lyryx with Open Texts

A First Course in Linear Algebra

K. Kuttler

10th Anniversary Edition

Lyryx Version
2023-A-D
(CC BY)

Lyryx with Open Texts

Open Texts

This open text is part of the comprehensive *Lyryx with Open Texts* project.

Experienced authors and Lyryx' own editorial team develop and update our open texts, which are made available at no cost to everyone.

Supplements

Additional instructor resources are available to *Lyryx with Open Texts* users.

Product dependent, these may include adaptable slides, videos, case studies, and solution manuals for both students and instructors.

Educational Software

Lyryx with Open Texts offers an advanced online homework and examination platform providing students with personalized feedback to guide their learning, and instructors with all the tools necessary to manage their course assessment.

This text has been enhanced with the *Engage* active learning app, "chunking" the content in small blocks, each with interactive questions.

Support

The Lyryx support team is available 7 days/week, providing prompt resolution to both student and instructor inquiries.

Lyryx with Open Texts provides comprehensive and customized solutions, including managing multiple sections, assistance with online homework and examinations, integrating with LMS, and much more!

A First Course in Linear Algebra

an Open Text

Distribution Version 2023 — Revision A-D

AUTHOR

Ken Kuttler, Brigham Young University

CONTRIBUTIONS

Ilijas Farah, York University

Christopher Leary, SUNY Geneseo

Lyryx Learning editorial group

Be a champion of OER!

Contribute suggestions for improvements, new content, or errata:

- A new topic
- A new example
- An interesting new question
- Any other suggestions to improve the material

Contact Lyryx at info@lyryx.com with your ideas.

A First Course in Linear Algebra

by Ken Kuttler — Distribution Version 2023 — Revision A-D

Attribution

To redistribute all of this book in its original form, please follow the guide below:

The front matter of the text should include a “License” page that includes the following statement.

> This text is *A First Course in Linear Algebra* by K. Kuttler and Lyryx Learning. View the text for free at `https://lyryx.com/first-course-linear-algebra/`

To redistribute part of this book in its original form, please follow the guide below. Clearly indicate which content has been redistributed

The front matter of the text should include a “License” page that includes the following statement.

> This text includes the following content from *A First Course in Linear Algebra* by K. Kuttler and Lyryx Learning. View the entire text for free at `https://lyryx.com/first-course-linear-algebra/`.
> <List of content from the original text>

The following must also be included at the beginning of the applicable content. Please clearly indicate which content has been redistributed from the Lyryx text.

> This chapter is redistributed from the original *A First Course in Linear Algebra* by K. Kuttler and Lyryx Learning. View the original text for free at `https://lyryx.com/first-course-linear-algebra/`.

To adapt and redistribute all or part of this book in its original form, please follow the guide below. Clearly indicate which content has been adapted/redistributed and summarize the changes made.

The front matter of the text should include a “License” page that includes the following statement.

> This text contains content adapted from the original *A First Course in Linear Algebra* by K. Kuttler and Lyryx Learning Inc. View the original text for free at `https://lyryx.com/first-course-linear-algebra/`.
> <List of content and summary of changes>

The following must also be included at the beginning of the applicable content. Please clearly indicate which content has been adapted from the Lyryx text.

> This chapter was adapted from the original *A First Course in Linear Algebra* by K. Kuttler and Lyryx Learning Inc. View the original text for free at `https://lyryx.com/first-course-linear-algebra/`.

Citation

Use the information below to create a citation:

Author: K. Kuttler
Contributions: I. Farah, C. Leary
Publisher: Lyryx Learning Inc.
Book title: A First Course in Linear Algebra
Book version: 2023A
Publication date: February 1, 2023
Location: Calgary, Alberta, Canada
Book URL: `https://lyryx.com/first-course-linear-algebra/`

For questions or comments please contact editorial@lyryx.com

A First Course in Linear Algebra

an Open Text

Revision History: Distribution Version 2023 — Revision A-D

Extensive edits, additions, and revisions have been completed by the editorial group at Lyryx Learning.
All new content (text and images) is released under the same license as noted above.

2023 A	• C. Leary: The entire text has been reviewed to improve the flow and logical organization, some proofs have been rewritten, and some notation made more consistent. A new feature "Looking under the Hood" has been introduced, providing additional insight into key results for the benefit of students interested in better understanding how the techniques actually work! • M. Fels: Various suggestions and new exercises have been incorporated.
2021 A	• Lyryx: Front matter has been updated including cover, Lyryx with Open Texts, copyright, and revision pages. Attribution page has been added. • Lyryx: Typo and other minor fixes have been implemented throughout.
2017 A	• Lyryx: Front matter has been updated including cover, copyright, and revision pages. • I. Farah: contributed edits and revisions, particularly the proofs in the Properties of Determinants II: Some Important Proofs section.
2016 B	• Lyryx: The text has been updated with the addition of subsections on Resistor Networks and the Matrix Exponential based on original material by K. Kuttler. • Lyryx: New example on Random Walks developed.
2016 A	• Lyryx: The layout and appearance of the text has been updated, including the title page and newly designed back cover.
2015 A	• Lyryx: The content was modified and adapted with the addition of new material and several images throughout. • Lyryx: Additional examples and proofs were added to existing material throughout.
2012 A	• **Original text** by K. Kuttler of Brigham Young University. That version is used under Creative Commons license CC BY (`https://creativecommons.org/licenses/by/3.0/`) made possible by funding from The Saylor Foundation's Open Textbook Challenge. See Elementary Linear Algebra for more information and the original version.

Table of Contents

Preface

A First Course in Linear Algebra

The text *A First Course in Linear Algebra* presents an introduction to the fascinating subject of linear algebra for students who have a reasonable grasp of basic algebra. It was originally written by Ken Kuttler who generously made it available under an open license. The text has since been extensively revised and adapted by various contributors under the guidance of the Lyryx Learning editorial group.

The 2023A version celebrates the 10^{th} anniversary of the text!

Overview

The major techniques of linear algebra are presented in detail, with proofs of important theorems provided. Various additional topics and applications of key concepts are explored in an effort to assist those students who are interested in continuing on with linear algebra connections to other fields, or to pursue the subject in advanced courses.

A new feature "Looking under the Hood" provides additional insight into key results for the benefit of students interested in better understanding how these techniques actually work! Those can be found throughout the book where appropriate, and can be omitted without loss of continuity.

Each chapter begins with a list of desired outcomes for students to achieve upon completing the chapter. Throughout the text, examples and diagrams are included to reinforce ideas and provide guidance on how to approach various problems. Students are encouraged to work through the suggested exercises provided at the end of each section, with selected solutions found at the end of the text.

Open License

As this is an open licensed text, everyone is encouraged to interact with the textbook through annotating, revising, and reusing to their advantage.

Lyryx with Open Texts

This open text is part of the *Lyryx with Open Texts* project. Authors and the Lyryx editorial team develop and update open texts which are made available at no cost to everyone. Various supplements, including slides and chapter exercises are available to *Lyryx with Open Texts* users.

Lyryx with Open Texts further offers educational software to support student learning and course assessment. The Lyryx online homework and examinations platform provides instructors with all the tools necessary to manage their course assessment. The Lyryx online problems are designed by experienced instructors, and the Lyryx marking algorithms capture their expertise to analyze and assess students' work, looking for common errors, consistent mistakes through multi-part questions, and arithmetic errors, to provide instant and informative personalized feedback.

This text has been enhanced with the *Engage* active learning app, "chunking" the content in small blocks each with interactive questions designed to encourage students in reading and learning course material!

Contact Lyryx at info@lyryx.com for inquiries.

Chapter 1

Systems of Equations

1.1 Systems of Equations, Geometry

Outcomes

A. *Relate the types of solution sets of a system of two (three) variables to the intersections of lines in a plane (the intersection of planes in three space)*

Welcome! We will begin our study of linear algebra by investigating methods of finding *solutions* to *systems* of *linear equations*. There are three highlighted terms in that last sentence, and it will be worthwhile to review what we mean by each of them. We'll start with the third term.

- A *linear equation* is an algebraic expression that includes an equals sign (hence, an equation), one or more variables (usually denoted by italicized letters like x and y and z or maybe subscripted letters like x_3 and z_{15}). In a linear equation these variables can be multiplied by numbers and then added to or subtracted from other such expressions or numbers, but no other operations are allowed. So the following are examples of linear equations:

$$x+3=5, \quad y=4x+7, \quad -3x+17y-42z=24601, \quad \frac{2}{3}x_1-\frac{5}{7}x_2+\pi x_3=3x_4-4x_5;$$

  while these expressions, although equations, are not linear:

$$y=x^2, \quad 3x+2xy=12, \quad \cos(2x+3y)=0.123.$$

  Again, in linear algebra we will be not be looking at these nonlinear equations.

- A *system* of linear equations is just a collection of one or more linear equations. So you are already an expert at solving systems of one linear equation. And you will soon be an expert at solving systems of more than one such equation.

- To *solve* a system of equations that involve the variables some set of variables means to find sets of numbers, one for each variable, so that when the numbers are substituted for the variables in the equations, every one of the equations is true. For example, the ordered pair of numbers $(x,y)=(3,1)$ is a solution to the system of linear equations

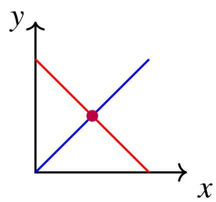
$$\begin{aligned} y&=2x-5 \\ 2x-4y&=2 \end{aligned}$$

  since it is the case that $1=2\cdot 3-5$ and $2\cdot 3-4\cdot 1=2$ are both true statements.

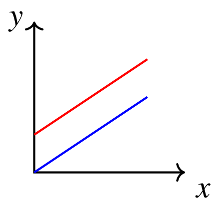
You probably remember that if we take an equation in the two variables x and y, for example the equation $2x+3y=6$, we can draw the graph of the equation in the coordinate plane. When we do that we are really just coloring in the collection of all of the solutions to the linear equation. Is $(0,2)$ on the graph of $2x+3y=6$? That means that the ordered pair of numbers $(x,y)=(0,2)$ is a solution to the equation. Since $(3,17)$ is not on the graph of the equation, then the ordered pair $(x,y)=(3,17)$ is not a solution to the equation. The graphs of linear equations allow us to tie together the algebraic object (the equation) with a geometric object (the graph of the equation) and use one to help inform us about the other. In this section we will concentrate on the geometric objects and use them to investigate what we can say about solutions to systems of equations in two or three variables.

Suppose you consider a system of two linear equations in the variables x and y. Each of these equations can be graphed as a straight line, and consider graphing both of these lines using the same set of axes. What would it mean if there exists a point of intersection between the two lines? This point, which lies on *both* graphs, gives x and y values for which both equations are true. In other words, this point gives the ordered pair (x,y) that satisfy both equations. If the point (x,y) is a point of intersection, we say that (x,y) is a **solution** to the system of two equations. In linear algebra, we often are concerned with finding the solution(s) to a system of equations, if such solutions exist. First, we consider graphical representations of solutions and later we will consider the algebraic methods for finding solutions.

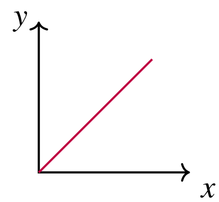
When looking for the intersection of two lines in a graph, several situations may arise. The following picture demonstrates the possible situations when considering two equations (two lines in the graph) involving two variables.



One Solution



No Solutions



Infinitely Many Solutions

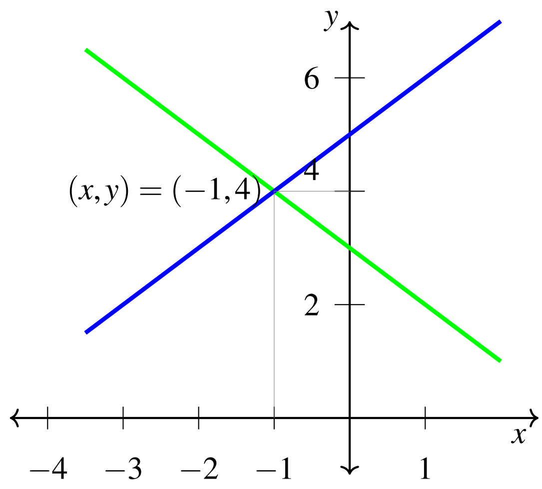
In the first diagram, there is a unique point of intersection, which means that there is only one (unique) solution to the two equations. In the second, there are no points of intersection and no solution. When no solution exists, this means that the two lines are parallel and they never intersect. The third situation which can occur, as demonstrated in diagram three, is that the two lines are really the same line. For example, $x+y=1$ and $2x+2y=2$ are equations which when graphed yield the same line. In this case there are infinitely many points which are solutions of these two equations, as every ordered pair which is on the graph of the line satisfies both equations. When considering linear systems of equations, there are always three types of solutions possible; exactly one (unique) solution, infinitely many solutions, or no solution.

Example 1.1: A Graphical Solution

Use a graph to find the solution to the following system of equations

$$\begin{aligned} x+y&=3 \\ y-x&=5 \end{aligned}$$

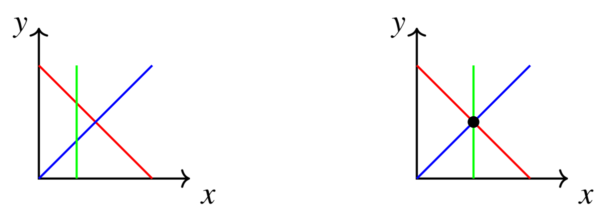
Solution. Through graphing the above equations and identifying the point of intersection, we can find the solution(s). Remember that we must have either one solution, infinitely many, or no solutions at all. The following graph shows the two equations, as well as the intersection. Remember, the point of intersection represents the solution of the two equations, or the (x,y) which satisfy both equations. In this case, there is one point of intersection at $(-1,4)$ which means we have one unique solution, $x=-1, y=4$.



♠

In the above example, we investigated the intersection point of two equations in two variables, x and y. Now we will consider the graphical solutions of three equations in two variables.

Consider a system of three equations in two variables. Again, these equations can be graphed as straight lines in the plane, so that the resulting graph contains three straight lines. Recall the three possible types of solutions; no solution, one solution, and infinitely many solutions. There are now more complex ways of achieving these situations, due to the presence of the third line. For example, you can imagine the case of three intersecting lines having no common point of intersection. Perhaps you can also imagine three intersecting lines which do intersect at a single point. These two situations are illustrated below.

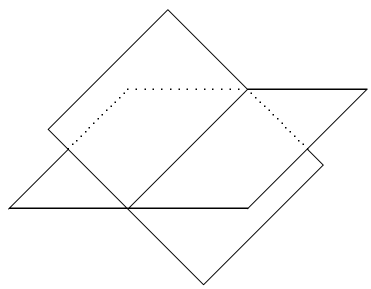


No Solution   One Solution

Consider the first picture above. While all three lines intersect with one another, there is no common point of intersection where all three lines meet at one point. Hence, there is no solution to the system of three equations. Remember, a solution is a point (x,y) which satisfies **all** three equations. In the case of the second picture, the lines intersect at a common point. This means that there is one solution to the three equations whose graphs are the given lines. You should take a moment now to draw the graph of a system which results in three parallel lines. Next, try the graph of three identical lines. Which type of solution is represented in each of these graphs?

We have now considered the graphical solutions of systems of two equations in two variables, as well as three equations in two variables. However, there is no reason to limit our investigation to equations in two variables. We will now consider equations in three variables.

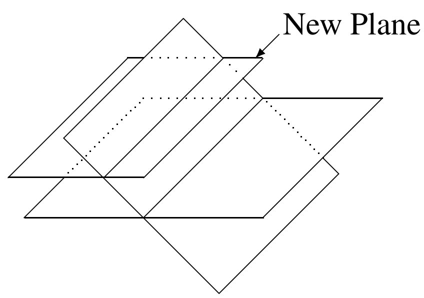
You may recall that linear equations in three variables, such as $2x+4y-5z=8$, represent a plane in three-space. Above, we were looking for intersections of lines in order to identify any possible solutions. When graphically solving systems of equations in three variables, we look for intersections of planes. These points of intersection give the (x,y,z) that satisfy all the equations in the system. What types of solutions are possible when working with three variables? Consider the following picture involving two planes, which are given by two equations in three variables.



Notice how these two planes intersect in a line. This means that the points (x,y,z) on this line satisfy both equations in the system. Since the line contains infinitely many points, this system has infinitely many solutions.

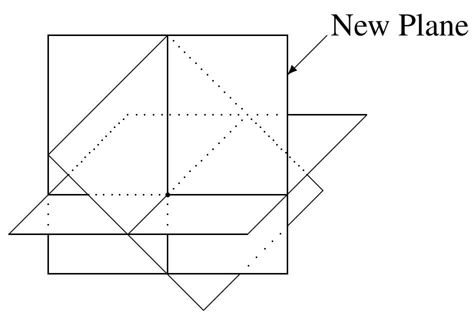
It could also happen that the two planes fail to intersect. However, is it possible to have two planes intersect at a single point? Take a moment to attempt drawing this situation, and convince yourself that it is not possible! This means that when we have only two equations in three variables, there is no way to have a unique solution! Hence, the types of solutions possible for two equations in three variables are no solution or infinitely many solutions.

Now imagine adding a third plane. In other words, consider three equations in three variables. What types of solutions are now possible? Consider the following diagram.

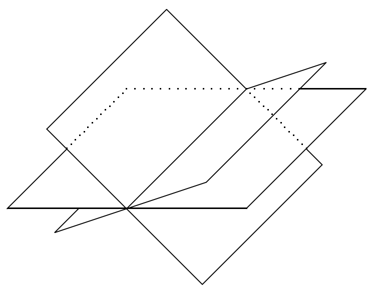


In this diagram, there is no point which lies in all three planes. There is no intersection between **all** planes so there is no solution. The picture illustrates the situation in which the line of intersection of the new plane with one of the original planes forms a line parallel to the line of intersection of the first two planes. However, in three dimensions, it is possible for two lines to fail to intersect even though they are not parallel. Such lines are called **skew lines**.

Recall that when working with two equations in three variables, it was not possible to have a unique solution. Is it possible when considering three equations in three variables? In fact, it is possible, and we demonstrate this situation in the following picture.



In this case, the three planes have a single point of intersection. Can you think of other types of solutions possible? Another is that the three planes could intersect in a line, resulting in infinitely many solutions, as in the following diagram.



We have now seen how three equations in three variables can have no solution, a unique solution, or intersect in a line resulting in infinitely many solutions. It is also possible that the three equations represent the same plane, which also leads to infinitely many solutions.

You can see that when working with equations in three variables, there are many more ways to achieve the different types of solutions than when working with two variables. It may prove enlightening to spend time imagining (and drawing) many possible scenarios, and you should take some time to try a few.

You should also take some time to imagine (and draw) graphs of systems in more than three variables. Equations like $x+y-2z+4w=8$ with more than three variables are often called **hyper-planes**. You may soon realize that it is tricky to draw the graphs of hyper-planes! Through the tools of linear algebra, we can algebraically examine these types of systems which are difficult to graph. In the following section, we will consider these algebraic tools.

1.2 Systems of Equations, Algebraic Procedures

Outcomes

A. *Use elementary operations to find the solution to a linear system of equations.*

B. *Given a matrix, use row operations to reduce it to row-echelon form and to reduced row-echelon form.*

C. *Determine whether a system of linear equations has no solution, a unique solution or an infinite number of solutions from its row-echelon form.*

D. *Solve a system of equations using Gaussian Elimination and Gauss-Jordan Elimination.*

E. *Model a physical system with linear equations and then solve.*

We have taken an in depth look at graphical representations of systems of equations, as well as how to find possible solutions graphically. Our attention now turns to working with systems algebraically.

Definition 1.2: System of Linear Equations

A ***system of linear equations*** *is a list of equations,*

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}$$

where a_{ij} and b_j are real numbers. The above is a system of m equations in the n variables, $x_1, x_2 \cdots, x_n$. Written more simply in terms of summation notation, the above can be written in the form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \ i = 1, 2, 3, \cdots, m$$

The relative size of m and n is not important here. Notice that we have allowed a_{ij} and b_j to be any real number. We can also call these numbers **scalars** . We will use this term throughout the text, so keep in mind that the term **scalar** just means that we are working with real numbers.

Now, suppose we have a system where $b_i = 0$ for all i. In other words every equation equals 0. This is a special type of system.

Definition 1.3: Homogeneous System of Equations

A system of equations is called ***homogeneous*** *if each equation in the system is equal to 0. A homogeneous system has the form*

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}$$

where a_{ij} are scalars and x_i are variables.

Recall from the previous section that our goal when working with systems of linear equations was to find the point of intersection of the equations when graphed. In other words, we looked for the solutions to the system. We now wish to find these solutions algebraically. We want to find values for $x_1, \cdots, x_n$ which solve all of the equations. If such a set of values exists, we call $(x_1, \cdots, x_n)$ the **solution set**.

Recall the above discussions about the types of solutions possible. We will see that systems of linear equations will have one unique solution, infinitely many solutions, or no solution. Consider the following definition.

Definition 1.4: Consistent and Inconsistent Systems

A system of linear equations is called ***consistent*** *if there exists at least one solution. It is called* ***inconsistent*** *if there is no solution.*

If you think of each equation as a condition which must be satisfied by the variables, consistent would mean there is some choice of variables which can satisfy **all** the conditions. Inconsistent would mean there is no choice of the variables which can satisfy all of the conditions.

The following sections provide methods for determining if a system is consistent or inconsistent, and finding solutions if they exist.

Elementary Operations

We begin this section with an example.

Example 1.5: Verifying an Ordered Pair is a Solution

Algebraically verify that $(x,y) = (-1,4)$ *is a solution to the following system of equations.*

$$\begin{array}{c} x+y=3 \\ y-x=5 \end{array}$$

Solution. By graphing these two equations and identifying the point of intersection, we previously found that $(x,y) = (-1,4)$ is the unique solution.

We can verify algebraically by substituting these values into the original equations, and ensuring that the equations hold. First, we substitute the values into the first equation and check that it equals 3.

$$x+y=(-1)+(4)=3$$

This equals 3 as needed, so we see that $(-1,4)$ is a solution to the first equation. Substituting the values into the second equation yields

$$y-x=(4)-(-1)=4+1=5$$

which is true. For $(x,y) = (-1,4)$ each equation is true and therefore, this is a solution to the system. ♠

Now, the interesting question is this: If you were not given these numbers to verify, how could you algebraically determine the solution? Linear algebra gives us the tools needed to answer this question.

The idea here is this: If we don't know the solution to this system of equations, let's take the system and trade it in for an easier, equivalent system of equations. We will say that two systems of equations are **equivalent** if they have the same solution set. We hope to take our system of equations and eventually find an equivalent system of equations that has a solution set that we can easily (or at least sort of easily) see.

The following basic operations are important tools that we will utilize.

Definition 1.6: Elementary Operations

Elementary operations *are those operations consisting of the following.*

1. *Interchange the order in which the equations are listed.*
2. *Multiply any equation by a nonzero number.*
3. *Add a multiple of one equation to another equation.*

It is important to note that none of these operations will change the set of solutions of the system of equations, as we prove below in Theorem 1.2. So, if we have a system of equations and apply one of these elementary operations, we will end up with a system of equations that is equivalent to the system that we started with. Elementary operations are the *key tool* we use in linear algebra to find solutions to systems of equations.

Consider the following example.

Example 1.7: Effects of an Elementary Operation

Show that the system

$$\begin{aligned} x+y &= 7 \\ 2x-y &= 8 \end{aligned}$$

has the same solution as the system

$$\begin{aligned} x+y &= 7 \\ -3y &= -6 \end{aligned}$$

Solution. Notice that the second system has been obtained by taking the second equation of the first system and adding -2 times the first equation, as follows:

$$2x-y+(-2)(x+y) = 8+(-2)(7)$$

By simplifying, we obtain

$$-3y = -6$$

which is the second equation in the second system. Now, from here we can solve for y and see that $y = 2$. Next, we substitute this value into the first equation as follows

$$x+y = x+2 = 7$$

Hence $x = 5$ and so $(x,y) = (5,2)$ is a solution to the second system. We want to check if $(5,2)$ is also a solution to the first system. We check this by substituting $(x,y) = (5,2)$ into the system and ensuring the equations are true.

$$\begin{aligned} x+y &= (5)+(2) = 7 \\ 2x-y &= 2(5)-(2) = 8 \end{aligned}$$

Hence, $(5,2)$ is also a solution to the first system. ♠

This example illustrates how an elementary operation applied to a system of two equations in two variables does not affect the solution set. However, a linear system may involve many equations and many variables and there is no reason to limit our study to small systems. For any size of system in any number of variables, the solution set is still the collection of solutions to the equations. In every case, the above operations of Definition 1.6 do not change the set of solutions to the system of linear equations.

In the following theorem, we use the notation E_i to represent an expression, while b_i denotes a constant.

Theorem 1.8: Elementary Operations and Solutions

Suppose you have a system of two linear equations

$$\begin{aligned} E_1 &= b_1 \\ E_2 &= b_2 \end{aligned} \tag{1.1}$$

Then the following systems have the same solution set as 1.1:

1. $$\begin{aligned} E_2 &= b_2 \\ E_1 &= b_1 \end{aligned} \tag{1.2}$$

2. $$\begin{aligned} E_1 &= b_1 \\ kE_2 &= kb_2 \end{aligned} \tag{1.3}$$

   for any scalar k, provided $k \neq 0$.

3. $$\begin{aligned} E_1 &= b_1 \\ E_2 + kE_1 &= b_2 + kb_1 \end{aligned} \tag{1.4}$$

   for any scalar k (including $k = 0$).

Before we proceed with the proof of Theorem 1.8, let us consider this theorem in context of Example 1.7. Then,

$$\begin{aligned} E_1 &= x+y, \quad b_1 = 7 \\ E_2 &= 2x-y, \quad b_2 = 8 \end{aligned}$$

Recall the elementary operations that we used to modify the system in the solution to the example. First, we added (-2) times the first equation to the second equation. In terms of Theorem 1.8, this action is given by

$$E_2 + (-2)E_1 = b_2 + (-2)b_1$$

or

$$2x - y + (-2)(x+y) = 8 + (-2)7$$

This gave us the second system in Example 1.7, given by

$$\begin{aligned} E_1 &= b_1 \\ E_2 + (-2)E_1 &= b_2 + (-2)b_1 \end{aligned}$$

From this point, we were able to find the solution to the system. Theorem 1.8 tells us that the solution we found is in fact a solution to the original system.

We will now prove Theorem 1.8.

Proof.

1. The proof that the systems 1.1 and 1.2 have the same solution set is as follows. Suppose that $(x_1, \cdots, x_n)$ is a solution to $E_1 = b_1, E_2 = b_2$. We want to show that this is a solution to the system in 1.2 above. This is clear, because the system in 1.2 is the original system, but listed in a different order. Changing the order does not effect the solution set, so $(x_1, \cdots, x_n)$ is a solution to 1.2.

2. Next we want to prove that the systems 1.1 and 1.3 have the same solution set. That is $E_1 = b_1, E_2 = b_2$ has the same solution set as the system $E_1 = b_1, kE_2 = kb_2$ provided $k \neq 0$. Let $(x_1, \cdots, x_n)$ be a solution of $E_1 = b_1, E_2 = b_2$,. We want to show that it is a solution to $E_1 = b_1, kE_2 = kb_2$. Notice that the only difference between these two systems is that the second involves multiplying the equation, $E_2 = b_2$ by the scalar k. Recall that when you multiply both sides of an equation by the same number, the sides are still equal to each other. Hence if $(x_1, \cdots, x_n)$ is a solution to $E_2 = b_2$, then it will also be a solution to $kE_2 = kb_2$. Hence, $(x_1, \cdots, x_n)$ is also a solution to 1.3.

   Similarly, let $(x_1, \cdots, x_n)$ be a solution of $E_1 = b_1, kE_2 = kb_2$. Then we can multiply the equation $kE_2 = kb_2$ by the scalar $1/k$, which is possible only because we have required that $k \neq 0$. Just as above, this action preserves equality and we obtain the equation $E_2 = b_2$. Hence $(x_1, \cdots, x_n)$ is also a solution to $E_1 = b_1, E_2 = b_2$.

3. Finally, we will prove that the systems 1.1 and 1.4 have the same solution set. We will show that any solution of $E_1 = b_1, E_2 = b_2$ is also a solution of 1.4. Then, we will show that any solution of 1.4 is also a solution of $E_1 = b_1, E_2 = b_2$. Let $(x_1, \cdots, x_n)$ be a solution to $E_1 = b_1, E_2 = b_2$. Then in particular it solves $E_1 = b_1$. Hence, it solves the first equation in 1.4. Similarly, it also solves $E_2 = b_2$. By our proof of 1.3, it also solves $kE_1 = kb_1$. Notice that if we add E_2 and kE_1, this is equal to $b_2 + kb_1$. Therefore, if $(x_1, \cdots, x_n)$ solves $E_1 = b_1, E_2 = b_2$ it must also solve $E_2 + kE_1 = b_2 + kb_1$.

   Now suppose $(x_1, \cdots, x_n)$ solves the system $E_1 = b_1, E_2 + kE_1 = b_2 + kb_1$. Then in particular it is a solution of $E_1 = b_1$. Again by our proof of 1.3, it is also a solution to $kE_1 = kb_1$. Now if we subtract these equal quantities from both sides of $E_2 + kE_1 = b_2 + kb_1$ we obtain $E_2 = b_2$, which shows that the solution also satisfies $E_1 = b_1, E_2 = b_2$.

♠

Stated simply, the above theorem shows that the elementary operations do not change the solution set of a system of equations.

We will now look at an example of a system of three equations and three variables. Similarly to the previous examples, the goal is to find values for x, y, z such that each of the given equations are satisfied when these values are substituted in.

Example 1.9: Solving a System of Equations with Elementary Operations

Find the solutions to the system,

$$\begin{array}{c} x+3y+6z=25 \\ 2x+7y+14z=58 \\ 2y+5z=19 \end{array} \tag{1.5}$$

Solution. We can relate this system to Theorem 1.8 above. In this case, we have

$$\begin{array}{cc} E_1 = x+3y+6z, & b_1 = 25 \\ E_2 = 2x+7y+14z, & b_2 = 58 \\ E_3 = 2y+5z, & b_3 = 19 \end{array}$$

Theorem 1.8 claims that if we do elementary operations on this system, we will not change the solution set. Therefore, we can solve this system using the elementary operations given in Definition 1.6. First, replace the second equation by (-2) times the first equation added to the second. This yields the system

$$\begin{array}{c} x+3y+6z=25 \\ y+2z=8 \\ 2y+5z=19 \end{array} \tag{1.6}$$

Now, replace the third equation with (-2) times the second added to the third. This yields the system

$$\begin{array}{c} x+3y+6z=25 \\ y+2z=8 \\ z=3 \end{array} \tag{1.7}$$

At this point, we can easily find the solution. Simply take $z=3$ and substitute this back into the previous equation to solve for y, and similarly to solve for x.

$$\begin{array}{c} x+3y+6(3)=x+3y+18=25 \\ y+2(3)=y+6=8 \\ z=3 \end{array}$$

The second equation is now

$$y+6=8$$

You can see from this equation that $y=2$. Therefore, we can substitute this value into the first equation as follows:

$$x+3(2)+18=25$$

By simplifying this equation, we find that $x=1$. Hence, the solution to this system is $(x,y,z)=(1,2,3)$. This process is called **back substitution**.

Alternatively, in 1.7 you could have continued as follows. Add (-2) times the third equation to the second and then add (-6) times the second to the first. This yields

$$\begin{array}{c} x+3y=7 \\ y=2 \\ z=3 \end{array}$$

Now add (-3) times the second to the first. This yields

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

a system which has the same solution set as the original system. This avoided back substitution and led to the same solution set. It is your decision which you prefer to use, as both methods lead to the correct solution, $(x,y,z) = (1,2,3)$. ♠

Gaussian Elimination

The work we did in the previous section will always find the solution to the system. In this section, we will explore a less cumbersome way to find the solutions. First, we will represent a linear system with an **augmented matrix.** A **matrix** is simply a rectangular array of numbers. The size or dimension of a matrix is defined as $m \times n$ where m is the number of rows and n is the number of columns. In order to construct an augmented matrix from a linear system, we create a **coefficient matrix** from the coefficients of the variables in the system, as well as a **constant matrix** from the constants. The coefficients from one equation of the system create one row of the augmented matrix.

For example, consider the linear system in Example 1.9

$$\begin{aligned} x + 3y + 6z &= 25 \\ 2x + 7y + 14z &= 58 \\ 2y + 5z &= 19 \end{aligned}$$

This system can be written as an augmented matrix, as follows

$$\left[\begin{array}{rrr|r} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right]$$

Notice that it has exactly the same information as the original system. Here it is understood that the first column contains the coefficients from x in each equation, in order, $\left[\begin{array}{c} 1 \\ 2 \\ 0 \end{array}\right]$. Similarly, we create a column from the coefficients on y in each equation, $\left[\begin{array}{c} 3 \\ 7 \\ 2 \end{array}\right]$ and a column from the coefficients on z in each equation, $\left[\begin{array}{r} 6 \\ 14 \\ 5 \end{array}\right]$. For a system of more than three variables, we would continue in this way constructing a column for each variable. Similarly, for a system with fewer than three variables, we simply construct a column for each variable.

Finally, we construct a column from the constants of the equations, $\left[\begin{array}{c} 25 \\ 58 \\ 19 \end{array}\right]$.

The rows of the augmented matrix correspond to the equations in the system. For example, the top row in the augmented matrix, $\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \end{array}\right]$ corresponds to the equation

$$x+3y+6z=25.$$

Consider the following definition.

Definition 1.10: Augmented Matrix of a Linear System

For a linear system of the form

$$\begin{gathered} a_{11}x_1+\cdots+a_{1n}x_n=b_1 \\ \vdots \\ a_{m1}x_1+\cdots+a_{mn}x_n=b_m \end{gathered}$$

where the x_i are variables and the a_{ij} and b_i are constants, the augmented matrix of this system is given by

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

Now, consider elementary operations in the context of the augmented matrix. The elementary operations in Definition 1.6 can be used on the rows just as we used them on equations previously. Changes to a system of equations as a result of an elementary operation are equivalent to changes in the augmented matrix resulting from the corresponding row operation. Note that Theorem 1.8 implies that any elementary row operations used on an augmented matrix will not change the solution to the corresponding system of equations. We now formally define elementary row operations. These are the *key tool* we will use to find solutions to systems of equations.

Definition 1.11: Elementary Row Operations

The ***elementary row operations*** *(also known as* ***row operations****) consist of the following*

1. *Switch two rows. The operation of taking a matrix A, switching row i and row j, and obtaining the matrix B will be denoted like this:*
$$A \xrightarrow{r_i \leftrightarrow r_j} B.$$
2. *Multiply a row by a nonzero number. To denote multiplying row i of matrix A by the nonzero number k and obtaining the matrix B, we will write*
$$A \xrightarrow{kr_i} B.$$
3. *Add a multiple of one row to another row. If take k times row i of the matrix A and add it to row j of A, producing the matrix B, we express that by writing*
$$A \xrightarrow{kr_i + r_j} B.$$

Recall how we solved Example 1.9. We can do the exact same steps as above, except now in the context of an augmented matrix and using row operations. The augmented matrix of this system is

$$M = \left[\begin{array}{rrr|r} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right]$$

Thus the first step in solving the system given by 1.5 would be to take (-2) times the first row of the augmented matrix and add it to the second row,

$$\left[\begin{array}{rrr|r} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right] \xrightarrow{-2r_1 + r_2} \left[\begin{array}{rrr|r} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 19 \end{array}\right]$$

Note how this corresponds to 1.6. Next take (-2) times the second row and add to the third,

$$\left[\begin{array}{rrr|r} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 19 \end{array}\right] \xrightarrow{-2r_2 + r_3} \left[\begin{array}{rrr|r} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This augmented matrix corresponds to the system

$$\begin{aligned} x + 3y + 6z &= 25 \\ y + 2z &= 8 \\ z &= 3 \end{aligned}$$

which is the same as 1.7. By back substitution you obtain the solution $x = 1, y = 2$, and $z = 3$.

Through a systematic procedure of row operations, we can simplify an augmented matrix and carry it to **row-echelon form** or **reduced row-echelon form**, which we define next. These forms are used to find the solutions of the system of equations corresponding to the augmented matrix.

In the following definitions, the term **leading entry** refers to the first nonzero entry of a row when scanning the row from left to right.

Definition 1.12: Row-Echelon Form

An augmented matrix is in ***row-echelon form*** *if*

1. *All nonzero rows are above any rows of zeros.*
2. *Each leading entry of a row is in a column to the right of the leading entries of any row above it.*
3. *Each leading entry of a row is equal to* 1.

We also consider another reduced form of the augmented matrix which has one further condition.

Definition 1.13: Reduced Row-Echelon Form

An augmented matrix is in ***reduced row-echelon form*** *if*

1. *All nonzero rows are above any rows of zeros.*
2. *Each leading entry of a row is in a column to the right of the leading entries of any rows above it.*
3. *Each leading entry of a row is equal to* 1.
4. *All entries in a column above and below a leading entry are zero.*

Notice that the first three conditions on a reduced row-echelon form matrix are the same as those for row-echelon form.

Hence, every reduced row-echelon form matrix is also in row-echelon form. The converse is not necessarily true; we cannot assume that every matrix in row-echelon form is also in reduced row-echelon form. However, it often happens that row-echelon form is sufficient to provide information about the solution of a system.

The following examples describe matrices in these various forms. As an exercise, take the time to carefully verify that they are in the specified form.

Example 1.14: Not in Row-Echelon Form

The following augmented matrices are not in row-echelon form (and therefore also not in reduced row-echelon form).

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{cc|r} 1 & 2 & 3 \\ 2 & 4 & -6 \\ 4 & 0 & 7 \end{array}\right], \quad \left[\begin{array}{ccc|c} 0 & 2 & 3 & 3 \\ 1 & 5 & 0 & 2 \\ 7 & 5 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Example 1.15: Matrices in Row-Echelon Form

The following augmented matrices are in row-echelon form, but not in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 6 & 5 & 8 & 2 \\ 0 & 0 & 1 & 2 & 7 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 3 & 5 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Notice that we could apply further row operations to these matrices to carry them to reduced row-echelon form. Take the time to try that on your own. Consider the following matrices, which are in reduced row-echelon form.

Example 1.16: Matrices in Reduced Row-Echelon Form

The following augmented matrices are in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

If we go through the trouble to reduce a matrix to row-echelon form, it becomes easy to identify the pivot positions and pivot columns of the matrix.

Definition 1.17: Pivot Position and Pivot Column

*A **pivot position** in a matrix is the location of a leading entry in an row-echelon form of the matrix.*
*A **pivot column** is a column that contains a pivot position.*

For example consider the following.

Example 1.18: Pivot Position

Let

$$A = \left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 4 & 10 \end{array}\right]$$

Where are the pivot positions and pivot columns of the augmented matrix A?

Solution. One row-echelon form of this matrix is

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is all we need in this example, but note that this matrix is not in reduced row-echelon form.

In order to identify the pivot positions in the original matrix, we look for the leading entries in an row-echelon form of the matrix. Here, the entry in the first row and first column, as well as the entry in the second row and second column are the leading entries. Hence, these locations are the pivot positions. We identify the pivot positions in the original matrix, as in the following:

$$\left[\begin{array}{rrr|r} \boxed{1} & 2 & 3 & 4 \\ 3 & \boxed{2} & 1 & 6 \\ 4 & 4 & 4 & 10 \end{array}\right]$$

Thus the pivot columns in the matrix are the first two columns. ♠

Row-Reducing a Matrix

The following is an algorithm for carrying a matrix to row-echelon form and reduced row-echelon form. You may wish to use this algorithm to carry the above matrix to row-echelon form or reduced row-echelon form yourself for practice.

The process we describe, called **row reducing a matrix**, will be a common thing to do for the rest of this text. It will seem that every time we want to do anything, the first step will be to find an appropriate matrix and row reduce it. That isn't quite true, but it is close. So you want to take the time to become very familiar with the process of row reduction.

In modern applications, row reduction is almost always carried out by using technology. There are several software packages, web sites, and calculators that can take a matrix and reduce it to reduced row-echelon form. It will be worth your while, however, to practice reducing at least smallish matrices by hand, if for no other reason than you might be asked to do so on an examination where you won't be allowed to use technology.

Algorithm 1.19: Gaussian and Gauss-Jordan Elimination

This algorithm provides a method for using row operations to convert an matrix A to row-echelon form (Gaussian Elimination) or to reduced row-echelon form (Gauss-Jordan Elimination).

1. *Starting from the left, find the first nonzero column of matrix A. Switch rows if needed to put a nonzero number at the top of this column. This is the current pivot column, and the position at the top of this column is the current pivot position.*

2. *Use row operations to make the entries below the current pivot position (in the current pivot column) equal to zero.*

3. *Ignoring the row containing the current pivot position and any rows above that row, repeat steps 1 and 2 with the remaining rows. Repeat the process until there are no more rows to modify.*

4. *Divide each nonzero row by the value of its leading entry, so that the leading entry becomes 1. The matrix will then be in row-echelon form. This concludes the process for Gaussian Elimination.*

The following step will carry the matrix from row-echelon form to reduced row-echelon form:

5. *Moving from right to left, use row operations to create zeros in the entries of the pivot columns which are above the pivot positions. The result will be a matrix in reduced row-echelon form. This concludes the algorithm for Gauss-Jordan Elimination.*

Most often we will apply this algorithm to an augmented matrix in order to find the solution to a system of linear equations. However, we can use this algorithm to compute the reduced row-echelon form of any matrix which could be useful in other applications.

Consider the following example of Algorithm 1.19.

Example 1.20: Finding Row-Echelon Form and Reduced Row-Echelon Form of a Matrix

Let

$$A = \left[\begin{array}{rrr} 0 & -5 & -4 \\ 1 & 4 & 3 \\ 5 & 10 & 7 \end{array}\right]$$

Find an row-echelon form of A. Then complete the process until A is in reduced row-echelon form.

Solution. In working through this example, we will use the steps outlined in Algorithm 1.19.

1. The first pivot column is the first column of the matrix, as this is the first nonzero column from the left. Hence the first pivot position is the one in the first row and first column. Switch the first two

rows to obtain a nonzero entry in the first pivot position, outlined in a box below.

$$\begin{bmatrix} 0 & -5 & -4 \\ 1 & 4 & 3 \\ 5 & 10 & 7 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} \boxed{1} & 4 & 3 \\ 0 & -5 & -4 \\ 5 & 10 & 7 \end{bmatrix}$$

2. Step two involves creating zeros in the entries below the current pivot position. The first entry of the second row is already a zero. All we need to do is add -5 times the first row to the third row. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 5 & 10 & 7 \end{bmatrix} \xrightarrow{-5r_1+r_3} \begin{bmatrix} 1 & 4 & 3 \\ 0 & \boxed{-5} & -4 \\ 0 & -10 & -8 \end{bmatrix}$$

3. Now ignore the top row, since it contains the current pivot position. The second column becomes our current pivot column and the pivot position (boxed above) already has a non-zero entry. Therefore, we need to create a zero below it. To do this, add -2 times the second row (of this matrix) to the third. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & -10 & -8 \end{bmatrix} \xrightarrow{-2r_2+r_3} \begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Now if we ignore all of the rows at and above the current pivot position, there are no non-zero columns and there are no more rows to modify.

4. Now, we need to create leading 1's in each row. The first row already has a leading 1 so no work is needed here. Multiply the second row by $-\frac{1}{5}$ to create a leading 1. The result is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{5}r_2} \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is now in row-echelon form.

5. Now create zeros in the entries above pivot positions in each column, in order to carry this matrix all the way to reduced row-echelon form. Notice that there is no pivot position in the third column so we do not need to create any zeros in this column! The column in which we need to create zeros is the second. To do so, add -4 times the second row to the first row. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-4r_2+r_1} \begin{bmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is now in reduced row-echelon form.

♠

A Word on Record Keeping

In the solution above you probably noticed that we kept track of the operations that we used to create our reduced matrix as we were going along. This may strike you as wasted effort and overly picky, but it will be useful to us later. Either use our notation, notation suggested by your instructor, or come up with your own, but do understand that the operations that we use to arrive at row-echelon form or reduced row-echelon form of a matrix will be important to us. And if you make a mistake (which you certainly will at some point), knowing what you were trying to do will help you chase down where the mistake happened.

The above algorithm gives you a simple way to obtain an row-echelon form and reduced row-echelon form of a matrix. The main idea is to do row operations in such a way as to end up with a matrix in row-echelon form or reduced row-echelon form. This process is important because the resulting matrix will allow you to describe the solutions to the corresponding linear system of equations in a meaningful way.

In the next example, we look at how to solve a system of equations using the corresponding augmented matrix.

Example 1.21: Finding the Solution to a System

Give the complete solution to the following system of equations

$$\begin{array}{c} 2x+4y-3z=-1 \\ 5x+10y-7z=-2 \\ 3x+6y+5z=9 \end{array}$$

Solution. The augmented matrix for this system is

$$\left[\begin{array}{rrr|r} 2 & 4 & -3 & -1 \\ 5 & 10 & -7 & -2 \\ 3 & 6 & 5 & 9 \end{array}\right]$$

In order to find the solution to this system, we wish to carry the augmented matrix to reduced row-echelon form. We will do so using Algorithm 1.19. Notice that the first column is nonzero, so this is our first pivot column. The first entry in the first row, 2, is the first leading entry and it is in the first pivot position. We will use row operations to create zeros in the entries below the 2.

This can be done by adding $-5/2$ times the first row to the second. This is perfectly fine but will introduce fractions which we try to avoid as long as possible. So instead we will do two operations: first multiply the second row by 2 and then add -5 times the first row to that new row. Thus together we are replacing the second row with -5 times the first row plus 2 times the second row. This yields

$$\left[\begin{array}{rrr|r} 2 & 4 & -3 & -1 \\ 5 & 10 & -7 & -2 \\ 3 & 6 & 5 & 9 \end{array}\right] \xrightarrow{2r_2} \left[\begin{array}{rrr|r} 2 & 4 & -3 & -1 \\ 10 & 20 & -14 & -4 \\ 3 & 6 & 5 & 9 \end{array}\right] \xrightarrow{-5r_1+r_2} \left[\begin{array}{rrr|r} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 6 & 5 & 9 \end{array}\right]$$

Now, using the same technique, replace the third row with -3 times the first row added to 2 times the third row. This yields

$$\left[\begin{array}{rrr|r} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 6 & 5 & 9 \end{array}\right] \xrightarrow{2r_3} \left[\begin{array}{rrr|r} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 6 & 12 & 10 & 18 \end{array}\right] \xrightarrow{-3r_1+r_3} \left[\begin{array}{rrr|r} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 19 & 21 \end{array}\right]$$

Now the entries in the first column below the pivot position are zeros. We now look for the second pivot column, which in this case is column three. Here, the 1 in the second row and third column is in the pivot position. We need to do just one row operation to create a zero below the 1.

Taking -19 times the second row and adding it to the third row yields

$$\left[\begin{array}{rrr|r} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 19 & 21 \end{array}\right] \xrightarrow{-19r_2+r_3} \left[\begin{array}{rrr|r} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

We could proceed with the algorithm to carry this matrix to row-echelon form or reduced row-echelon form. However, remember that we are looking for the solutions to the system of equations. Take another look at the third row of the matrix. Notice that it corresponds to the equation

$$0x+0y+0z=2$$

There is no solution to this equation because for all x,y,z, the left side will equal 0 and $0 \neq 2$. This shows there is no solution to the given system of equations. In other words, this system is inconsistent. ♠

The following is another example of how to find the solution to a system of equations by carrying the corresponding augmented matrix to reduced row-echelon form.

Example 1.22: An Infinite Set of Solutions

Give the complete solution to the system of equations

$$\begin{array}{c} 3x-y-5z=9 \\ y-10z=0 \\ -2x+y=-6 \end{array} \tag{1.8}$$

Solution. The augmented matrix of this system is

$$\left[\begin{array}{rrr|r} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ -2 & 1 & 0 & -6 \end{array}\right]$$

In order to find the solution to this system, we will carry the augmented matrix to reduced row-echelon form, using Algorithm 1.19. The first column is the first pivot column. We want to use row operations to create zeros beneath the first entry in this column, which is in the first pivot position. As in the last example, replace the third row with 2 times the first row added to 3 times the third row. This gives

$$\left[\begin{array}{rrr|r} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ -2 & 1 & 0 & -6 \end{array}\right] \xrightarrow{3r_3} \xrightarrow{2r_1+r_3} \left[\begin{array}{rrr|r} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 1 & -10 & 0 \end{array}\right],$$

where we have suppressed writing out the intermediate matrix.

Now, we have created zeros beneath the 3 in the first column, so we move on to the second pivot column (which is the second column) and repeat the procedure. Take -1 times the second row and add to the third row.

$$\left[\begin{array}{rrr|r} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 1 & -10 & 0 \end{array}\right] \xrightarrow{-1r_2+r_3} \left[\begin{array}{rrr|r} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The entry below the pivot position in the second column is now a zero. Notice that we have no more pivot columns because we have only two leading entries.

At this stage, we also want the leading entries to be equal to one. To do so, multiply the first row by $\frac{1}{3}$.

$$\left[\begin{array}{rrr|r} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}r_1} \left[\begin{array}{rrr|r} 1 & -\frac{1}{3} & -\frac{5}{3} & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is now in row-echelon form.

Let's continue with row operations until the matrix is in reduced row-echelon form. This involves creating zeros above the pivot positions in each pivot column. This requires only one step, which is to add $\frac{1}{3}$ times the second row to the first row.

$$\left[\begin{array}{rrr|r} 1 & -\frac{1}{3} & -\frac{5}{3} & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}r_2+r_1} \left[\begin{array}{rrr|r} 1 & 0 & -5 & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is in reduced row-echelon form, which you should verify using Definition 1.13. The equations corresponding to this reduced row-echelon form are

$$\begin{array}{c} x - 5z = 3 \\ y - 10z = 0 \end{array}$$

or

$$\begin{array}{c} x = 3 + 5z \\ y = 10z \end{array}$$

Observe that z is not restrained by any equation. In fact, z can equal any number. For example, we can let $z = t$, where we can choose t to be any number. In this context t is called a **parameter** . Therefore, the solution set of this system is

$$\begin{array}{c} x = 3 + 5t \\ y = 10t \\ z = t \end{array}$$

where t is arbitrary. The system has an infinite set of solutions which are given by these equations. For any value of t we select, x, y, and z will be given by the above equations. For example, if we choose $t = 4$ then the corresponding solution would be

$$\begin{array}{c} x = 3 + 5(4) = 23 \\ y = 10(4) = 40 \\ z = 4 \end{array}$$

♠

In Example 1.22 the solution involved one parameter. It may happen that the solution to a system involves more than one parameter, as shown in the following example.

Example 1.23: A Two Parameter Set of Solutions

Find the solution to the system

$$\begin{array}{c} x+2y-z+w=3 \\ x+y-z+w=1 \\ x+3y-z+w=5 \end{array}$$

Solution. The augmented matrix is

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{array}\right]$$

We wish to carry this matrix to row-echelon form. Here, we will outline the row operations used. However, make sure that you understand the steps in terms of Algorithm 1.19.

Take -1 times the first row and add to the second. Then take -1 times the first row and add to the third. This yields

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{array}\right] \xrightarrow{-1r_1+r_2} \xrightarrow{-1r_1+r_3} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 3 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right]$$

Now add the second row to the third row and multiply the second row by -1.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 3 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right] \xrightarrow{1r_2+r_3} \xrightarrow{-1r_2} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{1.9}$$

This matrix is in row-echelon form and we can see that x and y correspond to pivot columns, while z and w do not. Therefore, we will assign parameters to the variables z and w. Assign the parameter s to z and the parameter t to w. Then the first row yields the equation $x+2y-s+t=3$, while the second row yields the equation $y=2$. Since $y=2$, the first equation becomes $x+4-s+t=3$ showing that the solution is given by

$$\begin{aligned} x &= -1+s-t \\ y &= 2 \\ z &= s \\ w &= t \end{aligned}$$

It is customary to write this solution in the form

$$\left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right] = \left[\begin{array}{c} -1+s-t \\ 2 \\ s \\ t \end{array}\right] \tag{1.10}$$

♠

This example shows a system of equations with an infinite solution set which depends on two parameters. It can be less confusing in the case of an infinite solution set to first place the augmented matrix in reduced row-echelon form rather than just row-echelon form before seeking to write down the description of the solution.

In the above steps, this means we don't stop with our matrix in row-echelon form in equation 1.9. Instead we first place it in reduced row-echelon form as follows.

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then the solution is $y = 2$ from the second row and $x = -1 + z - w$ from the first. Thus letting $z = s$ and $w = t$, the solution is given by 1.10.

You can see here that there are two paths to the correct answer, which both yield the same answer. Hence, either approach may be used. The process which we first used in the above solution is called **Gaussian Elimination**. This process involves carrying the matrix to row-echelon form, converting back to equations, and using back substitution to find the solution. When you do row operations until you obtain reduced row-echelon form, the process is called **Gauss-Jordan Elimination**.

We have now found solutions for systems of equations with no solution and infinitely many solutions, with one parameter as well as two parameters. Recall the three types of solution sets which we discussed in the previous section; no solution, one solution, and infinitely many solutions. Each of these types of solutions could be identified from the graph of the system. It turns out that we can also identify the type of solution from the reduced row-echelon form of the augmented matrix.

- *No Solution:* In the case where the system of equations has no solution, the reduced row-echelon form of the augmented matrix will have a row of the form

  $$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right]$$

  This row indicates that the system is inconsistent and has no solution.

- *One Solution:* In the case where the system of equations has one solution, every column of the coefficient matrix is a pivot column. The following is an example of an augmented matrix in reduced row-echelon form for a system of equations with one solution.

  $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

- *Infinitely Many Solutions:* In the case where the system of equations has infinitely many solutions, the solution contains parameters. There will be columns of the coefficient matrix which are not pivot columns. The following are examples of augmented matrices in reduced row-echelon form for systems of equations with infinitely many solutions.

  $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  or

  $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \end{array}\right]$$

Uniqueness of the Reduced Row-Echelon Form

As we have seen in earlier sections, we know that every matrix can be brought into reduced row-echelon form by a sequence of elementary row operations. Here we will prove that the resulting matrix is unique; in other words, the resulting matrix in reduced row-echelon form does not depend upon the particular sequence of elementary row operations or the order in which they were performed.

Let A be the augmented matrix of a homogeneous system of linear equations in the variables $x_1, x_2, \cdots, x_n$ which is also in reduced row-echelon form. The matrix A divides the set of variables in two different types. We say that x_i is a **basic variable** whenever A has a leading 1 in column number i, in other words, when column i is a pivot column. Otherwise we say that x_i is a **free variable**.

Recall Example 1.23.

Example 1.24: Basic and Free Variables

Find the basic and free variables in the system

$$\begin{aligned} x+2y-z+w &= 3 \\ x+y-z+w &= 1 \\ x+3y-z+w &= 5 \end{aligned}$$

Solution. Recall from the solution of Example 1.23 that the row-echelon form of the augmented matrix of this system is given by

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

You can see that columns 1 and 2 are pivot columns. These columns correspond to variables x and y, making these the basic variables. Columns 3 and 4 are not pivot columns, which means that z and w are free variables.

We can write the solution to this system as

$$\begin{aligned} x &= -1+s-t \\ y &= 2 \\ z &= s \\ w &= t \end{aligned}$$

Here the free variables are written as parameters, and the basic variables are given by linear functions of these parameters. ♠

In general, all solutions can be written in terms of the free variables. In such a description, the free variables can take any values (they become parameters), while the basic variables become simple linear functions of these parameters. Indeed, a basic variable x_i is a linear function of *only* those free variables x_j with $j > i$. This leads to the following observation.

Proposition 1.25: Basic and Free Variables

If x_i is a basic variable of a homogeneous system of linear equations, then any solution of the system with $x_j = 0$ for all those free variables x_j with $j > i$ must also have $x_i = 0$.

Using this proposition, we prove a lemma which will be used in the proof of the main result of this section below.

Lemma 1.26: Solutions and the Reduced Row-Echelon Form of a Matrix

Let A and B be two distinct augmented matrices for two homogeneous systems of m equations in n variables, such that A and B are each in reduced row-echelon form. Then, the two systems do not have exactly the same solutions.

Proof. With respect to the linear systems associated with the matrices A and B, there are two cases to consider:

- Case 1: the two systems have the same basic variables
- Case 2: the two systems do not have the same basic variables

In case 1, the two matrices will have exactly the same pivot positions. However, since A and B are not identical, there is some row of A which is different from the corresponding row of B and yet the rows each have a pivot in the same column position. Let i be the index of this column position. Since the matrices are in reduced row-echelon form, the two rows must differ at some entry in a column $j > i$. Let these entries be a in A and b in B, where $a \neq b$. Since A is in reduced row-echelon form, if x_j were a basic variable for its linear system, we would have $a = 0$. Similarly, if x_j were a basic variable for the linear system of the matrix B, we would have $b = 0$. Since a and b are unequal, they cannot both be equal to 0, and hence x_j cannot be a basic variable for both linear systems. However, since the systems have the same basic variables, x_j must then be a free variable for each system. We now look at the solutions of the systems in which x_j is set equal to 1 and all other free variables are set equal to 0. For this choice of parameters, the solution of the system for matrix A has $x_i = -a$, while the solution of the system for matrix B has $x_i = -b$, so that the two systems have different solutions.

In case 2, there is a variable x_i which is a basic variable for one matrix, let's say A, and a free variable for the other matrix B. The system for matrix B has a solution in which $x_i = 1$ and $x_j = 0$ for all other free variables x_j. However, by Proposition 1.25 this cannot be a solution of the system for the matrix A. This completes the proof of case 2. ♠

Now, we say that the matrix B is **equivalent** to the matrix A provided that B can be obtained from A by performing a sequence of elementary row operations beginning with A. The importance of this concept lies in the following result.

Theorem 1.27: Equivalent Matrices

The two linear systems of equations corresponding to two equivalent augmented matrices have exactly the same solutions.

The proof of this theorem is left as an exercise.

Now, we can use Lemma 1.26 and Theorem 1.27 to prove the main result of this section.

Theorem 1.28: Uniqueness of the Reduced Row-Echelon Form

Every matrix A is equivalent to a unique matrix in reduced row-echelon form.

Proof. Let A be an $m \times n$ matrix and let B and C be matrices in reduced row-echelon form, each equivalent to A. It suffices to show that $B = C$.

Let A^+ be the matrix A augmented with a new rightmost column consisting entirely of zeros. Similarly, augment matrices B and C each with a rightmost column of zeros to obtain B^+ and C^+. Note that B^+ and C^+ are matrices in reduced row-echelon form which are obtained from A^+ by respectively applying the same sequence of elementary row operations which were used to obtain B and C from A.

Now, A^+, B^+, and C^+ can all be considered as augmented matrices of homogeneous linear systems in the variables $x_1, x_2, \cdots, x_n$. Because B^+ and C^+ are each equivalent to A^+, Theorem 1.27 ensures that all three homogeneous linear systems have exactly the same solutions. By Lemma 1.26 we conclude that $B^+ = C^+$. By construction, we must also have $B = C$. ♠

According to this theorem we can say that each matrix A has a unique reduced row-echelon form.

Rank and Homogeneous Systems

There is a special type of system which requires additional study. This type of system is called a homogeneous system of equations, which we defined above in Definition 1.3. Our focus in this section is to consider what types of solutions are possible for a homogeneous system of equations.

Consider the following definition.

Definition 1.29: Trivial Solution

Consider the homogeneous system of equations given by

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{gathered}$$

Then, $x_1 = 0, x_2 = 0, \cdots, x_n = 0$ is always a solution to this system. We call this the ***trivial solution****.*

If the system has a solution in which not all of the $x_1, \cdots, x_n$ are equal to zero, then we call this solution **nontrivial** . The trivial solution does not tell us much about the system, as it says that $0 = 0$! Therefore, when working with homogeneous systems of equations, we want to know when the system has a nontrivial solution.

Suppose we have a homogeneous system of m equations, using n variables, and suppose that $n > m$. In other words, there are more variables than equations. Then, it turns out that this system always has a nontrivial solution. Not only will the system have a nontrivial solution, but it also will have infinitely many solutions. It is also possible, but not required, to have a nontrivial solution if $n = m$ and $n < m$.

Consider the following example.

Example 1.30: Solutions to a Homogeneous System of Equations

Find the nontrivial solutions to the following homogeneous system of equations

$$\begin{gathered} 2x + y - z = 0 \\ x + 2y - 2z = 0 \end{gathered}$$

Solution. Notice that this system has $m = 2$ equations and $n = 3$ variables, so $n > m$. Therefore by our previous discussion, we expect this system to have infinitely many solutions.

The process we use to find the solutions for a homogeneous system of equations is the same process we used in the previous section. First, we construct the augmented matrix, given by

$$\left[\begin{array}{rrr|r} 2 & 1 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right]$$

Then, we carry this matrix to its reduced row-echelon form, given below.

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

So we see that x and y are the basic variables, while z is the free variable for our system. Let $z = t$, where t is any real number. Since the system of equations that corresponds to our row-reduced matrix is

$$\begin{array}{c} x = 0 \\ y - z = 0 \end{array},$$

our solution has the form

$$\begin{array}{c} x = 0 \\ y = z = t \end{array}$$

Hence this system has infinitely many solutions, with one parameter t. ♠

Suppose we were to write the solution to the previous example in another form. Specifically,

$$\begin{array}{c} x = 0 + 0t \\ y = 0 + t \\ z = 0 + t \end{array}$$

which can be conveniently written, using basic matrix arithmetic of addition and scalar multiplication as we wil see in the next chapter, in the form:

$$\left[\begin{array}{c} x \\ y \\ z \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] + t \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right]$$

Notice that we have constructed a column from the constants in the solution (all equal to 0), as well as a column corresponding to the coefficients on t in each equation. While we will discuss this form of solution more in further chapters, for now consider the column of coefficients of the parameter t. In this case, this is the column $\left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right]$.

There is a special name for this column, which is **basic solution**. The basic solutions of a homogeneous system of equations are columns constructed from the coefficients on parameters in the solution. We often denote basic solutions by X_1, X_2 etc., depending on how many solutions occur. Therefore, Example 1.30 has the basic solution $X_1 = \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right]$.

We explore this further in the following example.

Example 1.31: Basic Solutions of a Homogeneous System

Consider the following homogeneous system of equations.

$$\begin{array}{c} x + 4y + 3z = 0 \\ 3x + 12y + 9z = 0 \end{array}$$

Find the basic solutions to this system.

Solution. When we take the augmented matrix of this system and reduce it to reduced row-echelon form we obtain:

$$\left[\begin{array}{rrr|r} 1 & 4 & 3 & 0 \\ 3 & 12 & 9 & 0 \end{array}\right] \xrightarrow{-3r_1+r_2} \left[\begin{array}{rrr|r} 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

When written in equations, this last system is given by

$$x+4y+3z=0$$

Notice that only x corresponds to a pivot column. In this case, we will have two parameters, one for y and one for z. Let $y=s$ and $z=t$ for any numbers s and t. Then, our solution becomes

$$\begin{aligned} x &= -4s-3t \\ y &= s \\ z &= t \end{aligned}$$

which can be written as (again the constants in the solution are all equal to 0):

$$\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right] + s\left[\begin{array}{r} -4 \\ 1 \\ 0 \end{array}\right] + t\left[\begin{array}{r} -3 \\ 0 \\ 1 \end{array}\right]$$

You can see here that we have two columns of coefficients corresponding to parameters, specifically one for s and one for t. Therefore, this system has two basic solutions! These are

$$X_1 = \left[\begin{array}{r} -4 \\ 1 \\ 0 \end{array}\right], \qquad X_2 = \left[\begin{array}{r} -3 \\ 0 \\ 1 \end{array}\right]$$

♠

We now present a new definition.

Definition 1.32: Linear Combination

Let $X_1, \cdots, X_n, V$ be column matrices. Then V is said to be a ***linear combination*** *of the columns $X_1, \cdots, X_n$ if there exist scalars, $a_1, \cdots, a_n$ such that*

$$V = a_1X_1 + \cdots + a_nX_n$$

A remarkable result of this section is that a linear combination of the basic solutions to a homogeneous system of equations is again a solution to the system. Even more remarkable is that every solution can be written as a linear combination of these solutions. Therefore, if we take a linear combination of the two solutions to Example 1.31, this would also be a solution. For example, we could take the following linear combination

$$3\left[\begin{array}{r} -4 \\ 1 \\ 0 \end{array}\right] + 2\left[\begin{array}{r} -3 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{r} -18 \\ 3 \\ 2 \end{array}\right]$$

You should take a moment to verify that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}$$

is in fact a solution to the system in Example 1.31.

Also remarkable is that to write the general solution of a system of linear equations, one "only" needs to find one solution of the system together with the general solution of the associated homogeneous system. Here is an example of what this means.

Example 1.33: General Solution of a System

Consider the following homogeneous system of equations.

$$\begin{aligned} x + 4y + 3z &= 2 \\ 3x + 12y + 9z &= 6 \end{aligned}$$

Write the general solution of the system as the sum of a particular solution plus a linear combination of the basic solutions of the associated homogeneous system.

Solution. One can find using the normal process that the general solution to the system is of the form:

$$\begin{aligned} x &= 2 + -4s - 3t \\ y &= s \\ z &= t \end{aligned}$$

which can be written as (note here that the constants are not all 0!):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

You can verify here that $X_0 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution to the system (meaning it is one of the possible solutions), and the remaining corresponds to the linear combination of the two basic solutions of the associated homogeneous system from Example 1.31. ♠

It turns out that the general solution of a system of linear equations is always of that form, and this will be revisited in a later chapter.

Another way in which we can find out more information about the solutions of a homogeneous system is to consider the **rank** of the associated coefficient matrix. We now define what is meant by the rank of a matrix.

Definition 1.34: Rank of a Matrix

Let A be a matrix and consider any row-echelon form of A. Then, the number r of leading entries of A does not depend on the row-echelon form you choose, and is called the **rank** *of A. We denote it by* $\text{rank}(A)$.

Similarly, we could count the number of pivot positions (or pivot columns) to determine the rank of A.

Example 1.35: Finding the Rank of a Matrix

Consider the matrix

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 5 & 9 \\ 2 & 4 & 6 \end{array}\right]$$

What is its rank?

Solution. First, we need to find the reduced row-echelon form of A. Through the usual algorithm, we find that this is

$$\left[\begin{array}{rrr} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Here we have two leading entries, or two pivot positions, shown above in boxes.The rank of A is $r = 2$. ♠

Notice that we would have achieved the same answer if we had found the row-echelon form of A instead of the reduced row-echelon form.

Suppose we have a homogeneous system of m equations in n variables, and suppose that $n > m$. From our above discussion, we know that this system will have infinitely many solutions. If we consider the rank of the coefficient matrix of this system, we can find out even more about the solution. Note that we are looking at just the coefficient matrix, not the entire augmented matrix.

Theorem 1.36: Rank and Solutions to a Homogeneous System

Let A be the $m \times n$ coefficient matrix corresponding to a homogeneous system of equations, and suppose A has rank r. Then, the solution to the corresponding system has $n - r$ parameters.

Consider our above Example 1.31 in the context of this theorem. The system in this example has $m = 2$ equations in $n = 3$ variables. First, because $n > m$, we know that the system has a nontrivial solution, and therefore infinitely many solutions. This tells us that the solution will contain at least one parameter. The rank of the coefficient matrix can tell us even more about the solution! The rank of the coefficient matrix of the system is 1, as it has one leading entry in row-echelon form. Theorem 1.36 tells us that the solution will have $n - r = 3 - 1 = 2$ parameters. You can check that this is true in the solution to Example 1.31.

Notice that if $n = m$ or $n < m$, it is possible to have either a unique solution (which will be the trivial solution) or infinitely many solutions.

We are not limited to homogeneous systems of equations here. The rank of a matrix can be used to learn about the solutions of any system of linear equations. In the previous section, we discussed that a system of equations can have no solution, a unique solution, or infinitely many solutions. Suppose the system is consistent, whether it is homogeneous or not. The following theorem tells us how we can use the rank to learn about the type of solution we have.

Theorem 1.37: Rank and Solutions to a Consistent System of Equations

Let A be the $m \times (n+1)$ augmented matrix corresponding to a consistent system of equations in n variables, and suppose A has rank r. Then

1. *the system has a unique solution if $r = n$*
2. *the system has infinitely many solutions if $r < n$*

We will not present a formal proof of this, but consider the following discussions.

1. *No Solution* The above theorem assumes that the system is consistent, that is, that it has a solution. It turns out that it is possible for the augmented matrix of a system with no solution to have any rank r as long as $r > 1$. Therefore, we must know that the system is consistent in order to use this theorem!
2. *Unique Solution* Suppose $r = n$. Then, there is a pivot position in every column of the coefficient matrix of A. Hence, there is a unique solution.
3. *Infinitely Many Solutions* Suppose $r < n$. Then there are infinitely many solutions. There are fewer pivot positions (and hence fewer leading entries) than columns, meaning that not every column is a pivot column. The columns which are *not* pivot columns correspond to parameters. In fact, in this case we have $n - r$ parameters.

Balancing Chemical Reactions

The tools of linear algebra can also be used in the subject area of Chemistry, specifically for balancing chemical reactions.

Consider the chemical reaction

$$SnO_2 + H_2 \to Sn + H_2O$$

Here the elements involved are tin (Sn), oxygen (O), and hydrogen (H). A chemical reaction occurs and the result is a combination of tin (Sn) and water (H_2O). When considering chemical reactions, we want to investigate how much of each element we began with and how much of each element is involved in the result.

