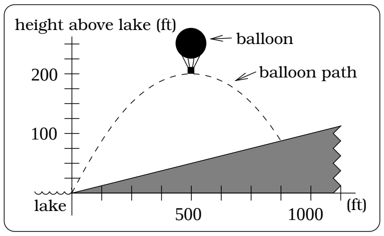


Figure 7.15: Visualizing.

Solution. In the coordinate system indicated, the origin is the takeoff point and the graph of $y = f(x)$ is the path of the balloon. Since $f(x)$ is a quadratic function with a negative leading coefficient, its graph will be a parabola which opens downward. The difficulty with this problem is that at any instant during the balloon's flight, the "height of the balloon above the ground" and the "height of the balloon above the lake level" are different! The picture in Figure 7.16 highlights this difference; consequently, two *different* functions will be needed to study these two different quantities.

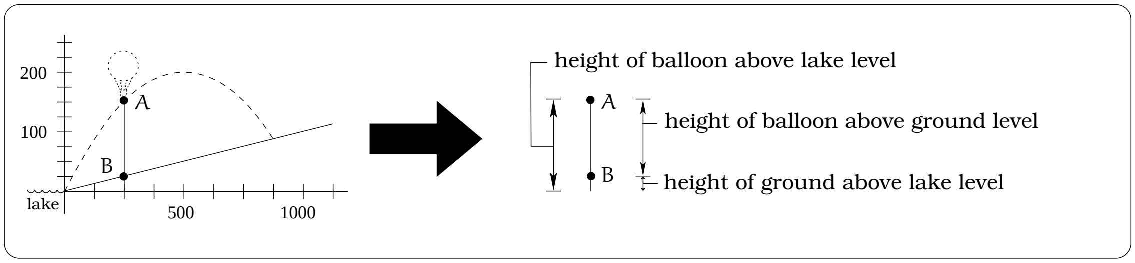

200
100
A
B
lake
500
1000
height of balloon above lake level
A
height of balloon above ground level
B
height of ground above lake level

Figure 7.16: The height of the balloon y as a function of x.

The function $y = f(x)$ keeps track of the height of the balloon above lake level at a given x location on the horizontal axis. The line ℓ with slope $m = \frac{2}{20} = \frac{1}{10}$ passing through the origin models the ground level. This says that the function

$$y = \frac{1}{10}x$$

keeps track of the height of the ground above lake level at a given x location on the horizontal axis.

We can determine the maximum height of the balloon above lake level by analyzing the parabolic graph of $y = f(x)$. Putting $f(x)$ in vertex form, via Fact 7.3.1,

$$f(x) = -\frac{2}{2500}(x - 500)^2 + 200.$$

The vertex of the graph of $y = f(x)$ is $(500, 200)$. This just tells us that the maximum height of the balloon above lake level is 200 feet. To find the landing point, we need to solve the system of equations

$$\left\{ \begin{array}{ll} y & = -\frac{2}{2500}x^2 + \frac{4}{5}x \\ y & = \frac{1}{10}x \end{array} \right\}.$$

As usual, plugging the second equation into the first and solving for x, we get

$$\begin{aligned} \frac{1}{10}x &= -\frac{2}{2500}x^2 + \frac{4}{5}x \\ x^2 &= 875x \\ x^2 - 875x &= 0 \\ x(x-875) &= 0 \end{aligned}$$

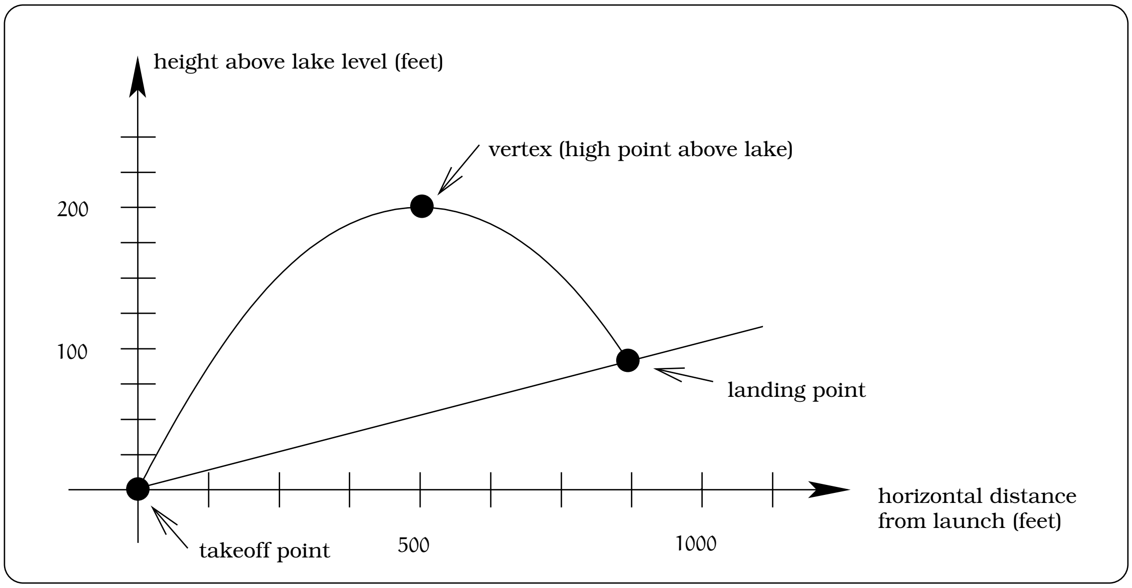


Figure 7.17: Locating the takeoff and landing points.

From the algebra, we see there are two solutions: $x = 0$ or $x = 875$; these correspond to the takeoff and landing points of the balloon, which are the two places the flight path and ground coincide. (Notice, if we had divided out x from the last equation, we would only get one solution; the tricky point is that we can't divide by zero!) The balloon lands at the position where $x = 875$ and to find the y coordinate of this landing point we plug $x = 875$ into our function for the balloon height above lake level: $y = f(875) = 87.5$ feet. So, the landing point has coordinates $(875,87.5)$.

Next, we want to study the height of the balloon above the ground. Let $y = g(x)$ be the function which represents the height of the balloon above the ground when the horizontal coordinate is x. We find

$$\begin{aligned} g(x) &= \left\{\begin{matrix}\text{height of the balloon}\\ \text{above lake level with}\\ \text{horizontal coordinate } x\end{matrix}\right\} - \left\{\begin{matrix}\text{elevation of ground}\\ \text{above lake level with}\\ \text{horizontal coordinate } x\end{matrix}\right\} \\ &= f(x) - g(x) \\ &= \underbrace{\left(-\frac{2}{2500}x^2 + \frac{4}{5}x\right)}_{\text{(balloon above lake level)}} \quad \underbrace{-}_{\text{minus}} \quad \underbrace{\left(\frac{1}{10}x\right)}_{\text{(ground above lake level)}} \\ &= -\frac{2}{2500}(x - 437.5)^2 + 153.12, \end{aligned}$$

Notice that $g(x)$ itself is a NEW quadratic function with a negative leading coefficient, so the graph of $y = g(x)$ will be a downward opening parabola. The vertex of this parabola will be $(437.5, 153.12)$, so the highest elevation of the balloon above the ground is 153.12 feet.

We can now sketch the graph of $g(x)$ and the horizontal line determined by $y = 50$ in a common coordinate system, as below. Finding where the balloon is 50 feet above the ground amounts to finding where these two graphs intersect. We need to now solve the system of equations

$$\left\{\begin{array}{ll} y & = -\frac{2}{2500}x^2 + \frac{7}{10}x \\ y & = 50 \end{array}\right\}.$$

Plug the second equation into the first and apply the quadratic formula to get $x = 796.54$ or 78.46. This tells us the two possible x coordinates when the balloon is 50 feet above the ground. In terms of the original coordinate system imposed, the two places where the balloon is 50 feet above the ground are $(78.46, 57.85)$ and $(796.54, 129.6)$.

$\square$

7.4.1 How many points determine a parabola?

We all recall from elementary geometry that two distinct points in the plane will uniquely determine a line; in fact, we used this to derive equations for lines in the plane. We could then ask if there is a similar characterization of parabolas.

Important Fact 7.4.3. *Let* $P = (x_1, y_1)$, $Q = (x_2, y_2)$ *and* $R = (x_3, y_3)$ *be three distinct non-collinear points in the plane such that the* x*-coordinates are all different. Then there exists a unique standard parabola passing through these three points. This parabola is the graph of a quadratic function* $y = f(x) = ax^2 + bx + c$ *and we can find these coefficients by simultaneously*

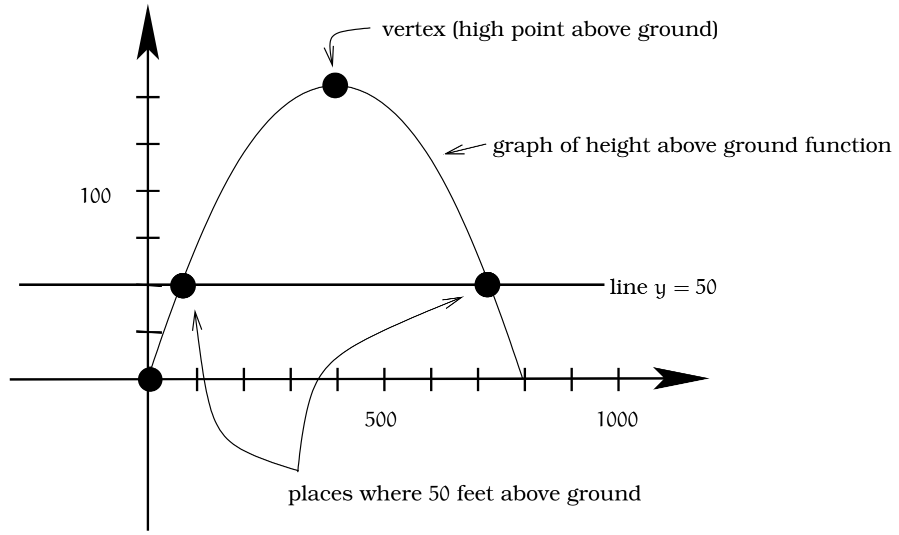

vertex (high point above ground)
graph of height above ground function
100
line $y = 50$
500
1000
places where 50 feet above ground

Figure 7.18: Finding heights above the ground.

solving the system of three equations and three unknowns obtained by assuming P, Q *and* R *are points on the graph of* $y = f(x)$*:*

$$\left\{ \begin{array}{ll} ax_1^2 + bx_1 + c & = y_1 \\ ax_2^2 + bx_2 + c & = y_2 \\ ax_3^2 + bx_3 + c & = y_3. \end{array} \right\}.$$

Example 7.4.4. *Assume the value of a particular house in Seattle has increased in value according to a quadratic function* $y = v(x)$*, where the units of* y *are in dollars and* x *represents the number of years the property has been owned. Suppose the house was purchased on January 1, 1970 and valued at \$50,000. In 1980, the value of the house on January 1 was \$80,000. Finally, on January 1, 1990 the value was \$200,000. Find the value function* $v(x)$*, determine the value on January 1, 1996 and find when the house will be valued at \$1,000,000.*

Solution. The goal is to explicitly find the value of the function $y = v(x)$. We are going to work in a xy-coordinate system in which the first coordinate of any point represents time and the second coordinate represents value. We need to decide what kind of units will be used. The x-variable, which represents time, will denote the number of years the house is owned. For the y-variable, which represents value, we could use dollars. But, instead, we will follow a typical practice in real estate and use the units of K, where K = \$1,000. For example, a house valued at \$235,600 would be worth 235.6 K. These will be the units we use, which essentially saves us from drowning in a sea of zeros!

We are given three pieces of information about the value of a particular house. This leads to three points in our coordinate system: $P = (0, 50)$, $Q = (10, 80)$ and $R = (20, 200)$. If we plot these points, they do not lie on a common line, so we know there is a unique quadratic function $v(x) = ax^2 + bx + c$ whose graph (which will be a parabola) passes through these three points. In order to find the coefficients a, b, and c, we need to solve the system of equations:

$$\left\{ \begin{array}{ll} a0^2 + b0 + c & = 50 \\ a(10)^2 + b(10) + c & = 80 \\ a(20)^2 + b(20) + c & = 200 \end{array} \right\},$$

which is equivalent to the system

$$\left\{ \begin{array}{ll} c & = 50 \\ 100a + 10b + c & = 80 \\ 400a + 20b + c & = 200 \end{array} \right\}.$$

Plugging $c = 50$ into the second two equations gives the system

$$\left\{ \begin{array}{ll} 100a + 10b & = 30 \\ 400a + 20b & = 150 \end{array} \right\} \tag{7.1}$$

Solve the first equation for a, obtaining $a = \frac{30-10b}{100}$, then plug this into the second equation to get:

$$\begin{aligned} 400\left(\frac{30-10b}{100}\right) + 20b &= 150 \\ 120 - 40b + 20b &= 150 \\ b &= -\frac{3}{2}. \end{aligned}$$

Now, plug $b = -\frac{3}{2}$ into the first equation of Equation 7.1 to get $100a + 10\left(-\frac{3}{2}\right) = 30$; i.e., $a = \frac{9}{20}$. We conclude that

$$y = v(x) = \frac{9}{20}x^2 - \frac{3}{2}x + 50,$$

keeping in mind the units here are K.

To find the value of the house on January 1, 1996, we simply note this is after $x = 26$ years of ownership. Plugging in, we get $y = v(26) = \frac{9}{20}(26)^2 - \frac{3}{2}26 + 50 = 315.2$; i.e., the value of the house is \$315,200. To find when the house will be worth \$1,000,000, we note that \$1,000,000 = 1,000 K and need to solve the equation

$$\begin{aligned} 1000 = v(x) &= \frac{9}{20}x^2 - \frac{3}{2}x + 50 \\ 0 &= \frac{9}{20}x^2 - \frac{3}{2}x - 950. \end{aligned}$$

By the quadratic formula,

$$\begin{aligned} x &= \frac{\frac{3}{2} \pm \sqrt{\left(-\frac{3}{2}\right)^2 - 4\left(\frac{9}{20}\right)(-950)}}{2\left(\frac{9}{20}\right)} \\ &= \frac{1.5 \pm \sqrt{1712.25}}{0.9} \\ &= 47.64 \text{ or } -44.31. \end{aligned}$$

Because x represents time, we can ignore the negative solution and so the value of the house will be \$1,000,000 after approximately 47.64 years of ownership. ☐

7.5 What's Needed to Build a Quadratic Model?

Back in Fact 4.7.1 on page 44, we highlighted the information required to determine a linear model. We now describe the quadratic model analog.

Important Facts 7.5.1. *A quadratic model is completely determined by:*

1. *Three distinct non-collinear points, or*
2. *The vertex and one other point on the graph.*

The first approach is just Fact 7.4.3. The second approach is based on the vertex form of a quadratic function. The idea is that we know any quadratic function $f(x)$ has the form

$$f(x) = a(x-h)^2 + k,$$

where (h, k) is the vertex. If we are given h and k, together with another point (x_0, y_0) on the graph, then plugging in gives this equation:

$$y_0 = a(x_0 - h)^2 + k.$$

The only unknown in this equation is a, which we can solve for using algebra. A couple of the exercises will depend upon these observations.

7.6 Summary

- A **quadratic** function is one of the form

  $$f(x) = ax^2 + bx + c.$$

  where $a \neq 0$.

- The graph of a quadratic function is a parabola which is symmetric about the vertical line through the highest (or lowest) point on the graph. This highest (or lowest) point is known as the **vertex** of the graph; its location is given by (h,k) where

  $$h = -\frac{b}{2a} \text{ and } k = f(h).$$

- If $a > 0$, then the vertex is the lowest (or minimum) point on the graph, and the parabola ”opens upward”. If $a < 0$, then the vertex is the highest (or maximum) point on the graph, and the parabola ”opens downward”.

- Every quadratic function can be expressed in the form

  $$f(x) = a(x - h)^2 + k$$

  where (h,k) is the vertex of the function’s graph.

7.7 Exercises

Problem 7.1. Write the following quadratic functions in vertex form, find the vertex, the axis of symmetry and sketch a rough graph.

(a) $f(x) = 2x^2 - 16x + 41$.

(b) $f(x) = 3x^2 - 15x - 77$.

(c) $f(x) = x^2 - \frac{3}{7}x + 13$.

(d) $f(x) = 2x^2$.

(e) $f(x) = \frac{1}{100}x^2$.

Problem 7.2. In each case, find a quadratic function whose graph passes through the given points:

(a) $(0,0)$, $(1,1)$ and $(3,-1)$.

(b) $(-1,1)$, $(1,-2)$ and $(3,4)$.

(c) $(2,1)$, $(3,2)$ and $(5,1)$.

(d) $(0,1)$, $(1,1)$ and $(1,3)$.

Problem 7.3. (a) Sketch the graph of the function $f(x) = x^2 - 3x + 4$ on the interval $-3 \le x \le 5$. What is the maximum value of $f(x)$ on that interval? What is the minimum value of $f(x)$ on that interval?

(b) Sketch the graph of the function $f(x) = x^2 - 3x + 4$ on the interval $2 \le x \le 7$. What is the maximum value of $f(x)$ on that interval? What is the minimum value of $f(x)$ on that interval?

(c) Sketch the graph of the function $g(x) = -(x+3)^2 + 3$ on the interval $0 \le x \le 4$. What is the maximum value of $g(x)$ on that interval? What is the minimum value of $g(x)$ on that interval?

Problem 7.4. If the graph of the quadratic function $f(x) = x^2 + dx + 3d$ has its vertex on the x-axis, what are the possible values of d? What if $f(x) = x^2 + 3dx - d^2 + 1$?

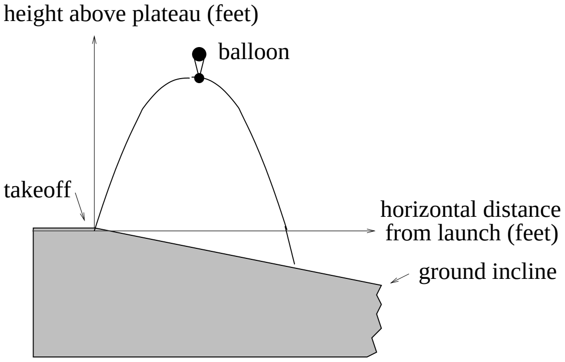
Problem 7.5. The initial price of *buzz.com* stock is \$10 per share. After 20 days the stock price is \$20 per share and after 40 days the price is \$25 per share. Assume that while the price of the stock is not zero it can be modeled by a quadratic function.

(a) Find the multipart function $s(t)$ giving the stock price after t days. If you buy 1000 shares after 30 days, what is the cost?

(b) To maximize profit, when should you sell shares? How much will the profit be on your 1000 shares purchased in (a)?

Problem 7.6. Sketch the graph of $y = x^2 - 2x - 3$. Label the coordinates of the x and y intercepts of the graph. In the same coordinate system, sketch the graph of $y = |x^2 - 2x - 3|$, give the multipart rule and label the x and y intercepts of the graph.

Problem 7.7. A hot air balloon takes off from the edge of a plateau. Impose a coordinate system as pictured below and assume that the path the balloon follows is the graph of the quadratic function $y = f(x) = -\frac{4}{2500}x^2 + \frac{4}{5}x$. The land drops at a constant incline from the plateau at the rate of 1 vertical foot for each 5 horizontal feet. Answer the following questions:


height above plateau (feet)
balloon
takeoff
horizontal distance from launch (feet)
ground incline

(a) What is the maximum height of the balloon above plateau level?

(b) What is the maximum height of the balloon above ground level?

(c) Where does the balloon land on the ground?

(d) Where is the balloon 50 feet above the ground?

Problem 7.8. (a) Suppose $f(x) = 3x^2 - 2$. Does the point $(1,2)$ lie on the graph of $y = f(x)$? Why or why not?

(b) If b is a constant, where does the line $y = 1 + 2b$ intersect the graph of $y = x^2 + bx + b$?

(c) If a is a constant, where does the line $y = 1 - a^2$ intersect the graph of $y = x^2 - 2ax + 1$?

(d) Where does the graph of $y = -2x^2 + 3x + 10$ intersect the graph of $y = x^2 + x - 10$?

Problem 7.9. Sylvia has an apple orchard. One season, her 100 trees yielded 140 apples per tree. She wants to increase her production by adding more trees to the orchard. However, she knows that for every 10 additional trees she plants, she will lose 4 apples per tree (i.e., the yield per tree will decrease by 4 apples). How many trees should she have in the orchard to maximize her production of apples?

Problem 7.10. Rosalie is organizing a circus performance to raise money for a charity. She is trying to decide how much to charge for tickets. From past experience, she knows that the number of people who will attend is a linear function of the price per ticket. If she charges 5 dollars, 1200 people will attend. If she charges 7 dollars, 970 people will attend. How much should she charge per ticket to make the most money?

Problem 7.11. A Norman window is a rectangle with a semicircle on top. Suppose that the perimeter of a particular Norman window is to be 24 feet. What should its dimensions be in order to maximize the area of the window and, therefore, allow in as much light as possible?

Problem 7.12. Jun has 300 meters of fencing to make a rectangular enclosure. She also wants to use some fencing to split the enclosure into two parts with a fence parallel to two of the sides. What dimensions should the enclosure have to have the maximum possible area?

Problem 7.13. You have $6000 with which to build a rectangular enclosure with fencing. The fencing material costs $20 per meter. You also want to have two paritions across the width of the enclosure, so that there will be three separated spaces in the enclosure. The material for the partitions costs $15 per meter. What is the maximum area you can achieve for the enclosure?

Problem 7.14. Steve likes to entertain friends at parties with "wire tricks." Suppose he takes a piece of wire 60 inches long and cuts it into two pieces. Steve takes the first piece of wire and bends it into the shape of a perfect circle. He then proceeds to bend the second piece of wire into the shape of a perfect square. Where should Steve cut the wire so that the total area of the circle and square combined is as small as possible? What is this minimal area? What should Steve do if he wants the combined area to be as *large* as possible?

Problem 7.15. Two particles are moving in the xy-plane. The move along straight lines at constant speed. At time t, particle A's position is given by

$$x = t + 2,\ y = \frac{1}{2}t - 3$$

and particle B's position is given by

$$x = 12 - 2t,\ y = 6 - \frac{1}{3}t.$$

(a) Find the equation of the line along which particle A moves. Sketch this line, and label A's starting point and direction of motion.

(b) Find the equation of the line along which particle B moves. Sketch this line on the same axes, and label B's starting point and direction of motion.

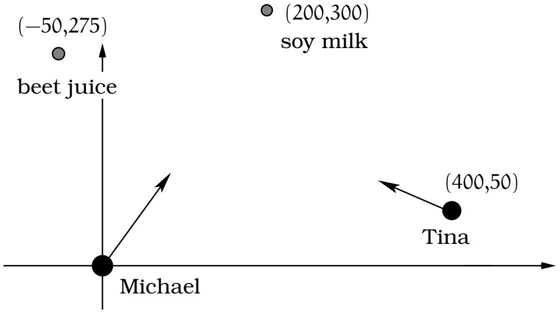
(c) Find the time (i.e., the value of t) at which the distance between A and B is minimal. Find the locations of particles A and B at this time, and label them on your graph.

Problem 7.16. Sven starts walking due south at 5 feet per second from a point 120 feet north of an intersection. At the same time Rudyard starts walking due east at 4 feet per second from a point 150 feet west of the intersection.

(a) Write an expression for the distance between Sven and Rudyard t seconds after they start walking.

(b) When are Sven and Rudyard closest? What is the minimum distance between them?

Problem 7.17. After a vigorous soccer match, Tina and Michael decide to have a glass of their favorite refreshment. They each run in a straight line along the indicated paths at a speed of 10 ft/sec. Parametrize the motion of Tina and Michael individually. Find when and where Tina and Michael are closest to one another; also compute this minimum distance.



Problem 7.18. Consider the equation: $\alpha x^2 + 2\alpha^2 x + 1 = 0$. Find the values of x that make this equation true (your answer will involve α). Find values of α that make this equation true (your answer will involve x).

Problem 7.19. For each of the following equations, find the value(s) of the constant α so that the equation has exactly one solution, and determine the solution for each value.

(a) $\alpha x^2 + x + 1 = 0$

(b) $x^2 + \alpha x + 1 = 0$

(c) $x^2 + x + \alpha = 0$

(d) $x^2 + \alpha x + 4\alpha + 1 = 0$

Problem 7.20. (a) Solve for t

$$s = 2(t-1)^2 + 1$$

(b) Solve for x

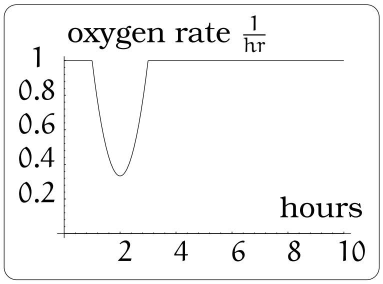
$$y = x^2 + 2x + 3$$

Chapter 8

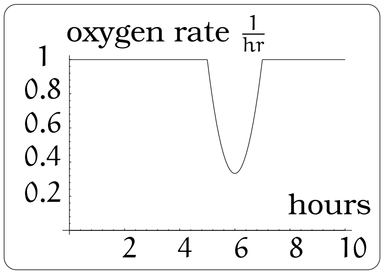
Composition

A new home takes its shape from basic building materials and the skillful use of construction tools. Likewise, we can build *new* functions from known functions through the application of analogous mathematical tools. There are five tools we want to develop: composition, reflection, shifting, dilation, arithmetic. We will handle composition in this section, then discuss the others in the following two sections.

To set the stage, let's look at a simple botany experiment. Imagine a plant growing under a particular steady light source. Plants continually give off oxygen gas to the environment at some rate; common units would be liters/hour. If we leave this plant unbothered, we measure that the plant puts out 1 liter/hour; so, the oxygen output is a steady constant rate. However, if we apply a flash of high intensity green light at the time $t = 1$ and measure the oxygen output of the plant, we obtain the plot in Figure 8.1(a).



(a) Flash at $t = 1$.



(b) Flash at $t = 5$.

Figure 8.1: Light flashes.

Using what we know from the previous section on quadratic functions, we can check that a reasonable model for the graph is this multipart function $f(t)$ (on the domain $0 \leq t \leq 10$):

$$f(t) = \begin{cases} 1 & \text{if } t \leq 1 \\ \frac{2}{3}t^2 - \frac{8}{3}t + 3 & \text{if } 1 \leq t \leq 3 \\ 1 & \text{if } 3 \leq t \end{cases}$$

Suppose we want to model the oxygen consumption when a green light pulse occurs at time $t = 5$ (instead of time $t = 1$), what is the mathematical model? For starters, it is pretty easy to believe that the graph for this new situation will look like the new graph in Figure 8.1(b).

But, can we somehow use the model $f(t)$ in hand (the known function) to build the model we want (the new function)? We will return in Example 8.2.4 to see the answer is yes; first, we need to develop the tool of *function composition*.

8.1 The Formula for a Composition

The basic idea is to start with two functions f and g and produce a new function called their *composition*. There are two basic steps in this process and we are going to focus on each separately. The first step is fairly mechanical, though perhaps somewhat unnatural. It involves combining the formulas for the functions f and g together to get a new formula; we will focus on that step in this subsection. The next step is of varying complexity and involves analyzing how the domains and ranges of f and g affect those of the composition; we defer that to the next subsection once we have the mechanics down.

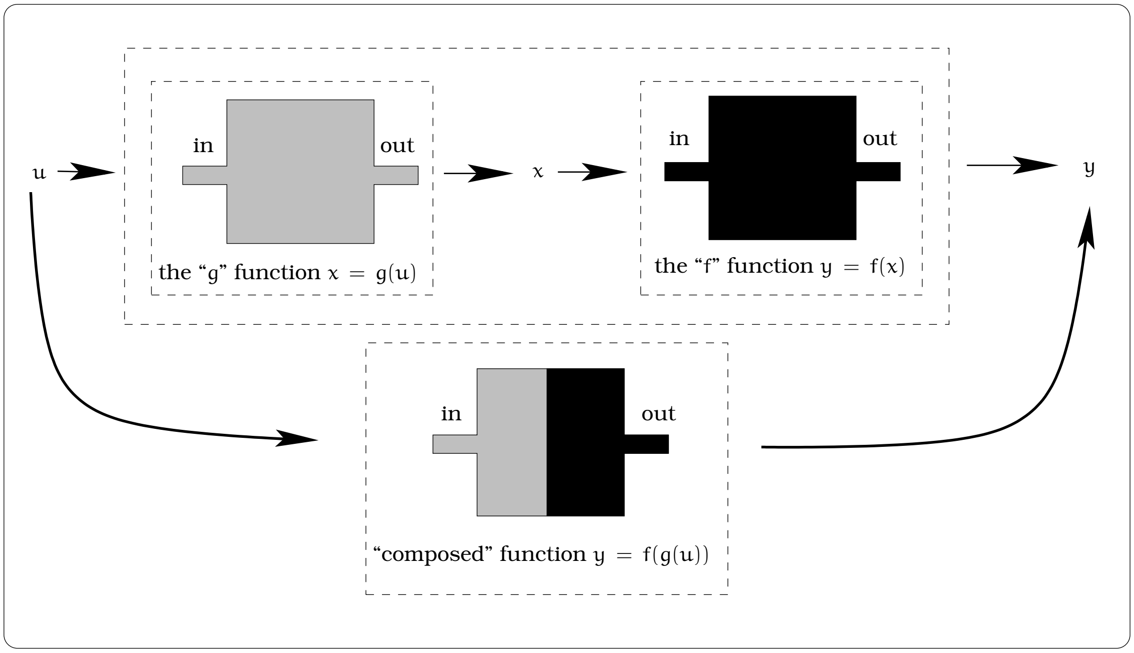

u
in
out
the "g" function $x = g(u)$
x
in
out
the "f" function $y = f(x)$
y
in
out
"composed" function $y = f(g(u))$

Figure 8.2: Visualizing a composite function $y = h(u) = f(g(u))$.

Here is a very common occurrence: We are handed a function $y = f(x)$, which means given an x value, the rule $f(x)$ produces a new y value. In addition, it may happen that the variable x is itself related to a third variable u by some *different* function equation $x = g(u)$. Given u, the rule $g(u)$ will produce a value of x; from this x we can use the rule $f(x)$ to produce a y value. In other words, we can regard y as a function depending on the *new* independent variable u. It is important to know the mechanics of working with this kind of setup. Abstractly, we have just described a situation where we take two functions and build a new function which "composes" the original ones together; schematically the situation looks like this:

Example 8.1.1. *A pebble is tossed into a pond. The radius of the first circular ripple is measured to increase at the constant rate of 2.3 ft/sec. What is the area enclosed by the leading ripple after 6 seconds have elapsed? How much time must elapse so that the area enclosed by the leading ripple is 300 square feet?*

Solution. We know that an object tossed into a pond will generate a series of concentric ripples, which grow steadily larger. We are asked questions that relate the area of the circular region bounded by the leading ripple and time elapsed.

Let r denote the radius of the leading ripple after t seconds; units of feet. The area A of a disc bounded by a leading ripple will be $A = A(r) = \pi r^2$. This exhibits A as a function in the variable r. However, the radius is changing with respect to time:

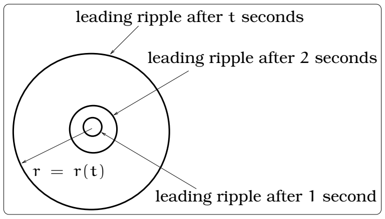

leading ripple after t seconds
leading ripple after 2 seconds
r = r(t)
leading ripple after 1 second

Figure 8.3: Concentric ripples.

$$r = r(t) = \text{radius after } t \text{ seconds} = \left(2.3\,\frac{\text{feet}}{\text{sec}}\right) t \ \text{ seconds} = 2.3t\,\text{feet}.$$

So, $r = r(t)$ is a function of t. In the expression $A = A(r)$, replace "r" by "$r(t)$," then

$$A = \pi(2.3t)^2 = 5.29\pi t^2.$$

The new function $a(t) = 5.29\pi t^2$ gives a precise relationship between area and time.

To answer our first question, $a(6) = 598.3\,\text{feet}^2$ is the area of the region bounded by the leading ripple after 6 seconds. On the other hand, if $a(t) = 300\,\text{ft}^2$;, $300 = 5.29\pi t^2$, so $t = \pm\sqrt{300/(5.29\pi)} = \pm 4.25$. Since t represents time, only the positive solution $t = 4.25$ seconds makes sense. □

We can formalize the key idea used in solving this problem, which is familiar from previous courses. Suppose that

$$y = f(x)$$

and that additionally the independent variable x is itself a function of a different independent variable t; i.e.,

$$x = g(t).$$

Then we can replace every occurrence of "x" in $f(x)$ by the expression "$g(t)$," thereby obtaining y as a function in the independent variable t. We usually denote this new function of t:

$$y = f(g(t)).$$

We refer to $f(g(t))$ as the *composition* of f and g or the *composite function*.

The process of forming the composition of two functions is a mechanical procedure. If you are handed the actual formulas for $y = f(x)$ and $x = g(t)$, then Procedure 8.1.2 is what you need.

Important Procedure 8.1.2. *To obtain the formula for* $f(g(t))$, *replace every occurrence of "*x*" in* $f(x)$ *by the expression "*$g(t)$*."*

Here are some examples of how to do this:

Examples 8.1.3. *Use the composition procedure in each of these cases.*

(i) If $y = f(x) = 2$ *and* $x = g(t) = 2t$, *then* $f(g(t)) = f(2t) = 2$.

(ii) If $y = 3x - 7$ *and* $x = g(t) = 4$, *then* $f(g(t)) = f(4) = 3 \cdot 4 - 7 = 5$.

(iii) If $y = f(x) = x^2 + 1$ *and* $x = g(t) = 2t - 1$, *then*

$$\begin{aligned} f(g(t)) &= f(2t-1) \\ &= (2t-1)^2 + 1 \\ &= 4t^2 - 4t + 2. \end{aligned}$$

(iv) If $y = f(x) = 2 + \sqrt{1 + (x-3)^2}$ *and* $x = g(t) = 2t^2 - 1$, *then*

$$\begin{aligned} f(g(t)) &= f(2t^2 - 1) \\ &= 2 + \sqrt{1 + (2t^2 - 1 - 3)^2} \\ &= 2 + \sqrt{4t^4 - 16t^2 + 17}. \end{aligned}$$

(v) If $y = f(x) = x^2$ *and* $x = g(t) = t + \heartsuit$, *then*

$$\begin{aligned} f(g(t)) &= f(t + \heartsuit) \\ &= (t + \heartsuit)^2 \\ &= t^2 + 2t\heartsuit + \heartsuit^2. \end{aligned}$$

It is natural to ask: What good is this whole business about compositions? One way to think of it is that we can use composite functions to break complicated functions into simpler parts. For example,

$$y = h(x) = \sqrt{x^2 + 1}$$

can be written as the composition $f(g(x))$, where $y = f(z) = \sqrt{z}$ and $z = g(x) = x^2 + 1$. Each of the functions f and g is "simpler" than the original h, which can help when studying h.

Examples 8.1.4. *Here we use composite functions to "simplify" a given function.*

(i) *The function* $y = \frac{1}{(x-3)^2+2}$ *can be written as a composition* $y = f(g(x))$, *where* $y = f(z) = \frac{1}{z^2+2}$ *and* $z = g(x) = x - 3$.

(ii) *The upper semicircle of radius* 2 *centered at (1,2) is the graph of the function* $y = 2 + \sqrt{4 - (x-1)^2}$. *This function can be written as a composition* $y = f(g(x))$, *where* $y = f(z) = 2 + \sqrt{4 - z^2}$ *and* $z = g(x) = x - 1$.

8.1.1 Some notational confusion

In our discussion above, we have used different letters to represent the domain variables of two functions we are composing. Typically, we have been writing: If $y = f(x)$ and $x = g(t)$, then $y = f(g(t))$ is the composition. This illustrates that the three variables t, x, and y can all be of different types. For example, t might represent time, x could be speed and y could be distance.

If we are given two functions that involve the same independent variable, like $f(x) = x^2$ and $g(x) = 2x+1$, then we can still form a new function $f(g(x))$ by following the same prescription as in Procedure 8.1.2:

Important Procedure 8.1.5. *To obtain the formula for* $f(g(x))$, *replace every occurrence of "*x*" in* $f(x)$ *by the expression "*$g(x)$*."*

For our example, this gives us:

$$f(g(x)) = f(2x+1) = (2x+1)^2.$$

Here are three other examples:

- If $f(x) = \sqrt{x}$, $g(x) = 2x^2 + 1$, then $f(g(x)) = \sqrt{2x^2+1}$.
- If $f(x) = \frac{1}{x}$, $g(x) = 2x + 1$, then $f(g(x)) = \frac{1}{2x+1}$.
- If $f(x) = x^2$, $g(x) = \triangle - x$, then $f(g(x)) = \triangle^2 - 2x\triangle + x^2$.

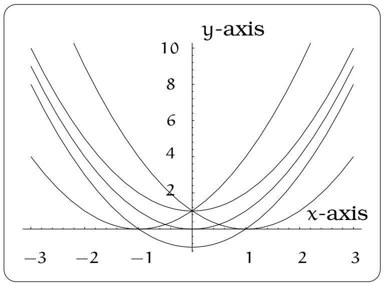


Figure 8.4: Sketching composite functions.

Example 8.1.6. *Let* $f(x) = x^2$, $g(x) = x + 1$ *and* $h(x) = x - 1$. *Find the formulas for* $f(g(x))$, $g(f(x))$, $f(h(x))$ *and* $h(f(x))$. *Discuss the relationship between the graphs of these four functions.*

Solution. If we apply Procedure 8.1.5, we obtain the composition formulas. The four graphs are given on the domain $-3 \leq x \leq 3$, together with the graph of $f(x) = x^2$.

$$\begin{aligned} f(g(x)) &= f(x+1) = (x+1)^2 \\ g(f(x)) &= g(x^2) = x^2 + 1 \\ f(h(x)) &= f(x-1) = (x-1)^2 \\ h(f(x)) &= h(x^2) = x^2 - 1. \end{aligned}$$

We can identify each graph by looking at its vertex:

- $f(x)$ has vertex (0,0)
- $f(g(x))$ has vertex (-1,0)
- $g(f(x))$ has vertex (0,1)
- $f(h(x))$ has vertex (1,0)
- $h(f(x))$ has vertex (0,-1)

Horizontal or vertical shifting of the graph of $f(x) = x^2$ gives the other four graphs: See Figure 8.4. □

8.2 Domain, Range, etc. for a Composition

A function is a "package" consisting of a rule, a domain of allowed input values, and a range of output values. When we start to compose functions, we sometimes need to worry about how the domains and ranges of the composing functions affect the composed function. First off, when you form the composition $f(g(x))$ of $f(x)$ and $g(x)$, the range values for $g(x)$ must lie within the domain values for $f(x)$. This may require that you modify the range values of $g(x)$ by changing its domain. The domain values for $f(g(x))$ will be the domain values for $g(x)$.

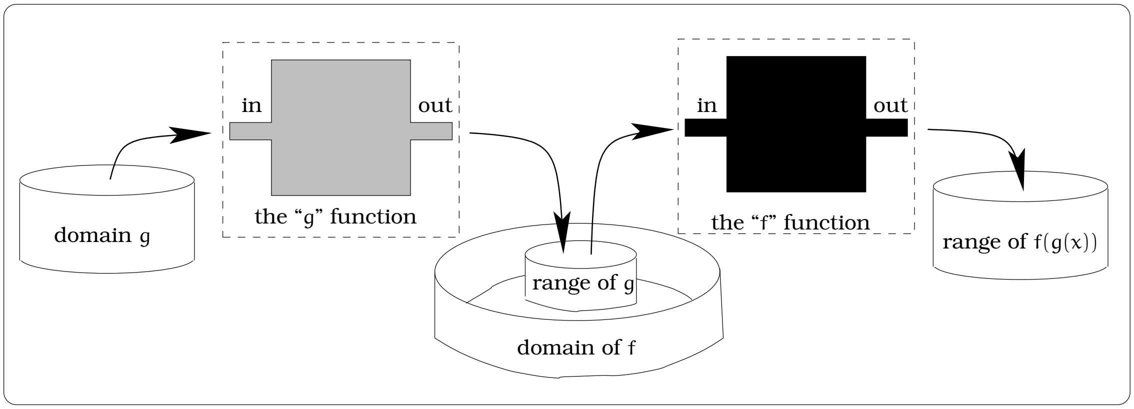


Figure 8.5: What is the domain and range of a composite function?

In practical terms, here is how one deals with the domain issues for a composition. This is a refinement of Procedure 8.1.5 on page 111.

Important Procedure 8.2.1. *To obtain the formula for* $f(g(x))$, *replace every occurrence of "*x*" in* $f(x)$ *by the expression "*$g(x)$*." In addition, if there is a condition on the domain of* f *that involves* x, *then replace every occurrence of "*x*" in that condition by the expression "*$g(x)$*."*

The next example illustrates how to use this principle.

Example 8.2.2. *Start with the function* $y = f(x) = x^2$ *on the domain* $-1 \leq x \leq 1$. *Find the rule and domain of* $y = f(g(x))$, *where* $g(x) = x - 1$.

Solution. We can apply the first statement in Procedure 8.2.1 to find the rule for $y = f(g(x))$:

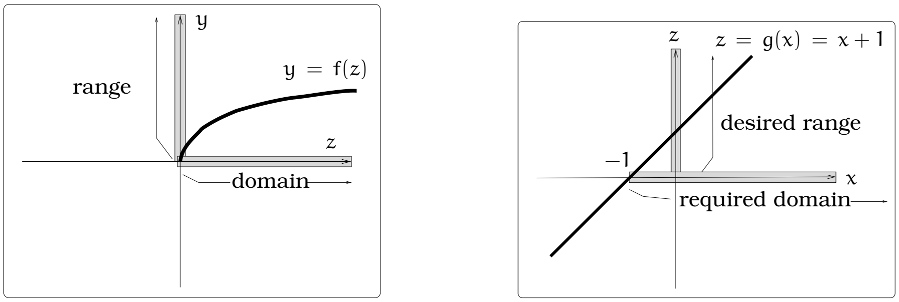
$$\begin{aligned} y &= f(g(x)) \\ &= f(x-1) = (x-1)^2 \\ &= x^2 - 2x + 1. \end{aligned}$$

To find the domain of $y = f(g(x))$, we apply the second statement in Procedure 8.2.1; this will require that we solve an inequality equation:

$$\begin{array}{rcl} -1 \leq & g(x) & \leq 1 \\ -1 \leq & x-1 & \leq 1 \\ 0 \leq & x & \leq 2 \end{array}$$

The conclusion is that $y = f(g(x)) = x^2 - 2x + 1$ on the domain $0 \leq x \leq 2$. □

Example 8.2.3. *Let* $y = f(z) = \sqrt{z}$, $z = g(x) = x + 1$. *What is the largest possible domain so that the composition* $f(g(x))$ *makes sense?*



(a) $y = \sqrt{z}$. (b) $z = x = 1$.

Figure 8.6: Finding the largest domain for $f(g(x))$.

Solution. The largest possible domain for $y = f(z)$ will consist of all non-negative real numbers; this is also the range of the function $f(z)$: See Figure 8.6(a).

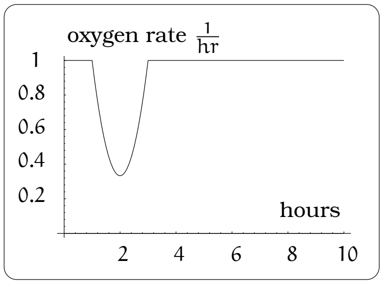
To find the largest domain for the composition, we try to find a domain of x-values so that the range of $z = g(x)$ is the domain of $y = f(z)$. So, in this case, we want the range of $g(x)$ to be all non-negative real numbers, denoted $0 \leq z$. We graph $z = g(x)$ in the xz-plane, mark the desired range $0 \leq z$ on the vertical z-axis, then determine which x-values would lead to points on the graph with second coordinates in this zone. We find that the domain of all x-values greater or equal to -1 (denoted $-1 \leq x$) leads to the desired range. In summary, the composition $y = f(g(x)) = \sqrt{x+1}$ is defined on the domain of x-values $-1 \leq x$. □

Let's return to the botany experiment that opened this section and see how composition of functions can be applied to the situation. Recall that plants continually give off oxygen gas to the environment at some rate; common units would be liters/hour.

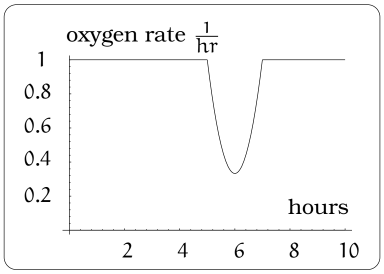
Example 8.2.4. *A plant is growing under a particular steady light source. If we apply a flash of high intensity green light at the time* $t = 1$ *and measure the oxygen output of the plant, we obtain the plot below and the mathematical model* $f(t)$.

$$f(t) = \begin{cases} 1 & \textit{if } t \leq 1 \\ \frac{2}{3}t^2 - \frac{8}{3}t + 3 & \textit{if } 1 \leq t \leq 3 \\ 1 & \textit{if } 3 \leq t \end{cases}$$

Now, suppose instead we apply the flash of high intensity green light at the time $t = 5$. *Verify that the mathematical model for this experiment is given by* $f(g(t))$, *where* $g(t) = t - 4$.



(a) Flash at $t = 1$.



(b) Flash at $t = 5$.

Figure 8.7: Applying light at time t.

Solution. Our expectation is that the plot for this new experiment will have the "parabolic dip" shifted over to occur starting at time $t = 5$ instead of at time $t = 1$. In other words, we expect the graph in Figure 8.7(b).

Our job is to verify that this graph is obtained from the function $f(g(t))$, where $g(t) = t - 4$. This is a new terrain for us, since we need to look at a composition involving a multipart function. Here is how to proceed: When we are calculating a composition involving a multipart function, we need to look at each of the parts separately, so there will be three cases to consider:

First part: $f(t) = 1$ when $t \leq 1$. To get the formula for $f(g(t))$, we now appeal to Procedure 8.2.1 and just replace every occurrence of t in $f(t)$ by $g(t)$. That gives us this NEW domain condition and function equation:

$$\begin{aligned} f(g(t)) &= f(t-4) = 1 \quad \text{when}\, t - 4 \leq 1 \\ &= 1 \quad \text{when}\, t \leq 5. \end{aligned}$$

Second part: $f(t) = \frac{2}{3}t^2 - \frac{8}{3}t + 3$ when $1 \leq t \leq 3$. We now appeal to Procedure 8.2.1 and just replace every occurrence of t in this function by $g(t)$. That gives us this NEW domain condition and function equation:

$$\begin{aligned} f(g(t)) &= f(t-4) = \frac{2}{3}(t-4)^2 - \frac{8}{3}(t-4) + 3 \quad \text{when}\, 1 \leq t - 4 \leq 3 \\ &= \frac{2}{3}t^2 - 8t + \frac{73}{3} \quad \text{when}\, 5 \leq t \leq 7. \end{aligned}$$

Third part: $f(t) = 1$ when $3 \leq t$. We now appeal to Procedure 8.2.1 and just replace every occurrence of t in this function by $g(t)$. That gives us this NEW domain condition and function equation:

$$\begin{aligned} f(g(t)) &= f(t-4) = 1 \quad \text{when}\, 3 \leq t - 4 \\ &= 1 \quad \text{when}\, 7 \leq t. \end{aligned}$$

The multipart rule for this composition can now be written down and using a graphing device you can verify its graph is the model for our experiment.

$$f(g(t)) = \begin{cases} 1 & \text{if } t \leq 5 \\ \frac{2}{3}t^2 - 8t + \frac{73}{3} & \text{if } 5 \leq t \leq 7 \\ 1 & \text{if } 7 \leq t \end{cases}$$

□

8.3 Exercises

Problem 8.1. For this problem, $f(t) = t - 1$, $g(t) = -t - 1$ and $h(t) = |t|$.

(a) Compute the multipart rules for $h(f(t))$ and $h(g(t))$ and sketch their graphs.

(b) Compute the multipart rules for $f(h(t))$ and $g(h(t))$ and sketch their graphs.

(c) Compute the multipart rule for $h(h(t)-1)$ and sketch the graph.

Problem 8.2. Write each of the following functions as a composition of two simpler functions: (There is more than one correct answer.)

(a) $y = (x - 11)^5$.

(b) $y = \sqrt[3]{1 + x^2}$.

(c) $y = 2(x-3)^5 - 5(x-3)^2 + \frac{1}{2}(x-3) + 11$.

(d) $y = \frac{1}{x^2+3}$.

(e) $y = \sqrt{\sqrt{x} + 1}$.

(f) $y = 2 - \sqrt{5 - (3x-1)^2}$.

Problem 8.3. (a) Let $f(x)$ be a linear function, $f(x) = ax + b$ for constants a and b. Show that $f(f(x))$ is a linear function.

(b) Find a function $g(x)$ such that $g(g(x)) = 6x - 8$.

Problem 8.4. Let $f(x) = \frac{1}{2}x + 3$.

(a) Sketch the graphs of $f(x)$,$f(f(x))$,$f(f(f(x)))$ on the interval $-2 \leq x \leq 10$.

(b) Your graphs should all intersect at the point (6,6). The value $x = 6$ is called a *fixed point* of the function $f(x)$ since $f(6) = 6$; that is, 6 is fixed - it doesn't move when f is applied to it. Give an explanation for why 6 is a fixed point for any function $f(f(f(...f(x)...)))$.

(c) Linear functions (with the exception of $f(x) = x$) can have at most one fixed point. Quadratic functions can have at most two. Find the fixed points of the function $g(x) = x^2 - 2$.

(d) Give a quadratic function whose fixed points are $x = -2$ and $x = 3$.

Problem 8.5. A car leaves Seattle heading east. The speed of the car in mph after m minutes is given by the function

$$C(m) = \frac{70m^2}{10 + m^2}.$$

(a) Find a function $m = f(s)$ that converts seconds s into minutes m. Write out the formula for the new function $C(f(s))$; what does this function calculate?

(b) Find a function $m = g(h)$ that converts hours h into minutes m. Write out the formula for the new function $C(g(h))$; what does this function calculate?

(c) Find a function $z = v(s)$ that converts mph s into ft/sec z. Write out the formula for the new function $v(C(m)$; what does this function calculate?

Problem 8.6. Compute the compositions $f(g(x))$, $f(f(x))$ and $g(f(x))$ in each case:

(a) $f(x) = x^2, g(x) = x + 3$.

(b) $f(x) = 1/x, g(x) = \sqrt{x}$.

(c) $f(x) = 9x + 2, g(x) = \frac{1}{9}(x - 2)$.

(d) $f(x) = 6x^2 + 5, g(x) = x - 4$.

(e) $f(x) = 4x^3 - 3, g(x) = \sqrt[3]{2x + 6}$

(f) $f(x) = 2x + 1, g(x) = x^3$.

(g) $f(x) = 3, g(x) = 4x^2 + 2x + 1$.

(h) $f(x) = -4, g(x) = 0$.

Problem 8.7. Let $y = f(z) = \sqrt{4 - z^2}$ and $z = g(x) = 2x + 3$. Compute the composition $y = f(g(x))$. Find the largest possible domain of x-values so that the composition $y = f(g(x))$ is defined.

Problem 8.8. Suppose you have a function $y = f(x)$ such that the domain of $f(x)$ is $1 \leq x \leq 6$ and the range of $f(x)$ is $-3 \leq y \leq 5$.

(a) What is the domain of $f(2(x-3))$?

(b) What is the range of $f(2(x-3))$?

(c) What is the domain of $2f(x) - 3$?

(d) What is the range of $2f(x) - 3$?

(e) Can you find constants B and C so that the domain of $f(B(x - C))$ is $8 \leq x \leq 9$?

(f) Can you find constants A and D so that the range of $Af(x) + D$ is $0 \leq y \leq 1$?

Problem 8.9. For each of the given functions $y = f(x)$, simplify the following expression so that h is no longer a factor in the denominator, then calculate the result of setting $h = 0$ in this simplified expression:

$$\frac{f(x+h) - f(x)}{h}.$$

(a) $f(x) = \frac{1}{x-1}$.

(b) $f(x) = (2x+1)^2$.

(c) $f(x) = \sqrt{25 - x^2}$.

Chapter 9

Inverse Functions

The experimental sciences are loaded with examples of functions relating time and some measured quantity. In this case, time represents our "input" and the quantity we are measuring is the "output." For example, maybe you have just mixed together some chemical reactants in a vessel. As time goes by, you measure the fraction of reactants remaining, tabulate your results, then sketch a graph as indicated in Figure 9.1.

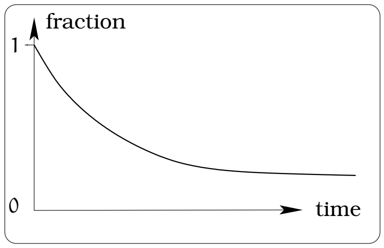


Figure 9.1: Fraction of reactants as a function of time.

Viewing the input value as "time" and the output value as "fraction of product," we could find a function $y = f(t)$ modeling this data. Using this function, you can easily compute the fraction of reactants remaining at any time in the future. However, it is probably just as interesting to know how to predict the time when a given fraction of reactants exists. In other words, we would like a *new* function that allows us to input a "fraction of reactants" and get out the "time" when this occurs. This "reverses" the input/output roles in the original function. Is there a systematic way to find the new function if we know $y = f(t)$? The answer is yes and depends upon the general theory of inverse functions.

9.1 Concept of an Inverse Function

Suppose you are asked to solve the following three equations for x*. How do you proceed?*

$$\begin{aligned}(x+2) &= 64\\(x+2)^2 &= 64\\(x+2)^3 &= 64.\end{aligned}$$

In the first equation, you add "-2" to each side, then obtain $x = 62$. In the third equation, you take the cube root of both sides of the equation, giving you $x + 2 = 4$, then subtract 2 getting $x = 2$. In second equation, you take a square root of both sides, BUT you need to remember **both** the positive and negative results when doing this. So, you are reduced down to $x + 2 = \pm 8$ or that $x = -10$ or 6. Why is it that in two of these cases you obtain a single solution, while in the remaining case there are two different answers? We need to sort this out, since the underlying ideas will surface when we address the *inverse circular functions* in Chapter 20.

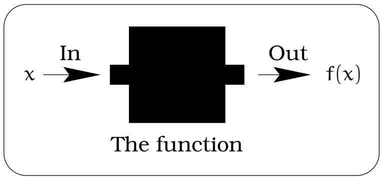


Figure 9.2: A function as a process.

Let's recall the conceptual idea of a function: A function is a process which takes a number x and outputs a new number $f(x)$. So far, we've only worked with this process from "left to right;" i.e., given x, we simply put it into a symbolic rule and out pops a new number $f(x)$. This is all pretty mechanical and straightforward.

9.1.1 An Example

Let's schematically interpret what happens for the specific concrete example $y = f(x) = 3x - 1$, when $x = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$: See Figure 9.3.

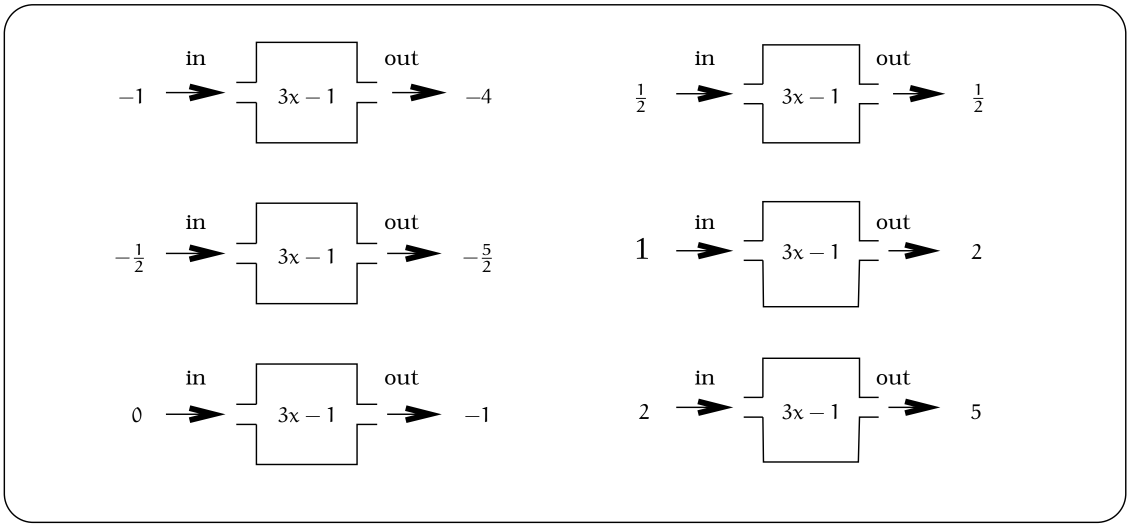


Figure 9.3: Function process $y = 3x - 1$.

We could try to understand the function process in this example in "reverse order," going "right to left;" namely, you might ask what x value can be run through the process so you end up with the number 11? This is somewhat like the "Jeopardy" game show: You know what the answer is, you want to find the question. For our example, if we start out with some given y values, then we can define a "reverse process" $x = \frac{1}{3}(y + 1)$, which returns the x value required so that $f(x) = y$: See Figure 9.4.

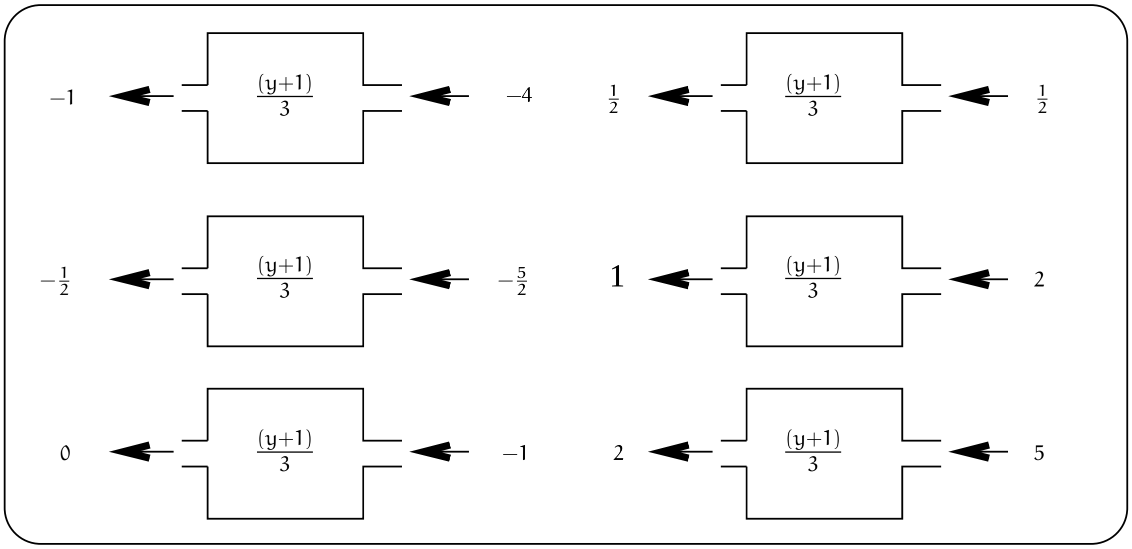


Figure 9.4: The reverse process $x = \frac{1}{3}(y+1)$.

9.1.2 A Second Example

If we begin with a linear function $y = f(x) = mx + b$, where $m \neq 0$, then we can always find a “reverse process” for the function. To find it, you must solve the equation $y = f(x)$ for x in terms of y:

$$\begin{aligned} y &= mx + b \\ y - b &= mx \\ \frac{1}{m}(y - b) &= x \end{aligned}$$

So, if $m = 3$ and $b = -1$, we just have the first example above.

For another example, suppose $y = -0.8x + 2$; then $m = -0.8$ and $b = 2$. In this case, the reverse process is $-1.25(y - 2) = x$. If we are given the value $y = 11$, we simply compute that $x = -11.25$; i.e., $f(-11.25) = 11$.

9.1.3 A Third Example

The previous examples hide a subtle point that can arise when we try to understand the “reverse process” for a given function. Suppose we begin with the function $y = f(x) = (x - 1)^2 + 1$. Figure 9.5 is a schematic of how the function works when we plug in $x = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 2$; what is being illustrated is a “forward process”, in that each input generates a unique output.

For this example, if we start out with some given y values, then we can try to define a “reverse process” $x =$??? which returns an x value required so that $f(x) = y$. Unfortunately, there is no way to obtain a single formula for this reverse process; Figure 9.6(a) shows what happens if you are given $y = 3$ and you try to solve for x.

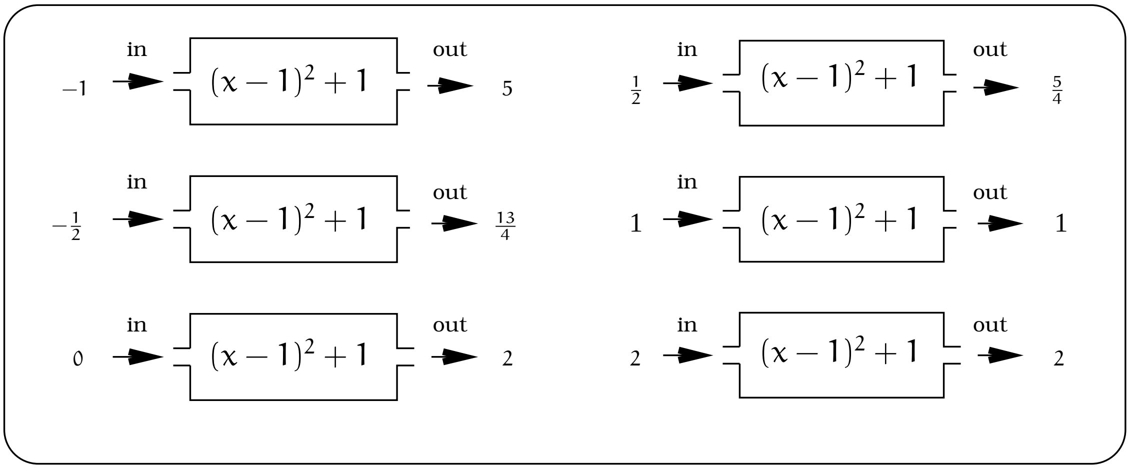
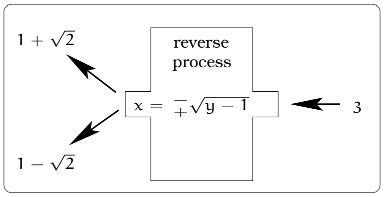


Figure 9.5: Function process $y = f(x) = (x-1)^2 + 1$.

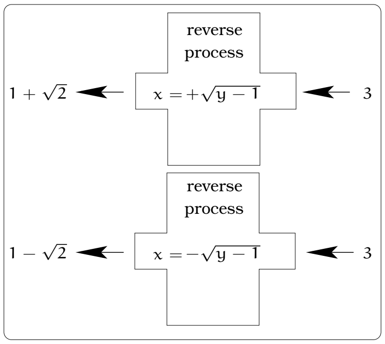
The conclusion is that the "reverse process" has two outputs. This violates the rules required for a function, so this is NOT a function. The solution is to create two new "reverse processes."

Each of these "reverse processes" has a unique output; in other words, each of these "reverse processes" defines a function.

So, given $y = 3$, there are TWO possible x values, namely $x = 1 \pm \sqrt{2}$, so that $f(1+\sqrt{2}) = 3$ and $f(1-\sqrt{2}) = 3$. In other words, the reverse process is not given by a single equation; there are TWO POSSIBLE reverse processes.



(a) Reverse process but not a function.



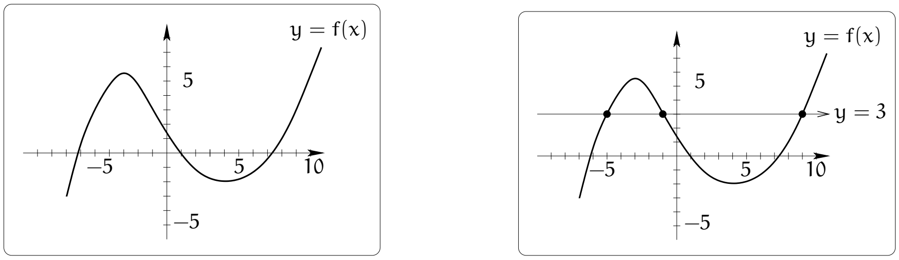
(b) Two new reverse processes that are functions.

Figure 9.6: What to do if a reverse process is not a function.

9.2 Graphical Idea of an Inverse

We have seen that finding inverses is related to solving equations. However so far, the discussion has been symbolic; we have pushed around a few equations and in the end generated some confusion. Let's use the tools of Chapter 6 to visualize what is going on here. Suppose we are given the graph of a function $f(x)$ as in Figure 9.7(a). What input x values result in an output value of 3? This involves finding all x such that $f(x) = 3$. Graphically, this means we are trying to find points on the graph of $f(x)$ so that their y-coordinates are 3. The easiest way to to do this is to draw the line $y = 3$ and find where it intersects the graph.

In Figure 9.7(b) we can see the points of intersection are $(-5,3)$, $(-1,3)$, and $(9,3)$. That means that $x = -5$, -1, 9 produce the output value 3; i.e., $f(-5) = f(-1) = f(9) = 3$.



(a) Given the function $y = f(x)$.

(b) What values of x give $f(x) = 3$?

Figure 9.7: Using the horizontal line $y = 3$ to find values on the x-axis.

This leads to our first important fact about the “reverse process” for a function:

Important Fact 9.2.1. *Given a number* c*, the* x *values such that* $f(x) = c$ *can be found by finding the* x*-coordinates of the intersection points of the graphs of* $y = f(x)$ *and* $y = c$*.*

Example 9.2.2. *Graph* $y = f(x) = x^2$ *and discuss the meaning of Fact 9.2.1 when* $c = 3, 1, 6$*.*

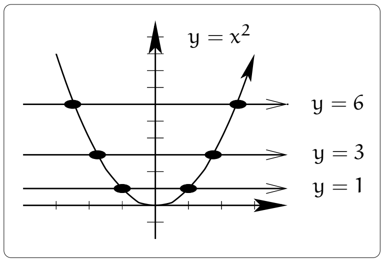


Figure 9.8: Graph of $y = f(x) = x^2$.

Solution. We graph $y = x^2$ and the lines $y = 1$, $y = 3$ and $y = 6$. Let’s use $c = 6$ as an example. We need to simultaneously solve the equations $y = x^2$ and $y = 6$. Putting these together, we get $x^2 = 6$ or $x = \pm\sqrt{6} \approx \pm 2.449$; i.e., $f(\pm 2.449) = 6$. If $c = 3$, we get $x = \pm 1.732$; i.e., $f(\pm 1.732) = 3$. Finally, if $c = 1$, we get $x = \pm 1$; i.e., $f(\pm 1) = 1$. $\square$

The pictures so far indicate another very important piece of information. For any number c, we can tell exactly “how many” input x values lead to the same output value c, just by counting the number of times the graphs of $y = f(x)$ and $y = c$ intersect.

Important Fact 9.2.3. *For any function* $f(x)$ *and any number* c*, the number of* x *values so that* $f(x) = c$ *is the number of times the graphs of* $y = c$ *and* $y = f(x)$ *intersect.*

Examples 9.2.4.

(i) If $f(x)$ *is a linear function* $f(x) = mx + b, m \neq 0$*, then the graph of* $f(x)$ *intersects a given horizontal line* $y = c$ *EXACTLY once; i.e., the equation* $c = f(x)$ *always has a unique solution.*

(ii) If $f(x) = d$ *is a constant function and* $c = d$*, then every input* x *value in the domain leads to the output value* c*. On the other hand, if* $c \neq d$*, then no input* x *value will lead to the output value* c*. For example, if* $f(x) = 1$ *and* $c = 1$*, then every real number can be input to produce an output of 1; if* $c = 2$*, then no input value of* x *will lead to an output of* 2*.*

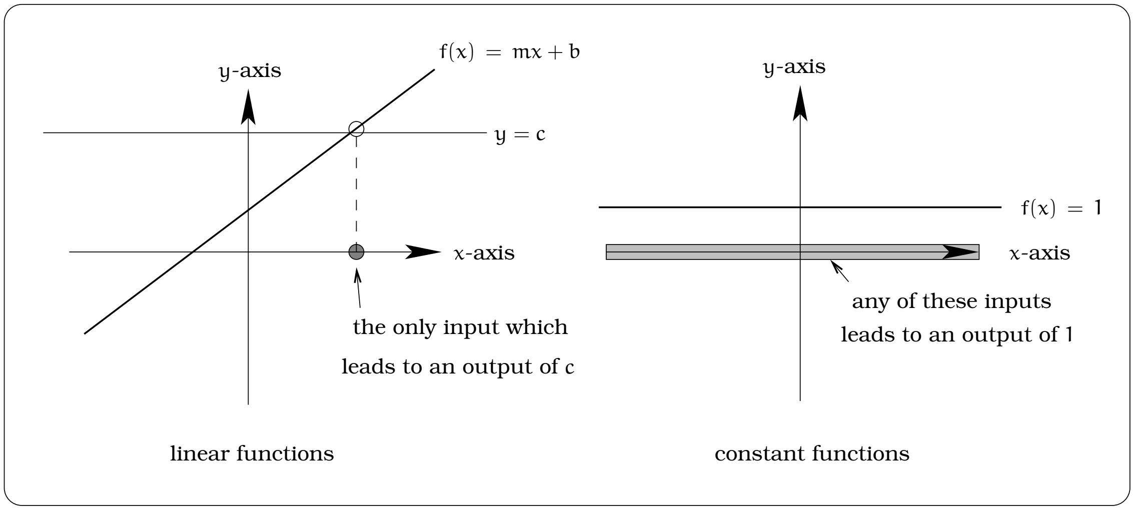

f(x) = mx + b
y-axis
y = c
x-axis
the only input which
leads to an output of c
linear functions
y-axis
f(x) = 1
x-axis
any of these inputs
leads to an output of 1
constant functions

Figure 9.9: Does a horizontal line $y = c$ intersect a curve once or more than once?