An important theory we will use here is the mass balance theory. It tells us that we cannot create or delete elements within a chemical reaction. For example, in the above expression, we must have the same number of oxygen, tin, and hydrogen on both sides of the reaction. Notice that this is not currently the case. For example, there are two oxygen atoms on the left and only one on the right. In order to fix this, we want to find numbers x,y,z,w such that

$$xSnO_2 + yH_2 \to zSn + wH_2O$$

where both sides of the reaction have the same number of atoms of the various elements.

This is a familiar problem. We can solve it by setting up a system of equations in the variables x,y,z,w. Thus you need

$$\begin{array}{rl} Sn: & x = z \\ O: & 2x = w \\ H: & 2y = 2w \end{array}$$

We can rewrite these equations as

$$\begin{array}{rl} Sn: & x - z = 0 \\ O: & 2x - w = 0 \\ H: & 2y - 2w = 0 \end{array}$$

The augmented matrix for this system of equations is given by

$$\left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 \end{array}\right]$$

The reduced row-echelon form of this matrix is

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{array}\right]$$

The solution is given by

$$\begin{array}{c} x - \frac{1}{2}w = 0 \\ y - w = 0 \\ z - \frac{1}{2}w = 0 \end{array}$$

which we can write as

$$\begin{aligned} x &= \tfrac{1}{2}t \\ y &= t \\ z &= \tfrac{1}{2}t \\ w &= t \end{aligned}$$

For example, let $w = 2$ and this would yield $x = 1, y = 2$, and $z = 1$. We can put these values back into the expression for the reaction which yields

$$SnO_2 + 2H_2 \rightarrow Sn + 2H_2O$$

Observe that each side of the expression contains the same number of atoms of each element. This means that it preserves the total number of atoms, as required, and so the chemical reaction is balanced.

Consider another example.

Example 1.38: Balancing a Chemical Reaction

Potassium is denoted by K, oxygen by O, phosphorus by P and hydrogen by H. Consider the reaction given by

$$KOH + H_3PO_4 \rightarrow K_3PO_4 + H_2O$$

Balance this chemical reaction.

Solution. We will use the same procedure as above to solve this problem. We need to find values for x, y, z, w such that

$$xKOH + yH_3PO_4 \rightarrow zK_3PO_4 + wH_2O$$

preserves the total number of atoms of each element.

Finding these values can be done by finding the solution to the following system of equations.

$$\begin{array}{ll} K: & x = 3z \\ O: & x + 4y = 4z + w \\ H: & x + 3y = 2w \\ P: & y = z \end{array}$$

The augmented matrix for this system is

$$\left[\begin{array}{rrrr|r} 1 & 0 & -3 & 0 & 0 \\ 1 & 4 & -4 & -1 & 0 \\ 1 & 3 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

and the reduced row-echelon form is

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The solution is given by

$$\begin{aligned} x - w &= 0 \\ y - \frac{1}{3}w &= 0 \\ z - \frac{1}{3}w &= 0 \end{aligned}$$

which can be written as

$$\begin{aligned} x &= t \\ y &= \frac{1}{3}t \\ z &= \frac{1}{3}t \\ w &= t \end{aligned}$$

Choose a value for t, say 3. Then $w = 3$ and this yields $x = 3, y = 1, z = 1$. It follows that the balanced reaction is given by

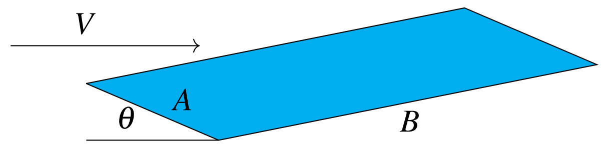
$$3KOH + 1H_3PO_4 \rightarrow 1K_3PO_4 + 3H_2O$$

Note that this results in the same number of atoms on both sides. ♠

Of course these numbers you are finding would typically be the number of moles of the molecules on each side. Thus three moles of KOH added to one mole of H_3PO_4 yields one mole of K_3PO_4 and three moles of H_2O.

Dimensionless Variables

This section shows how solving systems of equations can be used to determine appropriate dimensionless variables. It is only an introduction to this topic and considers a specific example of a simple airplane wing shown below. We assume for simplicity that it is a flat plane at an angle to the wind which is blowing against it with speed V as shown.



The angle θ is called the angle of incidence, B is the span of the wing and A is called the chord. Denote by l the lift. Then this should depend on various quantities like θ, V, B, A and so forth. Here is a table which indicates various quantities on which it is reasonable to expect l to depend.

Variable	Symbol	Units
chord	A	m
span	B	m
angle incidence	θ	$m^0 kg^0 \sec^0$
speed of wind	V	$m \sec^{-1}$
speed of sound	V_0	$m \sec^{-1}$
density of air	ρ	kgm^{-3}
viscosity	μ	$kg \sec^{-1} m^{-1}$
lift	l	$kg \sec^{-2} m$

Here m denotes meters, sec refers to seconds and kg refers to kilograms. All of these are likely familiar except for μ, which we will discuss in further detail now.

Viscosity is a measure of how much internal friction is experienced when the fluid moves. It is roughly a measure of how “sticky" the fluid is. Consider a piece of area parallel to the direction of motion of the fluid. To say that the viscosity is large is to say that the tangential force applied to this area must be large in order to achieve a given change in speed of the fluid in a direction normal to the tangential force. Thus

$$\mu\,(\text{area})\,(\text{velocity gradient}) = \text{ tangential force}$$

Hence

$$(\text{units on } \mu)\, m^2 \left(\frac{m}{\sec m}\right) = kg \sec^{-2} m$$

Thus the units on μ are

$$kg \sec^{-1} m^{-1}$$

as claimed above.

Returning to our original discussion, you may think that we would want

$$l = f(A, B, \theta, V, V_0, \rho, \mu)$$

This is very cumbersome because it depends on seven variables. Also, it is likely that without much care, a change in the units such as going from meters to feet would result in an incorrect value for l. The way to get around this problem is to look for l as a function of dimensionless variables multiplied by something which has units of force. It is helpful because first of all, you will likely have fewer independent variables and secondly, you could expect the formula to hold independent of the way of specifying length, mass and so forth. One looks for

$$l = f(g_1, \cdots, g_k) \rho V^2 AB$$

where the units on $\rho V^2 AB$ are

$$\frac{kg}{m^3}\left(\frac{m}{\sec}\right)^2 m^2 = \frac{kg \times m}{\sec^2}$$

which are the units of force. Each of these g_i is of the form

$$A^{x_1} B^{x_2} \theta^{x_3} V^{x_4} V_0^{x_5} \rho^{x_6} \mu^{x_7} \tag{1.11}$$

and each g_i is independent of the dimensions. That is, this expression must not depend on meters, kilograms, seconds, etc. Thus, placing in the units for each of these quantities, one needs

$$m^{x_1} m^{x_2} \left(m^{x_4} \sec^{-x_4}\right) \left(m^{x_5} \sec^{-x_5}\right) \left(kgm^{-3}\right)^{x_6} \left(kg \sec^{-1} m^{-1}\right)^{x_7} = m^0 kg^0 \sec^0$$

Notice that there are no units on θ because it is just the radian measure of an angle. Hence its dimensions consist of length divided by length, thus it is dimensionless. Then this leads to the following equations for the x_i.

$$\begin{array}{rc} m: & x_1 + x_2 + x_4 + x_5 - 3x_6 - x_7 = 0 \\ \sec: & -x_4 - x_5 - x_7 = 0 \\ kg: & x_6 + x_7 = 0 \end{array}$$

The augmented matrix for this system is

$$\left[\begin{array}{rrrrrrr|r} 1 & 1 & 0 & 1 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

The reduced row-echelon form is given by

$$\left[\begin{array}{rrrrrrr|r} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

and so the solutions are of the form

$$\begin{array}{rcl} x_1 & = & -x_2 - x_7 \\ x_3 & = & x_3 \\ x_4 & = & -x_5 - x_7 \\ x_6 & = & -x_7 \end{array}$$

Thus, in terms of vectors, the solution is

$$\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{array}\right] = \left[\begin{array}{c} -x_2 - x_7 \\ x_2 \\ x_3 \\ -x_5 - x_7 \\ x_5 \\ -x_7 \\ x_7 \end{array}\right]$$

Thus the free variables are x_2, x_3, x_5, x_7. By assigning values to these, we can obtain dimensionless variables by placing the values obtained for the x_i in the formula 1.11. For example, let $x_2 = 1$ and all the rest of the free variables are 0. This yields

$$x_1 = -1, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0$$

The dimensionless variable is then $A^{-1}B^1$. This is the ratio between the span and the chord. It is called the aspect ratio, denoted as AR. Next let $x_3 = 1$ and all others equal zero. This gives for a dimensionless quantity the angle θ. Next let $x_5 = 1$ and all others equal zero. This gives

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = -1, x_5 = 1, x_6 = 0, x_7 = 0$$

Then the dimensionless variable is $V^{-1}V_0^1$. However, it is written as V/V_0. This is called the Mach number $\mathscr{M}$. Finally, let $x_7 = 1$ and all the other free variables equal 0. Then

$$x_1 = -1, x_2 = 0, x_3 = 0, x_4 = -1, x_5 = 0, x_6 = -1, x_7 = 1$$

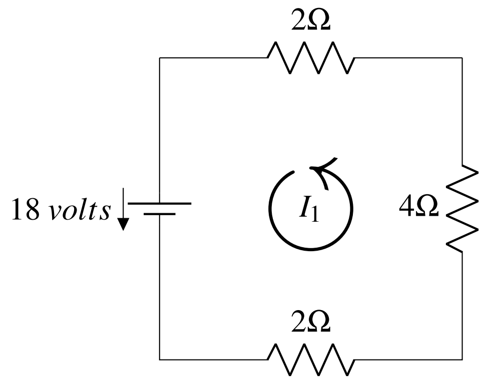
then the dimensionless variable which results from this is $A^{-1}V^{-1}\rho^{-1}\mu$. It is customary to write it as $\text{Re} = (AV\rho)/\mu$. This one is called the Reynold's number. It is the one which involves viscosity. Thus we would look for

$$l = f(\text{Re}, AR, \theta, \mathscr{M})\, kg \times m/\sec^2$$

This is quite interesting because it is easy to vary Re by simply adjusting the velocity or A but it is hard to vary things like μ or ρ. Note that all the quantities are easy to adjust. Now this could be used, along with wind tunnel experiments to get a formula for the lift which would be reasonable. You could also consider more variables and more complicated situations in the same way.

An Application to Resistor Networks

The tools of linear algebra can be used to study the application of resistor networks. An example of an electrical circuit is below.



The jagged lines ( ) denote resistors and the numbers next to them give their resistance in ohms, written as Ω. The voltage source ( ) causes the current to flow in the direction from the shorter of the two lines toward the longer (as indicated by the arrow). The current for a circuit is labelled I_k.

In the above figure, the current I_1 has been labelled with an arrow in the counter clockwise direction. This is an entirely arbitrary decision and we could have chosen to label the current in the clockwise direction. With our choice of direction here, we define a positive current to flow in the counter clockwise direction and a negative current to flow in the clockwise direction.

The goal of this section is to use the values of resistors and voltage sources in a circuit to determine the current. An essential theorem for this application is Kirchhoff's law.

Theorem 1.39: Kirchhoff's Law

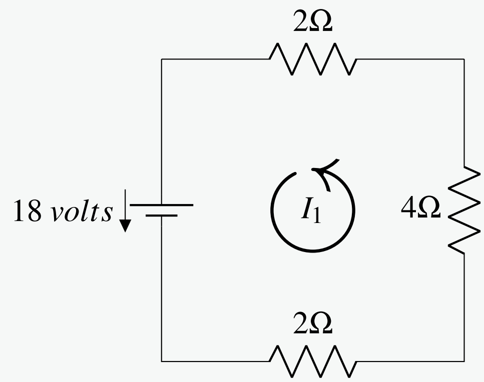
The sum of the resistance (R) times the amps (I) in the counter clockwise direction around a loop equals the sum of the voltage sources (V) in the same direction around the loop.

Kirchhoff's law allows us to set up a system of linear equations and solve for any unknown variables. When setting up this system, it is important to trace the circuit in the counter clockwise direction. If a resistor or voltage source is crossed against this direction, the related term must be given a negative sign.

We will explore this in the next example where we determine the value of the current in the initial diagram.

Example 1.40: Solving for Current

Applying Kirchhoff's Law to the diagram below, determine the value for I_1.



Solution. Begin in the bottom left corner, and trace the circuit in the counter clockwise direction. At the first resistor, multiplying resistance and current gives $2I_1$. Continuing in this way through all three resistors gives $2I_1 + 4I_1 + 2I_1$. This must equal the voltage source in the same direction. Notice that the direction of the voltage source matches the counter clockwise direction specified, so the voltage is positive.

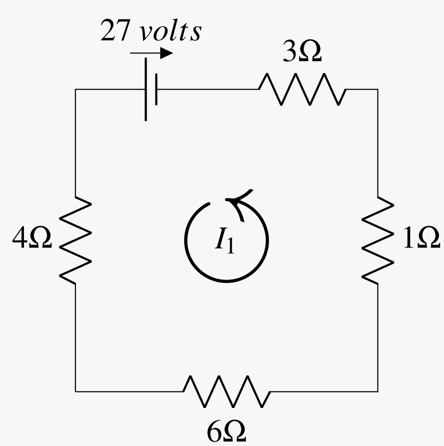
Therefore the equation and solution are given by

$$\begin{aligned} 2I_1 + 4I_1 + 2I_1 &= 18 \\ 8I_1 &= 18 \\ I_1 &= \frac{9}{4}A \end{aligned}$$

Since the answer is positive, this confirms that the current flows counter clockwise. ♠

Example 1.41: Solving for Current

Applying Kirchhoff's Law to the diagram below, determine the value for I_1.



Solution. Begin in the top left corner this time, and trace the circuit in the counter clockwise direction. At the first resistor, multiplying resistance and current gives $4I_1$. Continuing in this way through the four resistors gives $4I_1 + 6I_1 + 1I_1 + 3I_1$. This must equal the voltage source in the same direction. Notice that the direction of the voltage source is opposite to the counter clockwise direction, so the voltage is negative.

Therefore the equation and solution are given by

$$\begin{aligned} 4I_1 + 6I_1 + 1I_1 + 3I_1 &= -27 \\ 14I_1 &= -27 \\ I_1 &= -\frac{27}{14}A \end{aligned}$$

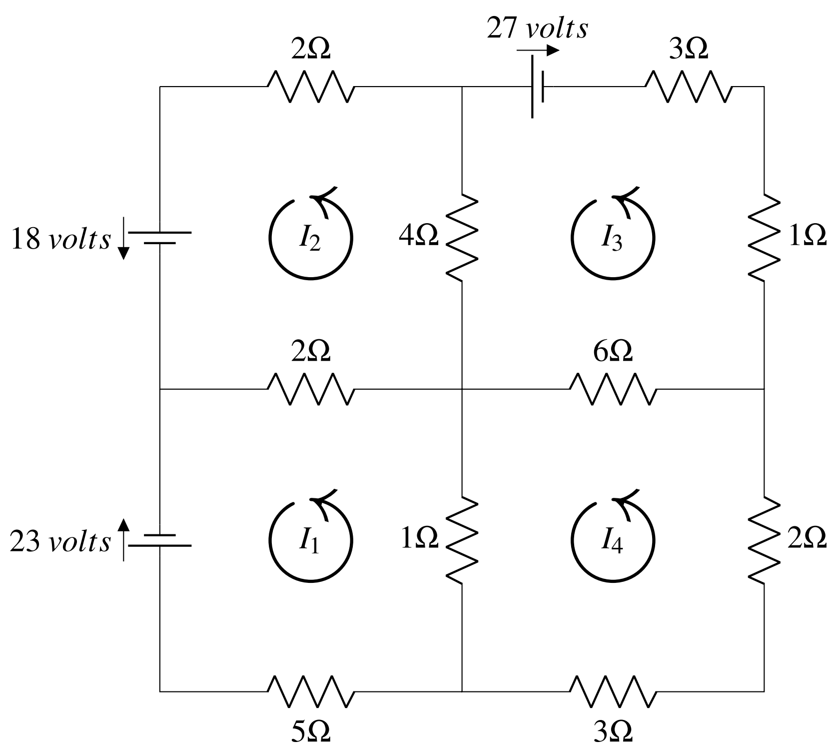
Since the answer is negative, this tells us that the current flows clockwise. ♠

A more complicated example follows. Two of the circuits below may be familiar; they were examined in the examples above. However as they are now part of a larger system of circuits, the answers will differ.

Example 1.42: Unknown Currents

The diagram below consists of four circuits. The current (I_k) in the four circuits is denoted by I_1, I_2, I_3, I_4. Using Kirchhoff's Law, write an equation for each circuit and solve for each current.

Solution. The circuits are given in the following diagram.



Starting with the top left circuit, multiply the resistance by the amps and sum the resulting products. Specifically, consider the resistor labelled 2Ω that is part of the circuits of I_1 and I_2. Notice that current I_2 runs through this in a positive (counter clockwise) direction, and I_1 runs through in the opposite (negative) direction. The product of resistance and amps is then $2(I_2 - I_1) = 2I_2 - 2I_1$. Continue in this way for each resistor, and set the sum of the products equal to the voltage source to write the equation:

$$2I_2 - 2I_1 + 4I_2 - 4I_3 + 2I_2 = 18$$

The above process is used on each of the other three circuits, and the resulting equations are:

Upper right circuit:

$$4I_3 - 4I_2 + 6I_3 - 6I_4 + I_3 + 3I_3 = -27$$

Lower right circuit:

$$3I_4 + 2I_4 + 6I_4 - 6I_3 + I_4 - I_1 = 0$$

Lower left circuit:

$$5I_1 + I_1 - I_4 + 2I_1 - 2I_2 = -23$$

Notice that the voltage for the upper right and lower left circuits are negative due to the clockwise direction they indicate.

The resulting system of four equations in four unknowns is

$$\begin{array}{rcl} 2I_2 - 2I_1 + 4I_2 - 4I_3 + 2I_2 & = & 18 \\ 4I_3 - 4I_2 + 6I_3 - 6I_4 + I_3 + I_3 & = & -27 \\ 2I_4 + 3I_4 + 6I_4 - 6I_3 + I_4 - I_1 & = & 0 \\ 5I_1 + I_1 - I_4 + 2I_1 - 2I_2 & = & -23 \end{array}$$

Simplifying and rearranging with variables in order, we have:

$$\begin{array}{rcl} -2I_1 + 8I_2 - 4I_3 & = & 18 \\ -4I_2 + 14I_3 - 6I_4 & = & -27 \\ -I_1 - 6I_3 + 12I_4 & = & 0 \\ 8I_1 - 2I_2 - I_4 & = & -23 \end{array}$$

The augmented matrix is

$$\left[\begin{array}{rrrr|r} -2 & 8 & -4 & 0 & 18 \\ 0 & -4 & 14 & -6 & -27 \\ -1 & 0 & -6 & 12 & 0 \\ 8 & -2 & 0 & -1 & -23 \end{array}\right]$$

The solution to this matrix is

$$\begin{array}{rcl} I_1 & = & -3A \\ I_2 & = & \frac{1}{4}A \\ I_3 & = & -\frac{5}{2}A \\ I_4 & = & -\frac{3}{2}A \end{array}$$

This tells us that currents I_1, I_3, and I_4 travel clockwise while I_2 travels counter clockwise. ♠

Chapter 2

Matrices

2.1 Matrix Addition and Scalar Multiplication

Outcomes

A. *Perform the matrix operations of matrix addition and scalar multiplication. Identify when these operations are not defined. Represent these operations in terms of the entries of a matrix.*

B. *Prove algebraic properties for matrix addition and scalar multiplication. Apply these properties to manipulate an algebraic expression involving matrices.*

You have now solved systems of equations by writing them in terms of an augmented matrix and then doing row operations on this augmented matrix. It turns out that matrices are important not only for systems of equations but also in many applications. In this chapter we will study matrices as objects of interest in their own right and build an algebra of matrices.

Recall that a **matrix** is a rectangular array of numbers. Several of them are referred to as **matrices**. For example, here is a matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix} \tag{2.1}$$

Recall that the size or dimension of a matrix is defined as $m \times n$ where m is the number of rows and n is the number of columns. The above matrix is a 3×4 matrix because there are three rows and four columns. You can remember the columns are like columns in a Greek temple. They stand upright while the rows lay flat like rows made by a tractor in a plowed field. When specifying the size of a matrix, you always list the number of rows before the number of columns.

Unsurprisingly, a matrix is said to be square if...

Definition 2.1: Square Matrix

A matrix A which has size $n \times n$ is called a **square matrix** *. In other words, A is a square matrix if it has the same number of rows and columns.*

There is some notation specific to matrices which we now introduce. We denote the columns of a matrix A by A_j as follows

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$$

Therefore, A_j is the j^{th} column of A, when counted from left to right.

The individual elements of the matrix are called **entries** or **components** of A. Elements of the matrix are identified according to their position. The $(\mathbf{i},\mathbf{j})$**-entry** of a matrix is the entry in the i^{th} row and j^{th} column. For example, in the matrix 2.1 above, 8 is in position $(2,3)$ (and is called the $(2,3)$-entry) because it is in the second row and the third column.

In order to remember which matrix we are speaking of, we will denote the entry in the i^{th} row and the j^{th} column of matrix A by a_{ij}. Then, we can write A in terms of its entries, as $A = \left[a_{ij}\right]$. Using this notation on the matrix in 2.1, $a_{23} = 8, a_{32} = -9, a_{12} = 2$, etc.

Among the collection of matrices, there are two that will be important to us as we build our matrix algebra. They will play roles analogous to the numbers 0 and 1.

Definition 2.2: The Zero Matrix

*The $m \times n$ **zero matrix** is the $m \times n$ matrix having every entry equal to zero. It is denoted by* 0.

One possible zero matrix is shown in the following example.

Example 2.3: The Zero Matrix

The 2×3 zero matrix is $0 = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$.

Note there is a 2×3 zero matrix, a 3×4 zero matrix, etc. In fact there is a zero matrix for every size!

The zero matrix will be the additive identity for the operation of matrix addition, in the same way that 0 is the additive identity for the operation of (ordinary) addition: $x + 0 = 0 + x = x$. Our second special matrix, the identity matrix, will be the multiplicative identity, once we get around to defining matrix multiplication in the next section.

Definition 2.4: The Identity Matrix

*The $n \times n$ **identity matrix** is the $n \times n$ matrix in which the entry at position ij is equal to 1 if $i = j$, and is equal to 0 if $i \neq j$.*
We denote the $n \times n$ identity matrix by y I_n. If the size is clear from context, we will denote the identity matrix by I.

Notice that the identity matrix is always a square matrix, and it has the property that there are ones down what we will call the **main diagonal** of the matrix and zeroes elsewhere.

Here are some identity matrices of various sizes.

$$\left[1\right], \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right], \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

The first is the 1×1 identity matrix, the second is the 2×2 identity matrix, and so on. By extension, you can likely see what the $n \times n$ identity matrix would be.

Toward an Algebra of Matrices

We are going to build a system for solving equations involving matrices, and since equations involve equals signs, we should probably be explicit about what we mean when we say that two matrices are equal. Although you may well be surprised that we are taking the time to write the following definition down, you probably will not be surprised at what the following definition says.

Definition 2.5: Equality of Matrices

Let A and B be two $m \times n$ matrices. Then $A = B$ means that for $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$, $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

In other words, two matrices are equal exactly when they are the same size and the corresponding entries are identical. Thus

$$\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right] \neq \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$$

because they are different sizes. Also,

$$\left[\begin{array}{cc} 0 & 1 \\ 3 & 2 \end{array}\right] \neq \left[\begin{array}{cc} 1 & 0 \\ 2 & 3 \end{array}\right]$$

because, although they are the same size, their corresponding entries are not identical.

Addition of Matrices

The algebra of matrices that we are building will include equations that involve the sum of two matrices. The notion of matrix addition is where we turn now.

When adding matrices, both matrices in the sum need have the same size. For example,

$$\left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & 2 \end{array}\right]$$

and

$$\left[\begin{array}{rrr} -1 & 4 & 8 \\ 2 & 8 & 5 \end{array}\right]$$

cannot be added, as one has size 3×2 while the other has size 2×3.

However, the addition

$$\left[\begin{array}{rrr} 4 & 6 & 3 \\ 5 & 0 & 4 \\ 11 & -2 & 3 \end{array}\right] + \left[\begin{array}{rrr} 0 & 5 & 0 \\ 4 & -4 & 14 \\ 1 & 2 & 6 \end{array}\right]$$

is possible.

The formal definition is as follows.

Definition 2.6: Addition of Matrices

Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be two $m \times n$ matrices. Then $A+B=C$ where C is the $m \times n$ matrix $C = \left[c_{ij}\right]$ defined by

$$c_{ij} = a_{ij} + b_{ij}$$

This definition tells us that when adding matrices, we simply add corresponding entries of the matrices. Please look carefully at what we are doing here. We are defining a new operation, matrix addition, in terms of a familiar operation, addition of numbers. Annoyingly, both of these operations are denoted by the same sign. Looking carefully at Definition 2.1, we see that the symbol $+$ appears twice. The first time it appears, in $A+B=C$, the symbol represents the new operation, matrix addition. The second time you see it, $c_{ij} = a_{ij} + b_{ij}$, the plus sign is referring to addition of real numbers. So the new operation is defined in terms of the old one. This will mean that many of the properties of (ordinary) addition will still hold when we are thinking of matrix addition. This will be a theme of which you should be aware. Matrix algebra will be sort of like ordinary algebra. However (and this is really important) there will be times when the parallel breaks down, so you will have to be careful. We will try to point out those pitfalls as they arise.

An example of matrix addition seems warranted here:

Example 2.7: Addition of Matrices of Same Size

Add the following matrices, if possible.

$$A = \left[\begin{array}{rrr} 1 & 2 & 3 \\ 1 & 0 & 4 \end{array}\right], \quad B = \left[\begin{array}{rrr} 5 & 2 & 3 \\ -6 & 2 & 1 \end{array}\right]$$

Solution. Notice that both A and B are of size 2×3. Since A and B are of the same size, the addition is possible. Using Definition 2.6, the addition is done as follows.

$$A+B = \left[\begin{array}{rrr} 1 & 2 & 3 \\ 1 & 0 & 4 \end{array}\right] + \left[\begin{array}{rrr} 5 & 2 & 3 \\ -6 & 2 & 1 \end{array}\right] = \left[\begin{array}{ccc} 1+5 & 2+2 & 3+3 \\ 1+-6 & 0+2 & 4+1 \end{array}\right] = \left[\begin{array}{rrr} 6 & 4 & 6 \\ -5 & 2 & 5 \end{array}\right]$$

♠

Note that when we write $A+B$ then we assume that both matrices are of equal size so that the operation is indeed possible.

A Look Under the Hood: Matrix Addition and Scalar Multiplication

We mentioned above that matrix addition is, in many ways, similar to addition of integers. Being precise about what we mean by that and actually establishing those claims is an integral part of what mathematicians do. Knowing the statements of what is true or not is essential to actually being able to competently do the computations involved in linear algebra. Understanding the proofs of those statements is one more step in your maturation as a mathematician, hence the phrase "Looking Under the Hood." You don't need to look under the hood to drive a car, but there is interesting stuff going on there, and sometimes a little knowledge ("How can I check whether I have enough oil?") can help make your life easier and keep you out of big, expensive problems.

Let us start by examining our new operation, matrix addition.

Proposition 2.8: Properties of Matrix Addition

Let A, B and C be matrices. Then, the following properties hold.

- *Commutative Law of Addition*
$$A + B = B + A \tag{2.2}$$
- *Associative Law of Addition*
$$(A + B) + C = A + (B + C) \tag{2.3}$$
- *Existence of an Additive Identity*
$$\begin{array}{c}\text{There exists a zero matrix } 0 \text{ such that} \\ A + 0 = A\end{array} \tag{2.4}$$
- *Existence of an Additive Inverse*
$$\begin{array}{c}\text{There exists a matrix } -A \text{ such that} \\ A + (-A) = 0\end{array} \tag{2.5}$$

Proof. Consider the Commutative Law of Addition given in 2.2. Let A, B, C, and D be matrices such that $A + B = C$ and $B + A = D$. We want to show that $D = C$. To do so, we will use the definition of matrix addition given in Definition 2.6. Now,

$$c_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = d_{ij}$$

Therefore, $C = D$ because the ij^{th} entries are the same for all i and j. Note that the conclusion follows from the commutative law of addition of numbers, which says that if a and b are two numbers, then $a + b = b + a$. The proof of the other results are similar, and are left as an exercise. ♠

We call the zero matrix in 2.4 the **additive identity**. Similarly, we call the matrix $-A$ in 2.5 the **additive inverse**. $-A$ is defined to equal $(-1)A = [-a_{ij}]$. In other words, every entry of A is multiplied by -1.

Scalar Multiplication of Matrices

Recall that we use the word *scalar* when referring to numbers. Therefore, *scalar multiplication of a matrix* is the multiplication of a matrix by a number. To illustrate this concept, consider the following example in which a matrix is multiplied by the scalar 3.

$$3\left[\begin{array}{rrrr} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{array}\right]=\left[\begin{array}{rrrr} 3 & 6 & 9 & 12 \\ 15 & 6 & 24 & 21 \\ 18 & -27 & 3 & 6 \end{array}\right]$$

The new matrix is obtained by multiplying every entry of the original matrix by the given scalar.

The formal definition of scalar multiplication is as follows.

Definition 2.9: Scalar Multiplication of Matrices

If $A=\left[a_{ij}\right]$ *and* k *is a scalar, then* $kA=\left[ka_{ij}\right]$.

Consider the following example.

Example 2.10: Effect of Multiplication by a Scalar

Find the result of multiplying the following matrix A *by* 7.

$$A=\left[\begin{array}{rr} 2 & 0 \\ 1 & -4 \end{array}\right]$$

Solution. By Definition 2.9, we multiply each element of A by 7. Therefore,

$$7A = 7\begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 7(2) & 7(0) \\ 7(1) & 7(-4) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 7 & -28 \end{bmatrix}$$

♠

Similarly to addition of matrices, there are several properties of scalar multiplication which hold. Establishing those results is this subsection's Look Under the Hood:

Proposition 2.11: Properties of Scalar Multiplication

Let A, B be matrices, and k, p be scalars. Then, the following properties hold.

- *Distributive Law over Matrix Addition*

$$k(A+B) = kA + kB$$

- *Distributive Law over Scalar Addition*

$$(k+p)A = kA + pA$$

- *Associative Law for Scalar Multiplication*

$$k(pA) = (kp)A$$

- *Rule for Multiplication by* 1

$$1A = A$$

The proof of this proposition is similar to the proof of Proposition 2.8 and is left an exercise to the reader.

2.2 Matrix Multiplication

Outcomes

A. *Perform the operations of multiplying a matrix times a vector and multiplying a matrix times a matrix. Identify when these operations are not defined. Represent these operations in terms of the entries of the matrix/vector.*

B. *Prove algebraic properties for matrix multiplication. Apply these properties to manipulate an algebraic expression involving matrices and/or vectors.*

The next important matrix operation we will explore is multiplication of matrices. The operation of matrix multiplication is one of the most important and useful of the matrix operations. Throughout this section, we will also demonstrate how matrix multiplication relates to linear systems of equations.

First, we define objects called vectors. Vectors and matrices go together like peanut butter and jelly, like Romeo and Juliet, like Yin and Yang, like... Well, you get the idea...

Definition 2.12: Vectors

Matrices of size $n \times 1$ are called **vectors**, *or occasionally n-***vectors**. *If X is such a matrix, then we write x_i to denote the entry of X in the i^{th} row of the matrix.*

Vectors will often, but not always, be named with lower case letters surmounted by an arrow, for example $\vec{v}$. Here are some examples of vectors:

$$\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \quad X = \begin{bmatrix} 0 \\ 1 \\ 9 \\ -3 \\ \frac{2}{3} \\ -\pi \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

In this chapter, we will again use the notion of linear combination of vectors as in Definition 9.10. In this context, a linear combination is a sum consisting of vectors multiplied by scalars. For example,

$$\left[\begin{array}{c} 50 \\ 122 \end{array}\right] = 7\left[\begin{array}{c} 1 \\ 4 \end{array}\right] + 8\left[\begin{array}{c} 2 \\ 5 \end{array}\right] + 9\left[\begin{array}{c} 3 \\ 6 \end{array}\right]$$

is a linear combination of three vectors.

It turns out that we can express any system of linear equations as an equation involving a linear combination of vectors. In fact, the vectors that we will use are just the columns of the corresponding augmented matrix!

Definition 2.13: The Vector Form of a System of Linear Equations

Suppose we have a system of equations given by

$$\begin{gathered} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{gathered}$$

We can express this system in ***vector form*** *which is as follows:*

$$x_1 \left[\begin{array}{c} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{array}\right] + x_2 \left[\begin{array}{c} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{array}\right] + \cdots + x_n \left[\begin{array}{c} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{array}\right] = \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_m \end{array}\right]$$

Notice that each vector used here is one column from the corresponding augmented matrix. There is one vector for each variable in the system, along with the constant vector.

The first important form of matrix multiplication is multiplying a matrix by a vector. Consider the product given by

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right] \left[\begin{array}{c} 7 \\ 8 \\ 9 \end{array}\right]$$

We will soon see that this equals

$$7\left[\begin{array}{c} 1 \\ 4 \end{array}\right] + 8\left[\begin{array}{c} 2 \\ 5 \end{array}\right] + 9\left[\begin{array}{c} 3 \\ 6 \end{array}\right] = \left[\begin{array}{c} 50 \\ 122 \end{array}\right]$$

In general terms,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}$$

Thus you take x_1 times the first column, add to x_2 times the second column, and finally x_3 times the third column. The above sum is a linear combination of the columns of the matrix. When you multiply a matrix on the left by a vector on the right, the numbers making up the vector are just the scalars to be used in the linear combination of the columns as illustrated above.

Here is the version to repeat to yourself until your brain turns to mush: **The product of a matrix and a vector is a linear combination of the columns of the matrix, where the weights come from the entries of the vector.**

Having established that, we should look at the formal definition of how to multiply an $m \times n$ matrix by an $n \times 1$ column vector.

Definition 2.14: Multiplication of Vector by Matrix

Let $A = \left[a_{ij}\right]$ be an $m \times n$ matrix and let X be an $n \times 1$ matrix given by

$$A = [A_1 \cdots A_n], \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Then the product AX is the $m \times 1$ column vector which equals the following linear combination of the columns of A:

$$x_1A_1 + x_2A_2 + \cdots + x_nA_n = \sum_{j=1}^{n} x_jA_j$$

If we write the columns of A in terms of their entries, they are of the form

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

Then, we can write the product AX as

$$AX = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Note that multiplication of an $m \times n$ matrix and an $n \times 1$ vector produces an $m \times 1$ vector.

Here is an example.

Example 2.15: A Vector Multiplied by a Matrix

Compute the product AX for

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & -2 \\ 2 & 1 & 4 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Solution. We will use Definition 2.14 to compute the product. Therefore, we compute the product AX as follows.

$$\begin{aligned} & 1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ = & \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ = & \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix} \end{aligned}$$

♠

Using the above operation, we can also write a system of linear equations in **matrix form**. In this form, we express the system as a matrix multiplied by a vector. Consider the following definition.

Definition 2.16: The Matrix Form of a System of Linear Equations

Suppose we have a system of equations given by

$$\begin{array}{c} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{array}$$

Then we can express this system in ***matrix form*** *as follows.*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The expression $AX = B$, called the matrix form of the corresponding system of linear equations. The matrix A is simply the coefficient matrix of the system, the vector X is the column vector constructed from the variables of the system, and finally the vector B is the column vector constructed from the constants of the system. It is important to note that any system of linear equations can be written in this form.

Notice that if we write a homogeneous system of equations in matrix form, it would have the form $AX = 0$, for the zero vector 0.

You can see from this definition that a vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

will satisfy the equation $AX = B$ only when the entries $x_1, x_2, \cdots, x_n$ of the vector X are solutions to the original system.

Now that we have examined how to multiply a matrix by a vector, we wish to consider the case where we multiply two matrices of more general sizes, although these sizes still need to be appropriate, as we will see. For example, in Example 2.15, we multiplied a 3×4 matrix by a 4×1 vector. We want to investigate how to multiply other sizes of matrices.

We have not yet given any conditions on when matrix multiplication is possible! For matrices A and B, in order to form the product AB, the number of columns of A must equal the number of rows of B. Consider a product AB where A has size $m \times n$ and B has size $n \times p$. Then, the product in terms of size of matrices is given by

$$(m \times \overbrace{n) \, (n}^{\text{these must match!}} \times p) = m \times p$$

Note the two outside numbers give the size of the product. One of the most important rules regarding matrix multiplication is the following. If the two middle numbers do not match, you cannot multiply the matrices!

When the number of columns of A equals the number of rows of B the two matrices are said to be **conformable** and the product AB is obtained as follows.

Definition 2.17: Multiplication of Two Matrices

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix of the form

$$B = [B_1 \cdots B_p]$$

where $B_1, \ldots, B_p$ are the $n \times 1$ columns of B. Then the $m \times p$ matrix AB is defined as follows:

$$AB = A\,[B_1 \ldots B_p] = [A(B_1) \ldots A(B_p)],$$

where $A(B_k)$ is the product of the matrix A and the vector B_k.

Consider the following example.

Example 2.18: Multiplying Two Matrices

Find AB if possible.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

Solution. The first thing you need to verify when calculating a product is whether the multiplication is possible. The first matrix has size 2×3 and the second matrix has size 3×3. The inside numbers are equal, so A and B are conformable matrices. According to the above discussion AB will be a 2×3 matrix. Definition 2.17 gives us a way to calculate each column of AB, as follows.

$$\left[\overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}}^{\text{First column}}, \overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}}^{\text{Second column}}, \overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}^{\text{Third column}} \right]$$

You know how to multiply a matrix times a vector, using Definition 2.14 for each of the three columns. Thus

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 9 & 3 \\ -2 & 7 & 3 \end{bmatrix}$$

♠

Since vectors are simply $n \times 1$ or $1 \times m$ matrices, we can also multiply a vector by another vector.

Example 2.19: Vector Times Vector Multiplication

If possible, compute the product $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}$.

Solution. In this case we are multiplying a matrix of size 3×1 by a matrix of size 1×4. The inside numbers match, so the product is defined. Note that the product will be a matrix of size 3×4. Using Definition 2.17, we can compute this product as follows

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix} = \left[\overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}}^{\text{First column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix}}^{\text{Second column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}}^{\text{Third column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}}^{\text{Fourth column}} \right]$$

You can use Definition 2.14 to verify that this product is

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

♠

Example 2.20: A Multiplication Which is Not Defined

Find BA if possible.

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Solution. First we check if the product is defined. This product is of the form $(3 \times 3)(2 \times 3)$. The inside numbers do not match and so we cannot perform the requested multiplication. ♠

In this case, we say that the multiplication is not defined. Notice that these are the same matrices which we used in Example 2.18. In this example, we tried to calculate BA instead of AB. This demonstrates another property of matrix multiplication. While the product AB may be be defined, we cannot assume that the product BA will also be defined. Therefore, it is important to always check that the product is defined before carrying out any calculations.

Earlier, we defined the zero matrix 0 to be the matrix (of appropriate size) containing zeros in all entries. Consider the following example for multiplication by the zero matrix.

Example 2.21: Multiplication by the Zero Matrix

Compute the product $A0$ for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and the 2×2 zero matrix given by

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution. In this product, we compute

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A0 = 0$. ♠

Notice that we could also multiply A by the 2×1 zero vector given by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The result would be the 2×1 zero vector. Therefore, it is always the case that $A0 = 0$, for an appropriately sized zero matrix or vector. So here we have another case of matrix algebra looking a lot like ordinary algebra: anything times zero is equal to zero times anything is equal to zero. With matrices, however, you do have to check that the matrices in the product are conformable, and that the matrix on the right hand side of the equal sign is of the appropriate size.

The ij^{th} Entry of a Product

In the previous section, we used the entries of a matrix to describe the action of matrix addition and scalar multiplication. We can also study matrix multiplication using the entries of matrices.

What is the ij^{th} entry of AB? It is the entry in the i^{th} row and the j^{th} column of the product AB.

Now if A is $m \times n$ and B is $n \times p$, then we know that the product AB has the form

$$\left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right] \left[\begin{array}{cccccc} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{array}\right]$$

The j^{th} column of AB is of the form

$$\left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right] \left[\begin{array}{c} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{array}\right]$$

which is an $m \times 1$ column vector. It is calculated by

$$b_{1j} \left[\begin{array}{c} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{array}\right] + b_{2j} \left[\begin{array}{c} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{array}\right] + \cdots + b_{nj} \left[\begin{array}{c} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{array}\right]$$

Therefore, the ij^{th} entry is the entry in row i of this vector. This is computed by

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

The following is the formal definition for the ij^{th} entry of a product of matrices.

Definition 2.22: The ij^{th} Entry of a Product

Let $A = \left[a_{ij}\right]$ be an $m \times n$ matrix and let $B = \left[b_{ij}\right]$ be an $n \times p$ matrix. Then AB is an $m \times p$ matrix and the (i, j)-entry of AB is defined as

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Another way to write this is

$$(AB)_{ij} = \left[\begin{array}{cccc} a_{i1} & a_{i2} & \cdots & a_{in} \end{array}\right] \left[\begin{array}{c} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{array}\right] = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

In other words, to find the (i, j)-entry of the product AB, or $(AB)_{ij}$, you multiply the i^{th} row of A, on the left by the j^{th} column of B. To express AB in terms of its entries, we write $AB = \left[(AB)_{ij}\right]$.

Consider the following example.

Example 2.23: The Entries of a Product

Compute AB if possible. If it is, find the $(3,2)$-entry of AB using Definition 2.22.

$$A = \left[\begin{array}{cc} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{array}\right], \quad B = \left[\begin{array}{ccc} 2 & 3 & 1 \\ 7 & 6 & 2 \end{array}\right]$$

Solution. First check if the product is defined. It is of the form $(3 \times 2)(2 \times 3)$ and since the inside numbers match, it is possible to do the multiplication. The result should be a 3×3 matrix. We can first compute AB:

$$\left[\left[\begin{array}{cc} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{array}\right]\left[\begin{array}{c} 2 \\ 7 \end{array}\right], \left[\begin{array}{cc} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{array}\right]\left[\begin{array}{c} 3 \\ 6 \end{array}\right], \left[\begin{array}{cc} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{array}\right]\left[\begin{array}{c} 1 \\ 2 \end{array}\right]\right]$$

where the commas separate the columns in the resulting product. Thus the above product equals

$$\left[\begin{array}{ccc} 16 & 15 & 5 \\ 13 & 15 & 5 \\ 46 & 42 & 14 \end{array}\right]$$

which is a 3×3 matrix as desired. Thus, the $(3,2)$-entry equals 42.

Now using Definition 2.22, we can find that the $(3,2)$-entry equals

$$\begin{aligned}\sum_{k=1}^{2} a_{3k}b_{k2} &= a_{31}b_{12} + a_{32}b_{22} \\ &= 2 \times 3 + 6 \times 6 = 42\end{aligned}$$

Consulting our result for AB above, this is correct!

You may wish to use this method to verify that the rest of the entries in AB are correct. ♠

Here is another example.

Example 2.24: Finding the Entries of a Product

Determine if the product AB is defined. If it is, find the $(2,1)$-entry of the product.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 7 & 6 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}$$

Solution. This product is of the form $(3 \times 3)(3 \times 2)$. The middle numbers match so the matrices are conformable and it is possible to compute the product.

We want to find the $(2,1)$-entry of AB, that is, the entry in the second row and first column of the product. We will use Definition 2.22, which states

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

In this case, $n = 3$, $i = 2$ and $j = 1$. Hence the $(2,1)$-entry is found by computing

$$(AB)_{21} = \sum_{k=1}^{3} a_{2k}b_{k1} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$$

Substituting in the appropriate values, this product becomes

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 7 & 6 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 1 \times 7 + 3 \times 6 + 2 \times 2 = 29$$

Hence, $(AB)_{21} = 29$.

You should take a moment to find a few other entries of AB. You can multiply the matrices to check that your answers are correct. The product AB is given by

$$AB = \begin{bmatrix} 13 & 13 \\ 29 & 32 \\ 0 & 0 \end{bmatrix}$$

♠

You will, of course, through trial and error and lots of practice, find the way to compute the product of two matrices that fits you best. But the short version of this subsection gives a quick and easy way to remember how to multiply two conformable matrices by hand:

To compute the ij^{th} entry of AB, take the i^{th} row of A and the j^{th} column of B. Multiply the entries componentwise, then add.

Properties of Matrix Multiplication

As pointed out above, it is sometimes possible to multiply matrices in one order but not in the other order. However, even if both AB and BA are defined, they may not be equal.

Example 2.25: Matrix Multiplication is Not Commutative

Compare the products AB and BA, for matrices $A = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right]$, $\quad B = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$

Solution. First, notice that A and B are both of size 2×2. Therefore, both products AB and BA are defined. The first product, AB is

$$AB = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right] \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] = \left[\begin{array}{cc} 2 & 1 \\ 4 & 3 \end{array}\right]$$

The second product, BA is

$$\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right] = \left[\begin{array}{cc} 3 & 4 \\ 1 & 2 \end{array}\right]$$

Therefore, $AB \neq BA$. ♠

This example illustrates that you cannot assume that $AB = BA$ even when both products are defined. We have stated several times that there are many ways in which matrix algebra is like ordinary the ordinary algebra of integers. But here we have probably the major difference between the two. Multiplication of numbers is **commutative**. Multiplication of matrices is not. So as you work with equations involving matrix algebra, the order in which you write your products will be important. This will be rather annoying until you get used to it, but do try to be careful.

But, in many other ways, matrix multiplication acts like regular multiplication. Notice that these properties hold only when the size of matrices are such that the products are defined.

Proposition 2.26: Properties of Matrix Multiplication

The following hold for matrices A, B, and C and for scalars r and s,

$$A\left(rB+sC\right)=r\left(AB\right)+s\left(AC\right) \tag{2.6}$$

$$\left(B+C\right)A=BA+CA \tag{2.7}$$

$$A\left(BC\right)=\left(AB\right)C \tag{2.8}$$

Proof. First we will prove 2.6. We will use Definition 2.22 and prove this statement using the ij^{th} entries of a matrix. Therefore,

$$\begin{aligned}\left(A\left(rB+sC\right)\right)_{ij} &= \sum_k a_{ik}\left(rB+sC\right)_{kj} = \sum_k a_{ik}\left(rb_{kj}+sc_{kj}\right) \\ &= r\sum_k a_{ik}b_{kj} + s\sum_k a_{ik}c_{kj} = r\left(AB\right)_{ij} + s\left(AC\right)_{ij} \\ &= \left(r\left(AB\right)+s\left(AC\right)\right)_{ij}\end{aligned}$$

Thus $A\left(rB+sC\right) = r(AB)+s(AC)$ as claimed.

The proof of 2.7 follows the same pattern and is left as an exercise.

Statement 2.8 is the associative law of multiplication. Using Definition 2.22,

$$\begin{aligned}\left(A\left(BC\right)\right)_{ij} &= \sum_k a_{ik}\left(BC\right)_{kj} = \sum_k a_{ik}\sum_l b_{kl}c_{lj} \\ &= \sum_l \left(AB\right)_{il}c_{lj} = \left(\left(AB\right)C\right)_{ij}.\end{aligned}$$

This proves 2.8. ♠

2.3 The Transpose

Outcomes

A. *Perform the operation of finding the transpose of a matrix and represent this operations in terms of the entries of the matrix.*

B. *Prove algebraic properties for matrix transposition. Apply these properties to manipulate an algebraic expression involving matrices.*

C. *Recognize symmetric and skew symmetric matrices.*

The matrix operations we have investigated to this point have had strong analogies to operations on the integers. In this short section we introduce the **transpose** of a matrix, which does not have a similar analogy, as it is tied to the shape of a matrix. An example will make this clear:

In order to find the transpose of, just for example, the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 2 & 6 \end{bmatrix},$$

all we will do is write the columns of A as the rows of the transpose of A, which we denote by A^T. Like this:

$$A^T = \begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \end{bmatrix}.$$

What happened? The first column of A became the first row of A^T and the second column became the second row. Thus the 3×2 matrix became a 2×3 matrix. The number 4 was in the first row and the second column and it ended up in the second row and first column.

The official definition of the transpose of a matrix is as follows.

Definition 2.27: The Transpose of a Matrix

Let A be an $m \times n$ matrix. Then A^T, the ***transpose*** *of A, denotes the $n \times m$ matrix given by*

$$A^T = \left[a_{ij}\right]^T = \left[a_{ji}\right]$$

The (i,j)-entry of A becomes the (j,i)-entry of A^T.

Consider the following example.

Example 2.28: The Transpose of a Matrix

Calculate A^T for the following matrix

$$A = \left[\begin{array}{rrr} 1 & 2 & -6 \\ 3 & 5 & 4 \end{array}\right]$$

Solution. By Definition 2.27, we know that for $A = \left[a_{ij}\right]$, $A^T = \left[a_{ji}\right]$. In other words, we switch the row and column location of each entry. The $(1,2)$-entry becomes the $(2,1)$-entry.

Thus,

$$A^T = \left[\begin{array}{rr} 1 & 3 \\ 2 & 5 \\ -6 & 4 \end{array}\right]$$

Notice that A is a 2×3 matrix, while A^T is a 3×2 matrix. ♠

The operation of transposing a matrix has the following important properties:

Lemma 2.29: Properties of Transposition

Let A be an $m \times n$ matrix, B an $n \times p$ matrix, and r and s scalars. Then

1. $\left(A^T\right)^T = A$
2. $(AB)^T = B^T A^T$
3. $(rA + sB)^T = rA^T + sB^T$

Proof. We prove 2. From Definition 2.27,

$$(AB)^T = \left[(AB)_{ij}\right]^T = \left[(AB)_{ji}\right] = \sum_k a_{jk} b_{ki} = \sum_k b_{ki} a_{jk}$$

$$= \sum_k [b_{ik}]^T [a_{kj}]^T = [b_{ij}]^T [a_{ij}]^T = B^T A^T$$

The proofs of Formulas 1 and 3 are left as exercises. ♠

Although you may have skimmed over that Look Under the Hood moment as we proved the second property of transposition, let us look at it a little more carefully. This is one of the times that the non-commutivity of matrix multiplication becomes important. If life were just and fair, one would hope that the transpose of a product would be the product of the transposes, like this: $(AB)^T = A^T B^T$. But that is not how the world works, as you should convince yourself by picking any random 2×3 matrix A and any random 3×1 matrix B. Compute $(AB)^T$ and try to compare it to $A^T B^T$ and see what goes wrong. *Where multiplication is concerned, order is important.* Even when the sizes of the matrices don't get in the way, order still matters. Try an example with two random 2×2 matrices A and B. Compare $(AB)^T$ and $A^T B^T$. Even though both are defined, are they equal? Sometimes they will be, but most likely they will not. Order is important.

The transpose of a matrix is related to other important topics. Consider the following definition.

Definition 2.30: Symmetric and Skew Symmetric Matrices

An $n \times n$ matrix A is said to be ***symmetric*** *if $A = A^T$. It is said to be* ***skew symmetric*** *if $A = -A^T$.*

We will explore these definitions in the following examples.

Example 2.31: Symmetric Matrices

Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Use Definition 2.30 to show that A is symmetric.

Solution. By Definition 2.30, we need to show that $A = A^T$. Now, using Definition 2.27,

$$A^T = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Hence, $A = A^T$, so A is symmetric. ♠

At this point you may be thinking to yourself, "Why is this sort of matrix called symmetric?" If you look at the matrix A in the last example and imagine a mirror placed on the main diagonal of A, you can see that there is mirror symmetry across the main diagonal: $a_{ij} = a_{ji}$. Hence the name.

Example 2.32: A Skew Symmetric Matrix

Let

$$A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$$

Show that A is skew symmetric.

Solution. By Definition 2.30,

$$A^T = \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

You can see that each entry of A^T is equal to -1 times the same entry of A. Hence, $A^T = -A$ and so by Definition 2.30, A is skew symmetric. ♠

Here, the mirror symmetry that we discussed after the last example is spoiled, but only by a minus sign. So for a symmetric matrix A we have $a_{ij} = a_{ji}$, but for a skew symmetric matrix A, we have $a_{ij} = -a_{ji}$. Notice that this forces the entries on the main diagonal of a skew symmetric matrix to be equal to 0.

2.4 The Identity Matrix and Matrix Inverses

Outcomes

A. *Define an invertible matrix and prove that the inverse of a matrix, when it exists, is unique.*

B. *Prove that a potential candidate for the inverse of a given matrix A is or is not equal to A^{-1}.*

C. *Prove that a non-invertible 2×2 matrix is not invertible.*

As you no doubt remember from Section 2.1, we defined the $n \times n$ identity matrix to be the matrix that has 1's on the main diagonal and 0's everywhere else. We mentioned that the identity matrix would play a role similar to the role that the number 1 plays with respect to matrix multiplication. It is time to expand on that idea a little further.

I_n is called the **identity matrix** because it is a **multiplicative identity** in the following sense.

Lemma 2.33: Multiplication by the Identity Matrix

Suppose A is an $m \times n$ matrix, I_n is the $n \times n$ identity matrix and I_m is the $m \times m$ identity matrix. Then $AI_n = A$ and $I_mA = A$.

Proof. The (i, j)-entry of AI_n is given by:

$$\sum_k a_{ik}\delta_{kj} = a_{ij},$$

where

$$\delta_{kj} = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}.$$

Thus the (i, j)-entry of AI_n is equal to a_{ij}, and so $AI_n = A$. The proof of the second claim is left as an exercise for you. ♠

Now think back to the happy days of your youth. When you were first learning about negative numbers, you found out that for any integer k, there was a special number called $-k$ such that the sum of k and $-k$ was equal to 0. So k had an *additive inverse*, a number which, when added to k, gave a result which was the additive identity. Notice that our matrices have the same property: Given any matrix A, there is a matrix (namely $-A$) which, when added to A, yields a matrix which is the additive identity.

Still thinking of the days when you were young, you knew that there was a multiplicative identity, the integer 1. But there were no multiplicative inverses because if you picked a random integer k (other than 1 or -1) there was no other integer j such that kj is equal to the multiplicative identity (i.e., 1). This made it very hard to solve equations like $3x = 5$ for x. The solution to this problem was to expand our collection of numbers to the rational numbers. If you worked in the world of the rational numbers, then every number x (except 0) had a multiplicative inverse, which you could call x^{-1}.

Now we are working with matrices, and we want to see how things develop here. The situation will turn out to be slightly more complicated than before, but there are plenty of parallels. This will turn out to be a good thing and a bad thing, as you will have to be careful not to assume that everything you remember to be true about numbers necessarily holds about matrices. But at this point, you are getting used to that.

Let's start by carefully defining what we mean when we say that a matrix A has an inverse. (Whenever we say inverse, you can safely assume we mean multiplicative inverse.) Notice that we are going to restrict ourselves to talking about square matrices.

Definition 2.34: Invertible Matrix

A square $n \times n$ matrix A is said to be **invertible** *if there is an $n \times n$ matrix B such that*

$$AB = BA = I_n.$$

We will say that such a matrix B is a **witness** *that A is invertible.*

For a couple of quick examples, notice that $A = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ is invertible, since if $B = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$, then $AB = BA = I$. Also notice that if $A = I_4$, then $B = I_4$ has the property that $AB = BA = I$. For a more complicated example, you can check that $B = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$ is a witness that $A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$ is invertible.

Suppose that a matrix A is invertible. This means that there is some witness matrix B such that $AB = BA = I$. But maybe there is another matrix C that also witnesses that A is invertible, so $AC = CA = I$. We will prove that this cannot happen, so if there is a witness that A is invertible, there is only one such witness.

Theorem 2.35: Uniqueness of Witnesses to Invertibility

Suppose A is an $n \times n$ invertible matrix. Suppose that $AB = BA = I$ and $AC = CA = I$. Then $B = C$.

Proof. In this proof, it is assumed that I is the $n \times n$ identity matrix. Let A, B, and C be $n \times n$ matrices such that $AB = BA = AC = CA = I$. We want to show that $B = C$. Now using properties we have seen, we get:

$$B = BI = B(AC) = (BA)C = IC = C$$

Hence, $B = C$ which is what we wanted to prove. ♠

Now that we know that an invertible matrix A has only one witness to its invertibility, we can give that witness a name. We will call the witness the inverse of A, and denote it by A^{-1}.

Definition 2.36: The Inverse of a Matrix

If A is an invertible matrix, the **inverse** *of A, denoted A^{-1}, is the unique matrix such that*

$$AA^{-1} = A^{-1}A = I_n$$

Notice that Theorem 2.35 justifies calling it *the* inverse of A, rather than *an* inverse of A.

Like many things in mathematics, although it may be hard to find the inverse of a matrix A (more on that soon), it is easy to check whether a given matrix is the inverse of A:

Example 2.37: Verifying the Inverse of a Matrix

Let $A = \left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array} \right]$. *Show* $\left[\begin{array}{rr} 2 & -1 \\ -1 & 1 \end{array} \right]$ *is the inverse of* A.

Solution. To check this, multiply

$$\left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array} \right] \left[\begin{array}{rr} 2 & -1 \\ -1 & 1 \end{array} \right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] = I$$

and

$$\left[\begin{array}{rr} 2 & -1 \\ -1 & 1 \end{array} \right] \left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array} \right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] = I$$

showing that this matrix is indeed the inverse of A. ♠

When we work with integers, no number, except for 1 and -1, has an inverse. When we work with rational numbers, every number, except for 0, has an inverse. Are matrices more like the integers or more like the rational numbers? It turns out that they are somewhere in between.

First, it is easy to convince yourself that the $n \times n$ zero matrix is not invertible, since for any matrix B, $0B = B0 = 0$. We've seen examples of matrices that are not equal to the identity but still have inverses. But some matrices besides the zero matrix also might not have an inverse. This is illustrated in the following example.

Example 2.38: A Nonzero Matrix With No Inverse

Let $A = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]$. *Show that* A *does not have an inverse.*

Solution. We will show that A has no inverse by assuming that A^{-1} does exist, and from that assumption deriving a contradiction.

So for our matrix A, if A^{-1} exists, then A^{-1} would have to be some 2×2 matrix

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$AA^{-1} = I.$$

So this means that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

But then if we look at the entries of these equal matrices, we see that $0 = a+c = 1$, which means that $0 = 1$, which is false. So our assumption that A^{-1} existed leads us to a contradiction, which means that A^{-1} can not exist.

♠

So, some matrices have inverses and others do not. In the next section we will outline a procedure that will find the inverse of A when it exists, and certify that A is not invertible when an inverse does not exist.

2.5 Finding the Inverse of a Matrix

Outcomes

A. *Use the matrix inversion algorithm to find the inverse of an $n \times n$ matrix, if that matrix exists.*

B. *Solve systems of n equations involving n unknowns using the matrix inverse.*

C. *Prove and use properties related to matrix inversion in analyzing algebraic expressions.*

In Example 2.37, we were given A^{-1} and asked to verify that this matrix was in fact the inverse of A. In this section, we explore how to find A^{-1}.

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

as in Example 2.37. In order to find A^{-1}, we need to find a matrix $\begin{bmatrix} x & z \\ y & w \end{bmatrix}$ such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can multiply these two matrices, and see that in order for this equation to be true, we must find the solution to the systems of equations,

$$\begin{aligned} x + y &= 1 \\ x + 2y &= 0 \end{aligned}$$

and

$$\begin{aligned} z + w &= 0 \\ z + 2w &= 1 \end{aligned}$$

Since we are already experts at solving systems of linear equations, we might as well put that skill to use. (That was the whole point of Chapter 1, right?) Writing the augmented matrix for these two systems gives

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right]$$

for the first system and

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 1 \end{array}\right] \tag{2.9}$$

for the second.

Let's solve the first system. Take -1 times the first row and add to the second to get

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right] \xrightarrow{-1r_1+r_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

Now take -1 times the second row and add to the first to get

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{-1r_2+r_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

Writing in terms of variables, this says $x = 2$ and $y = -1$.

Now solve the second system, 2.9 to find z and w. You will find that $z = -1$ and $w = 1$.

If we take the values found for x, y, z, and w and put them into our inverse matrix, we see that the inverse is

$$A^{-1} = \left[\begin{array}{cc} x & z \\ y & w \end{array}\right] = \left[\begin{array}{cc} 2 & -1 \\ -1 & 1 \end{array}\right]$$

After taking the time to solve the second system, you may have noticed that *exactly the same row operations were used to solve both systems*. In each case, the end result was something of the form $[I|X]$ where I is the identity and X gave a column of the inverse. In the above,

$$\left[\begin{array}{c} x \\ y \end{array}\right]$$

the first column of the inverse was obtained by solving the first system and then the second column

$$\left[\begin{array}{c} z \\ w \end{array}\right]$$

To simplify this procedure, we could have solved both systems at once. This is the key idea that will give us our algorithm for computing the inverse of a matrix. So, for our example above, we could have written

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

and row reduced until we obtained

$$\left[\begin{array}{rr|rr} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

and read off A^{-1} as the 2×2 matrix on the right of the vertical line.

This exploration motivates the following important algorithm.

Algorithm 2.39: Matrix Inverse Algorithm

Suppose A is an $n \times n$ matrix. To find A^{-1} if it exists, form the augmented $n \times 2n$ matrix

$$[A|I]$$

If possible do row operations until you obtain an $n \times 2n$ matrix of the form

$$[I|B]$$

When this has been done, $B = A^{-1}$. If it is impossible to row reduce to a matrix of the form $[I|B]$, then A has no inverse.

This algorithm shows how to find the inverse if it exists. It will also tell you if A does not have an inverse.

Consider the following example.

Example 2.40: Finding the Inverse

Let $A = \left[\begin{array}{rrr} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{array}\right]$. *Find A^{-1} if it exists.*

Solution. Set up the augmented matrix

$$[A|I] = \left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Now we row reduce, with the goal of obtaining the 3×3 identity matrix on the left hand side. First, take -1 times the first row and add to the second followed by -3 times the first row added to the third row. This yields

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1r_1+r_2} \xrightarrow{-3r_1+r_3} \left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right]$$

Then take 5 times the second row and add to -2 times the third row.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right] \xrightarrow{-2r_3} \xrightarrow{5r_2+r_3} \left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array}\right]$$

Next take the third row and add to -7 times the first row. This yields

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array}\right] \xrightarrow{-7r_1} \xrightarrow{1r_3+r_1} \left[\begin{array}{rrr|rrr} -7 & -14 & 0 & -6 & 5 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array}\right]$$

Now take $-\frac{7}{5}$ times the second row and add to the first row.

$$\left[\begin{array}{rrr|rrr} -7 & -14 & 0 & -6 & 5 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array}\right] \xrightarrow{-\frac{7}{5}r_2+r_1} \left[\begin{array}{rrr|rrr} -7 & 0 & 0 & 1 & -2 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array}\right]$$

Finally multiply the first row by $-1/7$, the second row by $-1/10$ and the third row by $1/14$ which yields

$$\left[\begin{array}{rrr|rrr} -7 & 0 & 0 & 1 & -2 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array}\right] \xrightarrow{-\frac{1}{7}r_1} \xrightarrow{-\frac{1}{10}r_2} \xrightarrow{\frac{1}{14}r_3} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array}\right]$$

Notice that the left hand side of this matrix is now the 3×3 identity matrix I_3. Therefore, the inverse is the 3×3 matrix on the right hand side, given by

$$\left[\begin{array}{rrr} -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array}\right]$$

♠

It may happen that through this algorithm, you discover that the left hand side cannot be row reduced to the identity matrix. Consider the following example of this situation.

Example 2.41: A Matrix Which Has No Inverse

Let $A = \left[\begin{array}{rrr} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \end{array}\right]$. *Find* A^{-1} *if it exists.*

Solution. Write the augmented matrix $[A|I]$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

and proceed to do row operations attempting to obtain $\left[I|A^{-1}\right]$. Take -1 times the first row and add to the second. Then take -2 times the first row and add to the third row.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1r_1+r_2} \xrightarrow{-2r_1+r_3} \left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 1 \end{array}\right]$$

Next add -1 times the second row to the third row.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

At this point, you can see there will be no way to obtain I on the left side of this augmented matrix. Hence, there is no way to complete this algorithm, and therefore the inverse of A does not exist. In this case, we say that A is not invertible. ♠

If the algorithm provides an inverse for the original matrix, it is always possible to check your answer. To do so, use the method demonstrated in Example 2.37. Check that the products AA^{-1} and $A^{-1}A$ both equal the identity matrix. Through this method, you can always be sure that you have calculated A^{-1} properly.

One way in which the inverse of a matrix is useful is to find the solution of a system of linear equations. Recall from Definition 2.16 that we can write a system of equations in matrix form, which is of the form $AX = B$. Suppose you find the inverse of the matrix A^{-1}. Then you could multiply both sides of this equation on the left by A^{-1} and simplify to obtain

$$\begin{aligned} \left(A^{-1}\right)AX &= A^{-1}B \\ \left(A^{-1}A\right)X &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Therefore we can find X, the solution to the system, by computing $X = A^{-1}B$. Note that once you have found A^{-1}, you can easily get the solution for different right hand sides (different B). It is always just $A^{-1}B$.

We will explore this method of finding the solution to a system in the following example.

Example 2.42: Using the Inverse to Solve a System of Equations

Consider the following system of equations. Use the inverse of a suitable matrix to give the solutions to this system.

$$\begin{aligned} x+z &= 1 \\ x-y+z &= 3 \\ x+y-z &= 2 \end{aligned}$$

Solution. First, we can write the system of equations in matrix form

$$AX = \left[\begin{array}{rrr} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right]\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} 1 \\ 3 \\ 2 \end{array}\right] = B \tag{2.10}$$

The inverse of the matrix

$$A = \left[\begin{array}{rrr} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right]$$

is

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Verifying this inverse is left as an exercise.

From here, the solution to the given system 2.10 is found by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -2 \\ -\frac{3}{2} \end{bmatrix}$$

♠

What if the right side, B, of 2.10 had been $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$? In other words, what would be the solution to

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}?$$

By the above discussion, the solution is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

This illustrates that for a system $AX = B$ where A^{-1} exists, it is easy to find the solution when the vector B is changed.

Let's gather together some properties of the inverse.

Theorem 2.43: Properties of the Inverse

Let A be an $n \times n$ matrix and I the usual identity matrix.

1. *I is invertible and $I^{-1} = I$*
2. *If A is invertible then so is A^{-1}, and $(A^{-1})^{-1} = A$*
3. *If A is invertible then so is A^k, and $(A^k)^{-1} = (A^{-1})^k$*
4. *If A is invertible and p is a nonzero real number, then pA is invertible and $(pA)^{-1} = \frac{1}{p}A^{-1}$*
5. *If A and B are invertible matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$*
6. *If $A_1, A_2, ..., A_k$ are invertible, then the product $A_1A_2 \cdots A_k$ is invertible, and $(A_1A_2 \cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1} \cdots A_2^{-1}A_1^{-1}$*
7. *If A is an invertible matrix, then $(A^T)^{-1} = (A^{-1})^T$*

These results are all established in the same way. There's a claim that some matrix is invertible and there is a candidate for what the inverse is. All we have to do is check that the proposed inverse works. For example, to prove (4), all we have to do is check that the matrix $\frac{1}{p}A^{-1}$ is, in fact, the inverse of the matrix pA. So just notice that

$$(pA)\left(\frac{1}{p}A^{-1}\right) = p \cdot \frac{1}{p}AA^{-1} = 1 \cdot I = I$$

and

$$\left(\frac{1}{p}A^{-1}\right)(pA) = \frac{a}{p} \cdot pA^{-1}A = 1 \cdot I = I,$$

and the result is established. The other claims are proven similarly.

We would be remiss if we didn't emphasize result (5) in the Theorem above. Notice the order of the matrices in $(AB)^{-1}$. Since we know that there is no reason to expect AB to be equal to BA, there is also no reason to expect $B^{-1}A^{-1}$ to equal $A^{-1}B^{-1}$. Try to be careful with the orders of the matrices you use in finding the inverses of products.

A Look Under the Hood: Inverses and Systems of Linear Equations

Recall back in Chapter 1 we said that a system of linear equations can have either no solutions, one solution, or an infinite number of solutions. Taking a look at what we have just accomplished, suppose that we have a system of equations $AX = B$ in which A^{-1} exists. Then our system only has one solution, namely the solution $X = A^{-1}B$. And we also just said that if we changed the right hand side of our system from B to some other vector B', again there would be only one solution. It seems that the number of solutions to our system depends only on the coefficients of the variables, not on the right hand side of the system. Although we cannot make that precise quite yet, we have established a small, but suggestive, proposition:

Proposition 2.44: Invertible Matrices and Systems

Suppose that A is an invertible matrix. Then the system of equations $AX = B$ has a unique solution.

On the other hand, if an $n \times n$ matrix A is not invertible, we know that we found that out when we tried to compute A^{-1} via our algorithm and we reached a point where we know that the reduced row-echelon form of A was not the identity matrix, as in Example 2.41. The only way our algorithm can fail is if, in the process of trying to row reduce A, we reach a point where we see a row of all zeros to the left of our vertical divider. This means that, if we were to try to solve the system $AX = B$, when we had transformed A to reduced row-echelon form there was a row of the matrix without a leading 1, and since there are as many rows as columns in A, that means that there is a column of A that is not a pivot column, so our solution to $AX = B$ must have a free variable, a parameter. And (whew) *this* means that the system $AX = B$ must have infinitely many solutions. Let's summarize:

Proposition 2.45: Systems with as Many Variables as Equations

Suppose that the system of equations $AX = B$ consists of n equations and n unknowns. Then either:

1. *The matrix of coefficients A is invertible and the system has a unique solution $X = A^{-1}B$, or*
2. *The matrix of coefficients A is not invertible and the system has infinitely many solutions.*

2.6 Elementary Matrices

Outcomes

A. *Identify elementary matrices and their inverses.*

B. *Recognize the relation between performing elementary row operations and left multiplying by elementary matrices.*

C. *Represent row reducing a matrix A to its reduced row-echelon form as left multiplying A by a matrix that is a product of elementary matrices.*

D. *Recognize that a matrix A is invertible if and only if it can be written as a product of elementary matrices.*

We now turn our attention to a special type of matrix called an **elementary matrix**. An elementary matrix is always a square matrix. Recall the row operations given in Definition 1.11. Any elementary matrix, which we often denote by E, is obtained from applying *one* row operation to the identity matrix of the same size.