9.2.1 One-to-one Functions

For a specified domain, *one-to-one functions* are functions with the property: Given any number c, there is *at most one* input x value in the domain so that $f(x) = c$. Among our examples thus far, linear functions (degree 1 polynomials) are always one-to-one. However, $f(x) = x^2$ is not one-to-one; we've already seen that it can have two values for some of its inverses. By Fact 9.2.3, we can quickly come up with what's called the horizontal line test.

Important Fact 9.2.5 (Horizontal Line Test)**.** *On a given domain of* x*-values, if the graph of some function* $f(x)$ *has the property that every horizontal line crosses the graph at most only once, then the function is one-to-one on this domain.*

Example 9.2.6. *By the horizontal line test, it is easy to see that* $f(x) = x^3$ *is one-to-one on the domain of all real numbers.*

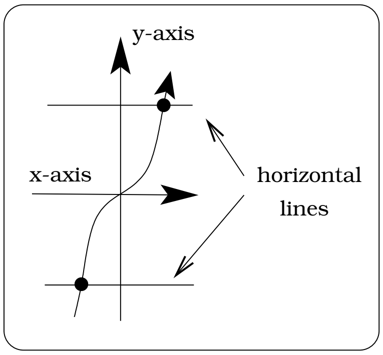


Figure 9.10: A one-to-one function $f(x) = x^3$.

Although it isn't common, it's quite nice when a function is one-to-one because we don't need to worry as much about the number of input x values producing the same output y value. In effect, this is saying that we can define a "reverse process" for the function $y = f(x)$ which will also be a function; this is the key theme of the next section.

9.3 Inverse Functions

Let's now come face to face with the problem of finding the "reverse process" for a given function $y = f(x)$. It is important to keep in mind that the domain and range of the function will both play an important role in this whole development. For example, Figure 9.11 shows the function $f(x) = x^2$ with three different domains specified and the corresponding range values.

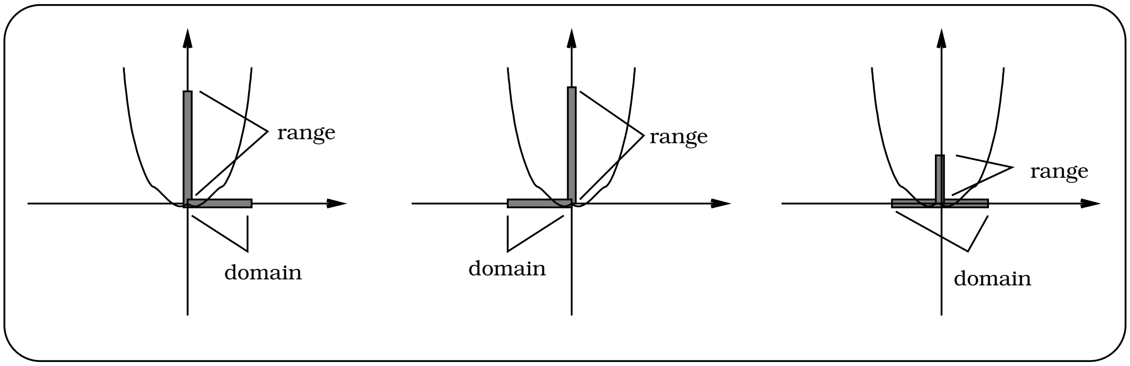


Figure 9.11: Possible domains for a given range.

These comments set the stage for a third important fact. Since the domain and range of the function and its inverse rule are going to be intimately related, we want to use notation that will highlight this fact. We have been using the letters x and y for the domain (input) and range (output) variables of $f(x)$ and the "reverse process" is going to reverse these roles. It then seems natural to simply write y (instead of c) for the input values of the "reverse process" and x for its output values.

Important Fact 9.3.1. *Suppose a function* $f(x)$ *is one-to-one on a domain of* x *values. Then define a NEW FUNCTION by the rule*

$$f^{-1}(y) = \text{the } x \text{ value so that } f(x) = y.$$

The domain of y *values for the function* $f^{-1}(y)$ *is equal to the range of the function* $f(x)$*.*

The rule defined here is the "reverse process" for the given function. It is referred to as the *inverse function* and we read $f^{-1}(y)$ as "...eff inverse of y...".

iii
CAUTION
!!!

Both the "domain" of $f(x)$ and the "rule" $f(x)$ have equal influence on whether the inverse rule is a function. Keep in mind, you do NOT get an inverse function automatically from functions that are not one-to-one!

9.3.1 Schematic Idea of an Inverse Function

Suppose that $f(x)$ is one-to-one, so that $f^{-1}(y)$ is a function. As a result, we can model $f^{-1}(y)$ as a black box. What does it do? If we put in y in the input side, we should get out the x such that $f(x) = y$.

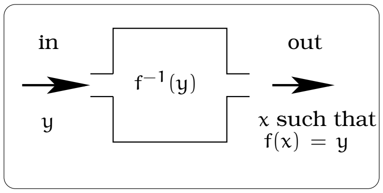


Figure 9.12: A new function $x = f^{-1}(y)$.

Now, let's try to unravel something very special that is happening on a symbolic level. What would happen if we plugged $f(a)$ into the inverse function for some number a? Then the inverse rule $f^{-1}(f(a))$ tells us that we want to find some x so that $f(x) = f(a)$. But, we already know $x = a$ works and since $f^{-1}(y)$ is a function (hence gives us unique answers), the output of $f^{-1}(f(a))$ is just a. Symbolically, this means we have Fact 9.3.2.

Important Fact 9.3.2. *For every* a *value in the domain of* $f(x)$*, we have*

$$f^{-1}(f(a)) = a. \tag{9.1}$$

This is better shown in the black box picture of Figure 9.13.

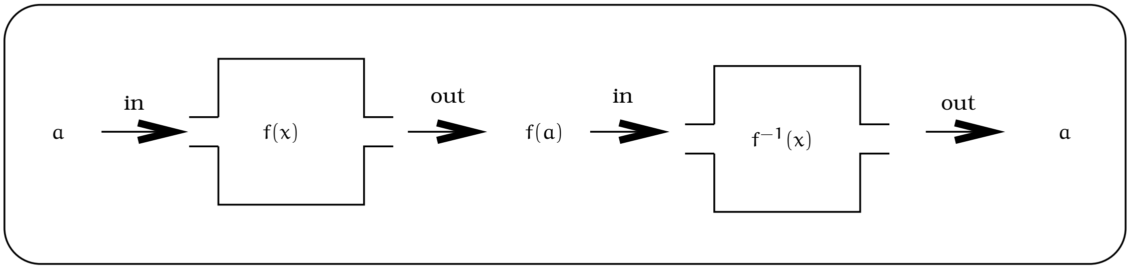


Figure 9.13: Visualizing $f^{-1}(f(a)) = a$.

A good way to get an idea of what an inverse function is doing is to remember that $f^{-1}(y)$ *reverses the process* of $f(x)$. We can think of $f^{-1}(y)$ as a "black box" running $f(x)$ backwards.

9.3.2 Graphing Inverse Functions

How can we get the graph of an inverse function? The idea is to manipulate the graph of our original one-to-one function in some prescribed way, ending up with the graph of $f^{-1}(y)$. This isn't as hard as it sounds, but some confusion with the variables enters into play. Remember that a typical point on the graph of a function $y = f(x)$ looks like $(x, f(x))$. Now let's take a look at the inverse function $x = f^{-1}(y)$. Given a number y in the domain of $f^{-1}(y)$, $y = f(x)$ for some x in the domain of $f(x)$; i.e., we are using the fact that the domain of f^{-1} equals the range of f. The function $f^{-1}(y)$ takes the number $f(x)$ and sends it to x, by Fact 9.3.2. So when $f(x)$ is the input value, x becomes the output value. Conclude a point on the graph of $f^{-1}(y)$ looks like $(f(x), x)$. It's similar to the graph of $y = f(x)$, only the x and y coordinates have reversed! What does that do to the graph? Essentially, you reorient the picture so that the positive x-axis and positive y-axis are interchanged. Figure 9.14 shows the process for the function $y = f(x) = x^3$ and its inverse function $x = f^{-1}(y) = \sqrt[3]{y}$. We place some $*$ symbols on the graph to help keep track of what is happening.

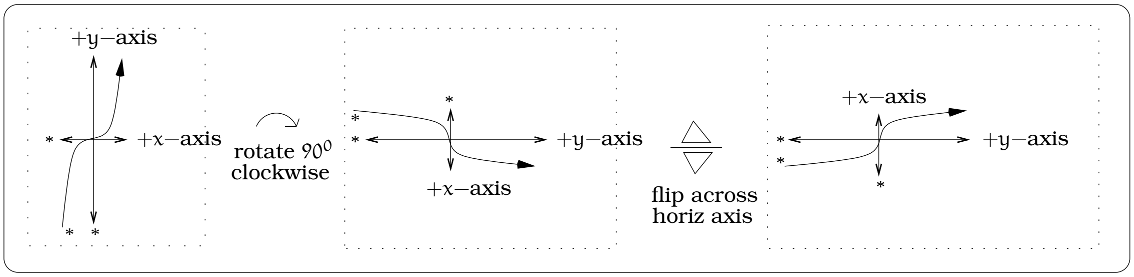


Figure 9.14: Graphically finding $x = f^{-1}(y) = \sqrt[3]{y}$.

9.4 Trying to Invert a Non one-to-one Function

Suppose we blindly try to show that $\sqrt{y}$ is the inverse function for $y = x^2$, without worrying about all of this one-to-one stuff. We'll start out with the number -7. If $f^{-1}(y) = \sqrt{y}$, then we know that

$$f^{-1}(f(-7)) = f^{-1}(49) = 7.$$

On the other hand, the formula in Fact 9.3.2 tells us that we must have

$$f^{-1}(f(-7)) = -7,$$

so we have just shown $7 = -7$! So clearly $f^{-1}(y) \neq \sqrt{y}$. Even if we try $f^{-1}(y) = -\sqrt{y}$, we produce a contradiction. It seems that if you didn't have to worry about negative numbers, things would be all right. Then you could say that $f^{-1}(y) = \sqrt{y}$. Let's try to see what this means graphically.

Let's set $f(x) = x^2$, **but** only for non-negative x-values. That means that we want to erase the graph to the left of the y-axis (so remember - no negative x-values allowed). The graph would then look like Figure 9.15.

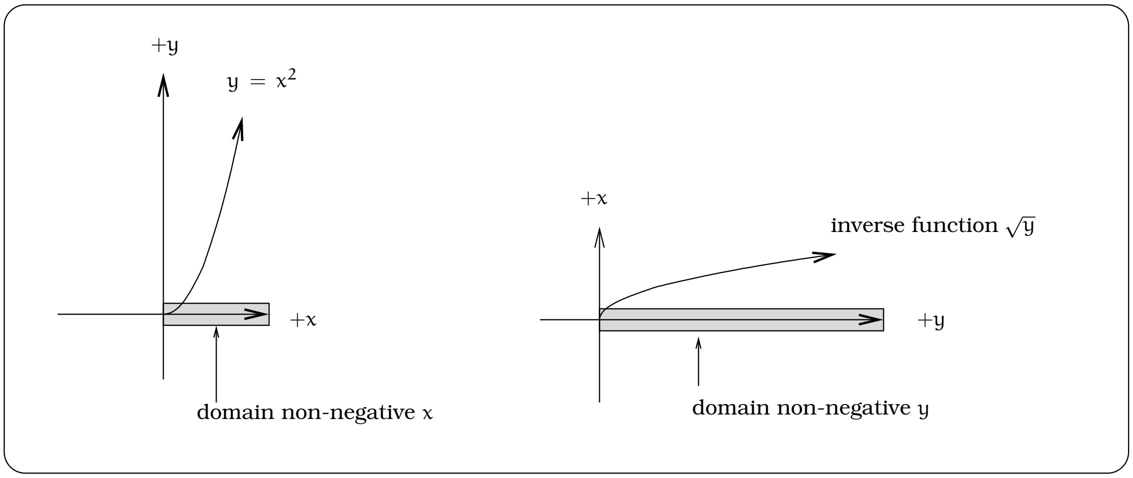

+y
y = x²
+x
domain non-negative x
+x
inverse function √y
+y
domain non-negative y

Figure 9.15: Restricting the domain: No negative x-values.

This is a now a one-to-one function! And now, one can see that its inverse function is $\sqrt{y}$. Similarly, we could have taken $f(x) = x^2$ but only for the non-positive x-values. In that case, $f^{-1}(y) = -\sqrt{y}$. In effect, we have split the graph of $y = x^2$ into two parts, each of which is the graph of a one-to-one function; Figure 9.16.

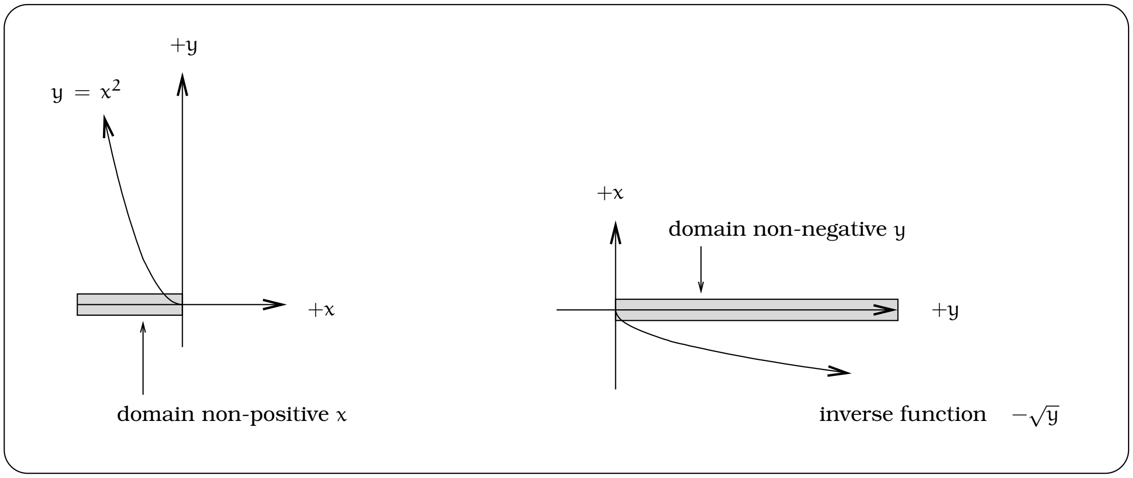

+y
y = x²
+x
domain non-positive x
+x
domain non-negative y
+y
inverse function −√y

Figure 9.16: Restricting the domain: No positive x-values.

It is precisely this splitting into two cases that leads us to multiple solutions of an equation like $x^2 = 5$. We obtain $x = \sqrt{5}$ and $x = -\sqrt{5}$; one solution comes from the side of the graph to the left of the y-axis, and the other from the right of the y-axis. This is because we have separate inverse functions for the left and right side of the graph of $y = x^2$.

9.5 Summary

- Two functions f and g are inverses if

  $$f(g(x)) = x \text{ and } g(f(x)) = x$$

  for all x in the domain of f and the domain of g.

- A function f is **one-to-one** if every equation

  $$f(x) = k$$

  has at most one solution. If there is a value of k such that the equation $f(x) = k$ has more than one solution, then f is not one-to-one.

- A function is one-to-one if every horizontal line intersects the function's graph at most once.

- A function has an inverse if the function is one-to-one, and every one-to-one function has an inverse.

- The domain of a function is the range of its inverse, and the range of a function is the domain of its inverse.

- The graph of a function and its inverse are mirror images of each other across the line $y = x$.

9.6 Exercises

Problem 9.1. Let $f(x) = \frac{2}{3x-4}$ on the largest domain for which the formula makes sense.

(a) Find the domain and range of $f(x)$, then sketch the graph.

(b) Find the domain, range and rule for the inverse function f^{-1}, then sketch its graph.

Problem 9.2. Find the inverse function of each of the following functions. Specify the domains of the inverse functions.

(a) $f(x) = \frac{1}{5}x + 8$

(b) $h(x) = \frac{5}{x+3}$

(c) $g(x) = 4\sqrt{3-x} - 7$

(d) $j(x) = \sqrt{x} + \sqrt{x-1}$

(e) $k(x) = \sqrt{16-x^2}, 0 \le x \le 4$

Problem 9.3. For this problem, $y = f(x) = 2x^2 - 3x - 1$ on the domain of all real numbers.

(a) Sketch the function graph and find the coordinates of the vertex $P = (a,b)$.

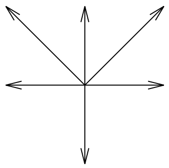
(b) Explain why $y = f(x)$ does not have an inverse function on the domain of all real numbers.

(c) Restrict $y = f(x)$ to the domain $\{a \le x\}$ and find the formula for the inverse function $f^{-1}(y)$. What are the domain and range of the inverse function?

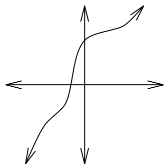
(d) Restrict $y = f(x)$ to the domain $\{x \le a\}$ and find the formula for the inverse function $f^{-1}(y)$. What are the domain and range of the inverse function?

Problem 9.4. Which of the following graphs are one-to-one? If they are not one-to-one, section the graph up into parts that are one-to-one.

A



B



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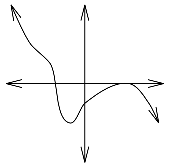

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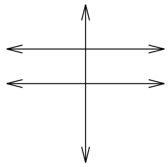


Problem 9.5. Show that, for every value of a, the function

$$f(x) = a + \frac{1}{x-a}$$

is its own inverse.

Problem 9.6. Clovis is standing at the edge of a cliff, which slopes 4 feet downward from him for every 1 horizontal foot. He launches a small model rocket from where he is standing. With the origin of the coordinate system located where he is standing, and the x-axis extending horizontally, the path of the rocket is described by the formula $y = -2x^2 + 120x$.

(a) Give a function $h = f(x)$ relating the height h of the rocket above the sloping ground to its x-coordinate.

(b) Find the maximum height of the rocket above the sloping ground. What is its x-coordinate when it is at its maximum height?

(c) Clovis measures its height h of the rocket above the sloping ground while it is going up. Give a function $x = g(h)$ relating the x-coordinate of the rocket to h.

(d) Does this function still work when the rocket is going down? Explain.

Problem 9.7. For each of the following functions: (1) sketch the function, (2) find the inverse function, and (3) sketch the inverse function. In each case, indicate the correct domains and ranges. (4) Finally, make sure you test each of the functions you propose as an inverse with the following compositions:

$$f(f^{-1}(x)) \overset{?}{=} x$$

and

$$f^{-1}(f(x)) \overset{?}{=} x.$$

(a) $f(x) = 3x - 2$

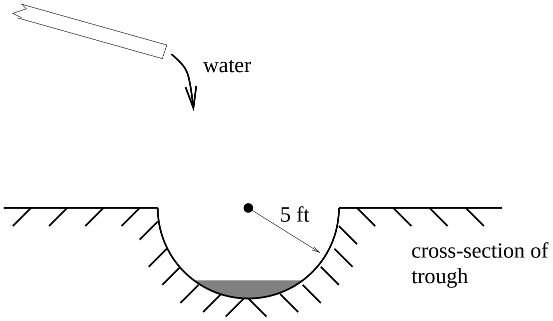
(b) $f(x) = \frac{1}{2}x + 5$

(c) $f(x) = -x^2 + 3, x \geq 0$

(d) $f(x) = x^2 + 2x + 5, x \leq -1$

(e) $f(x) = \sqrt{4 - x^2}$, $0 \leq x \leq 2$

Problem 9.8. A trough has a semicircular cross section with a radius of 5 feet. Water starts flowing into the trough in such a way that the depth of the water is increasing at a rate of 2 inches per hour.



(a) Give a function $w = f(t)$ relating the width w of the surface of the water to the time t, in hours. Make sure to specify the domain and compute the range too.

(b) After how many hours will the surface of the water have width of 6 feet?

(c) Give a function $t = f^{-1}(w)$ relating the time to the width of the surface of the water. Make sure to specify the domain and compute the range too.

Problem 9.9. A biochemical experiment involves combining together two protein extracts. Suppose a function $\phi(t)$ monitors the amount (nanograms) of extract A remaining at time t (nanoseconds). Assume you know these facts:

1. The function ϕ is invertible; i.e., it has an inverse function.
2. $\phi(0) = 6$, $\phi(1) = 5$, $\phi(2) = 3$, $\phi(3) = 1$, $\phi(4) = 0.5$, $\phi(10) = 0$.

(a) At what time do you know there will be 3 nanograms of extract A remaining?

(b) What is $\phi^{-1}(0.5)$ and what does it tell you?

(c) (True or False) There is exactly one time when the amount of extract A remaining is 4 nanograms.

(d) Calculate $\phi(\phi^{-1}(1)) =$

(e) Calculate $\phi^{-1}(\phi(6)) =$

(f) What is the domain and range of ϕ?

Chapter 10

Exponential Functions

If we start with a single yeast cell under favorable growth conditions, then it will divide in one hour to form two identical "daughter cells". In turn, after another hour, each of these daughter cells will divide to produce two identical cells; we now have four identical "granddaughter cells" of the original parent cell. Under ideal conditions, we can imagine how this "doubling effect" will continue:

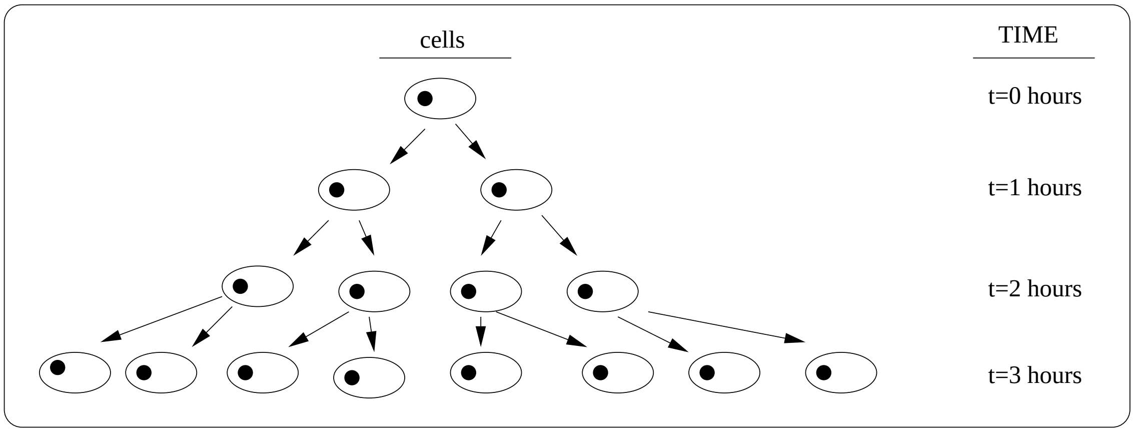


Figure 10.1: Observing cell growth.

The question is this: *Can we find a function of* t *that will predict (i.e. model) the number of yeast cells after* t *hours?* If we tabulate some data (as at right), the conclusion is that the formula

$$N(t) = 2^t$$

predicts the number of yeast cells after t hours. Now, let's make a very slight change. Suppose that instead of starting with a single cell, we begin with a population of 3×10^6 cells; a more realistic situation. If we assume

Total hours	**Number of yeast cells**
0	$1=2^0$
1	$2=2^1$
2	$4=2^2$
3	$8=2^3$
4	$16=2^4$
5	$32=2^5$
6	$64=2^6$

Table 10.1: Cell growth data.

that the population of cells will double every hour, then reasoning as above will lead us to conclude that the formula

$$N(t) = (3 \times 10^6)2^t$$

gives the population of cells after t hours. Now, as long as t represents a non-negative integer, we know how to calculate $N(t)$. For example, if $t = 6$, then

$$\begin{aligned} N(t) &= (3 \times 10^6)2^6 \\ &= (3 \times 10^6)(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \\ &= (3 \times 10^6)64 \\ &= 192 \times 10^6. \end{aligned}$$

The key point is that computing $N(t)$ only involves simple arithmetic. But what happens if we want to know the population of cells after 6.37 hours? That would require that we work with the formula

$$N(t) = (3 \times 10^6)2^{6.37}$$

and the rules of arithmetic do not suffice to calculate $N(t)$. We are stuck, since we must understand the meaning of an expression like $2^{6.37}$. In order to proceed, we will need to review the algebra required to make sense of raising a number (such as 2) to a non-integer power. We need to understand the precise meaning of expressions like: $2^{6.37}$, $2^{\sqrt{5}}$, $2^{-\pi}$, etc.

10.1 Functions of Exponential Type

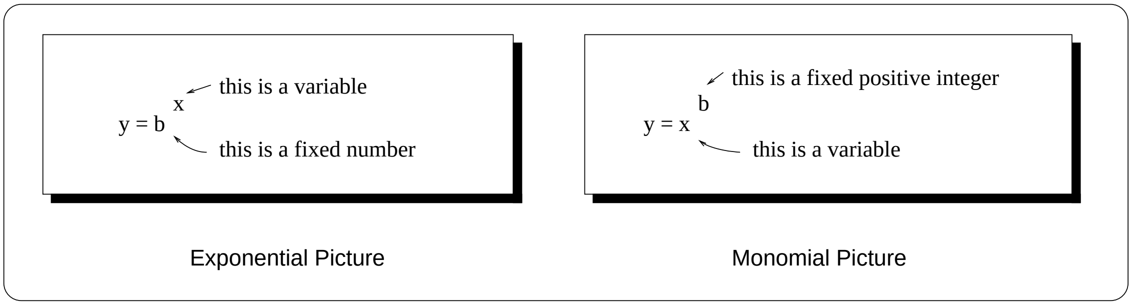


Figure 10.2: Viewing the difference between exponential and monomial functions.

On a symbolic level, the class of functions we are trying to motivate is easily introduced. We have already studied the monomials $y = x^b$, where x was our input variable and b was a fixed positive integer exponent. What happens if we turn this around, interchanging x and b, defining a new rule:

$$y = f(x) = b^x. \tag{10.1}$$

We refer to x as the *power* and b the *base*. An expression of this sort is called a function of *exponential type*. Actually, if your algebra is a bit rusty, it is easy to initially confuse functions of exponential type and monomials (see Figure 10.2).

10.1.1 Reviewing the Rules of Exponents

To be completely honest, making sense of the expression $y = b^x$ for *all* numbers x requires the tools of *Calculus*, but it is possible to establish a reasonable comfort level by handling the case when x is a rational number. If $b \geq 0$ and n is a positive integer (i.e. $n = 1, 2, 3, 4, \dots$), then we can try to solve the equation

$$t^n = b. \tag{10.2}$$

A solution t to this equation is called an nth root of b. This leads to complications, depending on whether n is even or odd. In the odd case, for any real number b, notice that the graph of $y = b$ will always cross the graph of $y = t^n$ exactly once, leading to one solution of (10.2).

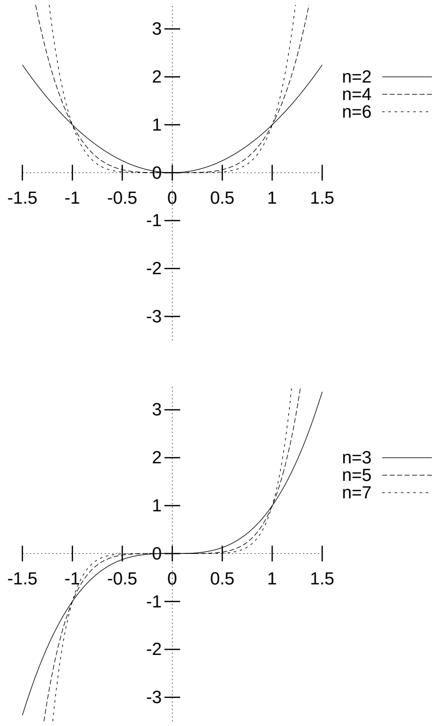


Figure 10.3: Even and odd monomials.

On the other hand, if n is even and $b < 0$, then the graph of $y = t^n$ will miss the graph of $y = b$, implying there are no solutions to the equation in (10.2). (There will be *complex* solutions to equations such as $t^2 = -1$, involving the imaginary complex numbers $\pm i = \pm\sqrt{-1}$, but we are only working with real numbers in this course.) Also, again in the case when n is even, it can happen that there are two solutions to (10.2). We do not want to constantly worry about this even/odd distinction, so we will henceforth **assume** $b > 0$. To eliminate possible ambiguity, we will single out a particular n^{th}-root; we define the symbols:

$$\sqrt[n]{b} = b^{\frac{1}{n}} = \text{ the largest real } n^{th} \text{ root of } b. \tag{10.3}$$

Thus, whereas ± 1 are both 4th-roots of 1, we have defined $\sqrt[4]{1} = 1$.

In order to manipulate $y = b^x$ for rational x, we need to recall some basic facts from algebra.

Important Facts 10.1.1 (Working with rational exponents)**.** *For all positive integers* p *and* q*, and any real number base* $b > 0$*, we have*

$$b^{\frac{p}{q}} = \left(\sqrt[q]{b}\right)^p = \sqrt[q]{b^p}.$$

For any rational numbers r *and* s*, and for all positive bases* a *and* b*:*

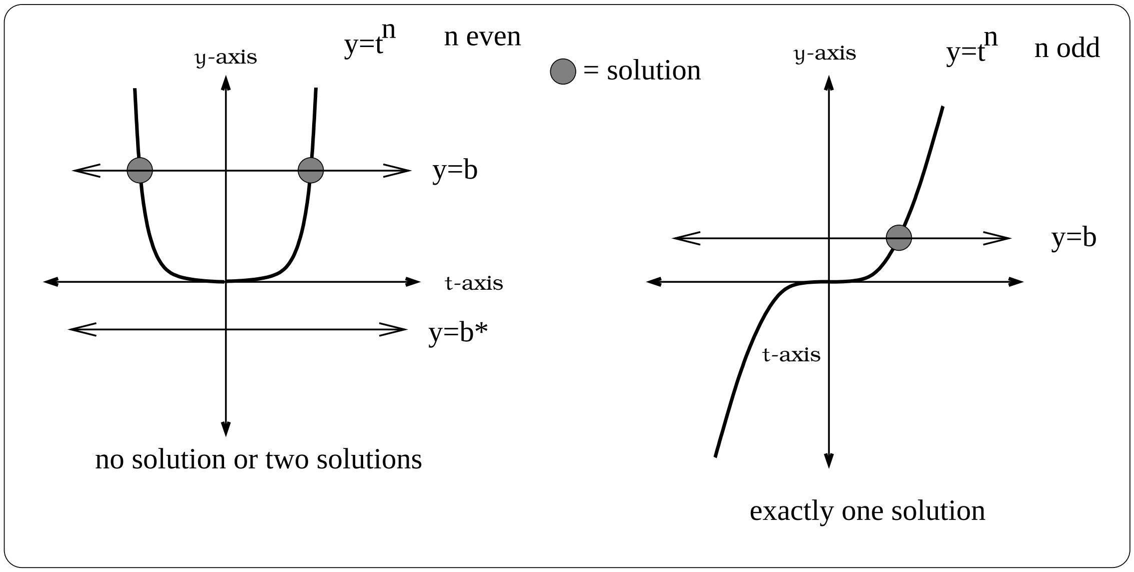


Figure 10.4: How many solutions to $t^n = b$?.

1. *Product of power rule:* $b^r b^s = b^{r+s}$
2. *Power of power rule:* $(b^r)^s = b^{rs}$
3. *Power of product rule:* $(ab)^r = a^r b^r$
4. *Zero exponent rule:* $b^0 = 1$
5. *Negative power rule:* $b^{-r} = \frac{1}{b^r}$

These rules have two important consequences, one theoretical and the other more practical. On the first count, recall that any rational number r can be written in the form $r = \frac{p}{q}$, where p and q are integers. Consequently, using these rules, we see that the expression $y = b^x$ defines a function of x, whenever x is a rational number. On the more practical side of things, using the rules we can calculate and manipulate certain expressions. For example,

$$\begin{aligned} 27^{\frac{2}{3}} &= \left(\sqrt[3]{27}\right)^2 = 3^2 \;= 9; \\ 8^{-\frac{5}{3}} &= \left(\sqrt[3]{8}\right)^{-5} = 2^{-5} = \frac{1}{2^5} = \frac{1}{32}. \end{aligned}$$

The sticky point which remains is knowing that $f(x) = b^x$ actually defines a function for *all real values of* x. This is **not** easy to verify and we are simply going to accept it as a fact. The difficulty is that we need the fundamentally new concept of a *limit*, which is the starting point of a *Calculus* course. Once we know the expression does define a function, we can also verify that the rules of Fact 10.1.1 carry through for all real exponent powers. Your calculator should have a "y to the x key", allowing you to calculate expressions such as $\pi^{\sqrt{2}}$ involving non-rational powers.

Here are the key modeling functions we will work with in this Chapter.

Definition 10.1.2. *A function of exponential type has the form*

$$A(x) = A_0 b^x,$$

for some $b > 0$, $b \neq 1$, *and* $A_0 \neq 0$.

We will refer to the formula in Definition 10.1.2 as the *standard exponential form*. Just as with standard forms for quadratic functions, we sometimes need to do a little calculation to put an equation in standard form. The constant A_0 is called the *initial value* of the exponential function; this is because if x represents time, then $A(0) = A_0 b^0 = A_0$ is the value of the function at time $x = 0$; i.e. the initial value of the function.

Example 10.1.3. *Write the equations* $y = 8^{3x}$ *and* $y = 7\left(\frac{1}{2}\right)^{2x-1}$ *in standard exponential form.*

Solution. In both cases, we just use the rules of exponents to maneuver the given equation into standard form:

$$\begin{aligned} y &= 8^{3x} \\ &= (8^3)^x \\ &= 512^x \end{aligned}$$

and

$$\begin{aligned} y &= 7\left(\frac{1}{2}\right)^{2x-1} \\ &= 7\left(\frac{1}{2}\right)^{2x}\left(\frac{1}{2}\right)^{-1} \\ &= 7\left(\left(\frac{1}{2}\right)^2\right)^x 2 \\ &= 14\left(\frac{1}{4}\right)^x \end{aligned}$$

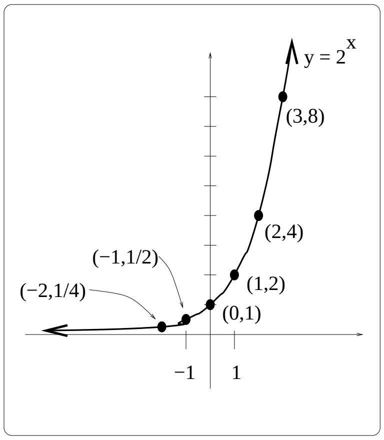
□

10.2 The Functions $y = A_0 b^x$

We know $f(x) = 2^x$ defines a function of x, so we can study basic qualitative features of its graph. The data assembled in the solution of the "Doubling Effect" beginning this Chapter, plus the rules of exponents, produce a number of points on the graph. This graph exhibits four key qualitative features that deserve mention:

x	2^x	**Point on the graph of** $y = 2^x$
⋮	⋮	⋮
-2	1/4	(-2, 1/4)
-1	1/2	(-1, 1/2)
0	1	(0, 1)
1	2	(1, 2)
2	4	(2, 4)
3	8	(3, 8)
⋮	⋮	⋮

(a) Data points from $y = 2^x$.



(b) Graph of $y = 2^x$.

Figure 10.5: Visualizing $y = 2^x$.

- The graph is always above the horizontal axis; i.e. the function values are always positive.
- The graph has y-intercept 1 and is increasing.
- The graph becomes closer and closer to the horizontal axis as we move left; i.e. the x-axis is a **horizontal asymptote** for the left-hand portion of the graph.
- The graph becomes higher and higher above the horizontal axis as we move to the right; i.e., the graph is *unbounded* as we move to the right.

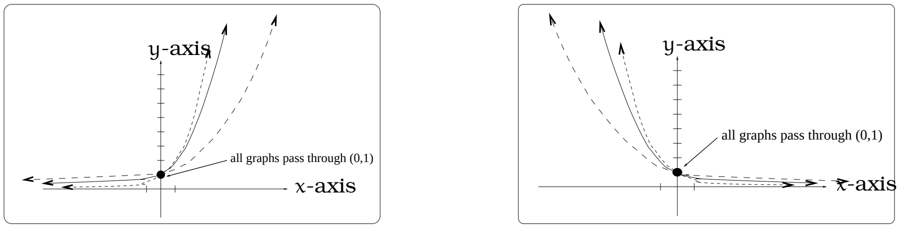
The special case of $y = 2^x$ is representative of the function $y = b^x$, but there are a few subtle points that need to be addressed. First, recall we are always assuming that our base $b > 0$. We will consider three separate cases: $b = 1$, $b > 1$, and $0 < b < 1$.

10.2.1 The case $b = 1$

In the case $b = 1$, we are working with the function $y = 1^x = 1$; this is not too exciting, since the graph is just a horizontal line. We will ignore this case.

10.2.2 The case $b > 1$

If $b > 1$, the graph of the function $y = b^x$ is qualitatively similar to the situation for $b = 2$, which we just considered. The only difference is the exact amount of "concavity" in the graph, but the four features highlighted above are still valid. Figure 10.6(a) indicates how these graphs compare for three different values of b. Functions of this type exhibit what is typically referred to as *exponential growth*; this codifies the fact that the function values grow rapidly as we move to the right along the x-axis.



(a) Graph of $y = b^x$, $b > 1$. (b) Graph of $y = b^x$, $0 < b < 1$.

Figure 10.6: Visualizing cases for b.

10.2.3 The case $0 < b < 1$

We can understand the remaining case $0 < b < 1$, by using the remarks above and our work in Chapter 13. First, with this condition on b, notice that $\frac{1}{b} > 1$, so the graph of $y = \left(\frac{1}{b}\right)^x$ is of the type in Figure 10.6(a). Now, using the rules of exponents:

$$y = \left(\frac{1}{b}\right)^{-x} = \left(\left(\frac{1}{b}\right)^{-1}\right)^x = b^x.$$

By the reflection principle, the graph of $y = \left(\frac{1}{b}\right)^{-x}$ is obtained by reflecting the graph of $y = (\frac{1}{b})^x$ about the y-axis. Putting these remarks together, if $0 < b < 1$, we conclude that the graph of $y = b^x$ will look like Figure 10.6(b). Notice, the graphs in Figure 10.6(b) share qualitative features, mirroring the features outlined previously, with the "asymptote" and "unbounded" portions of the graph interchanged. Graphs of this sort are often said to exhibit *exponential decay*, in the sense that the function values rapidly approach zero as we move to the right along the x-axis.

Important Facts 10.2.1 (Features of Exponential Type Functions)**.** *Let b be a positive real number, not equal to 1. The graph of $y = b^x$ has these four properties:*

1. *The graph is always above the horizontal axis.*
2. *The graph has y-intercept 1.*
3. *If $b > 1$ (resp. $0 < b < 1$), the graph becomes closer and closer to the horizontal axis as we move to the left (resp. move to the right); this says the x-axis is a horizontal asymptote for the left-hand portion of the graph (resp. right-hand portion of the graph).*

4. *If* $b > 1$ *(resp.* $0 < b < 1$*), the graph becomes higher and higher above the horizontal axis as we move to the right (resp. move to the left); this says that the graph is unbounded as we move to the right (resp. move to the left).*

If $A_0 > 0$*, the graph of the function* $y = A_0 b^x$ *is a vertically expanded or compressed version of the graph of* $y = b^x$*. If* $A_0 < 0$*, we additionally reflect about the* x*-axis.*

10.3 Piano Frequency Range

A sound wave will cause your eardrum to move back and forth. In the case of a so-called *pure tone*, this motion is modeled by a function of the form

$$d(t) = A\sin(2\pi f t),$$

where f is called the *frequency*, in units of "periods/unit time", called "Hertz" and abbreviated "Hz". The coefficient A is related to the actual displacement of the eardrum, which is, in turn, related to the loudness of the sound. A person can typically perceive sounds ranging from 20 Hz to 20,000 Hz.

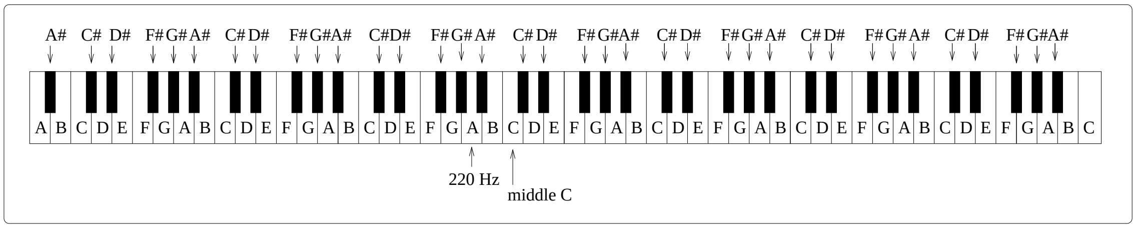

A# C# D# F# G# A# C# D# F# G# A# C# D# F# G# A# C# D# F# G# A# C# D# F# G# A# C# D# F# G# A# C# D# F# G# A#
A B C D E F G A B C D E F G A B C D E F G A B C D E F G A B C D E F G A B C D E F G A B C D E F G A B C
220 Hz
middle C

Figure 10.7: A piano keyboard.

A piano keyboard layout is shown in Figure 10.7. The white keys are labelled A, B, C, D, E, F, and G, with the sequence running from left to right and repeating for the length of the keyboard. The black keys fit into this sequence as "sharps", so that the black key between A and B is "A sharp", denoted $A^\#$. Thus, starting at any A key, the 12 keys to the right are A, $A^\#$, B, C, $C^\#$, D, $D^\#$, E, F, $F^\#$, G, and $G^\#$. The sequence then repeats. Notice that between some adjacent pairs of white keys there is no black key.

A piano keyboard is commonly tuned according to a rule requiring that each key (white and black) has a frequency $2^{1/12}$ times the frequency of the key to its immediate left. This makes the ratio of adjacent keys always the same ($2^{1/12}$), and it means that keys 12 keys apart have a ratio of frequencies exactly equal to 2 (since $(2^{1/12})^{12} = 2$). Two such keys are said to be an *o*ctave apart. Assuming that the key A below middle C has a frequency of 220 Hz, we can determine the frequency of every key on the keyboard. For instance, the A# to the right of this key has frequency $220 \times 2^{1/12} = 220 \times 1.059463094... \approx 233.08188\mathrm{Hz}$. The B to the right of this key has frequency $233.08188 \times 2^{1/12} \approx 246.94165\mathrm{Hz}$.

10.4 Exercises

Problem 10.1. Let's brush up on the required calculator skills. Use a calculator to approximate:

(a) 3^{π}

(b) $4^{2+\sqrt{5}}$

(c) π^{π}

(d) $5^{-\sqrt{3}}$

(e) 3^{π^2}

(f) $\sqrt{11^{\pi-7}}$

Problem 10.2. Put each equation in standard exponential form:

(a) $y = 3(2^{-x})$

(b) $y = 4^{-x/2}$

(c) $y = \pi^{\pi x}$

(d) $y = 1\left(\frac{1}{3}\right)^{3+\frac{x}{2}}$

(e) $y = \frac{5}{0.345^{2x-7}}$

(f) $y = 4(0.0003467)^{-0.4x+2}$

Problem 10.3. A colony of yeast cells is estimated to contain 10^6 cells at time $t = 0$. After collecting experimental data in the lab, you decide that the total population of cells at time t hours is given by the function

$$y = 10^6 e^{0.495105t}.$$

(a) How many cells are present after one hour?

(b) (True or False) The population of yeast cells will double every 1.4 hours.

(c) Cherie, another member of your lab, looks at your notebook and says : ...that formula is wrong, my calculations predict the formula for the number of yeast cells is given by the function

$$y = 10^6(2.042727)^{0.693147t}.$$

Should you be worried by Cherie's remark?

(d) Anja, a third member of your lab working with the same yeast cells, took these two measurements: 7.246×10^6 cells after 4 hours; 16.504×10^6 cells after 6 hours. Should you be worried by Anja's results? If Anja's measurements are correct, does your model over estimate or under estimate the number of yeast cells at time t?

Problem 10.4. (a) Find the frequency of middle C.

(b) Find the frequency of A above middle C.

(c) What is the frequency of the lowest note on the keyboard? Is there a way to solve this without simply computing the frequency of every key below A220?

(d) The Bosendorfer piano is famous, due in part, to the fact it includes additional keys at the left hand end of the keyboard, extending to the C below the bottom A on a standard keyboard. What is the lowest frequency produced by a Bosendorfer?

Problem 10.5. You have a chess board as pictured, with squares numbered 1 through 64. You also have a huge change jar with an unlimited number of dimes. On the first square you place one dime. On the second square you stack 2 dimes. Then you continue, always *doubling* the number from the previous square.

(a) How many dimes will you have stacked on the 10th square?

(b) How many dimes will you have stacked on the nth square?

(c) How many dimes will you have stacked on the 64th square?

(d) Assuming a dime is 1 mm thick, how high will this last pile be?

(e) The distance from the earth to the sun is approximately 150 million km. Relate the height of the last pile of dimes to this distance.

						63	64
						10	9
1	2	3					8

Problem 10.6. *Myoglobin* and *hemoglobin* are oxygen carrying molecules in the human body. Hemoglobin is found inside red blood cells, which flow from the lungs to the muscles through the bloodstream. Myoglobin is found in muscle cells. The function

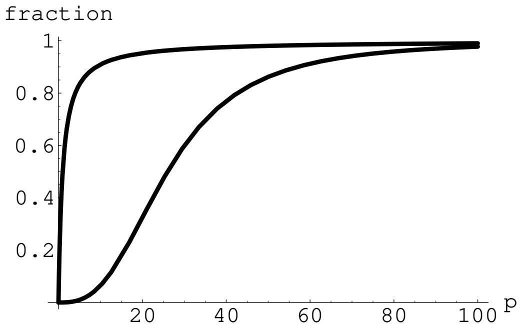
$$Y = M(p) = \frac{p}{1+p}$$

calculates the fraction of myoglobin saturated with oxygen at a given pressure p torrs. For example, at a pressure of 1 torr, $M(1) = 0.5$, which means half of the myoglobin (i.e. 50%) is oxygen saturated. (Note: More precisely, you need to use something called the "partial pressure", but the distinction is not important for this problem.) Likewise, the function

$$Y = H(p) = \frac{p^{2.8}}{26^{2.8} + p^{2.8}}$$

calculates the fraction of hemoglobin saturated with oxygen at a given pressure p.

(a) The graphs of $M(p)$ and $H(p)$ are given below on the domain $0 \le p \le 100$; which is which?



(b) If the pressure in the lungs is 100 torrs, what is the level of oxygen saturation of the hemoglobin in the lungs?

(c) The pressure in an active muscle is 20 torrs. What is the level of oxygen saturation of myoglobin in an active muscle? What is the level of hemoglobin in an active muscle?

(d) Define the efficiency of oxygen transport at a given pressure p to be $M(p) - H(p)$. What is the oxygen transport efficiency at 20 torrs? At 40 torrs? At 60 torrs? Sketch the graph of $M(p) - H(p)$; are there conditions under which transport efficiency is maximized (explain)?

Chapter 11

Exponential Modeling

Example 11.0.1. *A computer industry spokesperson has predicted that the number of subscribers to geton.com, an internet provider, will grow exponentially for the first 5 years. Assume this person is correct. If geton.com has 100,000 subscribers after 6 months and 750,000 subscribers after 12 months, how many subscribers will there be after 5 years?*

Solution. The solution to this problem offers a template for many exponential modeling applications. Since, we are assuming that the number of subscribers $N(x)$, where x represents years, is a function of exponential type,

$$N(x) = N_\circ b^x,$$

for some $N_\circ$ and $b > 1$. We are given two pieces of information about the values of $N(x)$:

$$\begin{aligned} N(0.5) &= 100{,}000; \text{ i.e., } N_0 b^{0.5} = 100{,}000, \text{ and} \\ N(1) &= 750{,}000; \text{ i.e., } \quad N_0 b = 750{,}000. \end{aligned}$$

We can use these two equations to solve for the two unknowns $N_\circ$ and b as follows: If we divide the second equation by the first, we get

$$\begin{aligned} &\frac{b^1}{b^{0.5}} = 7.5 \\ &b^{0.5} = b^{1/2} = \sqrt{b} = 7.5 \\ &\therefore b = 56.25. \end{aligned}$$

Plugging this value of b into either equation (say the first one), we can solve for $N_\circ$: $N_\circ = \frac{100{,}000}{(56.25)^{0.5}} = 13{,}333$. We conclude that the number of *geton.com* subscribers will be predicted by

$$N(x) = 13{,}333(56.25)^x.$$

In five years, we obtain $N(5) = 7{,}508{,}300{,}000{,}000$ subscribers, which exceeds the population of the Earth (which is between 5 and 6 billion)! □

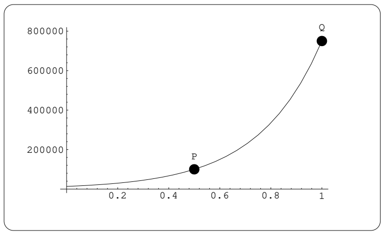


Figure 11.1: Finding the equation for $N(x) = N_0 b^x$.

There are two important conclusions we can draw from this problem. First, the given information provides us with two points on the graph of the function $N(x)$:

$$P = (0.5, 100{,}000)$$
$$Q = (1, 750{,}000).$$

More importantly, this example illustrates a very important principal we can use when modeling with functions of exponential type.

Important Fact 11.0.2. *A function of exponential type can be determined if we are given two data points on its graph.*

iii
CAUTION
!!!

When you use the above strategy to find the base b of the exponential model, make sure to write down a lengthy decimal approximation. As a rule of thumb, go for twice as many significant digits as you are otherwise using in the problem.

11.1 The Method of Compound Interest

You walk into a Bank with P_0 dollars (usually called principal), wishing to invest the money in a savings account. You expect to be rewarded by the Bank and paid interest, so how do you compute the total value of the account after t years?

The future value of the account is really a function of the number of years t elapsed, so we can write this as a function $P(t)$. Our goal is to see that $P(t)$ is a function of exponential type. In order to compute the future value of the account, the Bank provides any savings account investor with two important pieces of information:

r = annual (decimal) interest rate
n = the number of compounding periods per year

The number n tells us how many times each year the Bank will compute the total value $P(t)$ of the account. For example, if $n = 1$, the calculation is done at one-year intervals; if $n = 12$, the calculation is done each month, etc. The bank will compute the value of your account after a typical compounding period by using the *periodic rate of return* $\frac{r}{n}$. For example, if the interest rate percentage is 12% and the compounding period is monthly (i.e., $n = 12$), then the annual (decimal) interest rate is 0.12 and the periodic rate is $\frac{0.12}{12} = 0.01$.

iii
CAUTION
!!!

The number r always represents the decimal interest rate, which is a decimal between 0 and 1. If you are given the interest rate percentage (which is a positive number between 0 and 100), you need to convert to a decimal by dividing by 100.

11.1.1 Two Examples

Let's consider an example: $P_0 = \$1{,}000$ invested at the annual interest percentage of 8% compounded yearly, so $n = 1$ and $r = 0.08$. To compute the value $P(1)$ after one year, we will have

$$\begin{aligned} P(1) &= P_0 + (\text{periodic rate})P_0 \\ &= P_0 + rP_0 = P_0(1 + r) \\ &= \$1{,}000(1 + 0.08) = \$1{,}080. \end{aligned}$$

To compute the value after two years, we need to apply the periodic rate to the value of the account after one year:

$$\begin{aligned} P(2) &= P(1) + (\text{periodic rate})P(1) \\ &= P_0(1 + r) + rP_0(1 + r) = P_0(1 + r)^2 \\ &= \$1{,}000(1 + 0.08)^2 = \$1{,}166.40. \end{aligned}$$

Notice, the amount the Bank has paid after two years is \$166.40, which is slightly bigger than twice the \$80 paid after one year. To compute the value after three years, we need to apply the periodic rate to the value of the account after two years:

$$\begin{aligned} P(3) &= P(2) + (\text{periodic rate})P(2) \\ &= P_0(1 + r)^2 + rP_0(1 + r)^2 = P_0(1 + r)^3 \\ &= \$1{,}000(1 + 0.08)^3 = \$1{,}259.71. \end{aligned}$$

Again, notice the amount the Bank is paying after three years is \$259.71, which is slightly larger than three times the \$80 paid after one year. Continuing on in this way, to find the value after t years, we arrive at the formula

$$\begin{aligned} P(t) &= P_0(1 + r)^t \\ &= \$1{,}000(1.08)^t. \end{aligned}$$

In particular, after 5 and 10 years, the value of the account (to the nearest dollar) will be \$1,469 and \$2,159, respectively.

As a second example, suppose we begin with the same \$1,000 and the same annual interest percentage 8%, but now compound monthly, so $n = 12$ and $r = 0.08$. The value of the account after one compounding period is $P(1/12)$, since a month is one-twelfth of a year. Arguing as before, paying special attention that the periodic rate is now $\frac{r}{n} = \frac{0.08}{12}$, we have

$$\begin{aligned} P(1/12) &= P_0 + (\text{periodic rate})P_0 \\ &= P_0\left(1 + \frac{.08}{12}\right) \\ &= \$1{,}000(1 + 0.006667) = \$1{,}006.67. \end{aligned}$$

After two compounding periods, the value is $P(2/12)$,

$$\begin{aligned} P(2/12) &= P(1/12) + (\text{periodic rate})P(1/12) \\ &= P_0\left(1+\frac{.08}{12}\right) + \left(\frac{0.08}{12}\right)P_0\left(1+\frac{.08}{12}\right) \\ &= P_0\left(1+\frac{.08}{12}\right)^2 \\ &= \$1{,}000(1+0.006667)^2 = \$1{,}013.38. \end{aligned}$$

Continuing on in this way, after k compounding periods have elapsed, the value will be $P\left(\frac{k}{12}\right)$, which is computed as

$$P(k/12) = P_0\left(1+\frac{.08}{12}\right)^k.$$

It is possible to rewrite this formula to give us the value after t years, noting that t years will lead to $12t$ compounding periods; i.e., set $k = 12t$ in the previous formula:

$$P(t) = P_0\left(1+\frac{.08}{12}\right)^{12t}$$

For example, after 1, 5 and 10 years, the value of the account, to the nearest dollar, would be \$1,083, \$1,490, and \$2,220.

11.1.2 Discrete Compounding

The two examples above highlight a general formula for computing the future value of an account.

Important Fact 11.1.1 (Discrete compounding). *Suppose an account is opened with P_0 principal. If the decimal interest rate is r and the number of compounding periods per year is n, then the value $P(t)$ of the account after t years will be*

$$P(t) = P_0\left(1+\frac{r}{n}\right)^{nt}.$$

Notice, the future value $P(t)$ is a function of exponential type; the base is the number $\left(1+\frac{r}{n}\right)$, which will be greater than one. Since $P_0 > 0$, the graph will be qualitatively similar to the ones pictured in Figure 10.6(a).

Example 11.1.2. *At birth, your Uncle Hans secretly purchased a \$5,000 U.S. Savings Bond for \$2,500. The conditions of the bond state that the U.S. Government will pay a minimum annual interest rate of $r = 8.75\%$, compounded quarterly. Your Uncle has given you the bond as a gift, subject to the condition that you cash the bond at age 35 and buy a red Porsche.*

On your way to the Dealer, you receive a call from your tax accountant informing you of a 28% tax on the capital gain you realize through cashing in the bond; the capital gain is the selling price of the bond minus the purchase price. Before stepping onto the showroom floor, compute how much cash will you have on hand, after the U.S. Government shares in your profits.

Solution. The value of your bond after 35 years is computed by the formula in Fact 11.1.1, using $P_0 = \$2{,}500$, $r = 0.0875$, $n = 4$, and $t = 35$. Plugging this all in, we find that the selling price of the bond is

$$P(35) = \$2{,}500\left(1 + \frac{0.0875}{4}\right)^{4(35)} = \$51{,}716.42.$$

The capital gain will be \$51,716.42 - \$2,500 = \$49,216.42 and the tax due is \$(49,216.42)(0.28) = \$13,780.60. You are left with \$51,716.42 - \$13,780.60 = \$37,935.82. Better make that a used Porsche! □

11.2 The Number e and the Exponential Function

What happens to the future value of an investment of P_0 *dollars as the number of compounding periods is increased?* For example, return to our earlier example: $P_0 = \$1000$ and an annual interest percentage of 8%. After 1 year, the table below indicates the value of the investment for various compounding periods: yearly, quarterly, monthly, weekly, daily, and hourly.

n	**Compounding Period**	**Value after 1 year (to nearest dollar)**
1	yearly	$\$1{,}000(1 + 0.08)^1$ = \$1,080.00
4	quarterly	$\$1{,}000\left(1 + \frac{0.08}{4}\right)^4$= \$1,082.43
12	monthly	$\$1{,}000\left(1 + \frac{0.08}{12}\right)^{12}$ = \$1,083.00
52	weekly	$\$1{,}000\left(1 + \frac{0.08}{52}\right)^{52}$ = \$1,083.22
365	daily	$\$1{,}000\left(1 + \frac{0.08}{365}\right)^{365}$ = \$1,083.28
8,760	hourly	$\$1{,}000\left(1 + \frac{0.08}{8{,}760}\right)^{8{,}760}$ =\$1,083.29

We could continue on, considering "minute" and "second" compounding and what we will find is that the value will be at most \$1,083.29. This illustrates a general principal:

Important Fact 11.2.1. *Initially increasing the number of compounding periods makes a significant difference in the future value; however, eventually there appears to be a limiting value.*

Let's see if we can understand mathematically why this is happening. The first step is to recall the discrete compounding formula:

$$P(t) = P_0\left(1 + \frac{r}{n}\right)^{nt}.$$

If our desire is to study the effect of increasing the number of compounding periods, this means we want to see what happens to this formula as n gets BIG. To analyze this, it is best to rewrite the expression using a substitution trick: Set $z = \frac{n}{r}$, so that $n = rz$ and $\frac{r}{n} = \frac{1}{z}$. Plugging in, we have

$$\begin{aligned} P(t) &= P_0\left(1 + \frac{r}{n}\right)^{nt} \\ &= P_0\left(1 + \frac{1}{z}\right)^{rzt} \\ &= P_0\left(\left(1 + \frac{1}{z}\right)^{z}\right)^{rt}. \end{aligned} \tag{11.1}$$

So, since r is a fixed number and $z = \frac{n}{r}$, letting n get BIG is the same as letting z become BIG in (11.1). This all means we need to answer this new question: *What happens to the expression* $\left(1 + \frac{1}{z}\right)^z$ *as* z *becomes large?* On the one hand, the power in the expression is getting large; at the same time, the base is getting close to 1. This makes it very tricky to make quick predictions about the outcome. It is best to first tabulate some numerical data for the values of $y = g(z) = \left(1 + \frac{1}{z}\right)^z$ and look at a plot of this function graph on the domain $0.01 \leq z \leq 100$: See Figure 11.2.

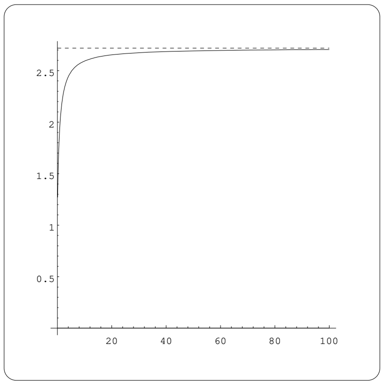
You can see from this plot, the graph of $y = \left(1 + \frac{1}{z}\right)^z$ approaches the "dashed" horizontal asymptote, as z becomes BIG. We will let the letter "e" represent the spot where this horizontal line crosses the vertical axis and $e \approx 2.7182818$. This number is only an approximation, since e is known to be an irrational number. What sets this irrational number apart from the ones you are familiar with (e.g. $\sqrt{2}$, π, etc.) is that defining the number e requires a "limiting" process. This will be studied a lot more in your *Calculus* course. The new number e is a positive number greater than 1, so we can study the function:

$$y = e^x. \tag{11.2}$$

Since $e > 1$, the graph will share the properties in Figure 10.6(a). This function is usually referred to as THE *exponential function.* Scientific calculators will have a key of the form "$\exp(x)$" or "e^x".

z	$\left(1+\frac{1}{z}\right)^z$
1	2
2	2.25
3	2.37037
4	2.4414
20	2.65329
100	2.70481
1000	2.71692
10^9	2.71828

(a) Data points for $\left(1+\frac{1}{z}\right)^z$.



(b) The graph of $\left(1+\frac{1}{z}\right)^z$.

Figure 11.2: What happens when z get very large?

11.2.1 Calculator drill

Plugging in $x = 1$, you can compute an approximation to e on your calculator; you should get $e = 2.7183$ to four decimal places. Make sure you can compute expressions like e^3, e^π, and $\sqrt{e} = e^{1/2}$; to four decimal places, you should get 20.0855, 23.1407, and 1.6487.

11.2.2 Back to the original problem...

We can now return to our future value formula (11.1) and conclude that as the number of compounding periods increases, the future value is approaching a limiting value:

$$P(t) = P_0\left(\left(1+\frac{1}{z}\right)^z\right)^{rt} \Longrightarrow P_0e^{rt}.$$

The right hand limiting formula $Q(t) = P_0e^{rt}$ computes the future value using what is usually referred to as *continuous compounding*. From the investors viewpoint, this is the best possible scheme for computing future value.

Important Fact 11.2.2 (Continuous compounding)**.** *The future value of* P_0 *dollars principal invested at an annual decimal interest rate of* r *under continuous compounding after* t *years is* $Q(t) = P_0e^{rt}$*; this value is always greater than the value of* $P_0\left(1+\frac{r}{n}\right)^{nt}$*, for any discrete compounding scheme. In fact,* P_0e^{rt} *is the limiting value.*

11.3 Exercises

Problem 11.1. In 1968, the U.S. minimum wage was \$1.60 per hour. In 1976, the minimum wage was \$2.30 per hour. Assume the minimum wage grows according to an exponential model $w(t)$, where t represents the time in years after 1960.

(a) Find a formula for $w(t)$.

(b) What does the model predict for the minimum wage in 1960?

(c) If the minimum wage was \$5.15 in 1996, is this above, below or equal to what the model predicts.

Problem 11.2. The town of Pinedale, Wyoming, is experiencing a population boom. In 1990, the population was 860 and five years later it was 1210.

(a) Find a linear model $l(x)$ and an exponential model $p(x)$ for the population of Pinedale in the year 1990+x.

(b) What do these models estimate the population of Pinedale to be in the year 2000?

Problem 11.3. In 1989, research scientists published a model for predicting the cumulative number of AIDS cases reported in the United States:

$$a(t) = 155\left(\frac{t-1980}{10}\right)^3, \quad \text{(thousands)}$$

where t is the year. This paper was considered a "relief", since there was a fear the correct model would be of exponential type. Pick two data points predicted by the research model $a(t)$ to construct a *new* exponential model $b(t)$ for the number of cumulative AIDS cases. Discuss how the two models differ and explain the use of the word "relief".

Problem 11.4. Define two new functions:

$$y = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

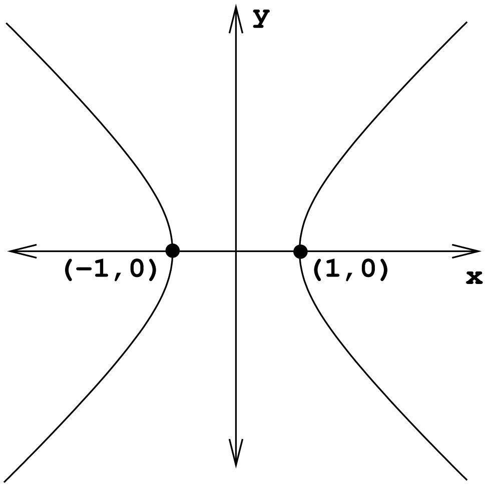
and

$$y = \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

These are called the basic *hyperbolic trigonometric functions*.

(a) Sketch rough graphs of these two functions.

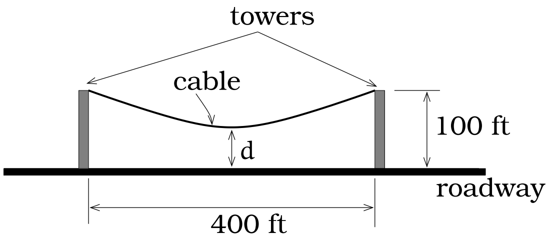
(b) The graph of the equation $x^2 - y^2 = 1$ is shown below; this is called the *unit hyperbola*. For any value a, show that the point $(x,y) = (\cosh(a), \sinh(a))$ is on the unit hyperbola. (Hint: Verify that $[\cosh(x)]^2 - [\sinh(x)]^2 = 1$, for all x.)



(c) A hanging cable is modeled by a portion of the graph of the function

$$y = a\cosh(\frac{x-h}{a}) + C,$$

for appropriate constants a, h and C. The constant h depends on how the coordinate system is imposed. A cable for a suspension bridge hangs from two 100 ft. high towers located 400 ft. apart. Impose a coordinate system so that the picture is symmetric about the y-axis and the roadway coincides with the x-axis. The hanging cable constant is $a = 500$ and $h = 0$. Find the minimum distance from the cable to the road.



Chapter 12

Logarithmic Functions

If we invest $P_0 = \$1{,}000$ *at an annual rate of* $r = 8\%$ *compounded continuously, how long will it take for the account to have a value of \$5000?*

The formula $P(t) = 1{,}000e^{0.08t}$ gives the value after t years, so we need to solve the equation:

$$\begin{aligned} 5{,}000 &= 1{,}000e^{0.08t} \\ 5 &= e^{0.08t}. \end{aligned}$$

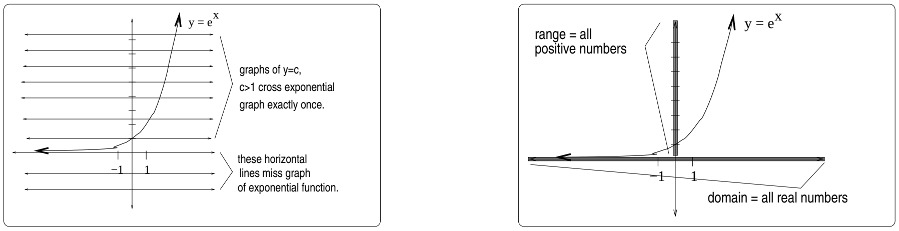
Unfortunately, algebraic manipulation will not lead to a further simplification of this equation; we are stuck! The required technique involves the theory of inverse functions. Assuming we can find the inverse function of $f(t) = e^t$, we can apply $f^{-1}(t)$ to each side of the equation and solve for t:

$$\begin{aligned} f^{-1}(5) &= f^{-1}(e^{0.08t}) = 0.08t \\ (12.5)f^{-1}(5) &= t \end{aligned}$$

The goal in this section is to describe the function f^{-1}, which is usually denoted by the symbol $f^{-1}(t) = \ln(t)$ and called the *natural logarithm* function. On your calculator, you will find a button dedicated to this function and we can now compute $\ln(5) = 1.60944$. Conclude that the solution is $t = 20.12$ years.

12.1 The Inverse Function of $y = e^x$

If we sketch a picture of the exponential function on the domain of all real numbers and keep in mind the properties in Fact 10.2.1, then every horizontal line above the x-axis intersects the graph of $y = e^x$ exactly once: See Figure 12.1(a). The range of the exponential function will consist of all possible y-coordinates of points on the graph. Using the graphical techniques of Chapter 6, we can see that the range of will be all POSITIVE real numbers: See Figure 12.1(b).


y = e^x
graphs of y=c,
c>1 cross exponential
graph exactly once.
these horizontal
lines miss graph
of exponential function.
−1
1
range = all
positive numbers
y = e^x
−1
1
domain = all real numbers

(a) Horizontal line test for $y = e^x$.

(b) The domain and range for $y = e^x$.

Figure 12.1: Properties needed to find the inverse of $f(x) = e^x$.

By the horizontal line test, this means the exponential function is one-to-one and the inverse rule $f^{-1}(c)$ will define a function

$$f^{-1}(c) = \begin{cases} \left(\begin{array}{l}\text{the unique solu-}\\ \text{tion of the equation}\\ c = e^x\end{array}\right), & \text{if } c > 0 \\ \text{(undefined)}, & \text{if } c \leq 0. \end{cases} \tag{12.1}$$

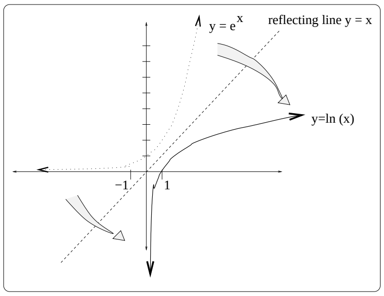


Figure 12.2: Visualizing the $y = \ln(x)$.

This inverse function is called the *natural logarithm* function, denoted $\ln(c)$. We can sketch the graph of the the natural logarithm as follows: First, by Fact 9.2.1, the domain of the function $\ln(y) = x$ is just the range of the exponential function, which we noted is all positive numbers. Likewise, the range of the function $\ln(y) = x$ is the domain of the exponential function, which we noted is all real numbers. Interchanging x and y, the graph of the natural logarithm function $y = \ln(x)$ can be obtained by flipping the graph of $y = e^x$ across the line $y = x$:

Important Facts 12.1.1 (Graphical features of natural log)**.** *The function* $y = \ln(x)$ *has these features:*

- *The largest domain is the set of positive numbers; e.g.* $\ln(-1)$ *makes no sense.*
- *The graph has* x*-intercept 1 and is increasing.*
- *The graph becomes closer and closer to the vertical axis as we approach* $x = 0$*; i.e. the* y*-axis is a vertical asymptote for the graph.*
- *The graph is unbounded as we move to the right.*

Any time we are working with an inverse function, symbolic properties are useful. Here are the important ones related to the natural logarithm.

Important Facts 12.1.2 (Natural log properties)**.** *We have the following properties:*

(a) *For any real number* x, $\ln(e^x) = x$.

(b) *For any positive number* x, $e^{\ln(x)} = x$.

(c) $\ln(b^t) = t\ln(b)$, *for* $b > 0$ *and* t *any real number;*

(d) $\ln(ba) = \ln(a) + \ln(b)$, *for all* $a, b > 0$;

(e) $\ln\left(\frac{b}{a}\right) = \ln(b) - \ln(a)$, *for all* $a, b > 0$.

The properties (c)-(e) are related to three of the rules of exponents in Facts 10.1.1. Here are the kinds of basic symbolic maneuvers you can pull off using these properties:

Examples 12.1.3.

(i) $\ln(8^3) = 3\ln(8) = 6.2383$; $\ln(6\pi) = \ln(6) + \ln(\pi) = 2.9365$; $\ln\left(\frac{3}{5}\right) = \ln(3) - \ln(5) = -0.5108$.

(ii) $\ln\left(\sqrt{x}\right) = \ln\left(x^{1/2}\right) = \frac{1}{2}\ln(x)$; $\ln\left(x^2 - 1\right) = \ln((x-1)(x+1)) = \ln(x-1) + \ln(x+1)$; $\ln\left(\frac{x^5}{x^2+1}\right) = \ln\left(x^5\right) - \ln\left(x^2+1\right) = 5\ln(x) - \ln\left(x^2+1\right)$.

Examples 12.1.4.
Given the equation $3^{x+1} = 12$, *we can solve for* x:

$$\begin{aligned} 3^{x+1} &= 12 \\ \ln\left(3^{x+1}\right) &= \ln(12) \\ (x+1)\ln(3) &= \ln(12) \\ x &= \frac{\ln(12)}{\ln(3)} - 1 = 1.2619. \end{aligned}$$

Example 12.1.5. *If \$2,000 is invested in a continuously compounding savings account and we want the value after 12 years to be \$130,000, what is the required annual interest rate? If, instead, the same \$2,000 is invested in a continuously compounding savings account with* $r = 6.4\%$ *annual interest, when will the exact account value be be \$130,000?*

Solution. In the first scenario,

$$\begin{aligned} 130{,}000 &= 2{,}000e^{12r} \\ 65 &= e^{12r} \\ \ln(65) &= \ln\left(e^{12r}\right) \\ \ln(65) &= 12r \\ r &= \frac{\ln(65)}{12} = 0.3479. \end{aligned}$$

This gives an annual interest rate of 34.79%. In the second scenario, we study the equation

$$\begin{aligned} 130{,}000 &= 2{,}000e^{(0.064)t} \\ 65 &= e^{(0.064)t} \\ t &= \frac{\ln(65)}{0.064} = 65.22. \end{aligned}$$

So, it takes over 65 years to accumulate \$130,000 under the second scheme. □

12.2 Alternate form for functions of exponential type

The standard model for an exponential function is $A(t) = A_0 b^t$, for some $b > 0$, $b \neq 1$, and $A_0 \neq 0$. Using the properties of the natural logarithm function,

$$b^t = \left(e^{\ln(b)}\right)^t = e^{t\ln(b)}.$$

This means that every function as in Definition 10.1.2 can be re-written using the exponential function e^t. Another way of saying this is that you really only need the function keys " e^t " and " $\ln(t)$ " on your calculator.

Important Fact 12.2.1 (Observation). *A function of exponential type can be written in the form*

$$A(t) = A_0 e^{at},$$

for some constants $A_0 \neq 0$ *and* $a \neq 0$.

By studying the sign of the constant a, we can determine whether the function exhibits exponential growth or decay. For example, given the function $A(t) = e^{at}$, if $a > 0$ (resp. $a < 0$), then the function exhibits exponential growth (resp. decay).

Examples 12.2.2.

(a) *The function* $A(t) = 200\left(2^t\right)$ *exhibits exponential growth and can be re-written as:*

$$A(t) = 200\left(e^{t\ln(2)}\right) = 200e^{0.69315t}$$

(b) *The function* $A(t) = 4e^{-0.2t}$ *exhibits exponential decay and can be re-written as:*

$$A(t) = 4e^{-0.2t} = 4\left(e^{-0.2}\right)^t = 4\left(0.81873^t\right).$$

12.3 The Inverse Function of $y = b^x$

For some topics in Chemistry and Physics (e.g. acid base equilibria and acoustics) it is useful to have on hand an inverse function for $y = b^x$, where $b > 0$ and $b \neq 1$. Just as above, we would show that $f(x) = b^x$ is one-to-one, the range is all positive numbers and obtain the graph using ideas in Figure 12.2. We will refer to the inverse rule as the *logarithm function base b*, denoted $\log_b(x)$, defined by the rule:

$$\log_b(c) = \begin{cases} \begin{pmatrix} \text{the unique solu-} \\ \text{tion of the equation} \\ c = b^x \end{pmatrix}, & \text{if } c > 0 \\ (\text{undefined}), & \text{if } c \leq 0. \end{cases}$$

We will need to consider two cases, depending on the magnitude of b: The important qualitative features of the logarithm function $y = \log_b(x)$ mirror Fact 12.1.1:

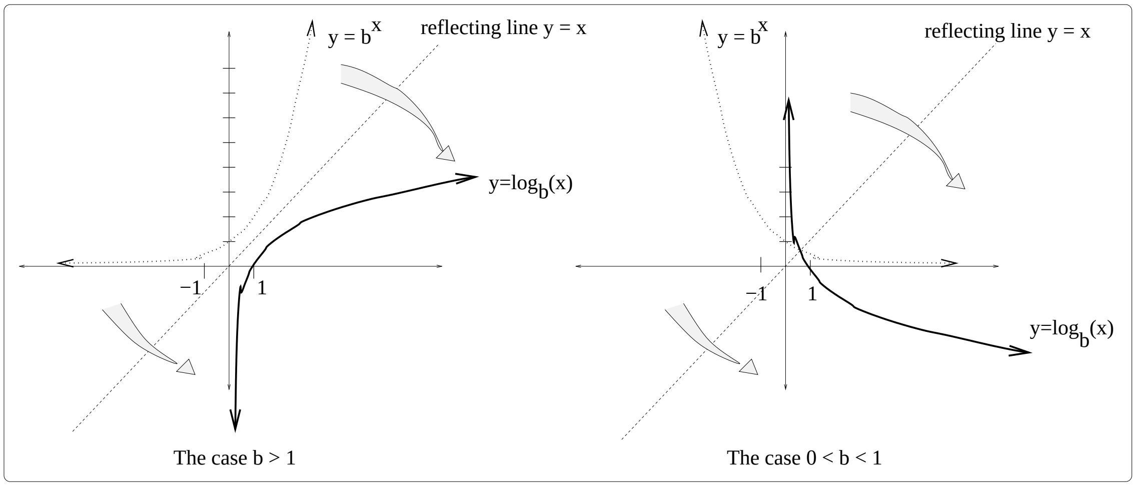


Figure 12.3: Cases to consider for b.

Important Facts 12.3.1 (Graphical features of general logs)**.** *The function* $y = \log_b(x)$ *has these features:*

- *The largest domain is the set of positive numbers; e.g.* $\log_b(-1)$ *is not defined.*
- *The graph has* x*-intercept 1 and is increasing if* $b > 1$ *(resp. decreasing if* $0 < b < 1$*).*
- *The graph becomes closer and closer to the vertical axis as we approach* $x = 0$*; this says the* y*-axis is a vertical asymptote for the graph.*

- *The graph is unbounded as we move to the right.*

Important Facts 12.3.2 (Log properties). *Fix a positive base* b, $b \neq 1$.

(a) *For any real number* x, $\log_b(b^x) = x$.

(b) *For any positive number* x, $b^{\log_b(x)} = x$.

(c) $\log_b(r^t) = t\log_b(r)$, *for* $r > 0$ *and* t *any real number;*

(d) $\log_b(rs) = \log_b(r) + \log_b(s)$, *for all* $r, s > 0$;

(e) $\log_b(\frac{r}{s}) = \log_b(r) - \log_b(s)$, *for all* $r, s > 0$.

It is common to simplify terminology and refer to the function $\log_b(x)$ as the *log base b* function, dropping the longer phrase "logarithm". Some scientific calculators will have a key devoted to this function. Other calculators may have a key labeled "log(x)", which is usually understood to mean the log base 10. However, many calculators only have the key "$\ln(x)$". This is not cause for alarm, since it is always possible to express $\log_b(x)$ in terms of the natural log function. Let's see how to do this, since it is a great application of the Log Properties listed in Fact 12.3.2.

Suppose we start with $y = \log_b(x)$. We will rewrite this in terms of the natural log by carrying out a sequence of algebraic steps below; make sure you see why each step is justified.

$$\begin{aligned} y &= \log_b(x) \\ b^y &= x \\ \ln(b^y) &= \ln(x) \\ y\ln(b) &= \ln(x) \\ y &= \frac{\ln(x)}{\ln(b)}. \end{aligned}$$

We have just verified a useful conversion formula:

Important Fact 12.3.3 (Log conversion formula). *For* x *a positive number and* $b > 0, b \neq 1$ *a base,*

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}.$$

For example,

$$\begin{aligned} \log_{10}(5) &= \frac{\ln(5)}{\ln(10)} &&= 0.699 \\ \log_{0.02}(11) &= \frac{\ln(11)}{\ln(0.02)} &&= -0.613 \\ \log_{20}\left(\frac{1}{2}\right) &= \frac{\ln(\frac{1}{2})}{\ln(20)} &&= -0.2314 \end{aligned}$$

The conversion formula allows one to proceed slightly differently when solving equations involving functions of exponential type. This is illustrated in the next example.

Example 12.3.4. *Ten years ago, you purchased a house valued at \$80,000. Your plan is to sell the house at some point in the future, when the value is at least \$1,000,000. Assume that the future value of the house can be computed using quarterly compounding and an annual interest rate of* 4.8%. *How soon can you sell the house?*

Solution. We can use the future value formula to obtain the equation

$$\begin{aligned} 1{,}000{,}000 &= 80{,}000\left(1+\frac{0.048}{4}\right)^{4t} \\ 12.5 &= (1.012)^{4t} \end{aligned}$$

Using the log base $b = 1.012$,

$$\begin{aligned} \log_{1.012}(12.5) &= \log_{1.012}\left((1.012)^{4t}\right) \\ \log_{1.012}(12.5) &= 4t \\ t &= \frac{\ln(12.5)}{4\ln(1.012)} = 52.934. \end{aligned}$$

Since you have already owned the house for 10 years, you would need to wait nearly 43 years to sell at the desired price. □

12.4 Measuring the Loudness of Sound

As we noted earlier, the reception of a sound wave by the ear gives rise to a vibration of the eardrum with a definite frequency and a definite amplitude. This vibration may also be described in terms of the variation of air pressure at the same point, which causes the eardrum to move. The perception that rustling leaves and a jet aircraft sound different involves two concepts: (1) the fact that the frequencies involved may differ; (2) the intuitive notion of “loudness”. This loudness is directly related to the force being exerted on the eardrum, which we refer to as the intensity of the sound. We can try to measure the intensity using some sort of scale. This becomes challenging, since the human ear is an amazing instrument, capable of hearing a large range of sound intensities. For that reason, a logarithmic scale becomes most useful. The *sound pressure level* β of a sound is defined by the equation

$$\beta = 10\log_{10}\left(\frac{I}{I_0}\right), \tag{12.2}$$

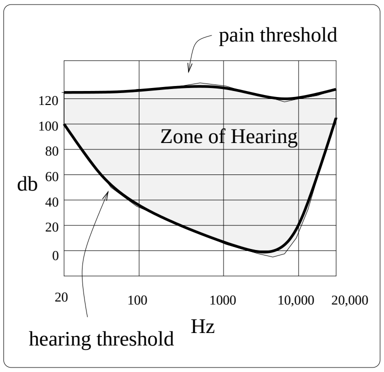
where I_0 is an arbitrary reference intensity which is taken to correspond with the average faintest sound which can be heard and I is the intensity of the sound being measured. The units used for β are called *decibels*, abbreviated “db”. (Historically, the units of loudness were called *bels*, in honor of Alexander Graham Bell, referring to the quantity $\log_{10}\left(\frac{I}{I_0}\right)$.) Notice, in the case of sound of intensity $I = I_0$, we have a sound pressure level of

$$\beta = 10\log_{10}\left(\frac{I_0}{I_0}\right) = 10\log_{10}(1) = 10(0) = 0.$$

We refer to any sound of intensity I_0 as having a sound pressure level at the *threshold of hearing*. At the other end of the scale, a sound of intensity the maximum the eardrum can tolerate has an average sound pressure level of about 120 db. The Table 12.4(a) gives a hint of the sound pressure levels associated to some common sounds.

Source of Noise	**Sound Pressure Level in db**
Threshold of pain	120
Riveter	95
Busy Street Traffic	70
Ordinary Conversation	65
Quiet Auto	50
Background Radio	40
Whisper	20
Rustle of Leaves	10
Threshold of Hearing	0

(a) Sources of noise levels.



(b) Graphing noise levels.

Figure 12.4: Considering noise levels.

It turns out that the above comments on the threshold of hearing and pain are really only averages and depend upon the frequency of the given sound. In fact, while the threshold of pain is on average close to 120 db across all frequencies between 20 Hz and 20,000 Hz, the threshold of hearing is much more sensitive to frequency. For example, for a tone of 20 Hz (something like the ground-shaking rumble of a passing freight train), the sound pressure level needs to be relatively high to be heard; 100 db on average. As the frequency increases, the required sound pressure level for hearing tends to drop down to 0 db around 2000 Hz. An examination by a hearing specialist can determine the precise sensitivities of your ear across the frequency range, leading to a plot of your “envelope of hearing”; a sample plot is given in Figure 12.4(b). Such a plot would differ from person to person and is helpful in isolating hearing problems.

Example 12.4.1. *A loudspeaker manufacturer advertises that their model no. 801 speaker produces a sound pressure level of 87 db when a reference test tone is applied. A competing speaker company advertises that* *their model X-1 speaker produces a sound pressure level of 93 db when fed the same test signal. What is the ratio of the two sound intensities produced by these speakers? If you wanted to find a speaker which produces a sound of intensity twice that of the no. 801 when fed the test signal, what is its sound pressure level?*

Solution. If we let I_1 and I_2 refer to the sound intensities of the two speakers reproducing the test signal, then we have two equations:

$$\begin{aligned} 87 &= 10\log_{10}\left(\frac{I_1}{I_0}\right) \\ 93 &= 10\log_{10}\left(\frac{I_2}{I_0}\right) \end{aligned}$$

Using log properties, we can solve the first equation for I_1:

$$\begin{aligned} 87 &= 10\log_{10}\left(\frac{I_1}{I_0}\right) = 10\log_{10}(I_1) - 10\log_{10}(I_0) \\ \log_{10}(I_1) &= 8.7 + \log_{10}(I_0) \\ 10^{\log_{10}(I_1)} &= 10^{8.7+\log_{10}(I_0)} \\ I_1 &= 10^{8.7}10^{\log_{10}(I_0)} = 10^{8.7}I_0. \end{aligned}$$

Similarly, we find that $I_2 = 10^{9.3}I_0$. This means that the ratio of the intensities will be

$$\frac{I_2}{I_1} = \frac{10^{9.3}I_0}{10^{8.7}I_0} = 10^{0.6} = 3.98.$$

This means that the test signal on the $X-1$ speaker produces a sound pressure level nearly 4 times that of the same test signal on the no. 801 speaker.

To finish the problem, imagine a third speaker which produces a sound pressure level β, which is twice that of the first speaker. If I_3 is the corresponding intensity of the sound, then as above, $I_3 = 10^{(\beta/10)}I_0$. We are assuming that $I_3 = 2I_1$, so this gives us the equation

$$\begin{aligned} I_1 &= \frac{1}{2}I_3 \\ 10^{8.7}I_0 &= \frac{1}{2}10^{(\beta/10)}I_0 \\ \log_{10}\left(10^{8.7}\right) &= \log_{10}\left(\frac{1}{2}10^{(\beta/10)}\right) \\ 8.7 &= \log_{10}\left(\frac{1}{2}\right) + \log_{10}\left(10^{(\beta/10)}\right) \\ 8.7 &= -0.30103 + \left(\frac{\beta}{10}\right) \\ 90 &= \beta \end{aligned}$$

So, the test signal on the third speaker must produce a sound pressure level of 90 db. $\square$

12.5 Exercises

Problem 12.1. These problems will help you develop your skills with logarithms.

(a) Compute: $\log_5 3$, $\log_e 11$, $\log_{\sqrt{2}} \pi$, $\log_2 10$, $\log_{10} 2$.

(b) Solve for x: $35 = e^x$, $\log_3 x = e$, $\log_3 5 = xe^3$.

(c) Solve each of these equations for x in terms of y: $y = 10^x$, $3y = 10^x$, $y = 10^{3x}$.

Problem 12.2. As light from the surface penetrates water, its intensity is diminished. In the clear waters of the Caribbean, the intensity is decreased by 15 percent for every 3 meters of depth. Thus, the intensity will have the form of a general exponential function.

(a) If the intensity of light at the water's surface is $I_\circ$, find a formula for $I(d)$, the intensity of light at a depth of d meters. Your formula should depend on $I_\circ$ and d.

(b) At what depth will the light intensity be decreased to 1% of its surface intensity?

Problem 12.3. Rewrite each function in the form $y = A_\circ e^{at}$, for appropriate constants $A_\circ$ and a.

(a) $y = 13(3^t)$

(b) $y = 2(\frac{1}{8})^t$

(c) $y = -7(1.567)^{t-3}$

(d) $y = -17(2.005)^{-t}$

(e) $y = 3(14.24)^{4t}$

Problem 12.4. (a) If you invest P_o dollars at 7% annual interest and the future value is computed by continuous compounding, how long will it take for your money to double?

(b) Suppose you invest P_o dollars at r% annual interest and the future value is computed by continuous compounding. If you want the value of the account to double in 2 years, what is the required interest rate?

(c) A rule of thumb used by many people to determine the length of time to double an investment is the *rule of 70*. The rule says it takes about $t = \frac{70}{r}$ years to double the investment. Graphically compare this rule to the one isolated in part b. of this problem.

Problem 12.5. The length of some fish are modeled by a *von Bertalanffy* growth function. For Pacific halibut, this function has the form

$$L(t) = 200\,(1 - 0.956\,e^{-0.18t})$$

where $L(t)$ is the length (in centimeters) of a fish t years old.

(a) What is the length of a new-born halibut at birth?

(b) Use the formula to estimate the length of a 6–year–old halibut.

(c) At what age would you expect the halibut to be 120 cm long?

(d) What is the practical (physical) significance of the number 200 in the formula for $L(t)$?

Problem 12.6. A *cancerous cell* lacks normal biological growth regulation and can divide continuously. Suppose a single mouse skin cell is cancerous and its *mitotic cell cycle* (the time for the cell to divide once) is 20 hours. The number of cells at time t grows according to an exponential model.

(a) Find a formula $C(t)$ for the number of cancerous skin cells after t hours.

(b) Assume a typical mouse skin cell is spherical of radius 50×10^{-4} cm. Find the combined volume of all cancerous skin cells after t hours. When will the volume of cancerous cells be 1 cm^3?

Problem 12.7. Your Grandfather purchased a house for \$55,000 in 1952 and it has increased in value according to a function $y = v(x)$, where x is the number of years owned. These questions probe the future value of the house under various mathematical models.

(a) Suppose the value of the house is \$75,000 in 1962. Assume $v(x)$ is a linear function. Find a formula for $v(x)$. What is the value of the house in 1995? When will the house be valued at \$200,000?

(b) Suppose the value of the house is \$75,000 in 1962 and \$120,000 in 1967. Assume $v(x)$ is a quadratic function. Find a formula for $v(x)$. What is the value of the house in 1995? When will the house be valued at \$200,000?

(c) Suppose the value of the house is \$75,000 in 1962. Assume $v(x)$ is a function of exponential type. Find a formula for $v(x)$. What is the value of the house in 1995? When will the house be valued at \$200,000?

Problem 12.8. Solve the following equations for x:

(a) $\log_3(5) = \log_2(x)$

(b) $10^{\log_2(x)} = 3$

(c) $3^{5^x} = 7$

(d) $\log_2(\ln(x)) = 3$

(e) $e^x = 10^5$

(f) $2^{3x+5} = 3^2$

Problem 12.9. A ship embarked on a long voyage. At the start of the voyage, there were 500 ants in the cargo hold of the ship. One week into the voyage, there were 800 ants. Suppose the population of ants is an exponential function of time.

(a) How long did it take the population to double?

(b) How long did it take the population to triple?

(c) When were there be 10,000 ants on board?

(d) There also was an exponentially-growing population of anteaters on board. At the start of the voyage there were 17 anteaters, and the population of anteaters doubled every 2.8 weeks. How long into the voyage were there 200 ants per anteater?

Problem 12.10. The populations of termites and spiders in a certain house are growing exponentially. The house contains 100 termites the day you move in. After 4 days, the house contains 200 termites. Three days after moving in, there are two times as many termites as spiders. Eight days after moving in, there were four times as many termites as spiders.

How long (in days) does it take the population of spiders to triple?

Problem 12.11. In 1987, the population of Mexico was estimated at 82 million people, with an annual growth rate of 2.5%. The 1987 population of the United States was estimated at 244 million with an annual growth rate of 0.7 %. Assume that both populations are growing exponentially.

(a) When will Mexico double its 1987 population?

(b) When will the United States and Mexico have the same population?

Problem 12.12. The cities of Abnarca and Bonipto have populations that are growing exponentially. In 1980, Abnarca had a population of 25,000 people. In 1990, its population was 29,000.

Bonipto had a population of 34,000 in 1980. The population of Bonipto doubles every 55 years.

(a) How long does it take the population of Abnarca to double?

(b) When will Abnarca's population equal that of Bonipto?

Chapter 13

Three Construction Tools

Sometimes the composition of two functions can be understood by graphical manipulation. When we discussed quadratic functions and parabolas in the previous section, certain key graphical maneuvers were laid out. In this section, we extend those graphical techniques to general function graphs.

13.1 A Low-tech Exercise

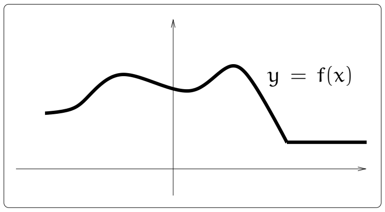


Figure 13.1: Start with some curve.

This section is all about building new functions from ones we already have in hand. This can be approached symbolically or graphically. Let's begin with a simple hands-on exercise involving the curve in Figure 13.1.

By the vertical line test, we know this represents the graph of a function $y = f(x)$. With this picture and a piece of bendable wire we can build an INFINITE number of new functions from the original function. Begin by making a "model" of this graph by bending a piece of wire to the exact shape of the graph and place it right on top of the curve. The wire model can be manipulated in a variety of ways: slide the model back and forth horizontally, up and down vertically, expand or compress the model horizontally or vertically.

Another way to build new curves from old ones is to exploit the built in symmetry of the xy-coordinate system. For example, imagine reflecting the graph of $y = f(x)$ across the x-axis or the y-axis.

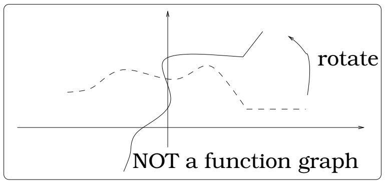
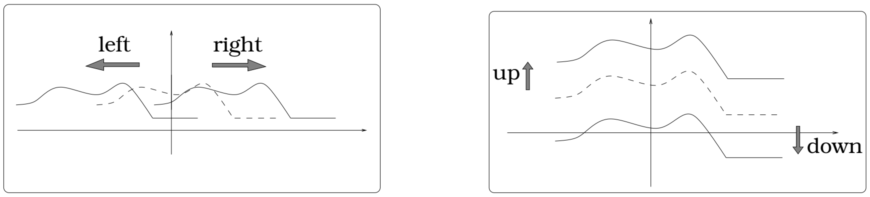


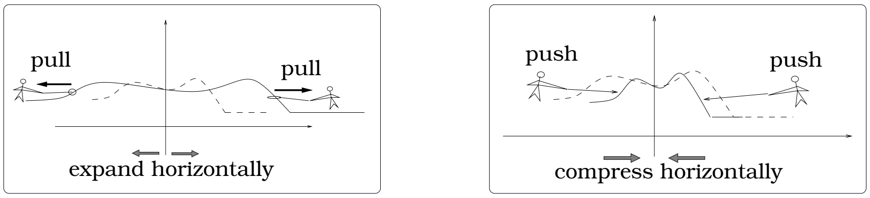
Figure 13.2: Rotating a curve.

In all of the above cases, we moved from the original wire model of our function graph to a *new curve* that (by the vertical line test) is the graph of a *new function*. The big caution in all this is that we are NOT ALLOWED to rotate or twist the curve; this kind of maneuver does lead to a new curve, but it may not be the graph of a function: See Figure 13.2.

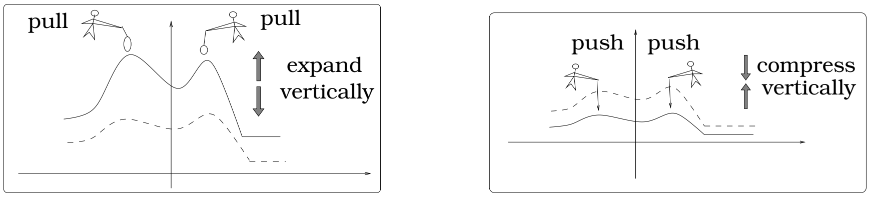
The pictures in Figure 13.3 highlight most of what we have to say in this section; the hard work remaining is a symbolic reinterpretation of these graphical operations.


left
right
up
down

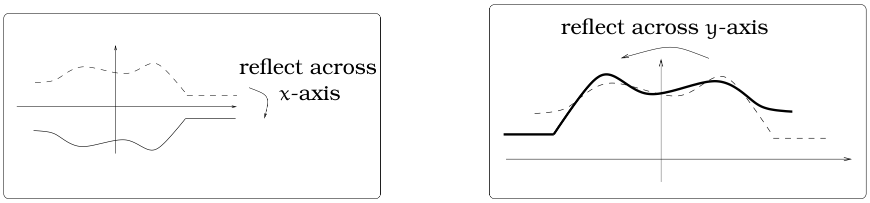
Horizontal shift. Vertical shift.


pull
pull
expand horizontally
push
push
compress horizontally

Horizontal expansion. Horizontal compression.


pull
pull
expand
vertically
push
push
compress
vertically

Vertical expansion. Vertical compression.


reflect across
x-axis
reflect across y-axis

Vertical reflection. Horizontal reflection.

Figure 13.3: Shifting, dilating, and reflecting $y = f(x)$.

13.2 Reflection

In order to illustrate the technique of *reflection*, we will use a concrete example:

Function:	$y = p(x) = 2x + 2$
Domain:	$-2 \leq x \leq 2$
Range:	$-2 \leq y \leq 6$

As we know, the graph of $y = p(x)$ on the domain $-2 \leq x \leq 2$ is a line of slope 2 with y-intercept 2, as pictured in Figure 13.4(a). Now, start with the function equation $y = p(x) = 2x + 2$ and replace every occurrence of "y" by "$-y$." This produces the new equation $-y = 2x+2$; or, equivalently,

$$y = q(x) = -2x - 2.$$

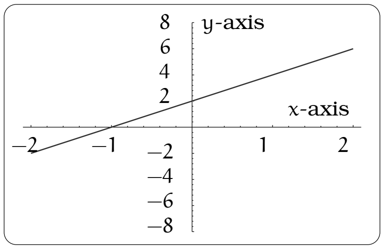
The domain is still $-2 \leq x \leq 2$, but the range will change; we obtain the new range by replacing "y" by "$-y$" in the original range: $-2 \leq -y \leq 6$; so $2 \geq y \geq -6$. The graph of this function is a DIFFERENT line; this one has slope -2 and y-intercept -2. We contrast these two curves in Figure 13.4(b), where $q(x)$ is graphed as the "dashed line" in the same picture with the original $p(x)$. Once we do this, it is easy to see how the graph of $q(x)$ is really just the original line reflected across the x-axis.

Next, take the original function equation $y = p(x) = 2x + 2$ and replace every occurrence of "x" by "$-x$." This produces a new equation $y = 2(-x) + 2$; or, equivalently,

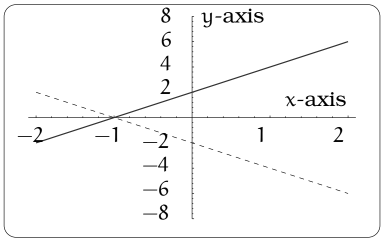
$$y = r(x) = -2x + 2.$$

The domain must also be checked by replacing "x" by "$-x$" in the original domain condition: $-2 \leq -x \leq 2$, so $2 \geq x \geq -2$. It just so happens in this case, the domain is unchanged. This is yet another DIFFERENT line; this one has slope -2 and y-intercept 2. We contrast these two curves in Figure 13.4(c), where $r(x)$ is graphed as the "dashed line" in the same picture with the original $p(x)$. Once we do this, it is easy to see how the graph of $r(x)$ is really just the original curve reflected across the y-axis.

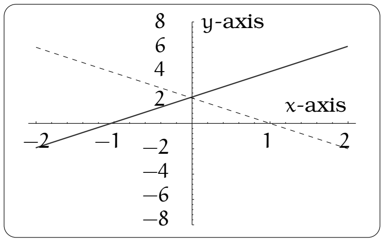
This example illustrates a general principle referred to as the *reflection principle*.



(a) Graph of $p(x) = 2x + 2$.



(b) Graph of $q(x) = -2x - 2$.



(c) Graph of $r(x) = -2x + 2$.

Figure 13.4: Reflecting $y = p(x)$.

Important Facts 13.2.1 (Reflection). *Let* $y = f(x)$ *be a function equation.*

(i) We can reflect the graph across the x*-axis and the resulting curve is the graph of the new function obtained by replacing "*y*" by "*$-y$*" in the original equation. The domain is the same as the domain for* $y = f(x)$*. If the range for* $y = f(x)$ *is* $c \leq y \leq d$*, then the range of* $-y = f(x)$ *is* $c \leq -y \leq d$*. In other words, the reflection across the* x*-axis is the graph of* $y = -f(x)$*.*

(ii) We can reflect the graph across the y*-axis and the resulting curve is the graph of the new function obtained by replacing "*x*" by "*$-x$*" in the original equation. The range is the same as the range for* $y = f(x)$*. If the domain for* $y = f(x)$ *is* $a \leq x \leq b$*, then the domain*

of $y = f(-x)$ *is* $a \leq -x \leq b$*. Using composition notation, the reflection across the* y*-axis is the graph of* $y = f(-x)$*.*

Example 13.2.2. *Consider the parallelogram-shaped region* $\mathcal{R}$ *with vertices* $(0,2)$*,* $(0,-2)$*,* $(1,0)$*, and* $(-1,0)$*. Use the reflection principle to find functions whose graphs bound* $\mathcal{R}$*.*

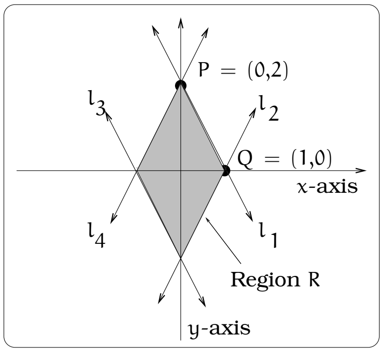


Figure 13.5: The region $\mathcal{R}$.

Solution. Here is a picture of the region $\mathcal{R}$: First off, using the two point formula for the equation of a line, we find that the line ℓ_1 passing through the points $P = (0,2)$ and $Q = (1,0)$ is the graph of the function $y = f_1(x) = -2x + 2$. By Fact 13.2.1 (i), ℓ_2 is the graph of the equation $-y = -2x + 2$, which we can write as the function $y = f_2(x) = 2x - 2$. By Fact 13.2.1 (ii) applied to ℓ_2, the line ℓ_3 is the graph of the function $y = f_3(x) = -2x - 2$. Finally, by Fact 13.2.1 (i) applied to ℓ_3, the line ℓ_4 is the graph of the equation $-y = -2x - 2$, which we can write as the function $y = f_4(x) = 2x + 2$. □

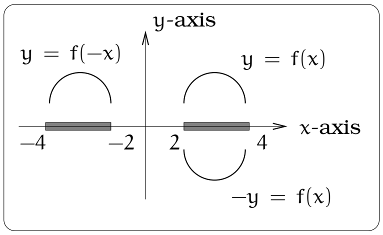


Figure 13.6: Reflecting the semicircle.

Figure 13.6 illustrates the fact that we need to be careful about the domain of the original function when using the reflection principle. For example, consider $y = f(x) = 1 + \sqrt{1-(x-3)^2}$. The largest possible domain of x-values is $2 \leq x \leq 4$ and the graph is an upper semicircle of radius 1 centered at the point $(3,1)$.

Reflection across the x-axis gives the graph of $y = -1 - \sqrt{1-(x-3)^2}$ on the same domain; reflection across the y-axis gives the graph of $y = 1 + \sqrt{1-(x+3)^2}$ on the new domain $-4 \leq x \leq -2$.

13.3 Shifting

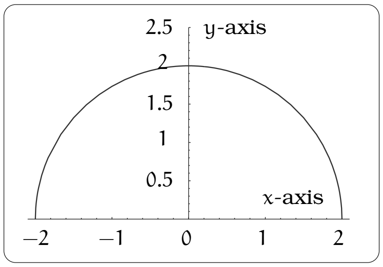


Figure 13.7: Graph of $y = \sqrt{4-x^2}$.

Let's start out with the function $y = f(x) = \sqrt{4-x^2}$, which has a largest possible domain $-2 \leq x \leq 2$. From Chapter 6, the graph of this equation is an upper semicircle of radius 2 centered at the origin $(0,0)$. Sliding the graph back and forth horizontally or vertically (or both), never rotating or twisting, we are led to the "dashed curves" below (contrasted with the original graph which is plotted with a solid curve). This describes some shifted curves on a pictorial level, but what are the underlying equations? For this example, we can use the fact that all of the shifted curves are still semicircles and Chapter 6 tells us how to find their equations.

The lower right-hand dashed semicircle is of radius 2 and is centered at $(3,0)$, so the corresponding equation must be $y = \sqrt{4-(x-3)^2}$. The upper left-hand dashed semicircle is of radius 2 and centered at $(0,3)$, so the corresponding equation must be $y = 3+\sqrt{4-x^2}$. The upper right-hand dashed semicircle is of radius 2 and centered at $(3,3)$, so the corresponding equation must be $y = 3+\sqrt{4-(x-3)^2}$.

Keeping this same example, we can continue this kind of shifting more generally by thinking about the effect of making the following three replacements in the equation $y = \sqrt{4-x^2}$:

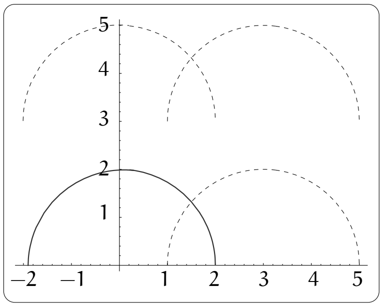


Figure 13.8: Shifting the upper semicircle.

$$\begin{aligned} x &\rightsquigarrow x-h \\ y &\rightsquigarrow y-k \\ (x \text{ and } y) &\rightsquigarrow (x-h \text{ and } y-k). \end{aligned}$$

These substitutions lead to three new equations, each the equation of a semicircle:

$$\begin{aligned} y &= \sqrt{4-(x-h)^2}; && \Leftarrow \begin{cases} \text{Upper semicircle with radius} \\ \text{2 and center } (h,0). \end{cases} \\ y-k &= \sqrt{4-x^2}; \text{ and,} && \Leftarrow \begin{cases} \text{Upper semicircle with radius} \\ \text{2 and center } (0,k). \end{cases} \\ y-k &= \sqrt{4-(x-h)^2}. && \Leftarrow \begin{cases} \text{Upper semicircle with radius} \\ \text{2 and center } (h,k). \end{cases} \end{aligned}$$

There are three potentially confusing points with this example:

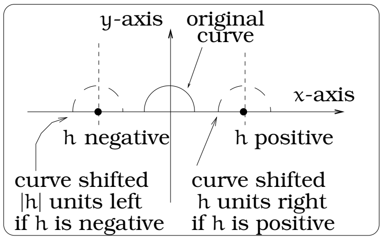

y-axis
original curve
x-axis
h negative
h positive
curve shifted |h| units left if h is negative
curve shifted h units right if h is positive

Figure 13.9: Potentially confusing points.

- Be careful with the sign (i.e., ±) of h and k. In Figure 13.9, if $h = 1$, we horizontally shift the semicircle 1 unit to the right; whereas, if $h = -1$, we horizontally shift the semicircle -1 units to the right. But, shifting -1 unit to the right is the same as shifting 1 unit to the left! In other words, if h is positive, then a horizontal shift by h will move the graph $|h|$ units to the right; if h is negative, then a horizontal shift by h will move the graph $|h|$ units to the left.

- If k is positive, then a vertical shift by k will move the graph $|k|$ units up; if k is negative, then a vertical shift by k will move the graph $|k|$ units down. These conventions insure that the “positivity” of h and k match up with “rightward” and “upward” movement of the graph.

- When shifting, the domain of allowed x-values may change.

This example illustrates an important general principle referred to as the *shifting principle.*

Important Facts 13.3.1 (Shifting)**.** *Let* $y = f(x)$ *be a function equation.*

(i) If we replace "x*" by "*$x - h$*" in the original function equation, then the graph of the resulting new function* $y = f(x - h)$ *is obtained by horizontally shifting the graph of* $f(x)$ *by* h*. If* h *is positive, the picture shifts to the right* h *units; if* h *is negative, the picture shifts to the left* h *units. If the domain of* $f(x)$ *is an interval* $a \leq x \leq b$*, then the domain of* $f(x - h)$ *is* $a \leq x - h \leq b$*. The range remains unchanged under horizontal shifting.*

(ii) If we replace "y*" by "*$y - k$*" in the original function equation, then the graph of the resulting new function* $y = f(x) + k$ *is obtained by vertically shifting the graph of* $f(x)$ *by* k*. If* k *is positive, the picture shifts upward* k *units; if* k *is negative, the picture shifts downward* k *units. If the range of* $f(x)$ *is an interval* $c \leq y \leq d$*, then the range of* $f(x) + k$ *is* $c \leq y - k \leq d$*. The domain remains unchanged under vertical shifting.*

13.4 Dilation

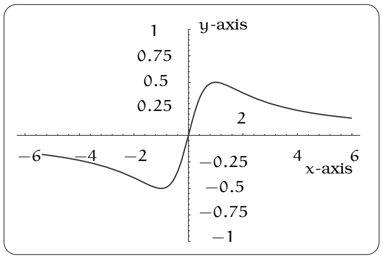


Figure 13.10: Graph of $y = f(x) = \frac{x}{x^2+1}$.

To introduce the next graphical principle we will look at the function

$$y = f(x) = \frac{x}{x^2 + 1}.$$

Using a graphing device, we have produced a plot of the graph on the domain $-6 \leq x \leq 6$. Figure 13.10 shows the curve has a high point H (like a "mountain peak") and a low point L (like a "valley"). Using a graphing device, we can determine that the high point is $H = (1, \frac{1}{2})$ (it lies on the line with equation $y = \frac{1}{2}$) and the low point is $L = \left(-1, -\frac{1}{2}\right)$ (it lies on the line with equation line $y = -\frac{1}{2}$), so the range is $-\frac{1}{2} \leq y \leq \frac{1}{2}$. Draw two new horizontal lines with equations $y = 2 \cdot \pm\frac{1}{2} = \pm 1$. Grab the high point H on the curve and uniformly pull straight up, so that the high point now lies on the horizontal line $y = 1$ at $(1, 1)$. Repeat this process by pulling L straight downward, so that the low point is now on the line $y = -1$ at $(-1, -1)$. We end up with the "stretched dashed curve" illustrated in Figure 13.11(a). In terms of the original function equation $y = \frac{x}{x^2+1}$, we are simply describing the graphical effect of multiplying the y-coordinate of every point on the curve by the positive number 2. In other words, the dashed curve is the graph of $y = \frac{2x}{x^2+1} = 2\left(\frac{x}{x^2+1}\right)$.

Next, draw the two horizontal lines with equations $y = \frac{1}{2} \cdot \pm\frac{1}{2} = \pm\frac{1}{4}$. Grab the high point H on the curve (in Figure 13.10) and uniformly push straight down, so that the high point now lies on the horizontal line $y = \frac{1}{4}$ at $(1, \frac{1}{4})$. Repeat this process at the low point L by pushing the curve straight upward, so that the low point is now on the line $y = -\frac{1}{4}$ at $(-1, -\frac{1}{4})$. We end up with the new ”dashed curve” illustrated in Figure 13.11(b). In terms of the original function equation $y = \frac{x}{x^2+1}$, we are simply describing the graphical effect of multiplying the y-coordinate of every point on the curve by the positive number $\frac{1}{2}$. In other words, the dashed curve is the graph of $y = \frac{x}{2(x^2+1)}$.

We could repeat this process systematically:

- Pick a positive number c.
- Draw the two horizontal lines $y = c \cdot \pm\frac{1}{2}$. If $c > 1$, then the graph of $y = \frac{c}{2}$ is parallel and above the graph of $y = \frac{1}{2}$. On the other hand, if $0 < c < 1$, then this new line is parallel and below $y = \frac{1}{2}$.
- Uniformly deform the original graph (in Figure 13.10) so that the new curve has it’s high and low points just touching $y = \pm\frac{c}{2}$. This will involve vertically stretching or compressing, depending on whether $1 < c$ or $0 < c < 1$, respectively. A number of possibilities are pictured in Figure 13.11(c).

We refer to each new dashed curve as a *vertical dilation* of the original (solid) curve. This example illustrates an important principle.

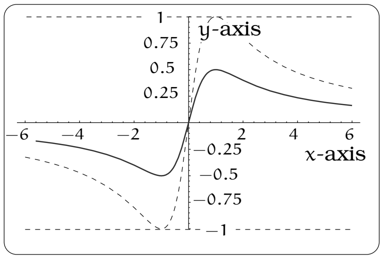
Important Facts 13.4.1 (Vertical dilation). *Let* $c > 0$ *be a positive number and* $y = f(x)$ *a function equation.*

(i) If we replace “ y ” by “ $\frac{y}{c}$ ” in the original equation, then the graph of the resulting new equation is obtained by vertical dilation of the graph of $y = f(x)$. *The domain of* x*-values is not affected.*

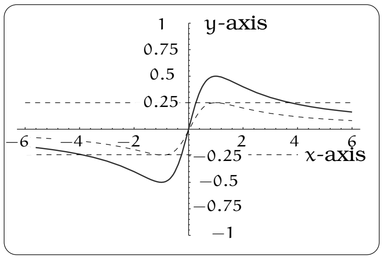
(ii) If $c > 1$, *then the graph of* $\frac{y}{c} = f(x)$ *(i.e.,* $y = cf(x)$*) is a vertically stretched version of the original graph.*

(iii) If $0 < c < 1$, *then the graph of* $\frac{y}{c} = f(x)$ *(i.e.,* $y = cf(x)$*) is a vertically compressed version of the original graph.*

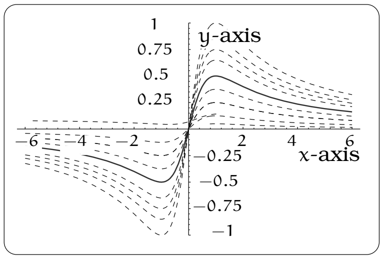
If we combine dilation with reflection across the x-axis, we can determine the graphical relationship between $y = f(x)$ and $y = cf(x)$, for *any* constant c. The key observation is that reflection across the x-axis corresponds to the case $c = -1$.



(a) Vertical expansion.



(b) Vertical compression.



(c) Many possibilities.

Figure 13.11: Dilating $y = f(x)$.

Example 13.4.2. *Describe the relationship between the graphs of*

$$\begin{aligned} y = f(x) &= \sqrt{1-(x+1)^2}, \\ y = -f(x) &= -\sqrt{1-(x+1)^2}, \textit{ and} \\ y = -4f(x) &= -4\sqrt{1-(x+1)^2}. \end{aligned}$$

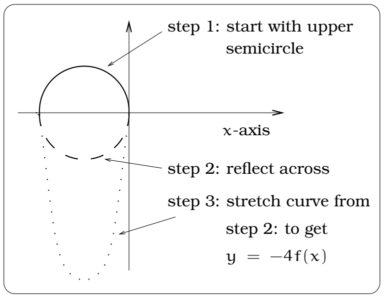
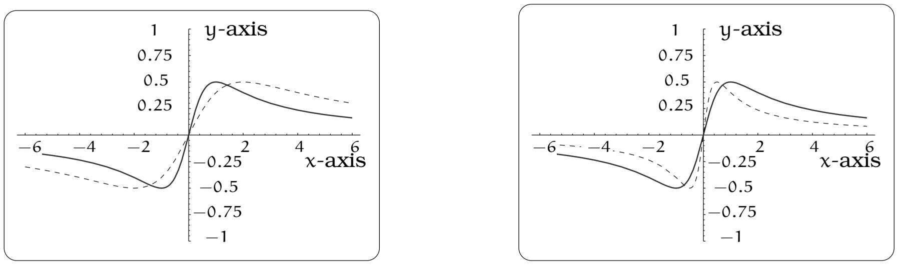

step 1: start with upper semicircle
x-axis
step 2: reflect across
step 3: stretch curve from step 2: to get
$y = -4f(x)$

Figure 13.12: Reflecting and dilating a lower semicircle.

Solution. The graph of $y = f(x)$ is an upper semicircle of radius 1 centered at the point $(-1,0)$. To obtain the picture of the graph of $y = -4f(x)$, we first reflect $y = f(x)$ across the x-axis; this gives us the graph of $y = -f(x)$. Then, we vertically dilate this picture by a factor of $c = 4$ to get the graph of $\frac{y}{4} = -f(x)$, which is the same as the graph of the equation $y = -4f(x)$. See Figure 13.12. □

Let's return to the original example $y = \frac{x}{x^2+1}$ and investigate a different type of dilation where the action is taking place in the horizontal direction (whereas it was in the vertical direction before). Grab the right-hand end of the graph (in Figure 13.10) and pull to the right, while at the same time pulling the left-hand end to the left. We can quantify this by stipulating that the high point $H = \left(1, \frac{1}{2}\right)$ of the original curve moves to the new location $\left(2, \frac{1}{2}\right)$ and the low point $L = \left(-1, -\frac{1}{2}\right)$ moves to the new location $\left(-2, -\frac{1}{2}\right)$.



(a) Stretching.

(b) Compressing.

Figure 13.13: Horizontally dilating $y = \frac{x}{x^2+1}$.

The result will be somewhat analogous to stretching a spring. By the same token, we could push the right-hand end to the left and push the left-hand end to the right, like compressing a spring. We can quantify this by stipulating that the high point $H = \left(1, \frac{1}{2}\right)$ of the original curve moves to the new location $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the low point $L = \left(-1, -\frac{1}{2}\right)$ moves to the new location $\left(-\frac{1}{2}, -\frac{1}{2}\right)$. These two situations are indicated in Figure 13.13. We refer to each of the dashed curves as a *horizontal dilation* of the original (solid) curve.

The tricky point is to understand what happens on the level of the original equation. In the case of the stretched graph in Figure 13.13(a), you can use a graphing device to verify that this looks like the graph of $y = \frac{x/2}{(x/2)^2+1}$; in other words, we replaced "x" by "$x/2$" in the original equation. In the case of the compressed graph in Figure 13.13(b), you can use a graphing device to verify that this looks like the graph of $y = \frac{2x}{(2x)^2+1}$; in other words, we replaced "x" by "$\frac{x}{1/2} = 2x$" in the original equation.

The process just described leads to a general principle.

Important Facts 13.4.3 (Horizontal dilation)**.** *Let $c > 0$ be a positive number and $y = f(x)$ a function equation.*

(i) If we replace "x" by "$\frac{x}{c}$" in the original function equation, then the graph of the resulting new function $y = f\left(\frac{x}{c}\right)$ is obtained by a horizontal dilation of the graph of $y = f(x)$. If the domain of $f(x)$ is $a \leq x \leq b$, then the domain of $y = f\left(\frac{x}{c}\right)$ is $a \leq \frac{x}{c} \leq b$.

(ii) If $c > 1$, then the graph of $y = f\left(\frac{x}{c}\right)$ is a horizontal stretch.

(iii) If $0 < c < 1$, then the graph of $y = f\left(\frac{x}{c}\right)$ is a horizontal compression.

13.5 Vertex Form and Order of Operations

Using the language of function compositions we can clarify our discussion in Example 7.1.2. Let's revisit that example:

Example 13.5.1. *The problem is to describe a sequence of geometric maneuvers that transform the graph of $y = x^2$ into the graph of $y = -3(x-1)^2 + 2$.*

Solution. The idea is to rewrite $y = -3(x-1)^2+2$ as a composition of $y = x^2$ with four other functions, each of which corresponds to a horizontal shift, vertical shift, reflection or dilation. Once we have done this, we can read off the order of geometric operations using the order of composition. Along the way, pay special attention to the exact order in which we will be composing our functions; this will make a big difference.

To begin with, we can isolate four key numbers in the equation:

$$\boxed{\overbrace{-}^{\textbf{Reflect}}\;\underbrace{3}_{\textbf{Dilate by 3}}\;(x-\overbrace{1}^{\textbf{Horizontal shift by } h=1})^2+\underbrace{2}_{\textbf{Vertical shift by } k=2}}$$

We want to use each number to define a *new* function, then compose these in the correct order. We will also give our starting function $y = x^2$ a specific name to make things definite:

$$\begin{aligned} f(x) &= x^2 & \qquad h(x) &= x-1 \\ v(x) &= x+2 & \qquad r(x) &= -x \\ d(x) &= 3x. \end{aligned}$$

Now, verify that

$$\begin{aligned}
v\left[r\left(d\left\{f\left[h(x)\right]\right\}\right)\right] &= v\left[r\left(d\left\{f[x-1]\right\}\right)\right] \\
&= v\left[r\left(d\{(x-1)^2\}\right)\right] \\
&= v\left[r(3(x-1)^2)\right] \\
&= v[-3(x-1)^2] \\
&= -3(x-1)^2+2 \\
&= -3x^2+6x-1.
\end{aligned}$$

□

13.6 Summary of Rules

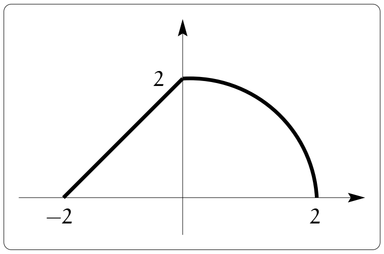


Figure 13.14: A multipart function.

For quick reference, we summarize the consequence of shifting and expanding symbolically and pictorially. The running example for Tables 13.1, 13.2, and 13.3 will be a multipart function $y = f(x)$ whose graph, seen in Figure 13.14, consists of a line segment and a quarter circle on the domain $-2 \leq x \leq 2$:

$$f(x) = \begin{cases} x+2 & \text{if } -2 \leq x \leq 0 \\ \sqrt{4-x^2} & \text{if } 0 \leq x \leq 2 \end{cases}$$

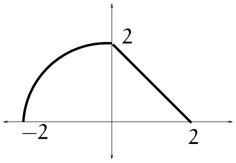
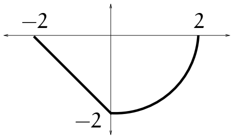
Reflection			
Symbolic Change	**New Equation**	**Graphical Consequence**	**Picture**
Replace x with $-x$.	$y = f(-x)$	A reflection across the y-axis.	
Replace $f(x)$ with $-f(x)$.	$y = -f(x)$	A reflection across the x-axis.	

Table 13.1: Reflecting $y = f(x)$.

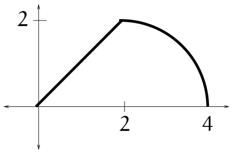
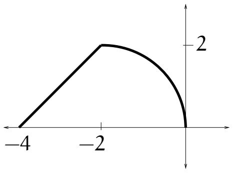
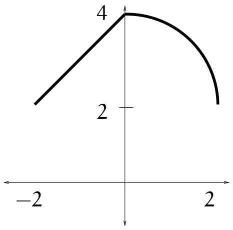
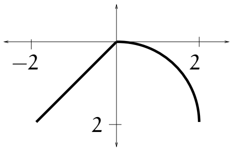
Shifting (Assume $c > 0$)			
Symbolic Change	**New Equation**	**Graphical Consequence**	**Picture**
Replace x with $(x - c)$.	$y = f(x - c)$	A shift to the right c units.	
Replace x with $(x + c)$.	$y = f(x + c)$	A shift to the left c units.	
Replace $f(x)$ with $(f(x) + c)$.	$y = f(x) + c$	A shift up c units.	
Replace $f(x)$ with $(f(x) - c)$.	$y = f(x) - c$	A shift down c units.	

Table 13.2: Shifting $y = f(x)$.

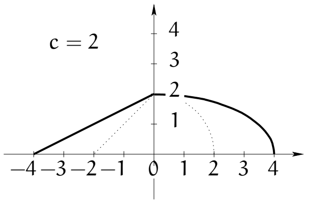
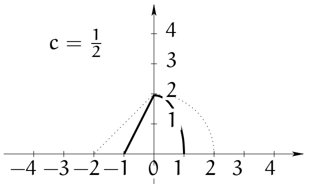
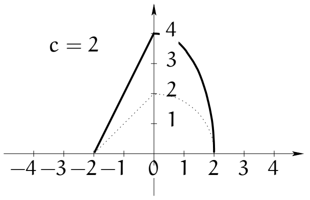
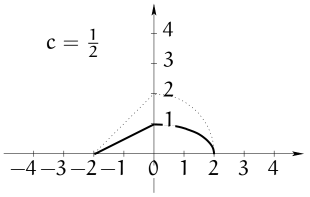
Dilation			
Symbolic Change	**New Equation**	**Graphical Consequence**	**Picture**
If $c > 1$, replace x with $\left(\frac{x}{c}\right)$.	$y = f\left(\frac{x}{c}\right)$	A horizontal expansion.	
If $0 < c < 1$, replace x with $\left(\frac{x}{c}\right)$.	$y = f\left(\frac{x}{c}\right)$	A horizontal compression.	
If $c > 1$, replace $f(x)$ with $(cf(x))$.	$y = cf(x)$	A vertical expansion.	
If $0 < c < 1$, replace $f(x)$ with $(cf(x))$.	$y = cf(x)$	A vertical compression.	

Table 13.3: Dilating $y = f(x)$.

13.7 Exercises

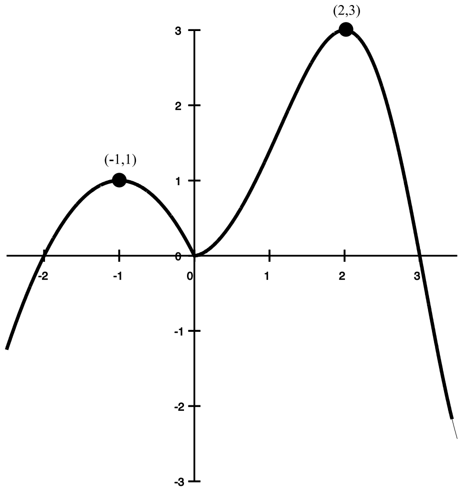
Problem 13.1. On a single set of axes, sketch a picture of the graphs of the following four equations: $y = -x+\sqrt{2}$, $y = -x-\sqrt{2}$, $y = x+\sqrt{2}$, and $y = x - \sqrt{2}$. These equations determine lines, which in turn bound a diamond shaped region in the plane.

(a) Show that the unit circle sits inside this diamond tangentially; i.e. show that the unit circle intersects each of the four lines exactly once.

(b) Find the intersection points between the unit circle and each of the four lines.

(c) Construct a diamond shaped region in which the circle of radius 1 centered at $(-2,-1)$ sits tangentially. Use the techniques of this section to help.

Problem 13.2. The graph of a function $y = f(x)$ is pictured with domain $-2.5 \le x \le 3.5$.



Sketch the graph of each of the new functions listed below.

(a) $g(x) = 2f(x+1)$

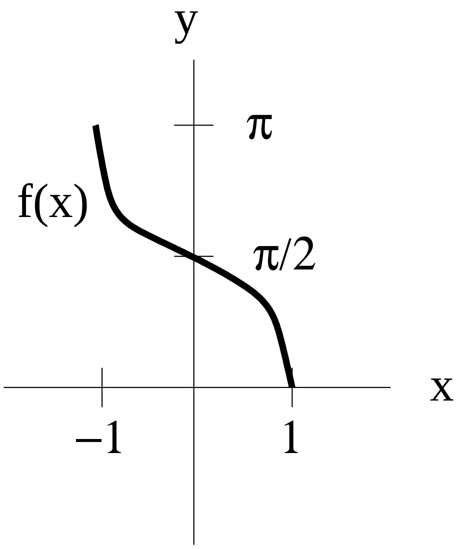
(b) $h(x) = \frac{1}{2}f(2x-1)$

(c) $j(x) = 5f(\frac{1}{3}x+2)-2$

Problem 13.3. The graph of a function $y = f(x)$ is pictured with domain $-1 \le x \le 1$. Sketch the graph of the new function

$$y = g(x) = \frac{1}{\pi}f(3x) - 0.5.$$

Find the largest possible domain of the function $y = \sqrt{g(x)}$.



Problem 13.4. (a) Each of the six functions $y = f(x)$ below can be written in the "standard form"

$$y = A|B(x-C)| + D,$$

for some constants A,B,C,D. Find these constants, describe the precise order of graphical operations involved in going from the graph of $y = |x|$ to the graph of $y = f(x)$ (paying close attention to the order), write out the multipart rule, sketch the graph and calculate the coordinates of the "vertex" of the graph.

(a1) $f(x) = |x-2|$

(a2) $f(x) = 2|x+3|$

(a3) $f(x) = |2x-1|$

(a4) $f(x) = |2(x-1)|$

(a5) $f(x) = 3|2x-1|+5$

(a6) $f(x) = -2|x+3|-1$

(b) Solve the following inequalities using your work in the previous part of this problem:

(b1) $|x-2| \le 3$

(b2) $1 \le 2|x+3| \le 5$

(b3) $y = 3|2x-1|+5 \ge 10$

(c) The graphs of $y = 3|2x-1|+5$ and $y = -|x-3|+10$ intersect to form a bounded region of the plane. Find the vertices of this region and sketch a picture.

Problem 13.5. Consider the function $y = f(x)$ with multipart definition

$$f(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ 2x+2 & \text{if } -1 \leq x \leq 0 \\ -x+2 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{if } x \geq 2 \end{cases}$$

(a) Sketch the graph of $y = f(x)$.

(b) Is $y = f(x)$ an even function? Is $y = f(x)$ an odd function? (A function $y = f(x)$ is called *even* if $f(x) = f(-x)$ for all x in the domain. A function $y = f(x)$ is called *odd* if $f(-x) = -f(x)$ for all x in the domain.)

(c) Sketch the reflection of the graph across the x-axis and y-axis. Obtain the resulting multipart equations for these reflected curves.

(d) Sketch the vertical dilations $y = 2f(x)$ and $y = \frac{1}{2}f(x)$.

(e) Sketch the horizontal dilations $y = f(2x)$ and $y = f(\frac{1}{2}x)$.

(f) Find a number $c > 0$ so that the highest point on the graph of the vertical dilation $y = cf(x)$ has y-coordinate 11.

(g) Using horizontal dilation, find a number $c > 0$ so that the function values $f(\frac{x}{c})$ are non-zero for all $-\frac{5}{2} < x < 5$.

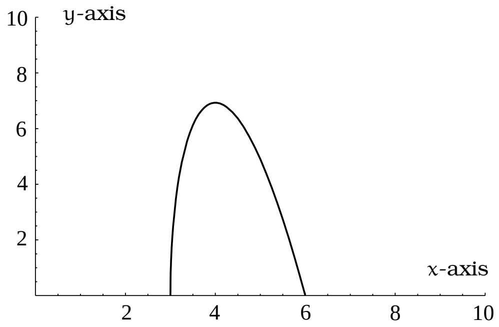
(h) Using horizontal dilation, find positive numbers $c, d > 0$ so that the function values $f(\frac{1}{c}(x-d))$ are non-zero precisely when $0 < x < 1$.

Problem 13.6. An *isosceles* triangle has sides of length x, x and y. In addition, assume the triangle has perimeter 12.

(a) Find the rule for a function that computes the area of the triangle as a function of x. Describe the largest possible domain of this function.

(b) Assume that the maximum value of the function $a(x)$ in (a) occurs when $x = 4$. Find the maximum value of $z = a(x)$ and $z = 2a(3x+3)+1$.

(c) The graph of $z = a(x)$ from part (a) is given below. Sketch the graph and find the rule for the function $z = 2a(3x+3)+1$; make sure to specify the domain and range of this new function.



Problem 13.7. Describe how each graph differs from that of $y = x^2$.

(a) $y = 2x^2$

(b) $y = x^2 - 5$

(c) $y = (x-4)^2$

(d) $y = (3x-12)^2$

(e) $y = 2(3x-12)^2 - 5$

Problem 13.8. In each case, start with the function $y = |x|$ and perform the operations described to the graph, *in the order specified.* Write out the resulting rule for the function and sketch the final graph you obtain.

(a) (1) horizontally compress by a factor of 2; (2) horizontally shift to the left by 2; (3) vertically stretch by a factor of 7; (4) vertically shift up 2 units.

(b) (1) horizontally stretch by a factor of 2; (2) horizontally shift to the right by 2; (3) vertically compress by a factor of 7; (4) vertically shift down 2 units.

(c) (1) horizontally shift to the right 2 units; (2) horizontally compress by a factor of 3.

Problem 13.9. (a) Begin with the function $y = f(x) = 2^x$.

(a1) Rewrite each of the following functions in standard exponential form: $f(2x)$, $f(x-1)$, $f(2x-1)$, $f(2(x-1))$, $3f(x)$, $3f(2(x-1))$.

- (a2) Is the function $3f(2(x-1))+1$ a function of exponential type?
- (a3) Sketch the graphs of $f(x), f(2x), f(2(x-1)), 3f(2(x-1))$ and $3f(2(x-1))+1$ in the same coordinate system and explain which graphical operation(s) (vertical shifting, vertical dilation, horizontal shifting, horizontal dilation) have been carried out.

(b) In general, explain what happens when you apply the four construction tools of Chapter 13 (vertical shifting, vertical dilation, horizontal shifting, horizontal dilation) to the standard exponential model $y = A_o b^x$. For which of the four operations is the resulting function still a standard exponential model?

Problem 13.10. Begin with a sketch of the graph of the function $y = 2^x$ on the domain of all real numbers. Describe how to use the "four tools" of Chapter 13 to obtain the graphs of these functions: $y = -2^x$, $y = 2^{-x}$, $y = 3(2^x)$, $y = \frac{1}{3}(2^x)$, $y = 3 + 2^x$, $y = 2^x - 2$, $y = 2^{x-2}$, $y = 2^{x+2}$, $y = 2^{3x}$, $y = 2^{x/3}$.

Chapter 14

Rational Functions

A *rational function* is a function of the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials. For example, the following are all rational functions.

$$f(x) = \frac{x}{3x+4} \quad g(x) = \frac{x^2+1}{3x-5} \quad h(x) = \frac{4x^5-4x^2-8}{x^3+x^2-x+1} \quad j(x) = \frac{x^6}{x^8+5x-\frac{1}{2}}$$

There is a huge variety of rational functions. In this course, we will concentrate our efforts exclusively toward understanding the simplest type of rational functions: *linear-to-linear rational functions*. Linear-to-linear rational functions are rational functions in which the numerator and the denominator are both linear polynomials. The following are linear-to-linear-rational functions.

$$k(x) = \frac{x}{3x+4} \quad m(x) = \frac{5x-6}{2x+1} \quad n(x) = \frac{0.34x-0.113}{x-1} \quad p(x) = \frac{4x+\frac{3}{4}}{\frac{5}{8}x-1.117}$$

The simplest example of a linear-to-linear rational function is $f(x) = \frac{1}{x}$ whose graph is shown in Figure 14.1. This is an important example for the study of this class of functions, as we shall see.

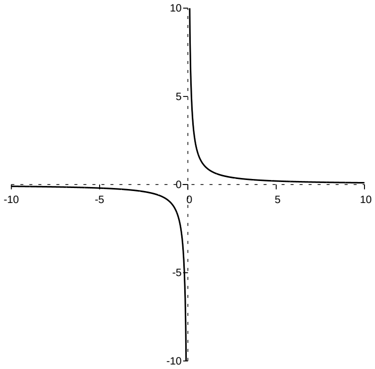


Figure 14.1: The graph of $f(x) = \frac{1}{x}$.

Let's consider the graph of this function, $f(x) = \frac{1}{x}$. We first begin by considering the domain of f. Since the numerator of $\frac{1}{x}$ is a constant, and the denominator is just x, the only way we can run into difficulty when evaluating this function is if we try to divide by zero; that is, the only value of x not in the domain of this function is zero. The function is defined for all x except $x = 0$. At $x = 0$, there must be a gap, or hole, in the graph.

To get the graph started we might simply give ourselves a point on the graph. For instance, we see that $f(1) = 1/1 = 1$, so the point (1,1) is on the graph. Then, if we use larger values of x, we see that $1/x$ becomes smaller as x grows. For instance, $f(2) = 1/2$, $f(10) = 1/10$, and $f(1000) = 1/1000$. In addition, we see that, other than the fact that $1/x > 0$ for positive x, there is nothing preventing us from making $1/x$ as small as we want simply by taking x big enough. Want $1/x$ to be less than 0.001? Just pick x bigger than 1000. Want $1/x$ to be less than 0.000001? Just use x bigger than 1,000,000.

What this means graphically is that as x gets bigger (starting from $x = 1$), the curve $y = 1/x$ gets closer and closer to the x-axis. As a result, we say that the x-axis is a *horizontal asymptote* for this function.

We see the same behavior for negative values of x. If x is large, and negative (think -1000, or -1000000), then $1/x$ is very small (i.e., close to zero), and it gets smaller the larger x becomes. Graphically, this means that as x gets large in the negative direction, the curve $y = 1/x$ gets closer and closer to the x-axis. We say that, in both the positive and negative directions, $y = 1/x$ is *asymptotic* to the x-axis.

A similar thing happens when we consider x near zero. If x is a small positive number (think 1/2, or 1/10, or 1/10000), then $1/x$ is a large positive number. What's more, if we think of x as getting closer to zero, $1/x$ gets bigger and bigger. Plus, there is no bound on how big we can make $1/x$ simply by taking x as close to zero as we need to.

For instance, can $1/x$ be as big as 10000? All you need to do is pick a positive x smaller than 1/10000.

Graphically, what this means is that as x approaches zero from the positive side, the y value gets larger and larger. As a result, the curve approaches the y-axis as y gets larger. We say that the y-axis is a *vertical asymptote* for the curve $y = 1/x$.

We see the same phenomenon as x approaches zero from the negative side: $y = 1/x$ gets larger in the negative direction (i.e., it gets more and more negative). The curve gets closer and closer to the negative y-axis as y becomes more and more negative. Again, we say that the $y - axis$ is a *vertical asymptote* for the curve $y = 1/x$.

It turns out that every *linear-to-linear* rational function has a graph that looks essentially the same as the graph of $y = 1/x$. Let's see why.

Consider the linear-to-linear rational function $f(x) = \frac{ax+b}{cx+d}$. If we divide $cx + d$ into $ax + b$, the result is

$$f(x) = \frac{ax+b}{cx+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cx+d}$$

which we can rewrite as

$$f(x) = \frac{a}{c} + \left(b - \frac{ad}{c}\right) \cdot \frac{1}{cx+d} = \frac{a}{c} + \left(b - \frac{ad}{c}\right) \cdot \frac{1}{c(x + \frac{d}{c})} = \frac{a}{c} + \left(\frac{bc - ad}{c^2}\right) \cdot \frac{1}{x + \frac{d}{c}}.$$

If we now let

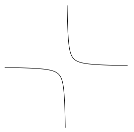
$$A = \frac{a}{c}\ ,\ B = \left(\frac{bc - ad}{c^2}\right),\ \text{and}\ C = \frac{d}{c}\ ,$$

then we can write

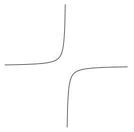
$$f(x) = A + B\frac{1}{x + C}.$$

If we let $g(x) = 1/x$ then we have shown that $f(x) = A + Bg(x + C)$, and so the graph of the function f is just a horizontally shifted, vertically shifted and vertically dilated version of the graph of g. Also, if B turned out to be negative the graph would be vertically flipped, too.

Why is that useful? It means that the graph of a linear-to-linear rational function can only take one of two forms. Either it looks like this:



Or like this:



We can sketch an accurate graph of a linear-to-linear rational function by sketching the asymptotes and then sketching in just one point on the graph. That will be enough information to nail down a decent sketch. We can always plot more points to give us more precision, but one point is enough to capture the essence of the graph.

Given a linear-to-linear function that we wish to graph, we must first determine the asymptotes. There will be one horizontal asymptote and one vertical asymptote.

The vertical asymptote will be a vertical line with equation $x = k$ where k is the one x value which is not in the domain of the function. That is, find the value of x which makes the denominator zero and that will tell you the vertical asymptote.

The horizontal asymptote is a little more involved. However, we can quickly get to a shortcut. The essence of a horizontal asymptote is that it describes what value the function is approximately equal to for very large values of x. To study what a linear-to-linear function is like when x is very large, we can perform the following algebraic manipulation:

$$f(x) = \frac{ax + b}{cx + d} = \frac{ax + b}{cx + d} \cdot \frac{1/x}{1/x} = \frac{a + \frac{b}{x}}{c + \frac{d}{x}}$$

While dividing by x is troublesome if x equals zero, here we are assuming x is very large, so it is certainly *not* zero.

Now, consider this last expression in the above equation. If x is very large, then

$$\frac{b}{x} \approx 0$$

(where ≈ means "is approximately") and likewise

$$\frac{d}{x} \approx 0.$$

Hence, when x is very large,

$$f(x) \approx \frac{a+0}{c+0} = \frac{a}{c}.$$

We can interpret this by saying that when x is very large, the function $f(x)$ is close to a constant, and that constant is $\frac{a}{c}$. Thus, the horizontal asymptote of $f(x) = \frac{ax+b}{cx+d}$ is the horizontal line $y = \frac{a}{c}$.