For example, the matrix

$$E = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$$

is the elementary matrix obtained from switching the two rows of the 2×2 identity matrix. The matrix

$$E = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

is the elementary matrix obtained from multiplying the second row of the 3×3 identity matrix by 17. The matrix

$$E = \left[\begin{array}{cc} 1 & 0 \\ -3 & 1 \end{array}\right]$$

is the elementary matrix obtained from adding -3 times the first row of I_2 to the second row.

You may construct an elementary matrix from any row operation, but remember that you can only apply one operation.

Here is the official definition.

Definition 2.46: Elementary Matrices and Row Operations

Let E be an $n \times n$ matrix. Then E is an ***elementary matrix*** *if it is the result of applying one row operation to the $n \times n$ identity matrix I_n.*
Those which involve switching rows of the identity matrix are called ***permutation matrices***.

Therefore, E constructed above by switching the two rows of I_2 is called a permutation matrix.

Elementary matrices can be used in place of row operations and therefore are very useful. It turns out that multiplying (on the left hand side) by an elementary matrix E will have the same effect as doing the row operation used to obtain E.

The following theorem is an important result which we will use throughout this text.

Theorem 2.47: Multiplication by an Elementary Matrix and Row Operations

To perform any of the three row operations on a matrix A it suffices to compute the product EA, where E is the elementary matrix obtained by using the desired row operation on the identity matrix.

Therefore, instead of performing row operations on a matrix A, we can row reduce through matrix multiplication with the appropriate elementary matrix. We will examine this theorem in detail for each of the three row operations given in Definition 1.11.

First, consider the following lemma.

Lemma 2.48: Action of Permutation Matrix

Let P^{ij} denote the elementary matrix which involves switching the i^{th} and the j^{th} rows. Then P^{ij} is a permutation matrix and

$$P^{ij}A = B$$

where B is obtained from A by switching the i^{th} and the j^{th} rows.

We will explore this idea more in the following example.

Example 2.49: Switching Rows with an Elementary Matrix

Let

$$P^{12} = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right], \quad A = \left[\begin{array}{cc} a & b \\ c & d \\ e & f \end{array} \right]$$

Find B where $B = P^{12}A$.

Solution. You can see that the matrix P^{12} is obtained by switching the first and second rows of the 3×3 identity matrix I.

Using our usual procedure, compute the product $P^{12}A = B$. The result is given by

$$B = \left[\begin{array}{cc} c & d \\ a & b \\ e & f \end{array} \right]$$

Notice that B is the matrix obtained by switching rows 1 and 2 of A. Therefore by multiplying A by P^{12}, the row operation which was applied to I to obtain P^{12} is applied to A to obtain B. ♠

Theorem 2.47 applies to all three row operations, and we now look at the row operation of multiplying a row by a scalar. Consider the following lemma.

Lemma 2.50: Multiplication by a Scalar and Elementary Matrices

Let $E(k,i)$ denote the elementary matrix corresponding to the row operation in which the i^{th} row is multiplied by the nonzero scalar, k. Then

$$E(k,i)A = B$$

where B is obtained from A by multiplying the i^{th} row of A by k.

Here is an example of using this lemma:

Example 2.51: Multiplication of a Row by 5 Using Elementary Matrix

Let

$$E(5,2) = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{array}\right], \quad A = \left[\begin{array}{cc} a & b \\ c & d \\ e & f \end{array}\right]$$

Find the matrix B where $B = E(5,2)A$

Solution. You can see that $E(5,2)$ is obtained by multiplying the second row of the identity matrix by 5.

Using our usual procedure for multiplication of matrices, we can compute the product $E(5,2)A$. The resulting matrix is given by

$$B = \left[\begin{array}{cc} a & b \\ 5c & 5d \\ e & f \end{array}\right]$$

Notice that B is obtained by multiplying the second row of A by the scalar 5. ♠

There is one last row operation to consider. The following lemma discusses the final operation of adding a multiple of a row to another row.

Lemma 2.52: Adding Multiples of Rows and Elementary Matrices

Let $E(k \times i + j)$ denote the elementary matrix obtained from I by adding k times the i^{th} row to the j^{th}. Then

$$E(k \times i + j)A = B$$

where B is obtained from A by adding k times the i^{th} row of A to the j^{th} row of A.

Consider the following example.

Example 2.53: Adding Two Times the First Row to the Last

Let

$$E(2\times 1+3)=\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{array}\right], \quad A=\left[\begin{array}{cc} a & b \\ c & d \\ e & f \end{array}\right]$$

Find B where $B=E(2\times 1+3)A$.

Solution. You can see that the matrix $E(2\times 1+3)$ was obtained by adding 2 times the first row of I to the third row of I.

Using our usual procedure, we can compute the product $E(2\times 1+3)A$. The resulting matrix B is given by

$$B=\left[\begin{array}{cc} a & b \\ c & d \\ 2a+e & 2b+f \end{array}\right]$$

You can see that B is the matrix obtained by adding 2 times the first row of A to the third row. ♠

Inverses of Elementary Matrices

Suppose we have applied a row operation to a matrix A. Consider the row operation required to return A to its original form, to undo the row operation. It turns out that this action is how we find the inverse of an elementary matrix E.

Consider the following theorem.

Theorem 2.54: Elementary Matrices and Inverses

Every elementary matrix is invertible and its inverse is also an elementary matrix.

In fact, the inverse of an elementary matrix is constructed by doing the *reverse* row operation on I. E^{-1} will be obtained by performing the row operation which would carry E back to I.

- If E is obtained by switching rows i and j, then E^{-1} is also obtained by switching rows i and j.
- If E is obtained by multiplying row i by the scalar k, then E^{-1} is obtained by multiplying row i by the scalar $\frac{1}{k}$.
- If E is obtained by adding k times row i to row j, then E^{-1} is obtained by adding $-k$ times row i to row j.

Consider the following example.

Example 2.55: Inverse of an Elementary Matrix

Let

$$E = \left[\begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right]$$

Find E^{-1}.

Solution. Consider the elementary matrix E given by

$$E = \left[\begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right]$$

Here, E is obtained from the 2×2 identity matrix by multiplying the second row by 2. In order to carry E back to the identity, we need to multiply the second row of E by $\frac{1}{2}$. Hence, E^{-1} is given by

$$E^{-1} = \left[\begin{array}{cc} 1 & 0 \\ 0 & \frac{1}{2} \end{array} \right]$$

We can verify that $EE^{-1} = I$. Take the product EE^{-1}, given by

$$EE^{-1} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right] \left[\begin{array}{cc} 1 & 0 \\ 0 & \frac{1}{2} \end{array} \right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

This equals I so we know that we have computed E^{-1} properly. ♠

Row Reduction via Elementary Matrices

Suppose an $m \times n$ matrix A is row reduced to its reduced row-echelon form. By tracking each row operation completed, this row reduction can be completed through multiplication by elementary matrices. Consider the following definition.

Definition 2.56: The Form $B = TA$

Let A be an $m \times n$ matrix and let B be the reduced row-echelon form of A. Then we can write $B = TA$ where T is the product of all elementary matrices representing the row operations done to A to obtain B.

Consider the following example.

Example 2.57: The Form $B = TA$

Let $A = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{array} \right]$. *Find B, the reduced row-echelon form of A and write it in the form $B = TA$.*

Solution. To find B, row reduce A. For each step, we will record the appropriate elementary matrix. First, switch rows 1 and 2.

$$\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{array}\right]$$

The resulting matrix is equivalent to finding the product of $P^{12} = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$ and A.

Next, add (-2) times row 1 to row 3.

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{array}\right] \xrightarrow{(-2)r_1 + r_3} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}\right]$$

This is equivalent to multiplying by the matrix $E(-2 \times 1 + 3) = \left[\begin{array}{rcc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{array}\right]$. Notice that the resulting matrix is B, the required reduced row-echelon form of A.

We can then write

$$\begin{aligned} B &= E(-2 \times 1 + 2)\left(P^{12}A\right) \\ &= \left(E(-2 \times 1 + 2)P^{12}\right)A \\ &= TA \end{aligned}$$

It remains to find the matrix T.

$$\begin{aligned} T &= E(-2 \times 1 + 2)P^{12} \\ &= \left[\begin{array}{rcc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{array}\right]\left[\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \\ &= \left[\begin{array}{crc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{array}\right] \end{aligned}$$

Notice in the above calculation that the *first* row operation performed (switching rows 1 and 2) corresponds to the elementary matrix that is on the *right* in the product.

We can verify that $B = TA$ holds for this matrix T:

$$\begin{aligned} TA &= \left[\begin{array}{crc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{array}\right]\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}\right] \\ &= B \end{aligned}$$

♠

While the process used in the above example is reliable and simple when only a few row operations are used, it becomes cumbersome in a case where many row operations are needed to carry A to B. The following theorem provides an alternate way to find the matrix T.

Theorem 2.58: Finding the Matrix T

Let A be an $m \times n$ matrix and let B be its reduced row-echelon form. Then $B = TA$ where T is an invertible $m \times m$ matrix found by forming the matrix $[A|I_m]$ and row reducing to $[B|T]$.

Let's revisit the above example using the process outlined in Theorem 2.58.

Example 2.59: The Form $B = TA$, Revisited

Let $A = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{array} \right]$. Using the process outlined in Theorem 2.58, find T such that $B = TA$.

Solution. First, set up the matrix $[A|I_m]$.

$$\left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array} \right]$$

Now, row reduce this matrix until the left side equals the reduced row-echelon form of A.

$$\begin{array}{c} \left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array} \right] \stackrel{r_1 \leftrightarrow r_2}{\longrightarrow} \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array} \right] \\ \stackrel{(-2)r_1 + r_3}{\longrightarrow} \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{array} \right] \end{array}$$

The left side of this matrix is B, and the right side is T. Comparing this to the matrix T found above in Example 2.57, you can see that the same matrix is obtained regardless of which process is used. ♠

Invertible Matrices and Elementary Matrices

Recall from Algorithm 2.39 that an $n \times n$ matrix A is invertible if and only if A can be carried to the $n \times n$ identity matrix using the usual row operations. This leads to an important consequence related to the above discussion.

Suppose A is an $n \times n$ invertible matrix. Then, set up the matrix $[A|I_n]$ as done above, and row reduce until it is of the form $[B|T]$. In this case, $B = I_n$ because A is invertible.

$$\begin{array}{rcl} B & = & TA \\ I_n & = & TA \\ T^{-1} & = & A \end{array}$$

Now suppose that $T = E_1 E_2 \cdots E_k$ where each E_i is an elementary matrix representing a row operation used to carry A to I. Then,

$$T^{-1} = (E_1 E_2 \cdots E_k)^{-1} = E_k^{-1} \cdots E_2^{-1} E_1^{-1}$$

Remember that if E_i is an elementary matrix, so too is E_i^{-1}. It follows that

$$\begin{array}{rcl} A & = & T^{-1} \\ & = & E_k^{-1} \cdots E_2^{-1} E_1^{-1} \end{array}$$

and A can be written as a product of elementary matrices.

Theorem 2.60: Product of Elementary Matrices

Let A be an $n \times n$ matrix. Then A is invertible if and only if it can be written as a product of elementary matrices.

Consider the following example.

Example 2.61: Product of Elementary Matrices

Let $A = \left[\begin{array}{rrr} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{array} \right]$. Write A as a product of elementary matrices.

Solution. We will use the process outlined in Theorem 2.58 to write A as a product of elementary matrices. We will set up the matrix $[A|I]$ and row reduce, recording each row operation as an elementary matrix.

First:

$$\left[\begin{array}{rrr|rrr} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right]$$

represented by the elementary matrix $E_1 = \left[\begin{array}{rrr} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right]$.

Secondly:

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-1)r_2+r_1} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

represented by the elementary matrix $E_2 = \left[\begin{array}{rrr} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$.

Finally:

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{2r_2+r_3} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$$

represented by the elementary matrix $E_3 = \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{array}\right]$.

Notice that the reduced row-echelon form of A is I. Hence $I = TA$ where T is the product of the above elementary matrices. It follows that $A = T^{-1}$. Since we want to write A as a product of elementary matrices, we wish to express T^{-1} as a product of elementary matrices.

$$\begin{aligned} T^{-1} &= (E_3E_2E_1)^{-1} \\ &= E_1^{-1}E_2^{-1}E_3^{-1} \\ &= \left[\begin{array}{rrr} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{rrr} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{array}\right] \\ &= A \end{aligned}$$

This gives A written as a product of elementary matrices. By Theorem 2.60 it follows that A is invertible. ♠

2.7 Two Theorems on Matrix Inverses

In this section, we will present two theorems which will clarify the concept of matrix inverses. The proofs of these theorems are somewhat technical and definitely are in the Look Under the Hood part of linear algebra. Because the results of the theorems are pretty important, we will present them first, and delay the proofs of the theorems until later in this section.

The first theorem tells us that if we have square matrices A and B such that $AB = I$, then automatically $BA = I$. Practically, this means that if we are given two matrices and asked if they are inverses of each other, we need only check the product AB. If it works out that $AB = I$, then we can conclude that $B = A^{-1}$ without checking the product BA. Here's the statement of the theorem:

Theorem 2.62: Unique Inverse of a Matrix

Suppose A and B are square matrices such that $AB = I$ where I is an identity matrix. Then it follows that $BA = I$. Thus both A and B are invertible and $B = A^{-1}$ and $A = B^{-1}$.

Our second major theorem of this section makes explicit something you probably noticed in Section 2.5.

Theorem 2.63: The Reduced Row-Echelon Form of an Invertible Matrix

For any matrix A the following conditions are equivalent:

- *A is invertible*
- *The reduced row-echelon form of A is an identity matrix*

A Look Under the Hood: Matrix Inverses

Let's prove both of these theorems. To get started we will need a lemma (a small result used to prove another result) that is based on our understanding of elementary matrices from the last section. Just to refresh your memory, remember that applying row operations is the same as left multiplying by elementary matrices, so if R is the reduced row-echelon form of a matrix A, then there are elementary matrices $E_1, E_2, \ldots, E_k$ such that $R = (E_k \cdots (E_2 (E_1 A))) = E_k \cdots E_2 E_1 A$. If we let E denote the product $E_k \cdots E_2 E_1$, then $R = EA$, where E is an invertible matrix.

Now we can state and prove our lemma:

Lemma 2.64: Invertible Matrix and Zeros

Suppose that A and B are matrices such that the product AB is an identity matrix. Then the reduced row-echelon form of A does not have a row of zeros.

Proof. Let R be the reduced row-echelon form of A. Then $R = EA$ for some invertible square matrix E as described above. By hypothesis $AB = I$ where I is an identity matrix, so we have a chain of equalities

$$R(BE^{-1}) = (EA)(BE^{-1}) = E(AB)E^{-1} = EIE^{-1} = EE^{-1} = I$$

If R would have a row of zeros, then so would the product $R(BE^{-1})$. But since the identity matrix I does not have a row of zeros, R cannot have one either. ♠

Having established this lemma, we can proceed to a proof of Theorem 2.62:

Proof. (of Theorem 2.62): We assume that we are given square matrices A and B such that $AB = I$. We must prove that $BA = I$.

We are assuming that $AB = I$, so by Lemma 2.64 we know that R, the reduced row-echelon form of A, does not have a row of zeros. But since A is square, R is square also, and as R is a square matrix in reduced row-echelon form which does not contain a row of zeros, R must be the identity matrix. (Take a minute and convince yourself of that.) So (again by the last section) there is an invertible matrix E such that $EA = R = I$.

Using the two facts that $AB = I$ and that $EA = I$, we can finish the proof with a chain of equalities. Remember that we are trying to prove that BA is equal to I:

$$\begin{aligned} BA = IBIA &= (EA)B(E^{-1}E)A \\ &= E(AB)E^{-1}(EA) \\ &= EIE^{-1}I \\ &= EE^{-1} = I. \end{aligned}$$

Since we have shown that $BA = I$, our proof is complete.

♠

Now we can prove Theorem 2.63. To show that the two given statements are equivalent, we will prove that each one implies the other.

Proof. (of Theorem 2.63).

First, assume that A is an invertible matrix. We must show that the reduced row-echelon form of A is the identity matrix. As A is invertible, we know by Theorem 2.60 that A can be written as a product of (invertible) elementary matrices:

$$A = E_1 E_2 \ldots E_k.$$

We left multiply both sides of this equation by the inverses of the E_i's, being careful about the order, and we get

$$E_k^{-1} \ldots E_2^{-1} E_1^{-1} A = \left(E_k^{-1} \ldots E_2^{-1} E_1^{-1}\right) E_1 E_2 \ldots E_k = I.$$

But since each E_i^{-1} is an elementary matrix, this equation shows that A can be row reduced to the identity matrix. Since the identity matrix is in reduced row-echelon form, this shows that the reduced row-echelon form of the matrix A is I, as needed for this direction.

To show the other direction, we assume that A's reduced row-echelon form is the identity matrix. We must show that A is invertible. Again representing the row reduction of A as a matrix product, we are given that $EA = I$, where E is a product of elementary matrices. But then by Theorem 2.62, this is enough to conclude that A is invertible, as needed.

Having shown that each of the two conditions of our theorem implies the other, we have shown that the two conditions are equivalent, as needed.

♠

Theorem 2.63 corresponds to Algorithm 2.39, which claims that A^{-1} is found by row reducing the augmented matrix $[A|I]$ to the form $\left[I|A^{-1}\right]$. This will be a matrix product $E\,[A|I]$ where E is a product of elementary matrices. By the rules of matrix multiplication, we have that $E\,[A|I] = [EA|EI] = [EA|E]$.

It follows that the reduced row-echelon form of $[A|I]$ is $[EA|E]$, where EA gives the reduced row-echelon form of A. By Theorem 2.63, if $EA \neq I$, then A is not invertible, and if $EA = I$, A is invertible. If $EA = I$, then by Theorem 2.62, $E = A^{-1}$. This proves that Algorithm 2.39 does in fact find A^{-1}.

2.8 LU Factorization

Outcomes

A. *Recognize upper triangular matrices and lower triangular matrices.*

B. *Use back- or forward-substitution to efficiently find solutions to the equation $AX = B$ when A is triangular.*

C. *When possible, find an LU factorization of a given matrix A either by direct computation or by the multiplier method.*

D. *In cases where the matrix A can be written as a product LU, efficiently use that LU factorization to find solutions to the matrix equation $AX = B$.*

When trying to solve a system of equations, we have developed an approach to the problem that is guaranteed to produce the solution to the system. We simply use Gaussian Elimination to produce an equivalent system of equations that is amenable to solution by back substitution. If the matrix of coefficients is invertible, you can find A^{-1} and then the solution to the system is $X = A^{-1}B$. We've practiced these techniques and we know that they produce the needed solution. What more could we want?

Well, one difficulty with the above method for solution is that it is computationally inefficient. That isn't going to matter too much when one is solving a system of 3 or (with a computer) 30 or 300 equations. But many problems that are actually solved in business and government settings involve thousands of equations, and then computational efficiency becomes quite important. In this section, we will introduce you to one method of efficiently solving square systems of equations, called LU Factorization.

Triangular Matrices

To begin to understand the appeal of this method, assume that we are trying to solve a system of n equations and n unknowns, $AX = Y$, and assume for the sake of argument that this system has a unique solution. If A happens to already be in row-echelon form then it is easy to find the solution Y by back substitution, as in this example:

Example 2.65: An Easy System to Solve

Find the solution to the system of equations

$$\begin{aligned} x+2y+3z &= 4 \\ y-2z &= 7 \\ 3z &= 6 \end{aligned}$$

Solution. By back substitution, the third equation tells us that $z = 2$, then the second equation tells us that $y = 7+2\cdot 2 = 11$, and then the first equation tells us that $x = \textit{(mumble, mumble)} = -24$. So the solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -24 \\ 11 \\ 2 \end{bmatrix}$. Nothing easier! ♠

The matrix A for the last example is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{bmatrix},$$

and such a matrix is called an upper triangular matrix. There are also lower triangular matrices. Here is the official definition:

Definition 2.66: Triangular Matrix

*An $n \times n$ matrix U is said to be an **upper triangular matrix** if every entry below the main diagonal is equal to 0. In other words, if $i > j$, then $u_{ij} = 0$.*
*An $n \times n$ matrix L is said to be a **lower triangular matrix** if every entry above the main diagonal is equal to 0. I.e., if $i < j$, then $l_{ij} = 0$.*
*An $n \times n$ matrix A is said to be a **triangular matrix** if it is either upper triangular or lower triangular.*

The short version is: Upper triangular matrices have 0's below the main diagonal. Lower triangular matrices have 0's above the main diagonal. If a matrix is both square and triangular (that sounds weird, but it is what we mean to say) then it looks triangular, but a matrix can be triangular without being square (that sounds better, right?) Take a minute and look at the following examples to make sure that the previous sentences make sense.

Example 2.67: Triangular Matrices

The following matrices are all triangular:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 0 & 4 \\ 0 & 1 & -1 & 7 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & -1 & 12 \\ 0 & 0 & -1 & 2 & 7 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 7 & 1 \end{bmatrix}$$

The first, second, and third matrices are upper triangular, while the first and fourth are lower triangular.

Example 2.65 involved showing that if U is an upper triangular matrix, then the system $UX = Y$ is easy to solve by back substitution. It is also easy to see[1] that if L is lower triangular, then the system $LY = B$ is easy to solve by forward substitution. The usefulness of the LU factorization that we are discussing in this section relies on these observations.

[1]This is math speak for "Make up an example on your own that verifies what is claimed next." Go ahead, do it. Write a system of equations that generates a lower triangular coefficient matrix and solve it.

Can We Find an LU Factorization?

We would like to solve a system $AX = B$, and our plan is going to be to factor A as a product of a lower triangular and upper triangular matrix, $A = LU$, where L has ones along the main diagonal. Lots of times this is doable, but not always. Partly because of this, we will emphasize the techniques of using LU factorization rather than looking for proofs in this section. It turns out that it takes about half as many operations to obtain an LU factorization as it does to find the reduced row echelon form. This makes using the LU factorization to solve the system an attractive method of attack, when the matrix A is factorable. Unfortunately, this is not always possible:

Example 2.68: A Matrix with no LU factorization

If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, *can we find a lower triangular matrix* L *with ones on the diagonal and an upper triangular matrix* U *such that* $A = LU$?

Solution. To do so you would need

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b \\ xa & xb+c \end{bmatrix}.$$

Therefore, $b = 1$ and $a = 0$. Also, from the bottom rows, $xa = 1$ which can't happen and have $a = 0$. Therefore, you can't write this matrix in the form LU. It has no LU factorization. This is what we mean above by saying the method lacks generality. ♠

Let's examine a couple of methods for finding the LU factorization, when it does exist.

Finding An LU Factorization By Direct Computation

Which matrices have an LU factorization? It turns out it is those whose row-echelon form can be achieved without switching rows. In other words matrices which only involve using row operations of type 2 or 3 to obtain the row-echelon form.

Example 2.69: An LU factorization

Find an LU *factorization of* $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 3 & 2 & 1 \\ 2 & 3 & 4 & 0 \end{bmatrix}$.

One way to find the LU factorization is to simply look for it directly. You need

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 3 & 2 & 1 \\ 2 & 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} \begin{bmatrix} a & d & h & j \\ 0 & b & e & i \\ 0 & 0 & c & f \end{bmatrix}.$$

Then multiplying these you get

$$\begin{bmatrix} a & d & h & j \\ xa & xd+b & xh+e & xj+i \\ ya & yd+zb & yh+ze+c & yj+iz+f \end{bmatrix}$$

and so you can now tell what the various quantities equal. From the first column, you need $a=1, x=1, y=2$. Now go to the second column. You need $d=2, xd+b=3$ so $b=1, yd+zb=3$ so $z=-1$. From the third column, $h=0, e=2, c=6$. Now from the fourth column, $j=2, i=-1, f=-5$. Therefore, an LU factorization is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 6 & -5 \end{bmatrix}.$$

You can check whether you got it right by simply multiplying these two.

LU Factorization via the Multiplier Method

Remember that for a matrix A to be written in the form $A=LU$, you must be able to reduce it to its row-echelon form without interchanging rows. The following procedure, called the *multiplier method*, gives a process for calculating the LU factorization of such a matrix A.

Example 2.70: LU factorization

Find an LU factorization for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & 3 & -2 \end{bmatrix}$$

Solution.

We take the matrix A and reduce it *only using our third elementary row operation: adding a multiple of one row to another*, keeping track of the operations we use to clear out the columns of A below the main diagonal:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & 3 & -2 \end{bmatrix} \xrightarrow{\boxed{-2}r_1+r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ -2 & 3 & -2 \end{bmatrix} \xrightarrow{\boxed{2}r_1+r_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 7 & 4 \end{bmatrix} \xrightarrow{\boxed{7}r_2+r_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & -31 \end{bmatrix}.$$

Notice that we have stopped our row reducing as soon as we have achieved an upper triangular matrix. This is our matrix U.

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & -31 \end{bmatrix}.$$

All we have to do is produce the lower triangular L.

To find L, you will notice that we have placed boxes around the *multipliers* that we have used in our row reduction. Notice that the -2 was used to create a 0 in position $(2,1)$ of our reduced matrix, the multiplier 2 got us the 0 in position $(3,1)$, and the 7 was used to clear out position $(3,2)$. To create the matrix L, start with the identity matrix and then put the *opposite of* each multiplier in the position of the matrix with which it is associated:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -7 & 1 \end{bmatrix}.$$

And that's it! You can check that we have found L and U such that

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & -31 \end{bmatrix}.$$

♠

Example 2.71: LU factorization

Find an LU factorization for the matrix $A = \begin{bmatrix} 3 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ -9 & -2 & 0 & -2 \\ 0 & 2 & 4 & 1 \end{bmatrix}$,

Solution. We reduce the given matrix A to an upper triangular form, only using our third row operation:

$$\begin{bmatrix} 3 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ -9 & -2 & 0 & -2 \\ 0 & 2 & 4 & 1 \end{bmatrix} \xrightarrow{3r_1+r_3} \begin{bmatrix} 3 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 4 & 1 \end{bmatrix} \xrightarrow{-1r_2+r_3} \begin{bmatrix} 3 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 2 & 4 & 1 \end{bmatrix} \xrightarrow{-2r_2+r_4} \begin{bmatrix} 3 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since our matrix is upper triangular, we have found the matrix U:

$$U = \begin{bmatrix} 3 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Looking at our row reduction, we see that our multipliers are 3, -1, and -2. Taking the identity matrix and inserting the opposite of the multipliers in the correct positions, we find that

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}.$$

And you can check that $A = LU$.

♠

One quick note to end this subsection. Suppose that the matrix A is not square, so that A is an $n \times m$ matrix. The matrix L will always be an $n \times n$ square matrix, since we construct it by adding non-zero entries to the $n \times n$ identity matrix. Our matrix U, on the other hand, will be an $n \times m$ matrix, since it is the result of row reducing A. All of our examples to this point have started with square A's, but you should be aware of the general case, as it will show up soon.

Solving Systems using LU Factorization

One reason people care about the LU factorization is it allows the quick solution of systems of equations. Here is an example.

Example 2.72: LU factorization to Solve Equations

Use LU factorization to solve the equation

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 2 \\ 4 & 3 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{array}\right]\left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right]=\left[\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right].$$

Solution.

Of course one way is to write the augmented matrix and grind away. However, this involves more row operations than the computation of the LU factorization and it turns out that the LU factorization can give the solution quickly. Here is how. You can (and probably should) check that the multiplier method discussed above yields the following as an LU factorization for the coefficient matrix.

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 2 \\ 4 & 3 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{array}\right]=\left[\begin{array}{ccc} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right]\left[\begin{array}{cccc} 1 & 2 & 3 & 2 \\ 0 & -5 & -11 & -7 \\ 0 & 0 & 0 & -2 \end{array}\right].$$

We are trying to solve the equation $AX=B$, where $B=\left[\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right]$. Notice that the following are equivalent:

$$\begin{aligned} AX &= B \\ (LU)X &= B \\ L(UX) &= B. \end{aligned}$$

Here's the idea that gives us the solution to our (relatively difficult) problem via two quickly computed (relatively easy) problems: Let $Y=UX$. Looking at our last equation above, we want to solve $LY=B$ for Y. Since L is lower triangular, this is easy. And then, once we are looking at Y, we can find X by simply solving the equation $UX=Y$ for X. And once again, as U is triangular, the solution by back substitution is quickly computed. Here are the details:

First, to solve $LY=B$, we need to solve

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right]\left[\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}\right]=\left[\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right]$$

which yields very quickly that $Y=\left[\begin{array}{c} 1 \\ -2 \\ 2 \end{array}\right]$.

Then we can find X by solving $UX = Y$. Thus in this case,

$$\left[\begin{array}{rrrr} 1 & 2 & 3 & 2 \\ 0 & -5 & -11 & -7 \\ 0 & 0 & 0 & -2 \end{array}\right]\left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right] = \left[\begin{array}{r} 1 \\ -2 \\ 2 \end{array}\right]$$

which yields

$$X = \left[\begin{array}{c} -\frac{3}{5} + \frac{7}{5}t \\ \frac{9}{5} - \frac{11}{5}t \\ t \\ -1 \end{array}\right], \quad t \in \mathbb{R}.$$

♠

A Look Under the Hood: A Justification for the Multiplier Method

Why does the multiplier method work for finding the LU factorization? Suppose A is a matrix which has the property that the row-echelon form for A may be achieved without switching rows. Thus every row which is replaced using this row operation in obtaining the row-echelon form may be modified by using a row which is above it.

Lemma 2.73: Multiplier Method and Triangular Matrices

Let L be a lower (upper) triangular $m \times m$ matrix which has ones down the main diagonal. Then L^{-1} also is a lower (upper) triangular matrix which has ones down the main diagonal. In the case that L is of the form

$$L = \begin{bmatrix} 1 & & & \\ a_1 & 1 & & \\ \vdots & & \ddots & \\ a_n & & & 1 \end{bmatrix} \tag{2.11}$$

where all entries are zero except for the left column and main diagonal, it is also the case that L^{-1} is obtained from L by simply multiplying each entry below the main diagonal in L with -1. The same is true if the single nonzero column is in another position.

Proof. Consider the usual setup for finding the inverse $\begin{bmatrix} L & I \end{bmatrix}$. Then each row operation done to L to reduce to row reduced echelon form results in changing only the entries in I below the main diagonal. In the special case of L given in 2.11 or the single nonzero column is in another position, multiplication by -1 as described in the lemma clearly results in L^{-1}. ♠

For a simple illustration of the last claim,

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & a & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -a & 1 \end{bmatrix}$$

Now let A be an $m \times n$ matrix, say

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and assume A can be row reduced to an upper triangular form using only row operation 3. Thus, in particular, $a_{11} \neq 0$. Multiply on the left by $E_1 =$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{m1}}{a_{11}} & 0 & \cdots & 1 \end{bmatrix}$$

This is the product of elementary matrices which make modifications in the first column only. It is equivalent to taking $-a_{21}/a_{11}$ times the first row and adding to the second. Then taking $-a_{31}/a_{11}$ times the first row and adding to the third and so forth. The quotients in the first column of the above matrix are the multipliers. Thus the result is of the form

$$E_1 A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a'_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a'_{m2} & \cdots & a'_{mn} \end{bmatrix}$$

By assumption, $a'_{22} \neq 0$ and so it is possible to use this entry to zero out all the entries below it in the matrix on the right by multiplication by a matrix of the form $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & E \end{bmatrix}$ where E is an $[m-1] \times [m-1]$ matrix of the form

$$E = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\frac{a'_{32}}{a'_{22}} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a'_{m2}}{a'_{22}} & 0 & \cdots & 1 \end{bmatrix}$$

Again, the entries in the first column below the 1 are the multipliers. Continuing this way, zeroing out the entries below the diagonal entries, finally leads to

$$E_{m-1}E_{m-2}\cdots E_1 A = U$$

where U is upper triangular. Each E_j has all ones down the main diagonal and is lower triangular. Now multiply both sides by the inverses of the E_j in the reverse order. This yields

$$A = E_1^{-1}E_2^{-1}\cdots E_{m-1}^{-1}U$$

By Lemma 2.73, this implies that the product of those E_j^{-1} is a lower triangular matrix having all ones down the main diagonal.

The above discussion and lemma gives the justification for the multiplier method. The expressions

$$\frac{-a_{21}}{a_{11}}, \frac{-a_{31}}{a_{11}}, \cdots, \frac{-a_{m1}}{a_{11}}$$

denoted respectively by $M_{21}, \cdots, M_{m1}$ to save notation which were obtained in building E_1 are the multipliers. Then according to the lemma, to find E_1^{-1} you simply write

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -M_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -M_{m1} & 0 & \cdots & 1 \end{bmatrix}$$

Similar considerations apply to the other E_j^{-1}. Thus L is a product of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -M_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -M_{m1} & 0 & \cdots & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & -M_{m[m-1]} & 1 \end{bmatrix}$$

each factor having at most one nonzero column, the position of which moves from left to right in scanning the above product of matrices from left to right. It follows from what we know about the effect of multiplying on the left by an elementary matrix that the above product is of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -M_{21} & 1 & \cdots & 0 & 0 \\ \vdots & -M_{32} & \ddots & \vdots & \vdots \\ -M_{[M-1]1} & \vdots & \cdots & 1 & 0 \\ -M_{M1} & -M_{M2} & \cdots & -M_{MM-1} & 1 \end{bmatrix}$$

In words, beginning at the left column and moving toward the right, you simply insert, into the corresponding position in the identity matrix, -1 times the multiplier which was used to zero out an entry in that position below the main diagonal in A, while retaining the main diagonal which consists entirely of ones. This is L.

Chapter 3

Determinants

3.1 Basic Techniques and Properties

Outcomes

A. *Evaluate the determinant of a square matrix using either Laplace Expansion or row operations.*

B. *Demonstrate the effects that row operations have on determinants.*

C. *Verify the following:*

   (a) *The determinant of a product of matrices is the product of the determinants.*

   (b) *The determinant of a matrix is equal to the determinant of its transpose.*

Cofactors and 2×2 Determinants

Let A be an $n \times n$ matrix. That is, let A be a square matrix. The **determinant** of A, denoted by $\det(A)$, is a very important number which we will explore throughout this section.

Let's start small.

Definition 3.1: Determinant of a One By One Matrix

Let $A = \left[\begin{array}{c} a \end{array} \right]$. *Then*
$$\det(A) = a$$

If A is a 2×2 matrix, the determinant is given by the following formula.

Definition 3.2: Determinant of a Two By Two Matrix

Let $A = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right]$. *Then*
$$\det(A) = ad - cb$$

The determinant is also often denoted by enclosing the matrix with two vertical lines. Thus
$$\det \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] = \left| \begin{array}{cc} a & b \\ c & d \end{array} \right| = ad - bc$$

The following is an example of finding the determinant of a 2×2 matrix.

Example 3.3: A Two by Two Determinant

Find $\det(A)$ *for the matrix* $A = \left[\begin{array}{rr} 2 & 4 \\ -1 & 6 \end{array} \right]$.

Solution. From Definition 3.2,
$$\det(A) = (2)(6) - (-1)(4) = 12 + 4 = 16$$

♠

The 2×2 determinant can be used to find the determinant of larger matrices. We will now explore how to find the determinant of a 3×3 matrix, using several tools including the 2×2 determinant.

We begin with the following definition.

Definition 3.4: The ij^{th} Minor of a Matrix

Let A be a 3×3 matrix. The ij^{th} ***minor*** *of A, denoted as $\text{minor}(A)_{ij}$, is the determinant of the 2×2 matrix which results from deleting the i^{th} row and the j^{th} column of A.*
In general, if A is an $n \times n$ matrix, then the ij^{th} minor of A is the determinant of the $n-1 \times n-1$ matrix which results from deleting the i^{th} row and the j^{th} column of A.

Hence, there is a minor associated with each entry of A. Consider the following example which demonstrates this definition.

Example 3.5: Finding Minors of a Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $\text{minor}(A)_{12}$ *and* $\text{minor}(A)_{23}$.

Solution. First we will find $\text{minor}(A)_{12}$. By Definition 3.4, this is the determinant of the 2×2 matrix which results when you delete the first row and the second column. This minor is given by

$$\text{minor}(A)_{12} = \det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

Using Definition 3.2, we see that

$$\det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = (4)(1) - (3)(2) = 4 - 6 = -2$$

Therefore $\text{minor}(A)_{12} = -2$.

Similarly, $\text{minor}(A)_{23}$ is the determinant of the 2×2 matrix which results when you delete the second row and the third column. This minor is therefore

$$\text{minor}(A)_{23} = \det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = -4$$

Finding the other minors of A is left as an exercise. ♠

The ij^{th} minor of a matrix A is used in another important definition, given next.

Definition 3.6: The ij^{th} Cofactor of a Matrix

Suppose A is an $n \times n$ matrix. The ij^{th} **cofactor**, *denoted by $\text{cof}(A)_{ij}$ is defined to be*

$$\text{cof}(A)_{ij} = (-1)^{i+j} \,\text{minor}(A)_{ij}$$

It is also convenient to refer to the cofactor of an entry of a matrix as follows. If a_{ij} is the ij^{th} entry of the matrix, then its cofactor is just $\text{cof}(A)_{ij}$.

Example 3.7: Finding Cofactors of a Matrix

Consider the matrix

$$A = \left[\begin{array}{rrr} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{array} \right]$$

Find $cof(A)_{12}$ *and* $cof(A)_{23}$.

Solution. We will use Definition 3.6 to compute these cofactors.

First, we will compute $\mathrm{cof}(A)_{12}$. Therefore, we need to find $\mathrm{minor}(A)_{12}$. This is the determinant of the 2×2 matrix which results when you delete the first row and the second column. Thus $\mathrm{minor}(A)_{12}$ is given by

$$\det \left[\begin{array}{rr} 4 & 2 \\ 3 & 1 \end{array} \right] = -2$$

Then,

$$\mathrm{cof}(A)_{12} = (-1)^{1+2} \mathrm{minor}(A)_{12} = (-1)^{1+2}(-2) = 2$$

Hence, $\mathrm{cof}(A)_{12} = 2$.

Similarly, we can find $\mathrm{cof}(A)_{23}$. First, find $\mathrm{minor}(A)_{23}$, which is the determinant of the 2×2 matrix which results when you delete the second row and the third column. This minor is therefore

$$\det \left[\begin{array}{rr} 1 & 2 \\ 3 & 2 \end{array} \right] = -4$$

Hence,

$$\mathrm{cof}(A)_{23} = (-1)^{2+3} \mathrm{minor}(A)_{23} = (-1)^{2+3}(-4) = 4$$

♠

You may wish to find the remaining cofactors for the above matrix. Remember that there is a cofactor for every entry in the matrix.

We have now established the tools we need to find the determinant of a 3×3 matrix.

Definition 3.8: The Determinant of a Three By Three Matrix

Let A *be a* 3×3 *matrix. We calculate* $\det(A)$ *by picking a row (or column) and taking the product of each entry in that row (column) with its cofactor and adding these products together.*
This process when applied to the i^{th} *row (column) is known as* ***expanding along the*** i^{th} ***row (column)*** *as is given by*

$$\det(A) = a_{i1} cof(A)_{i1} + a_{i2} cof(A)_{i2} + a_{i3} cof(A)_{i3}$$

When calculating the determinant, you can choose to expand any row or any column. Regardless of your choice, you will always get the same number which is the determinant of the matrix A. This method of evaluating a determinant by expanding along a row or a column is called **Laplace Expansion** or **Cofactor Expansion**.

Consider the following example.

Example 3.9: Finding the Determinant of a Three by Three Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $\det(A)$ *using the method of Laplace Expansion.*

Solution. First, we will calculate $\det(A)$ by expanding along the first column. Using Definition 3.8, we take the 1 in the first column and multiply it by its cofactor,

$$1(-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = (1)(1)(-1) = -1$$

Similarly, we take the 4 in the first column and multiply it by its cofactor, as well as with the 3 in the first column. Finally, we add these numbers together, as given in the following equation.

$$\det(A) = 1\overbrace{(-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}}^{\mathrm{cof}(A)_{11}} + 4\overbrace{(-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}}^{\mathrm{cof}(A)_{21}} + 3\overbrace{(-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}}^{\mathrm{cof}(A)_{31}}$$

Calculating each of these, we obtain

$$\det(A) = 1(1)(-1) + 4(-1)(-4) + 3(1)(-5) = -1 + 16 + -15 = 0$$

Hence, $\det(A) = 0$.

As mentioned in Definition 3.8, we can choose to expand along any row or column. Let's try expanding along the second row. Here, we take the 4 in the second row and multiply it to its cofactor, then add this to the 3 in the second row multiplied by its cofactor, and the 2 in the second row multiplied by its cofactor. The calculation is as follows.

$$\det(A) = 4\overbrace{(-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}}^{\mathrm{cof}(A)_{21}} + 3\overbrace{(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}}^{\mathrm{cof}(A)_{22}} + 2\overbrace{(-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}}^{\mathrm{cof}(A)_{23}}$$

Calculating each of these products, we obtain

$$\det(A) = 4(-1)(-2) + 3(1)(-8) + 2(-1)(-4) = 0$$

You can see that for both methods, we obtained $\det(A) = 0$. ♠

As mentioned above, we will always come up with the same value for $\det(A)$ regardless of the row or column we choose to expand along. You should try to compute the above determinant by expanding along other rows and columns. This is a good way to check your work, because you should come up with the same number each time!

We present this idea formally in the following theorem.

Theorem 3.10: The Determinant is Well Defined

Expanding the $n \times n$ matrix along any row or column always gives the same answer, which is the determinant.

We have now looked at the determinant of 2×2 and 3×3 matrices. It turns out that the method used to calculate the determinant of a 3×3 matrix can be used to calculate the determinant of any sized matrix. Notice that Definition 3.4, Definition 3.6 and Definition 3.8 can all be applied to a matrix of any size.

For example, the ij^{th} minor of a 4×4 matrix is the determinant of the 3×3 matrix you obtain when you delete the i^{th} row and the j^{th} column. Just as with the 3×3 determinant, we can compute the determinant of a 4×4 matrix by Laplace Expansion, along any row or column

Consider the following example.

Example 3.11: Determinant of a Four by Four Matrix

Find $\det(A)$ *where*

$$A = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 3 & 4 & 5 \\ 3 & 4 & 3 & 2 \end{array} \right]$$

Solution. As in the case of a 3×3 matrix, you can expand this along any row or column. Let's pick the third column. Then, using Laplace Expansion,

$$\det(A) = 3(-1)^{1+3} \left| \begin{array}{ccc} 5 & 4 & 3 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{array} \right| + 2(-1)^{2+3} \left| \begin{array}{ccc} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{array} \right| +$$

$$4(-1)^{3+3} \left| \begin{array}{ccc} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 3 & 4 & 2 \end{array} \right| + 3(-1)^{4+3} \left| \begin{array}{ccc} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 1 & 3 & 5 \end{array} \right|$$

Now, you can calculate each 3×3 determinant using Laplace Expansion, as we did above. You should complete these as an exercise and verify that $\det(A) = -12$. ♠

The following provides a formal definition for the determinant of an $n \times n$ matrix. You may wish to take a moment and consider the above definitions for 2×2 and 3×3 determinants in context of this definition.

Definition 3.12: The Determinant of an $n \times n$ Matrix

Let A be an $n \times n$ matrix where $n \geq 2$ and suppose the determinant of an $(n-1) \times (n-1)$ has been defined. Then

$$\det(A) = \sum_{j=1}^{n} a_{ij} cof(A)_{ij} = \sum_{i=1}^{n} a_{ij} cof(A)_{ij}$$

The first formula consists of expanding the determinant along the i^{th} row and the second expands the determinant along the j^{th} column.

Remember that we defined, back in Definition 3.2, the determinant of a 2×2 matrix as $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$. It would be a great exercise to check that this definition matches what we say above in Definition 3.12. So you should see if you take the time to expand the determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ across any row or any column, you always get $ad - bc$ as the value of the determinant.

In the following subsections, we will continue to explore some important properties and characteristics of the determinant. In the exposition, we will continue to illustrate our results and claims by examples, with the proofs gathered together in Section 3.1.

The Determinant of a Triangular Matrix

Recall triangular matrices, that we introduced in Definition 2.66. It turns out that for triangular matrices, the determinant can be calculated quite easily.

Theorem 3.13: Determinant of a Triangular Matrix

Let A be an upper or lower triangular matrix. Then $\det(A)$ is obtained by taking the product of the entries on the main diagonal.

The verification of this Theorem can be done by computing the determinant using Laplace Expansion along the first row or column.

Consider the following example.

Example 3.14: Determinant of a Triangular Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 77 \\ 0 & 2 & 6 & 7 \\ 0 & 0 & 3 & 33.7 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Find $\det(A)$.

Solution. From Theorem 3.13, it suffices to take the product of the elements on the main diagonal. Thus $\det(A) = 1 \times 2 \times 3 \times (-1) = -6$.

Without using Theorem 3.13, you could use Laplace Expansion. We will expand along the first column. This gives

$$\begin{aligned} \det(A) = \quad & 1 \begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{2+1} \begin{vmatrix} 2 & 3 & 77 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} + \\ & 0(-1)^{3+1} \begin{vmatrix} 2 & 3 & 77 \\ 2 & 6 & 7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{4+1} \begin{vmatrix} 2 & 3 & 77 \\ 2 & 6 & 7 \\ 0 & 3 & 33.7 \end{vmatrix} \end{aligned}$$

and the only nonzero term in the expansion is

$$1 \begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix}$$

Now find the determinant of this 3×3 matrix, by expanding along the first column to obtain

$$\begin{aligned} \det(A) &= 1 \times \left(2 \times \begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix} + 0(-1)^{2+1} \begin{vmatrix} 6 & 7 \\ 0 & -1 \end{vmatrix} + 0(-1)^{3+1} \begin{vmatrix} 6 & 7 \\ 3 & 33.7 \end{vmatrix} \right) \\ &= 1 \times 2 \times \begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix} \end{aligned}$$

Next use Definition 3.2 to find the determinant of this 2×2 matrix, which is just $3 \times -1 - 0 \times 33.7 = -3$. Putting all these steps together, we have

$$\det(A) = 1 \times 2 \times 3 \times (-1) = -6$$

which is just the product of the entries down the main diagonal of the original matrix! ♠

You can see that while both methods result in the same answer, Theorem 3.13 provides a much quicker method.

Now we will explore some important properties of determinants.

Properties of Determinants

There are many important properties of determinants. Since many of these properties involve the row operations discussed in Chapter 1, we recall that definition now.

Definition 3.15: Row Operations

The row operations consist of the following

1. *Switch two rows.*
2. *Multiply a row by a nonzero number.*
3. *Replace a row by a multiple of another row added to itself.*

We will now consider the effect of row operations on the determinant of a matrix. In future sections, we will see that using the following properties can greatly assist in finding determinants. This section will use the theorems as motivation to provide various examples of the usefulness of the properties.

The first theorem explains the effect on the determinant of a matrix when two rows are switched.

Theorem 3.16: Switching Rows

Let A be an $n \times n$ matrix and let B be a matrix which results from switching two rows of A. Then $\det(B) = -\det(A)$.

When we switch two rows of a matrix, the determinant is multiplied by -1. Consider the following example.

Example 3.17: Switching Two Rows

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and let* $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$. *Knowing that* $\det(A) = -2$, *find* $\det(B)$.

Solution. By Definition 3.2, $\det(A) = 1 \times 4 - 3 \times 2 = -2$. Notice that the rows of B are the rows of A but switched. By Theorem 3.16 since two rows of A have been switched, $\det(B) = -\det(A) = -(-2) = 2$. You can verify this using Definition 3.2. ♠

The next theorem demonstrates the effect on the determinant of a matrix when we multiply a row by a scalar.

Theorem 3.18: Multiplying a Row by a Scalar

Let A be an $n \times n$ matrix and let B be a matrix which results from multiplying some row of A by a scalar k. Then $\det(B) = k\det(A)$.

Notice that this theorem is true when we multiply *one* row of the matrix by k. If we were to multiply *two* rows of A by k to obtain B, we would have $\det(B) = k^2 \det(A)$. Suppose we were to multiply all n rows of A by k to obtain the matrix B, so that $B = kA$. Then, $\det(B) = k^n \det(A)$. This gives the next theorem.

Theorem 3.19: Scalar Multiplication

Let A and B be $n \times n$ matrices and k a scalar, such that $B = kA$. Then $\det(B) = k^n \det(A)$.

Consider the following example.

Example 3.20: Multiplying a Row by 5

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 10 \\ 3 & 4 \end{bmatrix}$. *Knowing that* $\det(A) = -2$, *find* $\det(B)$.

Solution. By Definition 3.2, $\det(A) = -2$. We can also compute $\det(B)$ using Definition 3.2, and we see that $\det(B) = -10$.

Now, let's compute $\det(B)$ using Theorem 3.18 and see if we obtain the same answer. Notice that the first row of B is 5 times the first row of A, while the second row of B is equal to the second row of A. By Theorem 3.18, $\det(B) = 5 \times \det(A) = 5 \times -2 = -10$.

You can see that this matches our answer above. ♠

Finally, consider the next theorem for the last row operation, that of adding a multiple of a row to another row.

Theorem 3.21: Adding a Multiple of a Row to Another Row

Let A be an $n \times n$ matrix and let B be a matrix which results from adding a multiple of a row to another row. Then $\det(A) = \det(B)$.

Therefore, when we add a multiple of a row to another row, the determinant of the matrix is unchanged. Note that if a matrix A contains a row which is a multiple of another row, $\det(A)$ will equal 0. To see this, suppose the first row of A is equal to -1 times the second row. By Theorem 3.21, we can add the first row to the second row, and the determinant will be unchanged. However, this row operation will result in a row of zeros. Using Laplace Expansion along the row of zeros, we find that the determinant is 0.

Consider the following example.

Example 3.22: Adding a Row to Another Row

Let $A = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right]$ *and let* $B = \left[\begin{array}{cc} 1 & 2 \\ 5 & 8 \end{array}\right]$. *Find* $\det(B)$.

Solution. By Definition 3.2, $\det(A) = -2$. Notice that the second row of B is two times the first row of A added to the second row. By Theorem 3.16, $\det(B) = \det(A) = -2$. As usual, you can verify this answer using Definition 3.2. ♠

Example 3.23: Multiple of a Row

Let $A = \left[\begin{array}{cc} 1 & 2 \\ 2 & 4 \end{array}\right]$. *Show that* $\det(A) = 0$.

Solution. Using Definition 3.2, the determinant is given by

$$\det(A) = 1 \times 4 - 2 \times 2 = 0$$

However notice that the second row is equal to 2 times the first row. Then by the discussion above following Theorem 3.21 the determinant will equal 0. ♠

Until now, our focus has primarily been on row operations. However, we can carry out the same operations with columns, rather than rows. The three operations outlined in Definition 3.15 can be done with columns instead of rows. In this case, in Theorems 3.16, 3.18, and 3.21 you can replace the word, "row" with the word "column".

There are several other major properties of determinants which do not involve row (or column) operations. The first is the determinant of a product of matrices.

Theorem 3.24: Determinant of a Product

Let A and B be two $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B)$$

In order to find the determinant of a product of matrices, we can simply take the product of the determinants.

Consider the following example.

Example 3.25: The Determinant of a Product

Compare $\det(AB)$ *and* $\det(A)\det(B)$ *for*

$$A = \left[\begin{array}{rr} 1 & 2 \\ -3 & 2 \end{array}\right], \quad B = \left[\begin{array}{rr} 3 & 2 \\ 4 & 1 \end{array}\right]$$

Solution. First compute AB, which is given by

$$AB = \left[\begin{array}{rr} 1 & 2 \\ -3 & 2 \end{array}\right]\left[\begin{array}{rr} 3 & 2 \\ 4 & 1 \end{array}\right] = \left[\begin{array}{rr} 11 & 4 \\ -1 & -4 \end{array}\right]$$

and so by Definition 3.2

$$\det(AB) = \det\left[\begin{array}{rr} 11 & 4 \\ -1 & -4 \end{array}\right] = -40$$

Now

$$\det(A) = \det\left[\begin{array}{rr} 1 & 2 \\ -3 & 2 \end{array}\right] = 8$$

and

$$\det(B) = \det\left[\begin{array}{rr} 3 & 2 \\ 4 & 1 \end{array}\right] = -5$$

Computing $\det(A) \times \det(B)$ we have $8 \times -5 = -40$. This is the same answer as above and you can see that $\det(A)\det(B) = 8 \times (-5) = -40 = \det(AB)$. ♠

Consider the next important property.

Theorem 3.26: Determinant of the Transpose

Let A *be a matrix where* A^T *is the transpose of* A. *Then,*

$$\det\left(A^T\right) = \det(A)$$

This theorem is illustrated in the following example.

Example 3.27: Determinant of the Transpose

Let

$$A = \left[\begin{array}{rr} 2 & 5 \\ 4 & 3 \end{array}\right]$$

Find $\det\left(A^T\right)$.

Solution. First, note that

$$A^T = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$$

Using Definition 3.2, we can compute $\det(A)$ and $\det\left(A^T\right)$. It follows that $\det(A) = 2 \times 3 - 4 \times 5 = -14$ and $\det\left(A^T\right) = 2 \times 3 - 5 \times 4 = -14$. Hence, $\det(A) = \det\left(A^T\right)$. ♠

The following provides an essential property of the determinant, as well as a useful way to determine if a matrix is invertible.

Theorem 3.28: Determinant of the Inverse

Let A be an $n \times n$ matrix. Then A is invertible if and only if $\det(A) \neq 0$. If this is true, it follows that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Consider the following example.

Example 3.29: Determinant of an Invertible Matrix

Let $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$. For each matrix, determine if it is invertible. If so, find the determinant of the inverse.

Solution. Consider the matrix A first. Using Definition 3.2 we can find the determinant as follows:

$$\det(A) = 3 \times 4 - 2 \times 6 = 12 - 12 = 0$$

By Theorem 3.28 A is not invertible.

Now consider the matrix B. Again by Definition 3.2 we have

$$\det(B) = 2 \times 1 - 5 \times 3 = 2 - 15 = -13$$

By Theorem 3.28 B is invertible and the determinant of the inverse is given by

$$\begin{aligned} \det\left(B^{-1}\right) &= \frac{1}{\det(B)} \\ &= \frac{1}{-13} \\ &= -\frac{1}{13} \end{aligned}$$

♠

Finding Determinants using Row Operations

Theorems 3.16, 3.18 and 3.21 illustrate how row operations affect the determinant of a matrix. In this section, we look at two examples where row operations are used to find the determinant of a large matrix. Recall that when working with large matrices, Laplace Expansion is effective but extremely time consuming, as there are in general many steps involved. This section provides useful tools for an alternative method. By first applying row operations, we can obtain a simpler matrix to which we apply Laplace Expansion.

While working through questions such as these, it is useful to record your row operations as you go along. Keep this in mind as you read through the next example.

Example 3.30: Finding a Determinant

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 1 & 2 & 3 \\ 4 & 5 & 4 & 3 \\ 2 & 2 & -4 & 5 \end{bmatrix}$$

Solution. We will use the properties of determinants outlined above to find $\det(A)$. First, add -5 times the first row to the second row. Then add -4 times the first row to the third row, and -2 times the first row to the fourth row, and call the result of all of this B. So we have

$$A \xrightarrow{-5r_1+r_2} \xrightarrow{-4r_1+r_3} \xrightarrow{-2r_1+r_4} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -9 & -13 & -17 \\ 0 & -3 & -8 & -13 \\ 0 & -2 & -10 & -3 \end{bmatrix} = B$$

Notice that the only row operation we have done so far is adding a multiple of a row to another row. Therefore, by Theorem 3.21, $\det(B) = \det(A)$.

At this stage, you could use Laplace Expansion to find $\det(B)$. However, we will continue with row operations to find an even simpler matrix to work with.

Add -3 times the third row to the second row. By Theorem 3.21 this does not change the value of the determinant. Then, multiply the fourth row by -3 to obtain a matrix C. Now our chain of transformations is

$$B \xrightarrow{-3r_3+r_2} \xrightarrow{-3r_4} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 11 & 22 \\ 0 & -3 & -8 & -13 \\ 0 & 6 & 30 & 9 \end{bmatrix} = C$$

Here, $\det(C) = -3\det(B)$, which means that $\det(B) = \left(-\frac{1}{3}\right)\det(C)$, and since $\det(A) = \det(B)$, we now have that $\det(A) = \left(-\frac{1}{3}\right)\det(C)$. Again, you could use Laplace Expansion here to find $\det(C)$. However, we will continue with row operations.

Take C, add 2 times the third row to the fourth row (no change in the determinant). Finally switch the third and second rows to obtain the matrix D:

$$C \xrightarrow{2r_3+r_4} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -8 & -13 \\ 0 & 0 & 11 & 22 \\ 0 & 0 & 14 & -17 \end{bmatrix} = D$$

That last row swap causes the determinant to be multiplied by -1. Thus $\det(C) = -\det(D)$. Hence, $\det(A) = \left(-\frac{1}{3}\right)\det(C) = \left(\frac{1}{3}\right)\det(D)$.

You could do more row operations or you could note that the determinant of D can be easily calculated by expanding along the first column. Then, expand the resulting 3×3 matrix also along the first column. This results in

$$\det(D) = 1(-3)\begin{vmatrix} 11 & 22 \\ 14 & -17 \end{vmatrix} = 1485$$

and so $\det(A) = \left(\frac{1}{3}\right)(1485) = 495$. ♠

You can see that by using row operations, we can simplify a matrix to the point where Laplace Expansion involves only a few steps. In Example 3.30, we also could have continued until the matrix was in upper triangular form, and taken the product of the entries on the main diagonal. Whenever computing the determinant, it is useful to consider all the possible methods and tools.

Consider the next example.

Example 3.31: Find the Determinant

Find the determinant of the matrix

$$A = \left[\begin{array}{rrrr} 1 & 2 & 3 & 2 \\ 1 & -3 & 2 & 1 \\ 2 & 1 & 2 & 5 \\ 3 & -4 & 1 & 2 \end{array} \right]$$

Solution. Once again, we will simplify the matrix through row operations. Add -1 times the first row to the second row. Next add -2 times the first row to the third and finally take -3 times the first row and add to the fourth row. This yields

$$B = \left[\begin{array}{rrrr} 1 & 2 & 3 & 2 \\ 0 & -5 & -1 & -1 \\ 0 & -3 & -4 & 1 \\ 0 & -10 & -8 & -4 \end{array} \right]$$

By Theorem 3.21, $\det(A) = \det(B)$.

Remember you can work with the columns also. Take -5 times the fourth column and add to the second column. This yields

$$C = \left[\begin{array}{rrrr} 1 & -8 & 3 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & -8 & -4 & 1 \\ 0 & 10 & -8 & -4 \end{array} \right]$$

By Theorem 3.21 $\det(A) = \det(C)$.

Now take -1 times the third row and add to the top row. This gives.

$$D = \left[\begin{array}{rrrr} 1 & 0 & 7 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & -8 & -4 & 1 \\ 0 & 10 & -8 & -4 \end{array} \right]$$

which by Theorem 3.21 has the same determinant as A.

Now, we can find $\det(D)$ by expanding along the first column as follows. You can see that there will be only one non zero term.

$$\det(D) = 1 \det \left[\begin{array}{rrr} 0 & -1 & -1 \\ -8 & -4 & 1 \\ 10 & -8 & -4 \end{array} \right] + 0 + 0 + 0$$

Expanding again along the first column, we have

$$\det(D) = 1 \left(0 + 8 \det \left[\begin{array}{rr} -1 & -1 \\ -8 & -4 \end{array} \right] + 10 \det \left[\begin{array}{rr} -1 & -1 \\ -4 & 1 \end{array} \right] \right) = -82$$

Now since $\det(A) = \det(D)$, it follows that $\det(A) = -82$. ♠

Remember that you can verify these answers by using Laplace Expansion on A. Similarly, if you first compute the determinant using Laplace Expansion, you can use the row operation method to verify.

A Look Under the Hood: Some Important Proofs about Determinants

In this section we provide proofs of many of the results from the last section concerning determinants and cofactors.

First we recall the definition of a determinant. If $A = \left[a_{ij}\right]$ is an $n \times n$ matrix, then $\det A$ is defined by computing the expansion along the first row:

$$\det A = \sum_{i=1}^{n} a_{1,i} \operatorname{cof}(A)_{1,i}. \tag{3.1}$$

If $n = 1$ then $\det A = a_{1,1}$.

The arguments that establish the various parts of the following lemma are straightforward. It would be a *good idea* for you to work them through on your own, as doing so will help you get used to working with the definitions that we will use in later proofs in this section.

Lemma 3.32: Determinants of Elementary Matrices

(1) Let E_{ij} be the elementary matrix obtained by interchanging ith and jth rows of I. Then $\det E_{ij} = -1$.
(2) Let E_{ik} be the elementary matrix obtained by multiplying the ith row of I by k. Then $\det E_{ik} = k$.
(3) Let E_{ijk} be the elementary matrix obtained by multiplying ith row of I by k and adding it to its jth row. Then $\det E_{ijk} = 1$.
(4) If E is an elementary matrix, then $\det E = \det E^T$.

Many of the proofs in section use the Principle of Mathematical Induction. This concept is discussed in Appendix A.2 and is reviewed here for convenience.

Suppose that we have some claim that is supposed to hold for every natural number n. For example, maybe we want to prove something is true for every $n \times n$ matrix. To use induction to establish the claim, we make two separate arguments:

First we check that the assertion is true for $n = 2$ (in this section, the case $n = 1$ is either completely trivial or meaningless). This is called establishing the *base case* of our proof.

Next we complete what is called the *induction step* of our proof. We assume that the assertion is true for the number $n - 1$ (where $n \geq 3$) and, given that assumption, which is called the *inductive hypothesis*, we prove that the assertion is true for the number n.

Once we have completed both of these steps, the Principle of Mathematical Induction tells us that we can conclude that our assertion is true for all $n \times n$ matrices for every $n \geq 2$.

To establish a bit of notation that will be useful to us, if A is an $n \times n$ matrix and $1 \leq j \leq n$, then the matrix obtained by removing 1st column and jth row from A will be denoted $A(j)$. Since $A(j)$ is an $n - 1 \times n - 1$ matrix, if they show up in the middle of a proof by induction, the inductive hypothesis will allow us some insight into the determinants of these matrices. Since these matrices are used in computation of cofactors $\text{cof}(A)_{1,i}$, for $1 \leq i \leq n$ when we are computing the determinant of A the inductive hypothesis will help us deduce properties of the determinant of A.

Don't worry, this will become clearer as we work through some of the proofs. Let's dive in.

Consider the following lemma.

Lemma 3.33

If A is an $n \times n$ matrix such that one of its rows consists of zeros, then $\det A = 0$.

Proof. We will prove this lemma using Mathematical Induction.

If $n = 2$ this is easy (check!).

Let $n \geq 3$ be such that every matrix of size $n - 1 \times n - 1$ with a row consisting of zeros has determinant equal to zero. Let i be such that the ith row of A consists of zeros. Then we have $a_{ij} = 0$ for $1 \leq j \leq n$.

Fix $j \in \{1, 2, \ldots, n\}$ such that $j \neq i$. Then matrix $A(j)$ used in computation of $\text{cof}(A)_{1,j}$ has a row consisting of zeros, and by our inductive hypothesis, $\det A(j) = 0$ and so $\text{cof}(A)_{1,j} = 0$.

On the other hand, if $j = i$ then $a_{1,j} = 0$. Therefore $a_{1,j}\text{cof}(A)_{1,j} = 0$ for all j and by (3.1) we have

$$\det A = \sum_{j=1}^{n} a_{1,j}\text{cof}(A)_{1,j} = 0$$

as each of the summands is equal to 0. ♠

Lemma 3.34

Assume A, B and C are $n \times n$ matrices that for some $1 \leq i \leq n$ satisfy the following.

1. *jth rows of all three matrices are identical, for $j \neq i$.*
2. *Each entry in the jth row of A is the sum of the corresponding entries in jth rows of B and C.*

Then $\det A = \det B + \det C$.

Proof. This is not difficult to check for $n = 2$ (do check it!).

Now assume that the statement of Lemma is true for $n-1 \times n-1$ matrices and fix A, B and C as in the statement. The assumptions state that we have $a_{l,j} = b_{l,j} = c_{l,j}$ for $j \neq i$ and for $1 \leq l \leq n$ and $a_{l,i} = b_{l,i} + c_{l,i}$ for all $1 \leq l \leq n$. Therefore $A(i) = B(i) = C(i)$, and $A(j)$ has the property that its ith row is the sum of ith rows of $B(j)$ and $C(j)$ for $j \neq i$ while the other rows of all three matrices are identical. Therefore by our inductive hypothesis we have $\det A(j) = \det B(j) + \det C(j)$, and so $\mathrm{cof}(A)_{1j} = \mathrm{cof}(B)_{1j} + \mathrm{cof}(C)_{1j}$ for $j \neq i$.

By (3.1) we have (using all equalities established above)

$$\begin{aligned}
\det A &= \sum_{l=1}^{n} a_{1,l}\mathrm{cof}(A)_{1,l} \\
&= \sum_{l \neq i} a_{1,l}(\mathrm{cof}(B)_{1,l} + \mathrm{cof}(C)_{1,l}) + (b_{1,i} + c_{1,i})\mathrm{cof}(A)_{1,i} \\
&= \det B + \det C
\end{aligned}$$

This proves that the assertion is true for all n and completes the proof. ♠

Theorem 3.35

Let A and B be $n \times n$ matrices.

1. *If A is obtained by interchanging ith and jth rows of B (with $i \neq j$), then* $\det A = -\det B$.
2. *If A is obtained by multiplying ith row of B by k then* $\det A = k \det B$.
3. *If two rows of A are identical then* $\det A = 0$.
4. *If A is obtained by multiplying ith row of B by k and adding it to jth row of B ($i \neq j$) then* $\det A = \det B$.

Proof. We prove all statements by induction. The case $n = 2$ is easily checked directly (and it is strongly suggested that you do check it).

We assume $n \geq 3$ and (1)–(4) are true for all matrices of size $n-1 \times n-1$.

(1) We first prove the case when $j = i+1$, i.e., we are interchanging two consecutive rows.

Let $l \in \{1, \ldots, n\} \setminus \{i, j\}$. Then $A(l)$ is obtained from $B(l)$ by interchanging two of its rows (draw a picture) and by our assumption

$$\mathrm{cof}(A)_{1,l} = -\mathrm{cof}(B)_{1,l}. \tag{3.2}$$

Now consider $a_{1,i}\mathrm{cof}(A)_{1,i}$. We have that $a_{1,i} = b_{1,j}$ and also that $A(i) = B(j)$. Since $j = i+1$, we have

$$(-1)^{1+j} = (-1)^{1+i+1} = -(-1)^{1+i}$$

and therefore $a_{1i}\mathrm{cof}(A)_{1i} = -b_{1j}\mathrm{cof}(B)_{1j}$ and $a_{1j}\mathrm{cof}(A)_{1j} = -b_{1i}\mathrm{cof}(B)_{1i}$. Putting this together with (3.2) into (3.1) we see that if in the formula for $\det A$ we change the sign of each of the summands we obtain the formula for $\det B$.

$$\det A = \sum_{l=1}^{n} a_{1l}\mathrm{cof}(A)_{1l} = -\sum_{l=1}^{n} b_{1l}\mathrm{cof}(B)_{1l} = -\det B.$$

We have therefore proved the case of (1) when $j = i+1$. In order to prove the general case, one needs the following fact. If $i < j$, then in order to interchange ith and jth row one can proceed by interchanging two adjacent rows $2(j-i)+1$ times: First swap ith and $i+1$st, then $i+1$st and $i+2$nd, and so on. After one interchanges $j-1$st and jth row, we have ith row in position of jth and lth row in position of $l-1$st for $i+1 \leq l \leq j$. Then proceed backwards swapping adjacent rows until everything is in place.

Since $2(j-i)+1$ is an odd number $(-1)^{2(j-i)+1} = -1$ and we have that $\det A = -\det B$.

(2) This is like (1)... but much easier. Assume that (2) is true for all $n-1 \times n-1$ matrices. We have that $a_{ji} = kb_{ji}$ for $1 \leq j \leq n$. In particular $a_{1i} = kb_{1i}$, and for $l \neq i$, the matrix $A(l)$ is obtained from $B(l)$ by multiplying one of its rows by k. Therefore $\mathrm{cof}(A)_{1l} = k\mathrm{cof}(B)_{1l}$ for $l \neq i$, and for all l we have $a_{1l}\mathrm{cof}(A)_{1l} = kb_{1l}\mathrm{cof}(B)_{1l}$. By (3.1), we have $\det A = k \det B$.

(3) This is a consequence of (1). If two rows of A are identical, then A is equal to the matrix obtained by interchanging those two rows and therefore by (1), $\det A = -\det A$. This implies $\det A = 0$.

(4) Assume (4) is true for all $n-1 \times n-1$ matrices and fix A and B such that A is obtained by multiplying ith row of B by k and adding it to jth row of B ($i \neq j$) then $\det A = \det B$. If $k = 0$ then $A = B$ and there is nothing to prove, so we may assume $k \neq 0$.

Let C be the matrix obtained by replacing the jth row of B by the ith row of B multiplied by k. By Lemma 3.34, we have that

$$\det A = \det B + \det C$$

and we 'only' need to show that $\det C = 0$. But ith and jth rows of C are proportional. If D is obtained by multiplying the jth row of C by $\frac{1}{k}$ then by (2) we have $\det C = \frac{1}{k}\det D$ (recall that $k \neq 0$). But ith and jth rows of D are identical, hence by (3) we have $\det D = 0$ and therefore $\det C = 0$. ♠

Theorem 3.36: The Determinant of the Product

Let A and B be two $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B)$$

Proof. If A is an elementary matrix of either type, then multiplying by A on the left has the same effect as performing the corresponding elementary row operation. Therefore the equality $\det(AB) = \det A \det B$ in this case follows by Lemma 3.32 and Theorem 3.35.

If C is the reduced row-echelon form of A then we can write $A = E_1 \cdot E_2 \cdot \cdots \cdot E_m \cdot C$ for some elementary matrices $E_1, \ldots, E_m$.

Now we consider two cases.

Assume first that $C = I$. Then $A = E_1 \cdot E_2 \cdot \dots \cdot E_m$ and $AB = E_1 \cdot E_2 \cdot \dots \cdot E_m B$. By applying the above equality m times, and then $m-1$ times, we have that

$$\begin{aligned} \det AB &= \det E_1 \det E_2 \cdot \det E_m \cdot \det B \\ &= \det(E_1 \cdot E_2 \cdot \dots \cdot E_m) \det B \\ &= \det A \det B. \end{aligned}$$

Now assume $C \neq I$. Since it is in reduced row-echelon form, its last row consists of zeros. But it is easy to check that if C's last row consists of zeros and the product CB is defined, then the last row of CB also consists of zeros. By Lemma 3.33 we have $\det C = \det(CB) = 0$ and therefore

$$\det A = \det(E_1 \cdot E_2 \cdot E_m) \cdot \det(C) = \det(E_1 \cdot E_2 \cdot E_m) \cdot 0 = 0$$

and also

$$\det AB = \det(E_1 \cdot E_2 \cdot E_m) \cdot \det(CB) = \det(E_1 \cdot E_2 \cdot \dots \cdot E_m) 0 = 0$$

hence $\det AB = 0 = \det A \det B$. ♠

The same 'machine' used in the previous proof will be used again.

Theorem 3.37

Let A be a matrix where A^T is the transpose of A. Then,

$$\det\left(A^T\right) = \det(A)$$

Proof. Note first that the conclusion is true if A is elementary by (4) of Lemma 3.32.

Let C be the reduced row-echelon form of A. Then we can write $A = E_1 \cdot E_2 \cdot \dots \cdot E_m C$. Then $A^T = C^T \cdot E_m^T \cdot \dots \cdot E_2^T \cdot E_1$. By Theorem 3.36 we have

$$\det(A^T) = \det(C^T) \cdot \det(E_m^T) \cdot \dots \cdot \det(E_2^T) \cdot \det(E_1).$$

By (4) of Lemma 3.32 we have that $\det E_j = \det E_j^T$ for all j. Also, $\det C$ is either 0 or 1 (depending on whether $C = I$ or not) and in either case $\det C = \det C^T$. Therefore $\det A = \det A^T$. ♠

The above discussions allow us to now prove Theorem 3.10. It is restated below.

Theorem 3.38

Expanding an $n \times n$ matrix along any row or column always gives the same result, which is the determinant.

Proof. We first show that the determinant can be computed along any row. The case $n = 1$ does not apply and thus let $n \geq 2$.

Let A be an $n \times n$ matrix and fix $j > 1$. We need to prove that

$$\det A = \sum_{i=1}^{n} a_{j,i} \operatorname{cof}(A)_{j,i}.$$

Let us prove the case when $j = 2$.

Let B be the matrix obtained from A by interchanging its 1st and 2nd rows. Then by Theorem 3.35 we have

$$\det A = -\det B.$$

Now we have

$$\det B = \sum_{i=1}^{n} b_{1,i} \operatorname{cof}(B)_{1,i}.$$

Since B is obtained by interchanging the 1st and 2nd rows of A we have that $b_{1,i} = a_{2,i}$ for all i and one can see that $\operatorname{minor}(B)_{1,i} = \operatorname{minor}(A)_{2,i}$.

Further,

$$\operatorname{cof}(B)_{1,i} = (-1)^{1+i} \operatorname{minor} B_{1,i} = -(-1)^{2+i} \operatorname{minor}(A)_{2,i} = -\operatorname{cof}(A)_{2,i}$$

hence $\det B = -\sum_{i=1}^{n} a_{2,i} \operatorname{cof}(A)_{2,i}$, and therefore $\det A = -\det B = \sum_{i=1}^{n} a_{2,i} \operatorname{cof}(A)_{2,i}$ as desired.

The case when $j > 2$ is very similar; we still have $\operatorname{minor}(B)_{1,i} = \operatorname{minor}(A)_{j,i}$ but checking that $\det B = -\sum_{i=1}^{n} a_{j,i} \operatorname{cof}(A)_{j,i}$ is slightly more involved.

Now the cofactor expansion along column j of A is equal to the cofactor expansion along row j of A^T, which is by the above result just proved equal to the cofactor expansion along row 1 of A^T, which is equal to the cofactor expansion along column 1 of A. Thus the cofactor cofactor along any column yields the same result.

Finally, since $\det A = \det A^T$ by Theorem 3.37, we conclude that the cofactor expansion along row 1 of A is equal to the cofactor expansion along row 1 of A^T, which is equal to the cofactor expansion along column 1 of A. Thus the proof is complete. ♠

3.2 Applications of the Determinant

Outcomes

A. *Use determinants to determine whether a matrix has an inverse, and evaluate the inverse using cofactors.*

B. *Apply Cramer's Rule to solve a 2×2 or a 3×3 linear system.*

C. *Given data points, find an appropriate interpolating polynomial and use it to estimate points.*

In this section we will examine three applications for the determinant of a matrix.

A Formula for the Inverse

Our first application will be to use the determinant of A to provide an alternative way to find A^{-1}. Our previous work has given us an algorithm, or method, of producing A^{-1}. Now we will have a formula that will generate the inverse of any invertible matrix A.

Recall the definition of the inverse of a matrix from Definition 2.36. We say that A^{-1}, an $n \times n$ matrix, is the inverse of A, also $n \times n$, if $AA^{-1} = I$ and $A^{-1}A = I$.

In order to find our formula for A^{-1}, we introduce two new matrices derived from A. They are similar in definition and closely related, so don't get them confused.

Remember from Definition 3.6, that the ij^{th} cofactor of a matrix is defined to be $(-1)^{i+j}\operatorname{minor}(A)_{ij}$, where $\operatorname{minor}(A)_{ij}$ is the determinant of the matrix that results from deleting row i and column j from the matrix A. We will gather up these cofactors into a matrix and give it a name:

Definition 3.39: The Cofactor Matrix

Let $A = \left[a_{ij}\right]$ be an $n \times n$ matrix. Then the ***cofactor matrix of*** *A, denoted $cof(A)$, is defined by $cof(A) = \left[cof(A)_{ij}\right]$ where $cof(A)_{ij}$ is the ij^{th} cofactor of A.*

Note that $\operatorname{cof}(A)_{ij}$ denotes the ij^{th} entry of the cofactor matrix.

Definition 3.40: The Adjugate

Let $A = \left[a_{ij}\right]$ be an $n \times n$ matrix. Then the ***adjugate of*** *A, denoted $adj(A)$, is defined by $adj(A) = (cof(A))^T$, the transpose of the cofactor matrix of A. The adjugate of A is also called the* ***classical adjoint of*** *A.*

Example 3.41: Cofactor Matrix and Adjugate Matrix

Find both the cofactor matrix and the adjugate of each of the following matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix}$$

Solution. For the two by two matrix, we find the matrix of cofactors is:

$$\operatorname{cof}(A) = \begin{bmatrix} (-1)^{1+1}\det[d] & (-1)^{1+2}\det[c] \\ (-1)^{2+1}\det[b] & (-1)^{2+2}\det[a] \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

and so $\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

For the larger matrix, we must compute 9 separate determinants, and then multiply them by either 1 or -1, to find the matrix of cofactors. You are invited to check that

$$\operatorname{cof}(B) = \begin{bmatrix} -3 & 3 & 9 \\ 6 & -4 & 2 \\ 1 & 6 & -3 \end{bmatrix}, \quad \operatorname{adj}(B) = \begin{bmatrix} -3 & 6 & 1 \\ 3 & -4 & 6 \\ 9 & 2 & -3 \end{bmatrix}.$$

♠

Now for the big result for this subsection. The following theorem provides a formula for A^{-1} using the determinant and adjugate of A.

Theorem 3.42: The Inverse and the Determinant

Let A be an $n \times n$ matrix. Then

$$A \; adj(A) = adj(A) A = \det(A) I$$

Moreover A is invertible if and only if $\det(A) \neq 0$. In this case we have:

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Notice that the first formula holds for any $n \times n$ matrix A, and in the case A is invertible we actually have a formula for A^{-1}.

Consider the following example.

Example 3.43: Find Inverse Using the Determinant

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

using the formula in Theorem 3.42.

Solution. According to Theorem 3.42,

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

First we will find the determinant of this matrix. Using Theorems 3.16, 3.18, and 3.21, we can first simplify the matrix through row operations. First, add -3 times the first row to the second row. Then add -1 times the first row to the third row to obtain

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & -8 \\ 0 & 0 & -2 \end{bmatrix}$$

By Theorem 3.21, $\det(A) = \det(B)$. By Theorem 3.13, $\det(B) = 1 \times -6 \times -2 = 12$. Hence, $\det(A) = 12$.

Now, we need to find $\operatorname{adj}(A)$. To do so, first we will find the cofactor matrix of A. This is given by

$$\operatorname{cof}(A) = \left[\begin{array}{rrr} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{array}\right]$$

Here, the ij^{th} entry is the ij^{th} cofactor of the original matrix A, as you can verify. Therefore, from Theorem 3.42, the inverse of A is given by

$$A^{-1} = \frac{1}{12}\left[\begin{array}{rrr} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{array}\right]^T = \left[\begin{array}{rrr} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right]$$

Remember that we can always verify our answer for A^{-1}. Compute the product AA^{-1} and $A^{-1}A$ and make sure each product is equal to I.

Compute $A^{-1}A$ as follows

$$A^{-1}A = \left[\begin{array}{rrr} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right]\left[\begin{array}{rrr} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{array}\right] = \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = I$$

You can verify that $AA^{-1} = I$ (or just quote Theorem 2.62) and hence we know that our answer is correct. ♠

We will look at another example of how to use this formula to find A^{-1}.

Example 3.44: Find the Inverse From a Formula

Find the inverse of the matrix

$$A = \left[\begin{array}{rrr} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & \frac{2}{3} & -\frac{1}{2} \end{array}\right]$$

using the formula given in Theorem 3.42.

Solution. First we need to find $\det(A)$. This step is left as an exercise and you should verify that $\det(A) = \frac{1}{6}$. The inverse is therefore equal to

$$A^{-1} = \frac{1}{(1/6)}\operatorname{adj}(A) = 6\operatorname{adj}(A)$$

We continue to calculate as follows. Here we show the 2×2 determinants needed to find the cofactors.

$$A^{-1} = 6 \begin{bmatrix} \begin{vmatrix} \frac{1}{3} & -\frac{1}{2} \\ \frac{2}{3} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} -\frac{1}{6} & -\frac{1}{2} \\ -\frac{5}{6} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} -\frac{1}{6} & \frac{1}{3} \\ -\frac{5}{6} & \frac{2}{3} \end{vmatrix} \\ -\begin{vmatrix} 0 & \frac{1}{2} \\ \frac{2}{3} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{5}{6} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} \frac{1}{2} & 0 \\ -\frac{5}{6} & \frac{2}{3} \end{vmatrix} \\ \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{vmatrix} \end{bmatrix}^T$$

Expanding all the 2×2 determinants, this yields

$$A^{-1} = 6 \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

Again, you can always check your work by multiplying $A^{-1}A$. If this product is equal to I, then Theorem 2.62 tells us that $AA^{-1} = I$, and so we will know that our computation is correct. Let's do so:

$$A^{-1}A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & \frac{2}{3} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This tells us that our calculation for A^{-1} is correct. ♠

The verification step is very important, as it is a simple way to check your work! If you multiply $A^{-1}A$ and you don't get the identity matrix, be sure to go back and double check each step. One common error is to forget to take the transpose of the cofactor matrix, so be sure to complete this step.

We will now prove Theorem 3.42.

Proof. (of Theorem 3.42) Recall that the (i, j)-entry of $\mathrm{adj}(A)$ is equal to $\mathrm{cof}(A)_{ji}$. Thus the (i, j)-entry of $B = A \cdot \mathrm{adj}(A)$ is :

$$B_{ij} = \sum_{k=1}^{n} a_{ik}\mathrm{adj}(A)_{kj} = \sum_{k=1}^{n} a_{ik}\mathrm{cof}(A)_{jk}$$

By the cofactor expansion theorem, we see that this expression for B_{ij} is equal to the determinant of the matrix obtained from A by replacing its jth row by $a_{i1}, a_{i2}, \dots a_{in}$ — i.e., its ith row.

If $i = j$ then this matrix is A itself and therefore $B_{ii} = \det A$. If on the other hand $i \neq j$, then this matrix has its ith row equal to its jth row, and therefore $B_{ij} = 0$ in his case. Thus we obtain:

$$A\ \mathrm{adj}(A) = \det(A) I$$

Similarly we can verify that:

$$\operatorname{adj}(A)A = \det(A)I \tag{3.3}$$

And this proves the first part of the theorem.

Further if A is invertible, then by Theorem 3.24 we have:

$$1 = \det(I) = \det\left(AA^{-1}\right) = \det(A)\det\left(A^{-1}\right)$$

and thus $\det(A) \neq 0$. Equivalently, if $\det(A) = 0$, then A is not invertible.

Finally if $\det(A) \neq 0$, then we can divide both sides of Equation 3.3 by $\det(A)$ and use the properties of matrix multiplication to obtain

$$\left(\frac{1}{\det(A)}\operatorname{adj}(A)\right)A = I,$$

and so Theorem 2.62 allows us to conclude that A is invertible and that:

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$$

This completes the proof. ♠

This method for finding the inverse of A is useful in many contexts. In particular, it is useful with complicated matrices where the entries are functions, rather than numbers.

Consider the following example.

Example 3.45: Inverse for Non-Constant Matrix

Suppose

$$A(t) = \left[\begin{array}{ccc} e^t & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{array}\right]$$

Show that $(A(t))^{-1}$ exists and then find it.

Solution. First note $\det(A(t)) = e^t(\cos^2 t + \sin^2 t) = e^t \neq 0$ so $(A(t))^{-1}$ exists.

The cofactor matrix is

$$C(t) = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & e^t\cos t & e^t\sin t \\ 0 & -e^t\sin t & e^t\cos t \end{array}\right]$$

and so the inverse is

$$\frac{1}{e^t}\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & e^t\cos t & e^t\sin t \\ 0 & -e^t\sin t & e^t\cos t \end{array}\right]^T = \left[\begin{array}{ccc} e^{-t} & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{array}\right]$$

♠

Cramer's Rule

Another context in which the formula given in Theorem 3.42 is important is **Cramer's Rule**. Recall that we can represent a system of linear equations in the form $AX = B$, where the solutions to this system are given by X. Cramer's Rule gives a formula for the solutions X in the special case that A is a square invertible matrix. Note this rule does not apply if you have a system of equations in which there is a different number of equations than variables (in other words, when A is not square), or when A is not invertible.

Suppose we have a system of equations given by $AX = B$, and we want to find solutions X which satisfy this system. Then recall that if A^{-1} exists,

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B \\ \left(A^{-1}A\right)X &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Hence, the solutions X to the system are given by $X = A^{-1}B$. Since we assume that A^{-1} exists, we can use the formula for A^{-1} given above. Substituting this formula into the equation for X, we have

$$X = A^{-1}B = \frac{1}{\det(A)}\operatorname{adj}(A)\,B$$

To compute x_i, the i^{th} entry of X, we would use the i^{th} row of the matrix A^{-1} and the entries b_j of B as follows:

$$x_i = \sum_{j=1}^{n} \frac{1}{\det(A)} \operatorname{adj}(A)_{ij} b_j = \frac{1}{\det(A)} \sum_{j=1}^{n} \operatorname{adj}(A)_{ij} b_j = \frac{1}{\det(A)} \sum_{j=1}^{n} \operatorname{cof}(A)_{j,i} b_j$$

where $\operatorname{adj}(A)_{ij}$ is the ij^{th} entry of $\operatorname{adj}(A)$.

If we look at this last sum, $\sum_{j=1}^{n} \operatorname{cof}(A)_{ji} b_j$, a little more closely and think about expanding determinants along the ith column of a matrix, you will see that our sum is equal to the determinant of the matrix

$$A_i = \left[\begin{array}{ccccc} * & \cdots & b_1 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & b_n & \cdots & * \end{array}\right],$$

where the $*$'s are supposed to represent the entries of the matrix A: Thus A_i is the matrix A with the ith column replaced with the entries of B.

So this gives us

$$x_i = \frac{1}{\det(A)} \det \left[\begin{array}{ccccc} * & \cdots & b_1 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & b_n & \cdots & * \end{array}\right]$$

where here the i^{th} column of A is replaced with the column vector $[b_1 \cdots, b_n]^T$. The determinant of this modified matrix is taken and divided by $\det(A)$. This formula is known as **Cramer's rule**.

We formally define this method now.

Procedure 3.46: Using Cramer's Rule

Suppose A is an $n \times n$ invertible matrix and we wish to solve the system $AX = B$ for $X = [x_1, \cdots, x_n]^T$. Then Cramer's rule says

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i is the matrix obtained by replacing the i^{th} column of A with the column matrix

$$B = \left[\begin{array}{c} b_1 \\ \vdots \\ b_n \end{array}\right].$$

We illustrate this procedure in the following example.

Example 3.47: Using Cramer's Rule

Find x, y, z if

$$\left[\begin{array}{rrr} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{array}\right] \left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right]$$

Solution. We will use method outlined in Procedure 3.46 to find the values for x, y, z which give the solution to this system. Let

$$B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

In order to find x, we calculate

$$x = \frac{\det(A_1)}{\det(A)}$$

where A_1 is the matrix obtained from replacing the first column of A with B.

Hence, A_1 is given by

$$A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & -3 & 2 \end{bmatrix}$$

Therefore,

$$x = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & -3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = \frac{1}{2}$$

Similarly, to find y we construct A_2 by replacing the second column of A with B. Hence, A_2 is given by

$$A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

Therefore,

$$y = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = -\frac{1}{7}$$

Similarly, A_3 is constructed by replacing the third column of A with B. Then, A_3 is given by

$$A_3 = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & -3 & 3 \end{bmatrix}$$

Therefore, z is calculated as follows.

$$z = \frac{\det(A_3)}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & -3 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = \frac{11}{14}$$

♠

Cramer's Rule gives you another tool to consider when solving a system of linear equations.

We can also use Cramer's Rule for some systems of non linear equations. Consider the following system where the matrix A has functions rather than numbers for entries.

Example 3.48: Use Cramer's Rule for Non-Constant Matrix

Solve for z if

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}$$

Solution. We are asked to find the value of z in the solution. We will solve using Cramer's rule. Thus

$$z = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & e^t \cos t & t \\ 0 & -e^t \sin t & t^2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{vmatrix}} = t\left((\cos t)\, t + \sin t\right) e^{-t}$$

♠

Polynomial Interpolation

In studying a set of data that relates variables x and y, it may be the case that we can find a polynomial to match our data. If such a polynomial can be established, it can be used to estimate values of x and y which have not been provided. As long as we are working with x values between our lowest and highest data values, this is called an *inerpolating polynomial*.

For example, the World Health Organization publishes data concerning the growth of children. In particular, relating the height of a child and the weight of the child. Since weight corresponds to volume, and volume seems like it should grow as the cube of the length, we might expect there to be a cubic polynomial that relates the two variables x, the height measured in centimeters, and y, the child's weight measured in kilograms. We will use data to find this cubic polynomial later in this section.

You are well aware of the fact that two points determine a line, so given two points (x_1, y_1) and (x_2, y_2), there is a unique linear equation $y = r_0 + r_1 x$ that passes through the two points. Similarly, three points determine a quadratic function, four points determine a cubic function, and in general n points in the plane (with distinct x-coordinates) determine a unique polynomial of degree $n-1$ that passes through the points. Our goal in this section is to show, given the points, how to use Cramer's Rule and the determinant of a matrix to find the coefficients (the r_i's) of the interpolating polynomial.

Consider the following example.

Example 3.49: Polynomial Interpolation

Given data points $(1,4), (2,9), (3,12)$, *find an interpolating polynomial* $p(x)$ *of degree at most* 2 *and then estimate the value of* y *corresponding to* $x = \frac{1}{2}$.

Solution. We want to find a polynomial given by

$$p(x) = r_0 + r_1 x + r_2 x^2$$

such that $p(1) = 4, p(2) = 9$ and $p(3) = 12$. To find this polynomial, we will set up a system of three linear equations and solve the system. By substituting in our three data points into our needed expression for $p(x)$, we see at we want to find values of r_1, r_2, and r_3 such that:

$$\begin{aligned} r_0 + \ \ r_1 + \ \ r_2 &= 4 \\ r_0 + 2r_1 + 4r_2 &= 9 \\ r_0 + 3r_1 + 9r_2 &= 12 \end{aligned} .$$

So this means that we need to solve the matrix equation $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}, \quad X = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 9 \\ 12 \end{bmatrix}.$$

Using Cramer's Rule from the last section, we see that

$$r_0 = \frac{\begin{vmatrix} 4 & 1 & 1 \\ 9 & 2 & 4 \\ 12 & 3 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix}} = \frac{-6}{2} = -3, \quad r_1 = \frac{\begin{vmatrix} 1 & 4 & 1 \\ 1 & 9 & 4 \\ 1 & 12 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix}} = \frac{16}{2} = 8, \quad r_2 = \frac{\begin{vmatrix} 1 & 1 & 4 \\ 1 & 2 & 9 \\ 1 & 3 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix}} = \frac{-2}{2} = -1,$$

Thus our interpolating polynomial is

$$p(x) = -3 + 8x - x^2$$

and our estimate for a y value corresponding to $x = 1/2$ would be $p(1/2) = 3/4$.

♠

You should notice that there was a fair bit of work involved in calculating those four determinants needed to apply Cramer's Rule. Of course one could solve the system of equations from that last example by writing the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 9 \\ 1 & 3 & 9 & 12 \end{array}\right]$$

and using row operations to find the equivalent matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

again finding the solution to the system to be $r_0 = -3, r_1 = 8, r_2 = -1$. For many calculations, finding the solution to a system either by row reducing or by finding the LU factorization will be quicker than using Cramer's Rule.

The procedure outlined above can be used for any number of data points, and any degree of polynomial. The steps are outlined below.

Procedure 3.50: Finding an Interpolating Polynomial

Suppose that distinct values of x and corresponding values of y are given, such that the actual relationship between x and y is unknown. Then, values of y can be estimated using an ***interpolating polynomial*** *$p(x)$. If given distinct $x_1, ..., x_n$ and the corresponding $y_1, ..., y_n$, the procedure to find $p(x)$ is as follows:*

1. *The desired polynomial $p(x)$ is given by*

$$p(x) = r_0 + r_1 x + r_2 x^2 + ... + r_{n-1} x^{n-1}$$

2. *Since it is required that $p(x_i) = y_i$ for all $i = 1, 2, ..., n$, we must find the values $r_0, r_1, \ldots, r_{n-1}$ that solve the following system of n linear equations in n unknowns:*

$$\begin{array}{c} r_0 + r_1 x_1 + r_2 x_1^2 + ... + r_{n-1} x_1^{n-1} = y_1 \\ r_0 + r_1 x_2 + r_2 x_2^2 + ... + r_{n-1} x_2^{n-1} = y_2 \\ \vdots \\ r_0 + r_1 x_n + r_2 x_n^2 + ... + r_{n-1} x_n^{n-1} = y_n \end{array}$$

3. *Set up the matrix equation $AX = B$,*

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}. \tag{3.4}$$

4. *Solving this system will result in a unique solution $r_0, r_1, \cdots, r_{n-1}$. Use these values to construct $p(x)$, and estimate the value of $p(a)$ for any $x = a$.*

This procedure motivates the following theorem.

Theorem 3.51: Polynomial Interpolation

Given n data points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ with the x_i distinct, there is a unique polynomial $p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$ such that $p(x_i) = y_i$ for $i = 1, 2, \cdots, n$. The resulting polynomial $p(x)$ is called the ***interpolating polynomial*** *for the data points.*

The proof of this theorem would take us too far afield at this point, but it is worth pointing out that the proof depends on the fact that if the x_i's are distinct, then the square coefficient matrix of Equation

3.4 is guaranteed to have a determinant that is not equal to zero. This means that the coefficient matrix is invertible, which guarantees a unique solution to our system of linear equations. A matrix of this form, where the rows of the matrix form a geometric progression starting with 1, is called a Vandermonde matrix.

We conclude this section with another example.

Example 3.52: Polynomial Interpolation

The WHO's growth chart for girls aged 0 to 2 years tells us that the mean weight for girls of given height is as follows:

Height x (cm)	Weight y (kg)
48	3
74	9
97	13
109	18

Find the cubic interpolating polynomial for this data set, and use that polynomial to estimate the mean weight of a girl whose height is 60 centimeters.

Solution. The desired polynomial $p(x)$ is given by:

$$p(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3$$

Using the given points, the system of equations we need to solve is

$$\begin{array}{r} r_0 + \ \ 48 r_1 + \ \ 48^2 r_2 + \ \ 48^3 r_3 = 3 \\ r_0 + \ \ 74 r_1 + \ \ 74^2 r_2 + \ \ 74^3 r_3 = 9 \\ r_0 + \ \ 97 r_1 + \ \ 97^2 r_2 + \ \ 97^3 r_3 = 13 \\ r_0 + 109 r_1 + 109^2 r_2 + 109^3 r_3 = 18 \end{array}.$$

The augmented matrix is given by:

$$\left[\begin{array}{cccc|c} 1 & 48 & 2304 & 110592 & 3 \\ 1 & 74 & 5476 & 405224 & 9 \\ 1 & 97 & 9409 & 912673 & 13 \\ 1 & 109 & 11881 & 1295029 & 18 \end{array}\right].$$

The solution of our system turns out to be (of course you should use technology to solve this system) $r_0 = -57.9275848, r_1 = 2.4144170, r_2 = -0.0302268, r_3 = 0.0001327$, and the interpolating cubic polynomial is

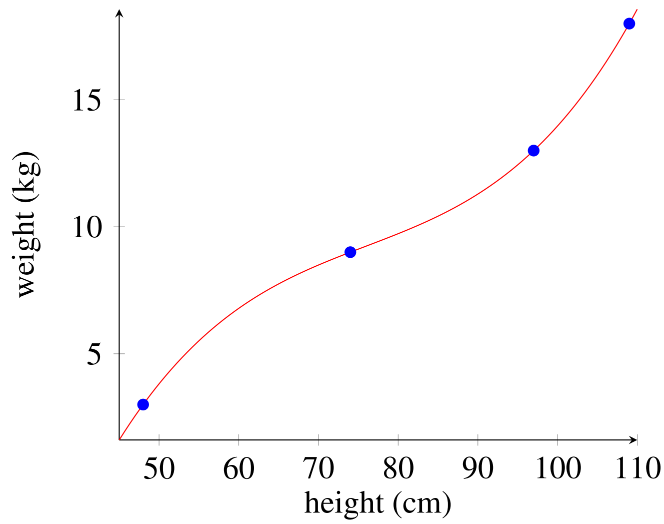
$$p(x) = -57.9275848 + 2.4144170x - 0.0302268x^2 + 0.0001327x^3.$$

Our predicted weight for a child of height 60 centimeters is

$$p(60) = -57.9275848 + 2.4144170(60) - 0.0302268(60)^2 + 0.0001327(60)^3 = 6.7892 \text{ kg}.$$

(In case you were wondering (and shame on you if you weren't!) the actual WHO predicted weight for our 60 cm child is 5.9 kilograms, so it looks as though our cubic model is in need of some tweaking!)

Four Data Points and an Interpolating Cubic Polynomial



♠

Chapter 4

$\mathbb{R}^n$

In the first three chapters of this book, we have concentrated on linear equations and matrices, with a focus on using matrix techniques to find solutions to systems of linear equations. Now our focus shifts to vectors, which we introduced earlier as $n \times 1$ matrices. This change of focus will give us new tools with which to describe and investigate the Cartesian plane and the three dimensional world in which we live. As a bonus, vectors will make it easy for us to generalize our intuition into higher dimensional settings and prepare us for deeper levels of understanding and analysis. We will start with a rather informal geometric introduction to the idea of a vector, and then make things formal in following sections.

4.1 Vectors in $\mathbb{R}^n$: Geometry

Outcomes

A. Represent a vector by an arrow, characterized by its length and its direction.

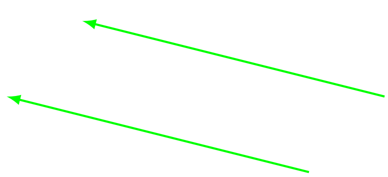
B. Given a geometric representation of a vector $\vec{v}$ and a real number k, sketch the vector $k\vec{v}$.

C. Given geometric representations of the vectors $\vec{u}$ and $\vec{v}$, sketch the vectors $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$.

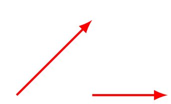
An Informal Introduction

We all experience force in our lives. All the time. You step on the scale in the morning to see the magnitude of the force that the earth exerts on your body. You push open a door. You feel the wind on your face. Maybe you catch a ball or feel the force that the seat of your car exerts on you as you drive through a tight turn. In all of these cases, there are two parts of the force, both of which are important. The magnitude of the force ("How could I possibly have gained three pounds?") and the direction of the force ("The wind is blowing from right to left, so my kite will probably end up entangled in that tree over there."). Vectors are the mathematician's objects that are characterized by their magnitude and direction. Understanding vectors helps us to understand the world. So let's dive in.

A good way to start our investigation of vectors is to think about arrows, which certainly have both magnitude (measured by the length of the arrow) and direction (indicated by the orientation of the arrow). If two arrows (vectors) have the same magnitude and the same direction, we will say that they are equal. Like this:



These two vectors are equal.



These two vectors are not equal.

Suppose your car is stuck in the snow, and you are with three friends. Leaving one of your friends to steer and work the gas, you and your other two friends jump out and push on the car, trying to get it unstuck. Each one of you exerts a force on the car, and the total force exerted by the three of you is the sum of your individual forces. So we will want to be able to add vectors together.

Perhaps, while pushing on your car, you remember that you have a can of spinach and you eat it, suddenly becoming three times as strong (look up the comic strip Popeye if that doesn't make sense to you). You push in the same direction as before, but the magnitude of the force that you exert has been increased by a factor of 3. We will want to be able to multiply a vector by a scalar so that we can model this (admittedly unlikely) situation.

So, we're thinking of vectors as corresponding to arrows, and now we will introduce the operations of vector addition and scalar multiplication. Let's look at the geometry of how these operations will be computed.

The Geometry of Scalar Multiplication

We are going to define a function that takes as input a vector $\vec{v}$ (that is the notation that we will almost always use for vectors from now on) and a scalar k and produce the vector $k\vec{v}$. There will be three cases, depending on whether k is positive, negative, or 0.

If k is a positive real number, then $k\vec{v}$ is the vector that points in the same direction as $\vec{v}$ and whose length is k times the length of $\vec{v}$. So $3\vec{v}$ will be three times as long as $\vec{v}$, while $\frac{2}{3}\vec{v}$ will be only $\frac{2}{3}$ as long as $\vec{v}$. Notice that $1\vec{v}$ is equal to $\vec{v}$, which is comforting.

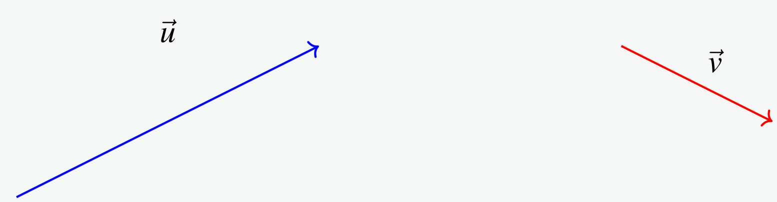
For our second case, if k is a negative number, then the direction of $k\vec{v}$ will be the *opposite* of the direction of $\vec{v}$, while the length of $k\vec{v}$ will be equal to $|k|$ times the length of $\vec{v}$.

Finally, if $k = 0$, then $k\vec{v}$ will be the vector that has length 0. We'll agree not to worry about the direction of the vector of length 0, which is called the **zero vector** and is denoted $\vec{0}$. Remember that there is only one zero vector.

An example may be helpful here:

Example 4.1: The Geometry of Scalar Multiplication

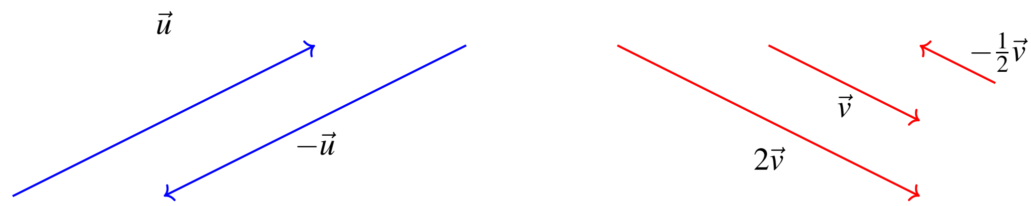
Consider the vectors $\vec{u}$ and $\vec{v}$ drawn below.



Draw $-\vec{u}$, $2\vec{v}$, and $-\frac{1}{2}\vec{v}$.

Solution.

In order to find $-\vec{u}$, we preserve the length of $\vec{u}$ and simply reverse the direction. For $2\vec{v}$, we double the length of $\vec{v}$, while preserving the direction. Finally $-\frac{1}{2}\vec{v}$ is found by taking half the length of $\vec{v}$ and reversing the direction. These vectors are shown in the following diagram.



♠

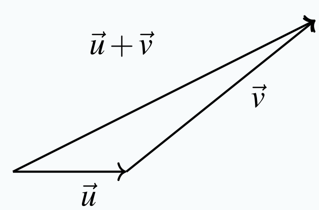
The Geometry of Vector Addition

We know that a vector is characterized by its length and it direction. This means that if we take a vector and move it around without changing either its length or direction we do not change the vector. That is going to be key in understanding the geometric representation of vector addition.

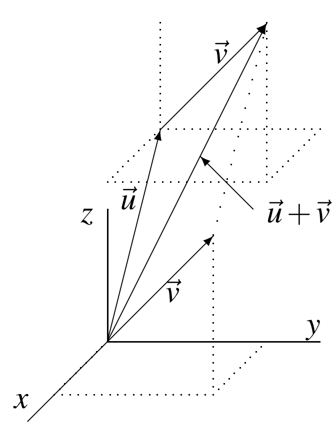
Suppose we have two vectors, $\vec{u}$ and $\vec{v}$. Each of these can be drawn geometrically by placing the tail of each vector at the same point. Now suppose we slide the vector $\vec{v}$ so that its tail sits at the point of $\vec{u}$. We know that this does not change the vector $\vec{v}$. Now, draw a new vector from the tail of $\vec{u}$ to the point of $\vec{v}$. This vector is $\vec{u}+\vec{v}$.

Definition 4.2: Geometry of Vector Addition

Let $\vec{u}$ and $\vec{v}$ be two vectors. Slide $\vec{v}$ so that the tail of $\vec{v}$ is on the point of $\vec{u}$. Then draw the arrow which goes from the tail of $\vec{u}$ to the point of $\vec{v}$. This arrow represents the vector $\vec{u}+\vec{v}$.



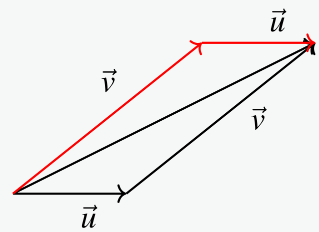
This definition is illustrated in the following picture, in which $\vec{u}+\vec{v}$ is shown for vectors that live in three-space.



Notice the parallelogram created by $\vec{u}$ and $\vec{v}$ in the above diagram. Then $\vec{u}+\vec{v}$ is the directed diagonal of the parallelogram determined by the two vectors $\vec{u}$ and $\vec{v}$. This immediately gives us that $\vec{u}+\vec{v}=\vec{v}+\vec{u}$:

Example 4.3: Vector Addition is Commutative

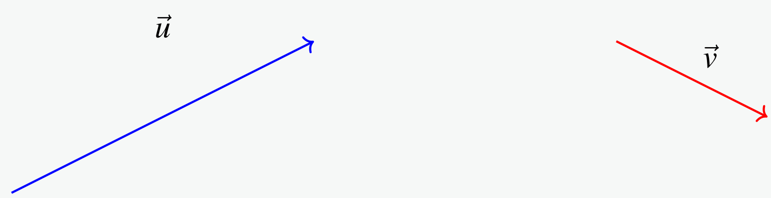
The vector $\vec{u}+\vec{v}$ is the same as the vector $\vec{v}+\vec{u}$:



When you have a vector $\vec{v}$, its additive inverse $-\vec{v}$ will be the vector which has the same magnitude as $\vec{v}$ but the opposite direction. When one writes $\vec{u}-\vec{v}$, the meaning is $\vec{u}+(-\vec{v})$ as with real numbers. The following example illustrates these definitions and conventions.

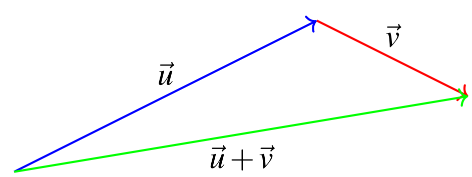
Example 4.4: Graphing Vector Addition

Consider the following picture of vectors $\vec{u}$ and $\vec{v}$.

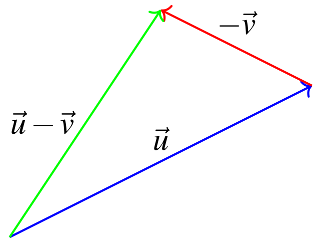


Sketch a picture of $\vec{u}+\vec{v},\vec{u}-\vec{v}$.

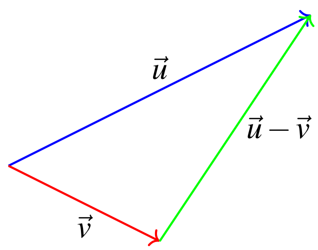
Solution. We will first sketch $\vec{u}+\vec{v}$. Begin by drawing $\vec{u}$ and then at the point of $\vec{u}$, place the tail of $\vec{v}$ as shown. Then $\vec{u}+\vec{v}$ is the vector which results from drawing a vector from the tail of $\vec{u}$ to the tip of $\vec{v}$.



Next consider $\vec{u}-\vec{v}$. This means $\vec{u}+(-\vec{v})$. From the above geometric description of vector addition, $-\vec{v}$ is the vector which has the same length but which points in the opposite direction to $\vec{v}$. Here is a picture.



An alternative way to draw the difference of two vectors is as follows: Suppose that we want to find the vector $\vec{u}-\vec{v}$. It would seem that if $\vec{w}$ is equal to that difference, so that $\vec{u}-\vec{v}=\vec{w}$, then we should have $\vec{v}+\vec{w}=\vec{u}$. So $\vec{u}-\vec{v}$ is the vector which, when added to $\vec{v}$, yields $\vec{u}$. This tells us that $\vec{u}-\vec{v}$ should be a vector that points from the tip of $\vec{v}$ to the tip of $\vec{u}$, when $\vec{u}$ and $\vec{v}$ emanate from the same point:



♠

4.2 Vectors in $\mathbb{R}^n$: Algebra

Outcomes

A. *Define the set $\mathbb{R}^n$.*

B. *Understand vector addition and scalar multiplication, algebraically.*

C. *Recognize when one vector is a linear combination of a set of vectors.*

A Not So Informal Introduction

In your previous mathematical work, you have dealt with the Cartesian plane $\mathbb{R}\times\mathbb{R}$, or $\mathbb{R}^2$. The major goal of this section is to tie your previous knowledge of points in the plane with our new notion of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ or $\mathbb{R}^n$.

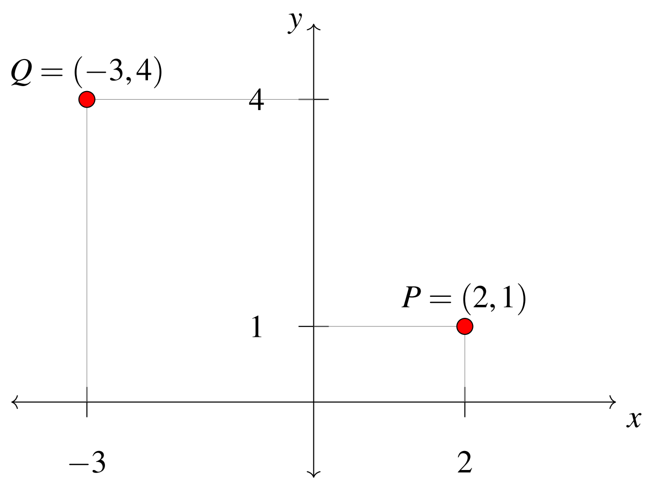
Most of our discussion in this section will happen in the plane, but the ideas generalize in a straightforward way to higher dimensional spaces. In the “forewarned is forearmed” school of pedagogy, let us just alert you to be very aware of the difference between a point in the plane, written horizontally and between parentheses, and a vector in $\mathbb{R}^2$, which is written vertically and between brackets:

$$\text{The point } P=(2,3) \text{ vs. the vector } \vec{p}=\begin{bmatrix}2\\3\end{bmatrix}.$$

In your previous work, when you worked with the plane $\mathbb{R}^2$ you considered it as the collection of ordered pairs of real numbers, of points:

$$\mathbb{R}^2 = \left\{ (x_1, x_2) : x_j \in \mathbb{R} \text{ for } j = 1, 2 \right\}$$

If we consider the familiar coordinate plane, with an x axis and a y axis, any point in this coordinate plane is identified by where it is located along the x axis, and also where it is located along the y axis. Consider as an example the following diagram.



Hence, every element in $\mathbb{R}^2$ is identified by two components, x and y, in the usual manner. The coordinates x, y (or x_1,x_2) uniquely determine a point in the plane. Note that while the definition uses x_1 and x_2 to label the coordinates and you may be used to x and y, these notations are equivalent.

We defined the notion of a vector in Definition 2.12: for any natural number n, an n-vector is simply an $n \times 1$ matrix. Up to this point, when we have been talking about vectors we have denoted them as if they were a matrix, so maybe we would talk about the vector X. From this point on, since vectors will be our point of interest, we will often label vectors as lower case letters or pairs of upper case letters surmounted by an arrow, for example

$$\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}.$$

Consider the following definition, which begins to tie together the notion of a point in n-space and an n-vector, and brings back the geometry of vectors introduced in the last section:

Definition 4.5: The Position Vector

Let $P = (p_1, \cdots, p_n)$ be the coordinates of a point in $\mathbb{R}^n$. Then the vector

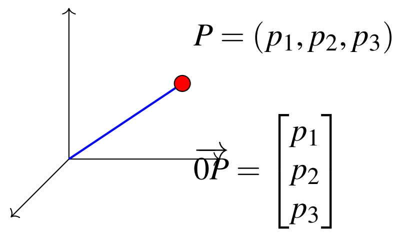
$$\overrightarrow{0P} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$

*is called the **position vector** of the point P.*

It is customary to think of $\overrightarrow{0P}$ as an arrow with its tail at $0 = (0, \cdots, 0)$ and its tip at P.

For this reason we may will talk about both the *point* $P = (p_1, \cdots, p_n) \in \mathbb{R}^n$ and the *vector* $\overrightarrow{0P} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \in \mathbb{R}^n$.

The connection between points and vectors is illustrated in the following picture for the special case of $\mathbb{R}^3$.



Thus every point P in $\mathbb{R}^n$ determines its position vector $\overrightarrow{0P}$. Conversely, every such position vector $\overrightarrow{0P}$ which has its tail at 0 and point at P determines the point P of $\mathbb{R}^n$.

Now suppose we are given two points, P, Q whose coordinates are $(p_1, \cdots, p_n)$ and $(q_1, \cdots, q_n)$ respectively. We can also determine the **position vector from** P **to** Q (also called the **vector from** P **to** Q) defined as follows.

$$\overrightarrow{PQ} = \begin{bmatrix} q_1 - p_1 \\ \vdots \\ q_n - p_n \end{bmatrix} = \overrightarrow{0Q} - \overrightarrow{0P}$$

Given a point in $\mathbb{R}^n$ named P, we will often use $\vec{p}$ to denote the position vector of point P. Notice that in this context, $\vec{p} = \overrightarrow{0P}$. If a point is referred to by an upper case letter, the position vector will usually be denoted by the corresponding lower case letter.

Think about the plane, $\mathbb{R}^2$. When you think about the plane as a collection of points, you should see a lot of dots. The point $P = (3,5)$ is a little dot, located 3 units right in the x-direction and five units up in the y-direction. The corresponding view of vectors is that the position vector $\overrightarrow{0P}$ is an arrow pointing from the origin to the point P. For our work with vectors in the plane (or in n-space), we will gather all of those vectors together and give them a name. Unfortunately, the name is $\mathbb{R}^n$, which is somewhat confusing at the start, as sometimes $\mathbb{R}^n$ will be best thought of as a bunch of points, and sometimes as a bunch of vectors. We will try to be careful about pointing out which is the appropriate view at any time.

We define real n-space to be the collection of n-vectors:

Definition 4.6: $\mathbb{R}^n$

The set $\mathbb{R}^n$ is defined to be the collection of n-vectors. So

$$\mathbb{R}^n = \{\vec{v} \mid \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \text{ where each } v_i \text{ is a real number}\}.$$

*The v_i's are called the **components** of the vector $\vec{v}$.*

You can think of the components of a vector as directions for obtaining the vector. Consider $n = 3$. Draw a vector with its tail at the point $(0,0,0)$ and its tip at the point (a,b,c). This vector it is obtained by starting at $(0,0,0)$, moving parallel to the x axis to $(a,0,0)$ and then from here, moving parallel to the y axis to $(a,b,0)$ and finally parallel to the z axis to (a,b,c). Observe that the same vector would result if you began at the point (d,e,f), moved parallel to the x axis to $(d+a,e,f)$, then parallel to the y axis to $(d+a,e+b,f)$, and finally parallel to the z axis to $(d+a,e+b,f+c)$. Here, the vector would have its tail sitting at the point determined by $A = (d,e,f)$ and its point at $B = (d+a,e+b,f+c)$. It is the **same vector** because it will point in the same direction and have the same length. It is like you took an actual arrow, and moved it from one location to another keeping it pointing the same direction.

Some important vectors that we will use include the **zero vector**, $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, and the so-called **standard basis vectors**

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

where $\vec{e}_i$ has a 1 as its ith component, but all other components are 0. In two special cases, $\mathbb{R}^2$ and $\mathbb{R}^3$, we will also denote the standard basis vectors by

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ or } \vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The Algebra of Scalar Multiplication

Since a vector is nothing more nor less than a matrix, we have already defined the algebraic operation scalar multiplication—you multiply a vector by a scalar exactly the same way as you multiply a (column) matrix times a scalar. Our goals now are to remind you of the definition, indicate that the algebraic definition matches our geometric definition from the last section, and then gather up some important results about scalar multiplication.

Scalar multiplication of vectors in $\mathbb{R}^n$ is defined as follows.

Definition 4.7: Scalar Multiplication of Vectors in $\mathbb{R}^n$

If $\vec{u} \in \mathbb{R}^n$ and $k \in \mathbb{R}$ is a scalar, then $k\vec{u} \in \mathbb{R}^n$ is defined by

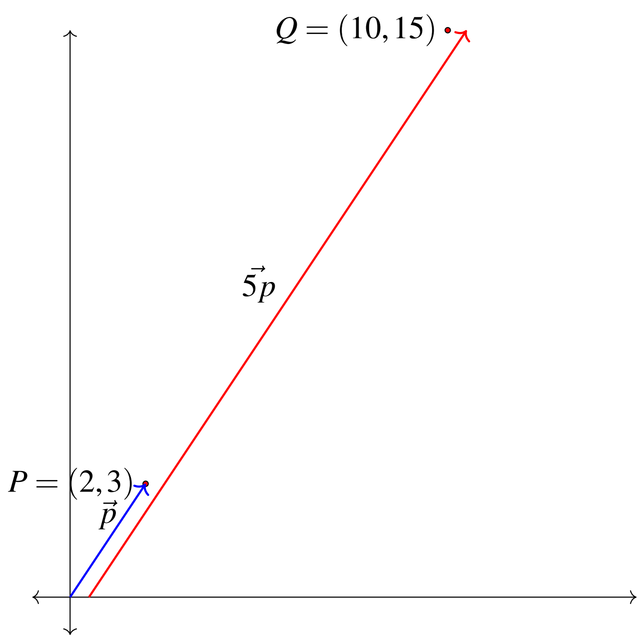
$$k\vec{u} = k \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ku_1 \\ \vdots \\ ku_n \end{bmatrix}$$

When we were working geometrically, we said that to multiply a vector $\vec{v}$ by a positive constant k would result in a vector with the same direction as $\vec{v}$, but with length scaled by a factor of k. Here's an example to indicate that our algebraic definition of scalar multiplication seems to work in the way it is supposed to:

Example 4.8: Scalar Multiplication

Suppose that $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. *Show that* $5\vec{v}$ *is 5 times as long as* $\vec{v}$ *and points in the same direction as* $\vec{v}$.

Solution. We need to compare the lengths and directions of the two vectors $\vec{p}$ and $5\vec{p}$. In this diagram, notice that $\vec{p}$ is the position vector corresponding to the point $P = (2,3)$, while $5\vec{p} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$ is the position vector corresponding to the point $Q = (10,15)$, offset slightly for clarity in the picture:



Since the points $(0,0)$, $(2,3)$, and $(10,15)$ are collinear, the direction of the vector $\vec{p}$ and the direction of the vector $5\vec{p}$ are the same. The distance from the origin to the point P, which is a reasonable interpretation of the length of the vector p, is $\sqrt{2^2+3^2} = \sqrt{13}$. The distance from the origin to Q is $\sqrt{10^2+15^2} = \sqrt{325} = 5\sqrt{13}$.

To summarize, the vector $5\vec{p}$ has the same direction as the vector $\vec{p}$ and is five times as long, so our algebraic definition of what happens when you multiply a vector by a scalar matches what we expect from our geometric description of the operation. ♠

Scalar multiplication of vectors satisfies several important properties. These are outlined in the following theorem.

Theorem 4.9: Properties of Scalar Multiplication

The following properties hold for vectors $\vec{u},\vec{v} \in \mathbb{R}^n$ and k,p scalars.

- *The Distributive Law over Vector Addition*

$$k\left(\vec{u}+\vec{v}\right) = k\vec{u}+k\vec{v}$$

- *The Distributive Law over Scalar Addition*

$$\left(k+p\right)\vec{u} = k\vec{u}+p\vec{u}$$

- *The Associative Law for Scalar Multiplication*

$$k\left(p\vec{u}\right) = \left(kp\right)\vec{u}$$

- *Rule for Multiplication by 1*

$$1\vec{u} = \vec{u}$$

As we proved these results earlier as Proposition 2.11, (actually, we left them as an exercise, but we know that you worked through the proof) we do not need to reprove them now.

The Algebra of Vector Addition

Once again, as a vector is nothing more than a matrix with one column, we have already know the algebra of vector addition:

Definition 4.10: Addition of Vectors in $\mathbb{R}^n$

If $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ then $\vec{u}+\vec{v} \in \mathbb{R}^n$ and is defined by

$$\begin{aligned} \vec{u}+\vec{v} &= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ &= \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} \end{aligned}$$

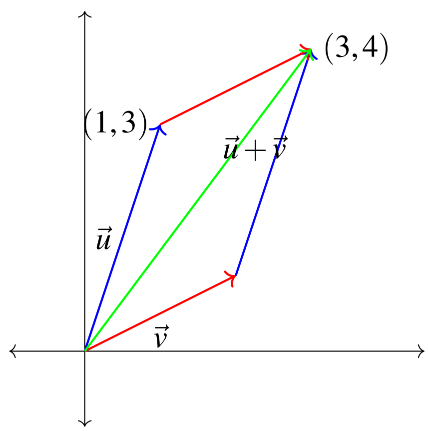
To add vectors, we simply add corresponding components. Therefore, in order to add vectors, they must be the same size.

To see how the algebraic definition corresponds to the geometric definition of vector addition, at least in $\mathbb{R}^2$, consider the following example.

Example 4.11: Vector Addition

Find the sum of $\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ *and* $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. *Show that the algebraic definition of vector addition corresponds to the geometric definition.*

Solution. Rapid mental calculation tells us that $\vec{u}+\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. A look at this diagram shows how our two definitions match.



When we slide the vector $\vec{v}$ so that its tail is at the point $(1,3)$, to find the point at the head of $\vec{u}+\vec{v}$ we have to add 2 to the x-coordinate and 1 to the y-coordinate, so the sum is the vector from the origin to the point $(3,4)$, as expected. ♠

The following theorem was established as Proposition 2.8.

Theorem 4.12: Properties of Vector Addition

The following properties hold for vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$.

- *The Commutative Law of Addition*

$$\vec{u}+\vec{v} = \vec{v}+\vec{u}$$

- *The Associative Law of Addition*

$$(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$$

- *The Existence of an Additive Identity*

$$\vec{u}+\vec{0} = \vec{u} \tag{4.1}$$

- *The Existence of an Additive Inverse*

$$\vec{u}+(-\vec{u}) = \vec{0}$$

The additive identity shown in equation 4.1 is the previously mentioned zero vector. You want to think of it as playing the role of the number 0. As was the case when we discussed matrices, $-\vec{u}$ is simply the vector $(-1)\vec{u}$.

Unsurprisingly, vector subtraction is defined as $\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$.

We conclude this section by reminding you of a crucial concept, first introduced in Definition 9.10, that combines vector addition and scalar multiplication.

Definition 4.13: Linear Combination

*A vector $\vec{v}$ is said to be a **linear combination** of the vectors $\vec{u}_1,\cdots,\vec{u}_n$ if there exist scalars, $a_1,\cdots,a_n$ such that*

$$\vec{v}=a_1\vec{u}_1+\cdots+a_n\vec{u}_n$$

For example,

$$3\left[\begin{array}{r} -4 \\ 1 \\ 0 \end{array}\right]+2\left[\begin{array}{r} -3 \\ 0 \\ 1 \end{array}\right]=\left[\begin{array}{r} -18 \\ 3 \\ 2 \end{array}\right].$$

Thus we can say that

$$\vec{v}=\left[\begin{array}{r} -18 \\ 3 \\ 2 \end{array}\right]$$

is a linear combination of the vectors

$$\vec{u}_1=\left[\begin{array}{r} -4 \\ 1 \\ 0 \end{array}\right] \text{ and } \vec{u}_2=\left[\begin{array}{r} -3 \\ 0 \\ 1 \end{array}\right]$$

4.3 Length of a Vector

Outcomes

A. *Find the length of a vector and the distance between two points in* $\mathbb{R}^n$.

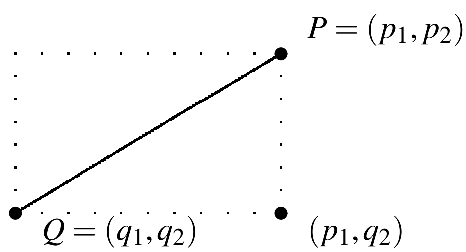
B. *Find the corresponding unit vector to a vector in* $\mathbb{R}^n$.

In this section, we explore what is meant by the length of a vector in $\mathbb{R}^n$. We develop this concept by first looking at the distance between two points in $\mathbb{R}^n$.

First, we will consider the concept of distance for $\mathbb{R}$, that is, for points in $\mathbb{R}^1$. Here, the distance between two points P and Q is given by the absolute value of their difference. We denote the distance between P and Q by $d(P,Q)$ which is defined as

$$d(P,Q) = \sqrt{(P-Q)^2} \tag{4.2}$$

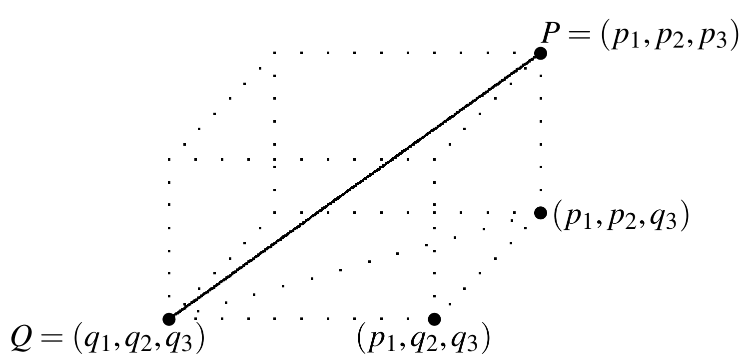
Consider now the case for $n=2$, demonstrated by the following picture.



There are two points $P=(p_1,p_2)$ and $Q=(q_1,q_2)$ in the plane. The distance between these points is shown in the picture as a solid line. Notice that this line is the hypotenuse of a right triangle which is half of the rectangle shown in dotted lines. We want to find the length of this hypotenuse which will give the distance between the two points. Note the lengths of the sides of this triangle are $|p_1-q_1|$ and $|p_2-q_2|$, the absolute value of the difference in these values. Therefore, the Pythagorean Theorem implies the length of the hypotenuse (and thus the distance between P and Q) equals

$$\left(|p_1-q_1|^2+|p_2-q_2|^2\right)^{1/2} = \left((p_1-q_1)^2+(p_2-q_2)^2\right)^{1/2} \tag{4.3}$$

Now suppose $n=3$ and let $P=(p_1,p_2,p_3)$ and $Q=(q_1,q_2,q_3)$ be two points in $\mathbb{R}^3$. Consider the following picture in which the solid line joins the two points and a dotted line joins the points (q_1,q_2,q_3) and (p_1,p_2,q_3).



Here, we need to use Pythagorean Theorem twice in order to find the length of the solid line. First, by the Pythagorean Theorem, the length of the dotted line joining (q_1, q_2, q_3) and (p_1, p_2, q_3) equals

$$\left((p_1 - q_1)^2 + (p_2 - q_2)^2\right)^{1/2}$$

while the length of the line joining (p_1, p_2, q_3) to (p_1, p_2, p_3) is just $|p_3 - q_3|$. Therefore, by the Pythagorean Theorem again, the length of the line joining the points $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ equals

$$\begin{aligned} &\left(\left(\left((p_1 - q_1)^2 + (p_2 - q_2)^2\right)^{1/2}\right)^2 + (p_3 - q_3)^2\right)^{1/2} \\ &= \left((p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2\right)^{1/2} \end{aligned} \tag{4.4}$$

This discussion motivates the following definition for the distance between points in $\mathbb{R}^n$.

Definition 4.14: Distance Between Points

Let $P = (p_1, \cdots, p_n)$ and $Q = (q_1, \cdots, q_n)$ be two points in $\mathbb{R}^n$. Then the distance between these points is defined as

$$\textit{distance between } P \textit{ and } Q = d(P, Q) = \left(\sum_{k=1}^{n} |p_k - q_k|^2\right)^{1/2}$$

*This is called the **distance formula**. We may also write $|P - Q|$ as the distance between P and Q.*

From the above discussion, you can see that Definition 4.14 holds for the special cases $n = 1, 2, 3$, as in Equations 4.2, 4.3, 4.4. In the following example, we use Definition 4.14 to find the distance between two points in $\mathbb{R}^4$.

Example 4.15: Distance Between Points

Find the distance between the points P and Q in $\mathbb{R}^4$, where P and Q are given by

$$P = (1, 2, -4, 6)$$

and

$$Q = (2, 3, -1, 0)$$

Solution. We will use the formula given in Definition 4.14 to find the distance between P and Q. Use the distance formula and write

$$d(P, Q) = \left((1-2)^2 + (2-3)^2 + (-4-(-1))^2 + (6-0)^2\right)^{\frac{1}{2}} = (47)^{\frac{1}{2}}$$

Therefore, $d(P, Q) = \sqrt{47}$.

♠

There are certain properties of the distance between points which are important in our study. These are outlined in the following theorem.

Theorem 4.16: Properties of Distance

Let P and Q be points in $\mathbb{R}^n$, and let the distance between them, $d(P,Q)$, be given as in Definition 4.14. Then, the following properties hold .

- $d(P,Q) = d(Q,P)$
- $d(P,Q) \geq 0$, *and equals 0 exactly when* $P = Q$.

There are many applications of the concept of distance. For instance, given two points, we can ask what collection of points are all the same distance between the given points. This is explored in the following example.

Example 4.17: The Plane Between Two Points

Describe the points in $\mathbb{R}^3$ which are at the same distance between $(1,2,3)$ and $(0,1,2)$.

Solution. Let $P = (p_1, p_2, p_3)$ be such a point. Therefore, P is the same distance from $(1,2,3)$ and $(0,1,2)$. Then by Definition 4.14,

$$\sqrt{(p_1-1)^2+(p_2-2)^2+(p_3-3)^2} = \sqrt{(p_1-0)^2+(p_2-1)^2+(p_3-2)^2}$$

Squaring both sides we obtain

$$(p_1-1)^2+(p_2-2)^2+(p_3-3)^2 = p_1^2+(p_2-1)^2+(p_3-2)^2$$

and so

$$p_1^2-2p_1+14+p_2^2-4p_2+p_3^2-6p_3 = p_1^2+p_2^2-2p_2+5+p_3^2-4p_3$$

Simplifying, this becomes

$$-2p_1+14-4p_2-6p_3 = -2p_2+5-4p_3$$

which can be written as

$$2p_1+2p_2+2p_3 = 9 \tag{4.5}$$

Therefore, the points $P = (p_1, p_2, p_3)$ which are the same distance from each of the given points are exactly the points that satisfy Equation 4.5. As we will see in Section 4.7, this equation define a plane in $\mathbb{R}^3$. ♠

We can now use our understanding of the distance between two points to define what is meant by the length of a vector. Consider the following definition.

Definition 4.18: Length of a Vector

Let $\vec{u} = [u_1 \cdots u_n]^T$ be a vector in $\mathbb{R}^n$. Then, the length of $\vec{u}$, written $\|\vec{u}\|$ is given by

$$\|\vec{u}\| = \sqrt{u_1^2 + \cdots + u_n^2}$$

This definition corresponds to Definition 4.14, if you consider the vector $\vec{u}$ to have its tail at the point $0 = (0, \cdots, 0)$ and its tip at the point $U = (u_1, \cdots, u_n)$. Then the length of $\vec{u}$ is equal to the distance between 0 and U, $d(0,U)$. In general, $d(P,Q) = \|\overrightarrow{PQ}\|$.

Consider Example 4.15. By Definition 4.18, we could also find the distance between P and Q as the length of the vector connecting them. Hence, if we were to draw a vector $\overrightarrow{PQ}$ with its tail at P and its point at Q, this vector would have length equal to $\sqrt{47}$.

We conclude this section with a new definition for the special case of vectors of length 1.

Definition 4.19: Unit Vector

Let $\vec{u}$ be a vector in $\mathbb{R}^n$. Then, we call $\vec{u}$ a **unit vector** *if it has length 1, that is if*

$$\|\vec{u}\| = 1$$

Let $\vec{v}$ be a vector in $\mathbb{R}^n$. Then, the vector $\vec{u}$ which has the same direction as $\vec{v}$ but length equal to 1 is the corresponding unit vector of $\vec{v}$. This vector is given by

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$$

We often use the term **normalize** to refer to this process. When we **normalize** a vector $\vec{v}$, we find unit vector that has the same direction as $\vec{v}$. Consider the following example.

Example 4.20: Finding a Unit Vector

Let $\vec{v}$ be given by

$$\vec{v} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

Find the unit vector $\vec{u}$ which has the same direction as $\vec{v}$.

Solution. We will use Definition 4.19 to solve this. Therefore, we need to find the length of $\vec{v}$ which, by Definition 4.18 is given by

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Using the corresponding values we find that

$$\|\vec{v}\| = \sqrt{1^2 + (-3)^2 + 4^2}$$

$$\begin{aligned} &= \sqrt{1+9+16} \\ &= \sqrt{26} \end{aligned}$$

In order to find $\vec{u}$, we divide $\vec{v}$ by $\sqrt{26}$. The result is

$$\begin{aligned} \vec{u} &= \frac{1}{\|\vec{v}\|}\vec{v} \\ &= \frac{1}{\sqrt{26}}\left[\begin{array}{r} 1 \\ -3 \\ 4 \end{array}\right] \\ &= \left[\begin{array}{r} \frac{1}{\sqrt{26}} \\ -\frac{3}{\sqrt{26}} \\ \frac{4}{\sqrt{26}} \end{array}\right] \end{aligned}$$

You can verify using the Definition 4.18 that $\|\vec{u}\| = 1$. ♠

4.4 The Dot Product

Outcomes

A. *Compute the dot product of vectors, and use this to compute vector projections.*

There are two ways of multiplying vectors which are of great importance in applications. The first of these is called the **dot product**. When we take the dot product of vectors, the result is a scalar. For this reason, the dot product is also called the **scalar product** . The definition is as follows.

Definition 4.21: Dot Product

*Let $\vec{u},\vec{v}$ be two vectors in $\mathbb{R}^n$. Then we define the **dot product** $\vec{u}\cdot\vec{v}$ as*

$$\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v}.$$

Notice that if

$$\vec{u} = \left[\begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_n \end{array}\right] \text{ and } \vec{v} = \left[\begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_n \end{array}\right],$$

then $\vec{u}\cdot\vec{v} = \sum_{k=1}^{n} u_k v_k$.

The dot product $\vec{u}\cdot\vec{v}$ is sometimes denoted as $(\vec{u},\vec{v})$ where a comma and two parentheses replace the dot. It can also be written as $\langle\vec{u},\vec{v}\rangle$ with angled brackets.

Consider the following example.

Example 4.22: Compute a Dot Product

Find $\vec{u} \cdot \vec{v}$ *for*

$$\vec{u} = \left[\begin{array}{r} 1 \\ 2 \\ 0 \\ -1 \end{array} \right], \quad \vec{v} = \left[\begin{array}{r} 0 \\ 1 \\ 2 \\ 3 \end{array} \right]$$

Solution. In using Definition 4.21 for computation it is usually easiest to compute

$$\vec{u} \cdot \vec{v} = \sum_{k=1}^{4} u_k v_k$$

This is given by

$$\begin{aligned} \vec{u} \cdot \vec{v} &= (1)(0) + (2)(1) + (0)(2) + (-1)(3) \\ &= 0 + 2 + 0 + -3 \\ &= -1 \end{aligned}$$

♠

With this definition, there are several important properties satisfied by the dot product.

Proposition 4.23: Properties of the Dot Product

Let k *and* p *denote scalars and* $\vec{u}, \vec{v}, \vec{w}$ *denote vectors. Then the dot product* $\vec{u} \cdot \vec{v}$ *satisfies the following properties.*

- $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
- $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v})$
- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{u} \cdot \vec{u} \geq 0$ *and equals zero if and only if* $\vec{u} = \vec{0}$

The proof is left as an exercise, but you should consider using the $\vec{u}^T \vec{v}$ definition of the dot product for the first two properties, and perhaps the $\sum u_k v_k$ version of the definition for the second two.

The last property above tells us that we can use the dot product to find the length of a vector:

Example 4.24: Length of a Vector

Find the length of

$$\vec{u} = \left[\begin{array}{r} 2 \\ 1 \\ 4 \\ 2 \end{array} \right]$$

That is, find $\|\vec{u}\|$.

Solution. By Proposition 4.23, $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$. Therefore, $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$. First, compute $\vec{u} \cdot \vec{u}$.

This is given by

$$\begin{aligned}
\vec{u} \cdot \vec{u} &= (2)(2)+(1)(1)+(4)(4)+(2)(2) \\
&= 4+1+16+4 \\
&= 25
\end{aligned}$$

Then,

$$\begin{aligned}
\|\vec{u}\| &= \sqrt{\vec{u} \cdot \vec{u}} \\
&= \sqrt{25} \\
&= 5
\end{aligned}$$

♠

You may wish to compare this to our previous definition of length, given in Definition 4.18.

The **Cauchy Schwarz inequality** is a fundamental inequality satisfied by the dot product. It is given in the following theorem.

Theorem 4.25: Cauchy Schwarz Inequality

The dot product satisfies the inequality

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\| \tag{4.6}$$

Furthermore equality is obtained if and only if one of $\vec{u}$ or $\vec{v}$ is a scalar multiple of the other.

Proof. First note that if $\vec{v} = \vec{0}$ both sides of 4.6 equal zero and so the inequality holds in this case. Therefore, it will be assumed in what follows that $\vec{v} \neq \vec{0}$.

Define a function of $t \in \mathbb{R}$ by

$$f(t) = (\vec{u} + t\vec{v}) \cdot (\vec{u} + t\vec{v})$$

Then by Proposition 4.23, $f(t) \geq 0$ for all $t \in \mathbb{R}$. Using Proposition 4.23 we can see

$$\begin{aligned}
f(t) &= \vec{u} \cdot (\vec{u} + t\vec{v}) + t\vec{v} \cdot (\vec{u} + t\vec{v}) \\
&= \vec{u} \cdot \vec{u} + t(\vec{u} \cdot \vec{v}) + t\vec{v} \cdot \vec{u} + t^2 \vec{v} \cdot \vec{v} \\
&= \|\vec{u}\|^2 + 2t(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2 t^2
\end{aligned}$$

(There are some details left out of the above, and you should fill them in. For example, the second line uses a distributive property that is not explicitly part of Proposition 4.23. How can we justify its use?)