Example 14.0.1. *Sketch the graph of the function* $f(x) = \frac{3x-1}{2x+7}$.

Solution. We begin by finding the asymptotes of f.

The denominator is equal to zero when $2x+7=0$, i.e., when $x = -7/2$. As a result, the vertical asymptote for this function is the vertical line $x = -7/2$.

By taking the ratio of the coefficients of x in the numerator and denominator, we can find that the horizontal asymptote is the horizontal line

$$y = \frac{3}{2}.$$

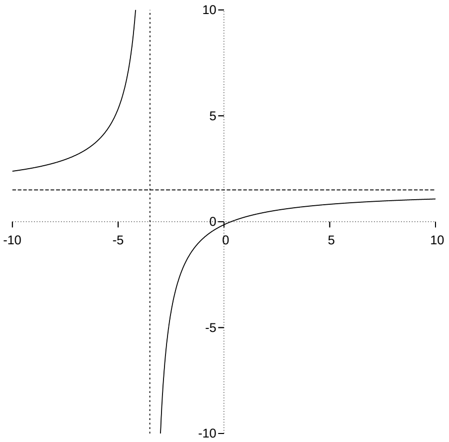


Figure 14.2: The graph of $f(x) = \frac{3x-1}{2x+7}$.

We then sketch these two asymptotes. The last thing we need is a single point. For instance, we may evaluate $f(0)$:

$$f(0) = \frac{-1}{7}$$

and so the point $(0, -1/7)$ is on the graph. With this information, we know that the curve lies *below* the horizontal asymptote to the right of the vertical asymptote, and consequently the curve lies *above* the horizontal asymptote to the left of the vertical asymptote.

We graph the result in Figure 14.2.

14.1 Modeling with Linear-to-linear Rational Functions

As we have done with other sorts of functions, such as linear and quadratic, we can also model using linear-to-linear rational functions. One reason for using this type of function is their asymptotic nature. Many changing quantities in the world continually increase or decrease, but with bounds on how large or small they can get. For instance, a population may steadily decrease, but a population can never be negative. Conversely, a population may steadily increase, but due to environmental and other factors we may hypothesize that the population will always stay below some upper bound. As a result, the population may "level off". This leveling off behavior is exemplified by the asymptotic nature of the linear-to-linear rational functions, and so this type of function provides a way to model such behavior.

Given any linear-to-linear rational function, we can always divide the numerator and the denominator by the coefficient of x in the denominator. In this way, we can always assume that the coefficient of x in the denominator of a function we seek is 1. This is illustrated in the next example.

Example 14.1.1. *Let* $f(x) = \frac{2x+3}{5x-7}$*. Then*

$$\begin{aligned} f(x) &= \frac{2x+3}{5x-7} \cdot \frac{\frac{1}{5}}{\frac{1}{5}} \\ &= \frac{\frac{2}{5}x + \frac{3}{5}}{x - \frac{7}{5}}. \end{aligned}$$

Thus, in general, when we seek a linear-to-linear rational function, we will be looking for a function of the form

$$f(x) = \frac{ax+b}{x+c}$$

and thus there are three parameters we need to determine.

Note that for a function of this form, the horizontal asymptote is $y = a$ and the vertical asymptote is $x = -c$.

Since these functions have three parameters (i.e., a, b and c), we will need three pieces of information to nail down the function.

There are essentially three types of modeling problems that require the determination of a linear-to-linear function. The three types are based on the kind of information given about the function. The three types are:

1. You know three points the the graph of the function passes through;
2. You know one of the function's asymptotes and two points the graph passes through;
3. You know both asymptotes and one point the graph passes through.

Notice that in all cases you know **three pieces of information**. Since a linear-to-linear function is determined by three parameters, this is exactly the amount of information needed to determine the function.

The worst case, in terms of the amount of algebra you need to do, is the first case. Let's look at an example of the algebra involved with this sort.

Example 14.1.2. *Find the linear-to-linear rational function* $f(x)$ *such that* $f(10) = 20$, $f(20) = 32$ *and* $f(25) = 36$*.*

Solution. Since $f(x)$ is a linear-to-linear rational function, we know

$$f(x) = \frac{ax+b}{x+c}$$

for constants a, b, and c. We need to find a, b and c.

We know three things.

First, $f(10) = 20$. So

$$f(10) = \frac{10a+b}{10+c} = 20,$$

which we can rewrite as

$$10a + b = 200 + 20c. \tag{14.1}$$

Second, $f(20) = 32$. So

$$f(20) = \frac{20a+b}{20+c} = 32,$$

which we can rewrite as

$$20a + b = 640 + 32c. \tag{14.2}$$

Third, $f(25) = 36$. So

$$f(25) = \frac{25a+b}{25+c} = 36,$$

which we can rewrite as

$$25a + b = 900 + 36c. \tag{14.3}$$

These three numbered equations are enough algebraic material to solve for a, b, and c. Here is one way to do that.

Subtract equation 14.1 from equation 14.2 to get

$$10a = 440 + 12c \tag{14.4}$$

and subtract equation 14.2 from equation 14.3 to get

$$5a = 260 + 4c \tag{14.5}$$

Note that we've eliminated b. Now multiply this last equation by 2 to get

$$10a = 520 + 8c$$

Subtract equation 14.4 from this to get

$$0 = 80 - 4c$$

which easily give us $c = 20$.

Plugging this value into equation 14.4, we can find $a = 68$, and then we can find $b = -80$.

Thus,

$$f(x) = \frac{68x - 80}{x + 20}.$$

We can check that we have done the algebra correctly by evaluating $f(x)$ at $x = 10$, $x = 20$ and $x = 25$. If we get $f(10) = 20$, $f(20) = 32$ and $f(25) = 36$, then we'll know our work is correct.

□

Algebraically, this was the worst situation of the three, since it required the most algebra. If, instead of knowing three points, we know one or both of the asymptotes, then we can easily find a and/or c, and so cut down on the amount of algebra needed. However, the method is essentially identical.

Let's now apply these ideas to a real world problem.

Example 14.1.3. *Clyde makes extra money selling tickets in front of the Safeco Field. The amount he charges for a ticket depends on how many he has. If he only has one ticket, he charges \$100 for it. If he has 10 tickets, he charges \$80 a piece. But if he has a large number of tickets, he will sell them for \$50 each. How much will he charge for a ticket if he holds 20 tickets?*

Solution. We want to give a linear-to-linear rational function relating the price of a ticket y to the number of tickets x that Clyde is holding. As we saw above, we can assume the function is of the form

$$y = \frac{ax+b}{x+c}$$

where a, b and c are numbers. Note that $y = a$ is the horizontal asymptote. When x is very large, y is close to 50. This means the line $y = 50$ is a horizontal asymptote. Thus $a = 50$ and

$$y = \frac{50x+b}{x+c}.$$

Next we plug in the point $(1,100)$ to get a linear equation in b and c.

$$\begin{aligned} 100 &= \frac{50 \cdot 1 + b}{1+c} \\ 100 \cdot (1+c) &= 50 + b \\ 50 &= b - 100c \end{aligned}$$

Similarly, plugging in $(10,80)$ and doing a little algebra (do it now!) gives another linear equation $300 = b - 80c$. Solving these two linear equations simultaneously gives $c = 12.5$ and $b = 1300$. Thus our function is

$$y = \frac{50x+1300}{x+12.5}$$

and, if Clyde holds 20 tickets, he will charge

$$y = \frac{50 \cdot 20 + 1300}{20 + 12.5} = \$70.77$$

per ticket. □

14.2 Summary

- Every linear-to-linear rational function has a graph which is a shifted, scaled version of the curve $y = 1/x$. As a result, they have one vertical asymptote, and one horizontal asymptote.
- Every linear-to-linear rational function f can be expressed in the form

  $$f(x) = \frac{ax+b}{x+c}.$$

  This function has horizontal asymptote $y = a$ and vertical asymptote $x = -c$.

14.3 Exercises

Problem 14.1. Give the domain of each of the following functions. Find the x- and y-intercepts of each function. Sketch a graph and indicate any vertical or horizontal asymptotes. Give equations for the asymptotes.

(a) $f(x) = \frac{2x}{x-1}$ (b) $g(x) = \frac{3x+2}{2x-5}$

(c) $h(x) = \frac{x+1}{x-2}$ (d) $j(x) = \frac{4x-12}{x+8}$

(e) $k(x) = \frac{8x+16}{5x-\frac{1}{2}}$ (f) $m(x) = \frac{9x+24}{35x-100}$

Problem 14.2. Oscar is hunting magnetic fields with his gauss meter, a device for measuring the strength and polarity of magnetic fields. The reading on the meter will increase as Oscar gets closer to a magnet. Oscar is in a long hallway at the end of which is a room containing an extremely strong magnet. When he is far down the hallway from the room, the meter reads a level of 0.2. He then walks down the hallway and enters the room. When he has gone 6 feet into the room, the meter reads 2.3. Eight feet into the room, the meter reads 4.4.

(a) Give a linear-to-linear rational model relating the meter reading y to how many feet x Oscar has gone into the room.

(b) How far must he go for the meter to reach 10? 100?

(c) Considering your function from part (a) and the results of part (b), how far into the room do you think the magnet is?

Problem 14.3. In 1975 I bought an old Martin ukulele for $200. In 1995 a similar uke was selling for $900. In 1980 I bought a new Kamaka uke for $100. In 1990 I sold it for $400.

(a) Give a linear model relating the price p of the Martin uke to the year t. Take $t = 0$ in 1975.

(b) Give a linear model relating the price q of the Kamaka uke to the year t. Again take $t = 0$ in 1975.

(c) When is the value of the Martin twice the value of the Kamaka?

(d) Give a function $f(t)$ which gives the ratio of the price of the Martin to the price of the Kamaka.

(e) In the long run, what will be the ratio of the prices of the ukuleles?

Problem 14.4. Isobel is producing and selling casette tapes of her rock band. When she had sold 10 tapes, her net profit was $6. When she had sold 20 tapes, however, her net profit had shrunk to $4 due to increased production expenses. But when she had sold 30 tapes, her net profit had rebounded to $8.

(a) Give a quadratic model relating Isobel's net profit y to the number of tapes sold x.

(b) Divide the profit function in part (a) by the number of tapes sold x to get a model relating average profit w per tape to the number of tapes sold.

(c) How many tapes must she sell in order to make $1.20 per tape in net profit?

Problem 14.5. Find the linear-to-linear function whose graph passes through the points $(1,1)$, $(5,2)$ and $(20,3)$. What is its horizontal asymptote?

Problem 14.6. Find the linear-to-linear function whose graph has $y = 6$ as a horizontal asymptote and passes through $(0,10)$ and $(3,7)$.

Problem 14.7. The more you study for a certain exam, the better your performance on it. If you study for 10 hours, your score will be 65%. If you study for 20 hours, your score will be 95%. You can get as close as you want to a perfect score just by studying long enough. Assume your percentage score is a linear-to-linear function of the number of hours that you study.

If you want a score of 80%, how long do you need to study?

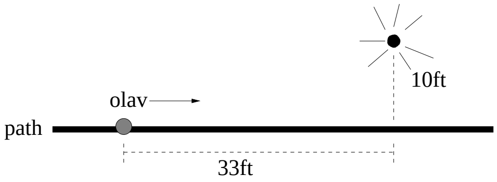
Problem 14.8. A street light is 10 feet above a straight bike path. Olav is bicycling down the path at a rate of 15 MPH. At midnight, Olav is 33 feet from the point on the bike path directly below the street light. (See the picture). The relationship between the intensity C of light (in candlepower) and the distance d (in feet) from the light source is given by $C = \frac{k}{d^2}$, where k is a constant depending on the light source.

(a) From 20 feet away, the street light has an intensity of 1 candle. What is k?

(b) Find a function which gives the intensity of the light shining on Olav as a function of time, in seconds.

(c) When will the light on Olav have maximum intensity?

(d) When will the intensity of the light be 2 candles?



Problem 14.9. For each of the following find the linear to linear function $f(x)$ satisfying the given requirements:

(a) $f(0) = 0, f(10) = 10, f(20) = 15$

(b) $f(0) = 10, f(5) = 4, f(20) = 3$

(c) $f(10) = 20, f(30) = 25$, and the graph of $f(x)$ has $y = 30$ as its horizontal asymptote

Problem 14.10. The number of customers in a local dive shop depends on the amount of money spent on advertising. If the shop spends nothing on advertising, there will be 100 customers/day. If the shop spends \$100, there will be 200 customers/day. As the amount spent on advertising increases, the number of customers/day increases and approaches (but never exceeds) 400 customers/day.

(a) Find a linear to linear rational function $y = f(x)$ that calculates the number y of customers/day if \$$x$ is spent on advertising.

(b) How much must the shop spend on advertising to have 300 customers/day.

(c) Sketch the graph of the function $y = f(x)$ on the domain $0 \leq x \leq 5000$.

(d) Find the rule, domain and range for the inverse function from part (c). Explain in words what the inverse function calculates.

Chapter 15

Measuring an Angle

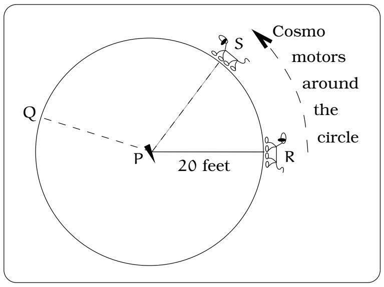


Figure 15.1: Cosmo the dog walking a circular path.

So far, the equations we have studied have an algebraic character, involving the variables x and y, arithmetic operations and maybe extraction of roots. Restricting our attention to such equations would limit our ability to describe certain natural phenomena. An important example involves understanding motion around a circle, and it can be motivated by analyzing a very simple scenario: Cosmo the dog, tied by a 20 foot long tether to a post, begins walking around a circle. A number of very natural questions arise:

Natural Questions 15.0.1. *How can we measure the angles* $\angle SPR$, $\angle QPR$, *and* $\angle QPS$? *How can we measure the arc lengths* $\mathrm{arc}(RS)$, *arc*(SQ) *and arc*(RQ)? *How can we measure the rate Cosmo is moving around the circle? If we know how to measure angles, can we compute the coordinates of* R, S, *and* Q? *Turning this around, if we know how to compute the coordinates of* R, S, *and* Q, *can we then measure the angles* $\angle SPR$, $\angle QPR$, *and* $\angle QPS$? *Finally, how can we specify the direction Cosmo is traveling?*

We will answer all of these questions and see how the theory which evolves can be applied to a variety of problems. The definition and basic properties of the *circular functions* will emerge as a central theme in this Chapter. The full problem-solving power of these functions will become apparent in our discussion of *sinusoidal functions* in Chapter 19.

The xy-coordinate system is well equipped to study straight line motion between two locations. For problems of this sort, the important tool is the distance formula. However, as Cosmo has illustrated, not all two-dimensional motion is along a straight line. In this section, we will describe how to calculate length along a circular arc, which requires a quick review of angle measurement.

15.1 Standard and Central Angles

An *angle* is the union of two rays emanating from a common point called the *vertex* of the angle. A typical angle can be dynamically generated by rotating a single ray from one position to another, sweeping *counterclockwise* or *clockwise*: See Figure 15.2. We often insert a curved arrow to indicate the direction in which we are sweeping out the angle. The ray ℓ_1 is called the *initial side* and ℓ_2 the *terminal side* of the angle $\angle AOB$.

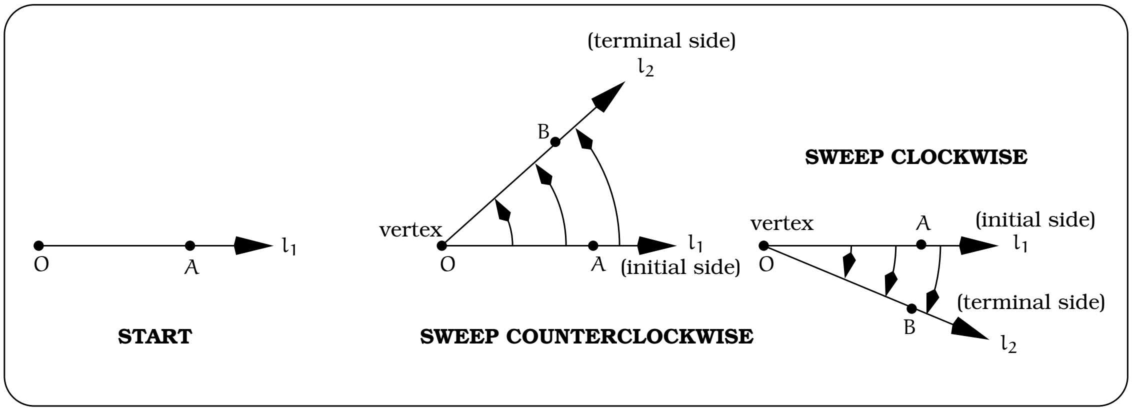


Figure 15.2: Angle $\angle AOB$.

Working with angles, we need to agree on a standard frame of reference for viewing them. Within the usual xy-coordinate system, imagine a model of $\angle AOB$ in Figure 15.2 constructed from two pieces of rigid wire, welded at the vertex. Sliding this model around inside the xy-plane will not distort its shape, only its position relative to the coordinate axis. So, we can slide the angle into position so that the initial side is coincident with the positive x-axis and the vertex is the origin. Whenever we do this, we say the angle is in *standard position*. Once an angle is in standard position, we can construct a circle centered at the origin and view our standard angle as cutting out a particular "pie shaped wedge" of the corresponding disc.

Notice, the shaded regions in Figure 15.3 depend on whether we sweep the angle counterclockwise or clockwise from the initial side. The portion of the "pie wedge" along the circle edge, which is an arc, is called the *arc subtended* by the angle. We can keep track of this arc using the notation $\mathrm{arc}(AB)$. A *central angle* is any angle with vertex at the center of a circle, but its initial side may or may not be the positive x-axis. For example, $\angle QPS$ in the Figure beginning this Chapter is a central angle which is not in standard position.

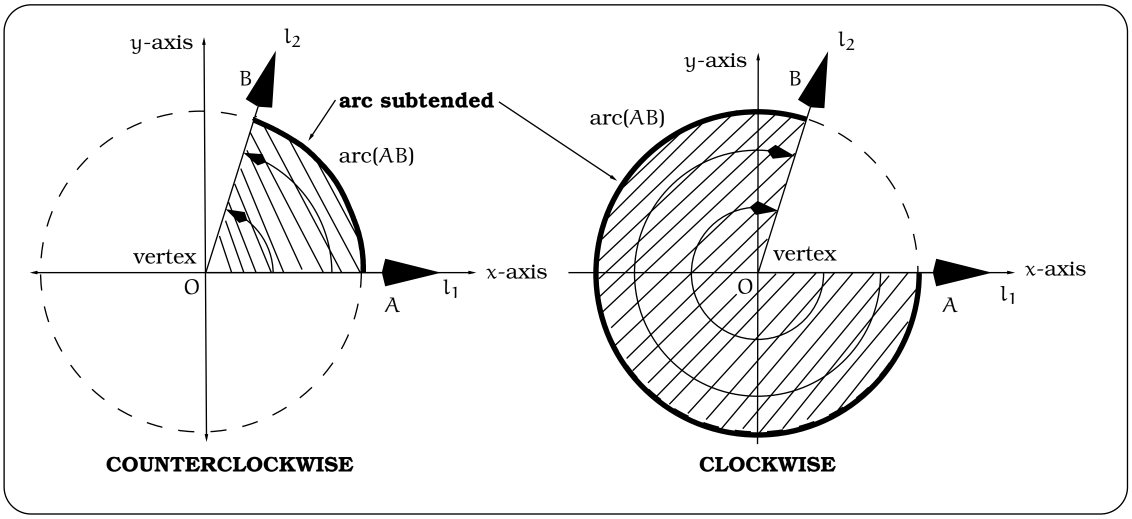


Figure 15.3: Standard angles and arcs.

15.2 An Analogy

To measure the dimensions of a box you would use a ruler. In other words, you use an instrument (the ruler) as a standard against which you measure the box. The ruler would most likely be divided up into either English units (inches) or metric units (centimeters), so we could express the dimensions in a couple of different ways, depending on the units desired.

By analogy, to measure the size of an angle, we need a standard against which any angle can be compared. In this section, we will describe two standards commonly used: the *degree method* and the *radian method* of angle measurement. The key idea is this: Beginning with a circular region, describe how to construct a “basic” pie shaped wedge whose interior angle becomes the standard unit of angle measurement.

15.3 Degree Method

Begin by drawing a circle of radius r, call it C_r, centered at the origin. Divide this circle into 360 equal sized pie shaped wedges, beginning with the the point $(r,0)$ on the circle; i.e. the place where the circle crosses the x-axis.

We will refer to the arcs along the outside edges of these wedges as *one-degree arcs*. Why 360 equal sized arcs? The reason for doing so is historically tied to the fact that the ancient Babylonians did so as they developed their study of astronomy. (There is actually an alternate system based on dividing the circular region into 400 equal sized wedges.) Any central angle which subtends one of these 360 equal sized arcs is

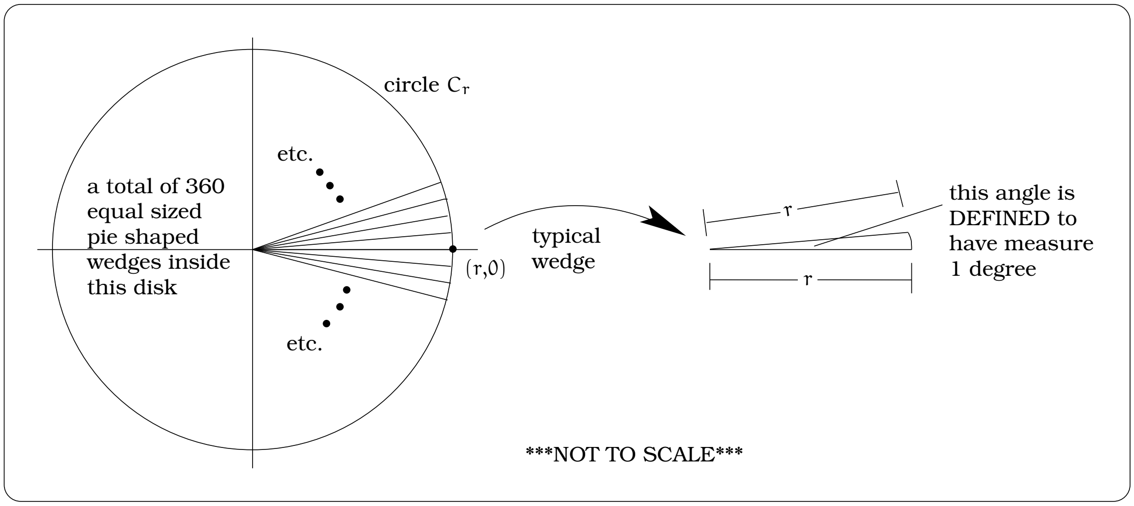

circle C_r
etc.
a total of 360 equal sized pie shaped wedges inside this disk
etc.
(r,0)
typical wedge
r
r
this angle is DEFINED to have measure 1 degree
NOT TO SCALE

Figure 15.4: Wedges as 1° arcs.

defined to have a measure of *one degree*, denoted 1°.

We can now use one-degree arcs to measure any angle: Begin by sliding the angle $\angle AOB$ into standard central position, as in Figure 15.3. Piece together consecutive one-degree arcs in a counterclockwise or clockwise direction, beginning from the initial side and working toward the terminal side, approximating the angle $\angle AOB$ to the nearest degree. If we are allowed to divide a one-degree arc into a fractional portion, then we could precisely determine the number m of one-degree arcs which consecutively fit together into the given arc. If we are sweeping counterclockwise from the initial side of the angle, m is defined to be the *degree measure* of the angle. If we sweep in a clockwise direction, then $-m$ is defined to be the *degree measure* of the angle. So, in Figure 15.3, the left-hand angle has positive degree measure while the right-hand angle has negative degree measure. Simple examples would be angles like the ones in Figure 15.5.

Notice, with our conventions, the rays determining an angle with measure -135° sit inside the circle in the same position as those for an angle of measure 225°; the minus sign keeps track of sweeping the positive x-axis clockwise (rather than counterclockwise).

We can further divide a one-degree arc into 60 equal arcs, each called a *one minute arc*. Each one-minute arc can be further divided into 60 equal arcs, each called a *one second arc*. This then leads to angle measures of one minute, denoted $1'$ and one second, denoted $1''$:

$$\begin{aligned} 1^\circ &= 60\,\text{minutes} \\ &= 3600\,\text{seconds}. \end{aligned}$$

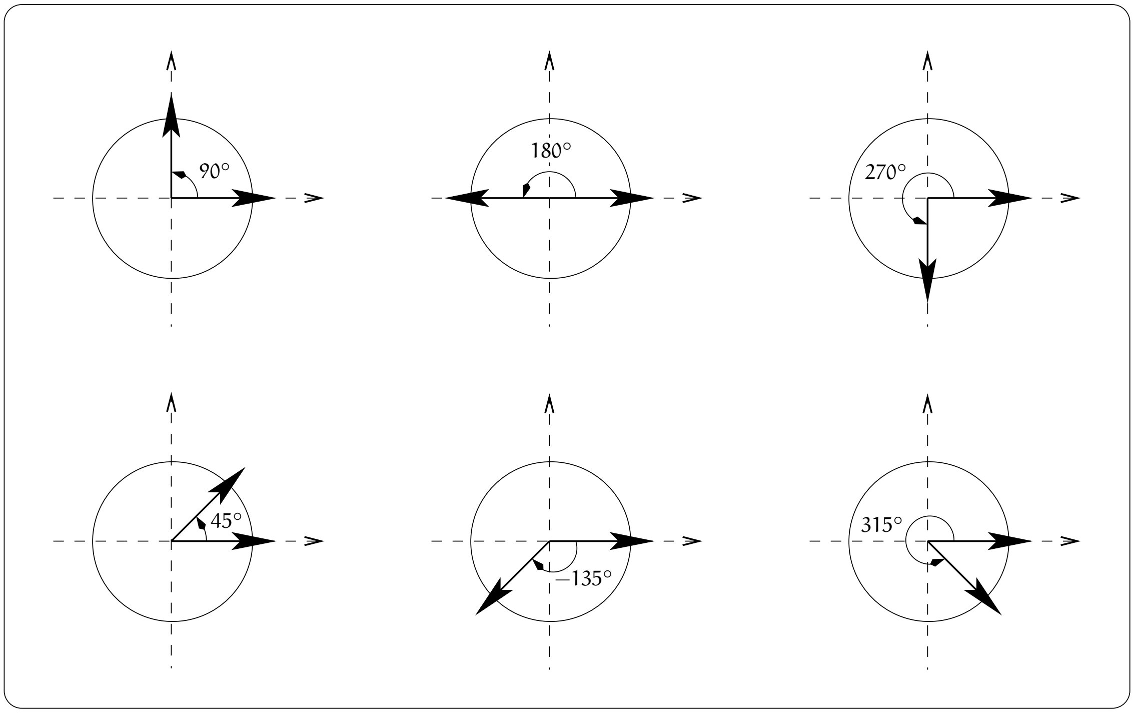


Figure 15.5: Examples of common angles.

For example, an angle of measure $\theta = 5$ degrees 23 minutes 18 seconds is usually denoted $5^\circ\, 23'\, 18''$. We could express this as a decimal of degrees:

$$\begin{aligned} 5^\circ\, 23'\, 18'' &= \left(5 + \frac{23}{60} + \frac{18}{3600}\right)^\circ \quad \leftarrow\text{In degrees!} \\ &= 5.3883^\circ. \end{aligned}$$

As another example, suppose we have an angle with measure 75.456° and we wish to convert this into degree/minute/second units. First, since $75.456^\circ = 75^\circ + 0.456^\circ$, we need to write 0.456° in minutes by the calculation:

$$0.456\,\text{degree} \times 60\,\frac{\text{minutes}}{\text{degree}} = 27.36'.$$

This tells us that $75.456^\circ = 75^\circ 27.36' = 75^\circ 27' + 0.36'$. Now we need to write $0.36'$ in seconds via the calculation:

$$0.36\,\text{minutes} \times 60\,\text{seconds/minute} = 21.6''.$$

In other words, $75.456^\circ = 75^\circ\, 27'\, 21.6''$.

Degree measurement of an angle is very closely tied to direction in the plane, explaining its use in map navigation. With some additional work, it is also possible to relate degree measure and lengths of circular arcs. To do this carefully, first go back to Figure 15.3 and recall the situation where an arc $\mathrm{arc}(AB)$ is subtended by the central angle $\angle AOB$. In this situation, the *arc length* of $\mathrm{arc}(AB)$, commonly denoted by the letter s, is the distance from A to B computed along the circular arc; keep in mind, this is NOT the same as the straight line distance between the points A and B.

For example, consider the six angles pictured above, of measures $90°$, $180°$, $270°$, $45°$, $-135°$, and $315°$. If the circle is of radius r and we want to compute the lengths of the arcs subtended by these six angles, then this can be done using the formula for the circumference of a circle (on the back of this text) and the following general principle:

Important Fact 15.3.1.

(length of a part) = (fraction of the part) × (length of the whole)

For example, the circumference of the entire circle of radius r is $2\pi r$; this is the "length of the whole" in the general principle. The arc subtended by a $90°$ angle is $\frac{90}{360} = \frac{1}{4}$ of the entire circumference; this is the "fraction of the part" in the general principle. The boxed formula implies:

$$s = \text{ arc length of the } 90° \text{ arc} = \left(\frac{1}{4}\right) 2\pi r = \frac{\pi r}{2}.$$

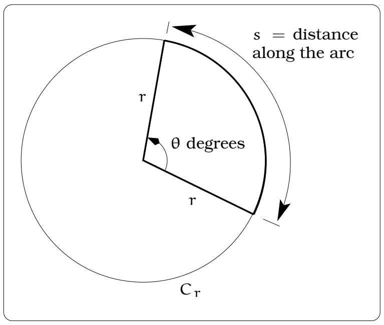


Figure 15.6: The definition of arc length.

Similarly, a $180°$ angle subtends an arc of length $s = \pi r$, a $315°$ angle subtends an arc of length $s = \left(\frac{315}{360}\right) 2\pi r = \frac{7\pi r}{4}$, etc. In general, we arrive at this formula:

Important Fact 15.3.2 (Arc length in degrees)**.** *Start with a central angle of measure θ degrees inside a circle of radius r. Then this angle will subtend an arc of length*

$$s = \left(\frac{2\pi}{360}\right) r\theta$$

15.4 Radian Method

The key to understanding degree measurement was the description of a "basic wedge" which contained an interior angle of measure $1°$; this was straightforward and familiar to all of us. Once this was done, we could proceed to measure any angle in degrees and compute arc lengths as in Fact 15.3.2. However, the formula for the length of an arc subtended by an angle measured in degrees is sort of cumbersome, involving the curious factor $\frac{2\pi}{360}$. Our next goal is to introduce an alternate angle measurement scheme called *radian measure* that begins with a different "basic wedge". As will become apparent, a big selling point of radian measure is that arc length calculations become easy.

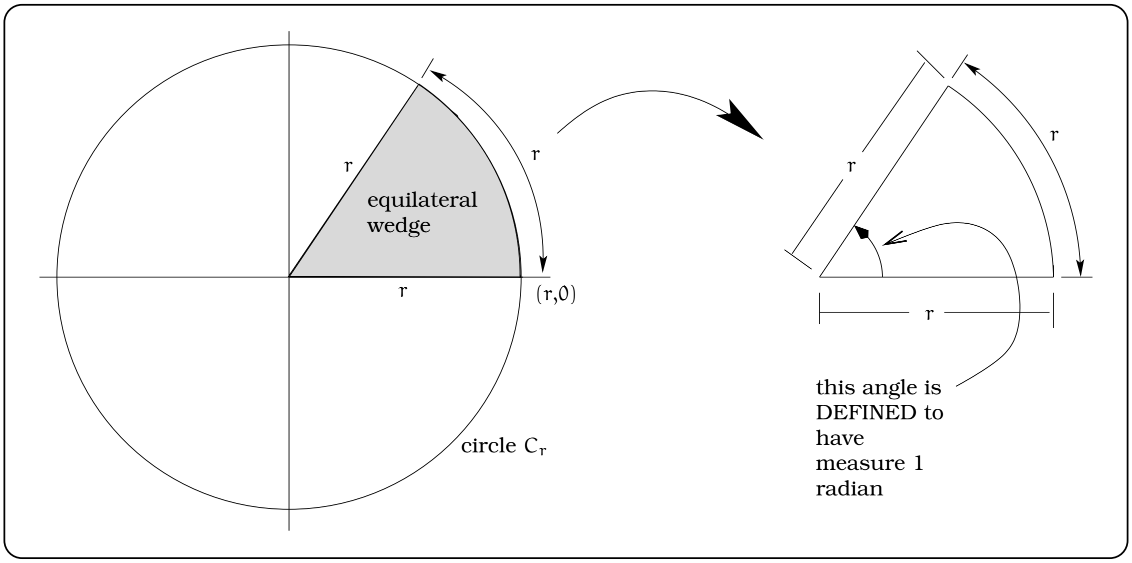


Figure 15.7: Constructing an equilateral wedge.

As before, begin with a circle C_r of radius r. Construct an *equilateral wedge* with all three sides of equal length r; see Figure 15.7. We define the measure of the interior angle of this wedge to be 1 *radian*.

Once we have defined an angle of measure 1 radian, we can define an angle of measure 2 radians by putting together two equilateral wedges. Likewise, an angle of measure $\frac{1}{2}$ radian is obtained by symmetrically dividing an equilateral wedge in half, etc.

Reasoning in this way, we can piece together equilateral wedges or fractions of such to compute the radian measure of any angle. It is important to notice an important relationship between the radian measure of an angle and arc length calculations. In the five angles pictured above, 1 radian, 2 radian, 3 radian, $\frac{1}{2}$ radian and $\frac{1}{4}$ radian, the length of the arcs subtended by these angles θ are r, $2r$, $3r$, $\frac{1}{2}r$, and $\frac{1}{4}r$. In other words, a pattern emerges that gives a very simple relationship between the length s of an arc and the radian measure of the subtended angle:

Important Fact 15.4.1 (Arc length in radians).
Start with a central angle of measure θ radians inside a circle of radius r. Then this angle will subtend an arc of length $s = \theta r$.

These remarks allow us to summarize the definition of the *radian measure* θ of $\angle AOB$ inside a circle of radius r by the formula:

$$\theta = \begin{cases} \frac{s}{r} & \text{if angle is swept counterclockwise} \\ -\frac{s}{r} & \text{if angle is swept clockwise} \end{cases}$$

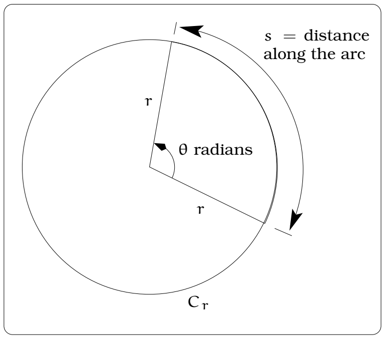


Figure 15.9: Defining arc length when angles are measured in radians.

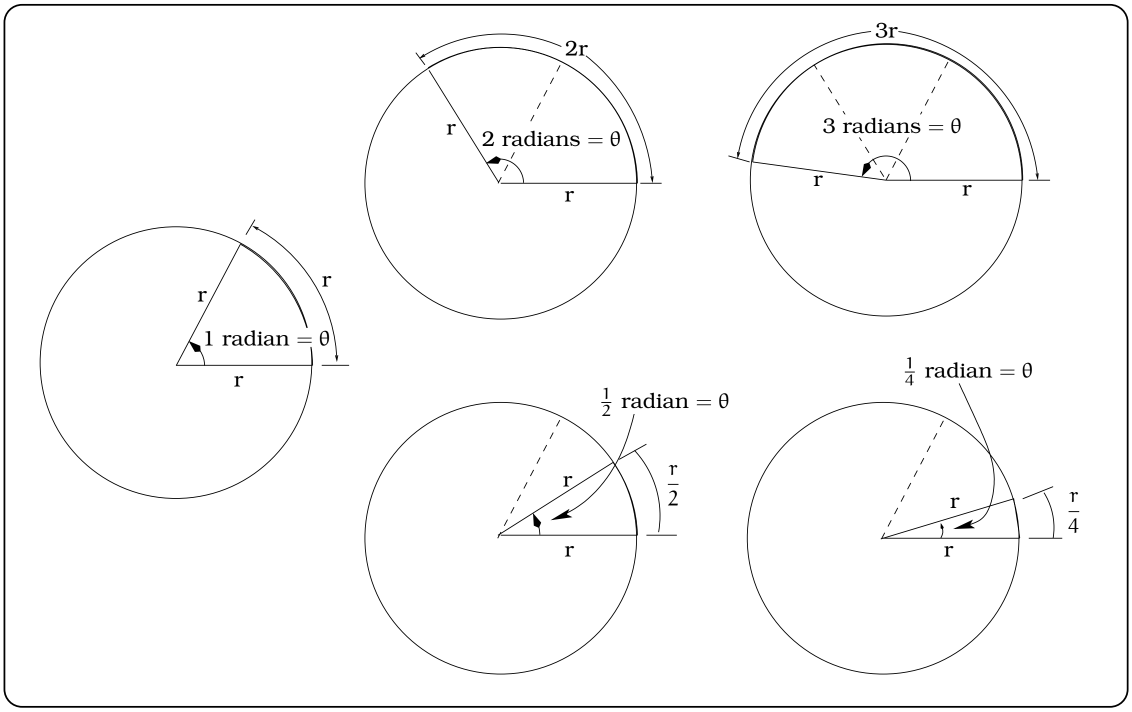


Figure 15.8: Measuring angles in radians.

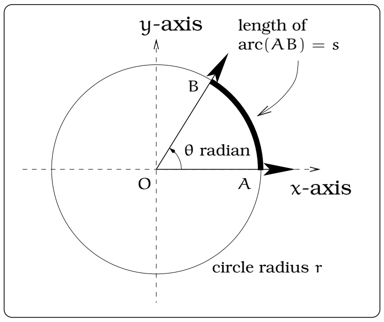


Figure 15.10: Arc length after imposing a coordinate system.

The units of θ are sometimes abbreviated as *rad*. It is important to appreciate the role of the radius of the circle C_r when using radian measure of an angle: An angle of radian measure θ will subtend an arc of length $|\theta|$ on the *unit circle*. In other words, radian measure of angles is **exactly the same** as arc length on the unit circle; we couldn't hope for a better connection!

The difficulty with radian measure versus degree measure is really one of familiarity. Let's view a few common angles in radian measure. It is easiest to start with the case of angles in central standard position within the unit circle. Examples of basic angles would be fractional parts of one complete revolution around the unit circle; for example, $\frac{1}{12}$ revolution, $\frac{1}{8}$ revolution, $\frac{1}{6}$ revolution, $\frac{1}{4}$ revolution, $\frac{1}{2}$ revolution and $\frac{3}{4}$ revolution. One revolution around the unit circle describes an arc of length 2π and so the subtended angle (1 revolution) is 2π radians. We can now easily find the radian measure of these six angles. For example, $\frac{1}{12}$ revolution would describe an angle of measure $(\frac{1}{12})2\pi$ rad=$\frac{\pi}{6}$ rad. Similarly, the other five angles pictured below have measures $\frac{\pi}{4}$ rad, $\frac{\pi}{3}$ rad, $\frac{\pi}{2}$ rad, π rad and $\frac{3\pi}{2}$ rad.

All of these examples have positive radian measure. For an angle with negative radian measure, such as $\theta = -\frac{\pi}{2}$ radians, we would locate B by rotating $\frac{1}{4}$ revolution *clockwise*, etc. From these calculations and our

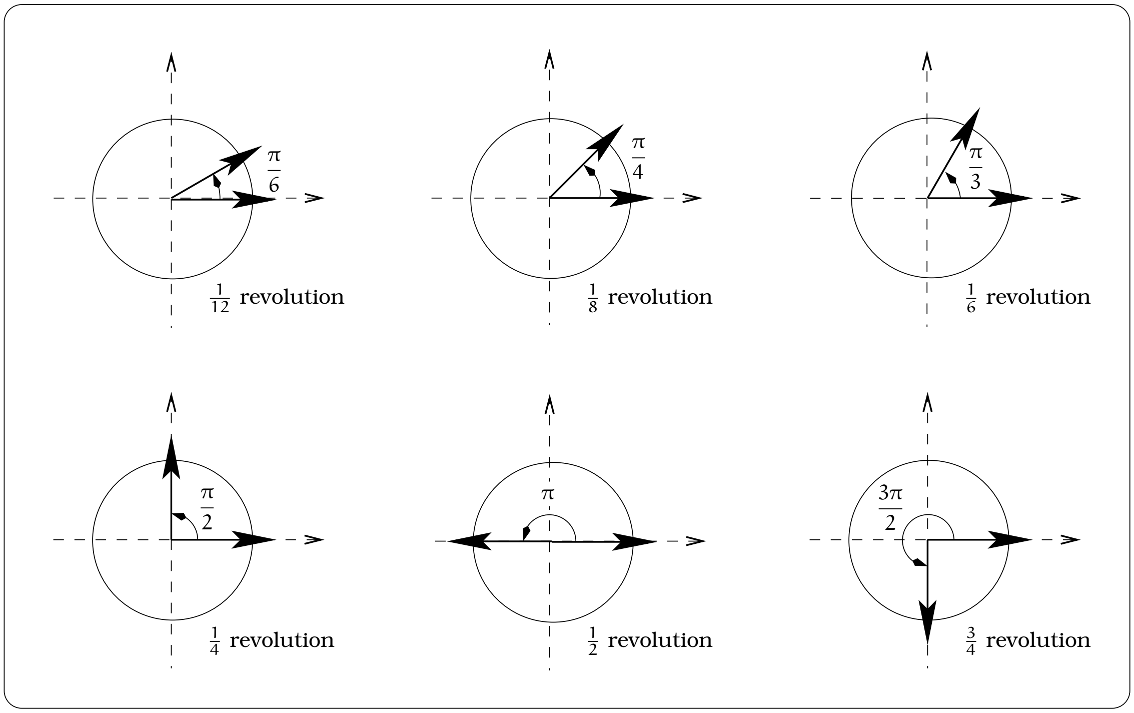

$\frac{\pi}{6}$
$\frac{1}{12}$ revolution
$\frac{\pi}{4}$
$\frac{1}{8}$ revolution
$\frac{\pi}{3}$
$\frac{1}{6}$ revolution
$\frac{\pi}{2}$
$\frac{1}{4}$ revolution
π
$\frac{1}{2}$ revolution
$\frac{3\pi}{2}$
$\frac{3}{4}$ revolution

Figure 15.11: Common angles measured in radians.

previous examples of degree measure we find that

$$180\,\text{degrees} = \pi\,\text{radians}. \tag{15.1}$$

Solving this equation for degrees or radians will provide conversion formulas relating the two types of angle measurement. The formula also helps explain the origin of the curious conversion factor $\frac{\pi}{180} = \frac{2\pi}{360}$ in Fact 15.3.2.

15.5 Areas of Wedges

The beauty of radian measure is that it is rigged so that we can easily compute lengths of arcs and areas of circular sectors (i.e. “pie-shaped regions”). This is a key reason why we will almost always prefer to work with radian measure.

Example 15.5.1. *If a 16 inch pizza is cut into 12 equal slices, what is the area of a single slice?*

This can be solved using a general principle:

$$(\text{Area of a part}) = (\text{area of the whole}) \times (\text{fraction of the part})$$

So, for our pizza:

$$\begin{aligned}(\text{area one slice}) &= (\text{area whole pie}) \times (\text{fraction of pie})\\ &= (8^2\pi)\left(\frac{1}{12}\right)\\ &= \frac{16\pi}{3}.\end{aligned}$$

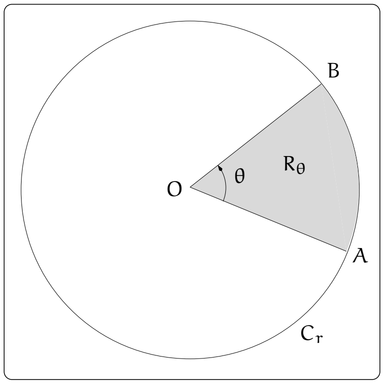


Figure 15.12: Finding the area of a "pie shaped wedge".

Let's apply the same reasoning to find the area of a circular sector. We know the area of the circular disc bounded by a circle of radius r is πr^2. Let R_θ be the "pie shaped wedge" cut out by an angle $\angle AOB$ with positive measure θ radians. Using the above principle

$$\begin{aligned}\text{area}(R_\theta) &= (\text{area of disc bounded by } C_r)\\ &\quad \times (\text{portion of disc accounted for by } R_\theta)\\ &= (\pi r^2)\left(\frac{\theta}{2\pi}\right) = \frac{1}{2}r^2\theta.\end{aligned}$$

For example, if $r = 3$ in. and $\theta = \frac{\pi}{4}$ rad, then the area of the pie shaped wedge is $\frac{9}{8}\pi$ sq. in.

Important Fact 15.5.2 (Wedge area)**.** *Start with a central angle with positive measure* θ *radians inside a circle of radius* r*. The area of the "pie shaped region" bounded by the angle is* $\frac{1}{2}r^2\theta$*.*

Example 15.5.3. *A water drip irrigation arm 1200 feet long rotates around a pivot* P *once every 12 hours. How much area is covered by the arm in one hour? in 37 minutes? How much time is required to drip irrigate 1000 square feet?*

Solution. The irrigation arm will complete one revolution in 12 hours. The angle swept out by one complete revolution is 2π radians, so after t hours the arm sweeps out an angle $\theta(t)$ given by

$$\theta(t) = \frac{2\pi \,\text{radians}}{12 \,\text{hours}} \times t \,\text{hours} = \frac{\pi}{6}t \,\text{radians}.$$

Consequently, by Important Fact 15.5.2 the area $A(t)$ of the irrigated region after t hours is

$$A(t) = \frac{1}{2}(1200)^2\theta(t) = \frac{1}{2}(1200)^2\frac{\pi}{6}t = 120{,}000\pi t \text{ square feet.}$$

After 1 hour, the irrigated area is $A(1) = 120{,}000\pi = 376{,}991$ sq. ft. Likewise, after 37 minutes, which is $\frac{37}{60}$ hours, the area of the irrigated region is $A(\frac{37}{60}) = 120{,}000\pi(\frac{37}{60}) = 232{,}500$ square feet. To answer the final question, we need to solve the equation $A(t) = 1000$; i.e., $120{,}000\pi t = 1000$, so

$$t = \left(\frac{1}{120\pi}\text{hours}\right) \times \left(\frac{3600 \text{ seconds}}{\text{hour}}\right) = 9.55 \text{ seconds.} \qquad \square$$

15.5.1 Chord Approximation

Our ability to compute arc lengths can be used as an estimating tool for distances between two points. Let's return to the situation posed at the beginning of this section: Cosmo the dog, tied by a 20 foot long tether to a post in the ground, begins at location R and walks counterclockwise to location S. Furthermore, let's suppose you are standing at the center of the circle determined by the tether and you measure the angle from R to S to be 5°; see the left-hand figure. Because the angle is small, notice that the straight line distance d from R to S is approximately the same as the arc length s subtended by the angle $\angle RPS$; the right-hand picture in Figure 15.13 is a blow-up:

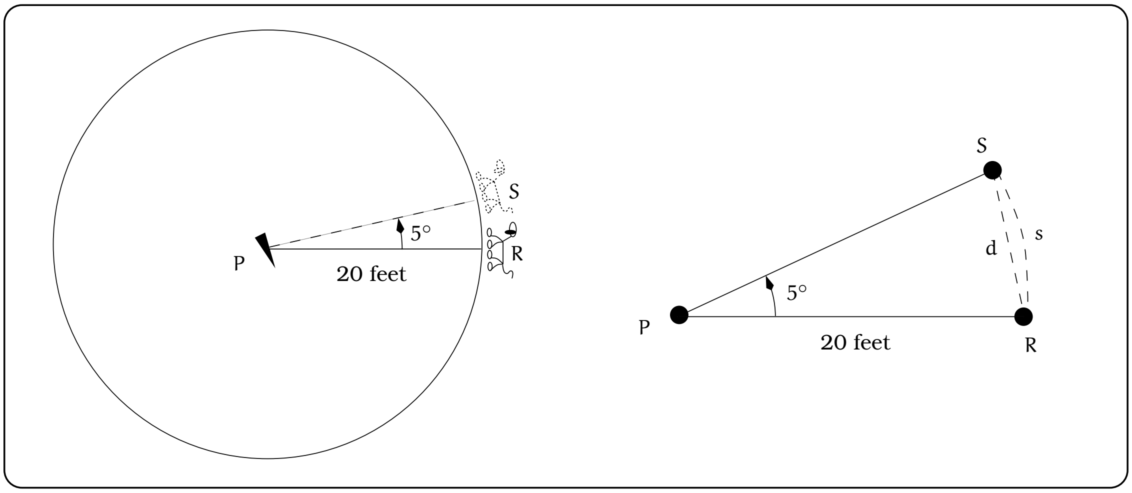


Figure 15.13: Using the arc length s to approximate the chord d.

Example 15.5.4. *Estimate the distance from* R *to* S.

Solution. We first convert the angle into radian measure via (15.1): $5^\circ = 0.0873$ radians. By Fact 15.3.2, the arc s has length 1.745 feet $= 20.94$ inches. This is approximately equal to the distance from R to S, since the angle is small. $\square$

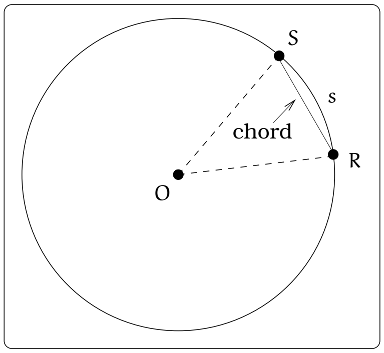


Figure 15.14: Chord approximation.

We call a line segment connecting two points on a circle a *chord* of the circle. The above example illustrates a general principal for approximating the length of any chord. A smaller angle will improve the accuracy of the arc length approximation.

Important Fact 15.5.5 (Chord Approximation)**.** *In Figure 15.14, if the central angle is small, then* $s \approx |\overline{RS}|$.

15.6 Great Circle Navigation

A basic problem is to find the shortest route between any two locations on the earth. We will review how to coordinatize the surface of the earth and recall the fact that the shortest path between two points is measured along a *great circle*.

View the earth as a sphere of radius $r = 3{,}960$ miles. We could slice the earth with a two-dimensional plane $\mathcal{P}_0$ which is both perpendicular to a line connecting the North and South poles and passes through the center of the earth. Of course, the resulting intersection will trace out a circle of radius $r = 3{,}960$ miles on the surface of the earth, which we call the *equator*. We call the plane $\mathcal{P}_0$ the *equatorial plane*. Slicing the earth with any other plane $\mathcal{P}$ parallel to $\mathcal{P}_0$, we can consider the right triangle pictured below and the angle θ:

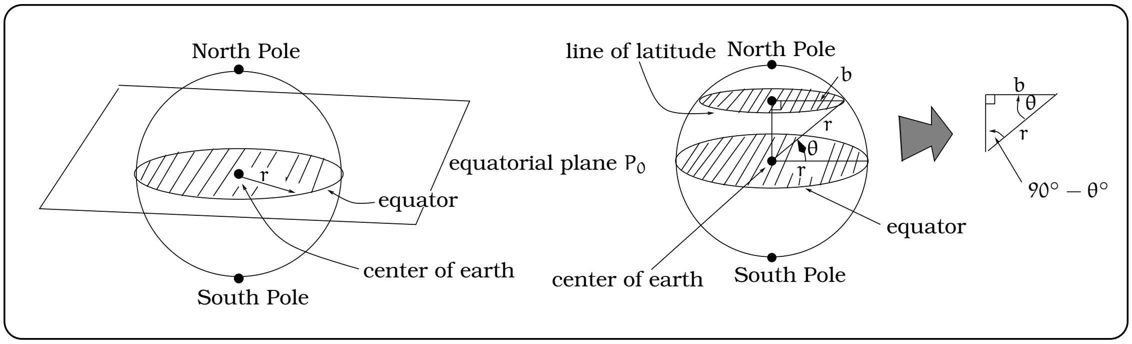


Figure 15.15: Measuring latitude.

Essentially two cases arise, depending on whether or not the plane $\mathcal{P}$ is above or below the equatorial plane. The plane $\mathcal{P}$ slices the surface of the earth in a circle, which we call a line of *latitude*. This terminology is somewhat incorrect, since these lines of latitude are actually circles on the surface of the earth, but the terminology is by now standard. Depending on whether this line of latitude lies above or below the equatorial plane, we refer to it as the θ° North line of latitude (denoted θ° N) or the θ° South line of latitude (denoted θ° S). Notice, the radius b of a line of latitude can vary from a maximum of 3,960 miles (in the case of $\theta = 0^\circ$), to a minimum of 0 miles, (when $\theta = 90^\circ$). When $b = 0$, we are at the North or South poles on the earth.

In a similar spirit, we could imagine slicing the earth with a plane $\mathcal{Q}$ which is perpendicular to the equatorial plane and passes through the center of the earth. The resulting intersection will trace out a circle of radius 3,960 miles on the surface of the earth, which is called a line of *longitude*. Half of a line of longitude from the North Pole to the South Pole is called a *meridian*. We distinguish one such meridian; the one which passes through Greenwich, England as the *Greenwich meridian*. Longitudes are measured using angles East or West of Greenwich. Pictured below, the longitude of A is θ. Because θ is east of Greenwich, θ measures longitude East, typically written θ° E; west longitudes would be denoted as θ° W. All longitudes are between 0° and 180°. The meridian which is 180° West (and 180° E) is called the *International Date Line*.

Introducing the grid of latitude and longitude lines on the earth amounts to imposing a coordinate system. In other words, any position on the earth can be determined by providing the longitude and latitude of the point. The usual convention is to list longitude first. For example, Seattle has coordinates 122.0333° W, 47.6° N. Since the labels "N and S" are attached to latitudes and the labels "E and W" are attached to longitudes, there is no ambiguity here. This means that Seattle is on the line of longitude 122.0333° West of the Greenwich meridian and on the line of latitude 47.6° North of the equator. In the figure below, we indicate the key angles $\psi = 47.6^\circ$ and $\theta = 122.0333^\circ$ by inserting the three indicated radial line segments.

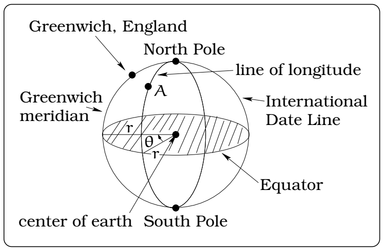

Greenwich, England
North Pole
line of longitude
A
Greenwich meridian
International Date Line
r
θ
r
Equator
center of earth
South Pole

Figure 15.16: The International Date Line.

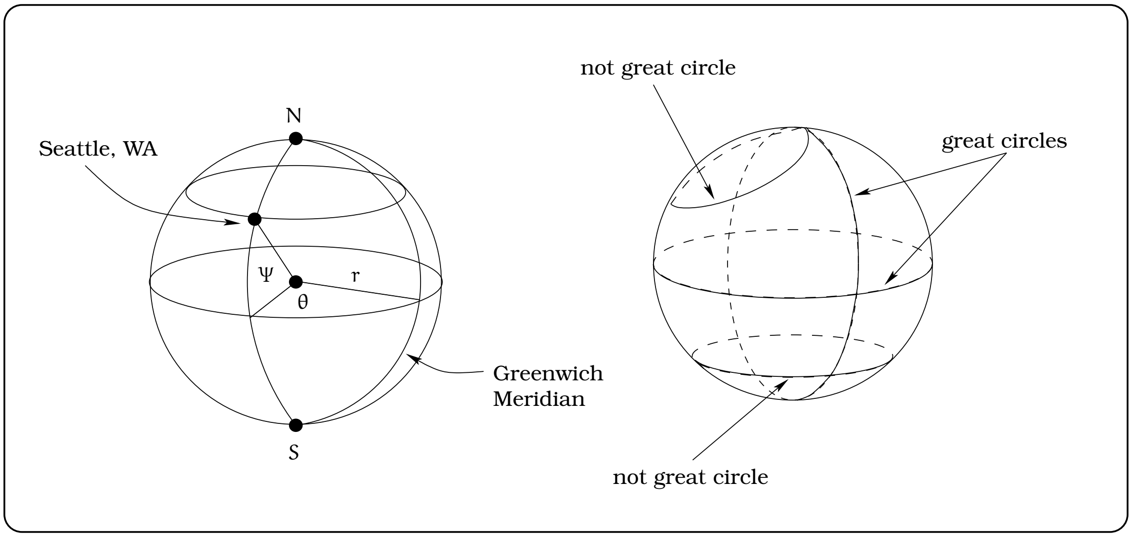

not great circle
N
Seattle, WA
great circles
Ψ
r
θ
Greenwich Meridian
S
not great circle

Figure 15.17: Distances along great circles.

Now that we have imposed a coordinate system on the earth, it is natural to study the distance between two locations. A *great circle* of a sphere is defined to be a circle lying on the sphere with the same center as the sphere. For example, the equator and any line of longitude are great circles. However, lines of latitude are not great circles (except the special case of the equator). Great circles are very important because they are used to find the shortest distance between two points on the earth. The important fact from geometry is summarized below.

Important Fact 15.6.1 (Great Circles). *The shortest distance between two points on the earth is measured along a great circle connecting them.*

Example 15.6.2. *What is the shortest distance from the North Pole to Seattle, WA ?*

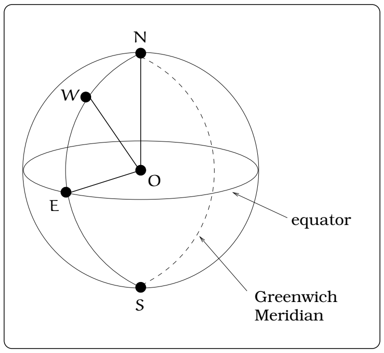


Figure 15.18: Distance between the North Pole and Seattle, Washington.

Solution. The line of longitude 122.0333° W is a great circle connecting the North Pole and Seattle. So, the shortest distance will be the arc length s subtended by the angle $\angle NOW$ pictured in Figure 15.18. Since the latitude of Seattle is 47.6°, the angle $\angle EOW$ has measure 47.6°. Since $\angle EON$ is a right angle (i.e., 90°), $\angle NOW$ has measure 42.4°. By Fact 15.4.1 and Equation 15.1,

$$\begin{aligned} s &= (3960\,\text{miles})(42.4^\circ)(0.01745\,\text{radians/degree}) \\ &= 2943.7\,\text{miles}, \end{aligned}$$

which is the shortest distance from the pole to Seattle. □

15.7 Summary

- $360^\circ = 2\pi$ radians
- A circular arc with radius r and angle θ has length s, with

  $$s = r\theta$$

  when θ is measured in radians.
- A circular wedge with radius r and angle θ has area A, with

  $$A = \frac{1}{2}r^2\theta$$

  when θ is measured in radians.

15.8 Exercises

Problem 15.1. Let $\angle AOB$ be an angle of measure θ.

(a) Convert $\theta = 13.4^\circ$ into degrees/ minutes/ seconds and into radians.

(b) Convert $\theta = 1^\circ 4' 44''$ into degrees and radians.

(c) Convert $\theta = 0.1$ radian into degrees and degrees/ minutes/ seconds.

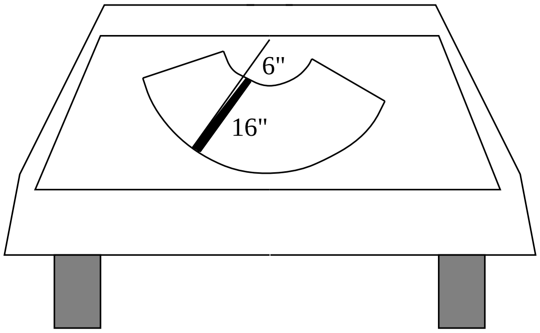
Problem 15.2. A *nautical mile* is a unit of distance frequently used in ocean navigation. It is defined as the length of an arc s along a great circle on the earth when the subtending angle has measure $1' =$ "one minute" $= 1/60$ of one degree. Assume the radius of the earth is 3,960 miles.

(a) Find the length of one nautical mile to the nearest 10 feet.

(b) A vessel which travels one nautical mile in one hours time is said to have the speed of one *knot*; this is the usual navigational measure of speed. If a vessel is traveling 26 knots, what is the speed in mph (miles per hour)?

(c) If a vessel is traveling 18 mph, what is the speed in knots?

Problem 15.3. The rear window wiper blade on a station wagon has a length of 16 inches. The wiper blade is mounted on a 22 inch arm, 6 inches from the pivot point.



(a) If the wiper turns through an angle of 110°, how much area is swept clean?

(b) Through how much of an angle would the wiper sweep if the area cleaned was 10 square inches?

(c) Suppose bug A lands on the end of the blade farthest from the pivot. Assume the wiper turns through an angle of 110°. In one cycle (back and forth) of the wiper blade, how far has the bug traveled?

(d) Suppose bug B lands on the end of the wiper blade closest to the pivot. Assume the wiper turns through an angle of 110°. In one cycle of the wiper blade, how far has the bug traveled?

(e) Suppose bug C lands on an intermediate location of the wiper blade. Assume the wiper turns through an angle of 110°. If bug C travels 28 inches after one cycle of the wiper blade, determine the location of bug C on the wiper blade.

Problem 15.4. A water treatment facility operates by dripping water from a 60 foot long arm whose end is mounted to a central pivot. The water then filters through a layer of charcoal. The arm rotates once every 8 minutes.

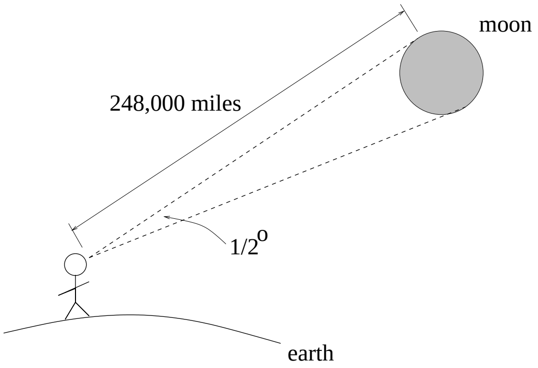
(a) Find the area of charcoal covered with water after 1 minute.

(b) Find the area of charcoal covered with water after 1 second.

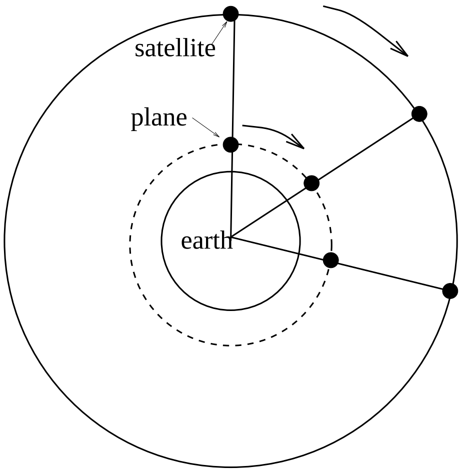
(c) How long would it take to cover 100 square feet of charcoal with water?

(d) How long would it take to cover 3245 square feet of charcoal with water?

Problem 15.5. Astronomical measurements are often made by computing the small angle formed by the extremities of a distant object and using the estimating technique in 15.5.1. In the picture below, the full moon is shown to form an angle of $\frac{1}{2}^\circ$ when the distance indicated is 248,000 miles. Estimate the diameter of the moon.



Problem 15.6. An aircraft is flying at the speed of 500 mph at an elevation of 10 miles above the earth, beginning at the North pole and heading South along the Greenwich meridian. A spy satellite is orbiting the earth at an elevation of 4800 miles above the earth in a circular orbit in the same plane as the Greenwich meridian. Miraculously, the plane and satellite always lie on the same radial line from the center of the earth. Assume the radius of the earth is 3960 miles.



(a) When is the plane directly over a location with latitude $74°30'18''$ N for the first time?

(b) How fast is the satellite moving?

(c) When is the plane directly over the equator and how far has it traveled?

(d) How far has the satellite traveled when the plane is directly over the equator?

Problem 15.7. Find the area of the sector of a circle of radius 11 inches if the measure θ of a central angle of this sector is:

(a) $\theta = 45°$

(b) $\theta = 80^{o}$

(c) $\theta = 3$ radians

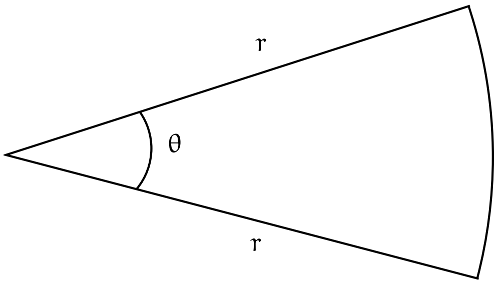
(d) $\theta = 2.46$ radians

(e) $\theta = 97^{o}23'3''$

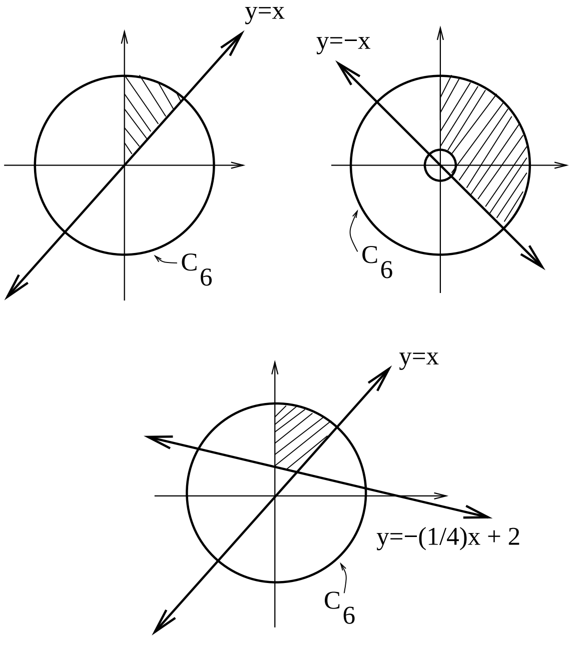
(f) $\theta = 35^{o}24'2''$

Problem 15.8. Matilda is planning a walk around the perimeter of Wedge Park, which is shaped like a circular wedge, as shown below. The walk around the park is 2.1 miles, and the park has an area of 0.25 square miles.

If θ is less than 90 degrees, what is the value of the radius, r?



Problem 15.9. Let C_6 be the circle of radius 6 inches centered at the origin in the xy-coordinate system. Compute the areas of the shaded regions in the picture below; the inner circle in the rightmost picture is the unit circle:



Chapter 16

Measuring Circular Motion

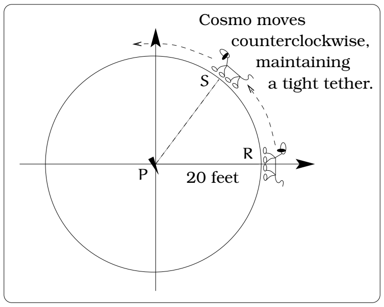


Figure 16.1: How fast is Cosmo moving?

If Cosmo the dog begins at location R *and walks counterclockwise, always connected to the post* P *by a tight tether, how can we measure Cosmo's speed?*

This is a "dynamic question" and requires that we discuss ways of measuring circular motion. In contrast, if we take a snapshot and ask to measure the specific angle $\angle RPS$, this is a "static question", which we answered in the previous section.

16.1 Different Ways to Measure Cosmo's Speed

If Cosmo starts at location R and arrives at location S after some amount of time, we could study

$$\omega = \frac{\text{measure } \angle RPS}{\text{time required to go from } R \text{ to } S}.$$

The funny Greek letter "ω" on the left of side of the equation is pronounced "oh-meg-a". We will refer to this as an *angular speed*. Typical units are $\frac{\text{degrees}}{\text{minute}}$, $\frac{\text{degrees}}{\text{second}}$, and $\frac{\text{radians}}{\text{minute}}$.

For example, if the angle swept out by Cosmo after 8 seconds is 40°, then Cosmo's angular speed is $\omega = \left(\frac{40^\circ}{8 \text{ seconds}}\right) = \left(\frac{5^\circ}{\text{sec}}\right)$. Using (15.1), we can convert to radian units and get $\omega = \frac{\pi}{36}\frac{\text{rad}}{\text{sec}} = \frac{5}{3}\pi\frac{\text{rad}}{\text{min}}$. This is a new example of a *rate* of change. If we are given ω in units of $\frac{\text{rad}}{\text{time}}$ or $\frac{\text{deg}}{\text{time}}$, we have

$$\theta = \omega t,$$

which computes the measure of the angle θ swept out after time t (i.e. the total change in the angle). Angular speed places emphasis upon the "size of the angle being swept out per unit time" by the moving object, starting from some initial position. We need to somehow indicate the direction in which the angle is being swept out. This can be done by indicating "clockwise" our "counterclockwise". Alternatively, we can adopt the convention that the positive rotational direction is counterclockwise, then insert a minus sign to indicate rotation clockwise. For example, saying that Cosmo is moving at an angular speed of $\omega = -\frac{\pi}{2}\,\frac{\text{rad}}{\text{sec}}$ means he is moving clockwise $\frac{\pi}{2}\,\frac{\text{rad}}{\text{sec}}$.

Another way to study the rate of a circular motion is to count the number of complete circuits of the circle per unit time. This sort of rate has the form

$$\frac{\text{Number of Revolutions}}{\text{Unit of Time}}$$

and we will also view this as an angular speed. If we take "minutes" to be the preferred unit of time, we arrive at the common measurement called *revolutions per minute*, usually denoted RPM or rev/min.