Now this means the graph of $y = f(t)$ is a parabola which opens up and either its vertex touches the t axis or else the entire graph is above the t axis. In the first case, there exists some t where $f(t) = 0$ and this requires $\vec{u} + t\vec{v} = \vec{0}$ so one vector is a multiple of the other. Then clearly equality holds in 4.6. In the case where $\vec{v}$ is not a multiple of $\vec{u}$, it follows $f(t) > 0$ for all t which says $f(t)$ has no real zeros and so from the quadratic formula,

$$(2(\vec{u} \cdot \vec{v}))^2 - 4\|\vec{u}\|^2\|\vec{v}\|^2 < 0$$

which is equivalent to $|\vec{u}\cdot\vec{v}| < ||\vec{u}||||\vec{v}||$. ♠

Notice that this proof was based only on the properties of the dot product listed in Proposition 4.23. This means that whenever an operation satisfies these properties, the Cauchy Schwarz inequality holds. There are many other instances of these properties besides vectors in $\mathbb{R}^n$.

The Cauchy Schwarz inequality provides another proof of the **triangle inequality** for distances in $\mathbb{R}^n$.

Theorem 4.26: Triangle Inequality

For $\vec{u},\vec{v} \in \mathbb{R}^n$

$$||\vec{u}+\vec{v}|| \leq ||\vec{u}|| + ||\vec{v}|| \tag{4.7}$$

and equality holds if and only if one of the vectors is a non-negative scalar multiple of the other. Also

$$\big|\,||\vec{u}|| - ||\vec{v}||\,\big| \leq ||\vec{u}-\vec{v}|| \tag{4.8}$$

Proof. By properties of the dot product and the Cauchy Schwarz inequality,

$$\begin{aligned}
||\vec{u}+\vec{v}||^2 &= (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) \\
&= (\vec{u}\cdot\vec{u}) + (\vec{u}\cdot\vec{v}) + (\vec{v}\cdot\vec{u}) + (\vec{v}\cdot\vec{v}) \\
&= ||\vec{u}||^2 + 2\,(\vec{u}\cdot\vec{v}) + ||\vec{v}||^2 \\
&\leq ||\vec{u}||^2 + 2\,|\vec{u}\cdot\vec{v}| + ||\vec{v}||^2 \\
&\leq ||\vec{u}||^2 + 2||\vec{u}||||\vec{v}|| + ||\vec{v}||^2 \qquad (4.9)\\
&= (||\vec{u}|| + ||\vec{v}||)^2 \qquad (4.10)
\end{aligned}$$

Hence,

$$||\vec{u}+\vec{v}||^2 \leq (||\vec{u}|| + ||\vec{v}||)^2$$

Taking square roots of both sides you obtain 4.7.

It remains to consider when equality occurs. First assume that $\vec{v} = k\vec{u}$ with $k \geq 0$. Then

$$\begin{aligned}
||\vec{u}+\vec{v}|| &= ||\vec{u}+k\vec{u}|| \\
&= ||(1+k)\vec{u}|| \\
&= |(1+k)|||\vec{u}|| \\
&= (1+k)||\vec{u}|| \\
&= 1||\vec{u}|| + k||\vec{u}|| \\
&= 1||\vec{u}|| + |k|||\vec{u}|| \\
&= ||\vec{u}|| + ||k\vec{u}|| \\
&= ||\vec{u}|| + ||\vec{v}||
\end{aligned}$$

and so equality holds.

To prove the converse of the equality claim, we assume that equality holds in Equation 4.7. We must prove that one of the vectors is a non-negative multiple of the other. To attack an easy case first, suppose $\vec{u} = \vec{0}$. Then no matter what $\vec{v}$ is, we know that $\vec{u} = 0\vec{v}$ and so $\vec{u}$ is a non-negative multiple of $\vec{v}$. The same argument holds if $\vec{v} = \vec{0}$. Therefore, we can assume that both vectors are nonzero. To get equality in 4.7 above, it must be the case that Inequality 4.9 be an actual equality. So it must be the case that $|\vec{u}\cdot\vec{v}| = \|\vec{u}\|\|\vec{v}\|$. For this to be true, we know from Theorem 4.25 that one of the vectors must be a multiple of the other. Say $\vec{v} = k\vec{u}$. If $k < 0$ then equality cannot occur in 4.7 because in this case

$$\vec{u}\cdot\vec{v} = k\|\vec{u}\|^2 < 0 < |k|\,\|\vec{u}\|^2 = |\vec{u}\cdot\vec{v}|$$

Therefore, $k \geq 0$.

To get the other form of the triangle inequality write

$$\vec{u} = \vec{u} - \vec{v} + \vec{v}$$

so

$$\begin{aligned}\|\vec{u}\| &= \|\vec{u} - \vec{v} + \vec{v}\| \\ &\leq \|\vec{u} - \vec{v}\| + \|\vec{v}\|\end{aligned}$$

Therefore,

$$\|\vec{u}\| - \|\vec{v}\| \leq \|\vec{u} - \vec{v}\| \tag{4.11}$$

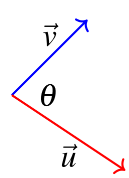
Similarly,

$$\|\vec{v}\| - \|\vec{u}\| \leq \|\vec{v} - \vec{u}\| = \|\vec{u} - \vec{v}\| \tag{4.12}$$

It follows from 4.11 and 4.12 that 4.8 holds. This is because $\left|\|\vec{u}\| - \|\vec{v}\|\right|$ equals the left side of either 4.11 or 4.12 and either way, $\left|\|\vec{u}\| - \|\vec{v}\|\right| \leq \|\vec{u} - \vec{v}\|$. ♠

The Geometric Significance of the Dot Product

Given two vectors, $\vec{u}$ and $\vec{v}$, the **included angle** is the angle between these two vectors which is given by θ such that $0 \leq \theta \leq \pi$. The dot product can be used to determine the included angle between two vectors. Consider the following picture where θ gives the included angle.



Proposition 4.27: The Dot Product and the Included Angle

Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^n$, and let θ be the included angle. Then the following equation holds.

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

In words, the dot product of two vectors equals the product of the magnitude (or length) of the two vectors multiplied by the cosine of the included angle. Note this gives a geometric description of the dot product which does not depend explicitly on the coordinates of the vectors.

Consider the following example.

Example 4.28: Find the Angle Between Two Vectors

Find the angle between the vectors given by

$$\vec{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

Solution. By Proposition 4.27,

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

Hence,

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

First, we can compute $\vec{u} \cdot \vec{v}$. By Definition 4.21, this equals

$$\vec{u} \cdot \vec{v} = (2)(3) + (1)(4) + (-1)(1) = 9$$

Then,

$$\begin{array}{c} \|\vec{u}\| = \sqrt{(2)(2) + (1)(1) + (1)(1)} = \sqrt{6} \\ \|\vec{v}\| = \sqrt{(3)(3) + (4)(4) + (1)(1)} = \sqrt{26} \end{array}$$

Therefore, the cosine of the included angle equals

$$\cos\theta = \frac{9}{\sqrt{26}\sqrt{6}} = 0.7205766...$$

With the cosine known, the angle can be determined by computing the inverse cosine of that angle, giving approximately $\theta = 0.76616$ radians. ♠

We can also use Proposition 4.27 to compute the dot product of two vectors.

Example 4.29: Using Geometric Description to Find a Dot Product

Let $\vec{u}, \vec{v}$ be vectors with $\|\vec{u}\| = 3$ and $\|\vec{v}\| = 4$. Suppose the angle between $\vec{u}$ and $\vec{v}$ is $\pi/3$. Find $\vec{u} \cdot \vec{v}$.

Solution. From the geometric description of the dot product in Proposition 4.27

$$\vec{u} \cdot \vec{v} = (3)(4)\cos(\pi/3) = 3 \times 4 \times 1/2 = 6$$

♠

Two nonzero vectors are said to be **perpendicular**, sometimes also called **orthogonal**, if the included angle is $\pi/2$ radians (90°).

Consider the following proposition.

Proposition 4.30: Perpendicular Vectors

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^n$. Then, $\vec{u}$ and $\vec{v}$ are said to be ***perpendicular*** *exactly when*

$$\vec{u} \cdot \vec{v} = 0$$

Proof. This follows directly from Proposition 4.27. First if the dot product of two nonzero vectors is equal to 0, this tells us that $\cos\theta = 0$ (this is where we need nonzero vectors). Thus $\theta = \pi/2$ and the vectors are perpendicular.

If on the other hand $\vec{v}$ is perpendicular to $\vec{u}$, then the included angle is $\pi/2$ radians. Hence $\cos\theta = 0$ and $\vec{u} \cdot \vec{v} = 0$. ♠

Consider the following example.

Example 4.31: Determine if Two Vectors are Perpendicular

Determine whether the two vectors,

$$\vec{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

are perpendicular.

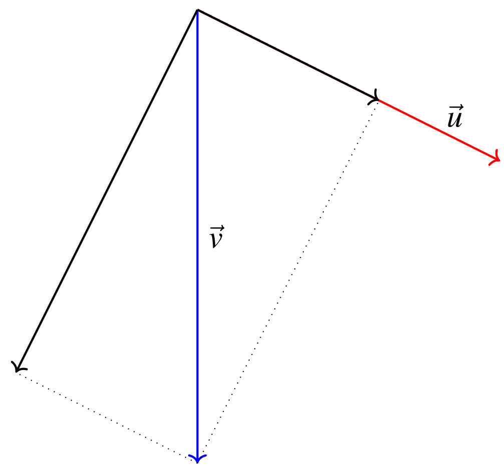
Solution. In order to determine if these two vectors are perpendicular, we compute the dot product. This is given by

$$\vec{u} \cdot \vec{v} = (2)(1) + (1)(3) + (-1)(5) = 0$$

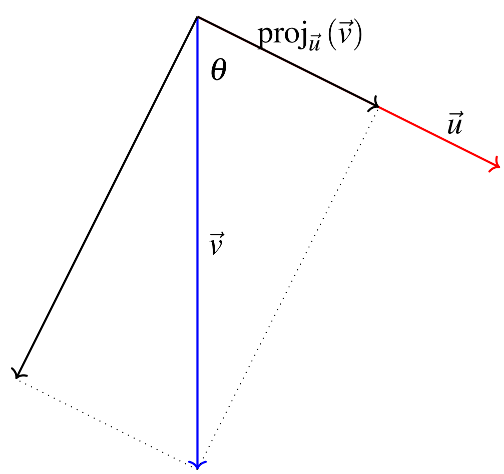
Therefore, by Proposition 4.30 these two vectors are perpendicular. ♠

Projections

Consider a box sitting on an inclined plane. The only force acting on the box is force of gravity, represented by a vector $\vec{v}$. We are interested in whether the box will slide down the inclined plane, and that will depend on whether the force exerted by $\vec{v}$ in the direction parallel to the plane is sufficient to overcome the starting friction between the box and the plane. If the angle of the plane is represented by the vector $\vec{u}$, we need to find how much of the vector $\vec{v}$ is pointing in the direction given by $\vec{u}$. The dot product will get us this vector, called the **projection of $\vec{v}$ onto $\vec{u}$**. In this section we develop a formula for this projection.



To motivate our formula, let θ be the angle between $\vec{u}$ and $\vec{v}$. For now, let's assume that $0 < \theta < \frac{\pi}{2}$. The vector we are looking for has the annoying, but descriptive, name $\text{proj}_{\vec{u}}(\vec{v})$.



We know the direction of the desired vector, $\text{proj}_{\vec{u}}(\vec{v})$, it is the same as the vector $\vec{u}$. All we need is the length. But from the above diagram and our (admittedly rusty, but still reliable) knowledge of right triangle trigonometry,

$$\frac{\|\text{proj}_{\vec{u}}(\vec{v})\|}{\|\vec{v}\|} = \cos\theta,$$

and since we can write $\cos\theta$ in terms of the dot product of $\vec{u}$ and $\vec{v}$, we have

$$\frac{\|\text{proj}_{\vec{u}}(\vec{v})\|}{\|\vec{v}\|} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|},$$

and so

$$\|\mathrm{proj}_{\vec{u}}(\vec{v})\| = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}.$$

Now our course is clear: to find our needed projection we can just take the vector $\vec{u}$, normalize it so that it has length 1, and then multiply it by $\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}$ to get a vector that has the correct direction and the correct length:

$$\mathrm{proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}\right)\left(\frac{\vec{u}}{\|\vec{u}\|}\right).$$

Let's gather that all up into an official definition:

Definition 4.32: Vector Projection

Let $\vec{u}$ and $\vec{v}$ be vectors. Then, the ***(orthogonal) projection of $\vec{v}$ onto $\vec{u}$*** *is given by*

$$\mathrm{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$$

Consider the following example of a projection.

Example 4.33: Find the Projection of One Vector Onto Another

Find $\mathrm{proj}_{\vec{u}}(\vec{v})$ if

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Solution. We can use the formula provided in Definition 4.32 to find $\mathrm{proj}_{\vec{u}}(\vec{v})$. First, compute $\vec{v}\cdot\vec{u}$. This is given by

$$\begin{aligned}
\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} &= (2)(1)+(3)(-2)+(-4)(1) \\
&= 2-6-4 \\
&= -8
\end{aligned}$$

Similarly, $\vec{u}\cdot\vec{u}$ is given by

$$\begin{aligned}
\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} &= (2)(2)+(3)(3)+(-4)(-4) \\
&= 4+9+16 \\
&= 29
\end{aligned}$$

Therefore, the projection is equal to

$$\begin{aligned}\mathrm{proj}_{\vec{u}}(\vec{v}) &= -\frac{8}{29}\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{16}{29} \\ -\frac{24}{29} \\ \frac{32}{29} \end{bmatrix}\end{aligned}$$

♠

A Look Under the Hood: Projection

Our derivation of the projection of $\vec{v}$ onto $\vec{u}$ contained a bit of a cheat, since we assumed that the angle between the two vectors was acute. To see how to find the formula without making that assumption, keep reading!

First, we show that there is only one way to write $\vec{v}$ as a sum of two vectors, one parallel to $\vec{u}$ and the other orthogonal to $\vec{u}$:

Theorem 4.34: Vector Projections

Let $\vec{v}$ and $\vec{u}$ be nonzero vectors. Then there exist unique vectors $\vec{v}_{||}$ and $\vec{v}_{\perp}$ such that

$$\vec{v} = \vec{v}_{||} + \vec{v}_{\perp} \tag{4.13}$$

where $\vec{v}_{||}$ is a scalar multiple of $\vec{u}$, and $\vec{v}_{\perp}$ is perpendicular to $\vec{u}$.

Proof. Suppose 4.13 holds and $\vec{v}_{||} = k\vec{u}$. Taking the dot product of both sides of 4.13 with $\vec{u}$ and using $\vec{v}_{\perp} \cdot \vec{u} = 0$, this yields

$$\begin{aligned}\vec{v} \cdot \vec{u} &= (\vec{v}_{||} + \vec{v}_{\perp}) \cdot \vec{u} \\ &= k\vec{u} \cdot \vec{u} + \vec{v}_{\perp} \cdot \vec{u} \\ &= k\|\vec{u}\|^2\end{aligned}$$

which requires $k = \vec{v} \cdot \vec{u}/\|\vec{u}\|^2$. Thus there can be no more than one vector $\vec{v}_{||}$. It follows $\vec{v}_{\perp}$ must equal $\vec{v} - \vec{v}_{||}$. This verifies there can be no more than one choice for both $\vec{v}_{||}$ and $\vec{v}_{\perp}$ and proves their uniqueness.

Now let

$$\vec{v}_{||} = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}\vec{u}$$

and let

$$\vec{v}_{\perp} = \vec{v} - \vec{v}_{||} = \vec{v} - \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}\vec{u}$$

Then $\vec{v}_{||} = k\vec{u}$ where $k = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}$. It only remains to verify $\vec{v}_{\perp} \cdot \vec{u} = 0$. But

$$\vec{v}_{\perp} \cdot \vec{u} = \vec{v} \cdot \vec{u} - \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}\vec{u} \cdot \vec{u}$$

$$\begin{aligned} &= \vec{v}\cdot\vec{u} - \vec{v}\cdot\vec{u} \\ &= 0 \end{aligned}$$

♠

Now, notice that the formula for $\vec{v}_{||}$ in the above is exactly the same as our formula for $\text{proj}_{\vec{u}}(\vec{v})$, and we have established that our formula works whether the angle between $\vec{u}$ and $\vec{v}$ is acute, obtuse, or right.

4.5 The Cross Product

Outcomes

A. *Compute the cross product and box product of vectors in $\mathbb{R}^3$.*

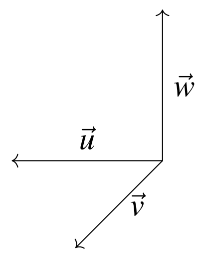
Recall that the dot product is one of two important products for vectors. The second type of product for vectors is called the cross product. It is important to note that the cross product is only defined in $\mathbb{R}^3$. First we discuss the geometric meaning and then a description in terms of coordinates is given, both of which are important. The geometric description is essential in order to understand the applications to physics and geometry while the coordinate description is necessary to compute the cross product.

Consider the following definition.

Definition 4.35: Right Hand System of Vectors

Three vectors, $\vec{u},\vec{v},\vec{w}$ form a right hand system if when you extend the fingers of your right hand along the direction of vector $\vec{u}$ and close them in the direction of $\vec{v}$, the thumb points roughly in the direction of $\vec{w}$.

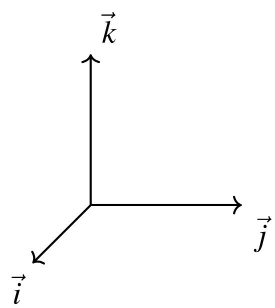
For an example of a right handed system of vectors, see the following picture.



In this picture the vector $\vec{w}$ points upwards from the plane determined by the other two vectors. Point the fingers of your right hand along $\vec{u}$, and close them in the direction of $\vec{v}$. Notice that if you extend the thumb on your right hand, it points in the direction of $\vec{w}$.

You should consider how a right hand system would differ from a left hand system. Try using your left hand and you will see that the vector $\vec{w}$ would need to point in the opposite direction.

Notice that the special vectors, $\vec{i}, \vec{j}, \vec{k}$ will always form a right handed system. If you extend the fingers of your right hand along $\vec{i}$ and close them in the direction $\vec{j}$, the thumb points in the direction of $\vec{k}$.



The following is the geometric description of the cross product. Recall that the dot product of two vectors results in a scalar. In contrast, the cross product results in a vector, as the product gives a direction as well as magnitude.

Definition 4.36: Geometric Definition of Cross Product

Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$. Then the ***cross product****, written $\vec{u} \times \vec{v}$, is defined by the following two rules.*

1. *Its length is $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$,*
   where θ is the included angle between $\vec{u}$ and $\vec{v}$.

2. *It is perpendicular to both $\vec{u}$ and $\vec{v}$, that is $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$, $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$,*
   and $\vec{u}, \vec{v}, \vec{u} \times \vec{v}$ form a right hand system.

The cross product of the special vectors $\vec{i}, \vec{j}, \vec{k}$ is as follows.

$$\begin{array}{ll} \vec{i} \times \vec{j} = \vec{k} & \vec{j} \times \vec{i} = -\vec{k} \\ \vec{k} \times \vec{i} = \vec{j} & \vec{i} \times \vec{k} = -\vec{j} \\ \vec{j} \times \vec{k} = \vec{i} & \vec{k} \times \vec{j} = -\vec{i} \end{array}$$

With this information, the following gives the coordinate description of the cross product.

Recall that the vector $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ can be written in terms of $\vec{i}, \vec{j}, \vec{k}$ as $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$.

Proposition 4.37: Coordinate Description of Cross Product

Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be two vectors. Then

$$\begin{aligned} \vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + \\ + (u_1v_2 - u_2v_1)\vec{k} \end{aligned} \tag{4.14}$$

Writing $\vec{u} \times \vec{v}$ in the usual way, it is given by

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

We now prove this proposition.

Proof. From the above table and the properties of the cross product listed,

$$\begin{aligned} \vec{u} \times \vec{v} &= \left(u_1\vec{i} + u_2\vec{j} + u_3\vec{k}\right) \times \left(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\right) \\ &= u_1v_2\vec{i} \times \vec{j} + u_1v_3\vec{i} \times \vec{k} + u_2v_1\vec{j} \times \vec{i} + u_2v_3\vec{j} \times \vec{k} + +u_3v_1\vec{k} \times \vec{i} + u_3v_2\vec{k} \times \vec{j} \\ &= u_1v_2\vec{k} - u_1v_3\vec{j} - u_2v_1\vec{k} + u_2v_3\vec{i} + u_3v_1\vec{j} - u_3v_2\vec{i} \\ &= (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k} \end{aligned} \tag{4.15}$$

♠

There is another version of 4.14 which may be easier to remember. We can express the cross product as the determinant of a matrix, as follows.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{4.16}$$

Expanding the determinant along the top row yields

$$\begin{aligned} &\vec{i}(-1)^{1+1}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} + \vec{j}(-1)^{2+1}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}(-1)^{3+1}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\ &= \vec{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \end{aligned}$$

Expanding these determinants leads to

$$(u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

which is the same as 4.15.

The cross product satisfies the following properties.

Proposition 4.38: Properties of the Cross Product

Let $\vec{u},\vec{v},\vec{w}$ be vectors in $\mathbb{R}^3$, and k a scalar. Then, the following properties of the cross product hold.

1. $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$, and $\vec{u}\times\vec{u} = \vec{0}$
2. $(k\vec{u})\times\vec{v} = k(\vec{u}\times\vec{v}) = \vec{u}\times(k\vec{v})$
3. $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v}+\vec{u}\times\vec{w}$
4. $(\vec{v}+\vec{w})\times\vec{u} = \vec{v}\times\vec{u}+\vec{w}\times\vec{u}$

Proof. Formula 1. follows immediately from the definition. The vectors $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ have the same magnitude, $|\vec{u}|\,|\vec{v}|\sin\theta$, and an application of the right hand rule shows they have opposite direction.

Formula 2. is proven as follows. If k is a non-negative scalar, the direction of $(k\vec{u})\times\vec{v}$ is the same as the direction of $\vec{u}\times\vec{v}, k(\vec{u}\times\vec{v})$ and $\vec{u}\times(k\vec{v})$. The magnitude is k times the magnitude of $\vec{u}\times\vec{v}$ which is the same as the magnitude of $k(\vec{u}\times\vec{v})$ and $\vec{u}\times(k\vec{v})$. Using this yields equality in 2. In the case where $k<0$, everything works the same way except the vectors are all pointing in the opposite direction and you must multiply by $|k|$ when comparing their magnitudes.

The distributive laws, 3. and 4., are much harder to establish. For now, it suffices to notice that if we know that 3. is true, 4. follows. Thus, assuming 3., and using 1.,

$$\begin{aligned}
(\vec{v}+\vec{w})\times\vec{u} &= -\vec{u}\times(\vec{v}+\vec{w}) \\
&= -(\vec{u}\times\vec{v}+\vec{u}\times\vec{w}) \\
&= \vec{v}\times\vec{u}+\vec{w}\times\vec{u}
\end{aligned}$$

♠

We will now look at an example of how to compute a cross product.

Example 4.39: Find a Cross Product

Find $\vec{u}\times\vec{v}$ for the following vectors

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Solution. Note that we can write $\vec{u},\vec{v}$ in terms of the special vectors $\vec{i},\vec{j},\vec{k}$ as

$$\begin{aligned}
\vec{u} &= \vec{i}-\vec{j}+2\vec{k} \\
\vec{v} &= 3\vec{i}-2\vec{j}+\vec{k}
\end{aligned}$$

We will use the equation given by 4.16 to compute the cross product.

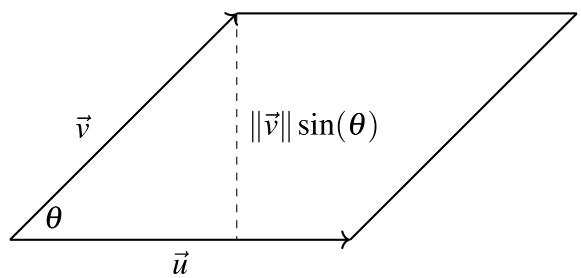
$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 3 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} \vec{k} = 3\vec{i} + 5\vec{j} + \vec{k}$$

We can write this result in the usual way, as

$$\vec{u} \times \vec{v} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

♠

An important geometrical application of the cross product is as follows. The size of the cross product, $\|\vec{u} \times \vec{v}\|$, is the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$, as shown in the following picture.



We examine this concept in the following example.

Example 4.40: Area of a Parallelogram

Find the area of the parallelogram determined by the vectors $\vec{u}$ and $\vec{v}$ given by

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Solution. Notice that these vectors are the same as the ones given in Example 4.39. Recall from the geometric description of the cross product, that the area of the parallelogram is simply the magnitude of $\vec{u} \times \vec{v}$. From Example 4.39,

$$\vec{u} \times \vec{v} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

Thus the area of the parallelogram is

$$\|\vec{u} \times \vec{v}\| = \sqrt{(3)(3) + (5)(5) + (1)(1)} = \sqrt{9 + 25 + 1} = \sqrt{35}$$

♠

We can also use this concept to find the area of a triangle determined by three points in $\mathbb{R}^3$. Consider the following example.

Example 4.41: Area of Triangle

Find the area of the triangle determined by the points $(1,2,3),(0,2,5),(5,1,2)$

Solution. This triangle is obtained by connecting the three points with lines. Picking $(1,2,3)$ as a starting point, there are two displacement vectors, $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$. Notice that if we add either of these vectors to the position vector of the starting point, the result is the position vectors of the other two points. Now, the area of the triangle is half the area of the parallelogram determined by these two displacement vectors. The required cross product is given by

$$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}$$

Taking the size of this vector gives the area of the parallelogram, given by

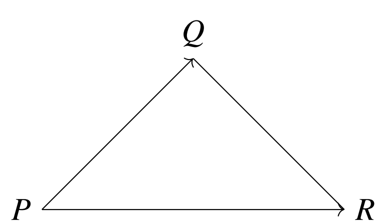
$$\sqrt{(2)(2)+(7)(7)+(1)(1)} = \sqrt{4+49+1} = \sqrt{54}$$

Hence the area of the triangle is $\frac{1}{2}\sqrt{54} = \frac{3}{2}\sqrt{6}$. ♠

In general, if you have three points in $\mathbb{R}^3, P, Q, R$, the area of the triangle is given by

$$\frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\|$$

Recall that $\overrightarrow{PQ}$ is the vector running from point P to point Q.



In the next section, we explore another application of the cross product.

The Box Product

Recall that we can use the cross product to find the the area of a parallelogram. It follows that we can use the cross product together with the dot product to find the volume of a parallelepiped.

We begin with a definition.

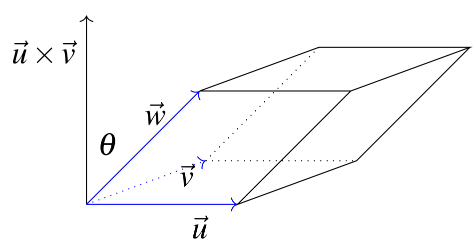
Definition 4.42: Parallelepiped

A parallelepiped determined by the three vectors, $\vec{u}, \vec{v}$, and $\vec{w}$ consists of

$$\{r\vec{u} + s\vec{v} + t\vec{w} : r, s, t \in [0, 1]\}$$

That is, if you pick three numbers, r, s, and t each in $[0, 1]$ and form $r\vec{u} + s\vec{v} + t\vec{w}$ then the collection of all such points makes up the parallelepiped determined by these three vectors.

The following is an example of a parallelepiped.



Notice that the base of the parallelepiped is the parallelogram determined by the vectors $\vec{u}$ and $\vec{v}$. Therefore, its area is equal to $\|\vec{u} \times \vec{v}\|$. The height of the parallelepiped is $\|\vec{w}\| \cos \theta$ where θ is the angle shown in the picture between $\vec{w}$ and $\vec{u} \times \vec{v}$. The volume of this parallelepiped is the area of the base times the height which is just

$$\|\vec{u} \times \vec{v}\| \|\vec{w}\| \cos \theta = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

This expression is known as the **box product** and is sometimes written as $[\vec{u}, \vec{v}, \vec{w}]$. You should consider what happens if you interchange the $\vec{v}$ with the $\vec{w}$ or the $\vec{u}$ with the $\vec{w}$. You can see geometrically from drawing pictures that this merely introduces a minus sign. In any case the box product of three vectors always equals either the volume of the parallelepiped determined by the three vectors or else -1 times this volume.

Proposition 4.43: The Box Product

Let $\vec{u}, \vec{v}, \vec{w}$ be three vectors in $\mathbb{R}^n$ that define a parallelepiped. Then the volume of the parallelepiped is the absolute value of the box product, given by

$$|(\vec{u} \times \vec{v}) \cdot \vec{w}|$$

Consider an example of this concept.

Example 4.44: Volume of a Parallelepiped

Find the volume of the parallelepiped determined by the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$$

Solution. According to the above discussion, pick any two of these vectors, take the cross product and then take the dot product of this with the third of these vectors. The result will be either the desired volume or -1 times the desired volume. Therefore by taking the absolute value of the result, we obtain the volume.

We will take the cross product of $\vec{u}$ and $\vec{v}$. This is given by

$$\vec{u} \times \vec{v} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -5 \\ 1 & 3 & -6 \end{vmatrix} = 3\vec{i} + \vec{j} + \vec{k} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

Now take the dot product of this vector with $\vec{w}$ which yields

$$\begin{aligned} (\vec{u} \times \vec{v}) \cdot \vec{w} &= \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \\ &= \left(3\vec{i} + \vec{j} + \vec{k}\right) \cdot \left(3\vec{i} + 2\vec{j} + 3\vec{k}\right) \\ &= 9 + 2 + 3 \\ &= 14 \end{aligned}$$

This shows the volume of this parallelepiped is 14 cubic units. ♠

There is a fundamental observation which comes directly from the geometric definitions of the cross product and the dot product.

Proposition 4.45: Order of the Product

Let $\vec{u}, \vec{v}$, and $\vec{w}$ be vectors. Then $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w})$.

Proof. This follows from observing that either $(\vec{u} \times \vec{v}) \cdot \vec{w}$ and $\vec{u} \cdot (\vec{v} \times \vec{w})$ both give the volume of the parallelepiped or they both give -1 times the volume. ♠

Recall that we can express the cross product as the determinant of a particular matrix. It turns out that the same can be done for the box product. Suppose you have three vectors, $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\vec{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$, and

$\vec{w} = \begin{bmatrix} g \\ h \\ i \end{bmatrix}$. Then the box product $\vec{u} \cdot (\vec{v} \times \vec{w})$ is given by the following.

$$\begin{aligned}
\vec{u} \cdot (\vec{v} \times \vec{w}) &= \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ d & e & f \\ g & h & i \end{vmatrix} \\
&= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\
&= \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}
\end{aligned}$$

To take the box product, you can simply take the determinant of the matrix which results by letting the rows be the components of the given vectors in the order in which they occur in the box product.

This follows directly from the definition of the cross product given above and the way we expand determinants. Thus the volume of a parallelepiped determined by the vectors $\vec{u}, \vec{v}, \vec{w}$ is just the absolute value of the above determinant.

4.6 Parametric Lines

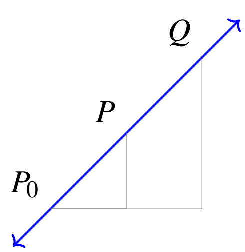
Outcomes

A. *Find the vector and parametric equations of a line.*

Having spent the first part of this chapter becoming familiar with vectors and some operations on vectors, now we will shift focus. The next couple of sections will use vectors to describe some familiar geometric objects—lines and planes. By examining these objects through the lens of linear algebra, we will be able to talk easily about lines in higher dimensional spaces, and then we will be able to generalize the idea of a plane in $\mathbb{R}^3$ to higher dimensional settings as well.

Let us consider lines. You are used to working with lines in the plane, and you are doubtless an expert at questions like, "Find an equation for the line that passes through the points $(1,7)$ and $(17,42)$." or "What is an equation of the line with slope -3 and y-intercept 7?" In all of these cases you were given two pieces of information and that sufficed to determine a unique line. By being a bit particular about how to think about the two needed pieces of information that we use to specify the line, we'll be able to generalize the notion of line to higher-dimensional spaces quite easily, but to do that we'll need to shift how we think about lines in $\mathbb{R}^2$ a bit. Slopes and intercepts are going to be out, points and direction vectors are going to be in.

Let P and P_0 be two different points in $\mathbb{R}^2$ which are contained in a line L. Our goal is to write an equation that characterizes the line L. Let $\vec{p}$ and $\vec{p}_0$ be the position vectors for the points P and P_0 respectively. Suppose that Q is an arbitrary point on L. Consider the following diagram.



Our goal is to be able to define Q in terms of P and P_0. Consider the vector $\overrightarrow{P_0P} = \vec{p} - \vec{p}_0$ which has its tail at P_0 and point at P. If we add $\vec{p} - \vec{p}_0$ to the position vector $\vec{p}_0$ for P_0, the sum would be a vector with its point at P. In other words,

$$\vec{p} = \vec{p}_0 + (\vec{p} - \vec{p}_0)$$

Now suppose we were to add $t(\vec{p} - \vec{p}_0)$ to $\vec{p}$ where t is some scalar. You can see that by doing so, we could find a vector with its point at Q. In other words, we can find t such that

$$\vec{q} = \vec{p}_0 + t(\vec{p} - \vec{p}_0)$$

This equation determines the line L in $\mathbb{R}^2$. The vector $\vec{p} - \vec{p}_0$ is called the **direction vector** of the line L. Our mantra is going to be: To find an equation for a line, we need a point P_0 on the line and a direction vector $\vec{d}$ for the line.

Example 4.46: Vector Equation of a Line in $\mathbb{R}^2$

Find an equation of the line L that passes through the points $(1,7)$ and $(17,42)$.

Solution. We need a point P_0 that is on the line, and since we are given two points on the line we have an embarrassment of riches. Arbitrarily, let's use $P_0 = (1,7)$. For the direction vector $\vec{d}$, we'll use the vector that points from P_0 to the other point, so $\vec{d} = \begin{bmatrix} 16 \\ 35 \end{bmatrix}$. Thus an equation for the line L is

$$\begin{aligned} \vec{q} &= \vec{p}_0 + t\vec{d} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 \\ 7 \end{bmatrix} + t \begin{bmatrix} 16 \\ 35 \end{bmatrix}. \end{aligned}$$

♠

Notice that the solution to the above example is just one more way of seeing the same line L. You already know other ways of writing the equation of L. For example if you wanted parametric equations for L you could take our solution and rewrite it as

$$\begin{aligned} x &= 1 + 16t \\ y &= 7 + 35t \end{aligned}$$

or you could take each of the above equations, solve them for t, and set them equal to get a familiar Cartesian equation for L, perhaps in slope-intercept form (making one of your previous teachers proud):

$$\begin{aligned} \frac{1}{16}x - \frac{1}{16} &= \frac{1}{35}y - \frac{7}{35} \\ \frac{35}{16}x - \frac{35}{16} + 7 &= y \\ y &= \frac{35}{16}x + \frac{77}{16} \end{aligned}$$

All of these are legitimate and correct ways to describe the line L from Example 4.46. But we will concentrate on the vector equation $\vec{q} = \vec{p}_0 + t\vec{d}$ as it generalizes quickly and easily to higher dimensions.

If you think about two points in $\mathbb{R}^3$, you can see that the vector pointing from one of the points to the other can serve as the direction vector $\vec{d}$ and that by adding multiples of $\vec{d}$ to the position vector of one of the points, you generate position vectors of all of the points on the line connecting the two points. The same concept works in higher dimensions, too, leading us to make the following definition:

Definition 4.47: Vector Equation of a Line

Suppose a line L in $\mathbb{R}^n$ contains the two different points P and P_0. Let $\vec{p}$ and $\vec{p}_0$ be the position vectors of these two points, respectively. Then, L is the collection of points Q which have the position vector $\vec{q}$ given by

$$\vec{q} = \vec{p}_0 + t(\vec{p} - \vec{p}_0)$$

where $t \in \mathbb{R}$.
*Let $\vec{d} = \vec{p} - \vec{p}_0$. Then $\vec{d}$ is a **direction vector for** L and a **vector equation** for L is given by*

$$\vec{q} = \vec{p}_0 + t\vec{d}, \quad t \in \mathbb{R}$$

Note that this definition agrees with the usual notion of a line in two dimensions and so this is consistent with earlier concepts. Consider now points in $\mathbb{R}^3$. If a point $P \in \mathbb{R}^3$ is given by $P = (x,y,z)$, $P_0 \in \mathbb{R}^3$ by $P_0 = (x_0, y_0, z_0)$, then we can write

$$\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} x_0 \\ y_0 \\ z_0 \end{array}\right] + t \left[\begin{array}{c} a \\ b \\ c \end{array}\right]$$

where $\vec{d} = \left[\begin{array}{c} a \\ b \\ c \end{array}\right]$. This is the vector equation of L written in **component form** .

The following theorem claims that such an equation is in fact a line.

Proposition 4.48: Algebraic Description of a Straight Line

Let $\vec{a}, \vec{b} \in \mathbb{R}^n$ with $\vec{b} \neq \vec{0}$. Then $\vec{x} = \vec{a} + t\vec{b}$, $t \in \mathbb{R}$, is a line.

Proof. Let $\vec{x_1}, \vec{x_2} \in \mathbb{R}^n$. Define $\vec{x_1} = \vec{a}$ and let $\vec{x_2} - \vec{x_1} = \vec{b}$. Since $\vec{b} \neq \vec{0}$, it follows that $\vec{x_2} \neq \vec{x_1}$. Then $\vec{a} + t\vec{b} = \vec{x_1} + t(\vec{x_2} - \vec{x_1})$. It follows that $\vec{x} = \vec{a} + t\vec{b}$ is a line containing the two different points X_1 and X_2 whose position vectors are given by $\vec{x_1}$ and $\vec{x_2}$ respectively. ♠

We can use the above discussion to find the equation of a line when given two distinct points. Consider the following example.

Example 4.49: A Line From Two Points

Find a vector equation for the line through the points $P_0 = (1,2,0,3)$ and $P = (2,-4,6,1)$.

Solution. We will use the definition of a line given above in Definition 4.47 to write this line in the form

$$\vec{q} = \vec{p_0} + t(\vec{p} - \vec{p_0})$$

Let $\vec{q} = \left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right]$. Then, we can find $\vec{p}$ and $\vec{p_0}$ by taking the position vectors of points P and P_0 respectively. Then,

$$\vec{q} = \vec{p_0} + t(\vec{p} - \vec{p_0})$$

can be written as

$$\left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right] = \left[\begin{array}{c} 1 \\ 2 \\ 0 \\ 3 \end{array}\right] + t \left[\begin{array}{r} 1 \\ -6 \\ 6 \\ -2 \end{array}\right], \ t \in \mathbb{R}$$

Here, the direction vector $\left[\begin{array}{r} 1 \\ -6 \\ 6 \\ -2 \end{array}\right]$ is obtained by $\vec{p} - \vec{p_0} = \left[\begin{array}{r} 2 \\ -4 \\ 6 \\ 1 \end{array}\right] - \left[\begin{array}{c} 1 \\ 2 \\ 0 \\ 3 \end{array}\right]$ as indicated above in Definition 4.47. ♠

Notice that in the above example we said that we found "a" vector equation for the line, not "the" equation. The reason for this terminology is that there are infinitely many different vector equations for the same line. To see this, replace t with another parameter, say $3s$. Then you obtain a different vector equation for the same line because the same set of points is obtained.

In Example 4.49, the vector given by $\left[\begin{array}{r} 1 \\ -6 \\ 6 \\ -2 \end{array}\right]$ is the direction vector defined in Definition 4.47. If we know the direction vector of a line, as well as a point on the line, we can find the vector equation.

Consider the following example.

Example 4.50: A Line From a Point and a Direction Vector

Find a vector equation for the line which contains the point $P_0 = (1,2,0)$ *and has direction vector* $\vec{d} = \left[\begin{array}{c} 1 \\ 2 \\ 1 \end{array}\right]$

Solution. We will use Definition 4.47 to write this line in the form $\vec{p} = \vec{p_0} + t\vec{d}$, $t \in \mathbb{R}$. We are given the direction vector $\vec{d}$. In order to find $\vec{p_0}$, we can use the position vector of the point P_0. This is given by $\left[\begin{array}{c} 1 \\ 2 \\ 0 \end{array}\right]$. Letting $\vec{p} = \left[\begin{array}{c} x \\ y \\ z \end{array}\right]$, the equation for the line is given by

$$\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} 1 \\ 2 \\ 0 \end{array}\right] + t \left[\begin{array}{c} 1 \\ 2 \\ 1 \end{array}\right], \; t \in \mathbb{R} \tag{4.17}$$

♠

We sometimes elect to write a line such as the one given in 4.17 in the form

$$\left.\begin{array}{l} x = 1 + t \\ y = 2 + 2t \\ z = t \end{array}\right\} \text{ where } t \in \mathbb{R} \tag{4.18}$$

This set of equations gives the same information as 4.17, and is called the **parametric equation of the line**.

Consider the following definition, which can easily be extended to $\mathbb{R}^n$:

Definition 4.51: Parametric Equation of a Line in $\mathbb{R}^3$

Let L be a line in $\mathbb{R}^3$ which has direction vector $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and goes through the point $P_0 = (x_0, y_0, z_0)$. Then, letting t be a parameter, we can write L as

$$\left.\begin{array}{l} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{array}\right\} \text{ where } t \in \mathbb{R}$$

*This is called a **parametric equation** of the line L.*

You can verify that the form discussed following Example 4.50 in equation 4.18 is of the form given in Definition 4.51.

There is one other form for a line which is useful, which is the **symmetric form**. Consider the line given by 4.18. You can solve for the parameter t to write

$$\begin{array}{l} t = x - 1 \\ t = \frac{y-2}{2} \\ t = z \end{array}$$

Therefore,

$$x - 1 = \frac{y-2}{2} = z$$

This is the **symmetric form** of the line.

In the following example, we look at how to take the equation of a line from symmetric form to parametric form.

Example 4.52: Change Symmetric Form to Parametric Form

*Suppose the **symmetric form of a line** is*

$$\frac{x-2}{3} = \frac{y-1}{2} = z + 3$$

Write the line in parametric form as well as vector form.

Solution. We want to write this line in the form given by Definition 4.51. This is of the form

$$\left.\begin{array}{l} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{array}\right\} \text{ where } t \in \mathbb{R}$$

Let $t = \frac{x-2}{3}, t = \frac{y-1}{2}$ and $t = z + 3$, as given in the symmetric form of the line. Then solving for x, y, z, yields

$$\left.\begin{array}{l} x = 2 + 3t \\ y = 1 + 2t \\ z = -3 + t \end{array}\right\} \text{ with } t \in \mathbb{R}$$

This is the parametric equation for this line.

Now, we want to write this line in the form given by Definition 4.47. This is the form

$$\vec{p} = \vec{p}_0 + t\vec{d}$$

where $t \in \mathbb{R}$. This equation becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}$$

♠

At this point we are experts at writing equations of lines, but there is much more to be said. As an example of a couple of applications to situations involving lines, we will find the angle between two lines and then find the distance from a point to a line.

When finding the angle between two lines, typically one would assume that the lines intersect. In some situations, however, it may make sense to ask this question when the lines do not intersect, such as the angle between the trajectories of two different objects. In any case we understand "the angle between two lines" to mean the smallest angle between (any of) their direction vectors. The only subtlety here is that if $\vec{u}$ is a direction vector for a line, then so is any multiple $k\vec{u}$, and thus we will find complementary angles among all angles between direction vectors for two lines, and we simply take the smaller of the two.

Example 4.53: Find the Angle Between Two Lines

Find the angle between the two lines

$$L_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

and

$$L_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Solution. You can verify that these lines do not intersect, but as discussed above this does not matter and we simply find the smallest angle between any directions vectors for these lines.

To do so we first find the angle between the direction vectors given above:

$$\vec{u} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

In order to find the angle, we solve the following equation for θ

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

to obtain $\cos\theta = -\frac{1}{2}$ and since we choose included angles between 0 and π we obtain $\theta = \frac{2\pi}{3}$.

Now the angles between any two direction vectors for these lines will either be $\frac{2\pi}{3}$ or its complement $\phi = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$. We choose the smaller angle, and therefore conclude that the angle between the two lines is $\frac{\pi}{3}$. ♠

For our second application, suppose that a line L and a point P are given such that P is not contained in L. Through the use of projections, we can determine the distance from P to L.

Example 4.54: Shortest Distance from a Point to a Line

Let $P = (1,3,5)$ be a point in $\mathbb{R}^3$, and let L be the line which goes through point $P_0 = (0,4,-2)$ with direction vector $\vec{d} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the shortest distance from P to the line L, and find the point Q on L that is closest to P.

Solution. In order to determine the shortest distance from P to L, we will first find the vector $\overrightarrow{P_0P}$ and then find the projection of this vector onto L. The vector $\overrightarrow{P_0P}$ is given by

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$$

Then, if Q is the point on L closest to P, it follows that

$$\begin{aligned} \overrightarrow{P_0Q} &= \text{proj}_{\vec{d}} \overrightarrow{P_0P} \\ &= \left(\frac{\overrightarrow{P_0P} \cdot \vec{d}}{\|\vec{d}\|^2} \right) \vec{d} \\ &= \frac{15}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \\ &= \frac{5}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

Now, the distance from P to L is given by

$$\|\overrightarrow{QP}\| = \|\overrightarrow{P_0P} - \overrightarrow{P_0Q}\| = \sqrt{26}$$

The point Q is found by adding the vector $\overrightarrow{P_0Q}$ to the position vector $\overrightarrow{0P_0}$ for P_0 as follows

$$\begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix} + \frac{5}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{10}{3} \\ \frac{17}{3} \\ \frac{4}{3} \end{bmatrix}$$

Therefore, $Q = \left(\frac{10}{3}, \frac{17}{3}, \frac{4}{3}\right)$.

4.7 Planes in $\mathbb{R}^3$, Hyperplanes in $\mathbb{R}^n$

Outcomes

A. *Find the vector and scalar equations of a plane in 3 and higher dimension.*

B. *Find the shortest distance between a point and a plane.*

Much like the above discussion with lines, vectors can be used to determine equations of planes in $\mathbb{R}^3$ in a way that generalizes nicely to define objects called **hyperplanes** in $\mathbb{R}^n$. We will focus on three-space, as it is easier to visualize than, say, $\mathbb{R}^{17}$.

Given a vector $\vec{n}$ in $\mathbb{R}^3$ and a point P_0, it is possible to find a **unique** plane which contains P_0 and is perpendicular to the given vector.

Definition 4.55: Normal Vector

Let $\vec{n}$ be a nonzero vector in $\mathbb{R}^3$. Then $\vec{n}$ is called a ***normal vector*** *to a plane if and only if*

$$\vec{n} \cdot \vec{v} = 0$$

for every vector $\vec{v}$ in the plane, where we say $\vec{v}$ is in *the plane if there are two points P_0 and P_1 such that P_0 and P_1 are on the plane and $\vec{v}$ is the vector pointing from P_0 to P_1.*

Notice this definition is saying that $\vec{n}$ is orthogonal (perpendicular) to every vector in the plane. Annoyingly, we now have three different words that all mean the same thing: perpendicular, orthogonal, and normal. Allow yourself a moment to curse your fate, then get used to it and on we go.

Consider a plane with normal vector given by $\vec{n}$, and containing a point P_0. Notice that this plane is unique. If P is an arbitrary point on this plane, then by definition the normal vector is orthogonal to the vector between P_0 and P. Letting $\overrightarrow{0P}$ and $\overrightarrow{0P_0}$ be the position vectors of points P and P_0 respectively, it follows that

$$\vec{n} \cdot (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0$$

or

$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$

The first of these equations gives the **vector equation** of the plane.

Definition 4.56: Vector Equation of a Plane in $\mathbb{R}^3$

Let $\vec{n}$ be a normal vector for a plane which contains a point P_0. If P is an arbitrary point on this plane, then a ***vector equation*** *of the plane is given by*

$$\vec{n} \cdot (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0$$

Notice that this equation can be used to determine if a point P is contained in a certain plane.

Example 4.57: A Point in a Plane

Let $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *be a normal vector for a plane which contains the point* $P_0 = (2,1,4)$. *Determine if the point* $P = (5,4,1)$ *is contained in this plane.*

Solution. By Definition 4.56, P is a point in the plane if it satisfies the equation

$$\vec{n} \cdot (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0$$

Given the above $\vec{n}$, P_0, and P, we have

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \left(\begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \right) = 3 + 6 - 9 = 0.$$

Hence the equation is satisfied and $P = (5,4,1)$ is contained in the plane. ♠

With vector equations for the plane in hand, let's examine a Cartesian form of the equation that is also very convenient. Suppose we are examining a plane containing the $P_0 = (x_0, y_0, z_0)$ and having normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Then an arbitrary point $P = (x,y,z)$ is on the plane exactly when the vector version of the equation of the plane is satisfied. That is:

$$\begin{aligned} \vec{n} \cdot (\overrightarrow{0P} - \overrightarrow{0P_0}) &= 0 \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \right) &= 0 \\ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \end{aligned}$$

This linear equation in three variables can be written as

$$ax + by + cz = ax_0 + by_0 + cz_0$$

Notice that since P_0 is given, $ax_0 + by_0 + cz_0$ is a known scalar, which we can call d. This equation becomes

$$ax + by + cz = d$$

Notice also that the coefficients of the variables are simply the coordinates of the normal vector $\vec{n}$.

Definition 4.58: Scalar Equation of a Plane in $\mathbb{R}^3$

*Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the normal vector for a plane which contains the point $P_0 = (x_0, y_0, z_0)$. Then if $P = (x, y, z)$ is an arbitrary point on the plane, the **scalar equation** of the plane is given by*

$$ax + by + cz = d$$

where $a, b, c, d \in \mathbb{R}$ and $d = ax_0 + by_0 + cz_0$.

Consider the following example.

Example 4.59: Finding an Equation of a Plane in $\mathbb{R}^3$

Find both a vector equation and a scalar equation of the plane containing $P_0 = (3, -2, 5)$ and orthogonal to $\vec{n} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$.

Solution. The above vector $\vec{n}$ is a normal vector for this plane. Using Definition 4.56, we can determine a vector equation for this plane.

$$\begin{aligned}
\vec{n} \cdot (\overrightarrow{0P} - \overrightarrow{0P_0}) &= 0 \\
\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \right) &= 0 \\
\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x-3 \\ y+2 \\ z-5 \end{bmatrix} &= 0
\end{aligned}$$

Using Definition 4.58, we can also determine a scalar equation of the plane.

$$-2x + 4y + 1z = -2(3) + 4(-2) + 1(5) = -9$$

♠

Here's another example about finding an equation of a plane. This time we won't be given a normal vector.

Example 4.60: Find an Equation of a Plane in $\mathbb{R}^3$ Given Three Points on the Plane

Find an equation of the plane that contains the points $P_0 = (1, 2, 3)$, $P_1 = (0, 1, 2)$, and $P_3 = (3, 0, 1)$.

Solution. We give two different solutions to this problem. You get to choose which you like best.

For our first solution, we know that an equation of the plane can be of the form $ax+by+cz=d$. We also know that our three points are on the plane, so they must satisfy the equation. So we are looking to find a, b, c, and d that solve this system of linear equations:

$$\begin{aligned} a+2b+3c-d&=0 \\ b+2c-d&=0\,. \\ 3a\qquad + \ c-d&=0 \end{aligned}$$

So we can just take the augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 3 & 0 & 1 & -1 & 0 \end{bmatrix}$$

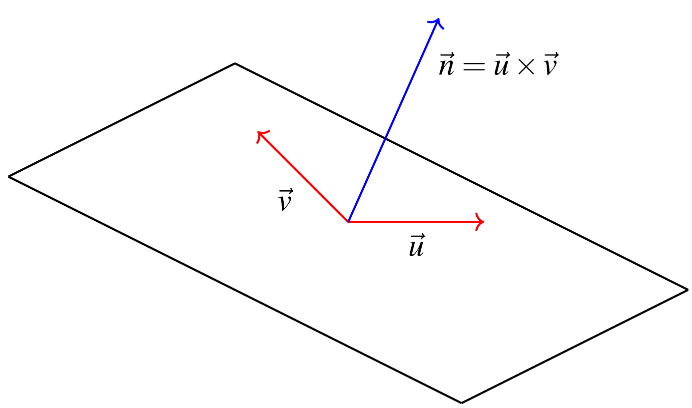
and reduce it to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

to get lots of solutions for a, b, c, and d. For example $a=0, b=1, c=-1, d=-1$ work, since all three of our points satisfy the equation $0x+y-z=-1$.

For our second solution, we will find a vector equation for the plane. To do this, we need two things: a point on the plane (no problem, we have three of them) and $\vec{n}$, a vector that is normal to the plane. We'll find the normal vector by taking advantage of the fact that we are working in $\mathbb{R}^3$ and so we can take the cross product of two vectors.

Suppose that we have two vectors $\vec{u}$ and $\vec{v}$ that both lie in the plane. Then if we take the cross product of $\vec{u}$ and $\vec{v}$ we will have a vector that it orthogonal to both $\vec{u}$ and $\vec{v}$ and, in fact, to every vector that lies in the plane! So we can use $\vec{u}\times\vec{v}$ as our normal vector.



For our situation, notice that the vectors $\overrightarrow{P_0P_1}=\begin{bmatrix}-1\\-1\\-1\end{bmatrix}$ and $\overrightarrow{P_0P_2}=\begin{bmatrix}2\\-2\\-2\end{bmatrix}$ both lie in the plane, so the vector

$$\vec{n}=\begin{bmatrix}-1\\-1\\-1\end{bmatrix}\times\begin{bmatrix}2\\-2\\-2\end{bmatrix}=\begin{bmatrix}0\\-4\\4\end{bmatrix}$$

is a normal vector for the plane. Then we can write our vector equation of the plane as

$$\begin{bmatrix} 0 \\ -4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix} = 0.$$

♠

In the same way that the projection of one vector onto another was a tool that we used to find the distance from a point to a line in Section 4.6, the projection will help us find the distance from a point to a plane.

Consider the following example.

Example 4.61: Shortest Distance From a Point to a Plane in $\mathbb{R}^3$

Find the shortest distance from the point $P = (3,2,3)$ to the plane given by $2x+y+2z = 2$, and find the point Q on the plane that is closest to P.

Solution. Pick an arbitrary point P_0 on the plane. Then, it follows that

$$\overrightarrow{QP} = \mathrm{proj}_{\vec{n}} \overrightarrow{P_0P}$$

and $\|\overrightarrow{QP}\|$ is the shortest distance from P to the plane. Further, the vector $\overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP}$ gives the necessary point Q.

From the above scalar equation, we have that $\vec{n} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Now, choose a simple point on the plane, for example $P_0 = (1,0,0)$ is simple enough and on the plane since it satisfies the given equation. Then, $\overrightarrow{P_0P} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$.

Next, compute

$$\begin{aligned} \overrightarrow{QP} &= \mathrm{proj}_{\vec{n}} \overrightarrow{P_0P} = \left(\frac{\overrightarrow{P_0P} \cdot \vec{n}}{\|\vec{n}\|^2} \right) \vec{n} \\ &= \frac{12}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \frac{4}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

Then, $\|\overrightarrow{QP}\| = 4$ so the shortest distance from P to the plane is 4.

Next, to find the point Q on the plane which is closest to P we have

$$\begin{aligned} \overrightarrow{0Q} &= \overrightarrow{0P} - \overrightarrow{QP} \\ &= \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore, $Q = (\frac{1}{3}, \frac{2}{3}, \frac{1}{3})$ is the desired point on the plane closest to P. ♠

Hyperplanes in $\mathbb{R}^n$

A plane is a two-dimensional flat object that lives in $\mathbb{R}^3$. This sentence is rough and not precise, but it pulls out some important characteristics of planes that we can generalize to surfaces that live inside higher (and lower) dimensional spaces. The important thing to notice for now is that the dimension of the plane (2) is one less than the dimension of the enclosing space (3). We'll take that idea and use it to talk about hyperplanes, which we will think of as $n-1$-dimensional flat objects that live in $\mathbb{R}^n$. We will reserve the word "plane" for the familiar 2-dimensional object that lives in $\mathbb{R}^3$.

Hyperplanes will be defined by a point and a normal vector, the same way that planes were. Given a vector $\vec{n} \in \mathbb{R}^n$ (we apologize that the dimension of the space and the name of the vector are both n, but it seems awkward to use another letter) and a point $P_0 \in \mathbb{R}^n$, they will define a hyperplane in the same way that a point and a normal vector define a plane in $\mathbb{R}^3$:

Definition 4.62: Vector Equation of a Hyperplane in $\mathbb{R}^n$

Let $\vec{n}$ be a vector in $\mathbb{R}^n$ and let P_0 be a point in $\mathbb{R}^n$. The hyperplane defined by $\vec{n}$ and P_0 is the set of points P such that

$$\vec{n} \cdot (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0.$$

*In this case, the vector $\vec{n}$ is said to be a normal vector for the hyperplane, and the above equation is called a **vector equation** for the hyperplane.*

For our first example, suppose that $n = 2$. Then a hyperplane in 2-space should be a flat 1-dimensional object; a line. And that's how it works out:

Example 4.63: A Line as a Hyperplane

Show that the line $y = 3x + 2$ is a hyperplane in $\mathbb{R}^2$.

Solution. Since $(0,2)$ and $(1,5)$ are both points on the line, a direction vector for the line is $\vec{d} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. A normal vector for the line has to be orthogonal to $\vec{d}$, and an easy way to construct such a vector is just to switch the components and slap a minus sign on one of them, so we'll look at $\vec{n} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Now, using the point $P_0 = (0,2)$, Definition 4.62 says that the vector equation of the hyperplane determined by $\vec{n}$ and P_0 is

$$\begin{aligned} \vec{n} \cdot (\overrightarrow{0P} - \overrightarrow{0P_0}) &= 0 \\ \begin{bmatrix} 3 \\ -1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) &= 0 \\ 3(x-0) + (-1)(y-2) &= 0 \\ 3x - y + 2 &= 0 \\ y &= 3x + 2 \end{aligned}$$

and that is the equation of the line with which we started. ♠

Someone once said that a mathematician is a person who can look at two different things and see how they are the same. That's what's going on here—in certain fundamental ways, a line and a plane are the same thing. And there are structures in higher dimensional spaces that relate to their space in exactly the same way that planes relate to $\mathbb{R}^3$. In fact all of our results and examples from earlier in this section apply to hyperplanes, including the example about the distance from a point to a hyperplane. All that changes is that there are more coordinates in the vectors.

Example 4.64: Equation of a Hyperplane

Find an equation of the hyperplane with normal vector $\vec{n} = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 4 \end{bmatrix}$ *that contains the point* $P_0 = (1,0,3,5)$.

Solution. Again using Definition 4.62, we have the equation of the hyperplane as

$$\begin{aligned}
\vec{n} \cdot (\overrightarrow{0P} - \overrightarrow{0P_0}) &= 0 \\
\begin{bmatrix} -1 \\ 3 \\ 2 \\ 4 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 3 \\ 5 \end{bmatrix} \right) &= 0 \\
(-1)(x-1) + 3(y-0) + 2(z-3) + 4(w-5) &= 0 \\
-x + 3y + 2z + 4w &= 25
\end{aligned}$$

♠

Notice that we have a nice pattern about linear equations defining hyperplanes:

$$\begin{aligned}
ax + by = c \quad & \text{defines a line, a hyperplane in } \mathbb{R}^2 \text{ with normal vector } \vec{n} = \begin{bmatrix} a \\ b \end{bmatrix} \\
ax + by + cz = d \quad & \text{defines a plane, a hyperplane in } \mathbb{R}^3 \text{ with normal vector } \vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\
ax + by + cz + dw = e \quad & \text{defines a hyperplane in } \mathbb{R}^4 \text{ with normal vector } \vec{n} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\
ax + by + cz + dw + ev = f \quad & \text{defines a hyperplane in } \mathbb{R}^5 \text{ with normal vector } \vec{n} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} \\
\vdots \quad & \vdots
\end{aligned}$$

We can now reconsider Example 4.61 in higher dimension, the techniques are very much the same.

Example 4.65: Shortest Distance From a Point to a Plane in $\mathbb{R}^4$

Find the shortest distance from the point $P = (1,-3,0,1)$ to the hyperplane given by $2x+y+2z-w=2$, and find the point Q on the hyperplane that is closest to P.

Solution. The solution strategy is exactly the same as before. Pick an arbitrary point P_0 on the hyperplane. Then, it follows that

$$\overrightarrow{QP} = \text{proj}_{\vec{n}}\overrightarrow{P_0P}$$

and $\|\overrightarrow{QP}\|$ is the shortest distance from P to the hyperplane. Further, the vector $\overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP}$ gives the necessary point Q.

From the above scalar equation, we have that $\vec{n} = \left[\begin{array}{c} 2 \\ 1 \\ 2 \\ -1 \end{array}\right]$. Now choose the (simple) point $P_0 = (1,0,0,0)$ on the hyperplane to obtain $\overrightarrow{P_0P} = \left[\begin{array}{c} 1 \\ -3 \\ 0 \\ 1 \end{array}\right] - \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ -3 \\ 0 \\ 1 \end{array}\right]$.

Next, compute

$$\begin{aligned}
\overrightarrow{QP} &= \text{proj}_{\vec{n}}\overrightarrow{P_0P} = \left(\frac{\overrightarrow{P_0P}\cdot\vec{n}}{\|\vec{n}\|^2}\right)\vec{n} \\
&= \frac{-4}{10}\left[\begin{array}{c} 2 \\ 1 \\ 2 \\ -1 \end{array}\right] = \frac{-2}{5}\left[\begin{array}{c} 2 \\ 1 \\ 2 \\ -1 \end{array}\right]
\end{aligned}$$

Then, $\|\overrightarrow{QP}\| = \frac{2}{5}\sqrt{10}$, and that is the shortest distance from P to the hyperplane.
Next, to find the point Q on the plane which is closest to P we have

$$\overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP} = \left[\begin{array}{c} 1 \\ -2 \\ 0 \\ 1 \end{array}\right] - \left(\frac{-2}{5}\right)\left[\begin{array}{c} 2 \\ 1 \\ 2 \\ -1 \end{array}\right] = \frac{1}{5}\left[\begin{array}{c} 9 \\ -13 \\ 4 \\ 3 \end{array}\right]$$

Therefore, $Q = (\frac{9}{5}, \frac{-13}{5}, \frac{4}{5}, \frac{3}{5})$ is the desired point on the hyperplane closest to the point P. ♠

4.8 Spanning and Linear Independence in $\mathbb{R}^n$

Outcomes

A. *Determine the span of a set of vectors, and determine if a vector is contained in a specified span.*

B. *Determine if a set of vectors is linearly independent.*

By generating all linear combinations of a set of vectors one can obtain various subsets of $\mathbb{R}^n$ which we call subspaces. For example what set of vectors in $\mathbb{R}^3$ generate the xy-plane? What is the smallest such set of vectors can you find? The tools of spanning, linear independence and basis are exactly what is needed to answer these and similar questions and are the focus of this section. The following definition is essential.

Definition 4.66: Subset

Let U and W be sets of vectors in $\mathbb{R}^n$. If all vectors in U are also in W, we say that U is a **subset** *of W, denoted*

$$U \subseteq W$$

Spanning Set of Vectors

We begin this section with a definition.

Definition 4.67: Span of a Set of Vectors

The collection of all linear combinations of a set of vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is known as the **span** *of these vectors and is written as* $\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$.

Consider the following example.

Example 4.68: Span of Vectors

Describe the span of the vectors $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ *and* $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \in \mathbb{R}^3$.

Solution. You can see that any linear combination of the vectors $\vec{u}$ and $\vec{v}$ yields a vector of the form $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ in the xy-plane.

Moreover every vector in the xy-plane is in fact such a linear combination of the vectors $\vec{u}$ and $\vec{v}$. That's because, for any real numbers x and y,

$$\left[\begin{array}{c} x \\ y \\ 0 \end{array}\right] = (-2x+3y)\left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right] + (x-y)\left[\begin{array}{c} 3 \\ 2 \\ 0 \end{array}\right]$$

Thus $\text{span}\{\vec{u},\vec{v}\}$ is precisely the xy-plane. ♠

You can convince yourself that no single vector can span the xy-plane. In fact, take a moment to consider what is meant by the span of a single vector.

However you can make the set larger if you wish. For example consider the larger set of vectors $\{\vec{u},\vec{v},\vec{w}\}$ where $\vec{w} = \left[\begin{array}{c} 4 \\ 5 \\ 0 \end{array}\right]$. Since the first two vectors already span the entire xy-plane, the span is once again precisely the xy-plane and nothing has been gained. Of course if you add a new vector such as $\vec{w} = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$ then our set of vectors does span a different space. What is the span of $\{\vec{u},\vec{v},\vec{w}\}$ in this case?

The distinction between the sets $\{\vec{u},\vec{v}\}$ and $\{\vec{u},\vec{v},\vec{w}\}$ will be made using the concept of linear independence in the next subsection.

Consider the vectors $\vec{u},\vec{v}$, and $\vec{w}$ discussed above. In the next example, we will show how to formally demonstrate that $\vec{w}$ is in the span of $\{\vec{u},\vec{v}\}$.

Example 4.69: Vector in a Span

Let $\vec{u} = \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right]$ *and* $\vec{v} = \left[\begin{array}{c} 3 \\ 2 \\ 0 \end{array}\right] \in \mathbb{R}^3$. *Show that* $\vec{w} = \left[\begin{array}{c} 4 \\ 5 \\ 0 \end{array}\right]$ *is in* $\text{span}\{\vec{u},\vec{v}\}$.

Solution. For a vector to be in $\text{span}\{\vec{u},\vec{v}\}$, it must be a linear combination of these vectors. If $\vec{w} \in \text{span}\{\vec{u},\vec{v}\}$, we must be able to find scalars a,b such that

$$\vec{w} = a\vec{u} + b\vec{v}$$

We proceed as follows.

$$\left[\begin{array}{c} 4 \\ 5 \\ 0 \end{array}\right] = a\left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right] + b\left[\begin{array}{c} 3 \\ 2 \\ 0 \end{array}\right]$$

This is equivalent to the following system of equations

$$\begin{array}{rcl} a+3b &=& 4 \\ a+2b &=& 5 \end{array}$$

We solve this system the usual way, constructing the augmented matrix and row reducing to find the reduced row-echelon form.

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 1 & 2 & 5 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|r} 1 & 0 & 7 \\ 0 & 1 & -1 \end{array}\right]$$

The solution is $a=7, b=-1$. This means that

$$\vec{w} = 7\vec{u} - \vec{v}$$

Therefore we can say that $\vec{w}$ is in span $\{\vec{u}, \vec{v}\}$. ♠

Linearly Independent Set of Vectors

We now turn our attention to the following question: what linear combinations of a given set of vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ yield the zero vector? Clearly $0\vec{u}_1 + 0\vec{u}_2 + \cdots + 0\vec{u}_k = \vec{0}$, but is it possible to have $\sum_{i=1}^{k} a_i\vec{u}_i = \vec{0}$ without all of the coefficients being zero?

You can create examples where this easily happens. For example if $\vec{u}_1 = \vec{u}_2$, then $1\vec{u}_1 - \vec{u}_2 + 0\vec{u}_3 + \cdots + 0\vec{u}_k = \vec{0}$, no matter the vectors $\{\vec{u}_3, \cdots, \vec{u}_k\}$. But sometimes it can be more subtle.

Example 4.70: Linearly Dependent Set of Vectors

Consider the vectors

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}, \text{ and } \vec{u}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

in $\mathbb{R}^3$.
Then verify that

$$1\vec{u}_1 + 0\vec{u}_2 + 1\vec{u}_3 + 2\vec{u}_4 = \vec{0}$$

You can see that the linear combination does yield the zero vector but has some non-zero coefficients. Thus we define a set of vectors to be *linearly dependent* if this happens.

Definition 4.71: Linearly Dependent Set of Vectors

A set of non-zero vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is said to be ***linearly dependent*** *if a linear combination of these vectors without all coefficients being zero does yield the zero vector.*

Note that if $\sum_{i=1}^{k} a_i \vec{u}_i = \vec{0}$ and some coefficient is non-zero, say $a_1 \neq 0$, then

$$\vec{u}_1 = \frac{-1}{a_1} \sum_{i=2}^{k} a_i \vec{u}_i$$

and thus $\vec{u}_1$ is in the span of the other vectors. And the converse clearly works as well, so we have shown the following proposition:

Proposition 4.72: Characterizing Linear Dependence

A set of vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is linearly dependent if and only if one of the vectors is a linear combination of the other vectors in the set.

In particular, you can show that the vector $\vec{u}_1$ in the above example is in the span of the vectors $\{\vec{u}_2, \vec{u}_3, \vec{u}_4\}$.

If a set of vectors is NOT linearly dependent, then it must be that any linear combination of these vectors which yields the zero vector must use all zero coefficients. This is a very important notion, and we give it its own name of *linear independence*.

Definition 4.73: Linearly Independent Set of Vectors

A set of non-zero vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is said to be ***linearly independent*** *if whenever*

$$\sum_{i=1}^{k} a_i \vec{u}_i = \vec{0}$$

it follows that each $a_i = 0$.

Notice that if any of the vectors u_i in the set $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is equal to the zero vector, then the set of vectors is automatically linearly dependent. Thus every vector in a linearly independent set of vectors must be non-zero.

To view this in a more familiar setting, form the $n \times k$ matrix A having these vectors as columns. Then all we are saying is that the set $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent precisely when $A\vec{x} = 0$ has only the trivial solution.

Here is an example.

Example 4.74: Linearly Independent Vectors

Consider the vectors $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, *and* $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ *in* $\mathbb{R}^3$. *Decide if the set* $\{\vec{u}, \vec{v}, \vec{w}\}$ *is linearly independent.*

Solution. So suppose that we have a linear combinations $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$. Then you can see that this can only happen with $a = b = c = 0$.

As mentioned above, you can equivalently form the 3×3 matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, and show that $A\vec{x} = 0$ has only the trivial solution.

Thus this means the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent. ♠

In terms of spanning, a set of vectors is linearly independent if it does not contain unnecessary vectors. That is, it does not contain a vector which is in the span of the others.

Thus we put all this together in the following important theorem.

Theorem 4.75: Linear Independence as a Linear Combination

Let $\{\vec{u}_1, \cdots, \vec{u}_k\}$ *be a collection of vectors in* $\mathbb{R}^n$. *Then the following are equivalent:*

1. *It is linearly independent, that is whenever*

$$\sum_{i=1}^{k} a_i \vec{u}_i = \vec{0}$$

   it follows that each coefficient $a_i = 0$.

2. *No vector is in the span of the others.*

3. *The system of linear equations* $A\vec{x} = 0$ *has only the trivial solution, where* A *is the* $n \times k$ *matrix having these vectors as columns.*

The last sentence of this theorem is useful as it allows us to use the reduced row-echelon form of a matrix to determine if a set of vectors is linearly independent. Let the vectors be columns of a matrix A. Find the reduced row-echelon form of A. If each column has a leading one, then it follows that the vectors are linearly independent.

Sometimes we refer to the condition regarding sums as follows: The set of vectors, $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent if and only if there is no nontrivial linear combination which equals the zero vector. A nontrivial linear combination is one in which not all the scalars equal zero. Similarly, a trivial linear combination is one in which all scalars equal zero.

Here is a detailed example in $\mathbb{R}^4$.

Example 4.76: Linear Independence

Determine whether the set of vectors given by

$$\left\{ \left[\begin{array}{c} 1 \\ 2 \\ 3 \\ 0 \end{array} \right], \left[\begin{array}{c} 2 \\ 1 \\ 0 \\ 1 \end{array} \right], \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ 2 \end{array} \right], \left[\begin{array}{c} 3 \\ 2 \\ 2 \\ 0 \end{array} \right] \right\}$$

is linearly independent. If it is linearly dependent, express one of the vectors as a linear combination of the others.

Solution. In this case the matrix of the corresponding homogeneous system of linear equations is

$$\left[\begin{array}{rrrr|r} 1 & 2 & 0 & 3 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 3 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array} \right]$$

The reduced row-echelon form is

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

and so every column is a pivot column and the corresponding system $A\vec{x} = 0$ only has the trivial solution. Therefore, these vectors are linearly independent and there is no way to obtain one of the vectors as a linear combination of the others. ♠

Consider another example.

Example 4.77: Linear Independence

Determine whether the set of vectors given by

$$\left\{ \left[\begin{array}{c} 1 \\ 2 \\ 3 \\ 0 \end{array} \right], \left[\begin{array}{c} 2 \\ 1 \\ 0 \\ 1 \end{array} \right], \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ 2 \end{array} \right], \left[\begin{array}{r} 3 \\ 2 \\ 2 \\ -1 \end{array} \right] \right\}$$

is linearly independent. If it is linearly dependent, express one of the vectors as a linear combination of the others.

Solution. Form the 4×4 matrix A having these vectors as columns:

$$A = \left[\begin{array}{rrrr} 1 & 2 & 0 & 3 \\ 2 & 1 & 1 & 2 \\ 3 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \end{array} \right]$$

Then by Theorem 4.75, the given set of vectors is linearly independent exactly if the system $A\vec{x} = 0$ has only the trivial solution.

The augmented matrix for this system and corresponding reduced row-echelon form are given by

$$\left[\begin{array}{rrrr|r} 1 & 2 & 0 & 3 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 3 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Not all the columns of the coefficient matrix are pivot columns and so the vectors are not linearly independent. In this case, we say the vectors are linearly dependent.

It follows that there are infinitely many solutions to $A\vec{x} = 0$, one of which is

$$\left[\begin{array}{r} 1 \\ 1 \\ -1 \\ -1 \end{array}\right]$$

Therefore we can write

$$1\left[\begin{array}{r} 1 \\ 2 \\ 3 \\ 0 \end{array}\right] + 1\left[\begin{array}{r} 2 \\ 1 \\ 0 \\ 1 \end{array}\right] - 1\left[\begin{array}{r} 0 \\ 1 \\ 1 \\ 2 \end{array}\right] - 1\left[\begin{array}{r} 3 \\ 2 \\ 2 \\ -1 \end{array}\right] = \left[\begin{array}{r} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

This can be rearranged as follows

$$1\left[\begin{array}{r} 1 \\ 2 \\ 3 \\ 0 \end{array}\right] + 1\left[\begin{array}{r} 2 \\ 1 \\ 0 \\ 1 \end{array}\right] - 1\left[\begin{array}{r} 0 \\ 1 \\ 1 \\ 2 \end{array}\right] = \left[\begin{array}{r} 3 \\ 2 \\ 2 \\ -1 \end{array}\right]$$

This gives the last vector as a linear combination of the first three vectors.

Notice that we could rearrange this equation to write any of the four vectors as a linear combination of the other three. ♠

When given a linearly independent set of vectors, we can determine if related sets are linearly independent.

Example 4.78: Related Sets of Vectors

Let $\{\vec{u}, \vec{v}, \vec{w}\}$ be an independent set of $\mathbb{R}^n$. Is $\{\vec{u}+\vec{v}, 2\vec{u}+\vec{w}, \vec{v}-5\vec{w}\}$ linearly independent?

Solution. Suppose $a(\vec{u}+\vec{v}) + b(2\vec{u}+\vec{w}) + c(\vec{v}-5\vec{w}) = \vec{0}_n$ for some $a, b, c \in \mathbb{R}$. Then

$$(a+2b)\vec{u} + (a+c)\vec{v} + (b-5c)\vec{w} = \vec{0}_n.$$

Since $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent,

$$a + 2b = 0$$

$$\begin{array}{rcl} a+c & = & 0 \\ b-5c & = & 0 \end{array}$$

This system of three equations in three variables has the unique solution $a=b=c=0$. Therefore, $\{\vec{u}+\vec{v}, 2\vec{u}+\vec{w}, \vec{v}-5\vec{w}\}$ is independent. ♠

The following corollary follows from the fact that if the augmented matrix of a homogeneous system of linear equations has more columns than rows, the system has infinitely many solutions.

Corollary 4.79: Linear Dependence in $\mathbb{R}^n$

Let $\{\vec{u}_1, \cdots, \vec{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. If $k>n$, then the set is linearly dependent (i.e. NOT linearly independent).

Proof. Form the $n \times k$ matrix A having the vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ as its columns and suppose $k>n$. Then A has rank $r \leq n < k$, so the system $A\vec{x}=0$ has a nontrivial solution and thus not linearly independent by Theorem 4.75. ♠

Example 4.80: Linear Dependence

Consider the vectors

$$\left\{ \left[\begin{array}{c} 1 \\ 4 \end{array} \right], \left[\begin{array}{c} 2 \\ 3 \end{array} \right], \left[\begin{array}{c} 3 \\ 2 \end{array} \right] \right\}$$

Are these vectors linearly independent?

Solution. This set contains three vectors in $\mathbb{R}^2$. By Corollary 4.79 these vectors are linearly dependent. In fact, we can write

$$(-1)\left[\begin{array}{c} 1 \\ 4 \end{array} \right] + (2)\left[\begin{array}{c} 2 \\ 3 \end{array} \right] = \left[\begin{array}{c} 3 \\ 2 \end{array} \right]$$

showing that this set is linearly dependent. ♠

The third vector in the previous example is in the span of the first two vectors. We could find a way to write this vector as a linear combination of the other two vectors. It turns out that the linear combination which we found is the **only** one, provided that the set is linearly independent.

Theorem 4.81: Unique Linear Combination

Let $U \subseteq \mathbb{R}^n$ be an independent set. Then any vector $\vec{x} \in \mathrm{span}(U)$ can be written uniquely as a linear combination of vectors of U.

Proof. To prove this theorem, we will show that two linear combinations of vectors in U that equal $\vec{x}$ must be the same. Let $U=\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$. Suppose that there is a vector $\vec{x} \in \mathrm{span}(U)$ such that

$$\begin{array}{rcl} \vec{x} & = & s_1\vec{u}_1 + s_2\vec{u}_2 + \cdots + s_k\vec{u}_k, \text{ for some } s_1, s_2, \ldots, s_k \in \mathbb{R}, \text{ and} \\ \vec{x} & = & t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots + t_k\vec{u}_k, \text{ for some } t_1, t_2, \ldots, t_k \in \mathbb{R}. \end{array}$$

Then $\vec{0}_n = \vec{x} - \vec{x} = (s_1 - t_1)\vec{u}_1 + (s_2 - t_2)\vec{u}_2 + \cdots + (s_k - t_k)\vec{u}_k$.

Since U is independent, the only linear combination that vanishes is the trivial one, so $s_i - t_i = 0$ for all i, $1 \leq i \leq k$.

Therefore, $s_i = t_i$ for all i, $1 \leq i \leq k$, and the representation is unique. Let $U \subseteq \mathbb{R}^n$ be an independent set. Then any vector $\vec{x} \in \mathrm{span}(U)$ can be written uniquely as a linear combination of vectors of U. ♠

Suppose that $\vec{u}, \vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^3$, and that $\{\vec{v}, \vec{w}\}$ is independent. Consider the set $\{\vec{u}, \vec{v}, \vec{w}\}$. When can we know that this set is independent? It turns out that this follows exactly when $\vec{u} \notin \mathrm{span}\{\vec{v}, \vec{w}\}$.

Example 4.82

Suppose that $\vec{u}, \vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^3$, and that $\{\vec{v}, \vec{w}\}$ is independent. Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent if and only if $\vec{u} \notin \mathrm{span}\{\vec{v}, \vec{w}\}$.

Solution. If $\vec{u} \in \mathrm{span}\{\vec{v}, \vec{w}\}$, then there exist $a, b \in \mathbb{R}$ so that $\vec{u} = a\vec{v} + b\vec{w}$. This implies that $\vec{u} - a\vec{v} - b\vec{w} = \vec{0}_3$, so $\vec{u} - a\vec{v} - b\vec{w}$ is a nontrivial linear combination of $\{\vec{u}, \vec{v}, \vec{w}\}$ that vanishes, and thus $\{\vec{u}, \vec{v}, \vec{w}\}$ is dependent.

Now suppose that $\vec{u} \notin \mathrm{span}\{\vec{v}, \vec{w}\}$, and suppose that there exist $a, b, c \in \mathbb{R}$ such that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}_3$. If $a \neq 0$, then $\vec{u} = -\frac{b}{a}\vec{v} - \frac{c}{a}\vec{w}$, and $\vec{u} \in \mathrm{span}\{\vec{v}, \vec{w}\}$, a contradiction. Therefore, $a = 0$, implying that $b\vec{v} + c\vec{w} = \vec{0}_3$. Since $\{\vec{v}, \vec{w}\}$ is independent, $b = c = 0$, and thus $a = b = c = 0$, i.e., the only linear combination of $\vec{u}, \vec{v}$ and $\vec{w}$ that vanishes is the trivial one.

Therefore, $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent. ♠

Consider the following useful theorem.

Theorem 4.83: Invertible Matrices

Let A be an invertible $n \times n$ matrix. Then the columns of A are independent and span $\mathbb{R}^n$. Similarly, the rows of A are independent and span the set of all $1 \times n$ vectors.

This theorem also allows us to determine if a matrix is invertible. If an $n \times n$ matrix A has columns which are independent, or span $\mathbb{R}^n$, then it follows that A is invertible. If it has rows that are independent, or span the set of all $1 \times n$ vectors, then A is invertible.

A Short Application to Chemistry

Here is a short example applying the concepts of spanning and linear independence to a question in chemistry.

When working with chemical reactions, there are sometimes a large number of reactions and some are in a sense redundant. Suppose you have the following chemical reactions.

$$\begin{gathered}
CO + \frac{1}{2}O_2 \to CO_2 \\
H_2 + \frac{1}{2}O_2 \to H_2O \\
CH_4 + \frac{3}{2}O_2 \to CO + 2H_2O \\
CH_4 + 2O_2 \to CO_2 + 2H_2O
\end{gathered}$$

There are four chemical reactions here but they are not independent reactions. There is some redundancy. What are the independent reactions? Is there a way to consider a shorter list of reactions? To analyze this situation, we can write the reactions in a matrix as follows

$$\begin{array}{c}
\begin{array}{cccccc} CO & O_2 & CO_2 & H_2 & H_2O & CH_4 \end{array} \\
\left[\begin{array}{cccccc}
1 & 1/2 & -1 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 1 & -1 & 0 \\
-1 & 3/2 & 0 & 0 & -2 & 1 \\
0 & 2 & -1 & 0 & -2 & 1
\end{array}\right]
\end{array}$$

Each row contains the coefficients of the respective elements in each reaction. For example, the top row of numbers comes from $CO + \frac{1}{2}O_2 - CO_2 = 0$ which represents the first of the chemical reactions.

We can write these coefficients in the following matrix

$$\left[\begin{array}{rrrrrr} 1 & 1/2 & -1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1 & -1 & 0 \\ -1 & 3/2 & 0 & 0 & -2 & 1 \\ 0 & 2 & -1 & 0 & -2 & 1 \end{array}\right]$$

Rather than listing all of the reactions as above, it would be more efficient to only list those which are independent by throwing out that which is redundant. We can use the concepts of the previous section to accomplish this.

First, take the reduced row-echelon form of the above matrix.

$$\left[\begin{array}{rrrrrr} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & 4 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The top three rows represent “independent" reactions which come from the original four reactions. One can obtain each of the original four rows of the matrix given above by taking a suitable linear combination of rows of this reduced row-echelon form matrix.

With the redundant reaction removed, we can consider the simplified reactions as the following equations

$$\begin{array}{c} CO+3H_2-1H_2O-1CH_4=0 \\ O_2+2H_2-2H_2O=0 \\ CO_2+4H_2-2H_2O-1CH_4=0 \end{array}$$

In terms of the original notation, these are the reactions

$$\begin{array}{c} CO+3H_2 \rightarrow H_2O+CH_4 \\ O_2+2H_2 \rightarrow 2H_2O \\ CO_2+4H_2 \rightarrow 2H_2O+CH_4 \end{array}$$

These three reactions provide an equivalent system to the original four equations. The idea is that, in terms of what happens chemically, you obtain the same information with the shorter list of reactions. Such a simplification is especially useful when dealing with very large lists of reactions which may result from experimental evidence.

4.9 Subspaces, Bases, and Dimension

Outcomes

A. *Understand the concepts of subspace, basis, and dimension.*

Suppose that S is a set of vectors. We will say that S is **closed under scalar multiplication** if, for any vector $\vec{v}$ that is an element of S and any $k \in \mathbb{R}$, the vector $k\vec{v}$ is also an element of S. Similarly, we will say that S is **closed under vector addition** if, for any vectors $\vec{u} \in S$ and $\vec{v} \in S$, it is also the case that $\vec{u}+\vec{v} \in S$.

Rather obviously each $\mathbb{R}^n$ is closed under both vector addition and scalar multiplication. More interestingly, there are some subsets of each $\mathbb{R}^n$ that are also closed under both of these operations. These special sets will be called subspaces, and examining subspaces will introduce us to the crucial ideas of a basis and the dimension of a subspace. By the end of this section, we will know exactly what it means to say that three-space is three dimensional. This is a rather dense section, but the ideas we introduce are crucial to your understanding of linear algebra.

We begin with a formal definition of what it means to say that a set of vectors is a subspace of $\mathbb{R}^n$:

Definition 4.84: Subspace

Let V be a nonempty collection of vectors in $\mathbb{R}^n$. Then V is called a subspace of $\mathbb{R}^n$ if for any scalar k and any vectors $\vec{u}$ and $\vec{v}$ in V,

- *$k\vec{u}$ is an element of V, (so V is closed under scalar multiplication) and*
- *$\vec{u}+\vec{v}$ is an element of V (and V is closed under vector addition).*

It is worth noting that if V is a subspace of $\mathbb{R}^n$, then any linear combination of vectors in V is also an element of V.

Notice that the subset $V = \left\{\vec{0}\right\}$ is a subspace of $\mathbb{R}^n$ (called the zero subspace), as is $\mathbb{R}^n$ itself. A subspace which is neither the zero subspace of $\mathbb{R}^n$ or the entire space $\mathbb{R}^n$, is referred to as a proper subspace.

A subspace is simply a set of vectors with the property that linear combinations of these vectors remain in the set. Geometrically in $\mathbb{R}^3$, it turns out that a subspace can be represented by either the origin as a single point, lines and planes which contain the origin, or the entire space $\mathbb{R}^3$.

Consider the following example of a line in $\mathbb{R}^3$.

Example 4.85: Subspace of $\mathbb{R}^3$

In $\mathbb{R}^3$, the line L through the origin that is parallel to the vector $\vec{d} = \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}$ has (vector) equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}$, $t \in \mathbb{R}$, so

$$L = \left\{ t\vec{d} \mid t \in \mathbb{R} \right\}.$$

Then L is a subspace of $\mathbb{R}^3$.

Solution. Using Definition 4.84 we can verify that L is a subspace of $\mathbb{R}^3$.

- Suppose $\vec{u} \in L$ and $k \in \mathbb{R}$ (k is a scalar). Then $\vec{u} = t\vec{d}$, for some $t \in \mathbb{R}$, so
$$k\vec{u} = k(t\vec{d}) = (kt)\vec{d}.$$
Since $kt \in \mathbb{R}$, $k\vec{u} \in L$; i.e., L is closed under scalar multiplication.
- Suppose $\vec{u}, \vec{v} \in L$. Then by definition, $\vec{u} = s\vec{d}$ and $\vec{v} = t\vec{d}$, for some $s,t \in \mathbb{R}$. Thus
$$\vec{u}+\vec{v} = s\vec{d}+t\vec{d} = (s+t)\vec{d}.$$

Since $s+t \in \mathbb{R}$, $\vec{u}+\vec{v} \in L$; i.e., L is closed under addition.

Since L satisfies both conditions of the definition, it follows that L is a subspace of $\mathbb{R}^3$. ♠

Note that there is nothing special about the vector $\vec{d}$ used in this example; the same proof works for any nonzero vector $\vec{d} \in \mathbb{R}^3$, so any line through the origin is a subspace of $\mathbb{R}^3$.

Here's an example of a subset of $\mathbb{R}^3$ that is not a subspace of $\mathbb{R}^3$:

Example 4.86: A Non-subspace of $\mathbb{R}^3$

Consider the plane P defined by the equation

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x-3 \\ y-1 \\ z-4 \end{bmatrix} = 0$$

is not a subspace of $\mathbb{R}^3$.

Solution. We must show either that P is not closed under vector addition or that P is not closed under scalar multiplication. So consider the vector

$$\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}.$$

Notice that $\vec{u} \in P$ but $0\vec{u} = \vec{0}$ is not an element of P. Thus P is not closed under scalar multiplication, and so P is not a subspace. ♠

It is worth noting that the above example shows us that any subspace of $\mathbb{R}^n$ must contain the zero vector. So if a subset doesn't contain the zero vector, it cannot be a subspace of $\mathbb{R}^n$.

More generally our definition implies that a subspace contains the span of any finite collection vectors in that subspace. It turns out that in $\mathbb{R}^n$, a subspace is exactly the span of finitely many of its vectors.

Theorem 4.87: Subspaces are Spans

Let V be a nonempty collection of vectors in $\mathbb{R}^n$. Then V is a subspace of $\mathbb{R}^n$ if and only if there exist vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in V with $k \leq n$ such that

$$V = \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$$

Furthermore, let W be another subspace of $\mathbb{R}^n$ and suppose $\{\vec{u}_1, \cdots, \vec{u}_k\} \in W$. Then it follows that V is a subset of W.

Note that since W is arbitrary, the statement that $V \subseteq W$ means that any other subspace of $\mathbb{R}^n$ that contains these vectors will also contain V.

Proof. We first show that if V is a subspace, then it can be written as $V = \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$. Pick a vector $\vec{u}_1$ in V. If $V = \text{span}\{\vec{u}_1\}$, then you have found your list of vectors and are done. If $V \neq \text{span}\{\vec{u}_1\}$, then there exists $\vec{u}_2$ a vector of V which is not in span$\{\vec{u}_1\}$. Notice that the set of vectors $\{\vec{u}_1, \vec{u}_2\}$ is linearly independent as $\vec{u}_2$ is not in span$\{\vec{u}_1\}$. Consider span$\{\vec{u}_1, \vec{u}_2\}$. If $V = $ span$\{\vec{u}_1, \vec{u}_2\}$, we are done. Otherwise, pick $\vec{u}_3$ not in span$\{\vec{u}_1, \vec{u}_2\}$. Continue this way. Note that since V is a subspace, these spans are each contained in V. The process must stop with $\vec{u}_k$ for some $k \leq n$ by Corollary 4.79, as each of the sets $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_j\}$ are linearly independend. Thus $V = $ span$\{\vec{u}_1, \cdots, \vec{u}_k\}$, as needed.

Now suppose $V = $ span$\{\vec{u}_1, \cdots, \vec{u}_k\}$, we must show this is a subspace. So let $\sum_{i=1}^{k} c_i \vec{u}_i$ and $\sum_{i=1}^{k} d_i \vec{u}_i$ be two vectors in V, and let a and b be two scalars. Then

$$a \sum_{i=1}^{k} c_i \vec{u}_i + b \sum_{i=1}^{k} d_i \vec{u}_i = \sum_{i=1}^{k} (ac_i + bd_i) \vec{u}_i$$

which is one of the vectors in span$\{\vec{u}_1, \cdots, \vec{u}_k\}$ and is therefore contained in V. This shows that span$\{\vec{u}_1, \cdots, \vec{u}_k\}$ has the properties of a subspace.

To prove that $V \subseteq W$, we prove that if $\vec{u}_i \in V$, then $\vec{u}_i \in W$.

Suppose $\vec{u} \in V$. Then $\vec{u} = a_1 \vec{u}_1 + a_2 \vec{u}_2 + \cdots + a_k \vec{u}_k$ for some $a_i \in \mathbb{R}$, $1 \leq i \leq k$. Since W contain each $\vec{u}_i$ and W is a subspace, it follows that $a_1 \vec{u}_1 + a_2 \vec{u}_2 + \cdots + a_k \vec{u}_k \in W$. ♠

Since the vectors $\vec{u}_i$ we constructed in the proof above are not in the span of the previous vectors (by definition), they must be linearly independent and thus we obtain the following corollary.

Corollary 4.88: Subspaces are Spans of Independent Sets of Vectors

If V is a subspace of $\mathbb{R}^n$, then there exist linearly independent vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in V with $k \leq n$ such that $V = $ span$\{\vec{u}_1, \cdots, \vec{u}_k\}$.

In summary, subspaces of $\mathbb{R}^n$ consist of spans of finite, linearly independent collections of vectors of $\mathbb{R}^n$. Such a collection of vectors is called a basis.

Definition 4.89: Basis of a Subspace

Let V be a subspace of $\mathbb{R}^n$. Then $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is a ***basis*** *for V if the following two conditions hold.*

1. *span$\{\vec{u}_1, \cdots, \vec{u}_k\} = V$*
2. *$\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent*

Note the plural of basis is ***bases****.*

So the short way of stating Corollary 4.88 is simply to say that every subspace of $\mathbb{R}^n$ has a basis consisting of n or fewer vectors.

The following is a simple but very useful example of a basis, called the standard basis.

Definition 4.90: Standard Basis of $\mathbb{R}^n$

Let $\vec{e}_i$ be the vector in $\mathbb{R}^n$ which has a 1 in the i^{th} entry and zeros elsewhere, that is the i^{th} column of the identity matrix. Then the collection $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$ and is called the standard basis of $\mathbb{R}^n$.

The main theorem about bases is not only they exist, but that they must be of the same size. To show this, we will need the the following fundamental result, called the Exchange Theorem, which has a proof that is technical, but mostly involves rewriting a sum using the commutative law of addition.

Theorem 4.91: Exchange Theorem

Suppose $\{\vec{u}_1,\cdots,\vec{u}_r\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, and each $\vec{u}_k$ is an element of $\text{span}\{\vec{v}_1,\cdots,\vec{v}_s\}$ Then $s \geq r$.
In words, spanning sets have at least as many vectors as linearly independent sets.

Proof. Since each $\vec{u}_j$ is in $\text{span}\{\vec{v}_1,\cdots,\vec{v}_s\}$, there exist scalars a_{ij} such that

$$\vec{u}_j = \sum_{i=1}^{s} a_{ij}\vec{v}_i$$

Suppose for a contradiction that $s < r$. Then the matrix $A = \left[a_{ij}\right]$ has fewer rows, s, than columns, r. Then the system $A\vec{x} = 0$ has a non trivial solution $\vec{d}$, that is there is a $\vec{d} \neq \vec{0}$ such that $A\vec{d} = \vec{0}$. In other words,

$$\sum_{j=1}^{r} a_{ij}d_j = 0,\ i = 1,2,\cdots,s$$

Therefore,

$$\begin{aligned}
\sum_{j=1}^{r} d_j\vec{u}_j &= \sum_{j=1}^{r} d_j \sum_{i=1}^{s} a_{ij}\vec{v}_i \\
&= \sum_{i=1}^{s}\left(\sum_{j=1}^{r} a_{ij}d_j\right)\vec{v}_i = \sum_{i=1}^{s} 0\vec{v}_i = \vec{0}
\end{aligned}$$

which contradicts the assumption that $\{\vec{u}_1,\cdots,\vec{u}_r\}$ is linearly independent, because not all the d_j are zero. Thus this contradiction indicates that $s \geq r$. ♠

We are now ready to show that any two bases are of the same size.

Theorem 4.92: Bases of a Subspace are of the Same Size

Let V be a subspace of $\mathbb{R}^n$ with two bases B_1 and B_2. Suppose B_1 contains s vectors and B_2 contains r vectors. Then $s = r$.

Proof. This follows right away from Theorem 4.91. Indeed observe that $B_1 = \{\vec{u}_1,\cdots,\vec{u}_s\}$ is a spanning set for V while $B_2 = \{\vec{v}_1,\cdots,\vec{v}_r\}$ is linearly independent, so $s \geq r$. Similarly $B_2 = \{\vec{v}_1,\cdots,\vec{v}_r\}$ is a spanning set for V while $B_1 = \{\vec{u}_1,\cdots,\vec{u}_s\}$ is linearly independent, so $r \geq s$. ♠

The following definition can now be stated.

Definition 4.93: Dimension of a Subspace

*Let V be a subspace of $\mathbb{R}^n$. Then the **dimension** of V, written $\dim(V)$ is defined to be the number of vectors in a basis.*

We immediately have

Theorem 4.94: Existence of Basis

Let V be a subspace of $\mathbb{R}^n$. Then $\dim(V) \leq n$, that is V contains a basis with at most n vectors.

Proof. By Corollary 4.88 we know that V is the span of a linearly independent set of k vectors with $k \leq n$. This set of vectors is a basis for V and thus the dimension of V is less than or equal to n. ♠

Now we can also establish that three space is three dimensional:

Corollary 4.95: Dimension of $\mathbb{R}^n$

The dimension of $\mathbb{R}^n$ is n.

Proof. You only need to exhibit a basis for $\mathbb{R}^n$ which has n vectors. Such a basis is the standard basis $\{\vec{e}_1, \cdots, \vec{e}_n\}$. ♠

Consider the following example.

Example 4.96: Basis of Subspace

Let

$$V = \left\{ \left[\begin{array}{c} a \\ b \\ c \\ d \end{array} \right] \in \mathbb{R}^4 \ : \ a - b = d - c \right\}.$$

Show that V is a subspace of $\mathbb{R}^4$, find a basis of V, and find $\dim(V)$.

Solution. The condition $a - b = d - c$ is equivalent to the condition $a = b - c + d$, so we may write

$$V = \left\{ \left[\begin{array}{c} b - c + d \\ b \\ c \\ d \end{array} \right] \ : \ b, c, d \in \mathbb{R} \right\} = \left\{ b \left[\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array} \right] + c \left[\begin{array}{c} -1 \\ 0 \\ 1 \\ 0 \end{array} \right] + d \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right] \ : \ b, c, d \in \mathbb{R} \right\}$$

This shows that V is a subspace of $\mathbb{R}^4$, since $V = \mathrm{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ where

$$\vec{u}_1 = \left[\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array} \right], \vec{u}_2 = \left[\begin{array}{c} -1 \\ 0 \\ 1 \\ 0 \end{array} \right], \vec{u}_3 = \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right]$$

Furthermore,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly independent, as can be seen by taking the reduced row-echelon form of the matrix whose columns are $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$.

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since every column of the reduced row-echelon form matrix has a leading one, the columns are linearly independent.

Therefore $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is linearly independent and spans V, so is a basis of V. Hence V has dimension three. ♠

We continue by stating further properties of a set of vectors in $\mathbb{R}^n$ relating the size of sets to their ability to span and/or be linearly independent.

Corollary 4.97: Linearly Independent and Spanning Sets in $\mathbb{R}^n$

The following properties hold in $\mathbb{R}^n$:

- *Suppose $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent. Then $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is a basis for $\mathbb{R}^n$.*
- *Suppose $\{\vec{u}_1, \cdots, \vec{u}_m\}$ spans $\mathbb{R}^n$. Then $m \geq n$.*
- *If $\{\vec{u}_1, \cdots, \vec{u}_n\}$ spans $\mathbb{R}^n$, then $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent.*

Proof. Assume first that $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent, and we need to show that this set spans $\mathbb{R}^n$. To do so, let $\vec{v}$ be a vector of $\mathbb{R}^n$, and we need to write $\vec{v}$ as a linear combination of $\vec{u}_i$'s. Consider the matrix A having the vectors $\vec{u}_i$ as columns:

$$A = \left[\begin{array}{ccc} \vec{u}_1 & \cdots & \vec{u}_n \end{array} \right]$$

By linear independence of the $\vec{u}_i$'s, the reduced row-echelon form of A is the identity matrix. Therefore the system $A\vec{x} = \vec{v}$ has a (unique) solution, so $\vec{v}$ is a linear combination of the $\vec{u}_i$'s.

To establish the second claim, suppose that $m < n$. Then letting $\vec{u}_{i_1}, \cdots, \vec{u}_{i_k}$ be the pivot columns of the matrix

$$\left[\begin{array}{ccc} \vec{u}_1 & \cdots & \vec{u}_m \end{array} \right]$$

it follows $k \leq m < n$ and these k pivot columns would be a basis for $\mathbb{R}^n$ having fewer than n vectors, contrary to Corollary 4.95.

Finally consider the third claim. If $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is not linearly independent, then replace this list with $\{\vec{u}_{i_1}, \cdots, \vec{u}_{i_k}\}$ where these are the pivot columns of the matrix

$$\left[\begin{array}{ccc} \vec{u}_1 & \cdots & \vec{u}_n \end{array} \right]$$

Then $\{\vec{u}_{i_1}, \cdots, \vec{u}_{i_k}\}$ spans $\mathbb{R}^n$ and is linearly independent, so it is a basis having fewer than n vectors again contrary to Corollary 4.95. ♠

Consider Corollary 4.97 together with Theorem 4.94. Let $\dim(V) = r$. Consider any linearly independent set of vectors chosen from V. If this set contains r vectors, then it is a basis for V. If it contains fewer than r vectors, then vectors can be added to the set to create a basis of V. Similarly, any spanning set of V which contains more than r vectors can have vectors removed to create a basis of V.

We illustrate this concept in the next example.

Example 4.98: Extending an Independent Set

Consider the set U given by

$$U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a - b = d - c \right\}$$

Then U is a subspace of $\mathbb{R}^4$ and $\dim(U) = 3$.
Then

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix} \right\},$$

is an independent subset of U. Therefore S can be extended to a basis of U.

Solution. To extend S to a basis of U, find a vector in U that is **not** in $\mathrm{span}(S)$.

$$\begin{bmatrix} 1 & 2 & ? \\ 1 & 3 & ? \\ 1 & 3 & ? \\ 1 & 2 & ? \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 3 & -1 \\ 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, S can be extended to the following basis of U:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\},$$

♠

Next we consider the case of removing vectors from a spanning set to result in a basis.

Theorem 4.99: Finding a Basis from a Span

Let W be a subspace. Also suppose that $W = span\{\vec{w}_1, \cdots, \vec{w}_m\}$. Then there exists a subset of $\{\vec{w}_1, \cdots, \vec{w}_m\}$ which is a basis for W.

Proof. Let S denote the set of positive integers such that for $k \in S$, there exists a subset of $\{\vec{w}_1, \cdots, \vec{w}_m\}$ consisting of exactly k vectors which is a spanning set for W. Thus $m \in S$. Pick the smallest positive integer in S. Call it k. Then there exists $\{\vec{u}_1, \cdots, \vec{u}_k\} \subseteq \{\vec{w}_1, \cdots, \vec{w}_m\}$ such that $\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\} = W$. We claim that $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is a linearly independent set of vectors. For suppose that

$$\sum_{i=1}^{k} c_i \vec{u}_i = \vec{0}$$

with not all of the $c_i = 0$. Then you could pick $c_j \neq 0$, divide by it and solve for $\vec{u}_j$ in terms of the others,

$$\vec{u}_j = \sum_{i \neq j} \left(-\frac{c_i}{c_j} \right) \vec{u}_i$$

Then you could delete $\vec{u}_j$ from the set and have the same span. Any linear combination involving $\vec{u}_j$ would equal one in which $\vec{u}_j$ is replaced with the above sum, showing that it could have been obtained as a linear combination of $\vec{u}_i$ for $i \neq j$. Thus $k-1 \in S$ contrary to the choice of k . Hence each $c_i = 0$ and so $\{\vec{u}_1, \cdots, \vec{u}_k\}$ both spans W and is linearly independent, making it a basis for W that is a subset of $\{\vec{w}_1, \cdots, \vec{w}_m\}$. ♠

The following example illustrates how to carry out this shrinking process to obtain a subset of a span of vectors which is linearly independent.

Example 4.100: Subset of a Span

Let W be the subspace

$$span\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 19 \\ -8 \\ 8 \end{bmatrix}, \begin{bmatrix} -6 \\ -15 \\ 6 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Find a basis for W which consists of a subset of the given vectors.

Solution. You can use the reduced row-echelon form to accomplish this reduction. Form the matrix which has the given vectors as columns.

$$\begin{bmatrix} 1 & 1 & 8 & -6 & 1 & 1 \\ 2 & 3 & 19 & -15 & 3 & 5 \\ -1 & -1 & -8 & 6 & 0 & 0 \\ 1 & 1 & 8 & -6 & 1 & 1 \end{bmatrix}$$

Then take the reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 5 & -3 & 0 & -2 \\ 0 & 1 & 3 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that columns 1, 2, and 5 are the pivot columns. It follows that a basis for W consists of the pivot columns of the original matrix:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

For example, notice in the reduced row-echelon formthat column 3 is equal to 5 times the first column plus 3 times the second column. If you look at the original matrix, the same relationship holds: the third column is equal to 5 times the first column plus 3 times the second column. In a similar fashion, you can check that our set of three vectors spans W and is linearly independent, making it a basis for W. ♠

Consider the following theorems regarding a subspace contained in another subspace.

Theorem 4.101: Subset of a Subspace

Let V and W be subspaces of $\mathbb{R}^n$, and suppose that $W \subseteq V$. Then $\dim(W) \leq \dim(V)$ with equality when $W = V$.

Theorem 4.102: Extending a Basis

Let W be any non-zero subspace of $\mathbb{R}^n$ and let $W \subseteq V$, where V is also a subspace of $\mathbb{R}^n$. Then every basis of W can be extended to a basis for V.

The proof is left as an exercise but proceeds as follows. Begin with a basis for $W, \{\vec{w}_1, \cdots, \vec{w}_s\}$ and add in vectors from V until you obtain a basis for V. Note that the process will stop because the dimension of V is no more than n.

Consider the following example.

Example 4.103: Extending a Basis

Let $V = \mathbb{R}^4$ and let

$$W = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Extend this basis of W to a basis of $\mathbb{R}^n$.

Solution. An easy way to do this is to take the reduced row-echelon form of the matrix

$$\left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{4.19}$$

Note how the given vectors were placed as the first two columns and then the matrix was extended in such a way that it is clear that the span of the columns of this matrix yield all of $\mathbb{R}^4$. Now determine the pivot columns. The reduced row-echelon form is

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right] \tag{4.20}$$

Therefore the pivot columns are

$$\left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 1 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \end{array}\right], \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array}\right]$$

and now this is an extension of the given basis for W to a basis for $\mathbb{R}^4$.

Why does this work? The columns of 4.19 obviously span $\mathbb{R}^4$. In fact the span of the first four is the same as the span of all six. ♠

Consider another example.

Example 4.104: Extending a Basis

Let W be the span of $\left\{ \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array}\right] \right\}$ *in $\mathbb{R}^4$. Let V consist of the span of the set of vectors*

$$\left\{ \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ 1 \end{array}\right], \left[\begin{array}{c} 7 \\ -6 \\ 1 \\ -6 \end{array}\right], \left[\begin{array}{c} -5 \\ 7 \\ 2 \\ 7 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}\right] \right\}$$

Find a basis for V which extends the basis for W.

Solution. Note that the above vectors are not linearly independent, but their span, denoted as V, is a subspace which does include the subspace W.

Using the process outlined in the previous example, form the following matrix

$$\left[\begin{array}{ccccc} 1 & 0 & 7 & -5 & 0 \\ 0 & 1 & -6 & 7 & 0 \\ 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & -6 & 7 & 1 \end{array}\right]$$

Next find its reduced row-echelon form

$$\left[\begin{array}{rrrrr} 1 & 0 & 7 & -5 & 0 \\ 0 & 1 & -6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

It follows that a basis for V consists of the first two vectors and the last.

$$\left\{ \left[\begin{array}{r} 1 \\ 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{r} 0 \\ 1 \\ 1 \\ 1 \end{array}\right], \left[\begin{array}{r} 0 \\ 0 \\ 0 \\ 1 \end{array}\right] \right\}$$

Thus V is of dimension 3 and it has a basis which extends the basis for W. ♠

4.10 Row Space, Column Space, and Null Space of a Matrix

Outcomes

A. *Find the row space, column space, and null space of a matrix.*

In this section we will consider an $m \times n$ matrix A and use that matrix to define certain subspaces related to that matrix. These will be very useful to us in later chapters as we consider linear transformations.

Definition 4.105: Row and Column Space

Let A be an $m \times n$ matrix. The **column space** *of A, written $col(A)$, is the span of the columns of A. Notice that $col(A)$ is a subspace of $\mathbb{R}^m$.*
The **row space** *of A, written $row(A)$, is the span of the rows of A. The row space of A is a subspace of $\mathbb{R}^n$.*

Using the reduced row-echelon form, we can obtain an efficient description of the row and column space of a matrix. Consider the following lemma.

Lemma 4.106: Effect of Row Operations on Row Space

Let A and B be $m \times n$ matrices such that A can be carried to B by elementary row $[$column$]$ operations. Then $row(A) = row(B)$ $[col(A) = col(B)]$.

Proof. We will prove that the above is true for row operations, which can be easily applied to column operations.

Let $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$ denote the rows of A.

- If B is obtained from A by a interchanging two rows of A, then A and B have exactly the same rows, so $\text{row}(B) = \text{row}(A)$.

- Suppose $p \neq 0$, and suppose that for some j, $1 \leq j \leq m$, B is obtained from A by multiplying row j by p. Then

$$\text{row}(B) = \text{span}\{\vec{r}_1, \ldots, p\vec{r}_j, \ldots, \vec{r}_m\}.$$

  Since

$$\{\vec{r}_1, \ldots, p\vec{r}_j, \ldots, \vec{r}_m\} \subseteq \text{row}(A),$$

  it follows that $\text{row}(B) \subseteq \text{row}(A)$. Conversely, since

$$\{\vec{r}_1, \ldots, \vec{r}_m\} \subseteq \text{row}(B),$$

  it follows that $\text{row}(A) \subseteq \text{row}(B)$. Therefore, $\text{row}(B) = \text{row}(A)$.

- Suppose $p \neq 0$, and suppose that for some i and j, $1 \leq i, j \leq m$, B is obtained from A by adding p time row j to row i. Without loss of generality, we may assume $i < j$.

  Then

$$\text{row}(B) = \text{span}\{\vec{r}_1, \ldots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \ldots, \vec{r}_j, \ldots, \vec{r}_m\}.$$

  Since

$$\{\vec{r}_1, \ldots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \ldots, \vec{r}_m\} \subseteq \text{row}(A),$$

it follows that $\mathrm{row}(B) \subseteq \mathrm{row}(A)$.

Conversely, since

$$\{\vec{r}_1, \ldots, \vec{r}_m\} \subseteq \mathrm{row}(B),$$

it follows that $\mathrm{row}(A) \subseteq \mathrm{row}(B)$. Therefore, $\mathrm{row}(B) = \mathrm{row}(A)$.

♠

Consider the following lemma.

Lemma 4.107: Row Space of a Row-Echelon Form Matrix

Let A be an $m \times n$ matrix and let R be its row-echelon form. Then the nonzero rows of R form a basis of $\mathrm{row}(R)$, and consequently of $\mathrm{row}(A)$.

This lemma suggests that we can examine the row-echelon form of a matrix in order to obtain the row space. Consider now the column space. The column space can be obtained by simply saying that it equals the span of all the columns. However, you can often get the column space as the span of fewer columns than this. A variation of the previous lemma provides a solution. Suppose A is row reduced to its row-echelon form R. Identify the pivot columns of R (columns which have leading ones), and take the corresponding columns of A. It turns out that this forms a basis of $\mathrm{col}(A)$.

Before proceeding to an example of this concept, we revisit the definition of rank.

Definition 4.108: Rank of a Matrix

Previously, we defined $\mathrm{rank}(A)$ to be the number of leading entries in the row-echelon form of A. Using an understanding of dimension and row space, we can now define rank as follows:

$$\mathrm{rank}(A) = \dim(\mathrm{row}(A))$$

Consider the following example.

Example 4.109: Rank, Column and Row Space

Find the rank of the following matrix and describe the column and row spaces.

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 3 & 7 & 8 & 6 & 6 \end{bmatrix}$$

Solution. The reduced row-echelon form of A is

$$R = \begin{bmatrix} 1 & 0 & -9 & 9 & 2 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the rank of A is 2, as A and R have the same row space and R's row space obviously has dimension 2.

Notice that the first two columns of R are pivot columns. By the discussion following Lemma 4.107, we find the corresponding columns of A, in this case the first two columns. Therefore a basis for $\mathrm{col}(A)$ is given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}$$

For example, consider the third column of the original matrix. It can be written as a linear combination of the first two columns of the original matrix as follows.

$$\begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix} = -9 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$$

What about an efficient description of the row space? By Lemma 4.107 we know that the nonzero rows of R create a basis of $\mathrm{row}(A)$. For the above matrix, the row space equals

$$\mathrm{row}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 & 0 & -9 & 9 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 5 & -3 & 0 \end{bmatrix} \right\}$$

♠

Notice that the column space of A is given as the span of columns of the original matrix, while the row space of A is the span of rows of the reduced row-echelon form of A.

Consider another example.

Example 4.110: Rank, Column and Row Space

Find the rank of the following matrix and describe the column and row spaces.

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 2 & 4 & 0 \end{bmatrix}$$

Solution. The reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{13}{2} \\ 0 & 1 & 0 & 2 & -\frac{5}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and so the rank is 3. The row space is given by

$$\mathrm{row}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{13}{2} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 2 & -\frac{5}{2} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -1 & \frac{1}{2} \end{bmatrix} \right\}$$

Notice that the first three columns of the reduced row-echelon form are pivot columns. The column space is the span of the first three columns in the **original matrix**,

$$\text{col}(A) = \text{span}\left\{ \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right], \left[\begin{array}{c} 2 \\ 3 \\ 2 \\ 3 \end{array}\right], \left[\begin{array}{c} 1 \\ 6 \\ 1 \\ 2 \end{array}\right] \right\}$$

♠

Consider the solution given above for Example 4.110, where the rank of A equals 3. Notice that the row space and the column space each had dimension equal to 3. It turns out that this is not a coincidence, and this essential result is referred to as the Rank Theorem and is given now. Recall that we defined $\text{rank}(A) = \dim(\text{row}(A))$.

Theorem 4.111: Rank Theorem

Let A be an $m \times n$ matrix. Then $\dim(\text{col}(A))$, the dimension of the column space, is equal to the dimension of the row space, $\dim(\text{row}(A))$.

The following statements all follow from the Rank Theorem.

Corollary 4.112: Results of the Rank Theorem

Let A be a matrix. Then the following are true:

1. $\text{rank}(A) = \text{rank}(A^T)$.
2. *For A of size $m \times n$, $\text{rank}(A) \leq m$ and $\text{rank}(A) \leq n$.*
3. *For A of size $n \times n$, A is invertible if and only if $\text{rank}(A) = n$.*
4. *For invertible matrices B and C of appropriate size, $\text{rank}(A) = \text{rank}(BA) = \text{rank}(AC)$.*

Consider the following example.

Example 4.113: Rank of the Transpose

Let

$$A = \left[\begin{array}{rr} 1 & 2 \\ -1 & 1 \end{array}\right]$$

Find $\text{rank}(A)$ and $\text{rank}(A^T)$.

Solution. To find $\text{rank}(A)$ we first row reduce to find the reduced row-echelon form.

$$A = \left[\begin{array}{rr} 1 & 2 \\ -1 & 1 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$$

Therefore the rank of A is 2. Now consider A^T given by

$$A^T = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

Again we row reduce to find the reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

You can see that $\text{rank}(A^T) = 2$, the same as $\text{rank}(A)$. ♠

We now define what is meant by the null space of a general $m \times n$ matrix.

Definition 4.114: Null Space, or Kernel, of A

Let A be an $m \times n$ matrix. The null space of A, also referred to as the kernel of A, is defined as follows.

$$null(A) = \left\{ \vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0} \right\}$$

It can also be referred to using the notation $\ker(A)$.

We will also discuss the image of A, denoted by $\text{im}(A)$. The image of A consists of the vectors of $\mathbb{R}^m$ which "get hit" by A. The formal definition is as follows.

Definition 4.115: Image of A

Let A be an $m \times n$ matrix. The image of A, written $\text{im}(A)$ is given by

$$\text{im}(A) = \{ A\vec{x} \in \mathbb{R}^m : \vec{x} \in \mathbb{R}^n \}$$

It turns out that the null space and image of A are both subspaces. Consider the following example.

Example 4.116: Null Space

Let A be an $m \times n$ matrix. Then the null space of A, $null(A)$ is a subspace of $\mathbb{R}^n$.

Solution.

- Let $\vec{x} \in \text{null}(A)$ and $k \in \mathbb{R}$. Then $A\vec{x} = \vec{0}_m$, so

  $$A(k\vec{x}) = k(A\vec{x}) = k\vec{0}_m = \vec{0}_m,$$

  and thus $k\vec{x} \in \text{null}(A)$.

- Let $\vec{x}, \vec{y} \in \text{null}(A)$. Then $A\vec{x} = \vec{0}_m$ and $A\vec{y} = \vec{0}_m$, so

  $$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}_m + \vec{0}_m = \vec{0}_m,$$

  and thus $\vec{x} + \vec{y} \in \text{null}(A)$.

Therefore by Definition 4.84, null(A) is a subspace of $\mathbb{R}^n$.

♠

The proof that im(A) is a subspace of $\mathbb{R}^m$ is similar and is left as an exercise to the reader.

We now wish to find a way to describe null(A) for a matrix A. However, finding null(A) is not new! There is just some new terminology being used, as null(A) is simply the solution to the system $A\vec{x} = \vec{0}$.

Theorem 4.117: Basis of null(A)

Let A be an $m \times n$ matrix such that rank$(A) = r$. Then the system $A\vec{x} = \vec{0}_m$ has $n - r$ basic solutions, providing a basis of null(A) with $\dim(\text{null}(A)) = n - r$.

Consider the following example.

Example 4.118: Null Space of A

Let

$$A = \left[\begin{array}{rrr} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & 3 \end{array} \right]$$

Find null(A) and im(A).

Solution. In order to find null(A), we simply need to solve the equation $A\vec{x} = \vec{0}$. This is the usual procedure of writing the augmented matrix, finding the reduced row-echelon form and then the solution. The augmented matrix and corresponding reduced row-echelon form are

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 3 & 3 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

The third column is not a pivot column, and therefore the solution will contain a parameter. The solution to the system $A\vec{x} = \vec{0}$ is given by

$$\left[\begin{array}{r} -3t \\ t \\ t \end{array} \right] : t \in \mathbb{R}$$

which can be written as

$$t \left[\begin{array}{r} -3 \\ 1 \\ 1 \end{array} \right] : t \in \mathbb{R}$$

Therefore, the null space of A is all multiples of this vector, which we can write as

$$\text{null}(A) = \text{span} \left\{ \left[\begin{array}{r} -3 \\ 1 \\ 1 \end{array} \right] \right\}$$

Finally $\text{im}(A)$ is just $\{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$ and hence consists of the span of all columns of A, that is $\text{im}(A) = \text{col}(A)$.

Notice from the above calculation that that the first two columns of the reduced row-echelon form are pivot columns. Thus the column space is the span of the first two columns in the **original matrix**, and we get

$$\text{im}(A) = \text{col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$$

♠

Here is a larger example, but the method is entirely similar.

Example 4.119: Null Space of A

Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & -1 & 1 & 3 & 0 \\ 3 & 1 & 2 & 3 & 1 \\ 4 & -2 & 2 & 6 & 0 \end{bmatrix}$$

Find the null space of A.

Solution. To find the null space, we need to solve the equation $A\vec{x} = 0$. The augmented matrix and corresponding reduced row-echelon form are given by

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 3 & 0 & 0 \\ 3 & 1 & 2 & 3 & 1 & 0 \\ 4 & -2 & 2 & 6 & 0 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrrrr|r} 1 & 0 & \frac{3}{5} & \frac{6}{5} & \frac{1}{5} & 0 \\ 0 & 1 & \frac{1}{5} & -\frac{3}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

It follows that the first two columns are pivot columns, and the next three correspond to parameters. Therefore, $\text{null}(A)$ is given by

$$\begin{bmatrix} \left(-\frac{3}{5}\right)s + \left(-\frac{6}{5}\right)t + \left(-\frac{1}{5}\right)r \\ \left(-\frac{1}{5}\right)s + \left(\frac{3}{5}\right)t + \left(-\frac{2}{5}\right)r \\ s \\ t \\ r \end{bmatrix} : s,t,r \in \mathbb{R}.$$

We write this in the form

$$s\begin{bmatrix} -\frac{3}{5} \\ -\frac{1}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{6}{5} \\ \frac{3}{5} \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -\frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 0 \\ 1 \end{bmatrix} : s,t,r \in \mathbb{R}.$$

In other words, the null space of this matrix equals the span of the three vectors above. Thus

$$\operatorname{null}(A) = \operatorname{span}\left\{ \begin{bmatrix} -\frac{3}{5} \\ -\frac{1}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{6}{5} \\ \frac{3}{5} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

♠

Notice also that the three vectors above are linearly independent and so the dimension of $\operatorname{null}(A)$ is 3. The following is true in general, the number of parameters in the solution of $A\vec{x} = 0$ equals the dimension of the null space. Recall also that the number of leading ones in the reduced row-echelon form equals the number of pivot columns, which is the rank of the matrix, which is the same as the dimension of either the column or row space.

Before we proceed to an important theorem, we first define what is meant by the nullity of a matrix.

Definition 4.120: Nullity

The dimension of the null space of a matrix is called the ***nullity****, denoted* $\dim(\operatorname{null}(A))$.

From our observation above we can now state an important theorem.

Theorem 4.121: Rank and Nullity

Let A be an $m \times n$ matrix. Then $\operatorname{rank}(A) + \dim(\operatorname{null}(A)) = n$.

Consider the following example, which we first explored above in Example 4.118

Example 4.122: Rank and Nullity

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & 3 \end{bmatrix}$$

Find $\operatorname{rank}(A)$ *and* $\dim(\operatorname{null}(A))$.

Solution. In the above Example 4.118 we determined that the reduced row-echelon form of A is given by

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the rank of A is 2. We also determined that the null space of A is given by

$$\operatorname{null}(A) = \operatorname{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Therefore the nullity of A is 1. It follows from Theorem 4.121 that $\text{rank}(A) + \dim(\text{null}(A)) = 2 + 1 = 3$, which is the number of columns of A. ♠

We conclude this section with two similar, and important, theorems.

Theorem 4.123

Let A be an $m \times n$ matrix. The following are equivalent.

1. $\text{rank}(A) = n$.
2. $\text{row}(A) = \mathbb{R}^n$, *i.e., the rows of A span $\mathbb{R}^n$.*
3. *The columns of A are independent in $\mathbb{R}^m$.*
4. *The $n \times n$ matrix $A^T A$ is invertible.*
5. *There exists an $n \times m$ matrix C so that $CA = I_n$.*
6. *If $A\vec{x} = \vec{0}_m$ for some $\vec{x} \in \mathbb{R}^n$, then $\vec{x} = \vec{0}_n$.*

Theorem 4.124

Let A be an $m \times n$ matrix. The following are equivalent.

1. $\text{rank}(A) = m$.
2. $\text{col}(A) = \mathbb{R}^m$, *i.e., the columns of A span $\mathbb{R}^m$.*
3. *The rows of A are independent in $\mathbb{R}^n$.*
4. *The $m \times m$ matrix AA^T is invertible.*
5. *There exists an $n \times m$ matrix C so that $AC = I_m$.*
6. *The system $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^m$.*

4.11 Orthogonality and the Gram Schmidt Process

Outcomes

A. *Determine if a given set of vectors is orthogonal or orthonormal.*

B. *Determine if a given matrix is orthogonal.*

C. *Given a linearly independent set of vectors, use the Gram-Schmidt Process to find orthogonal and orthonormal sets of vectors with the same span.*

Orthogonal and Orthonormal Sets

In this section, we examine what it means for vectors (and sets of vectors) to be orthogonal and orthonormal.

Recall from the properties of the dot product of vectors that two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$. Suppose a vector is orthogonal to every vector in a set that spans $\mathbb{R}^n$. What can be said about such a vector? This is the discussion in the following example.

Example 4.125: Orthogonal Vector to a Spanning Set

Let $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} \in \mathbb{R}^n$ and suppose $\mathbb{R}^n = span\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$. Furthermore, suppose that there exists a vector $\vec{u} \in \mathbb{R}^n$ for which $\vec{u} \cdot \vec{x}_j = 0$ for all j, $1 \leq j \leq k$. What type of vector is $\vec{u}$?

Solution. Write $\vec{u} = t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k$ for some $t_1, t_2, \ldots, t_k \in \mathbb{R}$ (this is possible because $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$

spans $\mathbb{R}^n$).

Then

$$\begin{aligned}
\|\vec{u}\|^2 &= \vec{u}\cdot\vec{u} \\
&= \vec{u}\cdot(t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k) \\
&= \vec{u}\cdot(t_1\vec{x}_1) + \vec{u}\cdot(t_2\vec{x}_2) + \cdots + \vec{u}\cdot(t_k\vec{x}_k) \\
&= t_1(\vec{u}\cdot\vec{x}_1) + t_2(\vec{u}\cdot\vec{x}_2) + \cdots + t_k(\vec{u}\cdot\vec{x}_k) \\
&= t_1(0) + t_2(0) + \cdots + t_k(0) = 0.
\end{aligned}$$

Since $\|\vec{u}\|^2 = 0$, $\|\vec{u}\| = 0$. We know that $\|\vec{u}\| = 0$ if and only if $\vec{u} = \vec{0}_n$. Therefore, $\vec{u} = \vec{0}_n$. In conclusion, the only vector orthogonal to every vector of a spanning set of $\mathbb{R}^n$ is the zero vector. ♠

We can now discuss what is meant by an orthogonal set of vectors.

Definition 4.126: Orthogonal Set of Vectors

*Let $\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_m\}$ be a set of vectors in $\mathbb{R}^n$. Then this set is called an **orthogonal set** if the following conditions hold:*

1. *$\vec{u}_i \cdot \vec{u}_j = 0$ for all $i \neq j$*
2. *$\vec{u}_i \neq \vec{0}$ for all i*

If we have an orthogonal set of vectors and normalize each vector so they have length 1, the resulting set is called an **orthonormal set** of vectors. They can be described as follows.

Definition 4.127: Orthonormal Set of Vectors

*A set of vectors, $\{\vec{w}_1, \cdots, \vec{w}_m\}$ is said to be an **orthonormal** set if*

$$\vec{w}_i \cdot \vec{w}_j = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Note that all orthonormal sets are orthogonal, but the reverse is not necessarily true since the vectors may not be normalized. In order to normalize the vectors, we simply need divide each one by its length.

Definition 4.128: Normalizing an Orthogonal Set

Normalizing an orthogonal set is the process of turning an orthogonal (but not orthonormal) set into an orthonormal set. If $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is an orthogonal subset of $\mathbb{R}^n$, then

$$\left\{ \frac{1}{\|\vec{u}_1\|}\vec{u}_1, \frac{1}{\|\vec{u}_2\|}\vec{u}_2, \ldots, \frac{1}{\|\vec{u}_k\|}\vec{u}_k \right\}$$

is an orthonormal set.

We illustrate this concept in the following example.

Example 4.129: Orthonormal Set

Consider the set of vectors given by

$$\{\vec{u}_1,\vec{u}_2\} = \left\{ \left[\begin{array}{r} 1 \\ 1 \end{array} \right], \left[\begin{array}{r} -1 \\ 1 \end{array} \right] \right\}$$

Show that it is an orthogonal set of vectors but not an orthonormal one. Find the corresponding orthonormal set.

Solution. One easily verifies that $\vec{u}_1 \cdot \vec{u}_2 = 0$ and $\{\vec{u}_1,\vec{u}_2\}$ is an orthogonal set of vectors. On the other hand one can compute that $\|\vec{u}_1\| = \|\vec{u}_2\| = \sqrt{2} \neq 1$ and thus it is not an orthonormal set.

Thus to find a corresponding orthonormal set, we simply need to normalize each vector. We will write $\{\vec{w}_1,\vec{w}_2\}$ for the corresponding orthonormal set. Then,

$$\begin{aligned} \vec{w}_1 &= \frac{1}{\|\vec{u}_1\|}\vec{u}_1 \\ &= \frac{1}{\sqrt{2}} \left[\begin{array}{r} 1 \\ 1 \end{array} \right] \\ &= \left[\begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array} \right] \end{aligned}$$

Similarly,

$$\begin{aligned} \vec{w}_2 &= \frac{1}{\|\vec{u}_2\|}\vec{u}_2 \\ &= \frac{1}{\sqrt{2}} \left[\begin{array}{r} -1 \\ 1 \end{array} \right] \\ &= \left[\begin{array}{r} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array} \right] \end{aligned}$$

Therefore the corresponding orthonormal set is

$$\{\vec{w}_1,\vec{w}_2\} = \left\{ \left[\begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array} \right], \left[\begin{array}{r} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array} \right] \right\}$$

You can verify that this set is orthogonal. ♠

Consider an orthogonal set of vectors in $\mathbb{R}^n$, written $\{\vec{w}_1,\cdots,\vec{w}_k\}$ with $k \leq n$. The span of these vectors is a subspace W of $\mathbb{R}^n$. If we could show that this orthogonal set is also linearly independent, we would have a basis of W. In fact, orthogonal sets of vectors are automatically linearly independent, a fact we show in the next theorem.

Theorem 4.130: Orthogonal Basis of a Subspace

Let $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$ be an orthogonal set of vectors in $\mathbb{R}^n$. Then this set is linearly independent and forms a basis for the subspace $W = span\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$.

Proof. To show that we have a linearly independent set of vectors, suppose a linear combination of these vectors equals $\vec{0}$, such as:

$$a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_k\vec{w}_k = \vec{0}, a_i \in \mathbb{R}$$

We need to show that all $a_i = 0$. To do so, take the dot product of each side of the above equation with the vector $\vec{w}_i$ and obtain the following.

$$\begin{array}{rcl}
\vec{w}_i \cdot (a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_k\vec{w}_k) & = & \vec{w}_i \cdot \vec{0} \\
a_1(\vec{w}_i \cdot \vec{w}_1) + a_2(\vec{w}_i \cdot \vec{w}_2) + \cdots + a_k(\vec{w}_i \cdot \vec{w}_k) & = & 0
\end{array}$$

Now since the set is orthogonal, $\vec{w}_i \cdot \vec{w}_m = 0$ for all $m \neq i$, so we have:

$$a_1(0) + \cdots + a_i(\vec{w}_i \cdot \vec{w}_i) + \cdots + a_k(0) = 0$$

$$a_i \|\vec{w}_i\|^2 = 0$$

Since the set is orthogonal, we know that $\|\vec{w}_i\|^2 \neq 0$. It follows that $a_i = 0$. Since the a_i was chosen arbitrarily, the set $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$ is linearly independent.

Finally since $W = \text{span}\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$, the set of vectors also spans W and therefore forms a basis of W.

♠

If an orthogonal set is a basis for a subspace, we call this an orthogonal basis. Similarly, if an orthonormal set is a basis, we call this an orthonormal basis. We already have an example of an orthonormal basis for $\mathbb{R}^n$, the standard basis $\{e_1, e_2, \ldots, e_n\}$. We will find many ways in which an arbitrary orthonormal basis is just as "nice" as the standard basis, hence our interest in finding/constructing orthonormal bases for subspaces.

We conclude this section with a discussion of Fourier expansions. Given any orthogonal basis B of $\mathbb{R}^n$ and an arbitrary vector $\vec{x} \in \mathbb{R}^n$, how do we express $\vec{x}$ as a linear combination of vectors in B? The solution is called the Fourier expansion of $\vec{x}$.

Theorem 4.131: Fourier Expansion

Let V be a subspace of $\mathbb{R}^n$ and suppose $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ is an orthogonal basis of V. Then for any $\vec{x} \in V$,

$$\vec{x} = \left(\frac{\vec{x}\cdot\vec{u}_1}{\|\vec{u}_1\|^2}\right)\vec{u}_1 + \left(\frac{\vec{x}\cdot\vec{u}_2}{\|\vec{u}_2\|^2}\right)\vec{u}_2 + \cdots + \left(\frac{\vec{x}\cdot\vec{u}_m}{\|\vec{u}_m\|^2}\right)\vec{u}_m.$$

This expression is called the Fourier expansion of $\vec{x}$, and

$$\frac{\vec{x}\cdot\vec{u}_j}{\|\vec{u}_j\|^2},$$

$j = 1, 2, \ldots, m$ are the Fourier coefficients.

If the set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ is an orthonormal *basis for V, the expression above simplifies so that*

$$\vec{x} = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\vec{u}_2 + \cdots + (\vec{x}\cdot\vec{u}_m)\vec{u}_m.$$

and the jth Fourier coefficient is simply $\vec{x}\cdot\vec{u}_j$.

Consider the following example.

Example 4.132: Fourier Expansion

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, *and* $\vec{u}_3 = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$, *and let* $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then $B = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis of $\mathbb{R}^3$.

Compute the Fourier expansion of $\vec{x}$, thus writing $\vec{x}$ as a linear combination of the vectors of B.

Solution. Since B is a basis (verify!) there is a unique way to express $\vec{x}$ as a linear combination of the vectors of B. Moreover since B is an orthogonal basis (verify!), then this can be done by computing the Fourier expansion of $\vec{x}$.

That is:

$$\vec{x} = \left(\frac{\vec{x}\cdot\vec{u}_1}{\|\vec{u}_1\|^2}\right)\vec{u}_1 + \left(\frac{\vec{x}\cdot\vec{u}_2}{\|\vec{u}_2\|^2}\right)\vec{u}_2 + \left(\frac{\vec{x}\cdot\vec{u}_3}{\|\vec{u}_3\|^2}\right)\vec{u}_3.$$

We readily compute:

$$\frac{\vec{x}\cdot\vec{u}_1}{\|\vec{u}_1\|^2} = \frac{2}{6}, \; \frac{\vec{x}\cdot\vec{u}_2}{\|\vec{u}_2\|^2} = \frac{3}{5}, \text{ and } \frac{\vec{x}\cdot\vec{u}_3}{\|\vec{u}_3\|^2} = \frac{4}{30}.$$

Therefore,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + \frac{2}{15}\begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}.$$

♠

Orthogonal Matrices

Recall that the process to find the inverse of a matrix was often cumbersome. In contrast, it was very easy to take the transpose of a matrix. Luckily for some special matrices, the transpose equals the inverse. When an $n \times n$ matrix has all real entries and its transpose equals its inverse, the matrix is called an **orthogonal matrix**.

The precise definition is as follows.

Definition 4.133: Orthogonal Matrices

*A real $n \times n$ matrix U is called an **orthogonal** matrix if $UU^T = U^T U = I$.*

Note since U is assumed to be a square matrix, it suffices to verify only one of these equalities $UU^T = I$ or $U^T U = I$ holds to guarantee that U^T is the inverse of U.

This may strike you as a rather odd definition, since our definition of orthogonal matrix does not immediately seem to have anything to do with the concept of orthogonality of vectors that we have been discussing. In fact, the ideas are closely bound, as we shall see.

First, let's try some examples just to make sure that we understand the definition of an orthogonal matrix.

Example 4.134: Orthogonal Matrix

Show the matrix

$$U = \left[\begin{array}{rr} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{array}\right]$$

is orthogonal.

Solution. All we need to do is verify (one of the equations from) the requirements of Definition 4.133.

$$UU^T = \left[\begin{array}{rr} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{array}\right] \left[\begin{array}{rr} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$$

Since $UU^T = I$, this matrix is orthogonal. ♠

Here is another example.

Example 4.135: Orthogonal Matrix

Let $U = \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{array}\right]$. *Is* U *orthogonal?*

Solution. Again the answer is yes and this can be verified simply by showing that $U^T U = I$:

$$\begin{aligned} U^T U &= \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{array}\right]^T \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{array}\right] \\ &= \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{array}\right] \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{array}\right] \\ &= \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \end{aligned}$$

♠

When we say that U is orthogonal, we are saying that $UU^T = I$, meaning that

$$\sum_j u_{ij} u^T_{jk} = \sum_j u_{ij} u_{kj} = \delta_{ik}$$

where δ_{ij} is the **Kronecker symbol** defined by

$$\delta_{ij} = \left\{ \begin{array}{c} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{array} \right.$$

In words, the product of the i^{th} row of U with the k^{th} row gives 1 if $i = k$ and 0 if $i \neq k$. The same is true of the columns because $U^T U = I$ also. Therefore,

$$\sum_j u_{ij}^T u_{jk} = \sum_j u_{ji} u_{jk} = \delta_{ik}$$

which says that the dot product of one column with another column gives 1 if the two columns are the same and 0 if the two columns are different.

More succinctly, this states that if $\vec{u}_1, \cdots, \vec{u}_n$ are the columns of U, an orthogonal matrix, then

$$\vec{u}_i \cdot \vec{u}_j = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

But this is exactly what it means to claim that the columns of U form an orthonormal set of vectors, and similarly for the rows. Thus a matrix is **orthogonal** if its rows (or columns) form an **orthonormal** set of vectors. Notice that the convention is to call such a matrix orthogonal rather than orthonormal (although this may make more sense!).

Proposition 4.136: Orthonormal Basis

The rows of an $n \times n$ orthogonal matrix form an orthonormal basis of $\mathbb{R}^n$. Further, any orthonormal basis of $\mathbb{R}^n$ can be used to construct an $n \times n$ orthogonal matrix.

Proof. Recall from Theorem 4.130 that an orthonormal set is linearly independent and forms a basis for its span. Since the rows of an $n \times n$ orthogonal matrix form an orthonormal set, they must be linearly independent. Now we have n linearly independent vectors, and it follows that their span equals $\mathbb{R}^n$. Therefore these vectors form an orthonormal basis for $\mathbb{R}^n$.

Suppose now that we have an orthonormal basis for $\mathbb{R}^n$. Since the basis will contain n vectors, these can be used to construct an $n \times n$ matrix, with each vector becoming a row. Therefore the matrix is composed of orthonormal rows, which by our above discussion, means that the matrix is orthogonal. Note we could also have construct a matrix with each vector becoming a column instead, and this would again be an orthogonal matrix. In fact this is simply the transpose of the previous matrix. ♠

Consider the following proposition.

Proposition 4.137: Determinant of Orthogonal Matrices

Suppose U is an orthogonal matrix. Then $\det(U) = \pm 1$.

Proof. This result follows from the properties of determinants. Recall that for any matrix A, $\det(A)^T = \det(A)$. Now if U is orthogonal, then:

$$(\det(U))^2 = \det\left(U^T\right)\det(U) = \det\left(U^T U\right) = \det(I) = 1$$

Therefore $(\det(U))^2 = 1$ and it follows that $\det(U) = \pm 1$. ♠

Orthogonal matrices are divided into two classes, proper and improper. The proper orthogonal matrices are those whose determinant equals 1 and the improper ones are those whose determinant equals

-1. The reason for the distinction is that the improper orthogonal matrices are sometimes considered to have no physical significance. These matrices cause a change in orientation which would correspond to material passing through itself in a non physical manner. Thus in considering which coordinate systems must be considered in certain applications, you only need to consider those which are related by a proper orthogonal transformation. Geometrically, the linear transformations determined by the proper orthogonal matrices correspond to the composition of rotations.

We conclude this section with two useful properties of orthogonal matrices.

Example 4.138: Product and Inverse of Orthogonal Matrices

Suppose A and B are orthogonal matrices. Then AB and A^{-1} both exist and are orthogonal.