For example, if Cosmo completes one trip around the circle every 2 minutes, then Cosmo is moving at a rate of $\frac{1}{2}$ RPM. If instead, Cosmo completes one trip around the circle every 12 seconds, then we could first express Cosmo's speed in units of *revolutions/second* as $\frac{1}{12}$ rev/second, then convert to RPM units:

$$\left(\frac{1}{12}\,\frac{\text{rev}}{\text{sec}}\right)\left(60\,\frac{\text{sec}}{\text{min}}\right) = 5\,\text{RPM}.$$

As a variation, if we measure that Cosmo completed $\frac{3}{7}$ of a revolution in 2 minutes, then Cosmo's angular speed is computed by

$$\frac{\frac{3}{7}\text{rev}}{2\,\text{min}} = \frac{3}{14}\,\text{RPM}.$$

The only possible ambiguity involves the direction of revolution: the object can move clockwise or counterclockwise.

The one shortcoming of using angular speed is that we are not directly keeping track of the distance the object is traveling. This is fairly easy to remedy. Returning to Figure 16.1, the circumference of the circle of motion is $2\pi(20) = 40\pi$ feet. This is the distance traveled per revolution, so we can now make conversions of angular speed into "distance traveled per unit time"; this is called the *linear speed*. If Cosmo is moving $\frac{1}{2}$ RPM, then he has a linear speed of

$$v = \left(\frac{1\ \text{rev}}{2\ \text{min}}\right)\left(\frac{40\pi\,\text{ft}}{\text{rev}}\right) = 20\pi\,\frac{\text{ft}}{\text{min}}.$$

Likewise, if Cosmo is moving $\frac{\pi}{7}\,\frac{\text{rad}}{\text{sec}}$, then

$$v = \left(\frac{\pi\,\text{rad}}{7\,\text{sec}}\right)\left(\frac{1\,\text{rev}}{2\pi\,\text{rad}}\right)\left(\frac{40\pi\,\text{ft}}{\text{rev}}\right) = \frac{20\pi}{7}\,\frac{\text{ft}}{\text{sec}}.$$

16.2 Different Ways to Measure Circular Motion

The discussion of Cosmo applies to circular motion of any object. As a matter of convention, we usually use the Greek letter ω to denote angular speed and v for linear speed. If an object is moving around a circle of radius r at a constant rate, then we can measure its speed in two ways:

- The angular speed
$$\omega = \frac{\text{revolutions}}{\text{unit time}} \text{ or } \frac{\text{degrees swept}}{\text{unit time}} \text{ or } \frac{\text{radians swept}}{\text{unit time}}.$$
- The linear speed
$$v = \frac{\text{distance traveled}}{\text{unit time}}.$$

Important Facts 16.2.1 (Measuring and converting)**.** *We can convert between angular and linear speeds using these facts:*

- *1 revolution = 360° = 2π radians.*
- *The circumference of a circle of radius r units is $2\pi r$ units.*

16.2.1 Three Key Formulas

We now know two general relationships for circular motion:

(i) $s = r\theta$, where s = arclength (a linear distance), r = radius of the circular path and θ =angle swept in RADIAN measure; this was the content of Fact 15.4.1 on page 197.

(ii) $\theta = \omega t$, where θ is the measure of an angle swept, ω = angular speed and t represents time elapsed.

Notice how the units work in these formulas. If $r = 20$ feet and $\theta = 1.3$ radians, then the arc length $s = 20(1.3)$ feet= 26 feet; this is the length of the arc of radius 20 feet that is subtending the angle θ. If $\omega = 3$ rad/second and $t = 5$ seconds, then $\theta = 3\frac{\text{rad}}{\text{seconds}} \times 5\,\text{seconds} = 15$ radians.

If we replace θ in $s = r\theta$ of (i) with $\theta = \omega t$ in (ii), then we get

$$s = r\omega t.$$

This gives us a relationship between arclength s (a distance) and time t. Plug in the fact that the linear speed is defined to be $v = \frac{\text{“distance”}}{t}$ and we get

$$v = \frac{s}{t} = \frac{r\omega t}{t} = r\omega.$$

All of these observations are summarized below.

Important Facts 16.2.2 (Three really useful formulas)**.** *If we measure angles θ in* ***radians*** *and ω in units of* ***radians*** *per unit time, we have these three formulas:*

$$s = r\theta \tag{16.1}$$
$$\theta = \omega t \tag{16.2}$$
$$v = r\omega \tag{16.3}$$

Example 16.2.3. *You are riding a stationary exercise bike and the speedometer reads a steady speed of 20 MPH (miles per hour). If the rear wheel is 28 inches in diameter, determine the angular speed of a location on the rear tire. A chalk mark on the tread reaches the lowest point in its rotation. Describe the location of the chalk mark after 1 second and 0.1 second.*

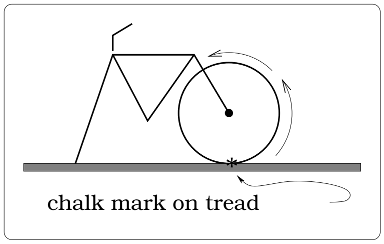


Figure 16.2: Where is the chalk mark after t seconds?

Solution. The tires will be rotating in a counterclockwise direction and the radius $r = \frac{1}{2}(28)$ inches $= 14$ inches. The other given quantity, 20 MPH, involves miles, so we need to decide which common units to work with. We will utilize inches.

If the speedometer reads 20 MPH, this is the linear speed of the rear tire. We need to convert this into an angular speed. First, the linear speed of the wheel is

$$\begin{aligned} v &= \left(20\,\frac{\text{miles}}{\text{hr}}\right)\left(5280\,\frac{\text{ft}}{\text{mile}}\right)\left(12\,\frac{\text{in}}{\text{ft}}\right)\left(\frac{1\text{ hr}}{60\text{ min}}\right)\left(1\,\frac{\text{min}}{60\text{ sec}}\right) \\ &= 352\,\frac{\text{in}}{\text{sec}}. \end{aligned}$$

Now, the angular speed ω of the wheel will be

$$\begin{aligned} \omega &= \frac{352\,\frac{\text{inches}}{\text{second}}}{2\pi(14)\,\frac{\text{inches}}{\text{revolution}}} \\ &\approx 4\,\frac{\text{revolution}}{\text{second}} \\ &= 240\,\text{RPM} \end{aligned}$$

It is then an easy matter to convert this to

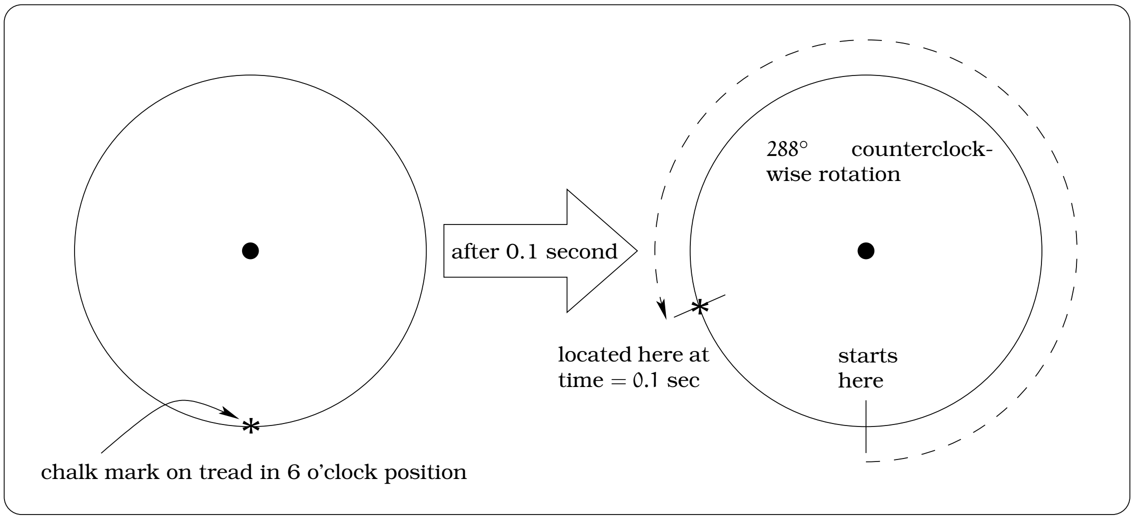
$$\begin{aligned} \omega &= \left(4\,\frac{\text{revolution}}{\text{second}}\right)\left(360\,\frac{\text{degrees}}{\text{revolution}}\right) \\ &= 1440\,\frac{\text{degrees}}{\text{second}}. \end{aligned}$$

If the chalk mark begins at the “6 o’clock” position (the place the tire touches the ground on the wheel), then after 1 second the chalk mark will go through 8 revolutions, so will be in the “6 o’clock” position again.

After 0.1 seconds, the chalk mark will have gone through an angle of

$$\left(1440\,\frac{\text{degrees}}{\text{sec}}\right)(0.1\,\text{sec}) = 144^\circ.$$

Keeping in mind that the rotation is counterclockwise, we can view the location of the chalk mark after 0.1 seconds as pictured below: □



after 0.1 second
288° counterclockwise rotation
located here at time = 0.1 sec
starts here
chalk mark on tread in 6 o'clock position

Figure 16.3: Computing the chalk mark's position after $t = 0.1$ sec.

We solved the previous problem using the "unit conversion method". There is an alternate approach available, which uses one of the formulas in Fact 16.2.2. Here is how you could proceed: First, as above, we know the linear speed is $v = 352$ in/sec. Using the $v = r\omega$ formula, we have

$$\begin{aligned} 352\,\frac{\text{in}}{\text{sec}} &= \omega(14\,\text{in}) \\ \omega &= 25.14286\,\frac{\text{rad}}{\text{sec}}. \end{aligned}$$

Notice how the units worked out in the calculation: the "time" unit comes from v and the "angular" unit will always be radians. As a comparison with the solution above, we can convert ω into RPM units:

$$\begin{aligned} \omega &= \left(25.14286\,\frac{\text{rad}}{\text{sec}}\right)\left(\frac{1\,\text{rev}}{2\pi\,\text{rad}}\right) \\ &\approx 4\,\frac{\text{rev}}{\text{sec}}. \end{aligned}$$

All of the problems in this section can be worked using either the "unit conversion method" or the $v = r\omega$ method.

16.3 Music Listening Technology

The technology of reproducing music has gone through a revolution since the early 1980's. The stereo long playing record (the *LP*) and the newer digital compact disc (the *CD*) are two methods of storing musical data for later reproduction in a home stereo system. These two technologies adopt different perspectives as to which notion of circular speed is best to work with.

Long playing stereo records are thin vinyl plastic discs of radius 6 inches onto which small spiral grooves are etched into the surface. We can approximately view this groove as a circle. The *LP* is placed on a flat 12 inch diameter platter which turns at a constant angular speed of $33\frac{1}{3}$ RPM. An arm on a pivot (called the *tone arm*) has a needle mounted on the end (called the *cartridge*), which is placed in the groove on the outside edge of the record. Because the grooves wobble microscopically from side-to-side, the needle will mimic this motion. In turn, this sets a magnet (mounted on the opposite end of the needle) into motion. This moving magnet sits inside a coil of wire, causing a small varying voltage; the electric signal is then fed to an amplifier and passed onto your speakers, reproducing music.

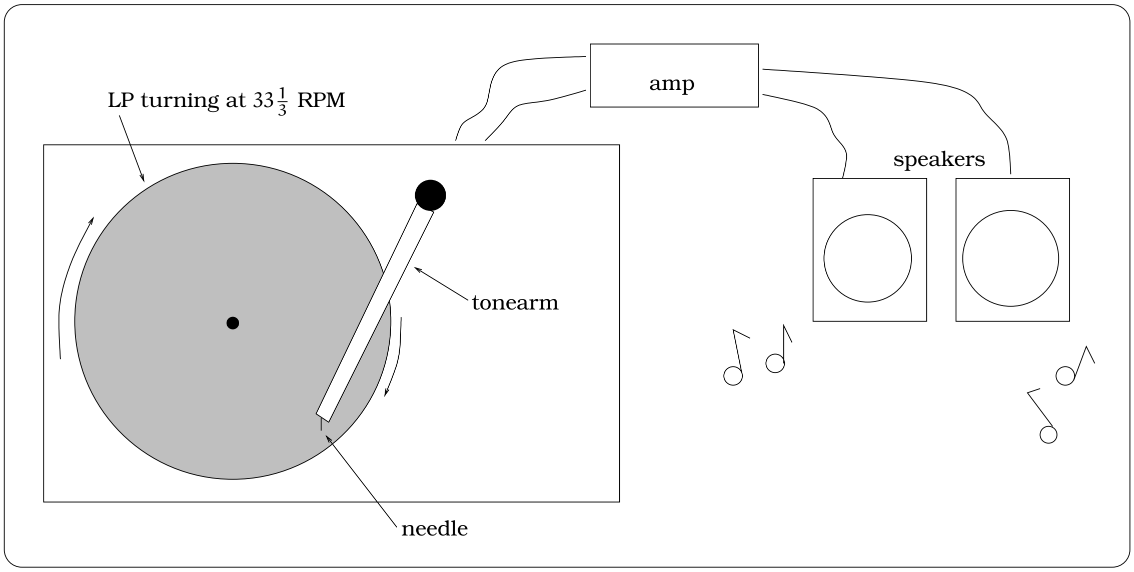


Figure 16.4: Reproducing music using analog technology.

This is known as *analog* technology (in contrast to *digital* technology) and is based upon the idea of maintaining a constant angular speed of $33\frac{1}{3}$ RPM for the storage medium (our LP). (45 RPM and 78 RPM records exist as well, but $33\frac{1}{3}$ RPM was the consumer standard for stereo music.) With an *LP*, the beginning of the record (the *lead-in groove*) would be on the outermost edge of the record and the end of the record (the *exit* *groove*) would be close to the center. Placing the needle in the lead-in groove, the needle gradually works its way to the exit groove. However, whereas the angular speed of the LP is a constant $33\frac{1}{3}$ RPM, the linear speed at the needle can vary quite a bit, depending on the needle location.

Example 16.3.1 (Analog LP's)**.** *The "lead-in groove" is 6 inches from the center of an LP, while the "exit groove" is 1 inch from the center. What is the linear speed (in inches per second) of the needle in the "lead-in groove"? What is the linear speed (in inches per second) of the needle in the "exit groove"? Find the location of the needle if the linear speed is* 10 *inches per second.*

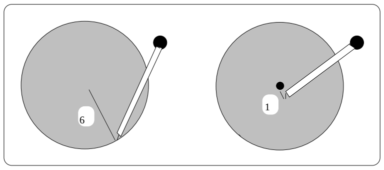


Figure 16.5: Lead-in and exit grooves.

Solution. We begin by converting the angular speed to radians:

$$\begin{aligned}\omega &= (33\frac{1}{3} \text{ revolutions/minute})(2\pi \text{ radians/revolution})\\ &= \frac{200\pi}{3} \text{ radians/minute}\\ &= \frac{10\pi}{9} \text{ radians/second.}\end{aligned}$$

The linear speed when the needle is 6 inches from the center (so $r = 6$ inches) is then

$$v_{=}r\omega = (6 \text{ inches})\left(\frac{10\pi}{9}\right) = \frac{20\pi}{3} \text{ inches/second} \approx 20.9439 \text{ inches/second.}$$

That is, every second, approximately 20.9439 inches of groove passes under the needle.

When the needle is 1 inch from the center (so $r = 1$ inch), the linear speed is

$$v = r\omega = (1 \text{ inch})\left(\frac{10\pi}{9}\right) = \frac{10\pi}{9} \text{ inches/second} \approx 3.49067 \text{ inches/second.}$$

One consequence of this difference in speed was a decrease in sound quality as the needle got closer to the center.

If $v = 10$ inches per second, then we can find the radius like this:

$$r = \frac{v}{\omega} = \frac{10 \text{ inches per second}}{10\pi/9 \text{ radians per second}} = \frac{9}{\pi} \text{ inches} \approx 2.8648 \text{ inches.}$$

$\square$

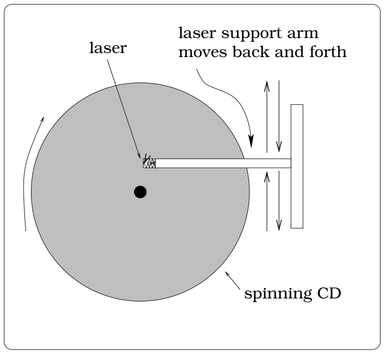


Figure 16.6: Reproducing music using digital technology.

In the early 1980's, the *digital compact disc* (CD) was introduced. A CD is a thin plastic disc of diameter 4.5 inches, which appears to the naked eye to have a shiny silver coating on one side. Upon microscopic examination one would find concentric circles of pits in the silver coating. This disc is placed in a *CD* player, which spins the disc. A laser located above the spinning disc will project onto the spinning disc and the pits in the silver coating will cause the reflected laser light to vary in intensity. A sensor detects this variation, converting it to a digital signal (the *analog to digital* or *AD conversion*). This is fed into a *digital to analog* or *DA conversion* device, which sends a signal to an amplifier which then send the signal to speakers, producing music.

The technology of CDs differs from that of LPs in two crucial ways. First, the circular motion of the spinning *CD* is controlled so that the target on the disc below the laser is always moving at a *constant linear speed* of $1.2\,\frac{\text{meters}}{\text{sec}} = 2{,}835\,\frac{\text{inches}}{\text{minute}}$. Secondly, the beginning location of the laser will be on the inside portion of the disc, working its way outward to the end. In this context, it makes sense to study how the angular speed of the CD is changing, as the laser position changes.

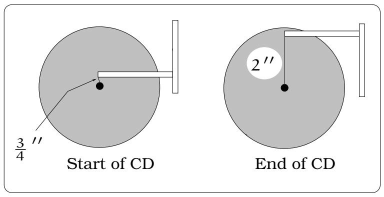


Figure 16.7: Computing the angular speed of a CD.

Example 16.3.2 (Digital CDs). *What is the angular speed (in RPM) of a CD if the laser is at the beginning, located $\frac{3}{4}$ inches from the center of the disc ? What is the angular speed (in RPM) of a CD if the laser is at the end, located 2 inches from the center of the disc ? Find the location of the laser if the angular speed is 350 RPM.*

Solution. We use the relationship $v = r\omega$. Let $\omega_{3/4}$ be the angular speed at the start and ω_2 the angular speed at the end of the *CD*.

From $v = r\omega$, we have $\omega = \frac{v}{r}$, and so

$$\omega_2 = \frac{2835\ \text{inches/minute}}{2\ \text{inches}} = 1417.5\ \text{radians/minute} = 225.602\ \text{RPM}$$

and

$$\omega_{3/4} = \frac{2835\ \text{inches/minute}}{0.75\ \text{inches}} = 3780\ \text{radians/minute} = 601.606\ \text{RPM}.$$

To answer the remaining question, we convert 350 RPM to 700π radians per minute. Then, the relation $r = \frac{v}{\omega}$ gives us

$$r = \frac{2835\ \text{inches/minute}}{700\pi\ \text{radians/minute}} \approx 1.289\ \text{inches}.$$

So, when the laser is 1.289 inches from the center, the CD is spinning at a rate of 350 RPM. ☐

16.4 Belt and Wheel Problems

The industrial revolution spawned a number of elaborate machines involving systems of belts and wheels. Computing the speed of various belts and wheels in such a system may seem complicated at first glance. The situation can range from a simple system of two wheels with a belt connecting them, to more elaborate designs. We call problems of this sort *belt and wheel problems*, or more generally, *connected wheel problems*. Solving problems of this type always uses the same strategy, which we will first highlight by way of an example.

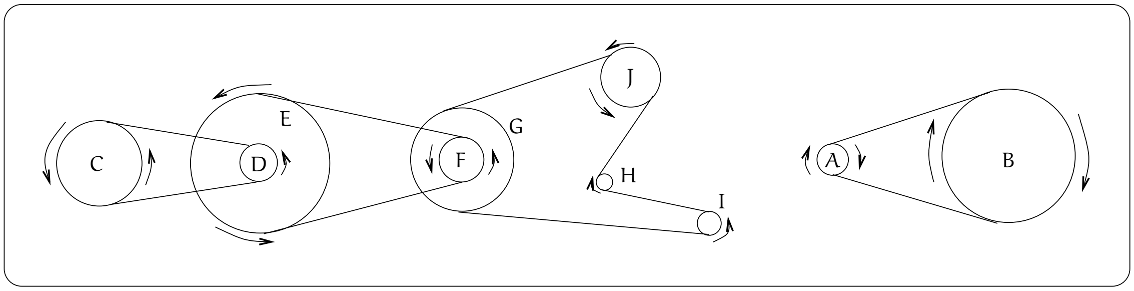
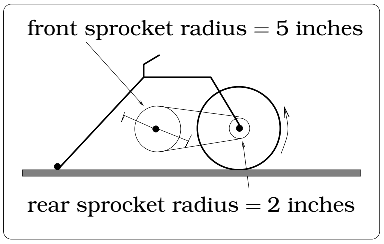
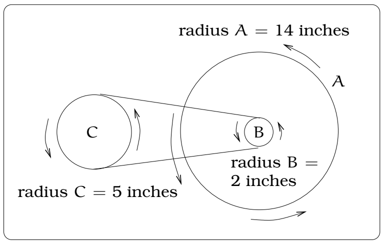


Figure 16.8: Two typical connected wheel scenarios.


front sprocket radius = 5 inches
rear sprocket radius = 2 inches

(a) A stationary exercise bike.



(b) A model of the bike's connected wheels.

Figure 16.9: Visualizing the connected wheels of an exercise bike.

Example 16.4.1. *You are riding a stationary exercise bike. Assume the rear wheel is 28 inches in diameter, the rear sprocket has radius 2 inches and the front sprocket has radius 5 inches. How many revolutions per minute of the front sprocket produces a (rear wheel) speed of 20 MPH on the bike?*

Solution. There are 3 wheels involved with a belt (the bicycle chain) connecting two of the wheels. In this problem, we are provided with the linear speed of wheel A (which is 20 MPH) and we need to find the angular speed of wheel C, the front sprocket.

Denote by v_A, v_B, and v_C the linear speeds of each of the wheels A, B, and C, respectively. Likewise, let ω_A, ω_B, and ω_C denote the angular speeds of each of the wheels. In addition, the chain connecting the wheels B and C will have a linear speed, which we will denote by v_{chain}. The strategy is broken into a sequence of steps which leads us from the known linear speed v_A to the angular speed ω_C of wheel C:

- Step 1: Given v_A, find ω_A. Use the fact $\omega_A = \frac{v_A}{r_A}$.

- Step 2: Observe $\omega_A = \omega_B$; this is because the wheel and rear sprocket are both rigidly mounted on a common axis of rotation.
- Step 3: Given ω_B, find v_B.

  Use the fact $v_B = r_B\omega_B = r_B\omega_A = \left(\frac{r_B}{r_A}\right) v_A$.
- Step 4: Observe $v_B = v_{chain} = v_C$. This is because the chain is directly connecting the two sprockets and assumed not to slip or stretch.
- Step 5: Given v_C, find ω_C.

  Use the fact $\omega_C = \frac{v_C}{r_C} = \frac{v_B}{r_C} = \left(\frac{r_B}{r_A r_C}\right) v_A$.

Saying that the speedometer reads 20 MPH is the same as saying that the linear speed of the rear wheel is $v_A = 20\,\text{MPH}$. (This means that, if the wheel were placed on the ground, and allowed to move, it would roll away at a speed of 20 MPH: the linear speed of a wheel of a (non-stationary) bicycle is the speed at which the bicycle travels.) Converting this into angular speed was carried out in our solution to Example 16.2.3 above; we found that $\omega_A = 240\,\text{RPM}$. Converting this to units of radians, we have

$$\omega_A = 240 \text{ RPM}\,(2\pi \text{ radians/revolution}) = 480\pi \text{ radians/minute.}$$

This completes Step 1 and so by Step 2, $\omega_A = \omega_B = 480\pi$ radians/minute. For Step 3, we use $v = r\omega$, to find

$$v_B = (2 \text{ inches})(480\pi \text{ radians/minute}) = 960\pi \text{ inches/minute.}$$

By Step 4, we may conclude that the linear speed of wheel C is

$$v_C = 960\pi \text{ inches/min.}$$

Finally, to carry out Step 5, we use $\omega = \dfrac{v}{r}$ to find

$$\omega_C = \frac{v_C}{r_C} = \frac{960\pi \text{ inches/min}}{5 \text{ inches}} = 192\pi \text{ radians/minute.}$$

Converting this to more common units, we have

$$\omega_C = (192\pi \text{ radians/minute})\left(\frac{1 \text{ revolution}}{2\pi\text{radians}}\right) = 96 \text{ RPM.}$$

In conclusion, the bike rider must pedal the front sprocket at the rate of 96 RPM. (The rate at which one pedals a bicycle is known as the *cadence*; 96 RPM is a fairly high, but not uncommon, cadence.) □

This example indicates the basic strategy used in all belt/wheel problems.

Important Facts 16.4.2 (Belt and Wheel Strategy)**.** *Three basic facts are used in all such problems:*

- *The relation* $v = r\omega$ *allows us to go from linear speed* v *to angular speed* ω*, and vice versa. It also allows us to find the radius of a wheel, given its linear and angular speeds.*
- *If two wheels are fastened rigidly to a common axle, then they have* ***the same angular speed****. (Caution: two wheels fastened to a common axle typically do not have the same linear speed, since they usually have different radii.)*
- *If two wheels are connected by a belt (or chain), the linear speed of the belt coincides with the linear speed of each wheel, and so the* ***linear speeds of the wheels will be equal****.*

16.5 Exercises

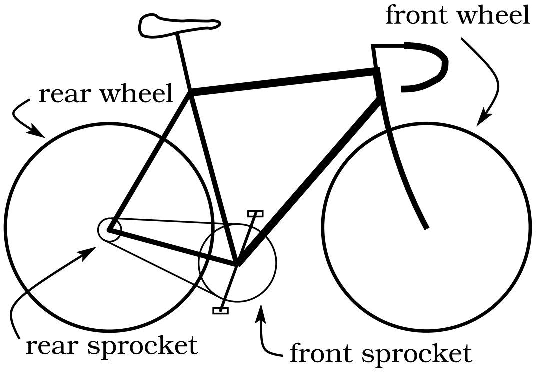
Problem 16.1. The restaurant in the Space Needle in Seattle rotates at the rate of one revolution per hour.

(a) Through how many radians does it turn in 100 minutes?

(b) How long does it take the restaurant to rotate through 4 radians?

(c) How far does a person sitting by the window move in 100 minutes if the radius of the restaurant is 21 meters?

Problem 16.2. You are riding a bicycle along a level road. Assume each wheel is 26 inches in diameter, the rear sprocket has a radius of 3 inches and the front sprocket has a radius of 7 inches. How fast do you need to pedal (in revolutions per minute) to achieve a speed of 35 mph?



Problem 16.3. Answer the following angular speed questions.

(a) A wheel of radius 22 ft. is rotating 11 RPM counterclockwise. Considering a point on the rim of the rotating wheel, what is the angular speed ω in rad/sec and the linear speed v in ft/sec?

(b) A wheel of radius 8 in. is rotating 15°/sec. What is the linear speed v, the angular speed in RPM and the angular speed in rad/sec?

(c) You are standing on the equator of the earth (radius 3960 miles). What is your linear and angular speed?

(d) An auto tire has radius 12 inches. If you are driving 65 mph, what is the angular speed in rad/sec and the angular speed in RPM?

Problem 16.4. Lee is running around the perimeter of a circular track at a rate of 10 ft/sec. The track has a radius of 100 yards. After 10 seconds, Lee turns and runs along a radial line to the center of the circle. Once he reaches the center, he turns and runs along a radial line to his starting point on the perimeter. Assume Lee does not slow down when he makes these two turns.

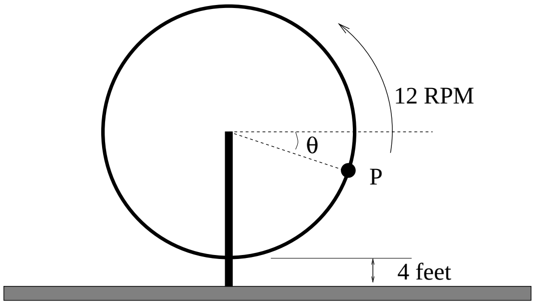
(a) Sketch a picture of the situation.

(b) How far has Lee traveled once he returns to his starting position?

(c) How much time will elapse during Lee's circuit?

(d) Find the area of the pie shaped sector enclosed by Lee's path.

Problem 16.5. John has been hired to design an exciting carnival ride. Tiff, the carnival owner, has decided to create the worlds greatest ferris wheel. Tiff isn't into math; she simply has a vision and has told John these constraints on her dream: (i) the wheel should rotate counterclockwise with an angular speed of 12 RPM; (ii) the linear speed of a rider should be 200 mph; (iii) the lowest point on the ride should be 4 feet above the level ground.

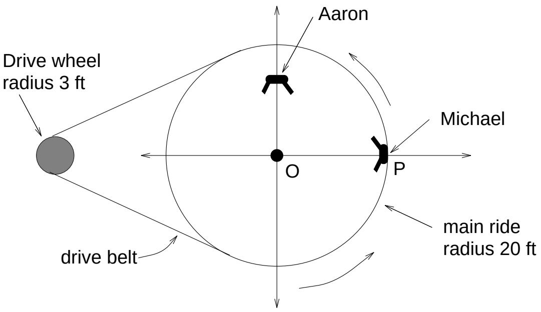


(a) Find the radius of the ferris wheel.

(b) Once the wheel is built, John suggests that Tiff should take the first ride. The wheel starts turning when Tiff is at the location P, which makes an angle θ with the horizontal, as pictured. It takes her 1.3 seconds to reach the top of the ride. Find the angle θ.

(c) Poor engineering causes Tiff's seat to fly off in 6 seconds. Describe where Tiff is located (an angle description) the instant she becomes a human missile.

Problem 16.6. Michael and Aaron are on the "UL-Tossum" ride at *Funworld*. This is a merry-go-round of radius 20 feet which spins counterclockwise 60 RPM. The ride is driven by a belt connecting the outer edge of the ride to a drive wheel of radius 3 feet:

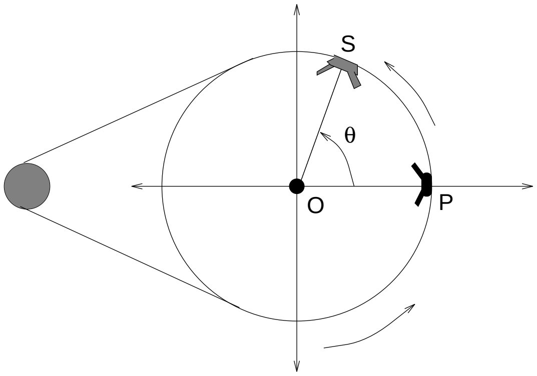


(a) Assume Michael is seated on the edge of the ride, as pictured. What is Michael's linear speed in mph and ft/sec?

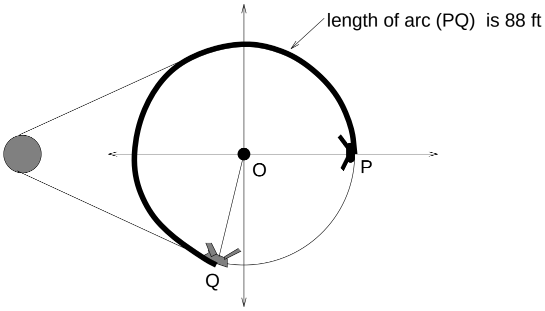
(b) What is the angular speed of the drive wheel in RPM?

(c) Suppose Aaron is seated 16 feet from the center of the ride. What is the angular speed of Aaron in RPM? What is the linear speed of Aaron in ft/sec?

(d) After 0.23 seconds Michael will be located at S as pictured. What is the angle $\angle POS$ in degrees? What is the angle $\angle POS$ in radians? How many feet has Michael traveled?

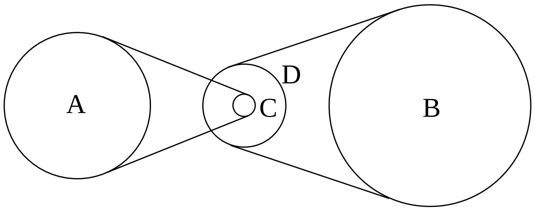


(e) Assume Michael has traveled 88 feet from the position P to a new position Q. How many seconds will this take? What will be the angle swept out by Michael?



Problem 16.7. You are riding a bicycle along a level road. Assume each wheel is 28 inches in diameter, the rear sprocket has radius 3 inches and the front sprocket has radius r inches. Suppose you are pedaling the front sprocket at the rate of $1.5\,\frac{\text{rev}}{\text{sec}}$ and your forward speed is 11 mph on the bike. What is the radius of the front sprocket?

Problem 16.8. You are designing a system of wheels and belts as pictured below. You want wheel A to rotate 20 RPM while wheel B rotates 42 RPM. Wheel A has a radius of 6 inches, wheel B has a radius of 7 inches and wheel C has a radius of 1 inch. Assume wheels C and D are rigidly fastened to the same axle. What is the radius r of wheel D?



Chapter 17

The Circular Functions

Suppose Cosmo begins at location R *and walks in a counterclockwise direction, always maintaining a tight 20 ft long tether. As Cosmo moves around the circle, how can we describe his location at any given instant?*

In one sense, we have already answered this question: The measure of $\angle RPS_1$ exactly pins down a location on the circle of radius 20 feet. But, we really might prefer a description of the horizontal and vertical coordinates of Cosmo; this would tie in better with the coordinate system we typically use. Solving this problem will require NEW functions, called the *circular functions*.

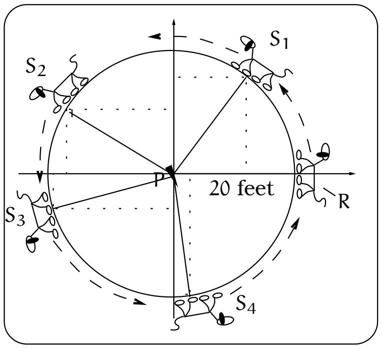


Figure 17.1: Cosmo moves counterclockwise maintaining a tight tether. Where's Cosmo?

17.1 Sides and Angles of a Right Triangle

Example 17.1.1. *You are preparing to make your final shot at the British Pocket Billiard World Championships. The position of your ball is as in Figure 17.2, and you must play the ball off the left cushion into the lower-right corner pocket, as indicated by the dotted path. For the big money, where should you aim to hit the cushion?*

Solution. This problem depends on two basic facts. First, the angles of entry and exit between the path the cushion will be equal. Secondly, the two obvious right triangles in this picture are similar triangles. Let x represent the distance from the bottom left corner to the impact point of the ball's path:

Properties of similar triangles tell us that the ratios of common sides are equal: $\frac{4}{5-x} = \frac{12}{x}$. If we solve this equation for x, we obtain $x = \frac{15}{4} = 3.75$ feet. $\square$

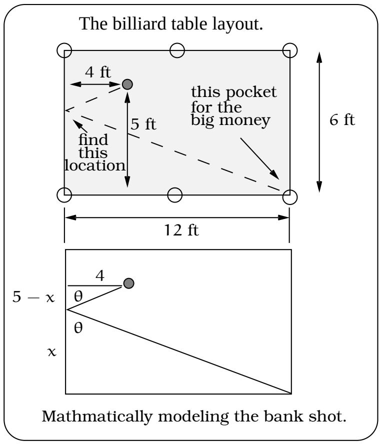

The billiard table layout.
4 ft
this pocket for the big money
5 ft
6 ft
find this location
12 ft
4
$5-x$
θ
θ
x
Mathmatically modeling the bank shot.

Figure 17.2: A pocket billiard banking problem.

This discussion is enough to win the tourney. But, of course, there are still other questions we can ask about this simple example: What is the angle θ? That is going to require substantially more work; indeed the bulk of this Chapter! It turns out, there is a lot of mathematical mileage in the idea of studying ratios of sides of right triangles. The first step, which will get the ball rolling, is to introduce new functions whose very definition involves relating sides and angles of right triangles.

17.2 The Trigonometric Ratios

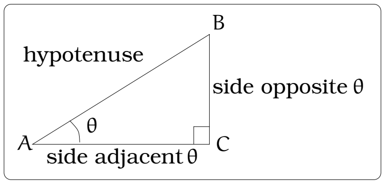


Figure 17.3: Labeling the sides of a right triangle.

From elementary geometry, the sum of the angles of any triangle will equal 180°. Given a right triangle $\triangle ABC$, since one of the angles is 90°, the remaining two angles must be *acute angles*; i.e., angles of measure between 0° and 90°. If we specify one of the acute angles in a right triangle $\triangle ABC$, say angle θ, we can label the three sides using this terminology. We then consider the following three ratios of side lengths, referred to as *trigonometric ratios*:

$$\sin(\theta) \overset{\text{def}}{=} \frac{\text{length of side opposite } \theta}{\text{length of hypotenuse}} \tag{17.1}$$

$$\cos(\theta) \overset{\text{def}}{=} \frac{\text{length of side adjacent } \theta}{\text{length of hypotenuse}} \tag{17.2}$$

$$\tan(\theta) \overset{\text{def}}{=} \frac{\text{length of side opposite } \theta}{\text{length of side adjacent to } \theta}. \tag{17.3}$$

For example, we have three right triangles in Figure 17.4; you can verify that the Pythagorean Theorem holds in each of the cases. In the left-hand triangle, $\sin(\theta) = \frac{5}{13}$, $\cos(\theta) = \frac{12}{13}$, $\tan(\theta) = \frac{5}{12}$. In the middle triangle, $\sin(\theta) = \frac{1}{\sqrt{2}}$, $\cos(\theta) = \frac{1}{\sqrt{2}}$, $\tan(\theta) = 1$. In the right-hand triangle, $\sin(\theta) = \frac{1}{2}$, $\cos(\theta) = \frac{\sqrt{3}}{2}$, $\tan(\theta) = \frac{1}{\sqrt{3}}$. The symbols "sin", "cos", and "tan" are abbreviations for the words *sine*, *cosine* and *tangent*, respectively. As we have defined them, the trigonometric ratios depend on the dimensions of the triangle. However, the same ratios are obtained for *any* right triangle with acute angle θ. This follows from the properties of similar triangles. Consider Figure 17.5. Notice $\triangle ABC$ and $\triangle ADE$ are similar. If we use $\triangle ABC$ to compute $\cos(\theta)$, then we find $\cos(\theta) = \frac{|\overline{AC}|}{|\overline{AB}|}$. On the other hand, if we use $\triangle ADE$, we obtain $\cos(\theta) = \frac{|\overline{AE}|}{|\overline{AD}|}$. Since the ratios of common sides of similar triangles must agree, we have $\cos(\theta) = \frac{|\overline{AC}|}{|\overline{AB}|} = \frac{|\overline{AE}|}{|\overline{AD}|}$,

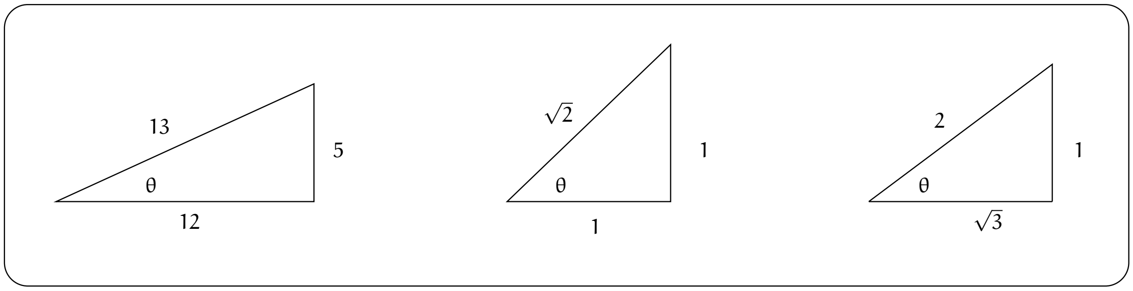


Figure 17.4: Computing trigonometric ratios for selected right triangles.

which is what we wanted to be true. The same argument can be used to show that sin(θ) and tan(θ) can be computed using *any* right triangle with acute angle θ.

Except for some "rigged" right triangles, it is not easy to calculate the trigonometric ratios. Before the 1970's, approximate values of sin(θ), cos(θ), tan(θ) were listed in long tables or calculated using a slide rule. Today, a scientific calculator saves the day on these computations. Most scientific calculators will give an approximation for the values of the trigonometric ratios. However, it is good to keep in mind we can compute the EXACT values of the trigonometric ratios when $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ radians or, equivalently, when $\theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$.

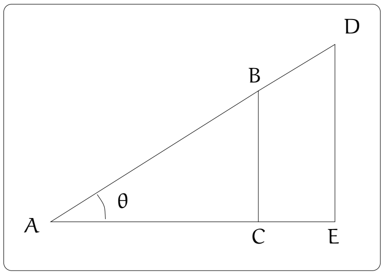


Figure 17.5: Applying trigonometric ratios to any right triangle.

Angle θ		**Trigonometric Ratio**		
Deg	**Rad**	**sin**(θ)	**cos**(θ)	**tan**(θ)
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	Undefined

Table 17.1: Exact Trigonometric Ratios

Some people make a big deal of “approximate” vs. “exact” answers; we won’t worry about it here, unless we are specifically asked for an exact answer. However, here is something we will make a big deal about:

iii
CAUTION
!!!

When computing values of $\cos(\theta)$, $\sin(\theta)$, and $\tan(\theta)$ on your calculator, make sure you are using the correct “angle mode” when entering θ; i.e. “degrees” or “radians”.

For example, if $\theta = 1^\circ$, then $\cos(1^\circ) = 0.9998$, $\sin(1^\circ) = 0.0175$, and $\tan(1^\circ) = 0.0175$. In contrast, if $\theta = 1$ radians, then $\cos(1) = 0.5403$, $\sin(1) = 0.8415$, and $\tan(1) = 1.5574$.

17.3 Applications

When confronted with a situation involving a right triangle where the measure of one acute angle θ and one side are known, we can solve for the remaining sides using the appropriate trigonometric ratios. Here is the key picture to keep in mind:

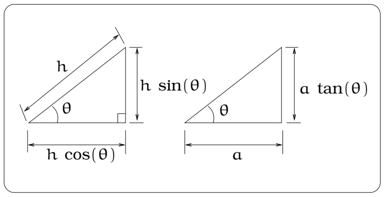


Figure 17.6: What do these ratios mean?

Important Facts 17.3.1 (Trigonometric ratios).
Given a right triangle, the trigonometric ratios relate the lengths of the sides as shown in Figure 17.6.

Example 17.3.2. *To measure the distance across a river for a new bridge, surveyors placed poles at locations* A, B *and* C. *The length* $|AB| = 100$ *feet and the measure of the angle* $\angle ABC$ *is* $31^\circ 18'$. *Find the distance to span the river. If the measurement of the angle* $\angle ABC$ *is only accurate within* $\pm 2'$, *find the possible error in* $|\overline{AC}|$.

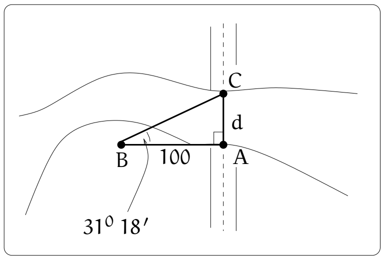


Figure 17.7: The distance spanning a river.

Solution. The trigonometric ratio relating these two sides would be the tangent and we can convert θ into decimal form, arriving at:

$$\tan(31^\circ 18') = \tan(31.3^\circ) = \frac{|\overline{AC}|}{|\overline{BA}|} = \frac{d}{100}$$

$$\text{therefore} \quad d = 60.8\,\text{feet}.$$

This tells us that the bridge needs to span a gap of 60.8 feet. If the measurement of the angle was in error by $+2'$, then $\tan(31^\circ 20') = \tan(31.3333^\circ) = 0.6088$ and the span is 60.88 ft. On the other hand, if the measurement of the angle was in error by $-2'$, then $\tan(31^\circ 16') = \tan(31.2667^\circ) = 0.6072$ and the span is 60.72 ft. □

Example 17.3.3. *A plane is flying 2000 feet above sea level toward a mountain. The pilot observes the top of the mountain to be* 18° *above the horizontal, then immediately flies the plane at an angle of* 20° *above horizontal. The airspeed of the plane is 100 mph. After 5 minutes, the plane is directly above the top of the mountain. How high is the plane above the top of the mountain (when it passes over)? What is the height of the mountain?*

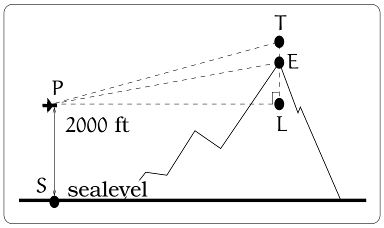


Figure 17.8: Flying toward a mountain.

Solution. We can compute the hypotenuse of $\triangle \mathrm{LPT}$ by using the speed and time information about the plane:

$$|\overline{\mathrm{PT}}| = (100\,\text{mph})(5\,\text{minutes})(1\,\text{hour}/60\,\text{minutes}) = \frac{25}{3}\,\text{miles}.$$

The definitions of the trigonometric ratios show:

$$|\overline{\mathrm{TL}}| = \frac{25}{3}\sin(20^\circ) = 2.850\,\text{miles, and}$$
$$|\overline{\mathrm{PL}}| = \frac{25}{3}\cos(20^\circ) = 7.831\,\text{miles.}$$

With this data, we can now find $|\overline{\mathrm{EL}}|$:

$$|\overline{\mathrm{EL}}| = |\overline{\mathrm{PL}}|\tan(18^\circ) = 2.544\,\text{miles.}$$

The height of the plane above the peak is $|\overline{\mathrm{TE}}| = |\overline{\mathrm{TL}}| - |\overline{\mathrm{EL}}| = 2.850 - 2.544 = 0.306\,\text{miles} = 1{,}616\,\text{feet}$. The elevation of the peak above sea level is given by: Peak elevation $=$ plane altitude $+ |\overline{\mathrm{EL}}| = |\overline{\mathrm{SP}}| + |\overline{\mathrm{EL}}| = 2{,}000 + (2.544)(5{,}280) = 15{,}432\,\text{feet}$. ☐

Example 17.3.4. *A Forest Service helicopter needs to determine the width of a deep canyon. While hovering, they measure the angle* $\gamma = 48^\circ$ *at position* B *(see picture), then descend 400 feet to position* A *and make two measurements of* $\alpha = 13^\circ$ *(the measure of* $\angle \mathrm{EAD}$*),* $\beta = 53^\circ$ *(the measure of* $\angle \mathrm{CAD}$*). Determine the width of the canyon to the nearest foot.*

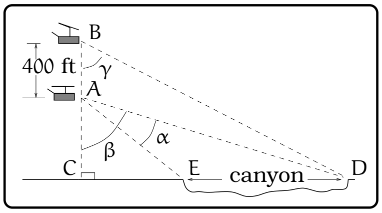


Figure 17.9: Finding the width of a canyon.

Solution. We will need to exploit three right triangles in the picture: $\triangle \mathrm{BCD}$, $\triangle \mathrm{ACD}$, and $\triangle \mathrm{ACE}$. Our goal is to compute $|\overline{\mathrm{ED}}| = |\overline{\mathrm{CD}}| - |\overline{\mathrm{CE}}|$, which suggests more than one right triangle will come into play.

The first step is to use $\triangle BCD$ and $\triangle ACD$ to obtain a system of two equations and two unknowns involving some of the side lengths; we will then solve the system. From the definitions of the trigonometric ratios,

$$\begin{aligned} |\overline{CD}| &= (400 + |\overline{AC}|)\tan(48^\circ) \\ |\overline{CD}| &= |\overline{AC}|\tan(53^\circ). \end{aligned}$$

Plugging the second equation into the first and rearranging we get

$$|\overline{AC}| = \frac{400\tan(48^\circ)}{\tan(53^\circ) - \tan(48^\circ)} = 2{,}053\,\text{feet}.$$

Plugging this back into the second equation of the system gives

$$|\overline{CD}| = (2053)\tan(53^\circ) = 2724\,\text{feet}.$$

The next step is to relate $\triangle ACD$ and $\triangle ACE$, which can now be done in an effective way using the calculations above. Notice that the measure of $\angle CAE$ is $\beta - \alpha = 40^\circ$. We have

$$|\overline{CE}| = |\overline{AC}|\tan(40^\circ) = (2053)\tan(40^\circ) = 1{,}723\,\text{feet}.$$

As noted above, $|\overline{ED}| = |\overline{CD}| - |\overline{CE}| = 2{,}724 - 1{,}723 = 1{,}001$ feet is the width of the canyon. $\square$

17.4 Circular Functions

If *Cosmo* is located somewhere in the first quadrant of Figure 17.1, represented by the location S, we can use the trigonometric ratios to describe his coordinates. Impose the indicated xy-coordinate system with origin at P and extract the pictured right triangle with vertices at P and S. The radius is 20 ft. and applying Fact 17.3.1 gives

$$S = (x, y) = (20\cos(\theta), 20\sin(\theta)).$$

Unfortunately, we run into a snag if we allow *Cosmo* to wander into the second, third or fourth quadrant, since then the angle θ is no longer acute.

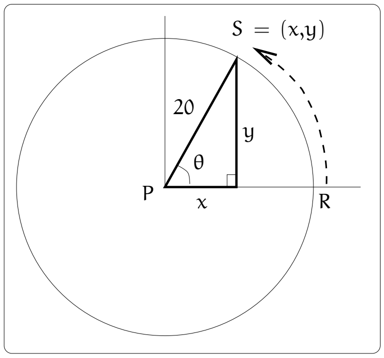


Figure 17.10: Cosmo on a circular path.

17.4.1 Are the trigonometric ratios functions?

Recall that $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ are defined for acute angles θ inside a right triangle. We would like to say that these three equations actually define functions where the variable is an angle θ. Having said this, it is natural to ask if these three equations can be extended to be defined for ANY angle θ. For example, we need to explain how $\sin\left(\frac{2\pi}{3}\right)$ is defined.

To start, we begin with the unit circle pictured in the xy-coordinate system. Let $\theta = \angle ROP$ be the angle in standard central position shown in Figure 17.11. If θ is positive (resp. negative), we adopt the convention that θ is swept out by counterclockwise (resp. clockwise) rotation of the initial side $\overline{OR}$. The objective is to find the coordinates of the point P in this figure. Notice that each coordinate of P (the x-coordinate and the y-coordinate) will depend on the given angle θ. For this reason, we need to introduce two new functions involving the variable θ.

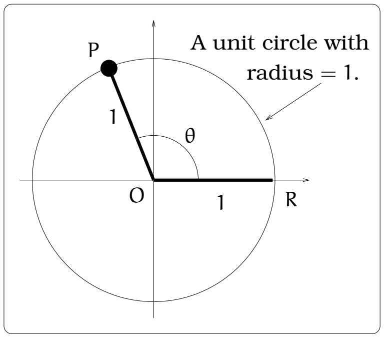


Figure 17.11: Coordinates of points on the unit circle.

Definition 17.4.1. *Let θ be an angle in standard central position inside the unit circle, as in Figure 17.11. This angle determines a point P on the unit circle. Define two new functions,* $\cos(\theta)$ *and* $\sin(\theta)$*, on the domain of all θ values as follows:*

$$\cos(\theta) \overset{def}{=} \textit{horizontal } x\textit{-coordinate of } P \textit{ on unit circle}$$
$$\sin(\theta) \overset{def}{=} \textit{vertical } y\textit{-coordinate of } P \textit{ on unit circle.}$$

We refer to $\sin(\theta)$ and $\cos(\theta)$ as the *basic circular functions*. Keep in mind that these functions have variables which are angles (either in degree or radian measure). These functions will be on your calculator. Again, BE CAREFUL to check the angle mode setting on your calculator ("degrees" or "radians") before doing a calculation.

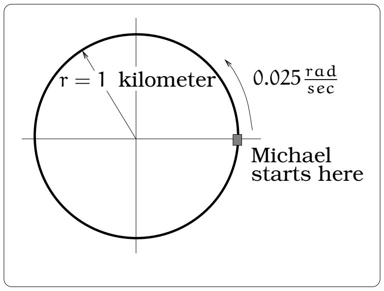


Figure 17.12: A circular driving track.

Example 17.4.2. *Michael is test driving a vehicle counterclockwise around a desert test track which is circular of radius 1 kilometer. He starts at the location pictured, traveling $0.025 \frac{rad}{sec}$. Impose coordinates as pictured. Where is Michael located (in xy-coordinates) after 18 seconds?*

Solution. Let $M(t)$ be the point on the circle of motion representing Michael's location after t seconds and $\theta(t)$ the angle swept out the by Michael after t seconds. Since we are given the angular speed, we get

$$\theta(t) = 0.025t \text{ radians.}$$

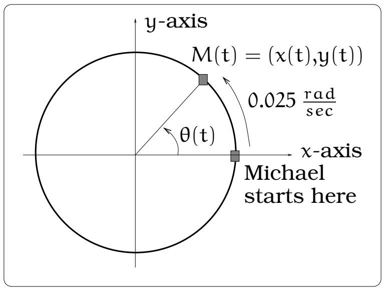


Figure 17.13: Modeling Michael's location.

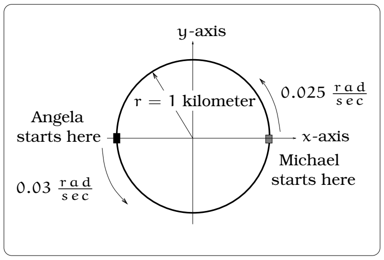
Since the angle $\theta(t)$ is in central standard position, we get

$$M(t) = (\cos(\theta(t)), \sin(\theta(t))) = (\cos(0.025t), \sin(0.025t)).$$

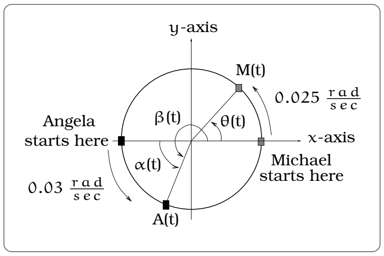
So, after 18 seconds Michael's location will be $M(18) = (0.9004, 0.4350)$. $\square$

¡¡¡ CAUTION !!!

Interpreting the coordinates of the point $P = (\mathbf{cos}(\theta), \mathbf{sin}(\theta))$ in Figure 17.11 only works if the angle θ is viewed in central standard position. You must do some additional work if the angle is placed in a different position; see the next Example.



(a) Angela and Michael on the same test track.



(b) Modeling the motion of Angela and Michael.

Figure 17.14: Visualizing motion on a circular track.

Example 17.4.3. *Both Angela and Michael are test driving vehicles counterclockwise around a desert test track which is circular of radius 1 kilometer. They start at the locations shown in Figure 17.14(a). Michael is traveling 0.025 rad/sec and Angela is traveling 0.03 rad/sec. Impose coordinates as pictured. Where are the drivers located (in xy-coordinates) after 18 seconds?*

Solution. Let $M(t)$ be the point on the circle of motion representing Michael's location after t seconds. Likewise, let $A(t)$ be the point on the circle of motion representing Angela's location after t seconds. Let $\theta(t)$ be the angle swept out the by Michael and $\alpha(t)$ the angle swept out by Angela after t seconds.

Since we are given the angular speeds, we get

$$\begin{aligned}\theta(t) &= 0.025t \text{ radians, and}\\ \alpha(t) &= 0.03t \text{ radians.}\end{aligned}$$

From the previous Example 17.4.2,

$$\begin{aligned}M(t) &= (\cos(0.025t), \sin(0.025t)), \quad \text{and}\\ M(18) &= (0.9004, 0.4350).\end{aligned}$$

Angela's angle $\alpha(t)$ is NOT in central standard position, so we must observe that $\alpha(t) + \pi = \beta(t)$, where $\beta(t)$ is in central standard position: See Figure 17.14(b). We conclude that

$$\begin{aligned}A(t) &= (\cos(\beta(t)), \sin(\beta(t)))\\ &= (\cos(\pi + 0.03t), \sin(\pi + 0.03t)).\end{aligned}$$

So, after 18 seconds Angela's location will be $A(18) = (-0.8577, -0.5141)$. $\square$

17.4.2 Relating circular functions and right triangles

If the point P on the unit circle is located in the first quadrant, then we can compute $\cos(\theta)$ and $\sin(\theta)$ using trigonometric ratios. In general, it's useful to relate right triangles, the unit circle and the circular functions. To describe this connection, given θ we place it in central standard position in the unit circle, where $\angle ROP = \theta$. Draw a line through P perpendicular to the x-axis, obtaining an inscribed right triangle. Such a right triangle has hypotenuse of length 1, vertical side of length labeled b and horizontal side of length labeled a. There are four cases: See Figure 17.16.

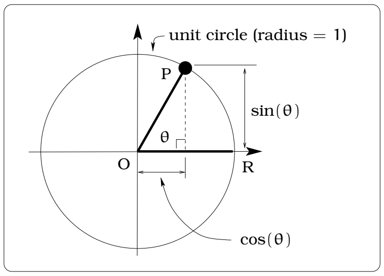


Figure 17.15: The point P in the first quadrant.

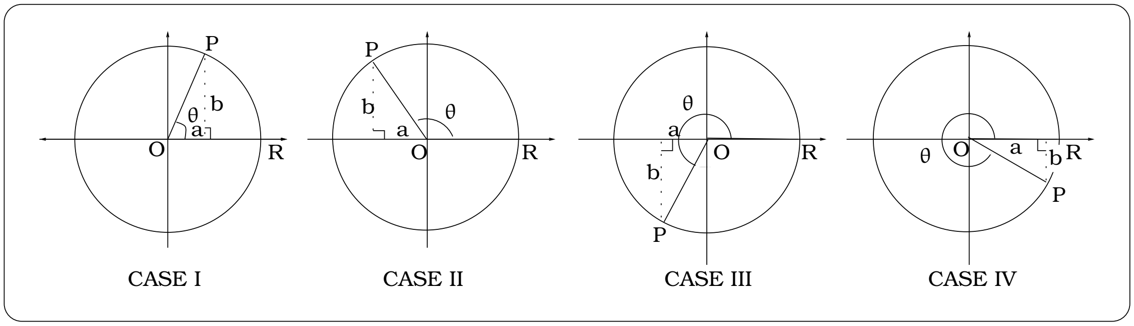


Figure 17.16: Possible positions of θ on the unit circle.

Case I has already been discussed, arriving at $\cos(\theta) = a$ and $\sin(\theta) = b$. In Case II , we can interpret $\cos(\theta) = -a, \sin(\theta) = b$. We can reason similarly in the other Cases III and IV, using Figure 17.16, and we arrive at this conclusion:

Important Facts 17.4.4 (Circular functions and triangles). *View θ as in Figure 17.16 and form the pictured inscribed right triangles. Then we can interpret $\cos(\theta)$ and $\sin(\theta)$ in terms of these right triangles as follows:*

$$\begin{aligned}
\textit{Case I}: &\quad \cos(\theta) = a, &\quad \sin(\theta) = b\\
\textit{Case II}: &\quad \cos(\theta) = -a, &\quad \sin(\theta) = b\\
\textit{Case III}: &\quad \cos(\theta) = -a, &\quad \sin(\theta) = -b\\
\textit{Case IV}: &\quad \cos(\theta) = a, &\quad \sin(\theta) = -b
\end{aligned}$$

17.5 What About Other Circles?

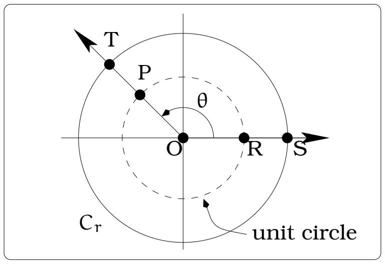


Figure 17.17: Points on other circles.

What happens if we begin with a circle C_r with radius r (possibly different than 1) and want to compute the coordinates of points on this circle?

The circular functions can be used to answer this more general question. Picture our circle C_r centered at the origin in the same picture with unit circle C_1 and the angle θ in standard central position for each circle. As pictured, we can view $\theta = \angle ROP = \angle SOT$. If $P = (x,y)$ is our point on the unit circle corresponding to the angle θ, then the calculation below shows how to compute coordinates on general circles:

$$\begin{aligned} P &= (x,y) \\ &= (\cos(\theta), \sin(\theta)) \in C_1 \quad \Leftrightarrow \quad x^2 + y^2 = 1 \\ &\qquad\qquad \Leftrightarrow \quad r^2x^2 + r^2y^2 = r^2 \\ &\qquad\qquad \Leftrightarrow \quad (rx)^2 + (ry)^2 = r^2 \\ &\qquad\qquad \Leftrightarrow \quad T = (rx, ry) \\ &\qquad\qquad\qquad\quad\; = (r\cos(\theta), r\sin(\theta)) \in C_r. \end{aligned}$$

Important Fact 17.5.1. *Let C_r be a circle of radius r centered at the origin and $\theta = \angle SOT$ an angle in standard central position for this circle, as in Figure 17.17. Then the coordinates of $T = (r\cos(\theta), r\sin(\theta))$.*

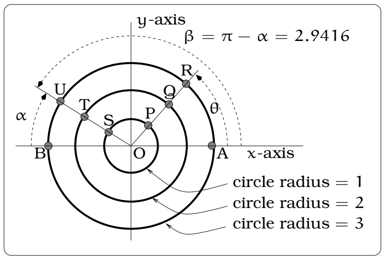


Figure 17.18: Coordinates of points on circles.

Examples 17.5.2. *Consider the picture below, with $\theta = 0.8$ radians and $\alpha = 0.2$ radians. What are the coordinates of the labeled points?*

Solution. The angle θ is in standard central position; α is a central angle, but it is not in standard position. Notice, $\beta = \pi - \alpha = 2.9416$ is an angle in standard central position which locates the same points U, T, S as the angle α. Applying Definition 17.4.1 on page 227:

$$\begin{aligned} P &= (\cos(0.8), \sin(0.8)) &&= (0.6967, 0.7174) \\ Q &= (2\cos(0.8), 2\sin(0.8)) &&= (1.3934, 1.4347) \\ R &= (3\cos(0.8), 3\sin(0.8)) &&= (2.0901, 2.1521) \\ S &= (\cos(2.9416), \sin(2.9416)) &&= (-0.9801, 0.1987) \\ T &= (2\cos(2.9416), 2\sin(2.9416)) &&= (-1.9602, 0.3973) \\ U &= (3\cos(2.9416), 3\sin(2.9416)) &&= (-2.9403, 0.5961). \end{aligned}$$

□

Example 17.5.3. *Suppose Cosmo begins at the position* R *in the figure, walking around the circle of radius 20 feet with an angular speed of* $\frac{4}{5}$ *RPM counterclockwise. After 3 minutes have elapsed, describe Cosmo's precise location.*

Solution. Cosmo has traveled $3\frac{4}{5} = \frac{12}{5}$ revolutions. If θ is the angle traveled after 3 minutes, $\theta = \left(\frac{12}{5}\text{rev}\right)\left(2\pi\,\frac{\text{radians}}{\text{rev}}\right) = \frac{24\pi}{5}$ radians $= 15.08$ radians. By (15.5.1), we have $x = 20\cos\left(\frac{24\pi}{5}\text{rad}\right) = -16.18$ feet and $y = 20\sin\left(\frac{24\pi}{5}\text{rad}\right) = 11.76$ feet. Conclude that Cosmo is located at the point $S = (-16.18, 11.76)$. Using (15.1), $\theta = 864° = 2(360°) + 144°$; this means that Cosmo walks counterclockwise around the circle two complete revolutions, plus 144°. □

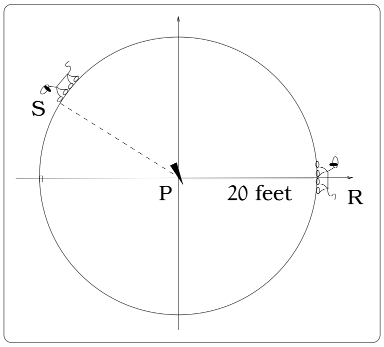
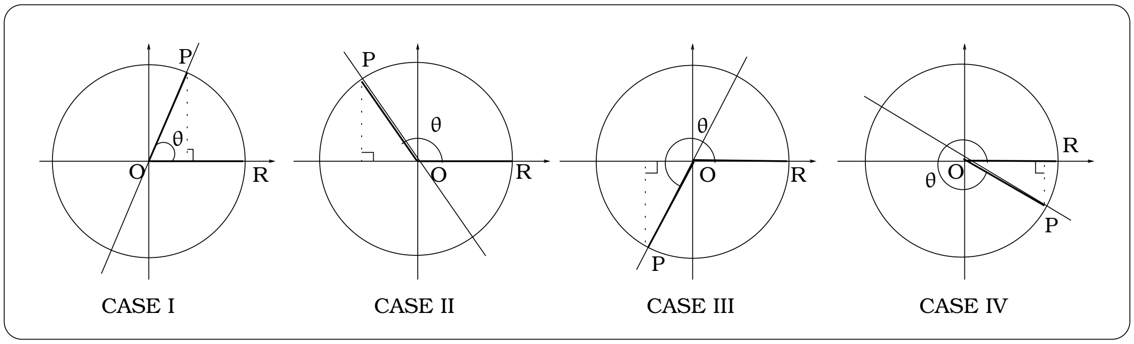


Figure 17.19: Where is Cosmo after 3 minutes?

17.6 Other Basic Circular Function

Given any angle θ, our constructions offer a concrete link between the cosine and sine functions and right triangles inscribed inside the unit circle: See Figure 17.20.



CASE I CASE II CASE III CASE IV

Figure 17.20: Computing the slope of a line using the function $\tan(\theta)$.

The slope of the hypotenuse of these inscribed triangles is just the slope of the line through $\overline{OP}$. Since $P = (\cos(\theta), \sin(\theta))$ and $O = (0, 0)$:

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{\sin(\theta)}{\cos(\theta)};$$

this would be valid as long as $\cos(\theta) \neq 0$. This calculation motivates a new circular function called the *tangent of* θ by the rule

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \quad \text{provided } \cos(\theta) \neq 0.$$

The only time $\cos(\theta) = 0$ is when the corresponding point P on the unit circle has x-coordinate 0. But, this only happens at the positions $(0, 1)$ and $(0, -1)$ on the unit circle, corresponding to angles of the form $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \cdots$. These are the cases when the inscribed right triangle would "degenerate" to having zero width and the line segment $\overline{OP}$ becomes vertical. In summary, we then have this general idea to keep in mind:

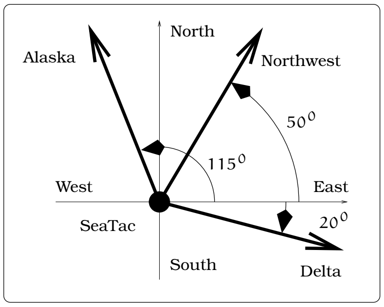
Important Fact 17.6.1. ***The slope of a line*** $= \tan(\theta)$, ***where*** θ ***is the angle the line makes with the*** x***-axis (or any other horizontal line)***

Three other commonly used circular functions come up from time to time. The *cotangent* function $y = \cot(\theta)$, the *secant* function $y = \sec(\theta)$ and the *cosecant* function $y = \csc(\theta)$ are defined by the formulas:

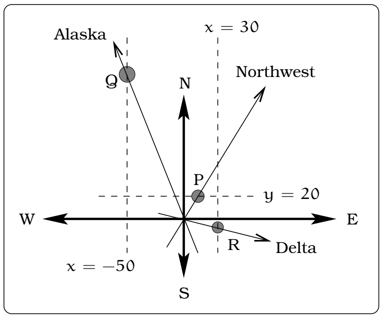
$$\sec(\theta) \overset{\text{def}}{=} \frac{1}{\cos(\theta)}, \qquad \csc(\theta) \overset{\text{def}}{=} \frac{1}{\sin(\theta)}, \qquad \cot(\theta) \overset{\text{def}}{=} \frac{1}{\tan(\theta)}.$$

Just as with the tangent function, one needs to worry about the values of θ for which these functions are undefined (due to division by zero). We will not need these functions in this text.

Example 17.6.2. *Three airplanes depart SeaTac Airport. A NorthWest flight is heading in a direction* 50° *counterclockwise from East, an Alaska flight is heading* 115° *counterclockwise from East and a Delta flight is heading* 20° *clockwise from East. Find the location of the Northwest flight when it is 20 miles North of SeaTac. Find the location of the Alaska flight when it is 50 miles West of SeaTac. Find the location of the Delta flight when it is 30 miles East of SeaTac.*



(a) The flight paths of three airplanes.



(b) Modeling the paths of each flight.

Figure 17.21: Visualizing and modeling departing airplanes.

Solution. We impose a coordinate system in Figure 17.21(a), where "East" (resp. "North") points along the positive x-axis (resp. positive y-axis). To solve the problem, we will find the equation of the three lines representing the flight paths, then determine where they intersect the appropriate horizontal or vertical line. The Northwest and Alaska directions of flight are angles in standard central position; the Delta flight direction will be -20°. We can imagine right triangles with their hypotenuses along the directions of flight, then using the tangent function, we have these three immediate conclusions:

$$\begin{aligned} \text{slope NW line} = \tan(50^\circ) &= 1.19, \\ \text{slope Alaska line} = \tan(115^\circ) &= -2.14, \text{ and} \\ \text{slope Delta line} = \tan(-20^\circ) &= -0.364. \end{aligned}$$

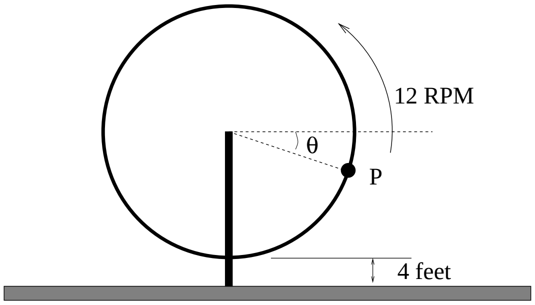
All three flight paths pass through the origin $(0,0)$ of our coordinate system, so the equations of the lines through the flight paths will be:

$$\begin{aligned} \text{NW flight}: &\quad y = 1.19x, \\ \text{Alaska flight}: &\quad y = -2.14x, \\ \text{Delta flight}: &\quad y = -0.364x. \end{aligned}$$

The Northwest flight is 20 miles North of SeaTac when $y = 20$; plugging into the equation of the line of flight gives $20 = 1.19x$, so $x = 16.81$ and the plane location will be $P = (16.81, 20)$. Similarly, the Alaska flight is 50 miles West of SeaTac when $x = -50$; plugging into the equation of the line of flight gives $y = (-2.14)(-50) = 107$ and the plane location will be $Q = (-50, 107)$. Finally, check that the Delta flight is at $R = (30, -10.92)$ when it is 30 miles East of SeaTac. $\square$

17.7 Exercises

Problem 17.1. John has been hired to design an exciting carnival ride. Tiff, the carnival owner, has decided to create the world's greatest ferris wheel. Tiff isn't into math; she simply has a vision and has told John these constraints on her dream: (i) the wheel should rotate counterclockwise with an angular speed of 12 RPM; (ii) the linear speed of a rider should be 200 mph; (iii) the lowest point on the ride should be 4 feet above the level ground. Recall, we worked on this in Exercise 16.5.