Solution. First we examine the product AB.

$$(AB)(AB)^T = (AB)(B^T A^T) = A(BB^T)A^T = AA^T = I$$

Since AB is square, $(AB)^T$ is the inverse of AB, so AB is invertible, and $(AB)^{-1} = (AB)^T$ Therefore, AB is orthogonal.

Next we show that $A^{-1} = A^T$ is also orthogonal.

$$(A^{-1})^{-1} = A = (A^T)^T = (A^{-1})^T$$

Therefore A^{-1} is also orthogonal. ♠

Gram-Schmidt Process

As mentioned earlier, working with an orthonormal or orthogonal basis is often easier than working with a run-of-the-mill off-the-shelf basis for a subspace V. So it will be convenient to have a method of trading in a random set of vectors for an orthogonal or orthonormal set of vectors with the same span. This section is devoted to that process, called the Gram-Schmidt Process.

The goal of the Gram-Schmidt process is to take a linearly independent set of vectors and transform it into an orthonormal set with the same span. The first objective is to construct an orthogonal set of vectors with the same span, since from there an orthonormal set can be obtained by simply dividing each vector by its length.

Algorithm 4.139: Gram-Schmidt Process

Let $\{\vec{u}_1,\cdots,\vec{u}_n\}$ be a linearly independent set of vectors in $\mathbb{R}^n$.
I: *Construct a new set of vectors $\{\vec{v}_1,\cdots,\vec{v}_n\}$ as follows:*

$$\begin{aligned}
\vec{v}_1 &= \vec{u}_1 \\
\vec{v}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\right)\vec{v}_1 \\
\vec{v}_3 &= \vec{u}_3 - \left(\frac{\vec{u}_3\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\right)\vec{v}_1 - \left(\frac{\vec{u}_3\cdot\vec{v}_2}{\|\vec{v}_2\|^2}\right)\vec{v}_2 \\
&\vdots \\
\vec{v}_n &= \vec{u}_n - \left(\frac{\vec{u}_n\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\right)\vec{v}_1 - \left(\frac{\vec{u}_n\cdot\vec{v}_2}{\|\vec{v}_2\|^2}\right)\vec{v}_2 - \cdots - \left(\frac{\vec{u}_n\cdot\vec{v}_{n-1}}{\|\vec{v}_{n-1}\|^2}\right)\vec{v}_{n-1}
\end{aligned}$$

II: *Now let $\vec{w}_i = \dfrac{\vec{v}_i}{\|\vec{v}_i\|}$ for $i = 1,\cdots,n$.*

Then

1. *$\{\vec{v}_1,\cdots,\vec{v}_n\}$ is an orthogonal set.*
2. *$\{\vec{w}_1,\cdots,\vec{w}_n\}$ is an orthonormal set.*
3. *$\text{span}\{\vec{u}_1,\cdots,\vec{u}_n\} = \text{span}\{\vec{v}_1,\cdots,\vec{v}_n\} = \text{span}\{\vec{w}_1,\cdots,\vec{w}_n\}$.*

Proof. The full proof of this algorithm is beyond the scope of this material, however here is an indication of the argument.

To show that $\{\vec{v}_1,\cdots,\vec{v}_n\}$ is an orthogonal set, let

$$a_2 = \frac{\vec{u}_2\cdot\vec{v}_1}{\|\vec{v}_1\|^2}$$

then:

$$\begin{aligned}
\vec{v}_1\cdot\vec{v}_2 &= \vec{v}_1\cdot(\vec{u}_2 - a_2\vec{v}_1) \\
&= \vec{v}_1\cdot\vec{u}_2 - a_2(\vec{v}_1\cdot\vec{v}_1) \\
&= \vec{v}_1\cdot\vec{u}_2 - \frac{\vec{u}_2\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\|\vec{v}_1\|^2 \\
&= (\vec{v}_1\cdot\vec{u}_2) - (\vec{u}_2\cdot\vec{v}_1) = 0
\end{aligned}$$

Now that you have shown that $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal set of vectors, use the same method as above to show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is also an orthogonal set, and so on.

To show that $\operatorname{span}\{\vec{u}_1, \cdots, \vec{u}_n\} = \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$, it suffices to show that each $\vec{v}_i$ can be written as a linear combination of the $\vec{u}_j$'s and each $\vec{u}_j$ can be written as a linear combination of the $\vec{v}_i$'s.

Finally defining $\vec{w}_i = \dfrac{\vec{v}_i}{\|\vec{v}_i\|}$ for $i = 1, \cdots, n$ does not affect orthogonality and yields vectors of length 1, hence an orthonormal set. You can also observe that it does not affect the span either and the proof would be complete. ♠

Let's become familiar with the Gram Schmidt Process by working through an example.

Example 4.140: Find Orthonormal Set with Same Span

Consider the set of vectors $\{\vec{u}_1, \vec{u}_2\}$ *given as in Example 4.68. That is*

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \in \mathbb{R}^3$$

Use the Gram-Schmidt algorithm to find an orthonormal set of vectors $\{\vec{w}_1, \vec{w}_2\}$ *having the same span.*

Solution. We already remarked that the set of vectors in $\{\vec{u}_1, \vec{u}_2\}$ is linearly independent, so we can proceed with the Gram-Schmidt algorithm:

$$\begin{aligned} \vec{v}_1 &= \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ \vec{v}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \right) \vec{v}_1 \\ &= \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} \end{aligned}$$

Now to normalize simply let

$$\vec{w}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\vec{w}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \left[\begin{array}{c} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{array}\right]$$

You can verify that $\{\vec{w}_1, \vec{w}_2\}$ is an orthonormal set of vectors having the same span as $\{\vec{u}_1, \vec{u}_2\}$, namely the XY-plane. ♠

In this example, we began with a linearly independent set and found an orthonormal set of vectors which had the same span. It turns out that if we start with a basis of a subspace and apply the Gram-Schmidt algorithm, the result will be an orthogonal basis of the same subspace. We examine this in the following example.

Example 4.141: Find a Corresponding Orthogonal Basis

Let

$$\vec{x}_1 = \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array}\right], \quad \vec{x}_2 = \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 1 \end{array}\right], \text{ and } \vec{x}_3 = \left[\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array}\right],$$

and let $U = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$. Use the Gram-Schmidt Process to construct an orthogonal basis B of U.

Solution. First $\vec{f}_1 = \vec{x}_1$.

Next,

$$\vec{f}_2 = \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 1 \end{array}\right] - \frac{2}{2}\left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}\right].$$

Finally,

$$\vec{f}_3 = \left[\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array}\right] - \frac{1}{2}\left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array}\right] - \frac{0}{1}\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} 1/2 \\ 1 \\ -1/2 \\ 0 \end{array}\right].$$

Therefore,

$$\left\{ \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}\right], \left[\begin{array}{c} 1/2 \\ 1 \\ -1/2 \\ 0 \end{array}\right] \right\}$$

is an orthogonal basis of U. However, it is sometimes more convenient to deal with vectors having integer entries, in which case we take

$$B = \left\{ \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}\right], \left[\begin{array}{r} 1 \\ 2 \\ -1 \\ 0 \end{array}\right] \right\}.$$

4.12 Orthogonal Projections and Least Squares Approximations

Outcomes

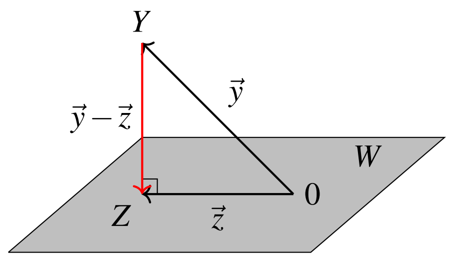
A. *Find the orthogonal projection of a vector onto a subspace.*

B. *Find the least squares approximation for a collection of points.*

An important use of the Gram-Schmidt Process is in finding the orthogonal projection of a vector onto a subspace, which is the focus of this section.

You may recall that a subspace of $\mathbb{R}^n$ is a set of vectors which contains the zero vector, and is closed under addition and scalar multiplication. Let's call such a subspace W. In particular, a hyperplane in $\mathbb{R}^n$ which contains the origin, $(0,0,\cdots,0)$, is a subspace of $\mathbb{R}^n$.

Suppose a point Y in $\mathbb{R}^n$ is not contained in W, then what point Z in W is closest to Y? Using the Gram-Schmidt Process, we can find such a point. Let $\vec{y},\vec{z}$ represent the position vectors of the points Y and Z respectively, with $\vec{y}-\vec{z}$ representing the vector connecting the two points Y and Z. It will follow that if Z is the point on W closest to Y, then $\vec{y}-\vec{z}$ will be perpendicular to W (can you see why?); in other words, $\vec{y}-\vec{z}$ is orthogonal to W (and to every vector contained in W) as in the following diagram.



The vector $\vec{z}$ is called the **orthogonal projection** of $\vec{y}$ on W. The definition is given as follows.

Definition 4.142: Orthogonal Projection

Let W be a subspace of $\mathbb{R}^n$, and Y be any point in $\mathbb{R}^n$. Then the orthogonal projection of Y onto W is given by

$$\vec{z} = proj_W(\vec{y}) = \left(\frac{\vec{y}\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\right)\vec{w}_1 + \left(\frac{\vec{y}\cdot\vec{w}_2}{\|\vec{w}_2\|^2}\right)\vec{w}_2 + \cdots + \left(\frac{\vec{y}\cdot\vec{w}_m}{\|\vec{w}_m\|^2}\right)\vec{w}_m$$

where $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_m\}$ is any orthogonal basis of W.

Therefore, in order to find the orthogonal projection, we must first find an orthogonal basis for the subspace. Note that one could use an orthonormal basis, but it is not necessary in this case since as you can see above the normalization of each vector is included in the formula for the projection.

Before we explore this further through an example, we show that the orthogonal projection does indeed yield a point Z (the point whose position vector is the vector $\vec{z}$ above) which is the point of W closest to Y.

Theorem 4.143: Approximation Theorem

Let W be a subspace of $\mathbb{R}^n$ and Y any point in $\mathbb{R}^n$. Let Z be the point whose position vector is the orthogonal projection of Y onto W.
Then, Z is the point in W closest to Y.

Proof. First Z is certainly a point in W since it is in the span of a basis of W.

To show that Z is the point in W closest to Y, we wish to show that $|\vec{y}-\vec{z}_1| > |\vec{y}-\vec{z}|$ for all $\vec{z}_1 \neq \vec{z} \in W$. We begin by writing $\vec{y}-\vec{z}_1 = (\vec{y}-\vec{z}) + (\vec{z}-\vec{z}_1)$. Now, the vector $\vec{y}-\vec{z}$ is orthogonal to W, and $\vec{z}-\vec{z}_1$ is contained in W. Therefore these vectors are orthogonal to each other. By the Pythagorean Theorem, we have that

$$\|\vec{y}-\vec{z}_1\|^2 = \|\vec{y}-\vec{z}\|^2 + \|\vec{z}-\vec{z}_1\|^2 > \|\vec{y}-\vec{z}\|^2$$

This follows because $\vec{z} \neq \vec{z}_1$ so $\|\vec{z}-\vec{z}_1\|^2 > 0$.

Hence, $\|\vec{y}-\vec{z}_1\|^2 > \|\vec{y}-\vec{z}\|^2$. Taking the square root of each side, we obtain the desired result. ♠

Consider the following example.

Example 4.144: Orthogonal Projection

Let W be the plane through the origin given by the equation $x-2y+z=0$.
Find the point in W closest to the point $Y=(1,0,3)$.

Solution. We must first find an orthogonal basis for W. Notice that W is characterized by all points (a,b,c) where $c = 2b - a$. In other words,

$$W = \left[\begin{array}{c} a \\ b \\ 2b-a \end{array}\right] = a\left[\begin{array}{r} 1 \\ 0 \\ -1 \end{array}\right] + b\left[\begin{array}{c} 0 \\ 1 \\ 2 \end{array}\right], \ a,b \in \mathbb{R}$$

We can thus write W as

$$\begin{aligned} W &= \operatorname{span}\{\vec{u}_1, \vec{u}_2\} \\ &= \operatorname{span}\left\{ \left[\begin{array}{r} 1 \\ 0 \\ -1 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 2 \end{array}\right] \right\} \end{aligned}$$

Notice that this span is a basis of W as it is linearly independent. We will use the Gram-Schmidt Process to convert this to an orthogonal basis, $\{\vec{w}_1, \vec{w}_2\}$. In this case, as we remarked it is only necessary to find an orthogonal basis, and it is not required that it be orthonormal.

$$\vec{w}_1 = \vec{u}_1 = \left[\begin{array}{r} 1 \\ 0 \\ -1 \end{array}\right]$$

$$\begin{aligned} \vec{w}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2 \cdot \vec{w}_1}{\|\vec{w}_1\|^2} \right) \vec{w}_1 \\ &= \left[\begin{array}{c} 0 \\ 1 \\ 2 \end{array}\right] - \left(\frac{-2}{2} \right) \left[\begin{array}{r} 1 \\ 0 \\ -1 \end{array}\right] \\ &= \left[\begin{array}{c} 0 \\ 1 \\ 2 \end{array}\right] + \left[\begin{array}{r} 1 \\ 0 \\ -1 \end{array}\right] \\ &= \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] \end{aligned}$$

Therefore an orthogonal basis of W is

$$\{\vec{w}_1, \vec{w}_2\} = \left\{ \left[\begin{array}{r} 1 \\ 0 \\ -1 \end{array}\right], \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] \right\}$$

We can now use this basis to find the orthogonal projection of the point $Y = (1,0,3)$ on the subspace W. We will write the position vector $\vec{y}$ of Y as $\vec{y} = \left[\begin{array}{c} 1 \\ 0 \\ 3 \end{array}\right]$. Using Definition 4.142, we compute the projection as follows:

$$\vec{z} = \operatorname{proj}_W(\vec{y})$$

$$
\begin{aligned}
&= \left(\frac{\vec{y}\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\right)\vec{w}_1 + \left(\frac{\vec{y}\cdot\vec{w}_2}{\|\vec{w}_2\|^2}\right)\vec{w}_2 \\
&= \left(\frac{-2}{2}\right)\left[\begin{array}{r} 1 \\ 0 \\ -1 \end{array}\right] + \left(\frac{4}{3}\right)\left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array}\right] \\
&= \left[\begin{array}{c} \frac{1}{3} \\ \frac{4}{3} \\ \frac{7}{3} \end{array}\right]
\end{aligned}
$$

Therefore the point Z on W closest to the point $(1,0,3)$ is $\left(\frac{1}{3},\frac{4}{3},\frac{7}{3}\right)$.

♠

Recall that the vector $\vec{y}-\vec{z}$ is perpendicular (orthogonal) to all the vectors contained in the plane W. Using a basis for W, we can in fact find all such vectors which are perpendicular to W. We call this set of vectors the **orthogonal complement** of W and denote it $W^{\perp}$.

Definition 4.145: Orthogonal Complement

Let W be a subspace of $\mathbb{R}^n$. Then the orthogonal complement of W, written $W^{\perp}$, is the set of all vectors $\vec{x}$ such that $\vec{x}\cdot\vec{z}=0$ for all vectors $\vec{z}$ in W.

$$W^{\perp} = \{\vec{x}\in\mathbb{R}^n \text{ such that } \vec{x}\cdot\vec{z}=0 \text{ for all } \vec{z}\in W\}$$

The orthogonal complement is defined as the set of all vectors which are orthogonal to all vectors in the original subspace. It turns out that it is sufficient that the vectors in the orthogonal complement be orthogonal to a spanning set of the original space.

Proposition 4.146: Orthogonal to Spanning Set

Let W be a subspace of $\mathbb{R}^n$ such that $W=\text{span}\{\vec{w}_1,\vec{w}_2,\cdots,\vec{w}_m\}$. Then $W^{\perp}$ is the set of all vectors which are orthogonal to each $\vec{w}_i$ in the spanning set.

The following proposition demonstrates that the orthogonal complement of a subspace is itself a subspace.

Proposition 4.147: The Orthogonal Complement

Let W be a subspace of $\mathbb{R}^n$. Then the orthogonal complement $W^{\perp}$ is also a subspace of $\mathbb{R}^n$.

Consider the following proposition.

Proposition 4.148: Orthogonal Complement of $\mathbb{R}^n$

The orthogonal complement of $\mathbb{R}^n$ is the set containing the zero vector:

$$(\mathbb{R}^n)^{\perp} = \left\{\vec{0}\right\}$$

Similarly,

$$\left\{\vec{0}\right\}^{\perp} = (\mathbb{R}^n)$$

Proof. Here, $\vec{0}$ is the zero vector of $\mathbb{R}^n$. Since $\vec{x}\cdot\vec{0} = 0$ for all $\vec{x} \in \mathbb{R}^n$, $\mathbb{R}^n \subseteq \{\vec{0}\}^{\perp}$. Since $\{\vec{0}\}^{\perp} \subseteq \mathbb{R}^n$, the equality follows, i.e., $\{\vec{0}\}^{\perp} = \mathbb{R}^n$.

Again, since $\vec{x}\cdot\vec{0} = 0$ for all $\vec{x} \in \mathbb{R}^n$, $\vec{0} \in (\mathbb{R}^n)^{\perp}$, so $\{\vec{0}\} \subseteq (\mathbb{R}^n)^{\perp}$. Suppose $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$. Since $\vec{x}\cdot\vec{x} = ||\vec{x}||^2$ and $\vec{x} \neq \vec{0}$, $\vec{x}\cdot\vec{x} \neq 0$, so $\vec{x} \notin (\mathbb{R}^n)^{\perp}$. Therefore $(\mathbb{R}^n)^{\perp} \subseteq \{\vec{0}\}$, and thus $(\mathbb{R}^n)^{\perp} = \{\vec{0}\}$. ♠

In the next example, we will look at how to find $W^{\perp}$.

Example 4.149: Orthogonal Complement

Let W be the plane through the origin given by the equation $x - 2y + z = 0$. Find a basis for the orthogonal complement of W.

Solution.

From Example 4.144 we know that we can write W as

$$W = \text{span}\{\vec{u}_1, \vec{u}_2\} = \text{span}\left\{ \left[\begin{array}{r} 1 \\ 0 \\ -1 \end{array}\right], \left[\begin{array}{r} 0 \\ 1 \\ 2 \end{array}\right] \right\}$$

In order to find $W^{\perp}$, we need to find all $\vec{x}$ which are orthogonal to every vector in this span.

Let $\vec{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right]$. In order to satisfy $\vec{x}\cdot\vec{u}_1 = 0$, the following equation must hold.

$$x_1 - x_3 = 0$$

In order to satisfy $\vec{x}\cdot\vec{u}_2 = 0$, the following equation must hold.

$$x_2 + 2x_3 = 0$$

Both of these equations must be satisfied, so we have the following system of equations.

$$\begin{array}{c} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \end{array}$$

To solve, set up the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

Using Gaussian Elimination, we find that $W^{\perp} = \text{span}\left\{\left[\begin{array}{r} 1 \\ -2 \\ 1 \end{array}\right]\right\}$, and hence $\left\{\left[\begin{array}{r} 1 \\ -2 \\ 1 \end{array}\right]\right\}$ is a basis for $W^{\perp}$. ♠

The following results summarize the important properties of the orthogonal projection.

Theorem 4.150: Orthogonal Projection

Let W be a subspace of $\mathbb{R}^n$, Y be any point in $\mathbb{R}^n$, and let Z be the point in W closest to Y. Then,

1. *The position vector $\vec{z}$ of the point Z is given by $\vec{z} = \text{proj}_W(\vec{y})$*
2. *$\vec{z} \in W$ and $\vec{y} - \vec{z} \in W^{\perp}$*
3. *$|Y - Z| < |Y - Z_1|$ for all $Z_1 \neq Z \in W$*

Consider the following example of this concept.

Example 4.151: Find a Vector Closest to a Given Vector

Let

$$\vec{x}_1 = \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array}\right], \quad \vec{x}_2 = \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 1 \end{array}\right], \quad \vec{x}_3 = \left[\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array}\right], \text{ and } \vec{y} = \left[\begin{array}{r} 4 \\ 3 \\ -2 \\ 5 \end{array}\right].$$

Find the vector in $W = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ closest to $\vec{y}$.

Solution. We first use the Gram-Schmidt Process to construct an orthogonal basis, B, of W. You can check that this step yields:

$$B = \left\{\left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}\right], \left[\begin{array}{r} 1 \\ 2 \\ -1 \\ 0 \end{array}\right]\right\}.$$

By Theorem 4.150,

$$\text{proj}_W(\vec{y}) = \frac{2}{2}\left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array}\right] + \frac{5}{1}\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}\right] + \frac{12}{6}\left[\begin{array}{r} 1 \\ 2 \\ -1 \\ 0 \end{array}\right] = \left[\begin{array}{r} 3 \\ 4 \\ -1 \\ 5 \end{array}\right]$$

is the vector in W closest to $\vec{y}$. ♠

Consider the next example.

Example 4.152: Vector Written as a Sum of Two Vectors

Let W be a subspace given by $W = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$, *and* $Y = (1,2,3,4)$.

Find the point Z in W closest to Y, and moreover write $\vec{y}$ as the sum of a vector in W and a vector in $W^{\perp}$.

Solution. From Theorem 4.143, the point Z in W closest to Y is given by $\vec{z} = \mathrm{proj}_W(\vec{y})$.

Notice that since the above vectors already give an orthogonal basis for W, we have:

$$\begin{aligned}
\vec{z} &= \mathrm{proj}_W(\vec{y}) \\
&= \left(\frac{\vec{y}\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\right)\vec{w}_1 + \left(\frac{\vec{y}\cdot\vec{w}_2}{\|\vec{w}_2\|^2}\right)\vec{w}_2 \\
&= \left(\frac{4}{2}\right)\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \left(\frac{10}{5}\right)\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix}
\end{aligned}$$

Therefore the point in W closest to Y is $Z = (2,2,2,4)$.

Now, we need to write $\vec{y}$ as the sum of a vector in W and a vector in $W^{\perp}$. This can easily be done as follows:

$$\vec{y} = \vec{z} + (\vec{y} - \vec{z})$$

since $\vec{z}$ is in W and as we have seen $\vec{y} - \vec{z}$ is in $W^{\perp}$.
The vector $\vec{y} - \vec{z}$ is given by

$$\vec{y} - \vec{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, we can write $\vec{y}$ as

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

♠

Example 4.153: Point in a Plane Closest to a Given Point

Find the point Z in the plane $3x+y-2z=0$ that is closest to the point $Y=(1,1,1)$.

Solution. The solution will proceed as follows.

1. Find a basis X of the subspace W of $\mathbb{R}^3$ defined by the equation $3x+y-2z=0$.
2. Orthogonalize the basis X to get an orthogonal basis B of W.
3. Find the projection on W of the position vector of the point Y.

We now begin the solution.

1. $3x+y-2z=0$ is a system of one equation in three variables. Putting the augmented matrix in reduced row-echelon form:
$$\left[\begin{array}{rrr|r} 3 & 1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & \frac{1}{3} & -\frac{2}{3} & 0 \end{array}\right]$$
gives general solution $x=-\frac{1}{3}s+\frac{2}{3}t$, $y=s$, $z=t$ for any $s,t\in\mathbb{R}$. Then
$$W=\mathrm{span}\left\{\left[\begin{array}{c} -\frac{1}{3} \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} \frac{2}{3} \\ 0 \\ 1 \end{array}\right]\right\}$$
Let $X=\left\{\left[\begin{array}{r} -1 \\ 3 \\ 0 \end{array}\right], \left[\begin{array}{r} 2 \\ 0 \\ 3 \end{array}\right]\right\}$. Then X is linearly independent and $\mathrm{span}(X)=W$, so X is a basis of W.

2. Use the Gram-Schmidt Process to get an orthogonal basis of W:
$$\vec{f}_1=\left[\begin{array}{r} -1 \\ 3 \\ 0 \end{array}\right] \text{ and } \vec{f}_2=\left[\begin{array}{r} 2 \\ 0 \\ 3 \end{array}\right]-\frac{-2}{10}\left[\begin{array}{r} -1 \\ 3 \\ 0 \end{array}\right]=\frac{1}{5}\left[\begin{array}{r} 9 \\ 3 \\ 15 \end{array}\right].$$
Therefore $B=\left\{\left[\begin{array}{r} -1 \\ 3 \\ 0 \end{array}\right], \left[\begin{array}{r} 3 \\ 1 \\ 5 \end{array}\right]\right\}$ is an orthogonal basis of W.

3. To find the point Z on W closest to $Y=(1,1,1)$, compute
$$\begin{aligned} \mathrm{proj}_W\left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] &= \frac{2}{10}\left[\begin{array}{r} -1 \\ 3 \\ 0 \end{array}\right]+\frac{9}{35}\left[\begin{array}{r} 3 \\ 1 \\ 5 \end{array}\right] \\ &= \frac{1}{7}\left[\begin{array}{r} 4 \\ 6 \\ 9 \end{array}\right]. \end{aligned}$$
Therefore, $Z=\left(\frac{4}{7},\frac{6}{7},\frac{9}{7}\right)$.

Least Squares Approximation

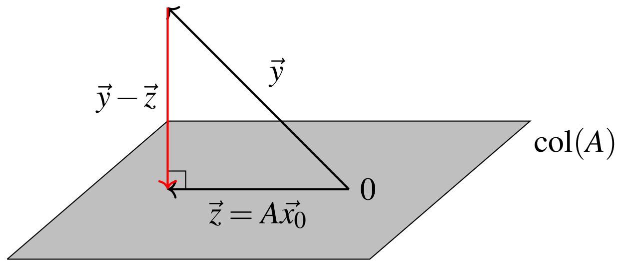
It should not be surprising to hear that many problems do not have a perfect solution, and in these cases the objective is always to try to do the best possible. This section will give us a method for finding, in at least one sense, the best possible solution.

For motivation, suppose that we are trying to find a vector $\vec{x}$ that is a solution to the equation

$$\left[\begin{array}{rr} 2 & 1 \\ -1 & 3 \\ 4 & 5 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right]=\left[\begin{array}{c} 2 \\ 1 \\ 1 \end{array}\right].$$

If you try some values for $\vec{x}$ you will start to get frustrated, so let's think about the problem differently. Every value $A\vec{x}$ is a linear combination of the columns of A, so the values that are possible for the product $A\vec{x}$ are exactly the elements of $\mathbb{R}^3$ that are in the column space of A. The column space of our A is pretty clearly 2-dimensional, as the columns of A form a linearly independent set. So the column space of A is this teeny tiny plane living in $\mathbb{R}^3$. There is some chance that our $\vec{y}$ value, $\left[\begin{array}{c} 2 \\ 1 \\ 1 \end{array}\right]$, is on that plane, but the odds are that it is not. In fact, if you row reduce the augmented matrix corresponding to our system you will see that there are no solutions to our problem, which means that $\vec{y}$ is not an element of the column space of A. But we aren't going to give up! Rather than throwing in the towel we will instead find a vector $\vec{z} \in \operatorname{col}(A)$ such that the system $A\vec{x}=\vec{z}$ *does* have a solution $\vec{x}_0$ and such that $\vec{z}=A\vec{x}_0$ is as close as possible to the vector $\vec{y}$. This solution $\vec{x}_0$ is what we will call the *least squares solution* to our original problem.

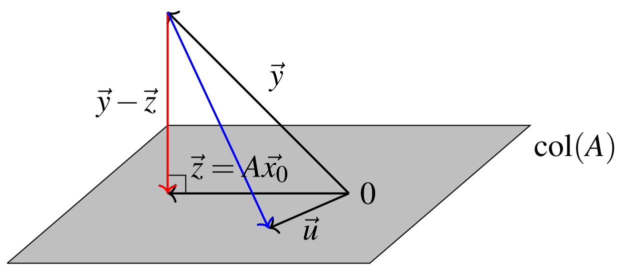
This diagram shows the situation.



This should look familiar to you: all this is saying is that we want $\vec{z}$ to be the orthogonal projection of $\vec{y}$ onto the subspace $\text{col}(A)$. In this section we will set out an algorithm that will find the least squares solution $\vec{x}_0$ and the projection $\vec{z}=A\vec{x}_0$.

We begin with a lemma.

Rephrasing Theorem 4.150 using the subspace $W=\text{col}(A)$ gives the equivalence of an orthogonality condition with a minimization condition. The following picture illustrates this orthogonality condition and geometric meaning of this theorem.



Theorem 4.154: Existence of Minimizers

Let $\vec{y}\in\mathbb{R}^m$ and let A be an $m\times n$ matrix.
Choose $\vec{z}\in W=col(A)$ given by $\vec{z}=proj_W(\vec{y})$, and let $\vec{x}_0\in\mathbb{R}^n$ be such that $\vec{z}=A\vec{x}_0$.
Then

1. $\vec{y}-A\vec{x}_0\in W^{\perp}$
2. $\|\vec{y}-A\vec{x}_0\|<\|\vec{y}-\vec{u}\|$ *for all* $\vec{u}\neq\vec{z}\in W$

We note a simple but useful observation.

Lemma 4.155: Transpose and Dot Product

Let A be an $m\times n$ matrix. Then

$$A\vec{x}\cdot\vec{y}=\vec{x}\cdot A^T\vec{y}$$

Proof. This follows from the definitions:

$$A\vec{x}\cdot\vec{y}=(A\vec{x})^T\vec{y}=\vec{x}^TA^T\vec{y}=x\cdot(A^T\vec{y}).$$

♠

The next corollary gives the technique of least squares.

Corollary 4.156: Least Squares and Normal Equation

A specific value of $\vec{x}$ which solves the problem of Theorem 4.154 is obtained by solving the equation

$$A^T A\vec{x} = A^T \vec{y} \tag{4.21}$$

Furthermore, there always exists a solution to this equation.
*The equation 4.21 is called the **normal equation** corresponding to the equation $A\vec{x} = \vec{y}y$.*

Proof. For $\vec{x}_0$ the minimizer of Theorem 4.154, $(\vec{y} - A\vec{x}_0) \cdot A\vec{u} = 0$ for all $\vec{u} \in \mathbb{R}^n$ and from Lemma 4.155, this is the same as saying

$$A^T (\vec{y} - A\vec{x}_0) \cdot \vec{u} = 0$$

for all $u \in \mathbb{R}^n$. This implies

$$A^T \vec{y} - A^T A\vec{x}_0 = \vec{0}.$$

and so

$$A^T \vec{y} = A^T A\vec{x}_0$$

Therefore, there is a solution to the equation of this corollary, and it solves the minimization problem of Theorem 4.154. ♠

Note that $\vec{x}_0$ might not be unique but $A\vec{x}$, the closest point of $A(\mathbb{R}^n)$ to $\vec{y}$ is unique as was shown in the above argument.

Consider the following example, continuing our discussion from the beginning of this subsection:

Example 4.157: Least Squares Solution to a System

Find a least squares solution to the system

$$\left[\begin{array}{rr} 2 & 1 \\ -1 & 3 \\ 4 & 5 \end{array}\right] \left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 2 \\ 1 \\ 1 \end{array}\right]$$

Solution. First, consider whether there exists a real solution. To do so, set up the augmnented matrix given by

$$\left[\begin{array}{rr|r} 2 & 1 & 2 \\ -1 & 3 & 1 \\ 4 & 5 & 1 \end{array}\right]$$

The reduced row-echelon form of this augmented matrix is

$$\left[\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

It follows that there is no real solution to this system. Therefore we wish to find the least squares solution. The normal equation is

$$\begin{aligned} A^T A\vec{x} &= A^T\vec{y} \\ \left[\begin{array}{rrr} 2 & -1 & 4 \\ 1 & 3 & 5 \end{array}\right]\left[\begin{array}{rr} 2 & 1 \\ -1 & 3 \\ 4 & 5 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] &= \left[\begin{array}{rrr} 2 & -1 & 4 \\ 1 & 3 & 5 \end{array}\right]\left[\begin{array}{c} 2 \\ 1 \\ 1 \end{array}\right] \end{aligned}$$

and so we need to solve the system

$$\left[\begin{array}{rr} 21 & 19 \\ 19 & 35 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{r} 7 \\ 10 \end{array}\right]$$

This is a familiar exercise and the solution is

$$\vec{x}_0 = \left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} \frac{5}{34} \\ \frac{7}{34} \end{array}\right]$$

♠

Consider another example.

Example 4.158: Least Squares Solution to a System

Find a least squares solution to the system

$$\left[\begin{array}{rr} 2 & 1 \\ -1 & 3 \\ 4 & 5 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 3 \\ 2 \\ 9 \end{array}\right]$$

Solution. First, consider whether there exists a real solution. To do so, set up the augmented matrix given by

$$\left[\begin{array}{rr|r} 2 & 1 & 3 \\ -1 & 3 & 2 \\ 4 & 5 & 9 \end{array}\right]$$

The reduced row-echelon form of this augmented matrix is

$$\left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

It follows that the system has a solution given by $x = y = 1$. However we can also use the normal equation and find the least squares solution.

$$\left[\begin{array}{rrr} 2 & -1 & 4 \\ 1 & 3 & 5 \end{array}\right]\left[\begin{array}{rr} 2 & 1 \\ -1 & 3 \\ 4 & 5 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{rrr} 2 & -1 & 4 \\ 1 & 3 & 5 \end{array}\right]\left[\begin{array}{c} 3 \\ 2 \\ 9 \end{array}\right]$$

Then

$$\left[\begin{array}{cc} 21 & 19 \\ 19 & 35 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right]=\left[\begin{array}{c} 40 \\ 54 \end{array}\right]$$

The least squares solution is

$$\vec{x}_0=\left[\begin{array}{c} x \\ y \end{array}\right]=\left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

which is the same as the exact solution found above. ♠

An important application of Corollary 4.156 is the problem of finding the least squares regression line in statistics. Suppose you are given points in the xy plane

$$\{(x_1,y_1),(x_2,y_2),\cdots,(x_n,y_n)\}$$

and you would like to find constants m and b such that the line $\vec{y}=m\vec{x}+b$ goes through all these points. Of course this will be impossible in general. Therefore, we try to find m,b such that the line will be as close as possible. The desired system is

$$\left[\begin{array}{cc} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{array}\right]\left[\begin{array}{c} m \\ b \end{array}\right]=\left[\begin{array}{c} y_1 \\ \vdots \\ y_n \end{array}\right]$$

which is of the form $A\vec{x}=\vec{y}$. It is desired to choose m and b to make

$$\left\| A\left[\begin{array}{c} m \\ b \end{array}\right]-\left[\begin{array}{c} y_1 \\ \vdots \\ y_n \end{array}\right]\right\|^2$$

as small as possible. According to Theorem 4.154 and Corollary 4.156, the best values for m and b occur as the solution to

$$A^TA\left[\begin{array}{c} m \\ b \end{array}\right]=A^T\left[\begin{array}{c} y_1 \\ \vdots \\ y_n \end{array}\right],\text{ where } A=\left[\begin{array}{cc} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{array}\right]$$

Thus, computing A^TA,

$$\left[\begin{array}{cc} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & n \end{array}\right]\left[\begin{array}{c} m \\ b \end{array}\right]=\left[\begin{array}{c} \sum_{i=1}^n x_iy_i \\ \sum_{i=1}^n y_i \end{array}\right]$$

Solving this system of equations for m and b (using Cramer's rule for example) yields:

$$m=\frac{-\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right)+\left(\sum_{i=1}^n x_iy_i\right)n}{\left(\sum_{i=1}^n x_i^2\right)n-\left(\sum_{i=1}^n x_i\right)^2}$$

and

$$b=\frac{-\left(\sum_{i=1}^n x_i\right)\sum_{i=1}^n x_iy_i+\left(\sum_{i=1}^n y_i\right)\sum_{i=1}^n x_i^2}{\left(\sum_{i=1}^n x_i^2\right)n-\left(\sum_{i=1}^n x_i\right)^2}.$$

Consider the following example.

Example 4.159: Least Squares Regression Line

Find the least squares regression line $\vec{y} = m\vec{x} + b$ for the following set of data points:

$$\{(0,1),(1,2),(2,2),(3,4),(4,5)\}$$

Solution. In this case we have $n = 5$ data points and we obtain:

$$\sum_{i=1}^{5} x_i = 10 \qquad \sum_{i=1}^{5} y_i = 14$$

$$\sum_{i=1}^{5} x_i y_i = 38 \quad \sum_{i=1}^{5} x_i^2 = 30$$

and hence

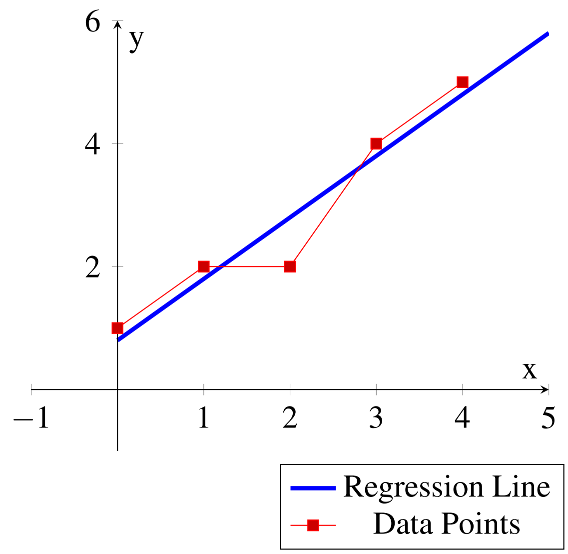
$$\begin{aligned} m &= \frac{-10*14+5*38}{5*30-10^2} = 1.00 \\ b &= \frac{-10*38+14*30}{5*30-10^2} = 0.80 \end{aligned}$$

The least squares regression line for the set of data points is:

$$\vec{y} = \vec{x} + .8$$

One could use this line to approximate other values for the data. For example for $x = 6$ one could use $y(6) = 6 + .8 = 6.8$ as an approximate value for the data.

The following diagram shows the data points and the corresponding regression line.



♠

One could clearly do a least squares fit for curves of the form $y = ax^2 + bx + c$ in the same way. In this case you want to solve as well as possible for a, b, and c the system

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

and one would use the same technique as above. Many other similar problems are important, including many in higher dimensions and they are all solved the same way.

Notice that the discussion preceding Example 4.159 provided (rather messy) formulas for m and b in the case when you want to find a least squares fit for a linear function. Those formulas are of absolutely no use if you want to fit a quadratic or a cubic. Perhaps it is better, then, to just remember to set up the matrix A for whatever degree polynomial you want to fit and then just use your linear algebra skills and solve the normal equation $A^T Ax = A^T y$ in order to find the coefficients for your least squares polynomial. Fewer disgusting formulas to memorize, and the algorithm works for polynomials of every degree.

4.13 Applications

Outcomes

A. *Apply the concepts of vectors in $\mathbb{R}^n$ to the applications of physics and work.*

Vectors and Physics

Suppose you push on something. Then, your push is made up of two components, how hard you push and the direction you push. This illustrates the concept of force.

Definition 4.160: Force

Force *is a vector. The magnitude of this vector is a measure of how hard it is pushing. It is measured in units such as Newtons or pounds or tons. The direction of this vector is the direction in which the push is taking place.*

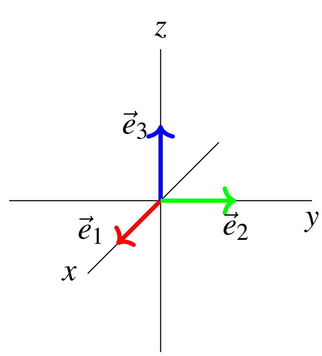
Vectors are used to model force and other physical vectors like velocity. As with all vectors, a vector modeling force has two essential ingredients, its magnitude and its direction.

Recall the special vectors which point along the coordinate axes. These are given by

$$\vec{e}_i = \left[\begin{array}{c} 0 \\ 0 \\ \vdots 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array}\right]$$

where the 1 is in the i^{th} slot and there are zeros in all the other spaces. The direction of $\vec{e}_i$ is referred to as the i^{th} direction.

Consider the following picture which illustrates the case of $\mathbb{R}^3$. Recall that in $\mathbb{R}^3$, we may refer to these vectors as $\vec{i}$, $\vec{j}$, and $\vec{k}$.



Given a vector $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, it follows that

$$\vec{u} = u_1\vec{e}_1 + \cdots + u_n\vec{e}_n = \sum_{k=1}^{n} u_i\vec{e}_i$$

What does addition of vectors mean physically? Suppose two forces are applied to some object. Each of these would be represented by a force vector and the two forces acting together would yield an overall force acting on the object which would also be a force vector known as the resultant. Suppose the two vectors are $\vec{u} = \sum_{k=1}^{n} u_i\vec{e}_i$ and $\vec{v} = \sum_{k=1}^{n} v_i\vec{e}_i$. Then the vector $\vec{u}$ involves a component in the i^{th} direction given by $u_i\vec{e}_i$, while the component in the i^{th} direction of $\vec{v}$ is $v_i\vec{e}_i$. Then the vector $\vec{u}+\vec{v}$ should have a component in the i^{th} direction equal to $(u_i + v_i)\vec{e}_i$. This is exactly what is obtained when the vectors, $\vec{u}$ and $\vec{v}$ are added.

$$\begin{aligned} \vec{u}+\vec{v} &= \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} \\ &= \sum_{i=1}^{n} (u_i+v_i)\vec{e}_i \end{aligned}$$

Thus the addition of vectors according to the rules of addition in $\mathbb{R}^n$ which were presented earlier, yields the appropriate vector which duplicates the cumulative effect of all the vectors in the sum.

Consider now some examples of vector addition.

Example 4.161: The Resultant of Three Forces

There are three ropes attached to a car and three people pull on these ropes. The first exerts a force of $\vec{F}_1 = 2\vec{i}+3\vec{j}-2\vec{k}$ Newtons, the second exerts a force of $\vec{F}_2 = 3\vec{i}+5\vec{j}+\vec{k}$ Newtons and the third exerts a force of $5\vec{i}-\vec{j}+2\vec{k}$ Newtons. Find the total force in the direction of $\vec{i}$.

Solution. To find the total force, we add the vectors as described above. This is given by

$$\begin{aligned} & (2\vec{i}+3\vec{j}-2\vec{k}) + (3\vec{i}+5\vec{j}+\vec{k}) + (5\vec{i}-\vec{j}+2\vec{k}) \\ = \ & (2+3+5)\vec{i} + (3+5+-1)\vec{j} + (-2+1+2)\vec{k} \\ = \ & 10\vec{i}+7\vec{j}+\vec{k} \end{aligned}$$

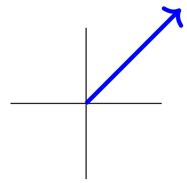
Hence, the total force is $10\vec{i}+7\vec{j}+\vec{k}$ Newtons. Therefore, the force in the $\vec{i}$ direction is 10 Newtons. ♠

Consider another example.

Example 4.162: Finding a Vector from Geometric Description

An airplane flies northeast at 100 miles per hour. Write this as a vector.

Solution. A picture of this situation follows.



Therefore, we need to find the vector $\vec{u}$ which has length 100 and direction as shown in this diagram. We can consider the vector $\vec{u}$ as the hypotenuse of a right triangle having equal sides, since the direction of $\vec{u}$ corresponds with the 45° line. The sides, corresponding to the $\vec{i}$ and $\vec{j}$ directions, should be each of length $100/\sqrt{2}$. Therefore, the vector is given by

$$\vec{u} = \frac{100}{\sqrt{2}}\vec{i} + \frac{100}{\sqrt{2}}\vec{j} = \begin{bmatrix} \frac{100}{\sqrt{2}} \\ \frac{100}{\sqrt{2}} \end{bmatrix}.$$

♠

This example also motivates the concept of **velocity**, defined below.

Definition 4.163: Speed and Velocity

The ***speed*** *of an object is a measure of how fast it is going. It is measured in units of length per unit time. For example, miles per hour, kilometers per minute, feet per second. The* ***velocity*** *is a vector having the speed as the magnitude but also specifying the direction.*

Thus the velocity vector in the above example is $\frac{100}{\sqrt{2}}\vec{i} + \frac{100}{\sqrt{2}}\vec{j}$, while the speed is 100 miles per hour.

Consider the following example.

Example 4.164: Position From Velocity and Time

The velocity of an airplane is $100\vec{i} + \vec{j} + \vec{k}$ *measured in kilometers per hour and at a certain instant of time its position is* $(1,2,1)$.
Find the position of this airplane one minute later.

Solution. Here imagine a Cartesian coordinate system in which the third component is altitude and the first and second components are measured on a line from West to East and a line from South to North.

Consider the vector $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, which is the initial position vector of the airplane. As the plane moves, the position vector changes according to the velocity vector. After one minute (considered as $\frac{1}{60}$ of an hour) the airplane has moved in the $\vec{i}$ direction a distance of $100 \times \frac{1}{60} = \frac{5}{3}$ kilometer. In the $\vec{j}$ direction it has moved $\frac{1}{60}$ kilometer during this same time, while it moves $\frac{1}{60}$ kilometer in the $\vec{k}$ direction. Therefore, the new displacement vector for the airplane is

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{5}{3} \\ \frac{1}{60} \\ \frac{1}{60} \end{bmatrix} = \begin{bmatrix} \frac{8}{3} \\ \frac{121}{60} \\ \frac{61}{60} \end{bmatrix}.$$

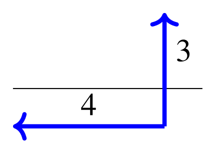
♠

Now consider an example which involves combining two velocities.

Example 4.165: Sum of Two Velocities

A certain river is one half kilometer wide with a current flowing at 4 kilometers per hour from East to West. A man swims directly toward the opposite shore from the South bank of the river at a speed of 3 kilometers per hour. How far down the river does he find himself when he has swam across? How far does he end up swimming?

Solution. Consider the following picture which demonstrates the above scenario.



First we want to know the total time of the swim across the river. The velocity in the direction across the river is 3 kilometers per hour, and the river is $\frac{1}{2}$ kilometer wide. It follows the trip takes $1/6$ hour or 10 minutes.

Now, we can compute how far downstream he will end up. Since the river runs at a rate of 4 kilometers per hour, and the trip takes $1/6$ hour, the distance traveled downstream is given by $4\left(\frac{1}{6}\right) = \frac{2}{3}$ kilometers.

The distance traveled by the swimmer is given by the hypotenuse of a right triangle. The two arms of the triangle are given by the distance across the river, $\frac{1}{2}$km, and the distance traveled downstream, $\frac{2}{3}$ km. Then, using the Pythagorean Theorem, we can calculate the total distance d traveled.

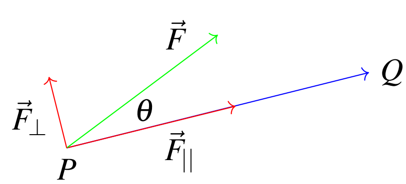
$$d = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{5}{6}\text{km}$$

Therefore, the swimmer travels a total distance of $\frac{5}{6}$ kilometers. ♠

Work

The mathematical concept of work is an application of vectors in $\mathbb{R}^n$. The physical concept of work differs from the notion of work employed in ordinary conversation. For example, suppose you were to slide a 150 pound weight off a table which is three feet high and shuffle along the floor for 50 yards, keeping the height always three feet and then deposit this weight on another three foot high table. The physical concept of work would indicate that the force exerted by your arms did no work during this project. The reason for this definition is that even though your arms exerted considerable force on the weight, the direction of motion was at right angles to the force they exerted. The only part of a force which does work in the sense of physics is the component of the force in the direction of motion.

Work is defined to be the magnitude of the component of this force times the distance over which it acts, when the component of force points in the direction of motion. In the case where the force points in exactly the opposite direction of motion work is given by (-1) times the magnitude of this component times the distance. Thus the work done by a force on an object as the object moves from one point to another is a measure of the extent to which the force contributes to the motion. This is illustrated in the following picture in the case where the given force contributes to the motion of the object from the point P to the point Q.



Recall that for any vector $\vec{u}$ in $\mathbb{R}^n$, we can write $\vec{u}$ as a sum of two vectors, as in

$$\vec{u} = \vec{u}_{||} + \vec{u}_{\perp}$$

For any force $\vec{F}$, we can write this force as the sum of a vector in the direction of the motion and a vector perpendicular to the motion. In other words,

$$\vec{F} = \vec{F}_{||} + \vec{F}_{\perp}$$

In the above picture the force, $\vec{F}$ is applied to an object which moves on the straight line from P to Q. There are two vectors shown, $\vec{F}_{||}$ and $\vec{F}_{\perp}$ and the picture is intended to indicate that when you add these two vectors you get $\vec{F}$. In other words, $\vec{F} = \vec{F}_{||} + \vec{F}_{\perp}$. Notice that $\vec{F}_{||}$ acts in the direction of motion and $\vec{F}_{\perp}$ acts perpendicular to the direction of motion. Only $\vec{F}_{||}$ contributes to the work done by $\vec{F}$ on the object as it moves from P to Q. $\vec{F}_{||}$ is called the component of the force in the direction of motion. From trigonometry, you see the magnitude of $\vec{F}_{||}$ should equal $\|\vec{F}\| \left|\cos\theta\right|$. Thus, since $\vec{F}_{||}$ points in the direction of the vector from P to Q, the total work done should equal

$$\|\vec{F}\|\|\overrightarrow{PQ}\|\cos\theta = \|\vec{F}\|\|\vec{q}-\vec{p}\|\cos\theta$$

Now, suppose the included angle had been obtuse. Then the work done by the force $\vec{F}$ on the object would have been negative because $\vec{F}_{||}$ would point in -1 times the direction of the motion. In this case, $\cos\theta$ would also be negative and so it is still the case that the work done would be given by the above formula. Thus from the geometric description of the dot product given above, the work equals

$$\|\vec{F}\|\|\vec{q}-\vec{p}\|\cos\theta = \vec{F}\cdot(\vec{q}-\vec{p})$$

This explains the following definition.

Definition 4.166: Work Done on an Object by a Force

Let $\vec{F}$ be a force acting on an object which moves from the point P to the point Q, which have position vectors given by $\vec{p}$ and $\vec{q}$ respectively. Then the **work** *done on the object by the given force equals $\vec{F}\cdot(\vec{q}-\vec{p})$.*

Consider the following example.

Example 4.167: Finding Work

Let $\vec{F} = \begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ Newtons. Find the work done by this force in moving from the point $(1,2,3)$ to the point $(-9,-3,4)$ along the straight line segment joining these points where distances are measured in meters.

Solution. First, compute the vector $\vec{q}-\vec{p}$, given by

$$\begin{bmatrix} -9 \\ -3 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -10 \\ -5 \\ 1 \end{bmatrix}.$$

According to Definition 4.166 the work done is

$$\begin{aligned}
\begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -10 \\ -5 \\ 1 \end{bmatrix} &= -20 + (-35) + (-3) \\
&= -58 \text{ Newton meters}
\end{aligned}$$

♠

Note that if the force had been given in pounds and the distance had been given in feet, the units on the work would have been foot pounds. In general, work has units equal to units of a force times units of a length. Recall that 1 Newton meter is equal to 1 Joule. Also notice that the work done by the force can be negative as in the above example.

Chapter 5

Linear Transformations

5.1 Linear Transformations

Outcomes

A. *Understand the definition of a linear transformation, and that all linear transformations are determined by matrix multiplication.*

Much of mathematics involves the study of functions, and in this chapter we are going to examine a certain class of functions, functions that behave particularly nicely. Without getting into too much detail, when we discuss a function we will always want to be aware of the *domain* of the function and the *codomain* of the function. Suppose that we are discussing a function whose name is f (always popular). Maybe f is the function returns the height of a person in centimeters. So the domain of f would be the collection of people and the codomain would be the collection of real numbers. Perhaps $f(\text{Pat}) = 152.73$ or something like that.

In most of your mathematical work to date, you have worked with functions whose domain has been $\mathbb{R}$, the collection of real numbers, and the codomain has also been the collection of real numbers. For example the cosine function is such a function.

But consider the function that adds two numbers together. This function has as its domain the collection of pairs of real numbers and has as its codomain the collection of real numbers. If we call this function g, we can explicitly define this function as follows:

$$\begin{aligned} g : \mathbb{R}^2 &\to \mathbb{R} \\ \begin{bmatrix} x \\ y \end{bmatrix} &\mapsto x + y \end{aligned}$$

You can see that we have specified the name of the function, g, the domain of the function, $\mathbb{R}^2$, the codomain of the function, $\mathbb{R}$, and the rule or formula for computing the value of the function, saying that the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ gets mapped to the real number $x + y$.

Here are some other functions with which you are familiar, written in this new, detailed style:

$$\begin{aligned} f : \text{The set of people} &\to \mathbb{R} \\ x &\mapsto \text{Height of } x \end{aligned} \qquad \begin{aligned} \text{Exp} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto e^x \end{aligned} \qquad \begin{aligned} T : \mathbb{R}^2 &\to \mathbb{R}^3 \\ \begin{bmatrix} x \\ y \end{bmatrix} &\mapsto \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

Functions of this last sort, where the domain is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$, will occupy us in this chapter. But the collection of all such functions is too vast and complicated for us in this course, so we will examine a well-behaved subset of these functions, the collection of Linear Transformations.

Matrix Multiplication and Linear Transformations

Recall that when we multiply an $m \times n$ matrix by an $n \times 1$ column vector, the result is an $m \times 1$ column vector. In this section we will discuss how, through matrix multiplication, an $m \times n$ matrix **transforms** an $n \times 1$ column vector into an $m \times 1$ column vector. This transformation is nothing more than a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$, which we will denote $T : \mathbb{R}^n \to \mathbb{R}^m$.

Consider the following example.

Example 5.1: A Function Which Transforms Vectors

Consider the matrix $A = \left[\begin{array}{ccc} 1 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right]$. *Show that left matrix multiplication by A transforms vectors in* $\mathbb{R}^3$ *into vectors in* $\mathbb{R}^2$.

Solution. First, recall that vectors in $\mathbb{R}^3$ are vectors of size 3×1, while vectors in $\mathbb{R}^2$ are of size 2×1. If we multiply A, which is a 2×3 matrix, by a 3×1 vector, the result will be a 2×1 vector. This what we mean when we say that A *transforms* vectors.

Now, for $\left[\begin{array}{c} x \\ y \\ z \end{array}\right]$ in $\mathbb{R}^3$, multiply on the left by the given matrix to obtain the new vector. This product looks like

$$\left[\begin{array}{ccc} 1 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] \left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} x+2y \\ 2x+y \end{array}\right]$$

The resulting product is a 2×1 vector which is determined by the choice of x and y. Here are some numerical examples.

$$\left[\begin{array}{ccc} 1 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] \left[\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right] = \left[\begin{array}{c} 5 \\ 4 \end{array}\right]$$

Here, the vector $\left[\begin{array}{c} 1 \\ 2 \\ 3 \end{array}\right]$ in $\mathbb{R}^3$ was transformed by the matrix into the vector $\left[\begin{array}{c} 5 \\ 4 \end{array}\right]$ in $\mathbb{R}^2$.

Here is another example:

$$\left[\begin{array}{ccc} 1 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right] \left[\begin{array}{r} 10 \\ 5 \\ -3 \end{array}\right] = \left[\begin{array}{c} 20 \\ 25 \end{array}\right]$$

♠

The idea is to define a function T_A which takes vectors in $\mathbb{R}^3$ (the domain) and delivers new vectors in $\mathbb{R}^2$ (the codomain). In this case, that function is multiplication by the matrix A, so the definition is

$$\begin{aligned} T_A : \mathbb{R}^3 &\to \mathbb{R}^2 \\ \vec{x} &\mapsto A\vec{x} \end{aligned}$$

Try to keep the function T_A separate in your mind from the matrix A. The matrix is used to define the function, but the matrix by itself is not the function—the matrix is just a rectangular array of numbers, not a function.

Notice the difference between T_A and $T_A(\vec{x})$. We know that T_A is the name of a function. But $T_A(\vec{x})$ is something different. $T(\vec{x})$ denotes the the value returned when the transformation T_A is applied to the vector $\vec{x}$. So $T_A(\vec{x})$ is a vector, not a function. You may have been sloppy about this in the past, talking about, for example, the function $\sin(x)$. But the *function* is not $\sin(x)$. Rather, $\sin(x)$ is a number, the value that the sine function returns when presented with the real number x. We will try to be careful about this notation in this text, and we hope you will be, too. It's all part of maturing as a mathematician.

The collection of functions defined by matrix multiplication in the way we have been discussing is called the collection of matrix transformations:

Definition 5.2: Matrix Transformation

A function $T:\mathbb{R}^n \to \mathbb{R}^m$ is said to be a ***matrix transformation*** *if there is an $m \times n$ matrix A such that*

$$T(\vec{x}) = A\vec{x}$$

for all $\vec{x} \in \mathbb{R}^n$.

In this case we will often say that T is the transformation determined by the matrix A.

Recall the property of matrix multiplication that states that for k and p scalars,

$$A(kB + pC) = kAB + pAC$$

In particular, for A an $m \times n$ matrix and B and C, $n \times 1$ vectors in $\mathbb{R}^n$, this formula holds.

In other words, this means that matrix transformations are examples of linear transformations, which we will now define.

Definition 5.3: Linear Transformation

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a function, where for each $\vec{x} \in \mathbb{R}^n, T(\vec{x}) \in \mathbb{R}^m$. Then T is a ***linear transformation*** *if whenever k is a scalar and $\vec{x}_1$ and $\vec{x}_2$ are vectors in $\mathbb{R}^n$ ($n \times 1$ vectors),*

1. *$T(\vec{x}_1 + \vec{x}_2) = T(\vec{x}_1) + T(\vec{x}_1)$, and*
2. *$T(k\vec{x}_1) = kT(\vec{x}_1)$*

One could amalgamate those together into a single equation, that is requiring that:

$$T(k\vec{x}_1 + k\vec{x}_2) = kT(\vec{x}_1) + kT(\vec{x}_2).$$

Clearly the two equations above imply the combined version, since

$$\begin{aligned} T(k\vec{x}_1 + k\vec{x}_2) &= T(k\vec{x}_1) + T(k\vec{x}_2) && \text{(Using the first equation with vectors } k\vec{x}_1 \text{ and } k\vec{x}_2\text{)} \\ &= kT(\vec{x}_1) + kT(\vec{x}_2) && \text{(Using the second equation twice)} \end{aligned}$$

Conversely choosing $k=1$ in the combined equation yields the first equation above, and choosing $\vec{x}_2=\vec{0}$ yields the second one.

The combined version can be useful when one wants to show that a particular function T is a linear transformation, it allows to verify a single equation instead of two. Consider the following example.

Example 5.4: Linear Transformation

Let T be a transformation defined by $T:\mathbb{R}^3\to\mathbb{R}^2$ is defined by

$$T\left[\begin{array}{c} x \\ y \\ z \end{array}\right]=\left[\begin{array}{c} x+y \\ x-z \end{array}\right] \text{ for all } \left[\begin{array}{c} x \\ y \\ z \end{array}\right]\in\mathbb{R}^3$$

Show that T is a linear transformation.

Solution. Using the combined equation, it suffices to show that $T\left(k\vec{x}_1+k\vec{x}_2\right)=kT\left(\vec{x}_1\right)+kT\left(\vec{x}_2\right)$ for all scalars k and vectors $\vec{x}_1,\vec{x}_2$. Let

$$\vec{x}_1=\left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array}\right], \quad \vec{x}_2=\left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array}\right]$$

Then

$$\begin{aligned}
T\left(k\vec{x}_1+k\vec{x}_2\right) &= T\left(k\left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array}\right]+k\left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array}\right]\right) \\
&= T\left(\left[\begin{array}{c} kx_1 \\ ky_1 \\ kz_1 \end{array}\right]+\left[\begin{array}{c} kx_2 \\ ky_2 \\ kz_2 \end{array}\right]\right) \\
&= T\left(\left[\begin{array}{c} kx_1+kx_2 \\ ky_1+ky_2 \\ kz_1+kz_2 \end{array}\right]\right) \\
&= \left[\begin{array}{c} (kx_1+kx_2)+(ky_1+ky_2) \\ (kx_1+kx_2)-(kz_1+kz_2) \end{array}\right] \\
&= \left[\begin{array}{c} (kx_1+ky_1)+(kx_2+ky_2) \\ (kx_1-kz_1)+(kx_2-kz_2) \end{array}\right] \\
&= \left[\begin{array}{c} kx_1+ky_1 \\ kx_1-kz_1 \end{array}\right]+\left[\begin{array}{c} kx_2+ky_2 \\ kx_2-kz_2 \end{array}\right] \\
&= k\left[\begin{array}{c} x_1+y_1 \\ x_1-z_1 \end{array}\right]+k\left[\begin{array}{c} x_2+y_2 \\ x_2-z_2 \end{array}\right] \\
&= kT\left(\vec{x}_1\right)+kT\left(\vec{x}_2\right)
\end{aligned}$$

Therefore T is a linear transformation. ♠

Two important examples of linear transformations are the zero transformation and identity transformation. The zero transformation defined by $T\left(\vec{x}\right)=\vec{0}$ for all $\vec{x}$ is an example of a linear transformation.

Similarly the identity transformation defined by $T(\vec{x}) = \vec{x}$ is also linear. Take the time to prove these using the method demonstrated in Example 5.4.

The argument above shows that every matrix transformation is a linear transformation:

Theorem 5.5: Matrix Transformations are Linear Transformations

Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be a function defined by $T(\vec{x}) = A\vec{x}$. Then T is a linear transformation.

It turns out that every linear transformation can be expressed as a matrix transformation, and thus linear transformations are exactly the same as matrix transformations. We will show this in the next section.

5.2 The Matrix of a Linear Transformation I

Outcomes

A. *Find the matrix of a linear transformation with respect to the standard basis.*

B. *Determine the action of a linear transformation on a vector in $\mathbb{R}^n$.*

In the examples in the last section, the action of the linear transformations was to multiply by a matrix. It turns out that this is always the case for linear transformations. If T is **any** linear transformation which maps $\mathbb{R}^n$ to $\mathbb{R}^m$, there is **always** an $m \times n$ matrix A with the property that

$$T(\vec{x}) = A\vec{x} \tag{5.1}$$

for all $\vec{x} \in \mathbb{R}^n$.

Establishing that fact is the main goal of this section.

Theorem 5.6: Matrix of a Linear Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then we can find a matrix A such that $T(\vec{x}) = A\vec{x}$. In this case, we say that T is determined *or* induced *or* represented *by the matrix A.*

We are going to establish this using the fundamental fact that the set $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$. Suppose $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation and you want to find the matrix that defines this linear transformation as described in Equation 5.1. Note that

$$\vec{x} = \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}\right] = x_1 \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array}\right] + x_2 \left[\begin{array}{c} 0 \\ 1 \\ \vdots \\ 0 \end{array}\right] + \cdots + x_n \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 1 \end{array}\right] = \sum_{i=1}^{n} x_i \vec{e}_i$$

where $\vec{e}_i$ is the i^{th} column of I_n, that is the $n \times 1$ vector which has zeros in every slot but the i^{th} and a 1 in this slot.

Then since T is linear,

$$\begin{aligned}
T(\vec{x}) &= \sum_{i=1}^{n} x_i T(\vec{e}_i) \\
&= \left[\begin{array}{ccc} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{array}\right] \left[\begin{array}{c} x_1 \\ \vdots \\ x_n \end{array}\right] \\
&= A \left[\begin{array}{c} x_1 \\ \vdots \\ x_n \end{array}\right]
\end{aligned}$$

The desired matrix is obtained from constructing the i^{th} column as $T(\vec{e}_i)$. Recall that the set $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is called the standard basis of $\mathbb{R}^n$. Therefore the matrix of T is found by applying T to the standard basis. We state this formally as the following theorem.

Theorem 5.7: Matrix of a Linear Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then the matrix A satisfying $T(\vec{x}) = A\vec{x}$ is given by

$$A = \left[\begin{array}{ccc} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{array}\right],$$

so the i^{th} column of A is the image, under the transformation T, of the i^{th} standard basis vector, $\vec{e}_i$.

We will say that the matrix A ***represents the linear transformation T with respect to the standard basis.***

Combining Theorem 5.7 with Theorem 5.5, we have the following fundamental result:

Corollary 5.8: Matrices and Linear Transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation.

Consider the following example.

Example 5.9: The Matrix of a Linear Transformation

Suppose T is a linear transformation, $T : \mathbb{R}^3 \to \mathbb{R}^2$ where

$$T\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} 1 \\ 2 \end{array}\right], T\left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right] = \left[\begin{array}{c} 9 \\ -3 \end{array}\right], T\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

Find the matrix A that represents T with respect to the standard basis.

Solution. By Theorem 5.7 we construct A as follows:

$$A = \left[\begin{array}{ccc} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{array}\right]$$

In this case, A will be a 2×3 matrix, so we need to find $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$. Luckily, we have been given these values so we can fill in A as needed, using these vectors as the columns of A. Hence,

$$A = \left[\begin{array}{rrr} 1 & 9 & 1 \\ 2 & -3 & 1 \end{array}\right]$$

♠

In this example, we were given the resulting vectors of $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$. Constructing the matrix A was simple, as we could simply use these vectors as the columns of A. The next example shows how to find A when we are not given the $T(\vec{e}_i)$ so clearly.

Example 5.10: The Matrix of Linear Transformation: Inconveniently Defined

Suppose T is a linear transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$ and

$$T\left[\begin{array}{c} 1 \\ 1 \end{array}\right] = \left[\begin{array}{c} 1 \\ 2 \end{array}\right], T\left[\begin{array}{r} 0 \\ -1 \end{array}\right] = \left[\begin{array}{c} 3 \\ 2 \end{array}\right]$$

Find the matrix A that represents T with respect to the standard basis.

Solution. By Theorem 5.7 to find this matrix, we need to determine the action of T on $\vec{e}_1$ and $\vec{e}_2$. In Example 9.90, we were given these resulting vectors. However, in this example, we have been given T

of two different vectors. How can we find out the action of T on $\vec{e}_1$ and $\vec{e}_2$? In particular for $\vec{e}_1$, suppose there exist x and y such that

$$\left[\begin{array}{c} 1 \\ 0 \end{array}\right] = x \left[\begin{array}{c} 1 \\ 1 \end{array}\right] + y \left[\begin{array}{r} 0 \\ -1 \end{array}\right] \tag{5.2}$$

Then, since T is linear,

$$T \left[\begin{array}{c} 1 \\ 0 \end{array}\right] = xT \left[\begin{array}{c} 1 \\ 1 \end{array}\right] + yT \left[\begin{array}{r} 0 \\ -1 \end{array}\right]$$

Substituting in values, this sum becomes

$$T \left[\begin{array}{c} 1 \\ 0 \end{array}\right] = x \left[\begin{array}{c} 1 \\ 2 \end{array}\right] + y \left[\begin{array}{c} 3 \\ 2 \end{array}\right] \tag{5.3}$$

Therefore, if we know the values of x and y which satisfy 5.2, we can substitute these into equation 5.3. By doing so, we find $T(\vec{e}_1)$ which is the first column of the matrix A.

We proceed to find x and y. We do so by solving 5.2, which can be done by solving the system

$$\begin{array}{c} x = 1 \\ x - y = 0 \end{array}$$

We see that $x = 1$ and $y = 1$ is the solution to this system. Substituting these values into equation 5.3, we have

$$T \left[\begin{array}{c} 1 \\ 0 \end{array}\right] = 1 \left[\begin{array}{c} 1 \\ 2 \end{array}\right] + 1 \left[\begin{array}{c} 3 \\ 2 \end{array}\right] = \left[\begin{array}{c} 1 \\ 2 \end{array}\right] + \left[\begin{array}{c} 3 \\ 2 \end{array}\right] = \left[\begin{array}{c} 4 \\ 4 \end{array}\right]$$

Therefore $\left[\begin{array}{c} 4 \\ 4 \end{array}\right]$ is the first column of A.

Computing the second column is done in the same way, and is left as an exercise.

The resulting matrix A is given by

$$A = \left[\begin{array}{cr} 4 & -3 \\ 4 & -2 \end{array}\right]$$

♠

This example illustrates a very long procedure for finding the matrix of A. While this method is reliable and will always result in the correct matrix A, the following procedure provides an alternative method.

Procedure 5.11: Finding the Matrix of Inconveniently Defined Linear Transformation

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Suppose there exist vectors $\{\vec{a}_1, \cdots, \vec{a}_n\}$ in $\mathbb{R}^n$ such that $\left[\begin{array}{ccc} \vec{a}_1 & \cdots & \vec{a}_n \end{array}\right]^{-1}$ exists, and

$$T(\vec{a}_i) = \vec{b}_i$$

Then the matrix that represents T with respect to the standard basis is

$$\left[\begin{array}{ccc} \vec{b}_1 & \cdots & \vec{b}_n \end{array}\right] \left[\begin{array}{ccc} \vec{a}_1 & \cdots & \vec{a}_n \end{array}\right]^{-1}$$

We will illustrate this procedure in the following example. You may also find it useful to work through Example 5.10 using this procedure.

Example 5.12: Matrix of a Linear Transformation Given Inconveniently

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and

$$T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the matrix of this linear transformation with respect to the standard basis.

Solution. By Procedure 5.11, $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1}$ and $B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

Then, Procedure 5.11 claims that the matrix of T is

$$C = BA^{-1} = \begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix}$$

Indeed you can first verify that $T(\vec{x}) = C\vec{x}$ for the 3 vectors above:

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

But more generally $T(\vec{x}) = C\vec{x}$ for any $\vec{x}$. To see this, let $\vec{y} = A^{-1}\vec{x}$ and then using linearity of T:

$$T(\vec{x}) = T(A\vec{y}) = T\left(\sum_i \vec{y}_i\vec{a}_i\right) = \sum \vec{y}_i T(\vec{a}_i) \sum \vec{y}_i\vec{b}_i = B\vec{y} = BA^{-1}\vec{x} = C\vec{x}$$

♠

Recall the dot product discussed earlier. Fix a vector $\vec{u} \in \mathbb{R}^n$ and consider the function $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\vec{v}) = \mathrm{proj}_{\vec{u}}(\vec{v})$ which takes a vector and maps it to its projection onto $\vec{u}$. It turns out that this function is a linear transformation, a result which follows from the properties of the dot product. This is shown as follows.

$$\mathrm{proj}_{\vec{u}}(k\vec{v} + p\vec{w}) = \left(\frac{(k\vec{v} + p\vec{w}) \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right)\vec{u}$$

$$
\begin{aligned}
&= k\left(\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}+p\left(\frac{\vec{w}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u} \\
&= k\ \mathrm{proj}_{\vec{u}}(\vec{v})+p\ \mathrm{proj}_{\vec{u}}(\vec{w})
\end{aligned}
$$

Consider the following example.