(a) Impose a coordinate system and find the coordinates $T(t) = (x(t), y(t))$ of Tiff at time t seconds after she starts the ride.

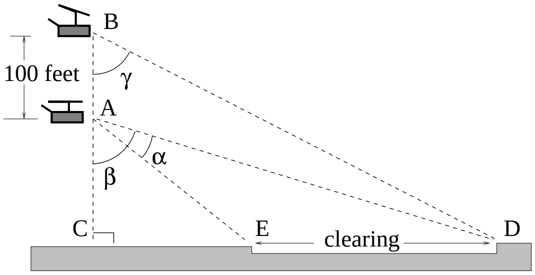
(b) Tiff becomes a human missile after 6 seconds on the ride. Find Tiff's coordinates the instant she becomes a human missile.

(c) Find the equation of the tangential line along which Tiff travels the instant she becomes a human missile. Sketch a picture indicating this line and her initial direction of motion along it when the seat detaches.

Problem 17.2. (a) Find the equation of a line passing through the point (-1,2) and making an angle of 13° with the x-axis. (Note: There are two answers; find them both.)

(b) Find the equation of a line making an angle of 8° with the y-axis and passing through the point $(1,1)$. (Note: There are two answers; find them both.)

Problem 17.3. The crew of a helicopter needs to land temporarily in a forest and spot a flat horizontal piece of ground (a clearing in the forest) as a potential landing site, but are uncertain whether it is wide enough. They make two measurements from A (see picture) finding $\alpha = 25^\circ$ and $\beta = 54^\circ$. They rise vertically 100 feet to B and measure $\gamma = 47^\circ$. Determine the width of the clearing to the nearest foot.



Problem 17.4. Marla is running clockwise around a circular track. She runs at a constant speed of 3 meters per second. She takes 46 seconds to complete one lap of the track. From her starting point, it takes her 12 seconds to reach the northermost point of the track.

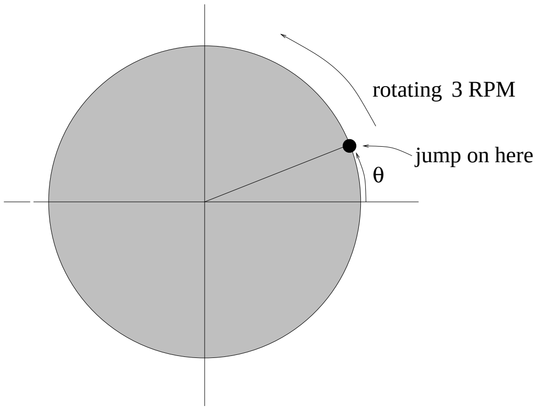
Impose a coordinate system with the center of the track at the origin, and the northernmost point on the positive y-axis.

(a) Give Marla's coordinates at her starting point.

(b) Give Marla's coordinates when she has been running for 10 seconds.

(c) Give Marla's coordinates when she has been running for 901.3 seconds.

Problem 17.5. A merry-go-round is rotating at the constant angular speed of 3 RPM counterclockwise. The platform of this ride is a circular disc of radius 24 feet. You jump onto the ride at the location pictured below.



(a) If $\theta = 34°$, then what are your xy-coordinates after 4 minutes?

(b) If $\theta = 20°$, then what are your xy-coordinates after 45 minutes?

(c) If $\theta = -14°$, then what are your xy-coordinates after 6 seconds? Draw an accurate picture of the situation.

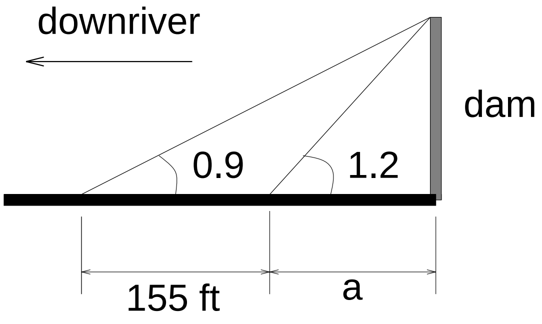
(d) If $\theta = -2.1$ rad, then what are your xy-coordinates after 2 hours and 7 seconds? Draw an accurate picture of the situation.

(e) If $\theta = 2.1$ rad, then what are your xy-coordinates after 5 seconds? Draw an accurate picture of the situation.

Problem 17.6. Shirley is on a ferris wheel which spins at the rate of 3.2 revolutions per minute. The wheel has a radius of 45 feet, and the center of the wheel is 59 feet above the ground. After the wheel starts moving, Shirley takes 16 seconds to reach the top of the wheel.

How high above the ground is she when the wheel has been moving for 9 minutes?

Problem 17.7. The top of the *Boulder Dam* has an angle of elevation of 1.2 radians from a point on the Colorado River. Measuring the angle of elevation to the top of the dam from a point 155 feet farther down river is 0.9 radians; assume the two angle measurements are taken at the same elevation above sea level. How high is the dam?

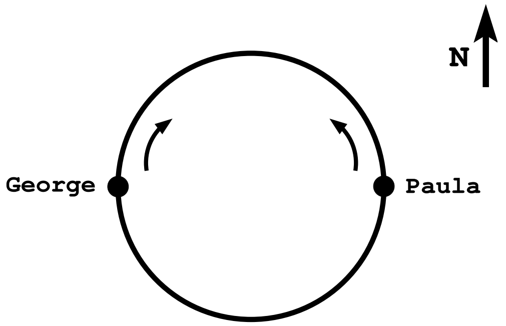


Problem 17.8. A radio station obtains a permit to increase the height of their radio tower on Queen Anne Hill by no more than 100 feet. You are the head of the Queen Anne Community Group and one of your members asks you to make sure that the radio station does not exceed the limits of the permit. After finding a relatively flat area nearby the tower (not necessarily the same altitude as the bottom of the tower), and standing some unknown distance away from the tower, you make three measurements all at the same height above sea level. You observe that the top of the old tower makes an angle of 39° above level. You move 110 feet away from the original measurement and observe that the old top of the tower now makes an angle of 34° above level. Finally, after the new construction is complete, you observe that the new top of the tower, from the same point as the second measurement was made, makes an angle of 40° above the horizontal. All three measurements are made at the same height above sea level and are in line with the tower. Find the height of the addition to the tower, to the nearest foot.

Problem 17.9. Charlie and Alexandra are running around a circular track with radius 60 meters. Charlie started at the westernmost point of the track, and, at the same time, Alexandra started at the northernmost point. They both run counterclockwise. Alexandra runs at 4 meters per second, and will take exactly 2 minutes to catch up to Charlie.

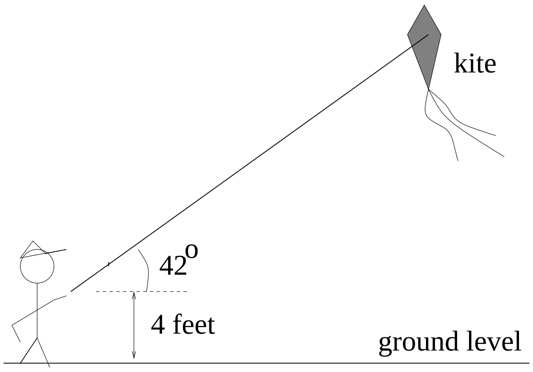
Impose a coordinate system, and give the x- and y-coordinates of Charlie after one minute of running.

Problem 17.10. George and Paula are running around a circular track. George starts at the westernmost point of the track, and Paula starts at the easternmost point. The illustration below shows their starting positions and running directions. They start running toward each other at constant speeds. George runs at 9 feet per second. Paula takes 50 seconds to run a lap of the track. George and Paula pass each other after 11 seconds.



After running for 3 minutes, how far east of his starting point is George?

Problem 17.11. A kite is attached to 300 feet of string, which makes a 42 degree angle with the level ground. The kite pilot is holding the string 4 feet above the ground.


kite
42°
4 feet
ground level

(a) How high above the ground is the kite?

(b) Suppose that power lines are located 250 feet in front of the kite flyer. Is any portion of the kite or string over the power lines?

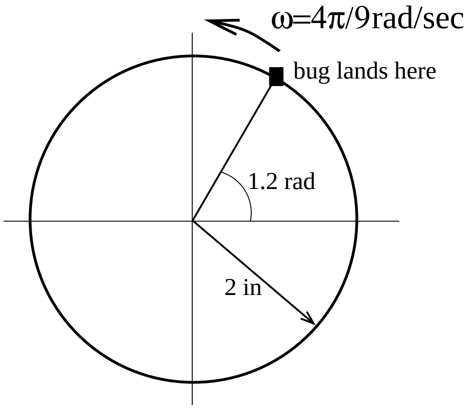
Problem 17.12. In the pictures below, a bug has landed on the rim of a jelly jar and is moving around the rim. The location where the bug initially lands is described and its angular speed is given. Impose a coordinate system with the origin at the center of the circle of motion. In each of the cases, answer these questions:

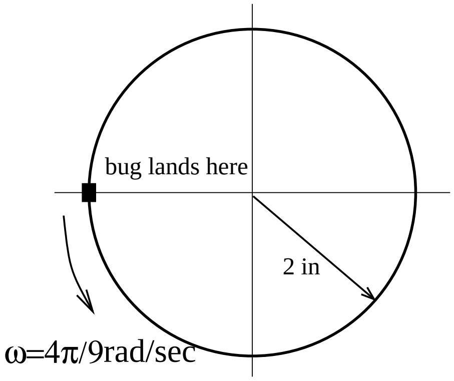
(a) Find an angle θ_0 in standard central position that gives the bugs initial location. (In some cases, this is the angle given in the picture; in other cases, you will need to do something.)

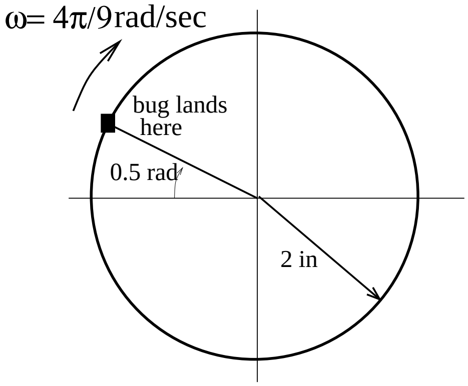
(b) The location angle of the bug at time t is given by the formula $\theta(t) = \theta_0 + \omega t$. Plug in the values for θ_0 and ω to explicitly obtain a formula for $\theta(t)$.

(c) Find the coordinates of the bug at time t.

(d) What are the coordinates of the bug after 1 second? After 0 seconds? After 3 seconds? After 22 seconds?


ω=4π/9rad/sec
bug lands here
1.2 rad
2 in


bug lands here
2 in
ω=4π/9rad/sec


ω= 4π/9rad/sec
bug lands here
0.5 rad
2 in

Chapter 18

Trigonometric Functions

Our definitions of the circular functions are based upon the unit circle. This makes it easy to visualize many of their properties.

18.1 Easy Properties of Circular Functions

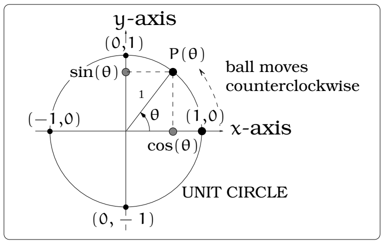
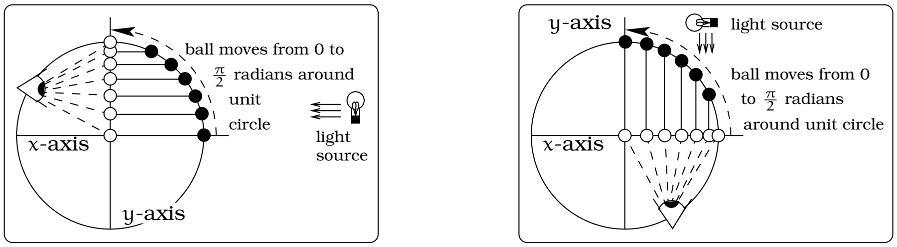

y-axis
(0, 1)
P(θ)
sin(θ)
ball moves
counterclockwise
1
θ
(−1, 0)
(1, 0)
x-axis
cos(θ)
UNIT CIRCLE
(0, −1)

Figure 18.1: Visualizing the range of sin(θ) and cos(θ).

How can we determine the range of function values for cos(θ) *and* sin(θ)*?* To begin with, recall the abstract definition for the range of a function $f(\theta)$:

$$\text{Range of } f = \{f(\theta) : \theta \text{ is in the domain}\}.$$

Using the unit circle constructions of the basic circular functions, it is easy to visualize the range of cos(θ) and sin(θ). Beginning at the position $(1, 0)$, imagine a ball moving counterclockwise around the unit circle. If we "freeze" the motion at any point in time, we will have swept out an angle θ and the corresponding position $P(\theta)$ on the circle will have coordinates $P(\theta) = (\cos(\theta), \sin(\theta))$.


ball moves from 0 to
$\frac{\pi}{2}$ radians around
unit
circle
light
source
x-axis
y-axis
y-axis
light source
ball moves from 0
to $\frac{\pi}{2}$ radians
around unit circle
x-axis

(a) What do you see on the y-axis?

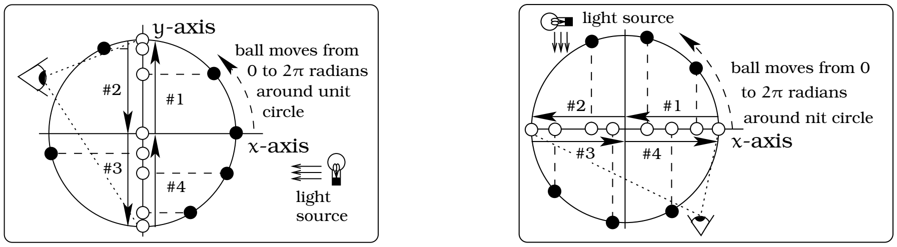
(b) What do you see on the x-axis?

Figure 18.2: Projecting the coordinates of points onto the y-axis and the x-axis.

By studying the coordinates of the ball as it moves in the first quadrant, we will be studying cos(θ) and sin(θ), for $0 \leq \theta \leq \pi/2$ radians.

We can visualize this very concretely. Imagine a light source as in Figure 18.2(a); then a shadow projects onto the vertical y-axis. The shadow locations you would see on the y-axis are precisely the values $\sin(\theta)$, for $0 \leq \theta \leq \pi/2$ radians. Similarly, imagine a light source as in Figure 18.2(b); then a shadow projects onto the horizontal x-axis. The shadow locations you would see on the x-axis are precisely the values $\cos(\theta)$, for $0 \leq \theta \leq \pi/2$ radians.

There are two visual conclusions: First, the function values of $\sin(\theta)$ vary from 0 to 1 as θ varies from 0 to $\pi/2$. Secondly, the function values of $\cos(\theta)$ vary from 1 to 0 as θ varies from 0 to $\pi/2$. Of course, we can go ahead and continue analyzing the motion as the ball moves into the second, third and fourth quadrant, ending up back at the starting position $(1, 0)$. See Figure 18.3.



(a) What do you see on the y-axis?

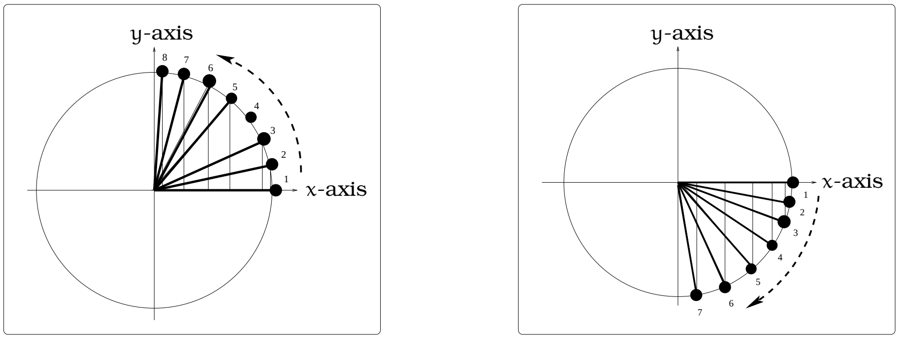
(b) What do you see on the x-axis?

Figure 18.3: Analyzing the values of the sine and cosine functions.

The conclusion is that after one complete counterclockwise rotation, the values of $\sin(\theta)$ and $\cos(\theta)$ range over the interval $[-1, 1]$. As the ball moves through the four quadrants, we have indicated the "order" in which these function values are assumed by labeling arrows #1 — #4: For example, for the sine function, look at Figure 18.3(a). The values of the sine function vary from 0 up to 1 while the ball moves through the first quadrant (arrow labeled #1), then from 1 down to 0 (arrow labeled #2), then from 0 down to -1 (arrow labeled #3), then from -1 up to 0 (arrow labeled #4).

What about the tangent function? We have seen that the tangent function computes the slope of the hypotenuse of an inscribed triangle. This means we can determine the range of values of $\tan(\theta)$ by investigating the possible slopes for these inscribed triangles. We will maintain the above model of a ball moving around the unit circle.

We look at two cases, each starting at $(1, 0)$. In the first quadrant, the ball moves counterclockwise and in the fourth quadrant it moves clockwise: In the first quadrant, we notice that these hypotenuse slopes are always non-negative, beginning with slope 0 (the degenerate right triangle when $\theta = 0$) then increasing. In fact, as the angle θ approaches $\pi/2$ radians, the ball is getting closer to the position $(0, 1)$ and the hypotenuse



(a) What happens to the slopes of these triangles?

(b) What happens to the slopes of these triangles?

Figure 18.4: Analyzing the values of the tangent function.

is approaching a vertical line. This tells us that as θ varies from 0 to $\pi/2$ (but not equal to $\pi/2$), these slopes attain all possible non-negative values. In other words, the range of values for $\tan(\theta)$ on the domain $0 \leq \theta < \pi/2$ will be $0 \leq z < \infty$. Similar reasoning shows that as the ball moves in the fourth quadrant, the slopes of the hypotenuses of the triangles are always non-positive, varying from 0 to ANY negative value. In other words, the range of values for $\tan(\theta)$ on the domain $-\pi/2 < \theta \leq 0$ will be $-\infty < z \leq 0$.

On your calculator, you can verify the visual conclusions we just established by studying the values of $\tan(\theta)$ for θ close (but not equal) to $\frac{\pi}{2}$ radians = $90°$:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\tan(89°) &=& 57.29 & \tan(-89°) &=& -57.29 \\
\tan(89.9°) &=& 572.96 & \tan(-89.9°) &=& -572.96 \\
\tan(89.99°) &=& 5729.58 & \tan(-89.99°) &=& -5729.58 \\
& \vdots & & & \vdots &
\end{array}$$

The fact that the values of the tangent function become arbitrarily large as we get close to $\pm\pi/2$ radians means the function output values are *unbounded*.

Important Fact 18.1.1 (Circular function values)**.** *For any angle* θ*, we always have* $-1 \leq \cos(\theta) \leq 1$ *and* $-1 \leq \sin(\theta) \leq 1$*. On domain* $0 \leq \theta \leq 2\pi$*, the range of both* $\cos(\theta)$ *and* $\sin(\theta)$ *is* $-1 \leq z \leq 1$*. In contrast, on the domain of all* θ *values for which tangent is defined, the range of* $\tan(\theta)$ *is all real numbers.*

For the sine and cosine functions, if the domain is not $0 \leq \theta \leq 2\pi$, then we need to consider the "periodic qualities" of the circular functions to determine the range. This is discussed below.

18.2 Identities

There are dozens of formulas that relate the values of two or more circular functions; these are usually lumped under the heading of *Trigonometric Identities*. In this course, we only need a couple frequently used identities.

If we take the point $P = (\cos(\theta), \sin(\theta))$ on the unit circle, corresponding to the standard central position angle θ, then recall the equation of the unit circle tells us $x^2+y^2 = 1$. But, since the x coordinate is $\cos(\theta)$ and the y coordinate is $\sin(\theta)$, we have $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. It is common notational practice to write $(\cos(\theta))^2 = \cos^2(\theta)$ and $(\sin(\theta))^2 = \sin^2(\theta)$. This leads to the **most important** of all trigonometric identities:

Important Fact 18.2.1 (Trigonometric identity)**.** *For any angle* θ*, we have the identity* $\cos^2(\theta) + \sin^2(\theta) = 1$.

Adding any multiple of 2π radians (or 360°) to an angle will not change the values of the circular functions. If we focus on radians for a moment, this says that knowing the values of $\cos(\theta)$ and $\sin(\theta)$ on the domain $0 \leq \theta \leq 2\pi$ determines the values for any other possible angle.

There is something very general going on here, so let's pause a moment to make a definition and then an observation.

Definition 18.2.2 (Periodic function)**.** *For* $c > 0$*, a function* $f(\theta)$ *is called* c*-periodic if two things are true:*

(i) $f(\theta + c) = f(\theta)$ *holds for all* θ*;*

(ii) There is no smaller d, $0 < d < c$, *such that* $f(\theta + d) = f(\theta)$ *holds for all* θ.

We usually call c *the period of the function.*

Using this new terminology, we conclude that the sine and cosine circular functions are 2π*-periodic*. In the case of the tangent circular function, it is also true that $\tan(\theta) = \tan(\theta + 2\pi n)$, for every integer n. However, referring back to the unit circle definitions of the circular functions, we have $\tan(\theta) = \tan(\theta + n\pi)$, for all integers n. If you take $n = 1$, then this tells us that the tangent circular function is π-periodic. We summarize this information below.

Important Fact 18.2.3 (Periodicity identity)**.** *For any angle* θ *and any integer* $n = 0, \pm 1, \pm 2, \pm 3, \dots$*, we have* $\cos(\theta) = \cos(\theta+2\pi n)$, $\sin(\theta) = \sin(\theta+2\pi n)$*, and* $\tan(\theta) = \tan(\theta + n\pi)$.

Next, we draw an angle θ and its negative in the same unit circle picture in standard central position. We have indicated the points P_θ and $P_{-\theta}$ used to define the circular functions. It is clear from the picture in Figure 18.5 that P_θ and $P_{-\theta}$ have the same x-coordinate, but the y-coordinates are negatives of one another. This gives the next identity:

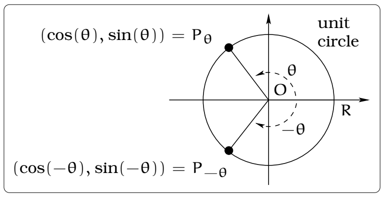

$(\cos(\theta), \sin(\theta)) = P_\theta$
unit circle
θ
O
R
$-\theta$
$(\cos(-\theta), \sin(-\theta)) = P_{-\theta}$

Figure 18.5: Visualizing a trigonometric identity.

Important Fact 18.2.4 (Even/Odd identity)**.** *For any angle* θ, $\sin(-\theta) = -\sin(\theta)$, *and* $\cos(-\theta) = \cos(\theta)$.

We can use the terminology of *even and odd functions* here. In this language, this result says that the cosine function is an even function and the sine function is an odd function.

Next, draw the angles θ and $\theta + \pi$ in the same unit circle picture in standard central position. We have indicated the corresponding points P_θ and $P_{\theta+\pi}$ on the unit circle and their coordinates in terms of the circular functions: From the picture in Figure 18.6, the x-coordinate of P_θ must be the "negative" of the x-coordinate of $P_{\theta+\pi}$ and similarly, the y-coordinate of P_θ must be the "negative" of the y-coordinate of $P_{\theta+\pi}$. This gives us the next identity:

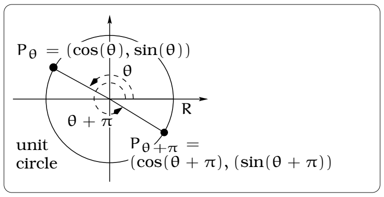

$P_\theta = (\cos(\theta), \sin(\theta))$
θ
R
$\theta + \pi$
unit circle
$P_{\theta+\pi} =$
$(\cos(\theta + \pi), (\sin(\theta + \pi))$

Figure 18.6: Visualizing Fact 18.2.5.

Important Fact 18.2.5 (Plus π identity)**.** *For any angle* θ, *we have* $\sin(\theta + \pi) = -\sin(\theta)$, *and* $\cos(\theta + \pi) = -\cos(\theta)$.

Important Fact 18.2.6. *For any angle* θ, *we have* $\sin(\pi - \theta) = \sin(\theta)$ *and* $\cos(\pi - \theta) = -\cos(\theta)$.

For example, we have $\sin\left(\frac{5\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. This calculation leads to a computational observation: Combining Table 17.1 with the previous two identities we can compute the EXACT value of $\cos(\theta)$, $\sin(\theta)$, and $\tan(\theta)$ at an angle θ which is a multiple of $30^\circ = \frac{\pi}{6}$ radians or $45^\circ = \frac{\pi}{4}$ radians. Here are some sample calculations together with a reference as to "why" each equality is valid:

Example 18.2.7.

(i)
$$\begin{aligned} \cos(-45^\circ) &= \cos(45^\circ) && \textit{Fact } 18.2.4 \textit{ on page } 241 \\ &= \frac{\sqrt{2}}{2} && \textit{Table } 17.1 \textit{ on page } 223 \end{aligned}$$

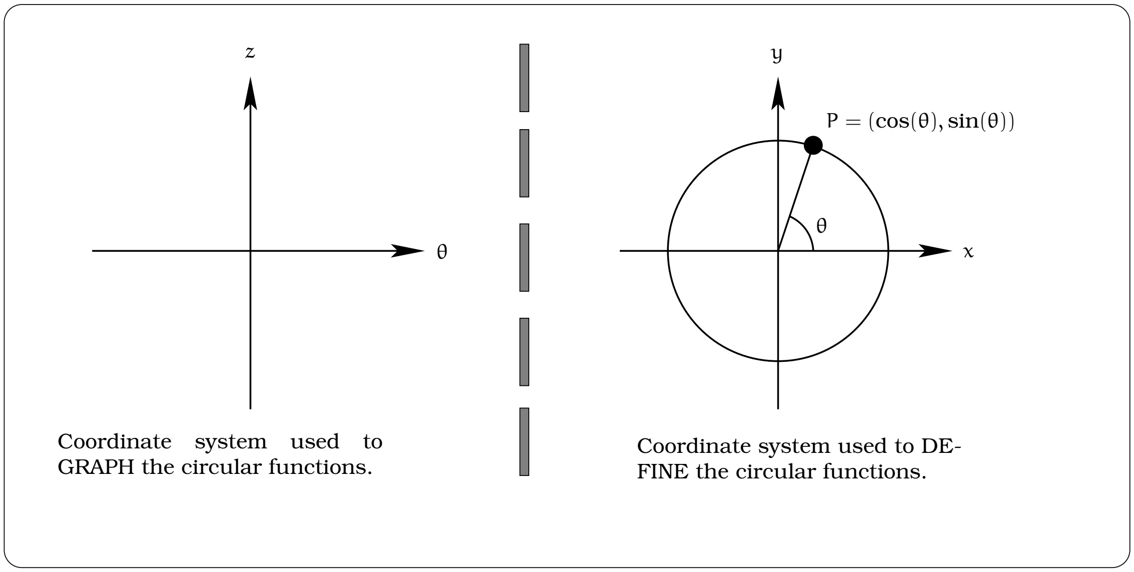
(ii)
$$\begin{aligned} \sin(225^\circ) &= \sin(45^\circ + 180^\circ) && \textit{Fact } 18.2.3 \textit{ on page } 240 \\ &= -\sin(45^\circ) && \textit{Fact } 18.2.5 \textit{ on page } 241 \\ &= -\frac{\sqrt{2}}{2} && \textit{Table } 17.1 \textit{ on page } 223 \end{aligned}$$

(iii)
$$\begin{aligned} \cos\left(\frac{2\pi}{3}\right) &= \cos\left(-\frac{\pi}{3} + \pi\right) && \textit{Fact } 18.2.3 \textit{ on page } 240 \\ &= -\cos\left(-\frac{\pi}{3}\right) && \textit{Fact } 18.2.5 \textit{ on page } 241 \\ &= -\cos\left(\frac{\pi}{3}\right) && \textit{Fact } 18.2.4 \textit{ on page } 241 \\ &= -\frac{1}{2} && \textit{Table } 17.1 \textit{ on page } 223 \end{aligned}$$

18.3 Graphs of Circular Functions

We have introduced three new functions of the variable θ and it is important to understand and interpret the pictures of their graphs. To do this, we need to settle on a coordinate system in which to work. The horizontal axis will correspond to the independent variable, so this should be the θ-axis. We will label the vertical axis, which corresponds to the dependent variable, the z-axis. With these conventions, beginning with any of the circular functions $z = \sin(\theta)$, $z = \cos(\theta)$, or $z = \tan(\theta)$, the graph will be a subset of the θz-coordinate system. Precisely, given a circular function $z = f(\theta)$, the graph consists of all pairs $(\theta, f(\theta))$, where θ varies over a domain of allowed values. We will record and discuss these graphs below; a graphing device will painlessly produce these for us!

There is a point of possible confusion that needs attention. We purposely did not use the letter "y" for the dependent variable of the circular functions. This is to avoid possible confusion with our construction of the sine and cosine functions using the unit circle. Since we viewed the unit circle inside the xy-coordinate system, the x-coordinates (resp. y-coordinates) of points on the unit circle are computed by $\cos(\theta)$ (resp. $\sin(\theta)$).



Coordinate system used to GRAPH the circular functions.

Coordinate system used to DEFINE the circular functions.

Figure 18.7: The $z\theta$ versus xy coordinate systems.

18.3.1 A matter of scaling

The first issue concerns scaling of the axes used in graphing the circular functions. As we know, the definition of radian measure is directly tied to the lengths of arcs subtended by angles in the unit circle:

Important Fact 18.3.1. *An angle of measure 1 radian inside the unit circle will subtend an arc of length 1.*

Since length is a good intuitive scaling quantity, it is natural to scale the θ-axis so that the length of 1 radian on the θ-axis (horizontal axis) is the same length as 1 unit on the vertical axis. For this reason, we will work primarily with radian measure when sketching the graphs of circular functions. If we need to work explicitly with degree measure for angles, then we can always convert radians to degrees using the fact: $360^\circ = 2\pi$ radians.

18.3.2 The sine and cosine graphs

Using Fact 18.1.1, we know that $-1 \leq \sin(\theta) \leq 1$ and $-1 \leq \cos(\theta) \leq 1$. Pictorially, this tells us that the graphs of $z = \sin(\theta)$ and $z = \cos(\theta)$ lie between the horizontal lines $z = 1$ and $z = -1$; i.e. the graphs lie inside the darkened band pictured in Figure 18.8.

By Fact 18.2.3, we know that the values of the sine and cosine repeat themselves every 2π radians. Consequently, if we know the graphs of the sine and cosine on the domain $0 \leq \theta \leq 2\pi$, then the picture will repeat for the interval $2\pi \leq \theta \leq 4\pi$, $-2\pi \leq \theta \leq 0$, etc.

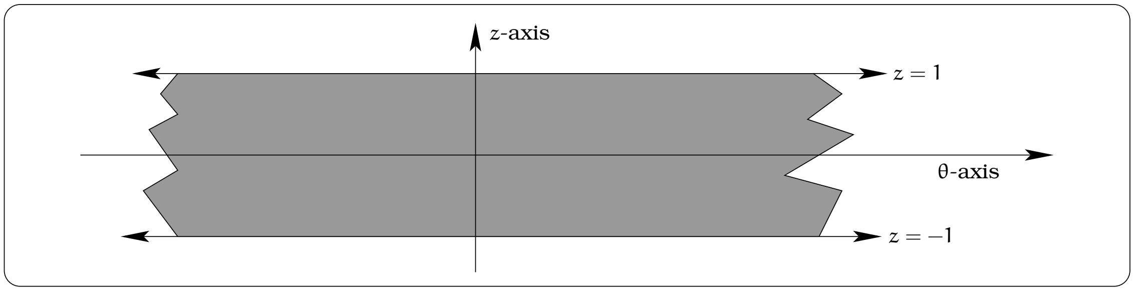

z-axis
z = 1
θ-axis
z = −1

Figure 18.8: Visualizing the range of $\sin(\theta)$ and $\cos(\theta)$.

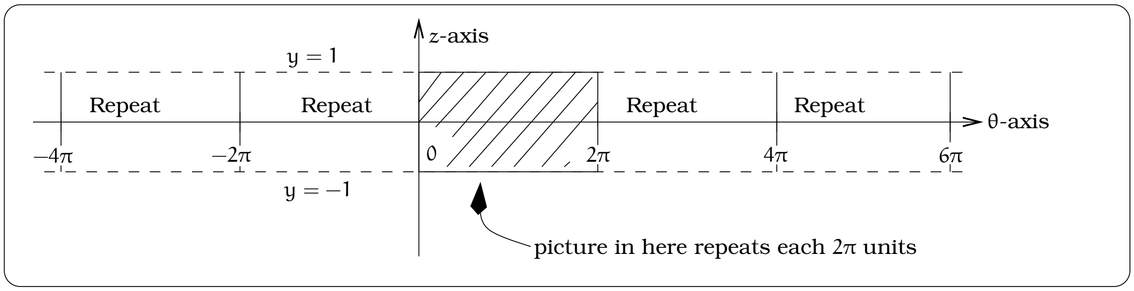

z-axis
y = 1
Repeat
Repeat
Repeat
Repeat
θ-axis
−4π
−2π
0
2π
4π
6π
y = −1
picture in here repeats each 2π units

Figure 18.9: On what intervals will the graph repeat?

Sketching the graph of $z = \sin(\theta)$ for $0 \leq \theta \leq 2\pi$ can be roughly achieved by plotting points. For example, $\left(\frac{\pi}{6}, \frac{1}{2}\right)$, $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$, $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$, and $\left(\frac{\pi}{2}, 1\right)$ lie on the graph, as do $\left(\frac{3\pi}{2}, -1\right)$, $\left(\frac{5\pi}{3}, -\frac{\sqrt{3}}{2}\right)$, $\left(\frac{7\pi}{4}, -\frac{\sqrt{2}}{2}\right)$, $\left(\frac{11\pi}{6}, -\frac{1}{2}\right)$, and $(2\pi, 0)$, etc. If we return to our analysis of the range of values for the sine function in Figure 18.2, it is easy to see where $\sin(\theta)$ is positive or negative; combined with Chapter 4, this tells us where the graph is above and below the horizontal axis (see Figure 18.10).

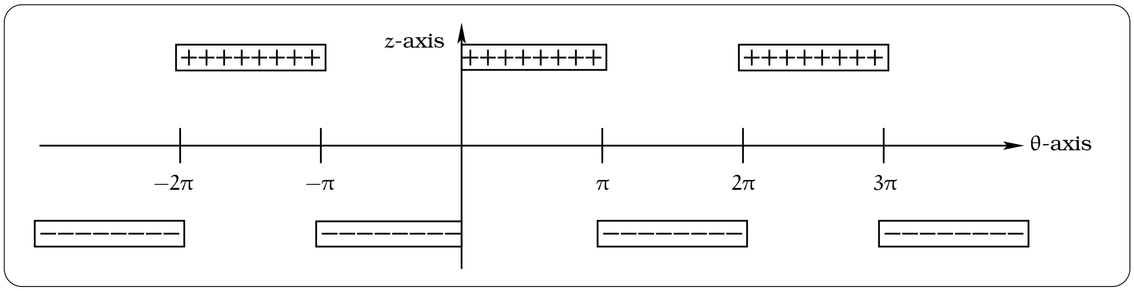

z-axis
++++++++
++++++++
++++++++
θ-axis
−2π
−π
π
2π
3π
−−−−−−−−
−−−−−−−−
−−−−−−−−
−−−−−−−−

Figure 18.10: Where is the graph positive or negative?

We now include a software plot of the graph of sine function, observing the three qualitative features just isolated: bounding, periodicity and sign properties (see Figure 18.11).

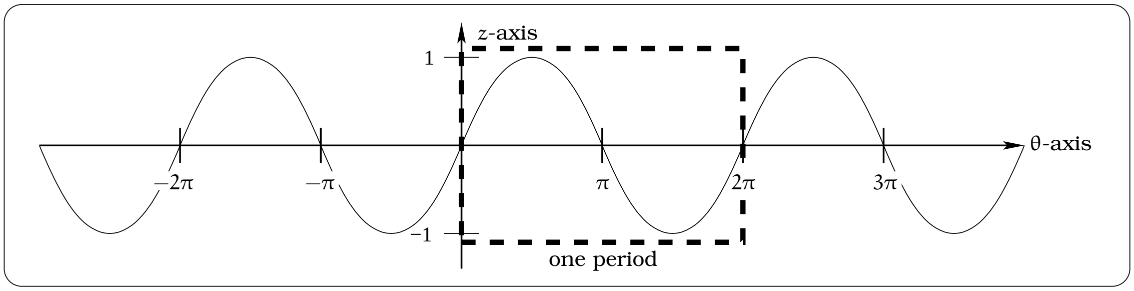


Figure 18.11: The graph of $z = \sin(\theta)$.

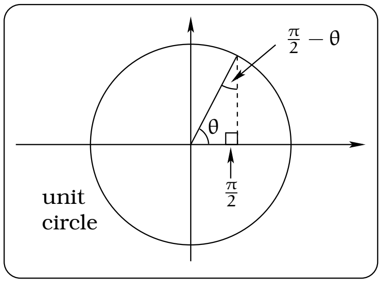


Figure 18.12: Visualizing the conversion identity.

We could repeat this analysis to arrive at the graph of the cosine. Instead, we will utilize an identity. Given an angle θ, place it in central standard position in the unit circle, as one of the four cases of Figure 17.16. For example, we have pictured Case I in this figure. Since the sum of the angles in a triangle is $180° = \pi$ radians, we know that θ, $\pi/2$, and $\frac{\pi}{2} - \theta$ are the three angles of the inscribed right triangle. From the picture in Figure 18.12, it then follows that

$$\begin{aligned}\cos(\theta) &= \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} \\ &= \frac{\text{side opposite to } \left(\frac{\pi}{2} - \theta\right)}{\text{hypotenuse}} \\ &= \sin\left(\frac{\pi}{2} - \theta\right).\end{aligned}$$

Using the same reasoning this identity is valid for all θ. This gives us another useful identity:

Important Fact 18.3.2 (Conversion identity)**.** *For any angle* θ, $\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$, *and* $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$.

This identity can be used to sketch the graph of the cosine function. First, we do a calculation using our new identity:

$$\begin{aligned}\cos(\theta) &= \cos(-\theta) && \text{Fact 18.2.4 on page 241} \\ &= \sin\left(\frac{\pi}{2} - (-\theta)\right) && \text{Fact 18.3.2 (above)} \\ &= \sin\left(\theta - \left(-\frac{\pi}{2}\right)\right) && \text{Since: } (a+b) = (a-(-b)) = (b-(-a))\end{aligned}$$

By the horizontal shifting principle in Fact 13.3.1 on page 170, the graph of $z = \cos(\theta)$ is obtained by horizontally shifting the graph of $z = \sin(\theta)$ by $\frac{\pi}{2}$ units to the **left**. Here is a plot of the graph of the cosine function: See Figure 18.13.

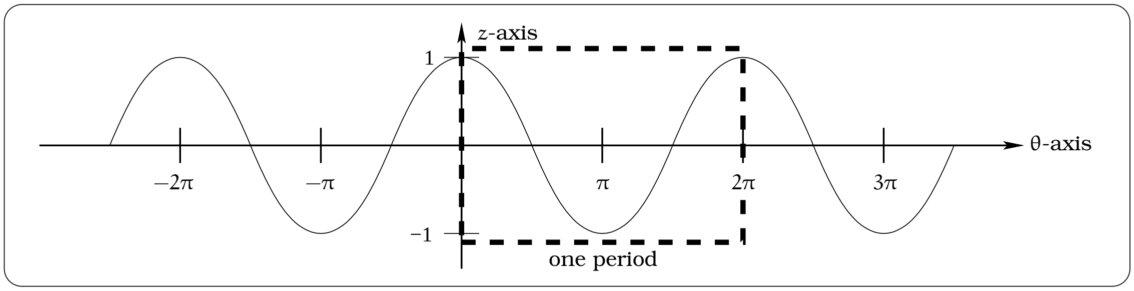


Figure 18.13: The graph of $z = \cos(\theta)$.

18.3.3 The tangent graph

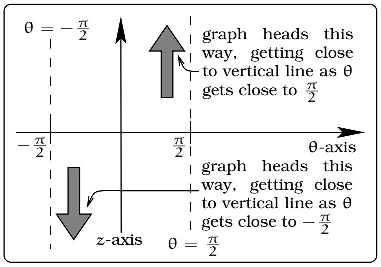

θ = −π/2
graph heads this way, getting close to vertical line as θ gets close to π/2
−π/2
π/2
θ-axis
graph heads this way, getting close to vertical line as θ gets close to −π/2
z-axis
θ = π/2

Figure 18.14: The behavior of $\tan(\theta)$ as θ approaches asymptotes.

As we have already seen, unlike the sine and cosine circular functions, the tangent function is NOT defined for all values of θ. Since $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, here are some properties we can immediately deduce:

- The function $z = \tan(\theta)$ is undefined if and only if $\theta = \frac{\pi}{2} + k\pi$, where $k = 0, \pm 1, \pm 2, \pm 3, \cdots$.
- The function $z = \tan(\theta) = 0$ if and only if $\theta = k\pi$, where $k = 0, \pm 1, \pm 2, \pm 3, \cdots$.
- By Fact 18.2.3, the tangent function is π-periodic, so the picture of the graph will repeat itself every π-units and it is enough to understand the graph when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.
- On the domain $0 \leq \theta < \frac{\pi}{2}$, $\tan(\theta) \geq 0$; on the domain $-\frac{\pi}{2} < \theta \leq 0$, $\tan(\theta) \leq 0$.

In the θz-coordinate system, the vertical lines $\theta = \frac{\pi}{2} + k\pi$, where $k = 0, \pm 1, \pm 2, \cdots$ will be *vertical asymptotes* for the graph of the tangent function. Using our slope interpretation in Figure 18.4, what becomes clear is this: As the values of θ get close to $\frac{\pi}{2}$, the graph is getting close to the vertical line $\theta = \frac{\pi}{2}$ AND becoming farther and farther away from the horizontal axis: To understand this numerically, first suppose θ is slightly smaller than $\frac{\pi}{2}$, say $\theta = \left(\frac{\pi}{2} - 0.1\right)$, $\left(\frac{\pi}{2} - 0.01\right)$, and $\left(\frac{\pi}{2} - 0.001\right)$. Then the calculation of $\tan(\theta)$ involves dividing a number very close to 1 by a very small positive number:

$$\begin{aligned}
\tan\left(\frac{\pi}{2} - 0.1\right) &= 9.9666, \\
\tan\left(\frac{\pi}{2} - 0.01\right) &= 99.9967, \text{ and} \\
\tan\left(\frac{\pi}{2} - 0.001\right) &= 1000.
\end{aligned}$$

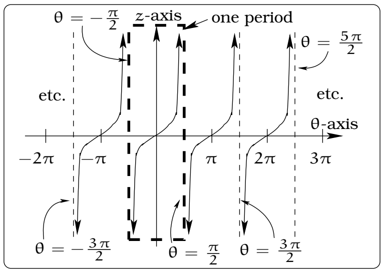


Figure 18.15: The graph of $z = \tan(\theta)$.

Conclude that as θ "approaches $\frac{\pi}{2}$ from below", the values of $\tan(\theta)$ are becoming larger and larger. This says that the function values become "unbounded". Likewise, imagine the case when θ is slightly bigger than $-\frac{\pi}{2}$, say $\theta = \left(-\frac{\pi}{2} + 0.1\right)$, $\left(-\frac{\pi}{2} + 0.01\right)$, and $\left(-\frac{\pi}{2} + 0.001\right)$. Then the calculation of $\tan(\theta)$ involves dividing a number very close to -1 by a very small positive number:

$$\begin{aligned}
\tan\left(-\frac{\pi}{2} + 0.1\right) &= -9.9666, \\
\tan\left(-\frac{\pi}{2} + 0.01\right) &= -99.9967, \text{ and} \\
\tan\left(-\frac{\pi}{2} + 0.001\right) &= -1000.
\end{aligned}$$

Conclude that as θ "approaches $-\frac{\pi}{2}$ from above", the values of $\tan(\theta)$ are becoming negative numbers of increasingly larger magnitude:

Again, this tells us the function values are becoming "unbounded". The graph of $z = \tan(\theta)$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ can be roughly achieved by combining the calculations as in Example 18.2.7 and the qualitative features highlighted. Figure 18.15 shows a software plot.

18.4 Trigonometric Functions

To become successful mathematical modelers, we must have wide variety of functions in our toolkit. As an illustration, the graph below might represent the height of the tide above some reference level over the course of several days. The curve drawn is clearly illustrating that the height of the tide is "periodic" as a function of time t; in other words, the behavior of the tide repeats itself as time goes by. However, if we try to model this periodic behavior, the only weapon at our disposal would be the circular functions and these require an angle variable, not a time variable such as t; we are stuck!

Modeling the tide graph requires the *trigonometric functions*, which lie at the heart of studying all kinds of periodic behavior. We have no desire to "reinvent the wheel", so let's use our previous work on the circular functions to define the trigonometric functions.

18.4.1 A Transition

Given a real number t*, is there a sensible way to define* $\cos(t)$ *and* $\sin(t)$*?* The answer is yes and depends on the ideas surrounding radian measure of angles. Given the positive real number t, we can certainly imagine an angle of measure t radians inside the unit circle (in standard position) and we know the arc subtended by this angle has length t (this is why we use the unit circle).

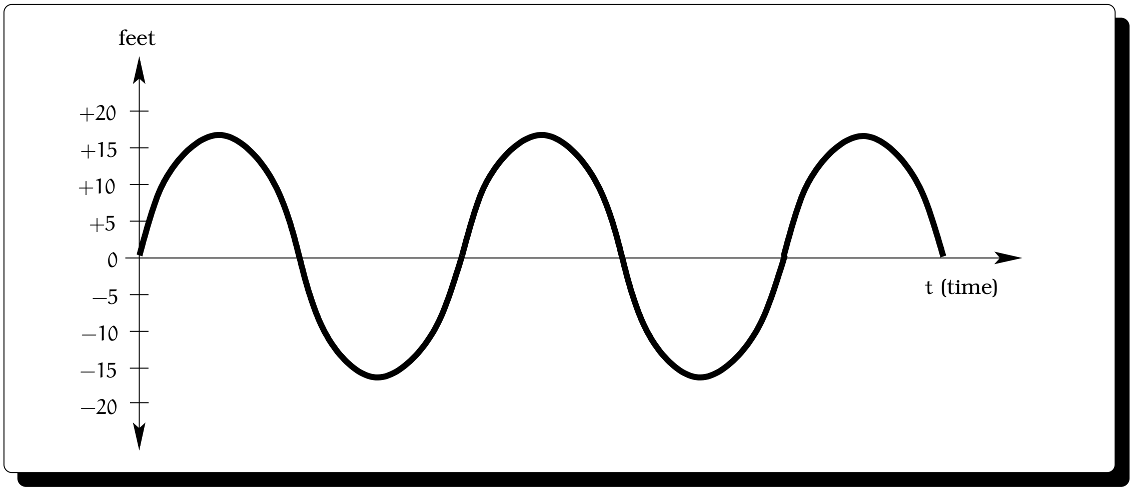


Figure 18.16: A periodic function with input variable t.

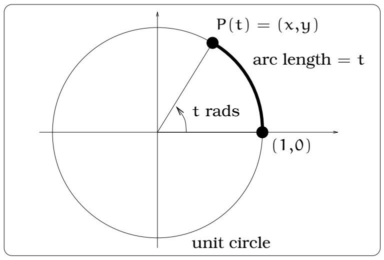


Figure 18.17: The circular functions with input variable t.

We already know that cos(t radians) and sin(t radians) compute the x and y coordinates of the point $P(t)$. In effect, we are just using the measure of the angle t to help us locate the point $P(t)$. An alternate way to locate $P(t)$ is to move along the circumference counterclockwise, beginning at $(1,0)$, until we have an arc of length t; that again puts us at the point $P(t)$. In the case of an angle of measure $-t$ radians, the point $P(-t)$ can be located by moving along the circumference clockwise, beginning at $(1,0)$, until we have an arc of length t.

Definition 18.4.1 (Trigonometric functions)**.** *Let* t *be a real number. We DEFINE the sine function* $y = \sin(t)$*, the cosine function* $y = \cos(t)$ *and the tangent function* $y = \tan(t)$ *by the rules*

$$\begin{aligned}
\sin(t) &\overset{def}{=} y\text{-}\textit{coordinate of } P(t) &= \sin(t\ \textit{radians})\\
\cos(t) &\overset{def}{=} x\text{-}\textit{coordinate of } P(t) &= \cos(t\ \textit{radians})\\
\tan(t) &\overset{def}{=} \frac{\sin(t)}{\cos(t)} &= \tan(t\ \textit{radians})
\end{aligned}$$

We refer to these as the basic *trigonometric functions*. If we are working with radian measure and t is a real number, then there is no difference between evaluating a trigonometric function at the real number t and evaluating the corresponding circular function at the angle of measure t radians.

Example 18.4.2. *Assume that the number of hours of daylight in Seattle during 1994 is given by the function* $d(t) = 3.7\sin\left(\frac{2\pi}{366}(t - 80.5)\right) + 12$*, where*

t *represents the day of the year and* $t = 0$ *corresponds to January 1. How many hours of daylight will there be on May 11?*

Solution. To solve the problem, you need to consult a calendar, finding every month has 31 days, except: February has 28 days and April, June, September and November have 30 days. May 11 is the $31+28+31+30+11 = 131^{\text{st}}$ day of the year. So, there will be $d(131) = 3.7\sin(2(50.5)\pi/366) + 12 = 14.82$ hours of daylight on May 11. $\square$

18.4.2 Graphs of trigonometric functions

The graphs of the trigonometric functions $y = \sin(t)$, $y = \cos(t)$, and $y = \tan(t)$ will look just like Figures 18.11, 18.13, and 18.15, except that the horizontal axis becomes the t-axis and the vertical axis becomes the y-axis.

18.4.3 Notation for trigonometric functions

In many texts, you will find the sine function written as $y = \sin t$; i.e. the parenthesis around the "t" are omitted. A similar comment applies to all of the trigonometric functions. We will **never** do this and the reasoning is simply this: Maintaining the parenthesis, as in $y = \sin(t)$, emphasizes the fact that we are dealing with a function and the "input values" are located between the parenthesis. For example, if we write the function $y = \sin(t^2+2t+1)$, it is crystal clear that the sine function is applied to the expression "$t^2 + 2t + 1$"; using the alternate notation yields the expression $y = \sin t^2 + 2t + 1$, which is interpreted to mean $y = (\sin t^2) + (2t + 1)$.

As a rule, whenever you see an expression involving sin$(\cdots)$, cos$(\cdots)$, or tan$(\cdots)$, we assume "$\cdots$" is in units of RADIANS, unless otherwise noted. When computing values on your calculator, MAKE SURE YOU ARE USING RADIAN MODE!

¡¡¡
CAUTION
!!!

18.5 Exercises

Problem 18.1. Work the following problems without using ANY calculators.

(a) Sketch $y = \sin(x)$.

(b) Sketch $y = \sin^2(x)$.

(c) Sketch $y = \dfrac{1}{1+\sin^2(x)}$.

Problem 18.2. Sketch the graphs of these functions:

(a) $f(t) = |\sin(t)|$.

(b) $f(t) = |\cos(t)|$.

(c) $f(t) = |\tan(t)|$.

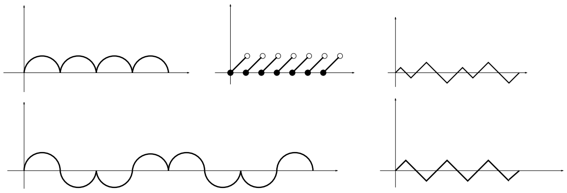
Problem 18.3. Solve the following:

(a) If $\cos(\theta) = \frac{24}{25}$, what are the two possible values of $\sin(\theta)$?

(b) If $\sin(\theta) = -0.8$ and θ is in the third quadrant of the xy plane, what is $\cos(\theta)$?

(c) If $\sin(\theta) = \frac{3}{7}$, what is $\sin(\frac{\pi}{2} - \theta)$?

Problem 18.4. These graphs represent periodic functions. Describe the period in each case.



Problem 18.5. Start with the equation $\sin(\theta) = \cos(\theta)$. Use the unit circle interpretation of the circular functions to find the solutions of this equation; make sure to describe your reasoning.

Chapter 19

Sinusoidal Functions

A migrating salmon is heading up a portion of the Columbia River. It's depth $d(t)$ *(in feet) below the water surface is measured and plotted over a 30 minute period, as a function of time* t *(minutes). What is the formula for* $d(t)$*?*

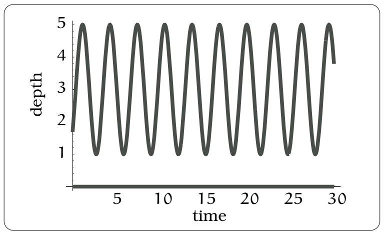


Figure 19.1: The depth of a salmon as a function of time.

In order to answer the question, we need to introduce an important new family of functions called the *sinusoidal functions*. These functions will play a central role in modeling any kind of periodic phenomena. The amazing fact is that almost any function you will encounter can be approximated by a sum of sinusoidal functions; a result that has far-reaching implications in all of our lives.

19.1 A special class of functions

Beginning with the trigonometric function $y = \sin(x)$*, what is the most general function we can build using the graphical techniques of shifting and stretching?*

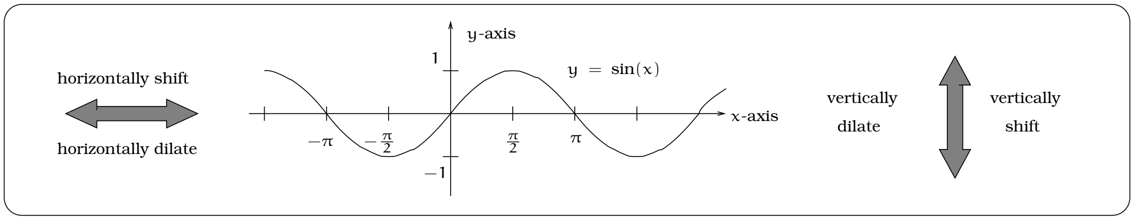


Figure 19.2: Visualizing the geometric operations available for curve sketching.

The graph of $y = \sin(x)$ can be manipulated in four basic ways: horizontally shift, vertically shift, horizontally dilate or vertically dilate. Each of these "geometric operations" corresponds to a simple change in the "symbolic formula" for the function, as discussed in Chapter 13.

If we vertically shift the graph by D units upward, the resulting curve would be the graph of the function $y = \sin(x)+D$; see Facts 13.3.1. Recall, the effect of the sign of D: If D is negative, the effect of shifting D units upward is the same as shifting $|D|$ units downward. Notice, the function $y = \sin(x) + D$ is still a periodic function, having the same period 2π as $y = \sin(x)$. Notice, whereas the graph of the function $y = \sin(x)$ oscillates between the horizontal lines $y = \pm 1$, the graph of $y = \sin(x) + D$ oscillates between $y = D \pm 1$. For this reason, we sometimes refer to the constant D as the *mean* of the function $y = \sin(x) + D$. In Figure 19.3, notice that the graph of $y = \sin(x)+D$ is symmetrically split by the horizontal "mean" line $y = D$.

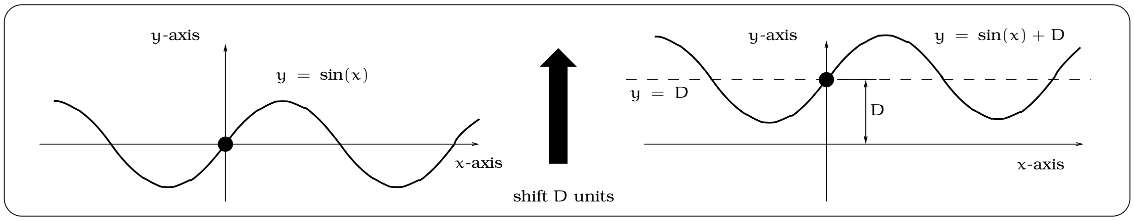


Figure 19.3: Interpreting the mean.

Next, consider the effect of horizontally shifting the graph of $y = \sin(x)$ by C units to the right. By Facts 13.3.1, the new curve is the graph of the function $y = \sin(x - C)$. Also, recall the effect of the sign of C: If C is negative, the effect of shifting C units right is the same as shifting $|C|$ units left. If the domain of $\sin(x)$ is $0 \leq x \leq 2\pi$, then the domain of $\sin(x - C)$ is $0 \leq x - C \leq 2\pi$, again by Facts 13.3.1. Rewriting this, the domain of $\sin(x - C)$ is $C \leq x \leq 2\pi + C$ and the graph will go through precisely one period on this domain. In other words, the new function $\sin(x - C)$ is still 2π-periodic. The constant C is usually called the *phase shift* of $y = \sin(x-C)$. Looking at Figure 19.4, it is possible to interpret C graphically: C will be a point where the graph crosses the horizontal axis *on its way up* from a minimum to a maximum.

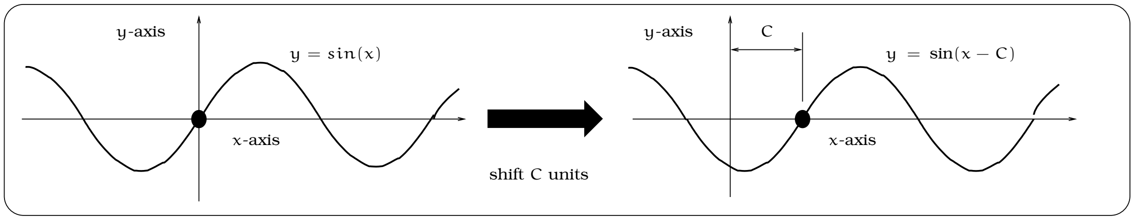


Figure 19.4: Interpreting the phase shift.

Vertically dilating the graph, either by vertical expansion or compression, leads to a new curve. The graph of this vertically dilated curve is $y = A\sin(x)$, for some positive constant A. Furthermore, if $A > 1$, the graph of $y = A\sin(x)$ is a vertically expanded version of $y = \sin(x)$, whereas, if $0 < A < 1$, then the graph of $y = A\sin(x)$ is a vertically compressed version of $y = \sin(x)$. Notice, the function $y = A\sin(x)$ is still 2π-periodic. What has changed is the band of oscillation: whereas the graph of the function $y = \sin(x)$ stays between the horizontal lines $y = \pm 1$, the graph of $y = A\sin(x)$ oscillates between the horizontal lines $y = \pm A$. We usually refer to A as the *amplitude* of the function $y = A\sin(x)$.

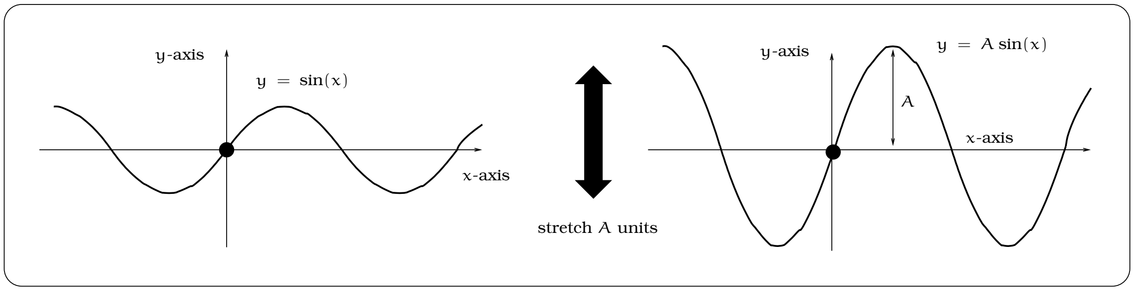


Figure 19.5: Interpreting the amplitude.

Finally, horizontally dilating the graph, either by horizontal expansion or compression, leads to a new curve. The equation of this horizontally dilated curve is $y = \sin(cx)$, for some constant $c > 0$. We know that $y = \sin(x)$ is a 2π-periodic function and observe that horizontally dilation still results in a periodic function, but the period will typically NOT be 2π. For future purposes, it is useful to rewrite the equation for the horizontally stretched curve in a way more directly highlighting the period. To begin with, once the horizontal stretching factor c is known, we could rewrite

$$c = \frac{2\pi}{B}, \quad \text{for some } B \neq 0.$$

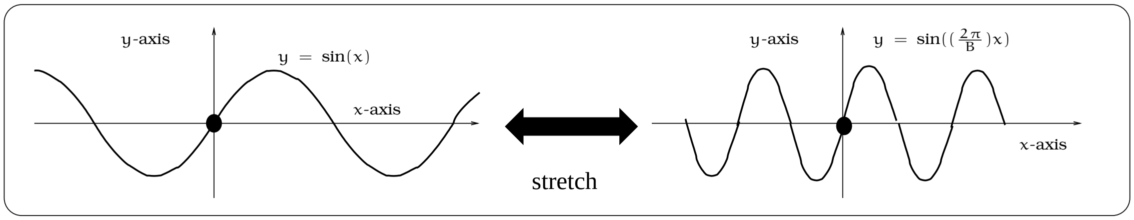


Figure 19.6: Interpreting the period.

Here is the point of this yoga with the horizontal dilating constant: If we let the values of x range over the interval $[0, B]$, then $\frac{2\pi}{B}x$ will range over the interval $[0,2\pi]$. In other words, the function $y = \sin\left(\frac{2\pi}{B}x\right)$ is B-periodic and we can read off the *period* of $y = \sin\left(\frac{2\pi}{B}x\right)$ by viewing the constant in this mysterious way. The four constructions outlined lead to a new family of functions.

Definition 19.1.1 (The Sinusoidal Function). *Let* A, B, C *and* D *be fixed constants, where* A *and* B *are both positive. Then we can form the new function*

$$y = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

which is called a sinusoidal function. The four constants can be interpreted graphically as indicated:

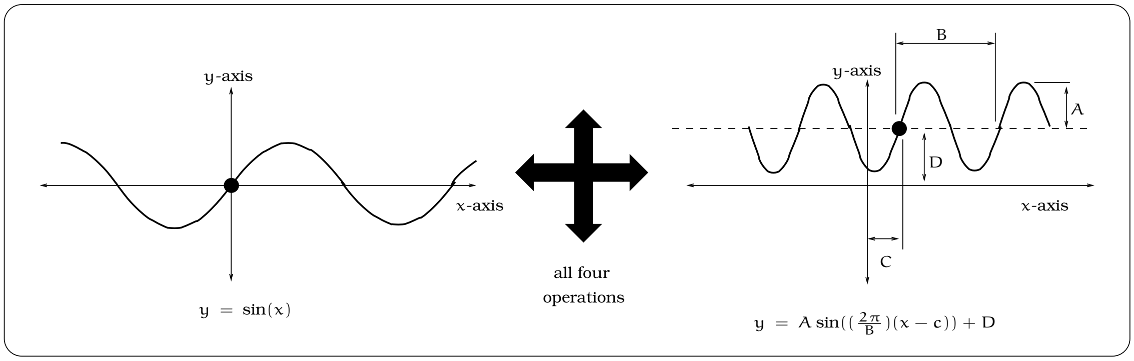


Figure 19.7: Putting it all together for the sinusoidal function.

19.1.1 How to roughly sketch a sinusoidal graph

Important Procedure 19.1.2. *Given a sinusoidal function in the standard form*

$$y = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

once the constants A, B, C, *and* D *are specified, any graphing device can produce an accurate graph. However, it is pretty straightforward to sketch a rough graph by hand and the process will help reinforce the graphical meaning of the constants* A, B, C, *and* D. *Here is a "five step procedure" one can follow, assuming we are given* A, B, C, *and* D. *It is a good idea to follow Example 19.1.3 as you read this procedure; that way it will seem a lot less abstract.*

1. *Draw the horizontal line given by the equation* $y = D$; *this line will "split" the graph of* $y = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D$ *into symmetrical upper and lower halves.*
2. *Draw the two horizontal lines given by the equations* $y = D \pm A$. *These two lines determine a horizontal strip inside which the graph of the*

sinusoidal function will oscillate. Notice, the points where the sinusoidal function has a maximum value lie on the line $y = D + A$. *Likewise, the points where the sinusoidal function has a minimum value lie on the line* $y = D - A$. *Of course, we do not yet have a prescription that tells us where these maxima (peaks) and minima (valleys) are located; that will come out of the next steps.*

3. *Since we are given the period* B, *we know these important facts: (1) The period* B *is the horizontal distance between two successive maxima (peaks) in the graph. Likewise, the period* B *is the horizontal distance between two successive minima (valleys) in the graph. (2) The horizontal distance between a maxima (peak) and the successive minima (valley) is* $\frac{1}{2}B$.

4. *Plot the point* (C, D). *This will be a place where the graph of the sinusoidal function will cross the mean line* $y = D$ *on its way up from a minima to a maxima. This is not the only place where the graph crosses the mean line; it will also cross at the points obtained from* (C, D) *by horizontally shifting by any integer multiple of* $\frac{1}{2}B$. *For example, here are three places the graph crosses the mean line:* $(C, D), (C + \frac{1}{2}B, D), (C + B, D)$

5. *Finally, midway between* (C, D) *and* $(C+\frac{1}{2}B, D)$ *there will be a maxima (peak); i.e. at the point* $(C + \frac{1}{4}B, D + A)$. *Likewise, midway between* $(C + \frac{1}{2}B, D)$ *and* $(C + B, D)$ *there will be a minima (valley); i.e. at the point* $(C + \frac{3}{4}B, D - A)$. *It is now possible to roughly sketch the graph on the domain* $C \leq x \leq C + B$ *by connecting the points described. Once this portion of the graph is known, the fact that the function is periodic tells us to simply repeat the picture in the intervals* $C + B \leq x \leq C + 2B$, $C - B \leq x \leq C$, *etc.*

To make sense of this procedure, let's do an explicit example to see how these five steps produce a rough sketch.

Example 19.1.3. *The temperature (in °C) of Adri-N's dorm room varies during the day according to the sinusoidal function* $d(t) = 6\sin\left(\frac{\pi}{12}(t-11)\right) + 19$, *where* t *represents hours after midnight. Roughly sketch the graph of* $d(t)$ *over a 24 hour period.. What is the temperature of the room at 2:00 pm? What is the maximum and minimum temperature of the room?*

Solution. We begin with the rough sketch. Start by taking an inventory of the constants in this sinusoidal function:

$$d(t) = 6\sin\left(\frac{\pi}{12}(t-11)\right) + 19 = A\sin\left(\frac{2\pi}{B}(t-C)\right) + D.$$

Conclude that $A = 6$, $B = 24$, $C = 11$, $D = 19$. Following the first four steps of the procedure outlined, we can sketch the lines $y = D = 19$, $y = D \pm A = 19 \pm 6$ and three points where the graph crosses the mean line (see Figure19.8).

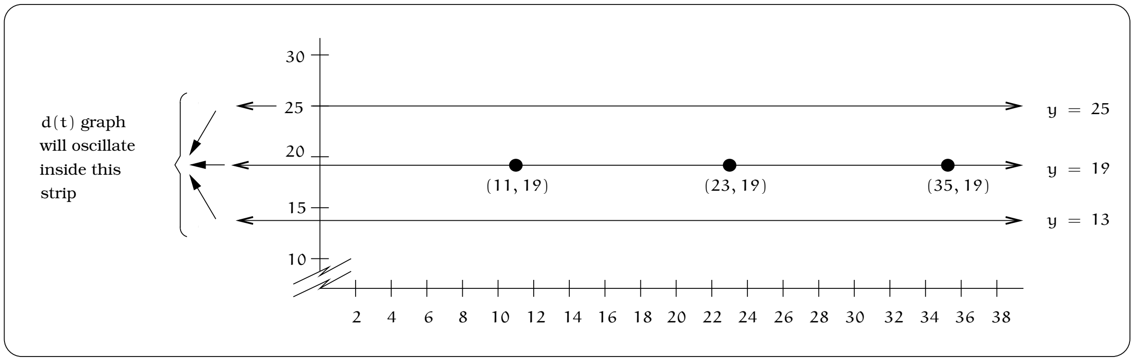


Figure 19.8: Sketching the mean D and amplitude A.

According to the fifth step in the sketching procedure, we can plot the maxima $(C+\frac{1}{4}B, D+A) = (17, 25)$ and the minima $(C+\frac{3}{4}B, D-A) = (29, 13)$. We then "connect the dots" to get a rough sketch on the domain $11 \le t \le 35$.

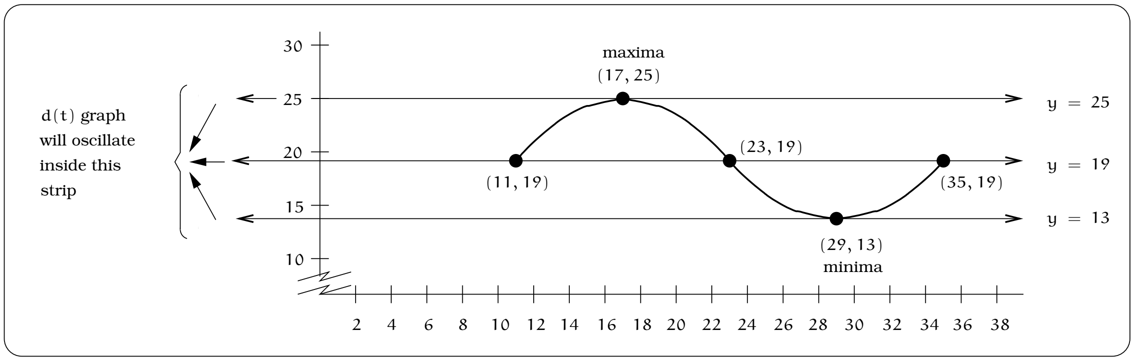


Figure 19.9: Visualizing the maximum and minimum over one period.

Finally, we can use the fact the function has period 24 to sketch the graph to the right and left by simply repeating the picture every 24 horizontal units.

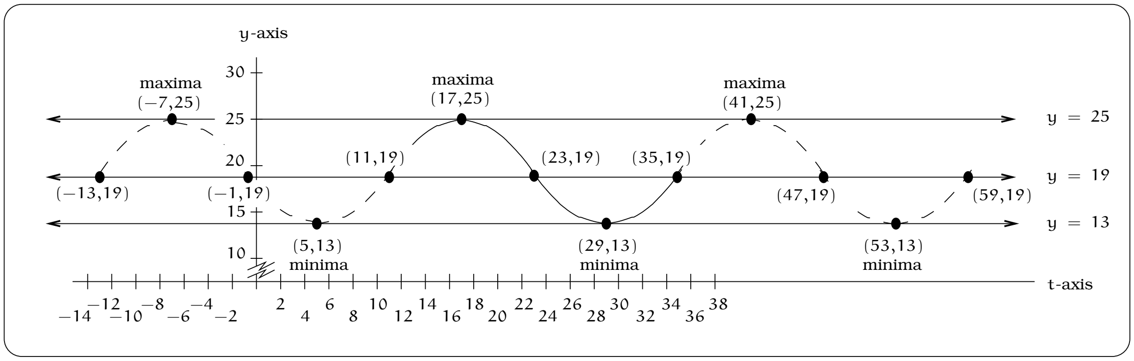


Figure 19.10: Repeat sketch for every full period.

We restrict the picture to the domain $0 \leq t \leq 24$ and obtain the computer generated graph pictured in Figure 19.11; as you can see, our rough graph is very accurate. The temperature at 2:00 p.m. is just $d(14) = 23.24°$ C. From the graph, the maximum value of the function will be $D + A = 25°$ C and the minimum value will be $D - A = 13°$ C.

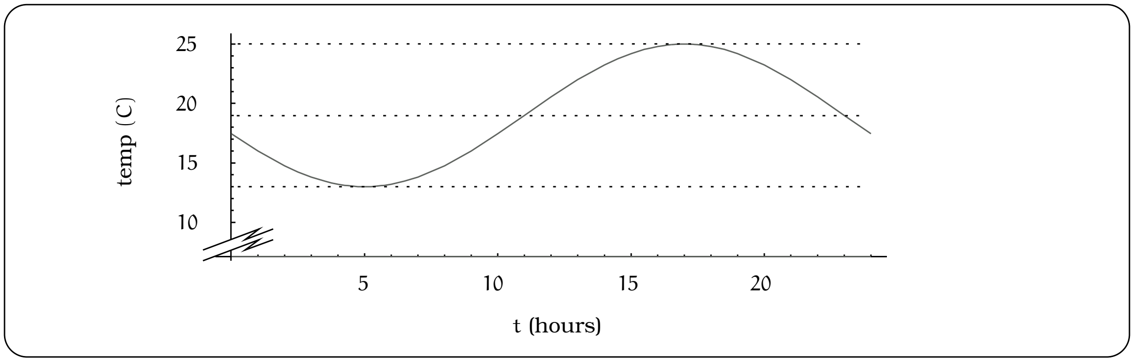


Figure 19.11: The computer generated solution.

□

19.1.2 Functions not in standard sinusoidal form

Any time we are given a trigonometric function written in the standard form

$$y = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

for constants A, B, C, and D (with A and B positive), the summary in Definition 19.1.1 tells us everything we could possibly want to know about the graph. But, there are two ways in which we might encounter a trigonometric type function that is not in this standard form:

- The constants A or B might be negative. For example, $y = -2\sin(2x-7) - 3$ and $y = 3\sin\left(-\frac{1}{2}x + 1\right) + 4$ are examples that fail to be in standard form.
- We might use the cosine function in place of the sine function. For example, something like $y = 2\cos(3x+1) - 2$ fails to be in standard sinusoidal form.

Now what do we do? Does this mean we need to repeat the analysis that led to Definition 19.1.1? It turns out that if we use our trig identities just right, then we can move any such equation into standard form and read off the amplitude, period, phase shift and mean. In other words, equations that fail to be in standard sinusoidal form for either of these two reasons will still define sinusoidal functions. We illustrate how this is done by way of some examples:

Examples 19.1.4.

(i) *Start with* $y = -2\sin(2x-7)-3$, *then here are the steps with reference to the required identities to put the equation in standard form:*

$$\begin{aligned} y &= -2\sin(2x-7) - 3 \\ &= 2\left(-\sin(2x-7)\right) - 3 \\ &= 2\sin(2x - 7 + \pi) + (-3) && \textit{Fact 18.2.5 on page 241} \\ &= 2\sin\left(\frac{2\pi}{\pi}\left(x - \left[\frac{7-\pi}{2}\right]\right)\right) + (-3). \end{aligned}$$

This function is now in the standard form of Definition 19.1.1, so it is a sinusoidal function with phase shift $C = \frac{7-\pi}{2} = 1.93$, *mean* $D = -3$, *amplitude* $A = 2$ *and period* $B = \pi$.

(ii) *Start with* $y = 3\sin(-\frac{1}{2}x+1)+4$, *then here are the steps with reference to the required identities to put the equation in standard form:*

$$\begin{aligned} y &= 3\sin\left(-\frac{1}{2}x + 1\right) + 4 \\ &= 3\sin\left(-\left(\frac{1}{2}x - 1\right)\right) + 4 \\ &= 3\left(-\sin\left(\frac{1}{2}x - 1\right)\right) + 4 && \textit{Fact 18.2.4 on page 241} \\ &= 3\sin\left(\frac{1}{2}x - 1 + \pi\right) + 4 && \textit{Fact 18.2.5 on page 241} \\ &= 3\sin\left(\frac{2\pi}{4\pi}(x - [2 - 2\pi])\right) + 4 \end{aligned}$$

This function is now in the standard form of Definition 19.1.1, so it is a sinusoidal function with phase shift $C = 2 - 2\pi$, *mean* $D = 4$, *amplitude* $A = 3$ *and period* $B = 4\pi$.

(iii) Start with $y = 2\cos(3x+1)-2$, *then here are the steps to put the equation in standard form. A key simplifying step is to use the identity:* $\cos(t) = \sin(\frac{\pi}{2} + t)$.

$$\begin{aligned} y &= 2\cos(3x+1) - 2 \\ &= 2\sin\left(\frac{\pi}{2} + 3x + 1\right) - 2 \\ &= 2\sin\left(3x - \left[-1 - \frac{\pi}{2}\right]\right) + (-2) \\ &= 2\sin\left(\frac{2\pi}{\left(\frac{2\pi}{3}\right)}\left(x - \frac{1}{3}\left[-1 - \frac{\pi}{2}\right]\right)\right) + (-2) \end{aligned}$$

This function is now in the standard form of Definition 19.1.1, so it is a sinusoidal function with phase shift $C = \frac{1}{3}[-1 - \frac{\pi}{2}]$, *mean* $D = -2$, *amplitude* $A = 2$ *and period* $B = \frac{2\pi}{3}$.

19.2 Examples of sinusoidal behavior

Problems involving sinusoidal behavior come in two basic flavors. On the one hand, we could be handed an explicit sinusoidal function

$$y = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D$$

and asked various questions. The answers typically require either direct calculation or interpretation of the constants. Example 19.1.3 is typical of this kind of problem. On the other hand, we might be told a particular situation is described by a sinusoidal function and provided some data or a graph. In order to further analyze the problem, we need a "formula", which means finding the constants A, B, C, and D. This is a typical scenario in a "mathematical modeling problem": the process of observing data, THEN obtaining a mathematical formula. To find A, take half the difference between the largest and smallest values of $f(x)$. The period B is most easily found by measuring the distance between two successive maxima (peaks) or minima (valleys) in the graph. The mean D is the average of the largest and smallest values of $f(x)$. The phase shift C (which is usually the most tricky quantity to get your hands on) is found by locating a "reference point". This "reference point" is a location where the graph crosses the mean line $y = D$ *on its way up* from a minimum to a maximum. The funny thing is that the phase shift C is NOT unique; there are an infinite number of correct choices. One choice that will work is $C = (x\text{-coordinate of a maximum}) - \frac{B}{4}$. Any other choice of C will differ from this one by a multiple of the period B.

$$\begin{aligned} A &= \frac{\text{max value } - \text{min value}}{2} \\ B &= \text{distance between two successive peaks (or valleys)} \\ C &= x\text{-coordinate of a maximum} - \frac{B}{4} \\ D &= \frac{\text{max value } + \text{min value}}{2}. \end{aligned}$$

Example 19.2.1. *Assume that the number of hours of daylight in Seattle is given by a sinusoidal function* $d(t)$ *of time. During 1994, assume the longest day of the year is June 21 with 15.7 hours of daylight and the shortest day is December 21 with 8.3 hours of daylight. Find a formula* $d(t)$ *for the number of hours of daylight on the* t^{th} *day of the year.*

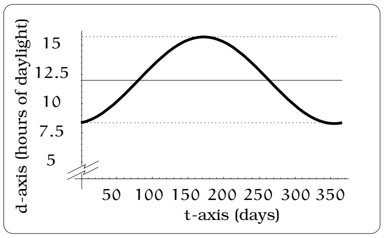


Figure 19.12: Hours of daylight in Seattle in 1994.

Solution. Because the function $d(t)$ is assumed to be sinusoidal, it has the form $y = A\sin\left(\frac{2\pi}{B}(t - C)\right) + D$, for constants A, B, C, and D. We simply need to use the given information to find these constants. The largest value of the function is 15.7 and the smallest value is 8.3. Knowing this, from the above discussion we can read off :

$$D = \frac{15.7 + 8.3}{2} = 12 \qquad A = \frac{15.7 - 8.3}{2} = 3.7.$$

To find the period, we need to compute the time between two successive maximum values of $d(t)$. To find this, we can simply double the time length of one-half period, which would be the length of time between successive maximum and minimum values of $d(t)$. This gives us the equation

$$B = 2(\text{days between June 21 and December 21}) = 2(183) = 366.$$

Locating the final constant C requires the most thought. Recall, the longest day of the year is June 21, which is day 172 of the year, so

$$C = (\text{day with max daylight}) - \frac{B}{4} = 172 - \frac{366}{4} = 80.5.$$

In summary, this shows that

$$d(t) = 3.7\sin\left(\frac{2\pi}{366}(t - 80.5)\right) + 12.$$

A rough sketch, following the procedure outlined above, gives this graph on the domain $0 \leq t \leq 366$; we have included the mean line $y = 12$ for reference. □

We close with the example that started this section.

Example 19.2.2. *The depth of a migrating salmon below the water surface changes according to a sinusoidal function of time. The fish varies between 1 and 5 feet below the surface of the water. It takes the fish 1.571 minutes to move from its minimum depth to its successive maximum depth. It is located at a maximum depth when* $t = 4.285$ *minutes. What is the formula for the function* $d(t)$ *that predicts the depth of the fish after* t *minutes? What was the depth of the salmon when it was first spotted? During the first 10 minutes, how many times will the salmon be exactly 4 feet below the surface of the water?*

Solution. We know that $d(t) = A\sin(\frac{2\pi}{B}(x - C)) + D$, for appropriate constants A, B, C, and D. We need to use the given information to extract these four constants. The amplitude and mean are easily found using the above formulas:

$$A = \frac{\text{max depth } - \text{min depth}}{2} = \frac{5-1}{2} = 2$$
$$D = \frac{\text{max depth } + \text{min depth}}{2} = \frac{5+1}{2} = 3.$$

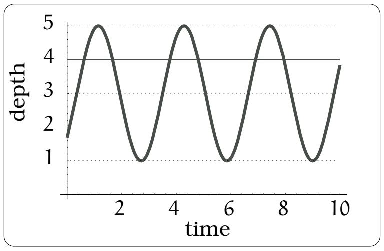


Figure 19.13: Depth of a migrating salmon.

The period can be found by noting that the information about the time between a successive minimum and maximum depth will be half of a period (look at the picture in Figure 19.13):

$$B = 2(1.571) = 3.142$$

Finally, to find C we

$$C = (\text{time of maximum depth}) - \frac{B}{4} = 4.285 - \frac{3.142}{4} = 3.50.$$

The formula is now

$$d(t) = 2\sin\left(\frac{2\pi}{3.142}(t - 3.5)\right) + 3 = 2\sin(2t - 7) + 3$$

The depth of the salmon when it was first spotted is just

$$d(0) = 2\sin(-7) + 3 = 1.686\,\text{feet}.$$

Finally, graphically, the last question amounts to determining how many times the graph of $d(t)$ crosses the line $y = 4$ on the domain [0,10]. This can be done using Figure 19.13. A simultaneous picture of the two graphs is given, from which we can see the salmon is exactly 4 feet below the surface of the water six times during the first 10 minutes.

$\square$

19.3 Summary

- A **sinusoidal** function is one of the form

  $$f(t) = A\sin\left(\frac{2\pi}{B}(t-C)\right) + D$$

  where A, B, C, and D are constants.

  - A is the *amplitude* of the function; this is half the vertical distance between a high point and a low point on its graph.
  - B is the *period* of the function; this is the horizontal distance between two consecutive high points (or low points) on its graph.
  - C is the *phase shift* of the function; it is multi-valued, but one choice for C is a value of t at which the function is increasing and equal to D.
  - D is the *mean value* of the function; it is the y-value of the horizontal line about which the graph of the function is balanced.

- The graph of a sinusoidal function is a shifted, scaled version of the graph of $y = \sin t$.

19.4 Exercises

Problem 19.1. Find the amplitude, period, a phase shift and the mean of the following sinusoidal functions.

(a) $y = \sin(2x - \pi) + 1$

(b) $y = 6\sin(\pi x) - 1$

(c) $y = 3\sin(x + 2.7) + 5.2$

(d) $y = 5.6\left(\sin\left(\frac{2}{3}x - 7\right) - 12.1\right)$

(e) $y = 2.1\sin\left(\frac{x}{\pi} + 44.3\right) - 9.8$

(f) $y = 3.9\left(\sin(22.34(x + 18)) - 11\right)$

(g) $y = 11.2\sin\left(\frac{5}{\pi}(x - 9.2)\right) + 8.3$

Problem 19.2. A weight is attached to a spring suspended from a beam. At time $t = 0$, it is pulled down to a point 10 cm above the ground and released. After that, it bounces up and down between its minimum height of 10 cm and a maximum height of 26 cm, and its height $h(t)$ is a sinusoidal function of time t. It first reaches a maximum height 0.6 seconds after starting.

(a) Follow the procedure outlined in this section to sketch a rough graph of $h(t)$. Draw at least two complete cycles of the oscillation, indicating where the maxima and minima occur.

(b) What are the mean, amplitude, phase shift and period for this function?

(c) Give four different possible values for the phase shift.

(d) Write down a formula for the function $h(t)$ in standard sinusoidal form; i.e. as in 19.1.1 on Page 254.

(e) What is the height of the weight after 0.18 seconds?

(f) During the first 10 seconds, how many times will the weight be exactly 22 cm above the floor? (Note: This problem does not require inverse trigonometry.)

Problem 19.3. A respiratory ailment called "Cheyne-Stokes Respiration" causes the volume per breath to increase and decrease in a sinusoidal manner, as a function of time. For one particular patient with this condition, a machine begins recording a plot of volume per breath versus time (in seconds). Let $b(t)$ be a function of time t that tells us the volume (in liters) of a breath that starts at time t. During the test, the smallest volume per breath is 0.6 liters and this first occurs for a breath that starts 5 seconds into the test. The largest volume per breath is 1.8 liters and this first occurs for a breath beginning 55 seconds into the test.

(a) Find a formula for the function $b(t)$ whose graph will model the test data for this patient.

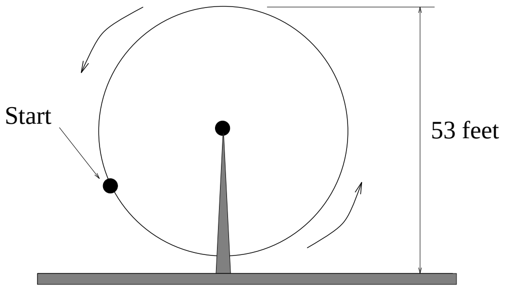
(b) If the patient begins a breath every 5 seconds, what are the breath volumes during the first minute of the test?

Problem 19.4. Suppose the high tide in Seattle occurs at 1:00 a.m. and 1:00 p.m. at which time the water is 10 feet above the height of low tide. Low tides occur 6 hours after high tides. Suppose there are two high tides and two low tides every day and the height of the tide varies sinusoidally.

(a) Find a formula for the function $y = h(t)$ that computes the height of the tide above low tide at time t. (In other words, $y = 0$ corresponds to low tide.)

(b) What is the tide height at 11:00 a.m.?

Problem 19.5. Your seat on a Ferris Wheel is at the indicated position at time $t = 0$.



Let t be the number of seconds elapsed after the wheel begins rotating counterclockwise. You find it takes 3 seconds to reach the top, which is 53 feet above the ground. The wheel is rotating 12 RPM and the diameter of the wheel is 50 feet. Let $d(t)$ be your height above the ground at time t.

(a) Argue that $d(t)$ is a sinusoidal function, describing the amplitude, phase shift, period and mean.

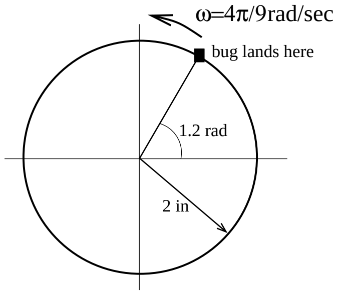
(b) When are the first and second times you are exactly 28 feet above the ground?

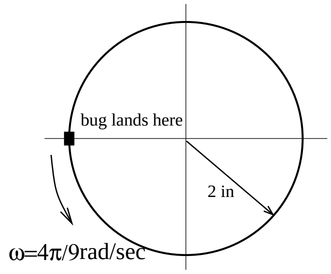
(c) After 29 seconds, how many times will you have been exactly 28 feet above the ground?

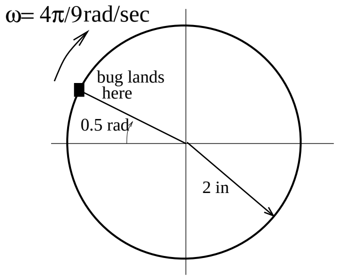
Problem 19.6. In Exercise 17.12, we studied the situation below: A bug has landed on the rim of a jelly jar and is moving around the rim. The location where the bug initially lands is described and its angular speed is given. Impose a coordinate system with the origin at the center of the circle of motion. In each of the cases, the earlier exercise found the coordinates $P(t)$ of the bug at time t. For each of the scenarios below, answer these two questions:

(a) Both coordinates of $P(t) = (x(t),y(t))$ are sinusoidal functions in the variable t. Sketch a rough graph of the functions $x(t)$ and $y(t)$ on the domain $0 \leq t \leq 9$.

(b) Use the graph sketches to help you find the the amplitude, mean, period and phase shift for each function. Write $x(t)$ and $y(t)$ in standard sinusoidal form.







Problem 19.7. The voltage output(in volts) of an electrical circuit at time t seconds is given by the function

$$V(t) = 2^{3\sin(5\pi t - 3\pi)+1}.$$

(a) What is the initial voltage output of the circuit?

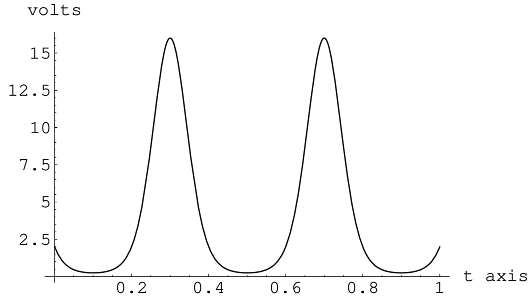
(b) Is the voltage output of the circuit ever equal to zero? Explain.

(c) The function $V(t) = 2^{p(t)}$, where $p(t) = 3\sin(5\pi t - 3\pi) + 1$. Put the sinusoidal function $p(t)$ in standard form and sketch the graph for $0 \leq t \leq 1$. Label the coordinates of the extrema on the graph.

(d) Calculate the maximum and minimum voltage output of the circuit.

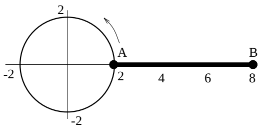
(e) During the first second, determine when the voltage output of the circuit is 10 volts.

(f) A picture of the graph of $y = V(t)$ on the domain $0 \le t \le 1$ is given; label the coordinates of the extrema on the graph.

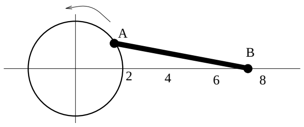


(g) Restrict the function $V(t)$ to the domain $0.1 \le t \le 0.3$; explain why this function has an inverse and find the formula for the inverse rule. Restrict the function $V(t)$ to the domain $0.3 \le t \le 0.5$; explain why this function has an inverse and find the formula for the inverse rule.

Problem 19.8. A six foot long rod is attached at one end A to a point on a wheel of radius 2 feet, centered at the origin. The other end B is free to move back and forth along the x-axis. The point A is at $(2,0)$ at time $t = 0$, and the wheel rotates counterclockwise at 3 rev/sec.



time t=0



time t > 0

(a) As the point A makes one complete revolution, indicate in the picture the direction and range of motion of the point B.

(b) Find the coordinates of the point A as a function of time t.

(c) Find the coordinates of the point B as a function of time t.

(d) What is the x-coordinate of the point B when $t = 1$? You should be able to find this two ways: with your function from part (c), and using some common sense (where is point A after one second?).

(e) Is the function you found in (c) a sinusoidal function? Explain.

Chapter 20

Inverse Circular Functions

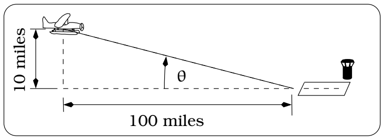


Figure 20.1: An aircraft decending toward an airport.

An aircraft is flying at an altitude 10 *miles above the elevation of an airport. If the airplane begins a steady descent 100 miles from the airport, what is the angle* θ *of descent?*

The only natural circular function we can use is $z = \tan(\theta)$, leading to the equation:

$$\tan(\theta) = \frac{10}{100} = \frac{1}{10}.$$

The problem is that this equation does not tell us the value of θ. Moreover, none of the equation solving techniques at our disposal (which all amount to algebraic manipulations) will help us solve the equation for θ. What we need is an *inverse function* $\theta = f^{-1}(z)$; then we could use the fact that $\tan^{-1}(\tan(\theta)) = \theta$ and obtain:

$$\theta = \tan^{-1}(\tan(\theta)) = \tan^{-1}\left(\frac{1}{10}\right).$$

Computationally, without even thinking about what is going on, any scientific calculator will allow us to compute values of an inverse circular function and leads to a solution of our problem. In this example, you will find $\theta = \tan^{-1}\left(\frac{1}{10}\right) = 5.71^\circ$. Punch this into your calculator and verify it!

20.1 Solving Three Equations

Example 20.1.1. *Find all values of* θ *(an angle) that make this equation true:* $\sin(\theta) = \frac{1}{2}$.

Solution. We begin with a graphical reinterpretation: the solutions correspond to the places where the graphs of $z = \sin(\theta)$ and $z = \frac{1}{2}$ intersect in the θz-coordinate system. Recalling Figure 18.11, we can picture these two graphs simultaneously as below:

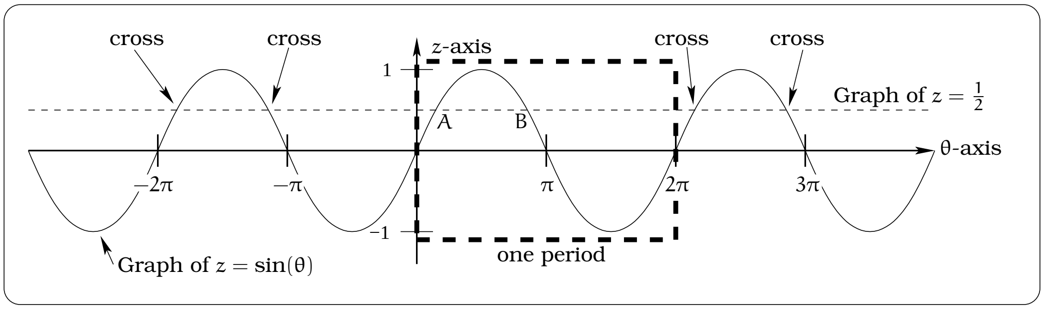


Figure 20.2: Where does $\sin(\theta)$ cross $z = \frac{1}{2}$?

The first thing to notice is that these two graphs will cross an infinite number of times, so there are infinitely many solutions to Example 20.1.1! However, notice there is a predictable spacing of the crossing points, which is just a manifestation of the periodicity of the sine function. In fact, if we can find the two crossing points labeled "A" and "B", then all other crossing points are obtained by adding multiples of 2π to either "A" or "B". By Table 17.1, $\theta = \frac{\pi}{6}$ radians is a special angle where we computed $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, which tells us that the crossing point labeled "A" is the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$. Using the identities in Facts 18.2.4 and 18.2.5, notice that

$$\begin{aligned}
\sin\left(\frac{5\pi}{6}\right) &= \sin\left(-\frac{\pi}{6} + \pi\right) \\
&= -\sin\left(-\frac{\pi}{6}\right) \\
&= -\left(-\sin\left(\frac{\pi}{6}\right)\right) \\
&= \sin\left(\frac{\pi}{6}\right) \\
&= \frac{1}{2}.
\end{aligned}$$

So, $\theta = \frac{5\pi}{6}$ is the only other angle θ between 0 and 2π such that Example 20.1.1 holds. This corresponds to the crossing point labeled "B", which has coordinates $\left(\frac{5\pi}{6}, \frac{1}{2}\right)$. In view of the remarks above, the crossing points come in two flavors:

$$\left(\frac{\pi}{6} + 2k\pi, \frac{1}{2}\right), \quad k = 0, \pm 1, \pm 2, \pm 3, \cdots, \text{ and}$$

$$\left(\frac{5\pi}{6} + 2k\pi, \frac{1}{2}\right), \quad k = 0, \pm 1, \pm 2, \pm 3, \cdots.$$

□

Taking this example as a model, we can tackle the more general problem: For a fixed real number c, describe the solution(s) of the equation

$c = f(\theta)$ for each of the circular functions $z = f(\theta)$. Studying solutions of these equations will force us to come to grips with three important issues:

- For what values of c does $f(\theta) = c$ have a solution?
- For a given value of c, how many solutions does $f(\theta) = c$ have?
- Can we restrict the domain so that the resulting function is one-to-one?

All of these questions must be answered before we can come to grips with any understanding of the inverse functions. Using the graphs of the circular functions, it is an easy matter to arrive at the following qualitative conclusions.

Important Fact 20.1.2. *None of the circular functions is one-to-one on the domain of all θ values. The equations $c = \sin(\theta)$ and $c = \cos(\theta)$ have a solution if and only if $-1 \leq c \leq 1$; if c is in this range, there are infinitely many solutions. The equation $c = \tan(\theta)$ has a solution for any value of c and there are infinitely many solutions.*

Example 20.1.3. *If two sides of a right triangle have lengths 1 and $\sqrt{3}$ as pictured below, what are the acute angles α and β?*

Solution. By the Pythagorean Theorem the remaining side has length

$$\sqrt{1+\left(\sqrt{3}\right)^2} = 2.$$

Since $\tan(\alpha) = \sqrt{3}$, we need to solve this equation for α. Graphically, we need to determine where $z = \sqrt{3}$ crosses the graph of the tangent function:

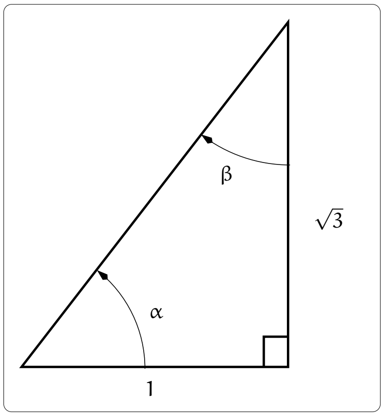


Figure 20.3: What are the values for α and β?

From Fact 20.1.2, there will be infinitely many solutions to our equation, but notice that there is **exactly one solution** in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and we can find it using Table 17.1:

$$\tan\left(\frac{\pi}{3}\right) = \tan(60^\circ) = \frac{\sin(60^\circ)}{\cos(60^\circ)} = \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = \sqrt{3}.$$

So, $\alpha = \frac{\pi}{3}$ radians $= 60^\circ$ is the only acute angle solution and $\beta = 180^\circ - 60^\circ - 90^\circ = 30^\circ$. ☐

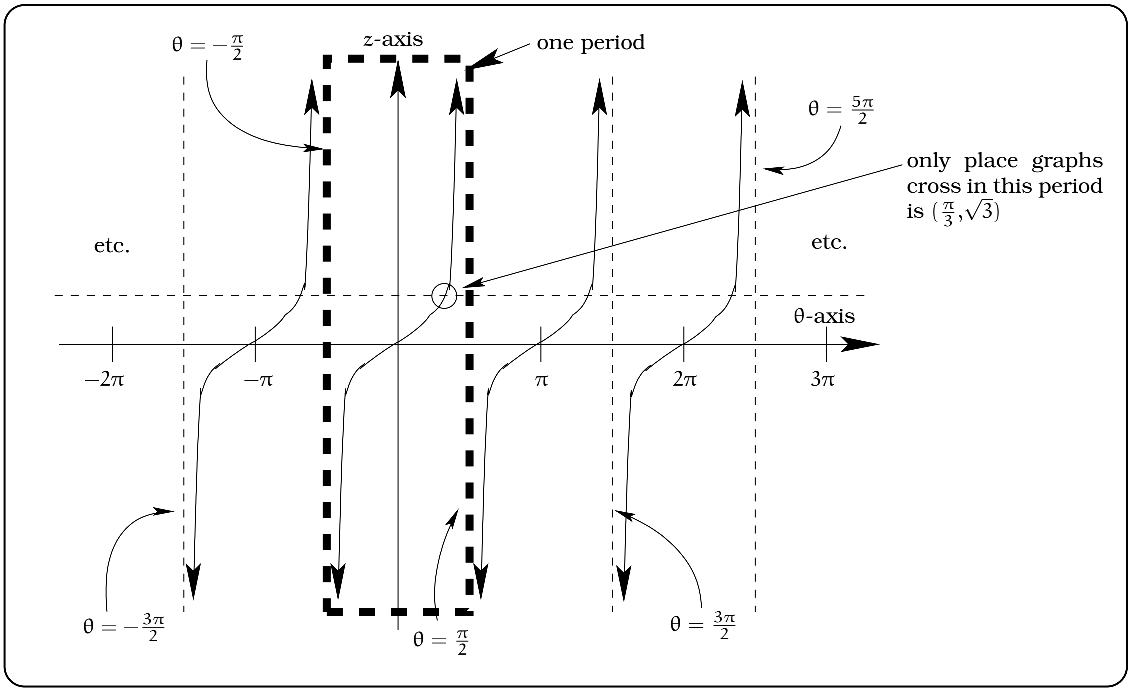


Figure 20.4: Where does the line $z = \sqrt{3}$ cross $z = \tan\theta$?

20.2 Inverse Circular Functions

Except for specially chosen angles, we have not addressed the serious problem of FINDING values of the inverse rules attached to the circular function equations. (Our previous examples were "rigged", so that we could use Table 17.1.) To proceed computationally, we need to obtain the *inverse circular functions*. If we were to proceed in a sloppy manner, then a first attempt at defining the inverse circular functions would be to write

$$\begin{aligned} &✘\ \sin^{-1}(z) = \text{solutions } \theta \text{ of the equation } z = \sin(\theta). \\ &✘\ \cos^{-1}(z) = \text{solutions } \theta \text{ of the equation } z = \cos(\theta). \qquad (20.1) \\ &✘\ \tan^{-1}(z) = \text{solutions } \theta \text{ of the equation } z = \tan(\theta). \end{aligned}$$

There are two main problems with these rules as they stand. First, to have a solution θ in the case of $\sin^{-1}(z)$ and $\cos^{-1}(z)$, we need to restrict z so that $-1 \le z \le 1$. Secondly, having made this restriction on z in the first two cases, there is no unique solution; rather, there are an infinite number of solutions. This means that the rules $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ as they now stand do **not** define functions. Given what we have reviewed about inverse functions, the only way to proceed is to restrict each circular function to a domain of θ values on which it becomes one-to-one, then we can appeal to Fact 9.3.1 and conclude the inverse function makes sense.

At this stage a lot of choice (flexibility) enters into determining the domain on which we should try to invert each circular function. In effect, there are an infinite number of possible choices. If $z = f(\theta)$ denotes one of the three circular functions, there are three natural criteria we use to guide the choice of a restricted domain, which we will call a *principal domain*:

- The domain of $f(\theta)$ should include the angles between 0 and $\frac{\pi}{2}$, since these are the possible acute angles in a right triangle.
- On the restricted domain, the function $f(\theta)$ should take on all possible values in the range of $f(\theta)$. In addition, the function should be one-to-one on this restricted domain.
- The function $f(\theta)$ should be "continuous" on this restricted domain; i.e. the graph on this domain could be traced with a pencil, without lifting it off the paper.

In the case of $z = \sin(\theta)$, the principal domain $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ satisfies our criteria and the picture is given below. Notice, we would **not** want to take the interval $0 \leq \theta \leq \pi$, since $z = \sin(\theta)$ doesn't achieve negative values on this domain; in addition, it's not one-to-one there.

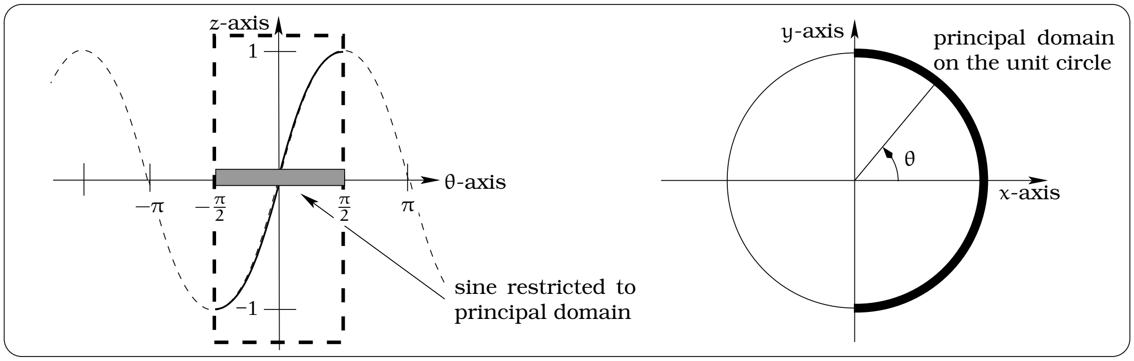

z-axis
1
−1
−π
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
θ-axis
sine restricted to principal domain
y-axis
principal domain on the unit circle
θ
x-axis

Figure 20.5: Principal domain for $\sin(\theta)$.

In the case of $z = \cos(\theta)$, the principal domain $0 \leq \theta \leq \pi$ satisfies our criteria and the picture is given below. Notice, we would **not** want to take the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, since $z = \cos(\theta)$ doesn't achieve negative values on this domain; in addition, it's not one-to-one there.

In the case of $z = \tan(\theta)$, the principal domain $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ satisfies our criteria and the picture is given below. Notice, we would **not** want to take the interval $0 \leq \theta \leq \pi$, since $z = \tan(\theta)$ does not have a continuous graph on this interval; in other words, we do not include the endpoints since $\tan(\theta)$ is undefined there.

Important Facts 20.2.1 (Inverse circular functions)**.** *Restricting each circular function to its principal domain, its inverse rule* $f^{-1}(z) = \theta$ *will define a function.*

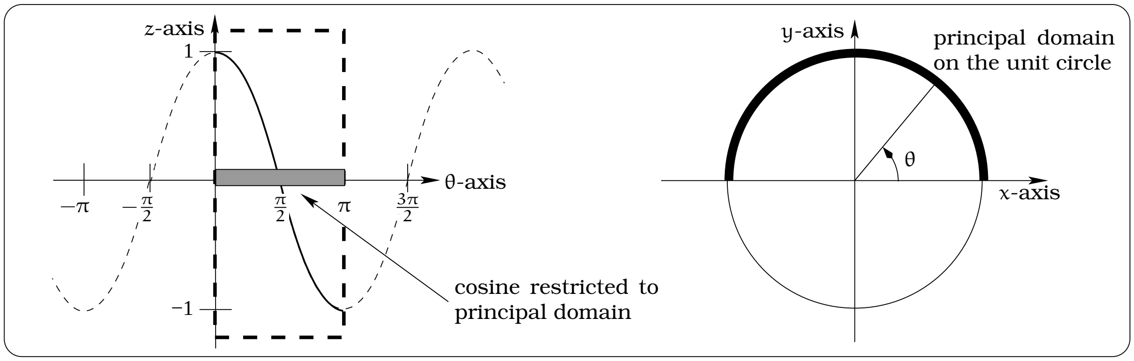

z-axis
1
−1
−π
−π/2
π/2
π
3π/2
θ-axis
cosine restricted to principal domain
y-axis
principal domain on the unit circle
θ
x-axis

Figure 20.6: Principal domain for $\cos(\theta)$.

(i) *If* $-1 \leq z \leq 1$*, then* $\sin^{-1}(z)$ *is the unique angle* θ *in the principal domain* $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ *with the property that* $\sin(\theta) = z$*.*

(ii) *If* $-1 \leq z \leq 1$*, then* $\cos^{-1}(z)$ *is the unique angle* θ *in the principal domain* $0 \leq \theta \leq \pi$ *with the property that* $\cos(\theta) = z$*.*

(iii) *For any real number* z*,* $\tan^{-1}(z)$ *is the unique angle* θ *in the principal domain* $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ *with the property that* $\tan(\theta) = z$*.*

We refer to the functions defined above as the *inverse circular functions*. These are sometimes referred to as the "arcsine", "arccosine" and "arctangent" functions, though we will not use that terminology. The inverse circular functions give us one solution for each of these equations:

$$\begin{aligned} c &= \cos(\theta) \\ c &= \sin(\theta) \\ c &= \tan(\theta); \end{aligned}$$

these are called the *principal solutions*. We also can refer to these as the *principal values* of the inverse circular function rules $\theta = f^{-1}(z)$.

iii CAUTION !!!

As usual, be careful with "radian mode" and "degree mode" when making calculations. For example, if your calculator is in "degree" mode and you type in "$\mathbf{tan}^{-1}(18)$", the answer given is "86.82". This means that an angle of measure $\theta = 86.82^\circ$ has $\mathbf{tan}(86.82) = 18$. If your calculator is in "radian" mode and you type in "$\mathbf{sin}^{-1}(0.9)$", the answer given is "1.12". This means that an angle of measure $\theta = 1.12$ radians has $\mathbf{sin}(1.12) = 0.9$.

There is a key property of the inverse circular functions which is useful in equation solving; it is just a direct translation of Fact 9.3.2 into our current context:

Important Facts 20.2.2 (Composition identities)**.** *We have the following equalities involving compositions of circular functions and their inverses:*

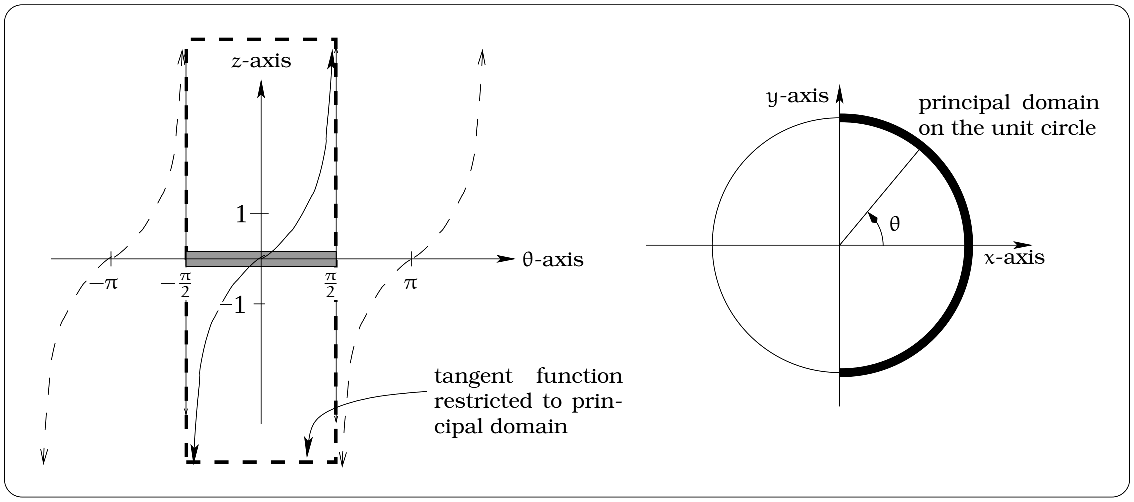

z-axis
1
−1
θ-axis
−π
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
tangent function restricted to principal domain
y-axis
principal domain on the unit circle
θ
x-axis

Figure 20.7: Principal domain for $\tan(\theta)$.

(a) If $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$*, then* $\sin^{-1}(\sin(\theta)) = \theta$.

(b) If $0 \leq \theta \leq \pi$*, then* $\cos^{-1}(\cos(\theta)) = \theta$.

(c) If $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$*, then* $\tan^{-1}(\tan(\theta)) = \theta$.

We have been very explicit about the allowed θ values for the equations in Fact 20.2.2. This is important and an Exercise will touch on this issue.

20.3 Applications

As a simple application of Fact 20.2.2, we can return to the beginning of this section and justify the reasoning used to find the angle of descent of the aircraft:

$$\theta = \tan^{-1}((\tan(\theta)) = \tan^{-1}\left(\frac{1}{10}\right) = 0.09967\,\mathrm{rad} = 5.71^\circ.$$

Let's look at some other applications.

Example 20.3.1. *Find two acute angles* θ *so that the following equation is satisfied:*

$$\frac{9}{4\cos^2(\theta)} = \frac{25^2}{16}\left(1 - \cos^2(\theta)\right).$$

Solution. Begin by multiplying each side of the equation by $\cos^2(\theta)$ and rearranging terms:

$$\begin{aligned} \frac{9}{4} &= \frac{25^2}{16}\left(1 - \cos^2(\theta)\right)\cos^2(\theta) \\ 0 &= \frac{25^2}{16}\cos^4(\theta) - \frac{25^2}{16}\cos^2(\theta) + \frac{9}{4}. \end{aligned}$$

To solve this equation for θ, we use what is called the technique of *substitution*. The central idea is to bring the quadratic formula into the picture by making the substitution $z = \cos^2(\theta)$:

$$0 = \frac{25^2}{16}z^2 - \frac{25^2}{16}z + \frac{9}{4}.$$

Applying the quadratic formula, we obtain

$$z = \frac{\frac{25^2}{16} \pm \sqrt{\left(\frac{25^2}{16}\right)^2 - 4\left(\frac{25^2}{16}\right)\left(\frac{9}{4}\right)}}{\frac{2(25^2)}{16}} = 0.9386 \text{ or } 0.06136.$$

We now use the fact that $z = \cos^2(\theta)$ and note the cosine of an acute angle is non-negative to conclude that

$$\begin{aligned} \cos^2(\theta) &= 0.9386 \quad \text{or} \quad \cos^2(\theta) = 0.06136 \\ \cos(\theta) &= 0.9688 \quad \text{or} \quad \cos(\theta) = 0.2477. \end{aligned}$$

Finally, we use the inverse cosine function to arrive at our two acute angle solutions:

$$\begin{aligned} \cos(\theta) = 0.9688 \quad &\Rightarrow \quad \theta = \cos^{-1}(0.9688) = 14.35^\circ \\ \cos(\theta) = 0.2477 \quad &\Rightarrow \quad \theta = \cos^{-1}(0.2477) = 75.66^\circ \end{aligned}$$

□

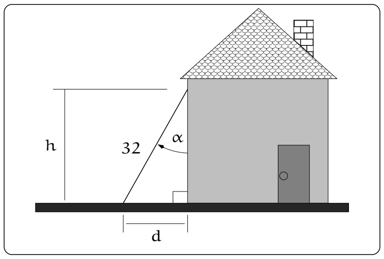
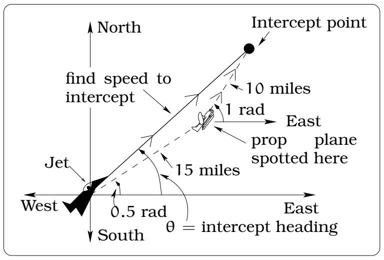


Figure 20.8: A ladder problem.

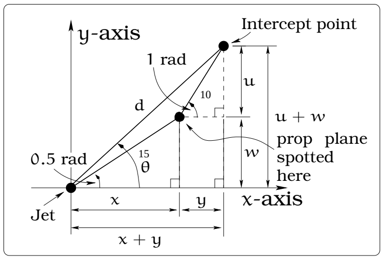
Example 20.3.2. *A 32 ft ladder leans against a building (as shown below) making an angle α with the wall. OSHA (Occupational Safety and Health Administration) specifies a "safety range" for the angle α to be $15^\circ \leq \alpha \leq 44^\circ$. If the base of the ladder is $d = 10$ feet from the house, is this a safe placement? Find the highest and lowest points safely accessible.*

Solution. If $d = 10$, then $\sin(\alpha) = \frac{10}{32}$, so the principal solution is $\alpha = \sin^{-1}\left(\frac{10}{32}\right) = 18.21^\circ$; this lies within the safety zone. From the picture, it is clear that the highest point safely reached will occur precisely when $\alpha = 15^\circ$ and as this angle increases, the height h decreases until we reach the lowest safe height when $\alpha = 44^\circ$. We need to solve two right triangles. If $\alpha = 15^\circ$, then $h = 32\cos(15^\circ) = 30.9\,\text{ft}$. If $\alpha = 44^\circ$, then $h = 32\cos(44^\circ) = 23.02\,\text{ft}$. □

Example 20.3.3. *A Coast Guard jet pilot makes contact with a small unidentified propeller plane 15 miles away at the same altitude in a direction 0.5 radians counterclockwise from East. The prop plane flies in the direction 1.0 radians counterclockwise from East. The jet has been instructed to allow the prop plane to fly 10 miles before intercepting. In what direction should the jet fly to intercept the prop plane? If the prop plane is flying 200 mph, how fast should the jet be flying to intercept?*


North
Intercept point
find speed to intercept
10 miles
1 rad
East
prop plane spotted here
Jet
15 miles
West
0.5 rad
East
South
θ = intercept heading

(a) The physical layout.


y-axis
Intercept point
1 rad
d
10
u
u + w
prop plane spotted here
0.5 rad
15
θ
w
x
y
x-axis
Jet
x + y

(b) Modeling the problem.

Figure 20.9: Visualizing the Coast Guard problem.

Solution. A picture of the situation is shown in Figure 20.9(a). After a look at the picture, three right triangles pop out and beg to be exploited. We highlight these in the Figure 20.9(b), by imposing a coordinate system and labeling the various sides of our triangles. We will work in radian units and label θ to be the required intercept heading.

We will first determine the sides $x + y$ and $u + w$ of the large right triangle. To do this, we have

$$\begin{aligned} x &= 15\cos(0.5) = 13.164\,\text{miles}, \\ y &= 10\cos(1.0) = 5.403\,\text{miles}, \\ w &= 15\sin(0.5) \ = 7.191\,\text{miles, and} \\ u &= 10\sin(1.0) \ = 8.415\,\text{miles}. \end{aligned}$$

We now have $\tan(\theta) = \frac{w+u}{x+y} = 0.8405$, so the principal solution is $\theta = 0.699$ radians, which is about $40.05°$. This is the only acute angle solution, so we have found the required intercept heading.

To find the intercept speed, first compute your distance to the intercept point, which is the length of the hypotenuse of the big right triangle: $d = \sqrt{(18.567)^2 + (15.606)^2} = 24.254$ miles. You need to travel this distance in the same amount of time T it takes the prop plane to travel 10 miles at 200 mph; i.e. $T = \frac{10}{200} = 0.05$ hours. Thus, the intercept speed s is

$$s = \frac{\text{distance traveled}}{\text{time } T \text{ elapsed}} = \frac{24.254}{0.05} = 485\,\text{mph}.$$

□

Later you will use an alternative approach to this problem using *velocity vectors*.

In certain applications, knowledge of the principal solutions for the equations $c = \cos(\theta)$, $c = \sin(\theta)$, and $c = \tan(\theta)$ is not sufficient. Here is a typical example of this, illustrating the reasoning required.

Example 20.3.4. *A rigid 14 ft pole is used to vault. The vaulter leaves and returns to the ground when the tip is 6 feet high, as indicated. What are the angles of the pole with the ground on takeoff and landing?*

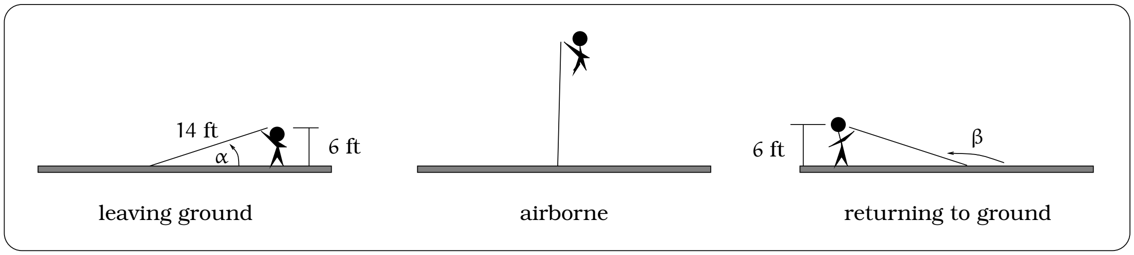


Figure 20.10: Various angles of a vaulter's pole.

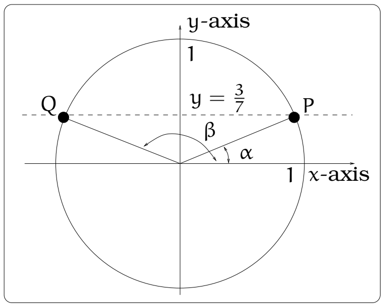


Figure 20.11: Modeling the problem with a unit circle.

Solution. From the obvious right triangles in the picture, we are interested in finding angles θ where $\sin(\theta) = \frac{6}{14} = \frac{3}{7}$. The idea is to proceed in three steps:

- Find the principal solution of the equation $\sin(\theta) = \frac{3}{7}$;
- Find **all** solutions of the equation $\sin(\theta) = \frac{3}{7}$;
- Use the constraints of the problem to find α and β among the set of all solutions.

Solving the equation $\sin(\theta) = \frac{3}{7}$ involves finding the points on the unit circle with y-coordinate equal to $\frac{3}{7}$. From the picture, we see there are two such points, labeled P and Q.

The coordinates of these points will be $P = \left(\cos(\alpha), \frac{3}{7}\right)$ and $Q = \left(\cos(\beta), \frac{3}{7}\right)$. Notice, α is the principal solution of our equation $\sin(\theta) = \frac{3}{7}$, since $0 \leq \alpha \leq 90°$; so $\alpha = \sin^{-1}\left(\frac{3}{7}\right) = 25.38°$. In general, the solutions come in two basic flavors:

$$\begin{aligned} \theta &= \alpha + 2k(180°) \\ &= 25.38° + 2k(180°), \\ &\text{or} \\ \theta &= \beta + 2k(180°), \end{aligned}$$

where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$. To find the angle β, we can use basic properties of the circular functions:

$$\sin(\beta) = \sin(\alpha) = -\sin(-\alpha) = \sin(180° - \alpha) = \sin(154.62°).$$

This tells us $\beta = 154.62°$. □

20.4 How to solve trigonometric equations

So far, our serious use of the inverse trigonometric functions has focused on situations that ultimately involve triangles. However, many trigonometric modeling problems have nothing to do with triangles and so we need to free ourselves from the necessity of relying on such a geometric picture. There are two general strategies for finding solutions to the equations $c = \sin(\theta), c = \cos(\theta)$, and $c = \tan(\theta)$:

- The first strategy is summarized in Procedure 20.4.1. This method has the advantage of offering a "prescription" for solving the equations; the disadvantage is you can lose intuition toward interpreting your answers.

- The second strategy is graphical in nature and is illustrated in Example 20.4.2 below. This method usually clarifies interpretation of the answers, but it does require more work since an essential step is to roughly sketch the graph of the trigonometric function (following the procedure of Chapter 19 or using a graphing device).

Each approach has its merits as you will see in the exercises.

Important Procedure 20.4.1. *To find ALL solutions to the equations* $c = \sin(\theta), c = \cos(\theta)$, *and* $c = \tan(\theta)$, *we can lay out a foolproof strategy.*

Step	***Sine case***	***Cosine case***	***Tangent case***
1. Find principal solution	$\theta = \sin^{-1}(c)$	$\theta = \cos^{-1}(c)$	$\theta = \tan^{-1}(c)$
2. Find symmetry solution	$\theta = -\sin^{-1}(c) + \pi$	$\theta = -\cos^{-1}(c)$	*not applicable*
3. Write out multiples of period $k = 0, \pm 1, \pm 2, \cdots$	$2k\pi$	$2k\pi$	$k\pi$
4. Obtain general principal solutions	$\theta = \sin^{-1}(c) + 2k\pi$	$\theta = \cos^{-1}(c) + 2k\pi$	$\theta = \tan^{-1}(c) + k\pi$
5. Obtain general symmetry solutions	$\theta = -\sin^{-1}(c) + \pi + 2k\pi$	$\theta = -\cos^{-1}(c) + 2k\pi$	*not applicable*

Example 20.4.2. *Assume that the number of hours of daylight in your hometown during 1994 is given by the function* $d(t) = 3.7 \sin\left(\frac{2\pi}{366}(t - 80.5)\right) + 12$, *where* t *represents the day of the year. Find the days of the year during which there will be approximately 14 hours of daylight?*[1]

Solution. To begin, we want to roughly sketch the graph of $y = d(t)$ on the domain $0 \leq t \leq 366$. If you apply the graphing procedure discussed in Chapter 19, you obtain the sinusoidal graph below on the larger domain $-366 \leq t \leq 732$. (The reason we use a larger domain is so that the "pattern" that will arise in the strategy described below is more evident. Ultimately, we will restrict our attention to the smaller domain $0 \leq t \leq 366$.) To determine when there will be 14 hours of daylight, we need to solve the equation $14 = d(t)$. Graphically, this amounts to finding

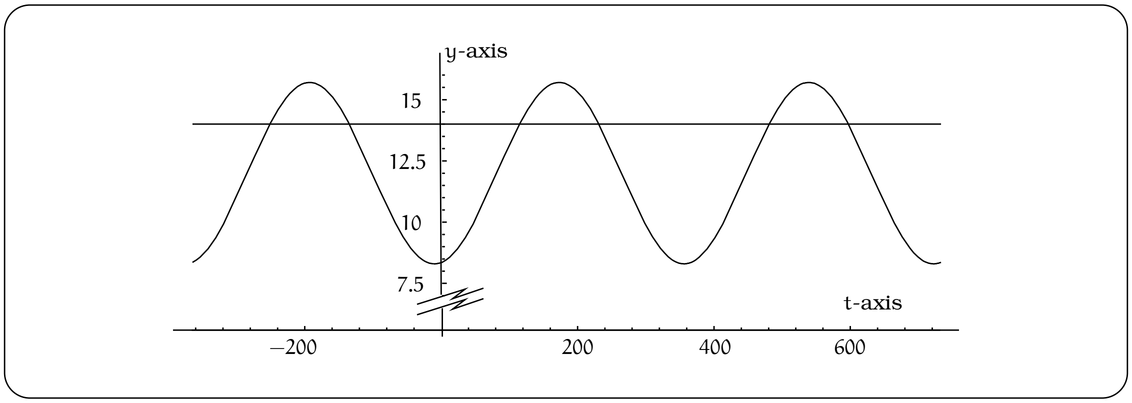


Figure 20.12: Where does the sinusoidal function $d(t)$ cross the line $y = 14$.

the places where the line $y = 14$ intersects the graph of $d(t)$. As can be seen in Figure 20.12, there are several such intersection points.

We now outline a "three step strategy" to find all of these intersection points (which amounts to solving the equation $14 = d(t)$):

1. **Principal Solution.** We will find one solution by using the inverse sine function. If we start with the function $y = \sin(t)$ on its principal domain $\frac{-\pi}{2} \leq t \leq \frac{\pi}{2}$, then we can compute the domain of $d(t) = 3.7\sin\left(\frac{2\pi}{366}(t - 80.5)\right) + 12$:

   $$\begin{aligned} -\frac{\pi}{2} &\leq \frac{2\pi}{366}(t - 80.5) \leq \frac{\pi}{2} \\ -11 &\leq t \leq 172. \end{aligned}$$

   Now, using the inverse sine function we can find the principal solution to the equation $14 = d(t)$:

   $$\begin{aligned} 14 &= 3.7\sin\left(\frac{2\pi}{366}(t - 80.5)\right) + 12 \qquad (20.2) \\ 0.54054 &= \sin\left(\frac{2\pi}{366}(t - 80.5)\right) \\ 0.57108 &= \sin^{-1}(0.54054) = \frac{\pi}{183}(t - 80.5) \\ t &= 113.8 \end{aligned}$$

   Notice, this answer is in the domain $-11 \leq t \leq 172$. In effect, we have found THE ONLY SOLUTION on this domain. Conclude that there will be about 14 hours of daylight on the 114th day of the year.

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[1] You can get the actual data from the naval observatory at this world wide web address: <http://tycho.usno.navy.mil/time.html>

2. **Symmetry Solution.** To find another solution to the equation $14 = d(t)$, we will use symmetry properties of the graph of $y = d(t)$. This is where having the graph of $y = d(t)$ is most useful. We know the a maxima on the graph occurs at the point $M = (172, 15.7)$; review Example 19.2.1 for a discussion of why this is the case. From the graph, we can see there are two symmetrically located intersection points on either side of M. The principal solution gives the intersection point $(113.8, 14)$. This point is 58.2 horizontal units to the left of M; see the picture below. So a symmetrically positioned intersection point will be $(172 + 58.2, 14) = (230.2, 14)$.

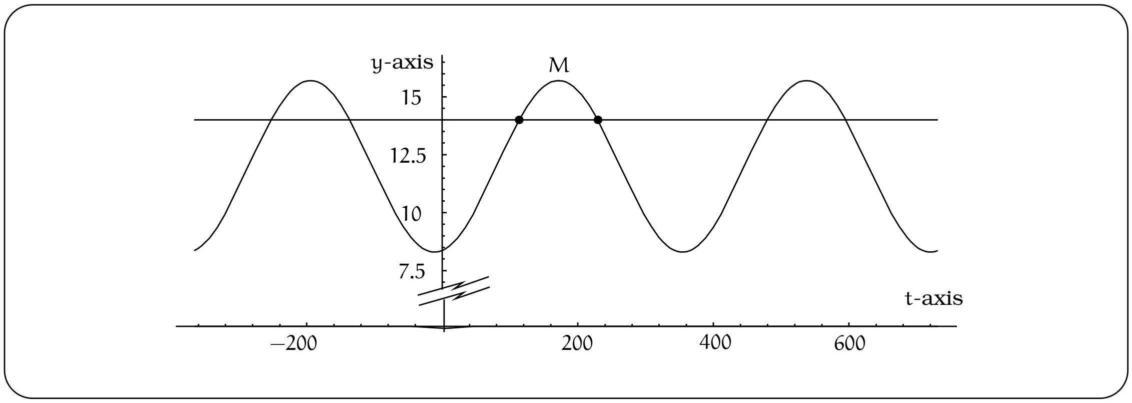


Figure 20.13: Finding the symetry solution.

In other words, $t = 230.2$ is a second solution to the equation $14 = d(t)$. We call this the symmetry solution.

3. **Other Solutions.** To find all other solutions of $14 = d(t)$, we add integer multiples of the period $B = 366$ to the t-coordinates of the principal and symmetric intersection points. On the domain $-366 \leq t \leq 732$ we get these six intersection points; refer to the picture of the graph:

$$(-252.2, 14),\ (113.8, 14),\ (479.8, 14),$$
$$(-135.8, 14),\ (230.2, 14),\ (596.2, 14).$$

So, on the domain $-366 \leq t \leq 732$ we have these six solutions to the equation $14 = d(t)$:

$$t = -252.2, -135.8, 113.8, 230.2, 479.8, 596.2.$$

To conclude the problem, we only are interested in solutions in the domain $0 \leq t \leq 366$, so the answers are $t = 113.8, 230.2$; i.e. on days 114 and 230 there will be about 14 hours of daylight. □

20.5 Summary

- The *inverse sine function*, $\sin^{-1} x$, is defined as the inverse of the sine function, $\sin x$, restricted to the domain $-\pi/2 \leq x \leq \pi/2$.
- Inverse cosine and inverse tangent are defined similarly.
- To find the solutions to an equation of the form

  $$f(t) = A \sin\left(\frac{2\pi}{B}(t - C)\right) + D = k,$$

  proceed as follows:

  1. Sketch the graph of the function $f(t)$, including a few periods and $t = C$.
  2. Use algebra and the inverse sine function ($\sin^{-1}$) to find *one* solution. This is the *principal* solution; it is the solution nearest to $t = C$. Call this solution P.
  3. Use the principal solution and your knowledge of the graph of the function to find the *symmetry* solution. Call this solution S.
  4. All solutions are then of the form

     $$P, P \pm B, P \pm 2B, P \pm 3B, \ldots$$

     or

     $$S, S \pm B, S \pm 2B, S \pm 3B, \ldots.$$

20.6 Exercises

Problem 20.1. Let's make sure we can handle the symbolic and mechanical aspects of working with the inverse trigonometric functions:

(a) Set your calculator to "radian mode" and compute to four decimal places:

  (a1) $\sin^{-1}(x)$, for $x = 0, 1, -1, \frac{\sqrt{3}}{2}, 0.657, \frac{-3}{11}, 2$.

  (a2) $\cos^{-1}(x)$, for $x = 0, 1, -1, \frac{\sqrt{3}}{2}, 0.657, \frac{-3}{11}, 2$.

  (a3) $\tan^{-1}(x)$, for $x = 0, 1, -1, \frac{\sqrt{3}}{2}, 0.657, \frac{-3}{11}, 2$.

(b) Redo part (a) with your calculator set in "degree mode".

(c) Find four values of x that satisfy the equation $5\sin(2x^2 + x - 1) = 2$.

(d) Find four solutions to the equation $5\tan(2x^2 + x - 1) = 2$

Problem 20.2. For each part of the problem below:

- Sketch the graphs of $f(x)$ and $g(x)$ on the same set of axes.
- Set $f(x) = g(x)$ and find the principal and symmetry solutions.
- Calculate at least two other solutions to $f(x) = g(x)$. Indicate them on your graph.

(a) $f(x) = \sin\left(x - \frac{\pi}{2}\right)$, $g(x) = \frac{1}{3}$.

(b) $f(x) = \sin\left(x + \frac{\pi}{6}\right)$, $g(x) = -1$.

(c) $f(x) = \sin(2x - 1)$, $g(x) = \frac{1}{4}$.

(d) $f(x) = 10\cos(2x + 1) - 5$, $g(x) = -1$.

Problem 20.3. Assume that the number of hours of daylight in New Orleans in 1994 is given by the function $D(x) = \frac{7}{3}\sin\left(\frac{2\pi}{365}x\right) + \frac{35}{3}$, where x represents the number of days after March 21.

(a) Find the number of hours of daylight on January 1, May 18 and October 5.

(b) On what days of the year will there be approximately 10 hours of daylight?

Problem 20.4. Hugo bakes world famous scones. The key to his success is a special oven whose temperature varies according to a sinusoidal function; assume the temperature (in degrees Fahrenheit) of the oven t minutes after inserting the scones is given by

$$y = s(t) = 15\sin\left(\frac{\pi}{5}t - \frac{3\pi}{2}\right) + 415$$

(a) Find the amplitude, phase shift, period and mean for $s(t)$, then sketch the graph on the domain $0 \leq t \leq 20$ minutes.

(b) What is the maximum temperature of the oven? Give all times when the oven achieves this maximum temperature during the first 20 minutes.

(c) What is the minimum temperature of the oven? Give all times when the oven achieves this minimum temperature during the first 20 minutes.

(d) During the first 20 minutes of baking, calculate the total amount of time the oven temperature is at least 410°F.

(e) During the first 20 minutes of baking, calculate the total amount of time the oven temperature is at most 425°F.

(f) During the first 20 minutes of baking, calculate the total amount of time the oven temperature is between 410°F and 425°F.

Problem 20.5. The temperature in Gavin's oven is a sinusoidal function of time. Gavin sets his oven so that it has a maximum temperature of 300°F and a minimum temperature of 240°. Once the temperature hits 300°, it takes 20 minutes before it is 300° again.

Gavin's cake needs to be in the oven for 30 minutes at temperatures at or above 280 °. He puts the cake into the oven when it is at 270° and rising. How long will Gavin need to leave the cake in the oven?

Problem 20.6. Maria started observing Elasticman's height at midnight. At 3 AM, he was at his shortest: only 5 feet tall. At 9 AM, he was at his tallest: 11 feet tall.

Elasticman's height is a sinusoidal function of time.

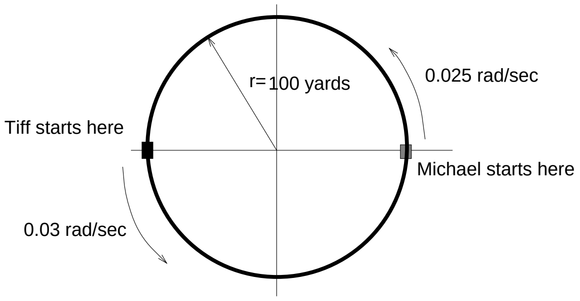
In the 24 hours after Maria began observing Elasticman, how much of the time will Elasticman be less than 6 feet tall?

Problem 20.7. Suppose

$$T(t) = 23\sin\left(\frac{2\pi}{24}(t-7)\right) + 66$$

is the temperature (in degrees Fahrenheit) at time t, where t is measured in hours after midnight on Sunday. You paint the exterior door to your house at 5 p.m. on Monday. The paint information states that 48 hours of 75° F drying time is required; i.e., you can only count time periods when the temperature is at least 75° F. When will the door be dry?

Problem 20.8. Tiffany and Michael begin running around a circular track of radius 100 yards. They start at the locations pictured. Michael is running 0.025 rad/sec counterclockwise and Tiffany is running 0.03 rad/sec counterclockwise. Impose coordinates as pictured.



(a) Where is each runner located (in xy-coordinates) after 8 seconds?

(b) How far has each runner traveled after 8 seconds?

(c) Find the angle swept out by Michael after t seconds.

(d) Find the angle swept out by Tiffany after t seconds.

(e) Find the xy-coordinates of Michael and Tiffany after t seconds.

(f) Find the first time when Michael's x-coordinate is -50.

(g) Find the first time when Tiffany's x-coordinate is -50.

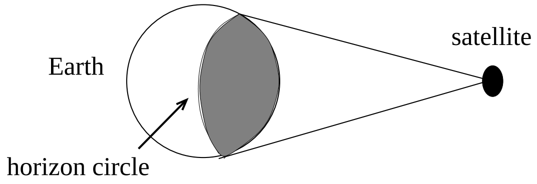
(h) Find when Tiffany passes Michael the first time.

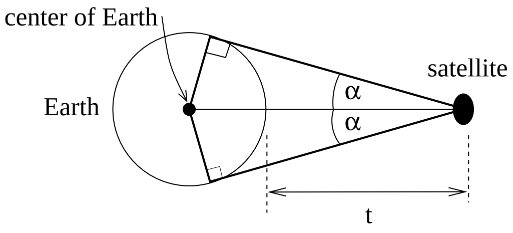
(i) Find where Tiffany passes Michael the first time.

(j) Find when Tiffany passes Michael the second time.

(k) Find where Tiffany passes Michael the second time.

Problem 20.9. A communications satellite orbits the earth t miles above the surface. Assume the radius of the earth is 3,960 miles. The satellite can only "see" a portion of the earth's surface, bounded by what is called a *horizon circle*. This leads to a two-dimensional cross-sectional picture we can use to study the size of the horizon slice:





CROSS−SECTION

(a) Find a formula for α in terms of t.

(b) If $t = 30{,}000$ miles, what is alpha? What percentage of the circumference of the earth is covered by the satellite? What would be the minimum number of such satellites required to cover the circumference?

(c) If $t = 1{,}000$ miles, what is alpha? What percentage of the circumference of the earth is covered by the satellite? What would be the minimum number of such satellites required to cover the circumference?

(d) Suppose you wish to place a satellite into orbit so that 20% of the circumference is covered by the satellite. What is the required distance t?

Problem 20.10. Answer the following questions:

(a) If $y = \sin(x)$ on the domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, what is the domain D and range R of $y = 2\sin(3x-1)+3$? How many solutions does the equation $4 = 2\sin(3x-1)+3$ have on the domain D and what are they?

(b) If $y = \sin(t)$ on the domain $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, what is the domain D and range R of $y = 8\sin(\frac{2\pi}{1.2}(t-0.3))+18$. How many solutions does the equation $22 = 8\sin(\frac{2\pi}{1.2}(t-0.3))+18$ have on the domain D and what are they?

(c) If $y = \sin(t)$ on the domain $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$, what is the domain D and range R of $y = 27\sin(\frac{2\pi}{366}(t-80.5))+45$. How many solutions does the equation $40 = 27\sin(\frac{2\pi}{366}(t-80.5))+45$ have on the domain D and what are they?

(d) If $y = \cos(x)$ on the domain $0 \leq x \leq \pi$, what is the domain D and range R of $y = 4\cos(2x+1)-3$? How many solutions does the equation $-1 = 4\cos(2x+1)-3$ have on the domain D and what are they?

(e) If $y = \tan(x)$ on the domain $-\frac{\pi}{2} < x < \frac{\pi}{2}$, what is the domain D and range R of $y = 2\tan(-x+5)+13$? How many solutions does the equation $100 = 2\tan(-x+5)+13$ have on the domain D and what are they?

Problem 20.11. Tiffany is a model rocket enthusiast. She has been working on a pressurized rocket filled with laughing gas. According to her design, if the atmospheric pressure exerted on the rocket is less than 10 pounds/sq.in., the laughing gas chamber inside the rocket will explode. Tiff worked from a formula $p = (14.7)e^{-h/10}$ pounds/sq.in. for the atmospheric pressure h miles above sea level. Assume that the rocket is launched at an angle of α above level ground at sea level with an initial speed of 1400 feet/sec. Also, assume the height (in feet) of the rocket at time t seconds is given by the equation $y(t) = -16t^2 + 1400\sin(\alpha)t$.

(a) At what altitude will the rocket explode?

(b) If the angle of launch is $\alpha = 12^o$, determine the minimum atmospheric pressure exerted on the rocket during its flight. Will the rocket explode in midair?

(c) If the angle of launch is $\alpha = 82^o$, determine the minimum atmospheric pressure exerted on the rocket during its flight. Will the rocket explode in midair?

(d) Find the largest launch angle α so that the rocket will not explode.

Problem 20.12. Let's make sure we can handle the symbolic and mechanical aspects of working with the inverse trigonometric functions:

(a) Find four solutions of $\tan(2x^2+x-1) = 5$.

(b) Solve for x: $\tan^{-1}(2x^2+x-1) = 0.5$

Appendix

Appendix A

Useful Formulas

Abbreviations

inch = in
feet = ft
yard = yd
mile = mi
millimeter = mm
centimeter = cm
meter = m
kilometer = km
second = sec = s
minute = min
hour = hr
year = yr
ounce = oz
pound = lb
gram = g
kilogram = kg
quart = qt
gallon = gal
milliliter = ml

liter = L
Joule = J
calorie = cal
atmosphere = atm
Coulomb = C
radian = rad

degree = deg
= °

miles per hour = mi/hr
= mph

feet per second = ft/sec
= ft/s

meters per second = m/sec
= m/s

revolutions per minute = rev/min
= RPM

Conversion Factors

Length

$$1\,\text{in} = 2.54\,\text{cm}$$

$$1\,\text{ft} = 0.3048\,\text{m}$$

$$1\,\text{mi} = 1.609344\,\text{km}$$

Volume

$$1\,\text{gal} = 3.7854\,\text{L}$$

$$1\,\text{qt} = 0.946353\,\text{L}$$

Energy

$$1\,\text{J} = 1\,\text{kg m}^2/\text{s}^2$$

$$1\,\text{cal} = 4.184\,\text{J}$$

Mass

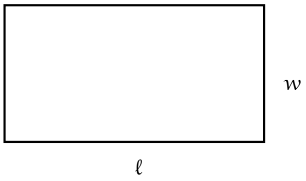
$$1\,\text{oz} = 28.3495\,\text{g}$$

$$1\,\text{lb} = 0.453592\,\text{kg}$$

Formulas from Plane and Solid Geometry

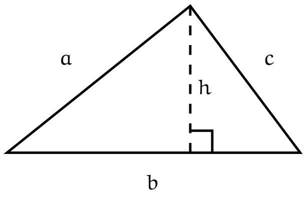
Rectangle

- Perimeter $= 2\ell + 2w$
- Area $= \ell w$



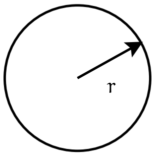
Triangle

- Perimeter $= a + b + c$
- Area $= \frac{1}{2}bh$



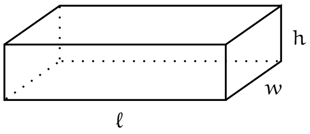
Circle

- Perimeter $= 2\pi r$
- Area $= \pi r^2$



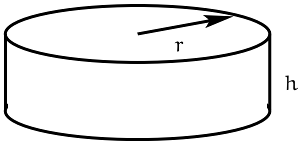
Rectanglular prism

- Surface Area $= 2(\ell w + \ell h + wh)$
- Volume $= \ell hw$



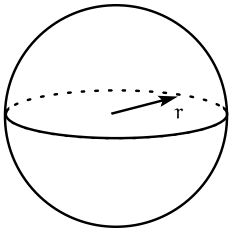
Right circular cylinder

- Surface Area $= 2\pi r^2 + 2\pi rh$
- Volume $= \pi r^2 h$



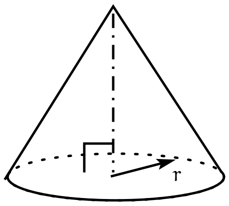
Sphere

- Surface Area = $4\pi r^2$
- Volume = $\frac{4}{3}\pi r^3$



Right circular cone

- Surface Area = $\pi r^2 + \pi r s$
- Volume = $\frac{1}{3}\pi r^2 h$



Constants

Avogadro's number = N = 6.022142×10^{23}
speed of light = c = 2.99792×10^8 m/s^2
density of water = 1 g/cm^3
mass of earth = 5.9736×10^{24} kg
earth's equatorial radius = 3,960 mi = 6.38×10^6 m
acceleration of gravity at earth's surface = 32 ft/sec^2 = 9.8 m/s^2

Algebra

- $a^x a^y = a^{x+y}$
- $(a^x)^y = a^{xy}$
- $a^0 = 1$
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
- $\frac{a^x}{a^y} = a^{x-y}$
- $\sqrt[n]{a} = a^{1/n}$
- $(a+b)(c+d) = ac + ad + bc + bd$
- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- Quadratic Formula: If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- Completing the Square: $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$

Trigonometry

- $\sin(-x) = -\sin x$
- $\cos(-x) = \cos x$
- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
- $\sin\left(\frac{\pi}{2} + x\right) = \cos x$
- $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$
- $\sin(\pi - x) = \sin x$
- $\cos(\pi - x) = -\cos x$
- $\sin(\pi + x) = -\sin x$
- $\cos(\pi + x) = -\cos x$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $\sin^2 x + \cos^2 x = 1$
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$

Appendix B

Answers

Answer 1.1 (b) 150 ft/sec. (c) Gina. (d) 6300 hours.

Answer 1.2 68.4444 km.

Answer 1.3 $r = 10.172$ cm for lead; $r = 16.433$ cm for aluminum.

Answer 1.4 The mass of the air is 9.740 million kg, more than the mass of the Tower.

Answer 1.5 (a) 5.5 min/mi = 5:30 pace. (b) $14\frac{2}{3}$ ft/sec. (c) Adrienne.

Answer 1.6 (a) (John's Salary) = \$56000 and (taxes) = \$0. (b) (John's Salary) $\leq$\$56000 and (taxes) = 0.15$\times$ (John's Salary). (c) (John's Salary) $\geq$ \$56000 and (John's taxes) $> 0.28\times$ (John's Salary). (d) 1500 $\leq$(number of 120 students each year) $\leq$ 1800. (e) (cost red Porsche) $> 3\times$ (cost F-150 pickup). (f) 2 hours $\leq$ (weekly study time per credit hour) $\leq$ 3 hours (g) 2$\times$ (number of happy math students) $> 5\times$ (number of happy chemistry students). (happy math students) + (happy chemistry students) $< \frac{1}{2}\times$ (number of cheerful biology students). (h) (Cady's high score) - (Cady's low score) = 10%. (Cady's final exam score) = 97%. (i) $(0.9999) \times \frac{1}{2}$ (total votes cast)$\leq$ (Tush votes) $\leq (1.0001) \times \frac{1}{2}$ (total votes cast).

Answer 1.7 Go for the 15 inch pie.

Answer 1.8 (a) $m_{90} = 151\text{kg}, m_{99} = 468\text{kg}, m_{99.9} = 1476\text{kg}$. (b) $v = 2.974 \times 10^8 \frac{\text{m}}{\text{sec}}$.

Answer 1.9 1080 pizzas sold in 4 hours. Profit reaches \$1000 at about $8:22$ pm.

Answer 1.10 Lee has 2.265 in^2 more pie.

Answer 1.11 About 5×10^6 times around the equator.

Answer 1.12 (a) Radii are about $r = 0.2416$, 2.317, 4.184 and 9.167 cm at the indicated times. (b) No.

Answer 1.13 (a) 140 million gallons per week; about 7300 million gallons per year. (b) 100 yds $\times$ 50 yds $\times$20 yds.

Answer 1.14 (a) $N = 12 + 0.45(60 - x)$. (b) Reduced competition for resources (eg. water, nutrients, etc.).

Answer 1.15 Formula simplifies to $\frac{r(1+x)}{1+2x}$. Rates are about $0.67r$, $0.6r$, $0.54r$ and $0.51r$ at the indicated times. Rates decrease over time, but will never be less than $0.5r$.

Answer 1.16 (a) Initial time = 7am, Initial temperature = 44°F, Final time = 10am, Final temperature = 50°F, rate of change = 2°F/hr. (b) 58°F. (c) Initial time = 4.5pm, Initial temperature = 54°F, Final time = 6.25pm, Final temperature = 26°F, rate of change = -16°F/hr.

Answer 1.17 (a) $t = 9$. (b) $a = 1/8$. (c) $x = -3a/4$. (d) $t > 3$. (e) $\frac{x+2}{x(x+1)}$.

Answer 2.1 (a) $d = \sqrt{2}$, $\Delta x = 1 = \Delta y = 1$. (b) $d = \sqrt{5}$, $\Delta x = -1$, $\Delta y = -2$. (c) $d = \sqrt{34}$, $\Delta x = 5$, $\Delta y = -3$. (d) $d = \sqrt{10t^2 + 2t + 1}$, $\Delta x = 3t$, $\Delta y = 1 + t$.

Answer 2.2 (a-c) True. (d) $\Delta x = s - a, \Delta y = t - b$. (e) $\Delta x = a - s, \Delta y = b - t$. (f) $\Delta x = 0$ means the points line on the same vertical line; $\Delta y = 0$ means the points line on the same horizontal line.

Answer 2.3 Just after 12:29 PM that afternoon.

Answer 2.4 (a) Erik= 6.818 mph, Ferry = 17.6 ft/sec. (b) Impose coordinates with Kingston the origin and units of miles on each axis; then Edmonds is located at $(6,0)$ and Erik's sailboat is at $(3,2)$. The table rows have these entries:

$(0,0)$, $(0.1,0)$, $(1.4,0)$, $(12t,0)$.
$(3,2)$, $(3,1.9432)$, $(3,1.2045)$, $(3, 2 - 6.818t)$
3.606, 3.491, 2.003, $\sqrt{(12t-3)^2 + (2 - 6.818t)^2}$ (c) Use coordinates as in (b), then when the ferry reaches $(3,0)$, Erik is at $(3,0.296)$. (d) CG vessel does not catch the ferry before Edmonds.

Answer 2.5 (a) $d(t) = (65.3)t$ (b) 227 minutes, 168.4 miles (c) $t = 80.86$ seconds.

Answer 2.6 (a) Allyson's coordinate position: $(0\,\text{ft}, 20\,\text{ft})$. Adrienne's coordinate position: $(-16\,\text{ft}, 0\,\text{ft})$. (b) After 2 seconds there will still be slack in the bungee cord. (c) $t \approx 2.34$. Use this time to find where Allyson and Adrienne are located. (d) $t = 5.5$ seconds.

Answer 2.7 Impose a coordinate system with origin at A and the shore the horizontal x axis; let x be the location on the shore where Brooke beaches. Equation $\frac{1}{2}\sqrt{25+x^2}+\frac{1}{4}(6-x)$ gives time to reach Kono's. $T(0) = 4$ hr, $T(6) = 3.9$ hr. Neither time will be the minimum time.

Answer 2.8 (b) $t = 2$; $t = \frac{5}{2}$. (c) $(3,3)$ (d) spider=$(\frac{7}{3},\frac{8}{3})$, ant=$(\frac{41}{3},\frac{4}{3})$. (e) 1.5 feet. (f) Spider reaches $(9,6)$ when $t = 4$; ant reaches $(9,6)$ when $t = 3$. (g) spider speed is $\sqrt{5}$ ft/min; ant speed is $\sqrt{8}$ ft/min.

Answer 2.9 49.92 mph (exactly).

Answer 2.10 141.46 miles. They are 300 miles apart at time 0.826 hr = 49.6 minutes.

Answer 2.11 (a) Final answer is correct, but second equality is wrong. (b) Final answer should be $4xy$; key fact is that $(x+y)^2 = x^2+2xy+y^2$, etc. (c) Answer and steps correct.

Answer 2.12 (a) $x = \pm\sqrt{\frac{1+\beta^2}{\alpha^2}}$ (b) $x = \frac{\alpha\beta}{\alpha+\beta}$ (c) $x = \frac{\beta}{\alpha\beta+1}$

Answer 2.13 (a) $5t^2+6t+5$ (b) $2t^2+4t$ (c) $\frac{2}{t^2-1}$ (d) $\sqrt{5t^2+4t+4}$

Answer 3.1 (a) $(x+3)^2+(y-4)^2 = 9$ (b) $(x-3)^2+\left(y+\frac{11}{3}\right)^2 = \frac{1}{16}$ (c) Draw a vertical and horizontal line through $(1,1)$. On the vertical line, a circle of radius 2 will have a center at either $(1,3)$ or $(1,-1)$. Likewise, the horizontal line will have circles at either $(-1,1)$ or $(3,1)$.

Eqn	C(h,k)	$(x-h)^2+(y-k)^2 = r^2$
1.	$(3,1)$	$(x-3)^2+(y-1)^2 = 4$
2.	$(1,3)$	$(x-1)^2+(y-3)^2 = 4$
3.	$(-1,1)$	$(x+1)^2+(y-1)^2 = 4$
4.	$(1,-1)$	$(x-1)^2+(y+1)^2 = 4$

(d) $(1,1)$, $(1,-3)$, $(1+\sqrt{3},0)$, $(0,-1-\sqrt{3})$

Answer 3.2 (a) center (3,-1), radius $2\sqrt{3}$ (b) center (-2,-3), radius 2 (c) center (-1/6,5/3), radius $\sqrt{203/12}$ (d) center (3/4, -1/2), radius $\sqrt{3}$

Answer 3.3 (a) Wet in 24.0 minutes. (b) Wet in 25.4154041 minutes.

Answer 3.4 (a) Imposing with x-axis along ground and y-axis along tower, wheel modeled by $x^2+(y-62)^2 = 60^2$. (b) 46.43 ft. to right of tower. (c) $(-24,7)$ and $(-24,117)$.

Answer 3.5 (a) Impose a coordinate system so that the tractor is at the origin at $t = 0$ seconds. With this coordinate system, the south edge of the sidewalk is modeled by $y_s = 100$; the north edge is modeled by, $y_n = 110$. (b) $t = 32$ minutes. (c) $t = 52$ minutes. (d) 20 minutes.

Answer 3.6 (a)] The equation for eastward travel from Kingston is $y = 8$. Southward travel along $x = -1$. (b) Boundary: $(x+2)^2+(y-10)^2 = 9$; Interior:$(x+2)^2+(y-10)^2 < 9$; Exterior:$(x+2)^2+(y-10)^2 > 9$. (c) 3.82 minutes. (d) Plug $x = -1$ into circle equation to find exit point. Use this to find when the ferry exits the radar zone. Be careful to reference all times relative to when the ferry departed Kingston. (e) 20.32 minutes.

Answer 3.7 (a) 6.92 seconds. (b) 7.67 seconds. (c) 38.16 seconds. (d) 7332 ft^2 < area < 7632 ft^2.

Answer 3.8 (a) $x = 11/5$. (b) no solutions. (c) $(-2,-2)$ and $(-2,4)$. (d) $(-1\pm\sqrt{2/5},3)$.

Answer 4.1 (a) $y = -\frac{5}{3}(x-1)-1$ (b) $y = 40(x+1)-2$ (c) $y = -2x-2$ (d) $y = 11$ (e) $m = \frac{3}{5}$, $y = \frac{3}{5}(x-1)+1$ (f) $y = 40x-14$ (g) $y = -\frac{3}{4}x+\frac{7}{4}$ (h) $y = x-1$

Answer 4.2 (a)$(\frac{2}{1+2\alpha},\frac{1+4\alpha}{1+2\alpha})$. (b) $\alpha = -2/5$. (c) $\alpha = -1/4$.

Answer 4.3 (a) area $= \frac{25}{13}$ (b) area $= -\frac{b^2}{2m}$ (c) $m = -\frac{1}{4}$

Answer 4.4 In some cases, answers in this problem are unique. In other cases, the answers are not unique. Here is a possible solution set:

Eqn	Slope	y-int	Pt on line	Pt on line
$y = 2x+1$	2	1	$(0,1)$	$(-\frac{1}{2},0)$
$y = -\frac{11}{4}x+\frac{17}{4}$	$-\frac{11}{4}$	$\frac{17}{4}$	$(3,-4)$	$(-1,7)$
$y = -2x+1$	-2	1	$(0,1)$	$(\frac{1}{2},0)$
$y = \frac{1}{2}x+1$	$\frac{1}{2}$	1	$(0,1)$	$(-2,0)$
$y = 1{,}000$	0	1,000	$(0,1{,}000)$	$(0,1{,}000)$
$y = 0$	0	0	$(0,0)$	$(1,0)$
$x = 3$	Undef	None	$(3,3)$	$(3,-2)$
$y = x-14$	1	-14	$(5,-9)$	$(0,-14)$

Answer 4.5 (a) $y = 6{,}850(x-1970)+38{,}000$ (b) $y = 8{,}000(x-1970)+8{,}400$ (d) Tabularize your answer:

Year	Seattle	Port Townsend
1983	\$127,050	\$112,400
1998	\$229,800	\$232,400

(e) 1995.74, \$214,313 (f) 1982.70, Sea = \$124,965 (g) 2008.78, Sea = \$303,661 (h) 1972.32, Sea = \$53,892 (i) No.

Answer 4.6 For these answers, we take the hole to be the origin of our coordinate system. Then, the ball's line of travel is $y = (0.667)(x-35)$. (a) $(-13.46,-32.31)$ (b) 3.19 seconds (c) 5.82 seconds (d) $(10.78,-16.17)$ at about $t = 6.1$ sec.

Answer 4.7 (a) Allyson at (0,70); Adrian at $(-44,0)$. Bungee is 83.1401 ft. long. (b) Occurs at time $t = 7.7546$ seconds and Allyson is at (0,77.546). Allyson's final location is 77.546 ft. from her starting point.

Answer 4.8 The lines of tangency have equations $y = \pm\frac{2}{\sqrt{5}}(x - 12)$. The non-visible portion of the y-axis is $-\frac{24\sqrt{5}}{5} \leq y \leq \frac{24\sqrt{5}}{5}$.

Answer 4.9 $C = \frac{5}{9}(F - 32)$ and $F = \frac{9}{5}C + 32$. In Oslo, the temperature is $-9.4°F$.

Answer 4.10 At the closest point, she will be 26.22471828 miles from Paris.

Answer 4.11 Impose coordinates with Angela's initial location as the origin. Angela is closest to Mary at $(18.8356,41.4383)$; this takes approximately 3.8 seconds.

Answer 4.12 (a) Impose coordinates with sprinkler initial location as the origin $(0,0)$. Line equation becomes $y = -\frac{1}{5}x + 100$. (b) Sprinkler is located at $(0,82.5)$ at time $t = 33$ minutes. Circular boundary of watered zone hits southern edge of sidewalk at the points $(-6.708,101.3416)$ and $(13.4386,97.3123)$. (c) $y = -\frac{1}{5}x + 110.198$.

Answer 4.13 (a) With the origin set at the statue, $x = 30-3.123475t$, $y = 2.49878t$. (b) The distance to the statue is $\sqrt{16t^2 - 187.4085t + 900}$ feet. (c) Margot will be 28 feet from the statue after 0.655672 seconds, and after 11.057360 seconds (assuming she continues past the point due north of the statue).

Answer 4.14 18.923076923 seconds

Answer 4.15 (a) $(-1 \pm \sqrt{7/3})/2 = (-3 \pm \sqrt{21})/6$. (b) $t = -0.463325, 0.863325$; or the exact answer would be $(1 \pm \sqrt{11})/5$. (c) $t = 0.2$. (d) No real number solutions.

Answer 4.16 (a) There are four answers: $x = \pm\sqrt{2+\sqrt{2}}$ and $x = \pm\sqrt{2-\sqrt{2}}$ (b) $y = 6 + 2\sqrt{5}$.

Answer 5.1 (a) $-2+h+2x$. (b) 2. (c) $h+2x$. (d) $-h-2x$. (e) $-\pi(h+2x)$ (f) $\frac{1}{\sqrt{h+x-1}+\sqrt{x-1}}$.

Answer 5.2 $g(x) = \frac{9}{5}x + 24$, $f(x) = \frac{4}{5}x + 4$, $v(x) = x + 20$, $v(5) = 25$, minimum $v(x) = 20$, maximum $v(x) = 40$.

Answer 5.3 For example, in (a), suppose Dave has constant speed v ft/min. Then the function $s = d(t) = vt$ will compute the distance Dave travels in t minutes. The graph would be a line with s-intercept 0 and slope v; the domain would be $0 \leq t \leq \frac{2400}{v}$; etc.

Answer 5.4

Answer 5.5 Several possible answers for each one.

Answer 5.6 (a) x-intercepts$= \frac{1}{6}(3\pm\sqrt{33})$. y-intercept $= -2$. (b) $(\frac{1}{6}(3-\sqrt{93}),5)$ and $(\frac{1}{6}(3+\sqrt{93}),5)$. (c) None. (d) Yes, No, No, Yes. (e) $(\sqrt{1+\sqrt{2}}, -2-3\sqrt{1+\sqrt{2}}+3(1+\sqrt{2}))$.

Answer 5.7 (b) $f(x) = 30000x + 50000\sqrt{(4-x)^2+1}$. (c) The table of values $(x,f(x))$ will be: $(0., 206155)$, $(0.5, 197003)$, $(1, 188114)$, $(1.5, 179629)$, $(2,171803)$, $(2.5, 165139)$, $(3., 160711)$, $(3.5, 160902)$, $(4,170000)$. Minimal cost occurs for some $2.5 < x < 3.5$; the exact answer is $x = \frac{13}{4}$, but we cannot solve this in our class since it requires the tools of Calculus.

Answer 5.8 (a) $-\frac{3}{2}, -\frac{5}{2}, \frac{x}{2}$, $\frac{1}{2}(\heartsuit - 3)$. (b) 0, 20, $6x + 2x^2$, $2\heartsuit^2 - 6\heartsuit$, $2(\heartsuit + \triangle - 3)(\heartsuit + \triangle)$. (c) Always $4\pi^2$.

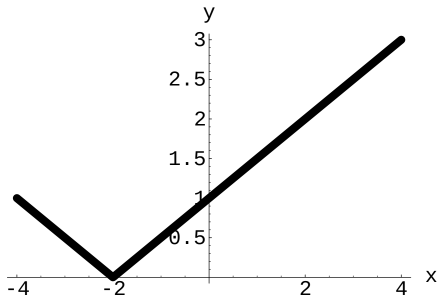
Answer 5.9 Use the vertical line test. For example, (a) is not a function, by the vertical line test; you can split it into two function graphs by slicing the ellipse symmetrically into upper and lower halves. On the other hand, (o) is a function, by the vertical line test; etc.

Answer 5.10 (a) $x = 5$. (b) $x = 4, 8$. (c) $x = 36$. (d) $\frac{242-\sqrt{56400}}{2}$

Answer 6.1 (a) 0, 2, 3. (b) $x = \pm 4$, $x = 0$, no solution. (c) Intersect at $(4,4)$ and $(-\frac{4}{3}, \frac{4}{3})$. Area is $\frac{16}{3}$.

Answer 6.2 (a)

$$|-0.5x - 1| = \begin{cases} -0.5x - 1 & \text{if } x \leq -2 \\ 0.5x + 1 & \text{if } x > -2 \end{cases}$$



Answer 6.3 (a) $x = 4$ and $x = -\frac{22}{3}$ (b) $x = 0.75$ (c) the equation has no solutions.

Answer 6.4 (a) $x = -7$ and $x = 3$ (b) $a = 3$ (c) $x = 8/3$

Answer 6.5 For $0 \leq x \leq 6$, the area is $\frac{1}{6}x(x + 18)$.

Answer 6.6 (a) The rule is

$$y = \begin{cases} 2x & \text{if } 0 \leq x \leq 10 \\ -x + 30 & \text{if } 10 \leq x \leq 30 \end{cases}$$

and the range is $0 \leq y \leq 20$.

(b) The rule for the area function $a(x)$ is

$$a(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 10 \\ -\frac{1}{2}x^2 + 30x - 150 & \text{if } 10 \leq x \leq 30 \end{cases}$$

(c) $x = 12.6795$ inches.

Answer 6.7 (a)

$$j(t) = \begin{cases} 0 & \text{if } t < -1 \\ 62.22(t+1) & \text{if } -1 \leq t \leq 3.5 \end{cases}$$

(b)

$$s(t) = \begin{cases} 280 & \text{if } t < 0 \\ 280 - 70t & \text{if } 0 \leq t \leq 4 \end{cases}$$

(c)

$$d(t) = \begin{cases} 280 & \text{if } t < -1 \\ 280 - 62.22(t+1) & \text{if } -1 \leq t < 0 \\ 217.78 - 132.22t & \text{if } 0 \leq t < 1.6471 \\ 132.22t - 217.78 & \text{if } 1.6471 \leq t < 3.5 \\ 70t & \text{if } 3.5 \leq t < 4 \end{cases}$$

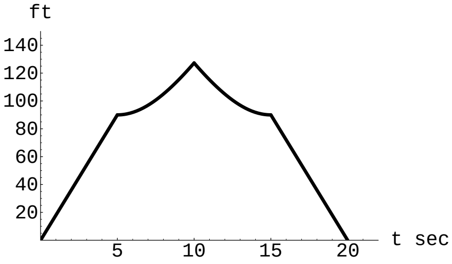
Answer 6.8 The distance to his starting point t seconds after he starts is

$$d(t) = \begin{cases} 10t & \text{if } 0 \leq t \leq 25, \\ \sqrt{250^2 + (12(t-25))^2} & \text{if } 25 \leq t \leq \frac{175}{3}, \\ \sqrt{400^2 + (250 - 9(t - \frac{175}{3}))^2} & \text{if } \frac{175}{3} \leq t \leq \frac{205}{3}. \end{cases}$$

Answer 6.9

$$y = d(t) = \begin{cases} 18t & \text{if } 0 \leq x \leq 5 \\ \sqrt{90^2 + 18^2(t-5)^2} & \text{if } 5 \leq x \leq 10 \\ \sqrt{90^2 + (90 - 18(t-10))^2} & \text{if } 10 \leq x \leq 15 \\ 90 - 18(t-15) & \text{if } 15 \leq x \leq 20 \end{cases}$$

Here is the graph of $d(t)$



Answer 6.10 (a) $v(x) = 2x^3 - 70x^2 + 500x$; degree 3. (b) $a(x) = 1000 - 50x - 2x^2$, degree 2. To get 600 sq. in. dimensions are: $6.3746 \times 7.2508 \times 18.6254$.

Answer 6.11 (a) 2 hours. (b) Impose coordinates with x axis the bottom of the ditch and the y axis the pictured centerline.

$$y = \begin{cases} 20 & \text{if } x \leq -40 \\ 10 + \sqrt{100 - (x+40)^2} & \text{if } -40 \leq x \leq -30 \\ 10 - \sqrt{100 - (x+20)^2} & \text{if } -30 \leq x \leq -20 \\ 0 & \text{if } -20 \leq x \leq 20 \\ 10 - \sqrt{100 - (x-20)^2} & \text{if } 20 \leq x \leq 30 \\ 10 + \sqrt{100 - (x-40)^2} & \text{if } 30 \leq x \leq 40 \\ 20 & \text{if } 40 \leq x \end{cases}$$

(c) $2(40 - \sqrt{91})$ feet. (d) 42 ft. wide: 0.3008 minutes. 50 ft. wide: 8.038 minutes. 73 ft. wide: 116.205 minutes.

Answer 6.12 (a) $0 \leq y \leq 6$. (b) Increasing: $-6 \leq x \leq -2$ and $2 \leq x \leq 4$. Decreasing: $-2 \leq x \leq 2$ and $4 \leq x \leq 6$.
(c)

$$y = \begin{cases} 2x + 12 & \text{if } -6 \leq x \leq -4 \\ 4 + \sqrt{4 - (x+2)^2} & \text{if } -4 \leq x \leq 0 \\ 4 - \sqrt{4 - (x-2)^2} & \text{if } 0 \leq x \leq 4 \\ -2x + 12 & \text{if } 4 \leq x \leq 6 \end{cases}$$

(d) $2 \leq y \leq 6$. (e) $2 \leq y \leq 4$.

Answer 6.13 (a) 0. (b) $a = \frac{13}{5}$, $b = -\frac{3}{5}$, $c = -3$.

Answer 7.1 (a) $2(x-4)^2 + 9$, Vertex: (4, 9), Axis: x = 4. (b) $3(x - 5/2)^2 - 383/4$, Vertex: (5/2, -383/4), Axis: x = 5/2. (c) $(x - 3/14)^2 + 2539/196$, Vertex: (3/14, 2539/196), Axis: x = 3/14. (d)$2(x-0)^2 - 0$, Vertex: (0, 0), Axis: x = 0. (e) $(1/100)(x-0)^2 - 0$, Vertex: (0, 0), Axis: x = 0.

Answer 7.2 (a) $y = -\frac{2}{3}x^2 + \frac{5}{3}x$. (b) $y = 1.125x^2 - 1.5x - 1.625$. (c) $y = -0.5x^2 + 3.5x - 4$. (d) No solution.

Answer 7.3 (a) maximum value is 22; minimum value is 1.75 (b) maximum value is 32; minimum value is 2 (c) maximum value is -6; minimum value is -46

Answer 7.4 In the first case, $d = 0$ or $d = 12$. In the second case, $d = \pm\frac{2}{\sqrt{13}}$.

Answer 7.5 (a) Multipart function: $s(t) = -\frac{1}{160}t^2 + \frac{5}{8}t + 10$ for $0 \leq t \leq 114.031$, $s(t) = 0$ for $t \geq 114.031$.; \$23,125. (b) Sell at time $t = 50$ days; \$2500.

Answer 7.6 The parabola has x intercepts at $-1, 3$ and y intercept at -3. The vertex of the parabola is $(1, -4)$.

$$|x^2 - 2x - 3| = \begin{cases} x^2 - 2x - 3 & \text{if } x \leq -1 \\ -x^2 + 2x + 3 & \text{if } -1 < x < 3 \\ x^2 - 2x - 3 & \text{if } x \geq 3 \end{cases}$$

Answer 7.7 (a) 100 ft. (b) 156.25 ft. (c) $(625, -125)$. (d) When x = 54.81 ft. or x = 570.19 ft.; i.e. at $(54.81, 39.04)$ and $(570.19, -64.04)$.

Answer 7.8 (a) No, since $f(1) = 1 \neq 2$. (b) The points $(1, 1+2b)$ and $(-(1+b), 1+2b)$. (c) Only the point $(a, 1-a^2)$. (d) The points $(-2.2701, -7.1168)$ and $(2.9368, 1.5612)$.

Answer 7.9 She should have 225 trees in the orchard.

Answer 7.10 She should charge \$7.72 to make the most money.

Answer 7.11 The radius of the circular part should be

$$\frac{24}{4+\pi} \approx 3.36059492 \text{ feet.}$$

The short side of the rectangular part should also be equal to this.

Answer 7.12 The enclosure should be 50 meters by 75 meters.

Answer 7.13 The maximum possible area is 4285.71 m^2.

Answer 7.14 To achieve the minimal combined area, cut the wire so that the pieces have lengths 26.394 in. and 33.606 in.; bend the 26.394 inch piece into a circle.

Answer 7.15 (a) $y = \frac{1}{2}x - 4$ (b) $y = 4 + \frac{1}{6}x$ (c) $t = 1350/349 \approx 3.8681948$.

Answer 7.16 (a) The distance between Sven and Rudyard is $\sqrt{41t^2 - 2400t + 36900}$ feet, where t is the number of seconds after they start moving. (b) They will be closest together after they have been moving for 29.268292 seconds, and will be 42.166916 feet apart at that time.

Answer 7.17 Michael $M(t) = (5.547t, 8.32t)$, Tina $T(t) = (400 - 8.944t, 50 + 4.472t)$. Study the distance SQUARED from $M(t)$ to $T(t)$. Michael and Tina will be closest when $t = 26.641$ sec; they are 54.3 ft at that instant.

Answer 7.18

$$x = \frac{-2\alpha^2 \pm \sqrt{4\alpha^4 - 4\alpha}}{2\alpha}$$

$$\alpha = \frac{-x^2 \pm \sqrt{x^4 - 8x}}{4x}$$

Answer 7.19 (d) There are two possible values for α: If $\alpha = 8+2\sqrt{17}$, then the unique solution of the equation will be $x = -4 - \sqrt{17}$. If $\alpha = 8 - 2\sqrt{17}$, then the unique solution of the equation will be $x = -4 + \sqrt{17}$.

Answer 7.20 (a) $t = \frac{1}{2}(2 \pm \sqrt{2}\sqrt{s-1})$. (b) $x = -1 \pm \sqrt{y-2}$.

Answer 8.1 (a)

$$h(f(t)) = \begin{cases} -t+1 & \text{if } t \le 1 \\ t-1 & \text{if } 1 \le t \end{cases}$$

$$h(g(t)) = \begin{cases} -t-1 & \text{if } t \le -1 \\ t+1 & \text{if } -1 \le t \end{cases}$$

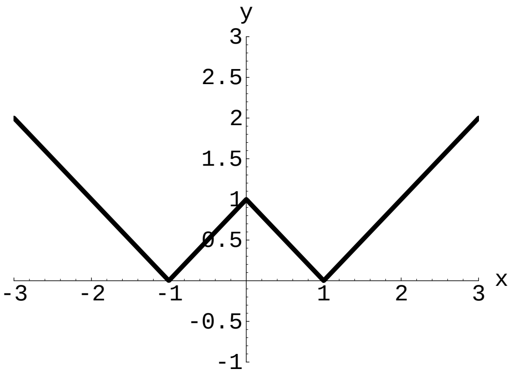
(b)

$$f(h(t)) = \begin{cases} -t-1 & \text{if } t \le 0 \\ t-1 & \text{if } 0 \le t \end{cases}$$

$$g(h(t)) = \begin{cases} t-1 & \text{if } t \le 0 \\ -t-1 & \text{if } 0 \le t \end{cases}$$

(c)

$$h(h(t)-1) = \begin{cases} -t-1 & \text{if } t \le -1 \\ t+1 & \text{if } -1 \le t \le 0 \\ -t+1 & \text{if } 0 \le t \le 1 \\ t-1 & \text{if } 1 \le t \end{cases}$$



Answer 8.2 (a) $y = f(g(x))$, if $f(x) = x^5$, $g(x) = x - 11$. (b) $y = f(g(x))$, if $f(x) = \sqrt[3]{x}$, $g(x) = 1 + x^2$. (c) $y = f(g(x))$, if $f(x) = 2x^5 - 5x^2 + (1/2)x + 11, g(x) = x - 3$. (d) $y = f(g(x))$, if $f(x) = (1/x)$, $g(x) = x^2 + 3$. (e) $y = f(g(x))$, if $f(x) = \sqrt{x}$, $g(x) = \sqrt{x} + 1$. (f) $y = f(g(x))$, if $f(x) = 2 - \sqrt{5 - x^2}$, $g(x) = 3x - 1$.

Answer 8.4 (c) $x = -1$ and $x = 2$

Answer 8.5 (a) $f(s) = \frac{s}{60}$;

$$C(f(s)) = \frac{70s^2}{10(60^2) + s^2}$$

$C(f(s))$ computes mph when you input seconds s. (b) $g(h) = 60h$;

$$C(g(h)) = \frac{252000h^2}{10 + 3600h^2}$$

$C(g(h))$ computes mph when you input hours h. (c) $v(s) = \frac{22}{15}s$;

$$v(C(m)) = \frac{308m^2}{30 + 3m^2}$$

$v(C(m))$ computes ft/sec when you input minutes m.

Answer 8.6 (a) $f(g(x)) = (x+3)^2$, $f(f(x)) = x^4$, $g(f(x)) = x^2 + 3$. (b) $f(g(x)) = \frac{1}{\sqrt{x}}$, $f(f(x)) = x$, $g(f(x)) = \frac{1}{\sqrt{x}}$. (c) $f(g(x)) = x$, $f(f(x)) = 81x + 20$, $g(f(x)) = x$. (d) $f(g(x)) = 6(x-4)^2 + 5$, $f(f(x)) = 6(6x^2+5)^2+5$, $g(f(x)) = 6x^2+1$. (e) $f(g(x)) = 8x+21$, $f(f(x)) = 4(4x^3 - 3)^3 - 3$, $g(f(x)) = 2x$. (f) $f(g(x)) = 2x^3 + 1$, $f(f(x)) = 4x+3$, $g(f(x)) = (2x+1)^3$. (g) $f(g(x)) = 3$, $f(f(x)) = 3$, $g(f(x)) = 43$. (h) $f(g(x)) = -4$, $f(f(x)) = -4$, $g(f(x)) = 0$.

Answer 8.7 $-5/2 \le x \le -1/2$

Answer 8.8 (a) $\frac{7}{2} \le x \le 6$. (b) $-3 \le y \le 5$. (c) $1 \le x \le 6$. (d) $-9 \le y \le 7$. (e) $B = 5, C = \frac{39}{5}$. (f) $A = \frac{1}{8}$, $D = \frac{3}{8}$.

Answer 8.9 (a) $\frac{-1}{(x-1)(x+h-1)}$, set $h = 0$ to get $\frac{-1}{(x-1)^2}$ (b) $4 + 4h + 8x$, set $h = 0$ to get $4 + 8x$. (c) $\frac{-h-2x}{\sqrt{25-x^2}+\sqrt{25-(x+h)^2}}$, set $h = 0$ to get $\frac{-x}{\sqrt{25-x^2}}$.

Answer 9.1 (a) domain of f =$\{x|x \ne \frac{4}{3}\}$; range=f= $\{y|y \ne 0\}$. (b) $f^{-1}(y) = \frac{2+4y}{3y}$.

Answer 9.2

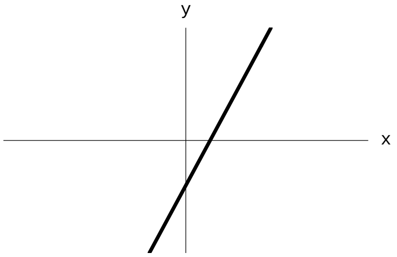
Answer 9.3 (c) $f^{-1}(y) = \frac{3+\sqrt{17+8y}}{4}$ on the domain $\{y|y \ge -\frac{17}{8}\}$.

Answer 9.4 Only (B) is one-to-one on the entire domain.

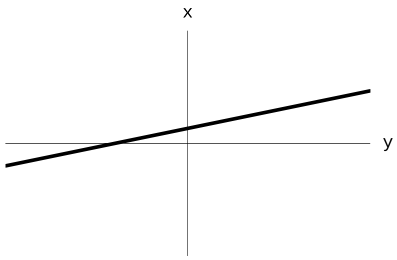
Answer 9.5

Answer 9.6 (a) $h = f(x) = -2x^2 + 124x$. (c) $x = g(h) = 31 - \frac{1}{2}\sqrt{3844 - 2h}$.

Answer 9.7 (a) (1)



(2) $x = f^{-1}(y) = \frac{y+2}{3}$ has domain and range all real numbers. (3)



(4) $f(f^{-1}(y)) = f(\frac{y+2}{3}) = 3(\frac{y+2}{3}) - 2 = y + 2 - 2 = y$; $f^{-1}(f(x)) = f^{-1}(3x-2) = \frac{3x-2+2}{3} = x$.

Answer 9.8 (a) $w = f(t) = 2\sqrt{25 - (\frac{t}{6} - 5)^2}$. Domain: $0 \leq t \leq 30$ hours; Range: $0 \leq w \leq 10$ ft. (b) $t = 6$ hours. (c) $t = f^{-1}(w) = 30 - 6\sqrt{25 - (\frac{w}{2})^2}$. Domain: $0 \leq w \leq 10$ ft.; Range: $0 \leq t \leq 30$ hours.

Answer 9.9 (a) 2 nanoseconds. (f) domain ϕ=$\{0 \leq t \leq 10\}$; range ϕ=$\{0 \leq y \leq 6\}$.

Answer 10.1 (a) 31.5443; (b) 355.1134; (c) 36.4622; (d) 0.0616; (e) 51,168; (f) 0.009794;

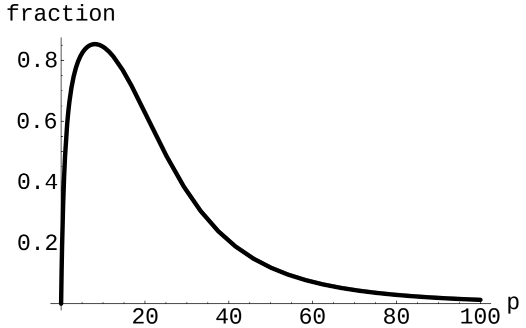
Answer 10.2 (a) $y = 3(\frac{1}{2})^x$ (b) $y = (\frac{1}{2})^x$. (d) $y = \frac{1}{27}(\frac{1}{\sqrt{3}})^x$.

Answer 10.3 (a) 1.64×10^6 cells. (b) True. (c) The two formulas are identical.

Answer 10.4 (a) 261.31 Hz. (b) 440 Hz. (c) 27.5 Hz. (d) 16.35 Hz.

Answer 10.5 (a) $2^9 = 512$. (b) 2^{n-1}. (c) $2^{63} = 9.2 \times 10^{18}$. (d) 9.2×10^{15} meters.

Answer 10.6 (a) $M(p)$ is the higher curve. (d)



Answer 11.1 (a) $w(t) = \$1.113(1.046408)^t$. (b) \$1.11. (c) It is below; should be \$5.70 by the model.

Answer 11.2 (a) $p(x) = 860(1.070674)^x$, $l(x) = 70x + 860$. (b) $p(10) = 1702, l(10) = 1560$.

Answer 11.3 If we use 1989 and 2000 in $a(t)$, we get two data points and a corresponding exponential model: $E(t) = 15.918(1.243301)^{(}t - 1980)$. The exponential model grows faster than the cubic model and eventually exceeds $a(t)$.

Answer 11.4 (c) The curve is modeled by $y = 500\cosh(\frac{x}{500}) - 440.536$ and the minimum height is 59.46 feet.

Answer 12.1 (a) 0.6826; 2.3979; 3.3030; 3.3219;0.3010. (b) 3.555; 19.8; 0.0729. (c) $x = \log_{10} y$; $x = \log_{10}(3y)$; $x = (1/3)\log_{10}(y)$.

Answer 12.2 (a) $I(d) = I_\circ(0.94727)^d$. (b) 85 meters.

Answer 12.3 (a) $y = 13e^{1.09861t}$. (b) $y = 2e^{-2.0794t}$.

Answer 12.4 (a) 9.9 years. (b) 34.66%.

Answer 12.5 (a) 8.8 cm. (b) 135 cm. (c) 4.84 yrs. (d) Maximum possible length of the halibut.

Answer 12.6 (a) $C(t) = 1.03526^t$.

Answer 12.7 (c) $v(x) = 55000(1.0315)^x$. Valued at \$200,000 during 1993.

Answer 12.8 (a) $x = 2.7606$. (b) 1.392. (c) 0.3552. (d) $x = e^8$. (e) $x = 11.513$. (f)-0.61. (g) $\sin(x) = -0.6931$; $x = -0.7658 + 2k\pi$ or $x = 3.9074 + 2k\pi$.

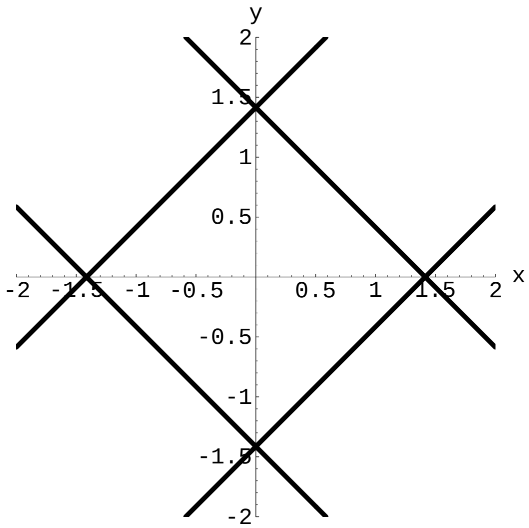
Answer 12.9 (d) 8.617 weeks.

Answer 12.10 31.699250014 days

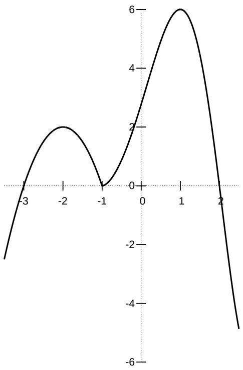
Answer 12.11 (a) 2015 (b) 2048.

Answer 12.12 (a) 46.701735 years (b) 137.3113631016 years after 1980

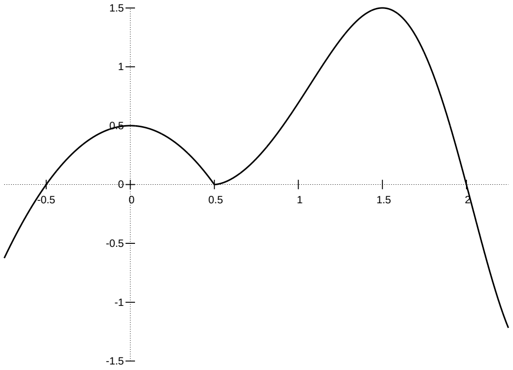
Answer 13.1 The graph is given below:



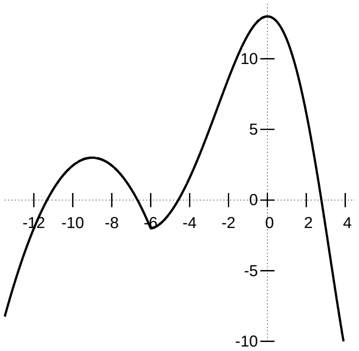
Answer 13.2 (a)



(b)



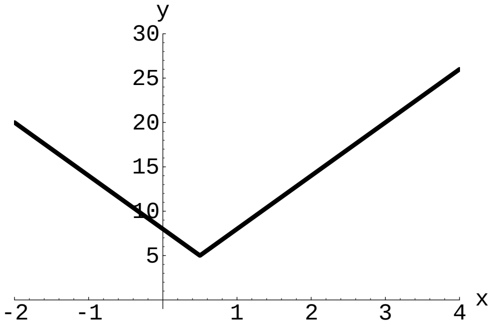
(c)



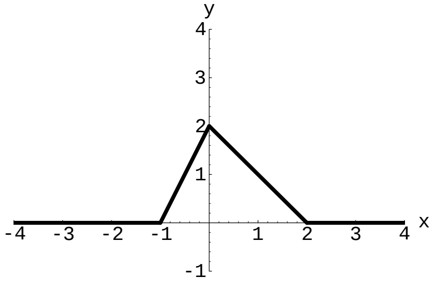
Answer 13.3 $-\frac{1}{3} \leq x \leq 0$.

Answer 13.4 (a5) (1) horizontally dilate (compress) by a factor of 2; (2) horizontal shift right by $\frac{1}{2}$; (3) vertical dilate (expand) by a factor of 3; vertical shift up by 5.

$$f(x) = \begin{cases} -6x + 8 & \text{if } x < \frac{1}{2} \\ 6x + 2 & \text{if } x \geq \frac{1}{2} \end{cases}$$

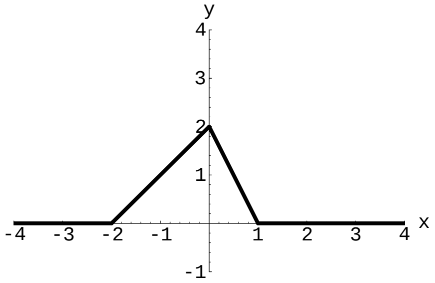


Answer 13.5 (a)

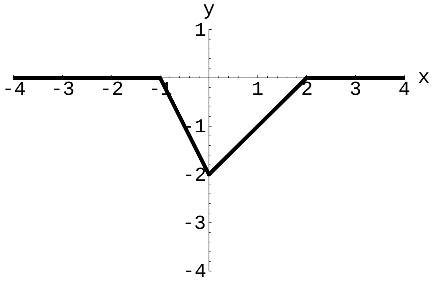


(b) No. (c)

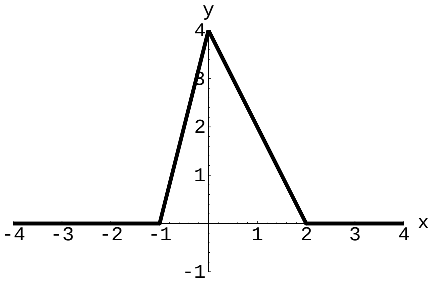
$$y = f(-x) = \begin{cases} 0 & \text{if } x \leq -2 \\ x + 2 & \text{if } -2 \leq x \leq 0 \\ -2x + 2 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$

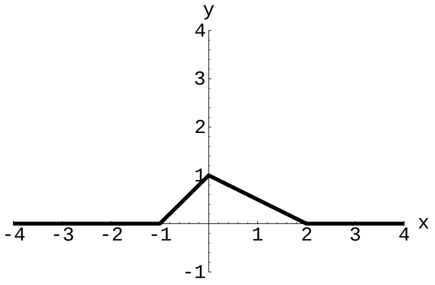


$$y = -f(x) \begin{cases} 0 & \text{if } x \leq -1 \\ -2x - 2 & \text{if } -1 \leq x \leq 0 \\ x - 2 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{if } x \geq 2 \end{cases}$$

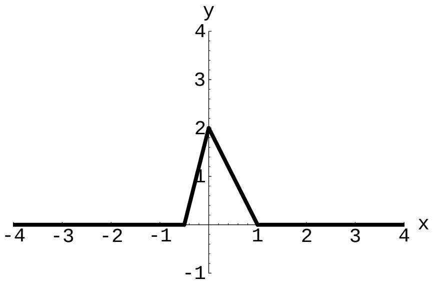


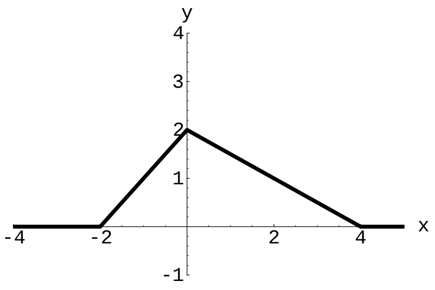
(d) Here are the graphs of $y = 2f(x)$ and $y = \frac{1}{2}f(x)$, respectively:





(e) Here are the graphs of $y = f(2x)$ and $y = f(\frac{1}{2}x)$, respectively:





(f) $c = \frac{11}{2}$. (g) $c = \frac{5}{2}$. (h) $c = d = \frac{1}{3}$.

Answer 13.6 (a) $a(x) = 2(6-x)\sqrt{3x-9}$ with domain $3 \leq x \leq 6$. (b) Max of $a(x)$ is $4\sqrt{3}$ and max of $2a(3x+3)+1$ is $8\sqrt{3}+1$ (when $x = \frac{1}{3}$). (c) $1 + 36(1-x)\sqrt{x}$ on the domain $0 \leq x \leq 1$. The range is all y values between 1 and $8\sqrt{3}+1$.

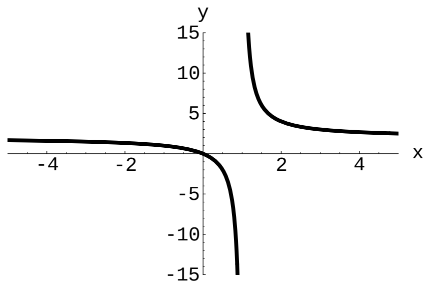
Answer 13.7 (c) Horizontally shift the graph of $y = x^2$ to the right 4 units.

Answer 13.8 (a) $y = 7|2x+4| + 2$.

Answer 13.9 (a1) $f(2x) = 4^x$. $f(2x-1) = \frac{1}{2}4^x$.

Answer 13.10 $y = \frac{1}{3}(2^x)$ is obtained by vertically dilating $y = 2^x$; it is vertically compressed. $y = 2^{x/3}$ is a horizontal dilation of $y = 2^x$; it is horizontally stretched.

Answer 14.1 (a) domain=$\{x|x \neq 1\}$; range=$\{y|y \neq 2\}$; zero at $x = 0$; horizontal asymptote $y = 2$; vertical asymptote $x = 1$; graph below:



Answer 14.2 (a) $y = \frac{0.2x-10.4}{x-10}$. (b) $x = 9.143$ ft.; $x = 9.916$ ft. (c) 10 ft.

Answer 14.3 (a) $m(t) = 35t + 200$. (b) $k(t) = 30t - 50$. (c) 1987. (d) $r(t) = \frac{35t+200}{30t-50}$. (e) $\frac{7}{6}$.

Answer 14.4 (a) $y = 0.03x^2 - 1.1x + 14$ (b) $w = 0.03x - 1.1 + \frac{14}{x}$ (c) 70 or $6\frac{2}{3}$

Answer 14.5 $f(x) = \dfrac{\frac{41}{11}x + \frac{35}{11}}{x + \frac{65}{11}} = \dfrac{41x+35}{11x+65}$. The horizontal asymptote is $y = \dfrac{41}{11}$.

Answer 14.6 $f(x) = \dfrac{6x+10}{x+1}$

Answer 14.7 You should study for 11.25 hours.

Answer 14.8 (a) $k = 400$ (b) $I(t) = \frac{400}{484t^2 - 1452t + 1189}$ (c) $t = 1.5$ (d) $t = 1.05$ and 1.95.

Answer 14.9

Answer 14.10 (a)$y = f(x) = \frac{400x+20000}{x+200}$. (b) $x = \$400$. (d)$x = f^{-1}(y) = \frac{200y-20000}{400-y}$. Domain $f^{-1} = \{100 \leq y \leq 388.46\}$, Range $f^{-1} = \{0 \leq x \leq 5000\}$. The inverse function takes the number of customers per day as an input value and gives the amount the shop spent on advertising as an output value.

Answer 15.1 (a) $13^\circ 24'$ or 0.233874 rads. (b) 1.0788 degs or .01882 rads. (c) 5.7296 degs or $5^\circ 43' 46.5''$.

Answer 15.2 (a) 6080 ft. (b) 29.95 mph. (c) 15.63 knots.

Answer 15.3 (a) 430 sq. in. (b) 2.56°. (c) 84.47 in. (d) 23.04 in. (e) 7.29 in.

Answer 15.4 (a) 1413.7 sq. ft. (c) 4.244 sec.

Answer 15.5 2164.208272472 miles.

Answer 15.6 (a) 2.147 hrs. (b) 1103 mph. (c) 12.47 hrs., 6236 miles. (d) 13760 miles.

Answer 15.7 (b) 84.47 sq. in. (c) 181.5 sq. in.

Answer 15.8 0.685078 miles.

Answer 15.9 Middle picture: shaded area= 12.537 sq. in.

Answer 16.1 (a) $10\pi/3$ rad = 10.47 radians. (b) $4/2\pi$ rev * 1 hour/rev = .64 hours = 38.2 minutes. (c) Using (2.2.2), $(21)(10\pi/3)$ = 219.91 meters.

Answer 16.2 194 RPM.

Answer 16.3 (a) $\omega = \frac{11\pi}{30}$ rad/sec, $\nu = \frac{121\pi}{15}$ ft/sec. (b) $\nu = 2.094$ in/sec, $\omega = 2.5$ RPM.

Answer 16.4 (b) 700 ft. (c) 70 sec. (d) 15000 sq. ft.

Answer 16.5 (a) $r = 233.427$ ft. (b) $\theta = 0.06283$ rad. (c) $2\pi/5$ rad counterclockwise from P.

Answer 16.6 (a) 40π ft/sec; 85.68 mph. (b) 400 RPM. (c) 60 RPM; 32π ft/sec. (d) 1.445 rad = 82.8°; 28.9 ft. (e) 0.7 sec; 1.4π rad.

Answer 16.7 4.4 inches.

Answer 16.8 $r = 2.45$ inches.

Answer 17.1 (a) If you impose coordinates with the center of the wheel at $(0, 237.427)$, then ground level coincides with the x-axis. (a) $T(t) = (x(t), y(t))$, where $x(t) = 233.427\cos(\frac{2\pi}{5}t - 0.06283)$ and $y(t) = 233.427\sin(\frac{2\pi}{5}t - 0.06283) + 237.427$. (b) $T(6) = (85.93, 454.45)$. (c) First find the slope of a radial line from the wheel center out to Tiff's launch point.

Answer 17.2 (a) $y = \pm 0.2309(x+1) + 2$

Answer 17.3 290 ft.

Answer 17.4 (a) (-21.91218, -1.498834) (b) (-5.92564, 21.14892) (c) (19.07064, -10.89497)

Answer 17.5 (a) $(19.9, 13.42)$. (b) $(22.55, 8.21)$. (c) $(-1.674, 23.942)$. (d) $(23.882, 2.375)$.

Answer 17.6 101.496936 feet above the ground.

Answer 17.7 The dam is 383 feet high.

Answer 17.8 108 ft.

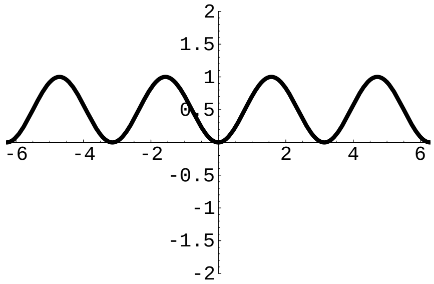
Answer 17.9 With the center of the track at the origin, and the northernmost point on the positive y-axis, Charlie's location after one minue of running is (59.84016,4.37666).

Answer 17.10 105.2718216 feet

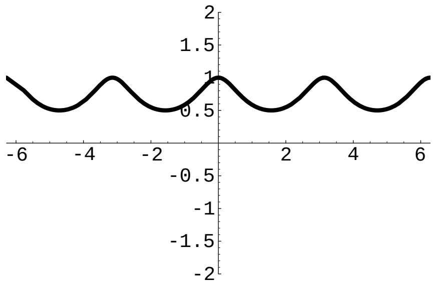
Answer 17.11 (a) 204.74 ft. (b) no.

Answer 17.12 Top right scenario: (a) $\theta_\circ = 1.2$ rad. (b)$\theta(t) = 1.2 + \frac{4\pi t}{9}$ (c) $b(t) = (2\cos(1.2 + \frac{4\pi t}{9}), 2\sin(1.2 + \frac{4\pi t}{9}))$. (d) $b(1) = (-1.710, 1.037)$. $b(0) = (0.725, 1.864)$. $b(3) = (1.252, -1.560)$. $b(22) = (1.753, 0.962)$.

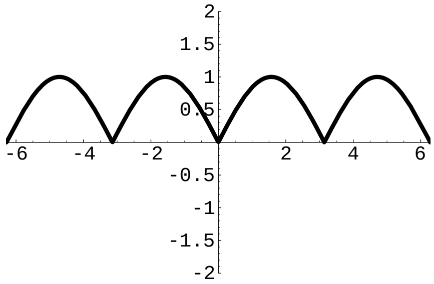
Answer 18.1 (b) $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$.



(c)



Answer 18.2 (a)



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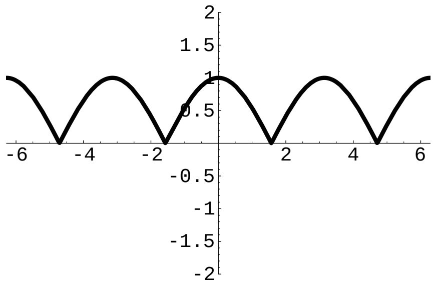
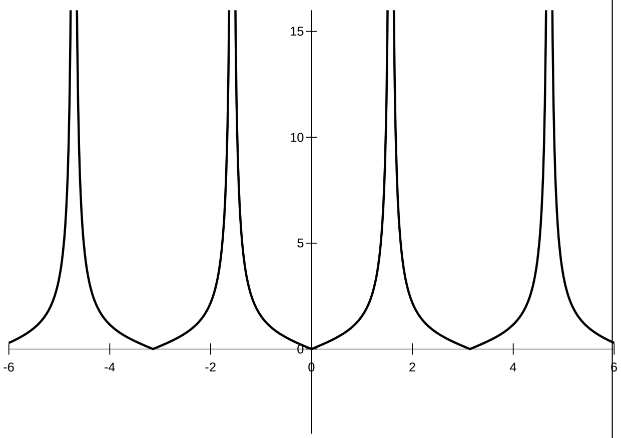
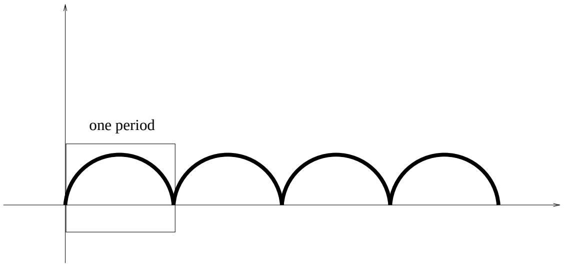


(c)



Answer 18.3 (a) 7/25 or -7/25. (b) -0.6. (c) $\pm\frac{\sqrt{40}}{7}$.

Answer 18.4 In the first case,



Answer 18.5 $\frac{\pi}{4}+2k\pi$ and $\frac{5\pi}{4}+2k\pi$, $k=0,\pm1,\pm2,\pm3,\ldots$.

Answer 19.1 (a) 1, π, $\frac{\pi}{2}$, 1. (b) 6, 2, 0, -1.

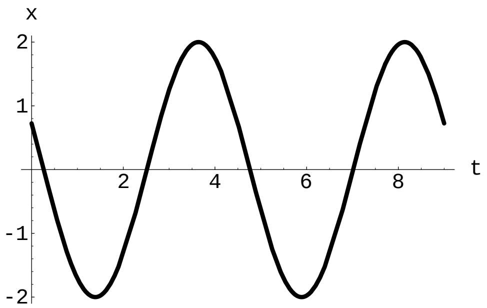
Answer 19.2 (d) $h(t)=8\sin(\frac{2\pi}{1.2}(t-0.3))+18$.

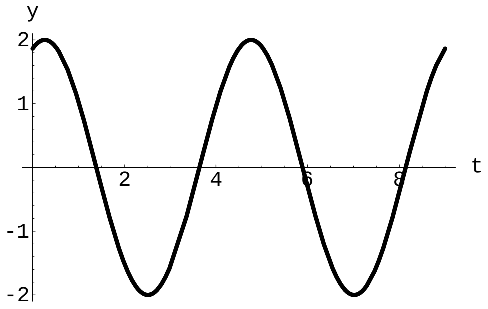
Answer 19.3 (a) $b(t)=0.6\sin(\frac{2\pi}{100}(t-30))+1.2$.

Answer 19.4 (a) $h(t)=5\sin(\frac{\pi}{6}(t-10)+5$, where t indicates hours after midnight. (b) 7.5 ft. above low tide.

Answer 19.5 (a) A = 25, B = 5 seconds, C = 1.75, D = 28. (b) t=1.75 and 4.25 seconds

Answer 19.6 First scenario: $x(t)=2\sin(\frac{2\pi}{4.5}(t-C_x))$, where $C_x=-(1.2+\frac{\pi}{2})(\frac{9}{4\pi})$; $y(t)=2\sin(\frac{2\pi}{4.5}(t-C_y))$, where $C_y=-(1.2)(\frac{9}{4\pi})$. Plots are below:





Answer 19.7 (a)2 volts. (b) Never zero since 2^x is positive for all x. (c)$p(t)=3\sin(\frac{2\pi}{2/5}(t-\frac{3}{5})+1$, so $A=3, D=1, B=2/5, C=3/5$. (d) $0.25\le V(t)\le 16$. (e)$t=0.65635+k(0.4)$ and $t=0.74365+k(0.4)$, $k=0,\pm1,\pm2,\ldots$ are ALL solutions. Four of these lie in the domain $0\le t\le 1$. (f) Maxima have coordinates $(0.3+k(0.4),16)$ and minima have coordinates $(0.1+k(0.4),0.25)$, where $k=0,\pm1,\pm2,\ldots$. (g) If we restrict $V(t)$ to $0.5\le t\le 0.7$, the inverse function has rule:

$$t=\frac{\arcsin(\frac{\ln(y)-\ln(2)}{3\ln(2)})+3\pi}{5\pi}.$$

If we restrict $V(t)$ to $0.5+k(0.4)\le t\le 0.7+k(0.4)$, the inverse function has rule:

$$t=k(0.4)+\frac{\arcsin(\frac{\ln(y)-\ln(2)}{3\ln(2)})+3\pi}{5\pi}.$$

In particular, if we restrict $V(t)$ to $0.1\le t\le 0.3$, the inverse function has rule:

$$t=-0.4+\frac{\arcsin(\frac{\ln(y)-\ln(2)}{3\ln(2)})+3\pi}{5\pi}.$$

If we restrict $V(t)$ to $0.7\le t\le 0.9$, the inverse function has rule:

$$t=\frac{-\arcsin(\frac{\ln(y)-\ln(2)}{3\ln(2)})+4\pi}{5\pi}.$$

If we restrict $V(t)$ to $0.7+k(0.4)\le t\le 0.9+k(0.4)$, the inverse function has rule:

$$t=k(0.4)+\frac{-\arcsin(\frac{\ln(y)-\ln(2)}{3\ln(2)})+4\pi}{5\pi}.$$

In particular, if we restrict $V(t)$ to $0.3\le t\le 0.5$, the inverse function has rule:

$$t=-0.4+\frac{-\arcsin(\frac{\ln(y)-\ln(2)}{3\ln(2)})+4\pi}{5\pi}.$$

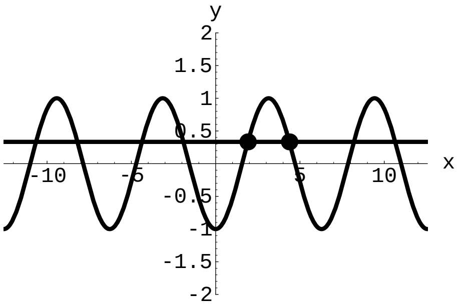
Answer 19.8 (c) On domain $t\ge 0$,

$$B=B(t)=2\cos(6\pi t)+\sqrt{36-4(\sin(6\pi t))^2}.$$

(e) 0.1023, 0.2310.

Answer 20.1 (a1) 0, 1.5708, −1.5708, 1.0472, 0.7168, −0.2762, not defined.

Answer 20.2 (a) Principal solution: $x = 1.9106$, symmetry solution: $x = 4.3726$; graph below with these two solutions graphically indicated:



Answer 20.3 (a) 9.39, 13.63, 11.09. (b) Feb. 3, Nov. 4.

Answer 20.4 (a) $A = 15, D = 415, B = 10, C = 15/2$. Note that $C = 10k + 15/2$, $k = 0, \pm 1, \pm 2, \ldots$ are also all valid choices for the phase shift. (b) maximum temperature= 430°F. (c) minimum temperature= 400°F. (d) 12.1635 minutes. (e) 14.6456 minutes. (f) 6.80907 minutes.

Answer 20.5 The cake should be in the oven for 67.572141357 minutes.

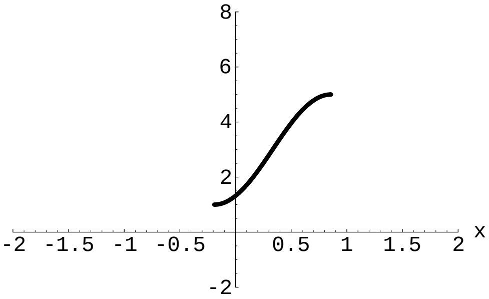
Answer 20.6 6.42529 hours

Answer 20.7 8.9286 hours of dry time each day.

Answer 20.8 The key fact to use over and over is this: $M(t)$=M's location after t seconds = $(100\cos(0.025t), 100\sin(0.025t))$; $T(t)$= T's location after t seconds = $(100\cos(0.03t + \pi), 100\sin(0.03t + \pi))$.

Answer 20.9 (a)$\alpha = arcsin[3960/(3960 + t)]$. (b) $\alpha = 6.696 degs$. The interior angle is 166.608 degs and so one satellite covers 46% of the circumference. Thus you need 3 (not two-point-something) satellites to cover the earth's circumference. (c)$\alpha = 52.976 deg$. The interior angle is 74.047 degs and so one satellite covers 20.57% of the circumference. Thus you need 5 satellites to cover the earth's circumference. (d) Get an equation for the interior angle in terms of t. Solve $2(90 - \arcsin[3960/(3960 + t)])$ = 20% of 360 degs = 72 degs. You'll get t = 934.83 miles.

Answer 20.10 (a) Domain: $\frac{1}{3} - \frac{\pi}{6} \leq x \leq \frac{1}{3} + \frac{\pi}{6}$; Range: $1 \leq y \leq 5$. One solution: $x = \frac{1}{3} + \frac{\pi}{18}$; graph is below:



Answer 20.11 (a) 3.852624 miles (a) $14.34 lb/in^2$; no explosion (b) $8.32 lb/in^2$; explosion (c) 54.6 degrees

Answer 20.12 (a) Many possible answers; for example: -1.9293, -1.3677, 0.8677, 1.4293.

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