Example 5.13: Matrix of a Projection Map

Let $\vec{u}=\begin{bmatrix}1\\2\\3\end{bmatrix}$ *and let* T *be the projection map* $T:\mathbb{R}^3\mapsto\mathbb{R}^3$ *defined by*

$$T(\vec{v})=\mathrm{proj}_{\vec{u}}(\vec{v})$$

for any $\vec{v}\in\mathbb{R}^3$.

1. *Does this transformation come from multiplication by a matrix?*
2. *If so, what is the matrix?*

Solution.

1. First, we have just seen that $T(\vec{v})=\mathrm{proj}_{\vec{u}}(\vec{v})$ is linear. Therefore by Theorem 5.6, we can find a matrix A such that $T(\vec{x})=A\vec{x}$.

2. The columns of the matrix for T are defined above as $T(\vec{e}_i)$. It follows that $T(\vec{e}_i)=\mathrm{proj}_{\vec{u}}(\vec{e}_i)$ gives the i^{th} column of the desired matrix. Therefore, we need to find

$$\mathrm{proj}_{\vec{u}}(\vec{e}_i)=\left(\frac{\vec{e}_i\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$$

For the given vector $\vec{u}$, this implies the columns of the desired matrix are

$$\frac{1}{14}\begin{bmatrix}1\\2\\3\end{bmatrix},\ \frac{2}{14}\begin{bmatrix}1\\2\\3\end{bmatrix},\ \frac{3}{14}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

which you can verify. Hence the matrix that represents T relative to the standard basis is

$$\frac{1}{14}\begin{bmatrix}1&2&3\\2&4&6\\3&6&9\end{bmatrix}$$

♠

5.3 Properties of Linear Transformations

Outcomes

A. *Use properties of linear transformations to solve problems.*

B. *Find the composite of transformations and the inverse of a transformation.*

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there are some important properties of T which will be examined in this section. Consider the following theorem.

Theorem 5.14: Properties of Linear Transformations

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation and let $\vec{x} \in \mathbb{R}^n$.

- *T preserves the zero vector.*

$$T(0\vec{x}) = 0T(\vec{x}). \text{ Hence } T(\vec{0}) = \vec{0}$$

- *T preserves the negative of a vector:*

$$T((-1)\vec{x}) = (-1)T(\vec{x}). \text{ Hence } T(-\vec{x}) = -T(\vec{x}).$$

- *T preserves linear combinations:*

$$\text{Let } \vec{x}_1, ..., \vec{x}_k \in \mathbb{R}^n \text{ and } a_1, ..., a_k \in \mathbb{R}.$$

$$\text{Then if } \vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + ... + a_k\vec{x}_k, \text{ it follows that}$$

$$T(\vec{y}) = T(a_1\vec{x}_1 + a_2\vec{x}_2 + ... + a_k\vec{x}_k) = a_1T(\vec{x}_1) + a_2T(\vec{x}_2) + ... + a_kT(\vec{x}_k).$$

These properties are useful in determining the action of a transformation on a given vector. Consider the following example.

Example 5.15: Linear Combination

Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^4$ be a linear transformation such that

$$T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix}, T\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix}$$

Find $T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$.

Solution. Using the third property in Theorem 9.54, we can find $T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$ by writing $\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$.

Therefore we want to find $a, b \in \mathbb{R}$ such that

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$

The necessary augmented matrix and resulting reduced row-echelon form are given by:

$$\left[\begin{array}{rr|r} 1 & 4 & -7 \\ 3 & 0 & 3 \\ 1 & 5 & -9 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $a = 1, b = -2$ and

$$\left[\begin{array}{r} -7 \\ 3 \\ -9 \end{array}\right] = 1 \left[\begin{array}{r} 1 \\ 3 \\ 1 \end{array}\right] + (-2) \left[\begin{array}{r} 4 \\ 0 \\ 5 \end{array}\right]$$

Now, using the third property above, we have

$$\begin{aligned} T \left[\begin{array}{r} -7 \\ 3 \\ -9 \end{array}\right] &= T \left(1 \left[\begin{array}{r} 1 \\ 3 \\ 1 \end{array}\right] + (-2) \left[\begin{array}{r} 4 \\ 0 \\ 5 \end{array}\right] \right) \\ &= 1T \left[\begin{array}{r} 1 \\ 3 \\ 1 \end{array}\right] - 2T \left[\begin{array}{r} 4 \\ 0 \\ 5 \end{array}\right] \\ &= \left[\begin{array}{r} 4 \\ 4 \\ 0 \\ -2 \end{array}\right] - 2 \left[\begin{array}{r} 4 \\ 5 \\ -1 \\ 5 \end{array}\right] \\ &= \left[\begin{array}{r} -4 \\ -6 \\ 2 \\ -12 \end{array}\right] \end{aligned}$$

Therefore, $T \left[\begin{array}{r} -7 \\ 3 \\ -9 \end{array}\right] = \left[\begin{array}{r} -4 \\ -6 \\ 2 \\ -12 \end{array}\right]$. ♠

Suppose two linear transformations act in the same way on $\vec{x}$ for all vectors. Then we say that these transformations are equal.

Definition 5.16: Equal Transformations

Let S and T be linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then $S = T$ if and only if for every $\vec{x} \in \mathbb{R}^n$,

$$S(\vec{x}) = T(\vec{x})$$

Suppose two linear transformations act on the same vector $\vec{x}$, first the transformation T and then a second transformation given by S. We can find the **composite** transformation that results from applying both transformations.

Definition 5.17: Composition of Linear Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations. Then the ***composite*** *of S and T is*

$$S \circ T : \mathbb{R}^k \mapsto \mathbb{R}^m$$

The action of $S \circ T$ is given by

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) \text{ for all } \vec{x} \in \mathbb{R}^k$$

Notice that the resulting vector will be in $\mathbb{R}^m$. Be careful to observe the order of transformations. We write $S \circ T$ but apply the transformation T first, followed by S.

Theorem 5.18: Composition of Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations such that T is induced by the matrix A and S is induced by the matrix B. Then $S \circ T$ is a linear transformation which is induced by the matrix BA.

Consider the following example.

Example 5.19: Composition of Transformations

Let T be a linear transformation induced by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

and S a linear transformation induced by the matrix

$$B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

Find the matrix of the composite transformation $S \circ T$. Then, find $(S \circ T)(\vec{x})$ for $\vec{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Solution. By Theorem 5.18, the matrix of $S \circ T$ is given by BA.

$$BA = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 2 & 0 \end{bmatrix}$$

To find $(S \circ T)(\vec{x})$, multiply $\vec{x}$ by BA as follows

$$\begin{bmatrix} 8 & 4 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \end{bmatrix}$$

To check, first determine $T(\vec{x})$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$$

Then, compute $S(T(\vec{x}))$ as follows:

$$\left[\begin{array}{cc} 2 & 3 \\ 0 & 1 \end{array}\right]\left[\begin{array}{c} 9 \\ 2 \end{array}\right]=\left[\begin{array}{c} 24 \\ 2 \end{array}\right]$$

♠

Consider a composite transformation $S \circ T$, and suppose that this transformation acted such that $(S \circ T)(\vec{x})=\vec{x}$. That is, the transformation S took the vector $T(\vec{x})$ and returned it to $\vec{x}$. In this case, S and T are inverses of each other. Consider the following definition.

Definition 5.20: Inverse of a Transformation

Let $T: \mathbb{R}^n \mapsto \mathbb{R}^n$ and $S: \mathbb{R}^n \mapsto \mathbb{R}^n$ be linear transformations. Suppose that for each $\vec{x} \in \mathbb{R}^n$,

$$(S \circ T)(\vec{x})=\vec{x}$$

and

$$(T \circ S)(\vec{x})=\vec{x}$$

Then, S is called an inverse of T and T is called an inverse of S. Geometrically, they reverse the action of each other.

The following theorem is crucial, as it claims that the above inverse transformations are unique.

Theorem 5.21: Inverse of a Transformation

Let $T: \mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear transformation induced by the matrix A. Then T has an inverse transformation if and only if the matrix A is invertible. In this case, the inverse transformation is unique and denoted $T^{-1}: \mathbb{R}^n \mapsto \mathbb{R}^n$. T^{-1} is induced by the matrix A^{-1}.

Consider the following example.

Example 5.22: Inverse of a Transformation

Let $T: \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a linear transformation induced by the matrix

$$A=\left[\begin{array}{cc} 2 & 3 \\ 3 & 4 \end{array}\right]$$

Show that T^{-1} exists and find the matrix B which it is induced by.

Solution. Since the matrix A is invertible, it follows that the transformation T is invertible. Therefore, T^{-1} exists.

You can verify that A^{-1} is given by:

$$A^{-1}=\left[\begin{array}{cc} -4 & 3 \\ 3 & -2 \end{array}\right]$$

Therefore the linear transformation T^{-1} is induced by the matrix A^{-1}. ♠

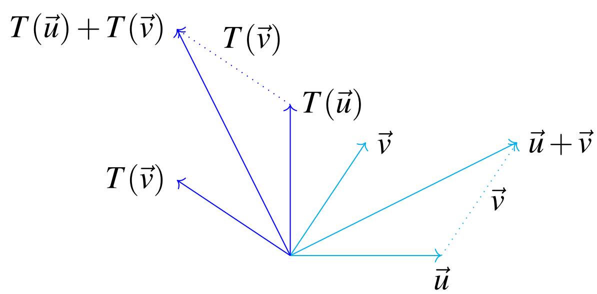
5.4 Special Linear Transformations in $\mathbb{R}^2$

Outcomes

A. *Find the matrix of rotations and reflections in $\mathbb{R}^2$ and determine the action of each on a vector in $\mathbb{R}^2$.*

In this section, we will examine some special examples of linear transformations mapping $\mathbb{R}^2$ to $\mathbb{R}^2$, including rotations and reflections. We will use the geometric descriptions of vector addition and scalar multiplication discussed earlier to show that a rotation of vectors through an angle and reflection of a vector across a line are examples of linear transformations.

More generally, denote a transformation given by a rotation by T. Why is such a transformation linear? Consider the following picture which illustrates a rotation. Let $\vec{u}, \vec{v}$ denote vectors.



Let's consider how to obtain $T(\vec{u} + \vec{v})$. Simply, you add $T(\vec{u})$ and $T(\vec{v})$. Here is why. If you add $T(\vec{u})$ to $T(\vec{v})$ you get the diagonal of the parallelogram determined by $T(\vec{u})$ and $T(\vec{v})$, as this action is our usual vector addition. Now, suppose we first add $\vec{u}$ and $\vec{v}$, and then apply the transformation T to $\vec{u}+\vec{v}$. Hence, we find $T(\vec{u}+\vec{v})$. As shown in the diagram, this will result in the same vector. In other words, $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$.

This is because the rotation preserves all angles between the vectors as well as their lengths. In particular, it preserves the shape of this parallelogram. Thus both $T(\vec{u})+T(\vec{v})$ and $T(\vec{u}+\vec{v})$ give the same vector. It follows that T distributes across addition of the vectors of $\mathbb{R}^2$.

Similarly, if k is a scalar, it follows that $T(k\vec{u}) = kT(\vec{u})$. Thus rotations are an example of a linear transformation by Definition 9.52.

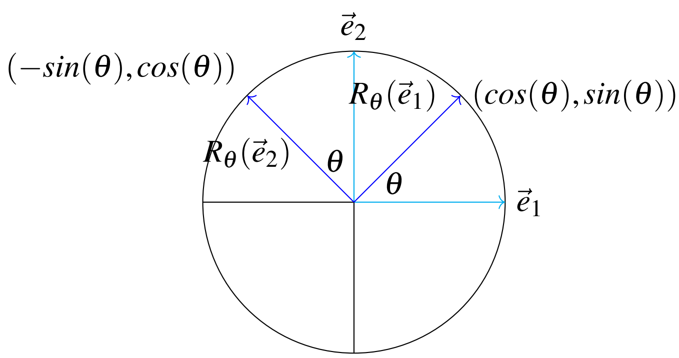
The following theorem gives the matrix of a linear transformation which rotates all vectors through an angle of θ.

Theorem 5.23: Rotation

Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by rotating vectors through an angle of θ. Then the matrix A that represents R_θ relative to the standard basis is given by

$$\left[\begin{array}{cc} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array}\right]$$

Proof. Let $\vec{e}_1 = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$ and $\vec{e}_2 = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$. These identify the geometric vectors which point along the positive x axis and positive y axis as shown.



From Theorem 5.7, we need to find $R_\theta(\vec{e}_1)$ and $R_\theta(\vec{e}_2)$, and use these as the columns of the matrix A of T. We can use the cosine and sine of the angle θ to find the coordinates of $R_\theta(\vec{e}_1)$ as shown in the above picture. The coordinates of $R_\theta(\vec{e}_2)$ also follow from trigonometry. Thus

$$R_\theta(\vec{e}_1) = \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array}\right],\ R_\theta(\vec{e}_2) = \left[\begin{array}{c} -\sin\theta \\ \cos\theta \end{array}\right]$$

Therefore, from Theorem 5.7,

$$A = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$$

We can also prove this algebraically without the use of the above picture. The definition of $(\cos(\theta), \sin(\theta))$ is as the coordinates of the point of $R_\theta(\vec{e}_1)$. Now the point of the vector $\vec{e}_2$ is exactly $\pi/2$ further along the unit circle from the point of $\vec{e}_1$, and therefore after rotation through an angle of θ the coordinates x and y of the point of $R_\theta(\vec{e}_2)$ are given by

$$(x,y) = (\cos(\theta+\pi/2), \sin(\theta+\pi/2)) = (-\sin\theta, \cos\theta)$$

♠

Consider the following example.

Example 5.24: Rotation in $\mathbb{R}^2$

Let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation through $\pi/2$. Find the matrix of $R_{\frac{\pi}{2}}$. Then, find $R_{\frac{\pi}{2}}(\vec{x})$ where $\vec{x} = \left[\begin{array}{r} 1 \\ -2 \end{array}\right]$.

Solution. By Theorem 5.23, the matrix of $R_{\frac{\pi}{2}}$ is given by

$$\left[\begin{array}{cc} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array}\right] = \left[\begin{array}{cc} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{array}\right] = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]$$

To find $R_{\frac{\pi}{2}}(\vec{x})$, we multiply the matrix of $R_{\frac{\pi}{2}}$ by $\vec{x}$ as follows

$$\left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array}\right]\left[\begin{array}{r} 1 \\ -2 \end{array}\right] = \left[\begin{array}{r} 2 \\ 1 \end{array}\right]$$

♠

We now look at an example of a linear transformation involving two angles.

Example 5.25: The Rotation Matrix of the Sum of Two Angles

Find the matrix of the linear transformation which is obtained by first rotating all vectors through an angle of ϕ and then through an angle θ. Hence the linear transformation rotates all vectors through an angle of $\theta+\phi$.

Solution. Let $R_{\theta+\phi}$ denote the linear transformation which rotates every vector through an angle of $\theta+\phi$. Then to obtain $R_{\theta+\phi}$, we first apply R_ϕ and then R_θ where R_ϕ is the linear transformation which rotates through an angle of ϕ and R_θ is the linear transformation which rotates through an angle of θ. Denoting the corresponding matrices by $A_{\theta+\phi}$, A_ϕ, and A_θ, it follows that for every $\vec{u}$

$$R_{\theta+\phi}(\vec{u}) = A_{\theta+\phi}\vec{u} = A_\theta A_\phi \vec{u} = R_\theta R_\phi(\vec{u})$$

Notice the order of the matrices here!

Consequently, you must have

$$A_{\theta+\phi} = \left[\begin{array}{cc} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{array}\right]$$

$$= \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]\left[\begin{array}{cc} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{array}\right] = A_\theta A_\phi$$

The usual matrix multiplication yields

$$\begin{aligned} A_{\theta+\phi} &= \left[\begin{array}{cc} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{array}\right] \\ &= \left[\begin{array}{cc} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{array}\right] \\ &= A_\theta A_\phi \end{aligned}$$

Don't these look familiar? They are the usual trigonometric identities for the sum of two angles derived here using linear algebra concepts.

♠

Here we have focused on rotations in two dimensions. However, you can consider rotations and other geometric concepts in any number of dimensions. This is one of the major advantages of linear algebra. You can break down a difficult geometrical procedure into small steps, each corresponding to multiplication by an appropriate matrix. Then by multiplying the matrices, you can obtain a single matrix which can give you numerical information on the results of applying the given sequence of simple procedures.

Linear transformations which reflect vectors across a line are a second important type of transformations in $\mathbb{R}^2$. You should draw a picture to convince yourself, geometrically, that reflecting across a line that passes through the origin is, in fact, a linear transformation. Once you have done that, consider the following theorem.

Theorem 5.26: Reflection

Let $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by reflecting vectors over the line $\vec{y} = m\vec{x}$. Then the matrix of Q_m relative to the standard basis is given by

$$\frac{1}{1+m^2}\left[\begin{array}{cc} 1-m^2 & 2m \\ 2m & m^2-1 \end{array}\right]$$

Consider the following example.

Example 5.27: Reflection in $\mathbb{R}^2$

Let $Q_2 : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection over the line $\vec{y} = 2\vec{x}$. Then Q_2 is a linear transformation. Find the matrix of Q_2. Then, find $Q_2(\vec{x})$ where $\vec{x} = \left[\begin{array}{c} 1 \\ -2 \end{array}\right]$.

Solution. By Theorem 5.26, the matrix of Q_2 is given by

$$\frac{1}{1+m^2}\left[\begin{array}{cc} 1-m^2 & 2m \\ 2m & m^2-1 \end{array}\right] = \frac{1}{1+(2)^2}\left[\begin{array}{cc} 1-(2)^2 & 2(2) \\ 2(2) & (2)^2-1 \end{array}\right] = \frac{1}{5}\left[\begin{array}{cc} -3 & 8 \\ 8 & 3 \end{array}\right]$$

To find $Q_2(\vec{x})$ we multiply $\vec{x}$ by the matrix of Q_2 as follows:

$$\frac{1}{5}\begin{bmatrix} -3 & 8 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{19}{5} \\ \frac{2}{5} \end{bmatrix}$$

♠

Consider the following example which incorporates a reflection as well as a rotation of vectors.

Example 5.28: Rotation Followed by a Reflection

Find the matrix of the linear transformation which is obtained by first rotating all vectors through an angle of $\pi/6$ and then reflecting through the x axis.

Solution. By Theorem 5.23, the matrix of the transformation which involves rotating through an angle of $\pi/6$ is

$$\begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix}$$

Reflecting across the x axis is the same action as reflecting vectors over the line $\vec{y} = m\vec{x}$ with $m = 0$. By Theorem 5.26, the matrix for the transformation which reflects all vectors through the x axis is

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} = \frac{1}{1+(0)^2}\begin{bmatrix} 1-(0)^2 & 2(0) \\ 2(0) & (0)^2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Therefore, the matrix of the linear transformation which first rotates through $\pi/6$ and then reflects through the x axis is given by

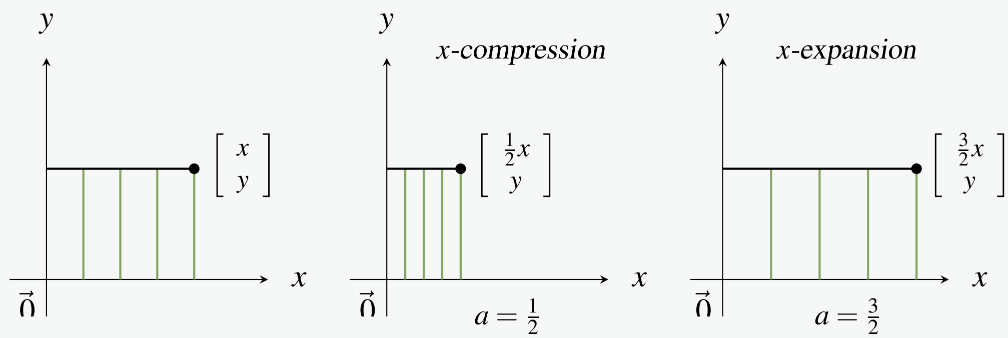
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{bmatrix}$$

♠

Here are two more examples of geometric transformations which are actually matrix transformations.

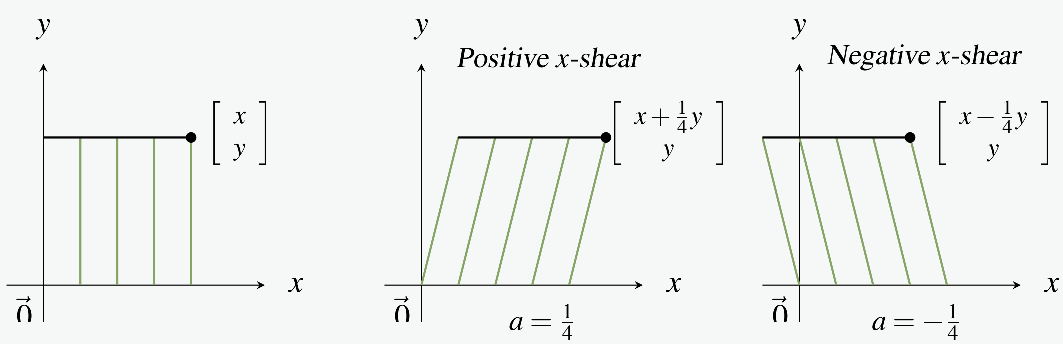
Example 5.29: Expansion and Compression

*If $a > 0$, the matrix transformation $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ y \end{bmatrix}$ induced by the matrix $A = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ is called an x-**expansion** of $\mathbb{R}^2$ if $a > 1$, and an x-**compression** if $0 < a < 1$. The names follow from their geometric interpretation as shown in the diagram below. Similarly, if $b > 0$ the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$ gives rise to y-**expansions** and y-**compressions**.*



Example 5.30: Shear

*If a is a number, the matrix transformation $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ay \\ y \end{bmatrix}$ induced by the matrix $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ is called an x-**shear** of $\mathbb{R}^2$ (**positive** if $a > 0$ and **negative** if $a < 0$). Its effect is illustrated below when $a = \frac{1}{4}$ and $a = -\frac{1}{4}$.*



5.5 One to One and Onto Transformations

Outcomes

A. *Determine if a linear transformation is onto or one to one.*

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. We define the **range** or **image** of T as the set of vectors of $\mathbb{R}^m$ which are of the form $T(\vec{x})$ (equivalently, $A\vec{x}$) for some $\vec{x} \in \mathbb{R}^n$. It is common to write $T\mathbb{R}^n$, $T(\mathbb{R}^n)$, or $\mathrm{Im}(T)$ to denote the range of T.

Lemma 5.31: Range of a Matrix Transformation

Let A be an $m \times n$ matrix where $A_1, \cdots, A_n$ denote the columns of A. Then, for a vector $\vec{x} = \left[\begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right]$ in $\mathbb{R}^n$,

$$A\vec{x} = \sum_{k=1}^{n} x_k A_k$$

Therefore, $A(\mathbb{R}^n)$ is the collection of all linear combinations of these products.

Proof. This follows from the definition of matrix multiplication. ♠

This section is devoted to studying two important types of linear transformations, called one to one transformations and onto transformations. We define them now.

Definition 5.32: One to One

A linear transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is called ***one to one*** *(often written as $1-1$) or* ***injective*** *if whenever $\vec{x}_1 \neq \vec{x}_2 \in \mathbb{R}^n$ it follows that :*

$$T\left(\vec{x}_1\right) \neq T\left(\vec{x}_2\right)$$

Equivalently, if $T\left(\vec{x}_1\right) = T\left(\vec{x}_2\right)$, then $\vec{x}_1 = \vec{x}_2$. Thus, T is one to one if it never takes two different vectors to the same vector.

The second important property a linear transformation may have is called being onto, or surjective.

Definition 5.33: Onto

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then T is called ***onto*** *or* ***surjective*** *if for every $\vec{x}_2 \in \mathbb{R}^m$ there exists some $\vec{x}_1 \in \mathbb{R}^n$ such that $T\left(\vec{x}_1\right) = \vec{x}_2$.*

We often call a linear transformation which is one-to-one an **injection**. Similarly, a linear transformation which is onto is often called a **surjection**.

The following proposition is an important result.

Proposition 5.34: One to One

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$.

Proof. We need to prove two things here. First, we will prove that if T is one to one, then $T(\vec{x}) = \vec{0}$ implies that $\vec{x} = \vec{0}$. Second, we will show that if $T(\vec{x}) = \vec{0}$ implies that $\vec{x} = \vec{0}$, then it follows that T is one to one. Recall that a linear transformation has the property that $T(\vec{0}) = \vec{0}$.

Suppose first that T is one to one and consider $T(\vec{0})$.

$$T(\vec{0}) = T\left(\vec{0} + \vec{0}\right) = T(\vec{0}) + T(\vec{0})$$

and so, adding the additive inverse of $T(\vec{0})$ to both sides, one sees that $T(\vec{0}) = \vec{0}$. If $T(\vec{x}) = \vec{0}$ it must be the case that $\vec{x} = \vec{0}$ because it was just shown that $T(\vec{0}) = \vec{0}$ and T is assumed to be one to one.

Now assume that if $T(\vec{x}) = \vec{0}$, then it follows that $\vec{x} = \vec{0}$. If $T(\vec{v}) = T(\vec{u})$, then

$$T(\vec{v}) - T(\vec{u}) = T\left(\vec{v} - \vec{u}\right) = \vec{0}$$

which shows that $\vec{v} - \vec{u} = 0$. In other words, $\vec{v} = \vec{u}$, and T is one to one. ♠

Suppose that $T : \mathbb{R}^n \to^m$ is a linear transformation and suppose that A is the matrix that represents T relative to the standard basis. Then Proposition 5.34 tells us that if $A = \left[\begin{array}{ccc} A_1 & \cdots & A_n \end{array}\right]$ then A is one to one if and only if whenever

$$\sum_{k=1}^{n} c_k A_k = \vec{0}$$

it follows that each scalar $c_k = 0$.

We will now take a look at an example of a one to one and onto linear transformation.

Example 5.35: A One to One and Onto Linear Transformation

Suppose

$$T\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right]$$

Then, $T : \mathbb{R}^2 \to \mathbb{R}^2$ *is a linear transformation. Is* T *onto? Is it one to one?*

Solution. Recall that because T can be expressed as matrix multiplication, we know that T is a linear transformation. We will first check whether the linear transformation T is an onto transformation. So suppose $\left[\begin{array}{c} a \\ b \end{array}\right] \in \mathbb{R}^2$. Does there exist $\left[\begin{array}{c} x \\ y \end{array}\right] \in \mathbb{R}^2$ such that $T\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} a \\ b \end{array}\right]$? If so, then since $\left[\begin{array}{c} a \\ b \end{array}\right]$ is an arbitrary vector in $\mathbb{R}^2$, it will follow that T is onto.

This question is familiar to you. It is asking whether there is a solution to the equation

$$\left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} a \\ b \end{array}\right]$$

This is the same thing as asking for a solution to the following system of equations.

$$\begin{array}{c} x + y = a \\ x + 2y = b \end{array}$$

Set up the augmented matrix and row reduce.

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 2 & b \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2a-b \\ 0 & 1 & b-a \end{array}\right] \tag{5.4}$$

You can see from this point that the system has a solution. Therefore, we have shown that for any a,b, there is a $\left[\begin{array}{c} x \\ y \end{array}\right]$ such that $T\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} a \\ b \end{array}\right]$. Thus T is onto.

Now we want to know if T is one to one. By Proposition 5.34 it is enough to show that $A\vec{x} = \vec{0}$ implies $\vec{x} = \vec{0}$. Consider the system $A\vec{x} = \vec{0}$ given by:

$$\left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

This is the same as the system given by

$$\begin{array}{c} x + y = 0 \\ x + 2y = 0 \end{array}$$

We need to show that the only solution to this system is $x = 0$ and $y = 0$. By setting up the augmented matrix and row reducing, we end up with

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

This tells us that $x = 0$ and $y = 0$. Returning to the original system, this says that if

$$\left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

then

$$\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

In other words, $A\vec{x} = \vec{0}$ implies that $\vec{x} = \vec{0}$. By Proposition 5.34, A is one to one, and so T is also one to one.

We also could have seen that T is one to one from our above solution for onto. By looking at the matrix given by 5.4, you can see that there is a **unique** solution given by $x = 2a - b$ and $y = b - a$. Therefore, there is only one vector, specifically $\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 2a-b \\ b-a \end{array}\right]$ such that $T\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} a \\ b \end{array}\right]$. Hence by Definition 5.32, T is one to one. ♠

Example 5.36: An Onto Transformation

Let $T : \mathbb{R}^4 \mapsto \mathbb{R}^2$ be a linear transformation defined by

$$T\left[\begin{array}{c} a \\ b \\ c \\ d \end{array}\right] = \left[\begin{array}{c} a+d \\ b+c \end{array}\right] \text{ for all } \left[\begin{array}{c} a \\ b \\ c \\ d \end{array}\right] \in \mathbb{R}^4$$

Prove that T is onto but not one to one.

Solution. You can prove that T is in fact linear.

To show that T is onto, let $\left[\begin{array}{c} x \\ y \end{array}\right]$ be an arbitrary vector in $\mathbb{R}^2$. Taking the vector $\left[\begin{array}{c} x \\ y \\ 0 \\ 0 \end{array}\right] \in \mathbb{R}^4$ we have

$$T\left[\begin{array}{c} x \\ y \\ 0 \\ 0 \end{array}\right] = \left[\begin{array}{c} x+0 \\ y+0 \end{array}\right] = \left[\begin{array}{c} x \\ y \end{array}\right]$$

This shows that T is onto.

By Proposition 5.34 T is one to one if and only if $T(\vec{x}) = \vec{0}$ implies that $\vec{x} = \vec{0}$. Observe that

$$T\left[\begin{array}{c} 1 \\ 0 \\ 0 \\ -1 \end{array}\right] = \left[\begin{array}{c} 1+-1 \\ 0+0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

There exists a nonzero vector $\vec{x}$ in $\mathbb{R}^4$ such that $T(\vec{x}) = \vec{0}$. It follows that T is not one to one. ♠

The above example demonstrates a method to determine if a linear transformation T is one to one or onto, but the method was sort of haphazard—there isn't a nice procedure that generalizes to other situations. Fortunately, it turns out that the matrix A that represents T with respect to the standard basis can tell us whether T is injective or surjective or both or neither.

Theorem 5.37: Matrix of a One to One or Onto Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation represented by the $m \times n$ matrix A. Then T is one to one if and only if the rank of A is n. T is onto if and only if the rank of A is m.

Consider Example 5.36. Above we showed that T was onto but not one to one. We can now use this theorem to determine this fact about T.

Example 5.38: An Onto Transformation

Let $T : \mathbb{R}^4 \mapsto \mathbb{R}^2$ be a linear transformation defined by

$$T \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+d \\ b+c \end{bmatrix} \text{ for all } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$$

Prove that T is onto but not one to one.

Solution. Using Theorem 5.37 we can show that T is onto but not one to one from the matrix of T. Recall that to find the matrix A of T, we apply T to each of the standard basis vectors $\vec{e}_i$ of $\mathbb{R}^4$. The result is the 2×4 matrix A given by

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Fortunately, this matrix is already in reduced row-echelon form. The rank of A is 2. Therefore by the above theorem T is onto but not one to one. ♠

Compositions

If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^k$ are both linear transformations, we can think about the composition of these two functions, which is denoted $S \circ T$. Here's how the composition is defined:

$$\begin{aligned} S \circ T : \mathbb{R}^n &\to \mathbb{R}^k \\ \vec{x} &\mapsto S(T(\vec{x})) \end{aligned}$$

So to compute the value of the composition $S \circ T$ applied to the vector $\vec{x}$, first you compute $T(\vec{x})$, and then you compute $S(T(\vec{x}))$. Notice that $T(\vec{x}) \in \mathbb{R}^m$, so it makes sense to apply the linear transformation S to that vector.

It turns out that if both T and S are linear transformations, then the composition $S \circ T$ is also a linear transformation. We know that T is represented by an $m \times n$ matrix A and S is represented by a $m \times k$ matrix B. We also know that *some* matrix represents $S \circ T$ relative to the standard basis. Fortunately, there is an easy way to find that matrix—it is simply the matrix product BA, since

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) = S(A\vec{x}) = B(A\vec{x}) = (BA)\vec{x}.$$

This is one of the best things about our definition of matrix multiplication—we can represent composition by multiplication.

We'll finish this section by examining some of the ways that taking compostions effects injectivity and surjectivity of linear transformations.

Example 5.39: Composite of Onto Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \mapsto \mathbb{R}^k$ be linear transformations. Show that if T and S are onto, then $S \circ T$ is onto.

Solution. Let $\vec{z} \in \mathbb{R}^k$. Since S is onto, there exists a vector $\vec{y} \in \mathbb{R}^m$ such that $S(\vec{y}) = \vec{z}$. Furthermore, since T is onto, there exists a vector $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{y}$. Thus

$$\vec{z} = S(\vec{y}) = S(T(\vec{x})) = (ST)(\vec{x}),$$

showing that for each $\vec{z} \in \mathbb{R}^k$ there exists and $\vec{x} \in \mathbb{R}^n$ such that $(ST)(\vec{x}) = \vec{z}$. Therefore, $S \circ T$ is onto. ♠

The next example shows the same concept with regards to one-to-one transformations.

Example 5.40: Composite of One to One Transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations. Prove that if T and S are one to one, then $S \circ T$ is one-to-one.

Solution. To prove that $S \circ T$ is one to one, we need to show that if $S(T(\vec{v})) = \vec{0}$ it follows that $\vec{v} = \vec{0}$. Suppose that $S(T(\vec{v})) = \vec{0}$. Since S is one to one, it follows that $T(\vec{v}) = \vec{0}$. Similarly, since T is one to one, it follows that $\vec{v} = \vec{0}$. Hence $S \circ T$ is one to one. ♠

Here's a chance for another look under the hood. Notice that nowhere in the last two examples did we use the fact that our functions were linear transformations. So our arguments show that compositions of injections are injections whether or not the functions involved are linear transformations. And the composition of surjections is a surjection. So, for example, the function $f(x) = e^x$ is an injection and the function $g(x) = x^3$ is also an injection. Therefore the function $h(x) = (g \circ f)(x) = g(f(x)) = [e^x]^3$ is also an injection.

5.6 Isomorphisms

Outcomes

A. *Determine if a linear transformation is an isomorphism.*

B. *Determine if two subspaces of $\mathbb{R}^n$ are isomorphic.*

Recall the definition of a linear transformation. Let V and W be two subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. A mapping $T : V \to W$ is called a **linear transformation** or **linear map** if it preserves the algebraic operations of addition and scalar multiplication. Specifically, if a, b are scalars and $\vec{x}, \vec{y}$ are vectors,

$$T(a\vec{x} + b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$$

Consider the following important definition.

Definition 5.41: Isomorphism

Suppose that V is a subspace of $\mathbb{R}^n$ and that W is a subspace of $\mathbb{R}^n$. A linear map $T : V \to W$ is called an ***isomorphism from*** *V* ***to*** *W if the following two conditions are satisfied.*

- *T is one to one. That is, if $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$.*
- *T is onto. That is, if $\vec{w} \in W$, there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.*

Consider the following example of an isomorphism.

Example 5.42: Isomorphism

Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be defined by

$$T\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} x+y \\ x-y \end{array}\right]$$

Show that T is an isomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Solution. To prove that T is an isomorphism we must show

1. T is a linear transformation;
2. T is one to one;
3. T is onto.

We proceed as follows.

1. T is a linear transformation:

   Let k, p be scalars.

$$\begin{array}{rcl} T\left(k\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right] + p\left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right) & = & T\left(\left[\begin{array}{c} kx_1 \\ ky_1 \end{array}\right] + \left[\begin{array}{c} px_2 \\ py_2 \end{array}\right]\right) \\ & = & T\left(\left[\begin{array}{c} kx_1 + px_2 \\ ky_1 + py_2 \end{array}\right]\right) \\ & = & \left[\begin{array}{c} (kx_1 + px_2) + (ky_1 + py_2) \\ (kx_1 + px_2) - (ky_1 + py_2) \end{array}\right] \\ & = & \left[\begin{array}{c} (kx_1 + ky_1) + (px_2 + py_2) \\ (kx_1 - ky_1) + (px_2 - py_2) \end{array}\right] \\ & = & \left[\begin{array}{c} kx_1 + ky_1 \\ kx_1 - ky_1 \end{array}\right] + \left[\begin{array}{c} px_2 + py_2 \\ px_2 - py_2 \end{array}\right] \\ & = & k\left[\begin{array}{c} x_1 + y_1 \\ x_1 - y_1 \end{array}\right] + p\left[\begin{array}{c} x_2 + y_2 \\ x_2 - y_2 \end{array}\right] \\ & = & kT\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]\right) + pT\left(\left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right) \end{array}$$

   Therefore T is linear.

2. T is one to one:

   We need to show that if $T(\vec{x}) = \vec{0}$ for a vector $\vec{x} \in \mathbb{R}^2$, then it follows that $\vec{x} = \vec{0}$. Let $\vec{x} = \left[\begin{array}{c} x \\ y \end{array}\right]$.

$$T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left[\begin{array}{c} x+y \\ x-y \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

   This provides a system of equations given by

$$x + y = 0$$

$$x - y = 0$$

You can verify that the solution to this system if $x = y = 0$. Therefore

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and T is one to one.

3. T is onto:

   Let a, b be scalars. We want to check if there is always a solution to

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

   This can be represented as the system of equations

$$\begin{aligned} x+y &= a \\ x-y &= b \end{aligned}$$

   Setting up the augmented matrix and row reducing gives

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{a+b}{2} \\ 0 & 1 & \frac{a-b}{2} \end{array}\right]$$

   This has a solution for all a, b and therefore T is onto.

Therefore T is an isomorphism. ♠

If there is an isomorphism from V to W, the idea is that V and W are supposed to have the same shape, as the Greek roots of the word, iso-, meaning equal or identical, and -morphe, meaning form or shape. This is one of the most important words in mathematics, since seeing when two things have the same shape lets you use what you know about one of the things to deduce properties about the other thing. Different subfields of mathematics have different definitions of what an isomorphism is, as they are interested in emphasizing different aspects of the "shape" of an object. For us, we are mostly interested in the dimension of the subspace—what bases might look like. As we have seen, if you know what happens to a basis of V, you know what happens to any vector in V. We will prove in Theorem 5.47 that there is an isomorphism from V to W if and only if they have the same dimension. This means (roughly) that there is only one kind of 3-dimensional space, since every 3-dimensional space "looks like" $\mathbb{R}^3$.

One might expect that if V has the same shape as W, then W should have the same shape as V. Translating that, this says that if there is an isomorphism mapping V to W, then there should be an isomorphism mapping W to V. Our next result gives us such an isomorphism by looking at inverses. Thus we will be justified in saying that if there is an isomorphism mapping V to W then the subspaces V and W are **isomorphic.**

Proposition 5.43: Inverse of an Isomorphism

Suppose that T is a subspace of $\mathbb{R}^n$ and W is a subspace of $\mathbb{R}^m$. Suppose that $T : V \to W$ is an isomorphism. Then $T^{-1} : W \to V$ is also an isomorphism.

Proof. Let T be an isomorphism. We must show that the function T^{-1} is a linear transformation that is both surjective and injective.

To show that T^{-1} is a linear transformation, fix vectors $\vec{w}_1$ and $\vec{w}_2$ in W and fix scalars a and b. We must show that

$$T^{-1}(a\vec{w}_1 + b\vec{w}_2) = aT^{-1}(\vec{w}_1) + bT^{-1}(\vec{w}_2).$$

Since T is onto, we know there are vectors $\vec{v}_1$ and $\vec{v}_2$, both elements of $\mathbb{R}^n$, such that $\vec{w}_1 = T(\vec{v}_1)$ and $\vec{w}_2 = T(\vec{v}_2)$. So we must show the following:

$$T^{-1}(aT(\vec{v}_1) + bT(\vec{v}_2)) = aT^{-1}(T(\vec{v}_1) + bT^{-1}(T(\vec{v}_2)).$$

As T and T^{-1} are inverses of each other, we can simplify the right hand side of this equation, so we need only show that

$$T^{-1}(aT(\vec{v}_1) + bT(\vec{v}_2)) = a\vec{v}_1 + b\vec{v}_2.$$

This equation is of the form $T^{-1}(y) = x$. Since T^{-1} is the inverse of T, this is equivalent to the equation $y = T(x)$. So (finally) to show that T^{-1} is a linear transformation, all we must do is prove that

$$aT(\vec{v}_1) + bT(\vec{v}_2) = T(a\vec{v}_1 + b\vec{v}_2).$$

But this is exactly what it means to say that T is a linear transformation. Since we have assumed that T is a linear transformation, we can conclude that T^{-1} is also a linear transformation.

To finish showing that T^{-1} is an isomorphism, we must show that T^{-1} is both onto and one to one. Fortunately, both of these arguments are shorter and easier.

To show that $T^{-1} : W \to V$ is onto. Fix $\vec{v} \in V$. Notice that $T^{-1}(T(\vec{v}) = \vec{v}$, and so we have found an element of W (namely, $T(\vec{v})$) that is mapped to $\vec{v}$. Thus T^{-1} is onto.

To show that T^{-1} is one to one, it suffices to show that if $T^{-1}(\vec{w}) = \vec{0}$, then $\vec{w} = \vec{0}$. So assume that $T^{-1}(\vec{w}) = \vec{0}$. Then

$$\vec{w} = T(T^{-1}(\vec{w}) = T(\vec{0}) = \vec{0},$$

as T is a linear transformation. But this means that we have shown that T^{-1} is injective, and this finishes the proof that T^{-1} is an isomorphism. ♠

Another important result is that the composition of multiple isomorphisms is also an isomorphism.

Proposition 5.44: Composition of Isomorphisms

Let $T : V \to W$ and $S : W \to Z$ be isomorphisms where V, W, Z are subspaces of $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^k$, respectively. Then $S \circ T : V \to Z$ defined by $(S \circ T)(\vec{v}) = S(T(\vec{v}))$ is also an isomorphism.

Proof. Suppose $T : V \to W$ and $S : W \to Z$ are isomorphisms. Why is $S \circ T$ a linear map? For a, b scalars,

$$\begin{aligned} S \circ T (a\vec{v}_1 + b(\vec{v}_2)) &= S(T(a\vec{v}_1 + b\vec{v}_2)) = S(aT\vec{v}_1 + bT\vec{v}_2) \\ &= aS(T\vec{v}_1) + bS(T\vec{v}_2) = a(S \circ T)(\vec{v}_1) + b(S \circ T)(\vec{v}_2) \end{aligned}$$

Hence $S \circ T$ is a linear map. If $(S \circ T)(\vec{v}) = \vec{0}$, then $S(T(\vec{v})) = \vec{0}$ and it follows from the fact that S is an injection and Proposition 5.34 that $T(\vec{v}) = \vec{0}$ and hence by the same proposition again, $\vec{v} = \vec{0}$. Thus $S \circ T$ is one to one. It remains to verify that $S \circ T$ is onto. Let $\vec{z} \in Z$. Then since S is onto, there exists $\vec{w} \in W$ such that $S(\vec{w}) = \vec{z}$. Also, since T is onto, there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. It follows that $S(T(\vec{v})) = \vec{z}$ and so $S \circ T$ is also onto. ♠

Consider two subspaces V and W, and suppose there exists an isomorphism mapping one to the other. In this way the two subspaces are related, which we can write as $V \sim W$. Then the previous two propositions together claim that $\sim$ is an equivalence relation. That is: $\sim$ satisfies the following conditions:

- $V \sim V$
- If $V \sim W$, it follows that $W \sim V$
- If $V \sim W$ and $W \sim Z$, then $V \sim Z$

We leave the verification of these conditions as an exercise.

Consider the following example.

Example 5.45: Matrix Isomorphism

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\vec{x}) = A(\vec{x})$ where A is an invertible $n \times n$ matrix. Then T is an isomorphism.

Solution. The reason for this is that, since A is invertible, the only vector it sends to $\vec{0}$ is the zero vector. Hence if $A(\vec{x}) = A(\vec{y})$, then $A(\vec{x} - \vec{y}) = \vec{0}$ and so $\vec{x} = \vec{y}$. It is onto because if $\vec{y} \in \mathbb{R}^n$, $A\left(A^{-1}(\vec{y})\right) = \left(AA^{-1}\right)(\vec{y})$ $= \vec{y}$. ♠

In fact, all isomorphisms from $\mathbb{R}^n$ to $\mathbb{R}^n$ can be expressed as $T(\vec{x}) = A(\vec{x})$ where A is an invertible $n \times n$ matrix. One simply considers the matrix whose i^{th} column is $T\vec{e}_i$, which is the matrix that represents the transformation T with respect to the standard basis.

Recall that a basis of a subspace V is a set of linearly independent vectors which span V. The following fundamental lemma describes the relation between bases and isomorphisms.

Lemma 5.46: Mapping Bases

Let $T : V \to W$ be a linear transformation where V, W are subspaces of $\mathbb{R}^n$. If T is one to one, then it has the property that if $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent, so is $\{T(\vec{u}_1), \cdots, T(\vec{u}_k)\}$.
More generally, T is an isomorphism if and only if whenever $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, it follows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis for W.

Proof. First suppose that T is a one to one linear transformation and assume that $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent. It is required to show that $\{T(\vec{u}_1), \cdots, T(\vec{u}_k)\}$ is also linearly independent. Suppose then that

$$\sum_{i=1}^{k} c_i T(\vec{u}_i) = \vec{0}$$

Then, since T is linear,

$$T\left(\sum_{i=1}^{n} c_i \vec{u}_i\right) = \vec{0}$$

Since T is one to one, it follows that

$$\sum_{i=1}^{n} c_i \vec{u}_i = 0$$

Now the fact that $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent implies that each $c_i = 0$. Hence $\{T(\vec{u}_1), \cdots, T(\vec{u}_n)\}$ is linearly independent.

Now suppose that T is an isomorphism and $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V. It was just shown that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent. It remains to verify that $\text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\} = W$. If $\vec{w} \in W$, then since T is onto there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. Since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis, it follows that there exists scalars $\{c_i\}_{i=1}^{n}$ such that

$$\sum_{i=1}^{n} c_i \vec{v}_i = \vec{v}.$$

Hence,

$$\vec{w} = T(\vec{v}) = T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i T(\vec{v}_i)$$

It follows that $\text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\} = W$ showing that this set of vectors is a basis for W.

Next suppose that T is a linear transformation which takes a basis to a basis. This means that if $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, it follows $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis for W. Then if $w \in W$, there exist scalars c_i such that $w = \sum_{i=1}^{n} c_i T(\vec{v}_i) = T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right)$ showing that T is onto. If $T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \vec{0}$ then $\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$ and since the vectors $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ are linearly independent, it follows that each $c_i = 0$. Since $\sum_{i=1}^{n} c_i \vec{v}_i$ is a typical vector in V, this has shown that if $T(\vec{v}) = \vec{0}$ then $\vec{v} = \vec{0}$ and so T is also one to one. Thus T is an isomorphism. ♠

The following theorem illustrates a very useful idea for defining an isomorphism. Basically, if you know what it does to a basis, then you can construct the isomorphism.

Theorem 5.47: Isomorphic Subspaces

Suppose V and W are two subspaces of $\mathbb{R}^n$. Then the two subspaces are isomorphic if and only if they have the same dimension. In the case that the two subspaces have the same dimension, then for a linear map $T : V \to W$, the following are equivalent.

1. *T is one to one.*
2. *T is onto.*
3. *T is an isomorphism.*

Proof. Suppose first that these two subspaces have the same dimension. Let a basis for V be $\{\vec{v}_1, \cdots, \vec{v}_n\}$ and let a basis for W be $\{\vec{w}_1, \cdots, \vec{w}_n\}$. Now define T as follows.

$$T(\vec{v}_i) = \vec{w}_i$$

for $\sum_{i=1}^n c_i \vec{v}_i$ an arbitrary vector of V,

$$T\left(\sum_{i=1}^n c_i \vec{v}_i\right) = \sum_{i=1}^n c_i T \vec{v}_i = \sum_{i=1}^n c_i \vec{w}_i.$$

It is necessary to verify that this is well defined. Suppose then that

$$\sum_{i=1}^n c_i \vec{v}_i = \sum_{i=1}^n \hat{c}_i \vec{v}_i$$

Then

$$\sum_{i=1}^n (c_i - \hat{c}_i) \vec{v}_i = \vec{0}$$

and since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis, $c_i = \hat{c}_i$ for each i. Hence

$$\sum_{i=1}^n c_i \vec{w}_i = \sum_{i=1}^n \hat{c}_i \vec{w}_i$$

and so the mapping is well defined. Also if a, b are scalars,

$$\begin{aligned}
T\left(a \sum_{i=1}^n c_i \vec{v}_i + b \sum_{i=1}^n \hat{c}_i \vec{v}_i\right) &= T\left(\sum_{i=1}^n (a c_i + b \hat{c}_i) \vec{v}_i\right) = \sum_{i=1}^n (a c_i + b \hat{c}_i) \vec{w}_i \\
&= a \sum_{i=1}^n c_i \vec{w}_i + b \sum_{i=1}^n \hat{c}_i \vec{w}_i \\
&= aT\left(\sum_{i=1}^n c_i \vec{v}_i\right) + bT\left(\sum_{i=1}^n \hat{c}_i \vec{v}_i\right)
\end{aligned}$$

Thus T is a linear transformation.

Now if

$$T\left(\sum_{i=1}^n c_i \vec{v}_i\right) = \sum_{i=1}^n c_i \vec{w}_i = \vec{0},$$

then since the $\{\vec{w}_1, \cdots, \vec{w}_n\}$ are independent, each $c_i = 0$ and so $\sum_{i=1}^n c_i \vec{v}_i = \vec{0}$ also. Hence T is one to one. If $\sum_{i=1}^n c_i \vec{w}_i$ is a vector in W, then it equals

$$\sum_{i=1}^n c_i T(\vec{v}_i) = T\left(\sum_{i=1}^n c_i \vec{v}_i\right)$$

showing that T is also onto. Hence T is an isomorphism and so V and W are isomorphic.

Next suppose $T : V \mapsto W$ is an isomorphism, so these two subspaces are isomorphic. Then for $\{\vec{v}_1, \cdots, \vec{v}_n\}$ a basis for V, it follows that a basis for W is $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ showing that the two subspaces have the same dimension.

Now suppose the two subspaces have the same dimension. Consider the three claimed equivalences.

First consider the claim that 1. $\Rightarrow$ 2. If T is one to one and if $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, then $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent. If it is not a basis, then it must fail to span W. But then there would exist $\vec{w} \notin \text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ and it follows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n), \vec{w}\}$ would be linearly independent which is impossible because there exists a basis for W of n vectors.

Hence $\text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\} = W$ and so $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis. If $\vec{w} \in W$, there exist scalars c_i such that

$$\vec{w} = \sum_{i=1}^{n} c_i T(\vec{v}_i) = T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right)$$

showing that T is onto. This shows that 1. $\Rightarrow$ 2.

Next consider the claim that 2. $\Rightarrow$ 3. Since 2. holds, it follows that T is onto. It remains to verify that T is one to one. Since T is onto, there exists a basis of the form $\{T(\vec{v}_i), \cdots, T(\vec{v}_n)\}$. Then it follows that $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is linearly independent. Suppose

$$\sum_{i=1}^{n} c_i \vec{v}_i = \vec{0}$$

Then

$$\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$$

Hence each $c_i = 0$ and so, $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V. Now it follows that a typical vector in V is of the form $\sum_{i=1}^{n} c_i \vec{v}_i$. If $T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \vec{0}$, it follows that

$$\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$$

and so, since $\{T(\vec{v}_i), \cdots, T(\vec{v}_n)\}$ is independent, it follows each $c_i = 0$ and hence $\sum_{i=1}^{n} c_i \vec{v}_i = \vec{0}$. Thus T is one to one as well as onto and so it is an isomorphism.

If T is an isomorphism, it is both one to one and onto by definition so 3. implies both 1. and 2. ♠

Note the interesting way of defining a linear transformation in the first part of the argument by describing what it does to a basis and then "extending it linearly" to the entire subspace.

Example 5.48: Isomorphic Subspaces

Let $V = \mathbb{R}^3$ and let

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Show that V and W are isomorphic.

Solution. First observe that these subspaces are both of dimension 3 and so they are isomorphic by Theorem 5.47. The three vectors which span W are easily seen to be linearly independent by making them the columns of a matrix and row reducing to the reduced row-echelon form.

You can exhibit an isomorphism of these two spaces as follows.

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, T(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, T(\vec{e}_3) = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

and extend linearly. Recall that the matrix of this linear transformation is just the matrix having these vectors as columns. Thus the matrix of this isomorphism is

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

You should check that multiplication on the left by this matrix does reproduce the claimed effect resulting from an application by T. ♠

Consider the following example.

Example 5.49: Finding the Matrix of an Isomorphism

Let $V = \mathbb{R}^3$ and let

$$W = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

Let $T : V \to W$ be defined as follows.

$$T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

Find the matrix of this isomorphism T with respect to the standard basis.

Solution. First note that the vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

are indeed a basis for $\mathbb{R}^3$ as can be seen by making them the columns of a matrix and using the reduced row-echelon form.

Now recall the matrix of T is a 4×3 matrix A which gives the same effect as T. Thus, from the way we multiply matrices,

$$A \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

Hence,

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 2 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

Note how the span of the columns of this new matrix must be the same as the span of the vectors defining W. ♠

This idea of defining a linear transformation by what it does on a basis works for linear maps which are not necessarily isomorphisms.

Example 5.50: Finding the Matrix of an Isomorphism

Let $V = \mathbb{R}^3$ and let W denote

$$span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Let $T : V \to W$ be defined as follows.

$$T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

Find the matrix of this linear transformation.

Solution. Note that in this case, the three vectors which span W are not linearly independent. Nevertheless the above procedure will still work. The reasoning is the same as before. If A is this matrix, then

$$A \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

and so

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The columns of this last matrix are obviously not linearly independent. ♠

5.7 The Kernel And Image Of A Linear Map

Outcomes

A. *Describe the kernel and image of a linear transformation, and find a basis for each.*

In this section we will consider the case where the linear transformation is not necessarily an isomorphism. First consider the following important definition.

Definition 5.51: Kernel and Image

Let V and W be subspaces of $\mathbb{R}^n$ and let $T: V \to W$ be a linear transformation. Then the image of T, denoted as $\mathrm{im}(T)$, is defined to be the set

$$\mathrm{im}(T) = \{T(\vec{v}) : \vec{v} \in V\}$$

In words, it consists of all vectors in W which equal $T(\vec{v})$ for some $\vec{v} \in V$.
The kernel of T, written $\ker(T)$, consists of all $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$. That is,

$$\ker(T) = \left\{\vec{v} \in V : T(\vec{v}) = \vec{0}\right\}$$

The kernel of T is also called the ***null space*** *of T.*

It follows that $\mathrm{im}(T)$ and $\ker(T)$ are subspaces of W and V respectively.

Proposition 5.52: Kernel and Image as Subspaces

Let V, W be subspaces of $\mathbb{R}^n$ and let $T : V \to W$ be a linear transformation. Then $\ker(T)$ is a subspace of V and $\operatorname{im}(T)$ is a subspace of W.

Proof. First consider $\ker(T)$. It is necessary to show that if $\vec{v}_1, \vec{v}_2$ are vectors in $\ker(T)$ and if a, b are scalars, then $a\vec{v}_1 + b\vec{v}_2$ is also in $\ker(T)$. But

$$T(a\vec{v}_1 + b\vec{v}_2) = aT(\vec{v}_1) + bT(\vec{v}_2) = a\vec{0} + b\vec{0} = \vec{0}$$

Thus $\ker(T)$ is a subspace of V.

Next suppose $T(\vec{v}_1), T(\vec{v}_2)$ are two vectors in $\operatorname{im}(T)$. Then if a, b are scalars,

$$aT(\vec{v}_2) + bT(\vec{v}_2) = T(a\vec{v}_1 + b\vec{v}_2)$$

and this last vector is in $\operatorname{im}(T)$ by definition. ♠

We will now examine how to find the kernel and image of a linear transformation and describe a basis of each.

Example 5.53: Kernel and Image of a Linear Transformation

Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a-b \\ c+d \end{bmatrix}$$

Then T is a linear transformation. Find a basis for $\ker(T)$ and $\operatorname{im}(T)$.

Solution. You can verify that T is a linear transformation.

First we will find a basis for $\ker(T)$. To do so, we want to find a way to describe all vectors $\vec{x} \in \mathbb{R}^4$ such that $T(\vec{x}) = \vec{0}$. Let $\vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ be such a vector. Then

$$T\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a-b \\ c+d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The values of a, b, c, d that make this true are given by solutions to the system

$$\begin{array}{rcl} a-b & = & 0 \\ c+d & = & 0 \end{array}$$

The solution to this system is $a=s, b=s, c=t, d=-t$ where s,t are scalars. We can describe $\ker(T)$ as follows.

$$\ker(T)=\left\{\left[\begin{array}{r} s \\ s \\ t \\ -t \end{array}\right]\right\}=\operatorname{span}\left\{\left[\begin{array}{r} 1 \\ 1 \\ 0 \\ 0 \end{array}\right],\left[\begin{array}{r} 0 \\ 0 \\ 1 \\ -1 \end{array}\right]\right\}$$

Notice that this set is linearly independent and therefore forms a basis for $\ker(T)$.

We move on to finding a basis for $\operatorname{im}(T)$. We can write the image of T as

$$\operatorname{im}(T)=\left\{\left[\begin{array}{c} a-b \\ c+d \end{array}\right]\right\}$$

We can write this in the form

$$\operatorname{span}=\left\{\left[\begin{array}{r} 1 \\ 0 \end{array}\right],\left[\begin{array}{r} -1 \\ 0 \end{array}\right],\left[\begin{array}{r} 0 \\ 1 \end{array}\right],\left[\begin{array}{r} 0 \\ 1 \end{array}\right]\right\}$$

This set is clearly not linearly independent. By removing unnecessary vectors from the set we can create a linearly independent set with the same span. This gives a basis for $\operatorname{im}(T)$ as

$$\operatorname{im}(T)=\operatorname{span}\left\{\left[\begin{array}{r} 1 \\ 0 \end{array}\right],\left[\begin{array}{r} 0 \\ 1 \end{array}\right]\right\}$$

♠

Recall that a linear transformation T is called one to one if and only if $T(\vec{x})=\vec{0}$ implies $\vec{x}=\vec{0}$. Using the concept of kernel, we can state this theorem in another way.

Theorem 5.54: One to One and Kernel

Let T be a linear transformation where $\ker(T)$ is the kernel of T. Then T is one to one if and only if $\ker(T)$ consists of ***only*** *the zero vector.*

A major result is the relation between the dimension of the kernel and dimension of the image of a linear transformation. In the previous example $\ker(T)$ had dimension 2, and $\operatorname{im}(T)$ also had dimension of 2. Consider the following theorem.

Theorem 5.55: Dimension of Kernel and Image

Let $T: V \rightarrow W$ be a linear transformation where V, W are subspaces of $\mathbb{R}^n$. Suppose the dimension of V is m. Then

$$m=\operatorname{dim}(\ker(T))+\operatorname{dim}(\operatorname{im}(T))$$

Proof. From Proposition 5.52, $\operatorname{im}(T)$ is a subspace of W. We know that there exists a basis for $\operatorname{im}(T)$, written $\{T(\vec{v}_1), \cdots, T(\vec{v}_r)\}$. Similarly, there is a basis for $\ker(T), \{\vec{u}_1, \cdots, \vec{u}_s\}$. Then if $\vec{v} \in V$, there exist scalars c_i such that

$$T(\vec{v})=\sum_{i=1}^r c_i T(\vec{v}_i)$$

Hence $T\left(\vec{v} - \sum_{i=1}^{r} c_i \vec{v}_i\right) = 0$. It follows that $\vec{v} - \sum_{i=1}^{r} c_i \vec{v}_i$ is in $\ker(T)$. Hence there are scalars a_i such that

$$\vec{v} - \sum_{i=1}^{r} c_i \vec{v}_i = \sum_{j=1}^{s} a_j \vec{u}_j$$

Hence $\vec{v} = \sum_{i=1}^{r} c_i \vec{v}_i + \sum_{j=1}^{s} a_j \vec{u}_j$. Since $\vec{v}$ is arbitrary, it follows that

$$V = \operatorname{span}\{\vec{u}_1, \cdots, \vec{u}_s, \vec{v}_1, \cdots, \vec{v}_r\}$$

If the vectors $\{\vec{u}_1, \cdots, \vec{u}_s, \vec{v}_1, \cdots, \vec{v}_r\}$ are linearly independent, then it will follow that this set is a basis for the m-dimensional subspace V. Suppose then that

$$\sum_{i=1}^{r} c_i \vec{v}_i + \sum_{j=1}^{s} a_j \vec{u}_j = 0$$

Apply T to both sides to obtain

$$\sum_{i=1}^{r} c_i T(\vec{v}_i) + \sum_{j=1}^{s} a_j T(\vec{u})_j = \sum_{i=1}^{r} c_i T(\vec{v}_i) = 0$$

Since $\{T(\vec{v}_1), \cdots, T(\vec{v}_r)\}$ is linearly independent, it follows that each $c_i = 0$. Hence $\sum_{j=1}^{s} a_j \vec{u}_j = 0$ and so, since the $\{\vec{u}_1, \cdots, \vec{u}_s\}$ are linearly independent, it follows that each $a_j = 0$ also. Therefore $\{\vec{u}_1, \cdots, \vec{u}_s, \vec{v}_1, \cdots, \vec{v}_r\}$ is a basis for V and so

$$m = s + r = \dim(\ker(T)) + \dim(\operatorname{im}(T))$$

♠

The above theorem leads to the next corollary.

Corollary 5.56

Let $T : V \to W$ be a linear transformation where V, W are subspaces of $\mathbb{R}^n$. Suppose the dimension of V is m. Then

$$\dim(\ker(T)) \leq m$$

$$\dim(\operatorname{im}(T)) \leq m$$

This follows directly from the fact that $m = \dim(\ker(T)) + \dim(\operatorname{im}(T))$.

Consider the following example.

Example 5.57

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$$

Let $\operatorname{im}(T) = W$. Show that T is an isomorphism from $\mathbb{R}^2$ to W. Find a 2×3 matrix A such that the restriction of multiplication by A to W equals T^{-1}.

Solution. Since the two columns of the above matrix are linearly independent, we conclude that $\dim(\mathrm{im}(T)) = 2$ and therefore $\dim(\ker(T)) = 2 - \dim(\mathrm{im}(T)) = 2 - 2 = 0$ by Theorem 5.55. Then by Theorem 5.54 it follows that T is one to one.

Thus T is an isomorphism of $\mathbb{R}^2$ and the two dimensional subspace of $\mathbb{R}^3$ which is the span of the columns of the given matrix. Now in particular,

$$T(\vec{e}_1) = \left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right],\ T(\vec{e}_2) = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$$

Thus

$$T^{-1}\left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right] = \vec{e}_1,\ T^{-1}\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] = \vec{e}_2$$

Extend T^{-1} to all of $\mathbb{R}^3$ by defining

$$T^{-1}\left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right] = \vec{e}_1$$

Notice that the set of vectors

$$\left\{\left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right]\right\}$$

is linearly independent, so T^{-1} can be extended linearly to yield a linear transformation defined on $\mathbb{R}^3$. The requested matrix of T^{-1}, denoted by A, needs to satisfy

$$A\left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]$$

and so

$$A = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]\left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]^{-1} = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

Note that

$$\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right] = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$$

$$\left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$$

so the restriction to W of matrix multiplication by this matrix A yields T^{-1}. ♠

5.8 The General Solution of a Linear System

Outcomes

A. *Use linear transformations to determine the particular solution and general solution to a system of equations.*

It turns out that we can use linear transformations as a way to think about solving systems of linear equations. Indeed given a system of linear equations of the form $A\vec{x} = \vec{b}$, one may rephrase this as $T(\vec{x}) = \vec{b}$ where T is the linear transformation defined by $T(\vec{x}) = A\vec{x}$. With this in mind consider the following definition.

Definition 5.58: Particular Solution of a System of Equations

Suppose a linear system of equations can be written in the form

$$T(\vec{x}) = \vec{b}$$

If $T(\vec{x}_p) = \vec{b}$, then $\vec{x}_p$ is called a ***particular solution*** *of the linear system.*

Recall that a system of equations $A\vec{x} = \vec{b}$ is called homogeneous if $\vec{b} = \vec{0}$. Suppose we represent a homogeneous system of equations by $T(\vec{x}) = \vec{0}$. As discussed in Section 5.7, the $\vec{x}$ for which $T(\vec{x}) = \vec{0}$ form the the null space, or kernel, of T.

We may also refer to the kernel of T as the **solution space** of the equation $T(\vec{x}) = \vec{0}$. Since we can write $T(\vec{x}) = \vec{0}$ as $A\vec{x} = \vec{0}$, you have been solving such equations for quite some time.

We have spent a lot of time finding solutions to systems of equations in general, as well as homogeneous systems. Suppose we look at a system given by $A\vec{x} = \vec{b}$, and consider the related homogeneous system. By this, we mean that we replace $\vec{b}$ by $\vec{0}$ and look at $A\vec{x} = \vec{0}$. It turns out that there is a very important relationship between the solutions of the original system and the solutions of the associated homogeneous system. In the following theorem, we use linear transformations to denote a system of equations. Remember that $T(\vec{x}) = A\vec{x}$.

Theorem 5.59: Particular Solution and General Solution

Suppose $\vec{x}_p$ is a solution to the linear system given by

$$T(\vec{x}) = \vec{b}$$

Then if $\vec{y}$ is any solution to $T(\vec{x}) = \vec{b}$, there exists $\vec{x}_0 \in \ker(T)$ such that

$$\vec{y} = \vec{x}_p + \vec{x}_0$$

Hence, every solution to the linear system can be written as a sum of a particular solution, $\vec{x}_p$, and a solution $\vec{x}_0$ to the associated homogeneous system given by $T(\vec{x}) = \vec{0}$.

Proof. Let $\vec{y}$ be any solution to $T(\vec{x}) = \vec{b}$ and consider $\vec{y} - \vec{x}_p = \vec{y} + (-1)\vec{x}_p$. Then $T(\vec{y} - \vec{x}_p) = T(\vec{y}) - T(\vec{x}_p)$. Since $\vec{y}$ and $\vec{x}_p$ are both solutions to the system, it follows that $T(\vec{y}) = \vec{b}$ and $T(\vec{x}_p) = \vec{b}$.

Hence, $T(\vec{y}) - T(\vec{x}_p) = \vec{b} - \vec{b} = \vec{0}$. Let $\vec{x}_0 = \vec{y} - \vec{x}_p$. Then, $T(\vec{x}_0) = \vec{0}$ so $\vec{x}_0$ is a solution to the associated homogeneous system and so $\vec{x}_0 \in \ker(T)$. Then notice that $\vec{x}_p + \vec{x}_0 = \vec{x}_p + (\vec{y} - \vec{x}_p) = \vec{y}$, and our proof is complete. ♠

Sometimes people remember the above theorem in the following form. The solutions to the system $T(\vec{x}) = \vec{b}$ are given by $\vec{x}_p + \ker(T)$ where $\vec{x}_p$ is a particular solution to $T(\vec{x}) = \vec{b}$.

For now, we have been speaking about the kernel or null space of a linear transformation T. However, we know that every linear transformation T is determined by some matrix A. Therefore, we can also speak about the null space of a matrix. Consider the following example.

Example 5.60: The Null Space of a Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix}$$

Find $\mathrm{null}(A)$. Equivalently, find the solutions to the system of equations $A\vec{x} = \vec{0}$.

Solution. We are asked to find $\left\{\vec{x} : A\vec{x} = \vec{0}\right\}$. In other words we want to solve the system, $A\vec{x} = \vec{0}$. Let

$\vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$. Then this amounts to solving

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is the linear system

$$\begin{array}{c} x + 2y + 3z = 0 \\ 2x + y + z + 2w = 0 \\ 4x + 5y + 7z + 2w = 0 \end{array}$$

To solve, set up the augmented matrix and row reduce to find the reduced row-echelon form.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 4 & 5 & 7 & 2 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{3} & \frac{4}{3} & 0 \\ 0 & 1 & \frac{5}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This yields $x = \frac{1}{3}z - \frac{4}{3}w$ and $y = \frac{2}{3}w - \frac{5}{3}z$. Since $\operatorname{null}(A)$ consists of the solutions to this system, it consists vectors of the form,

$$\begin{bmatrix} \frac{1}{3}z - \frac{4}{3}w \\ \frac{2}{3}w - \frac{5}{3}z \\ z \\ w \end{bmatrix} = z \begin{bmatrix} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -\frac{4}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{bmatrix}$$

♠

Consider the following example.

Example 5.61: A General Solution

The ***general solution*** *of a linear system of equations is the set of all possible solutions. Find the general solution to the linear system,*

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 25 \end{bmatrix}$$

given that $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ *is one solution.*

Solution. Note the matrix of this system is the same as the matrix in Example 5.60. Therefore, from Theorem 5.59, you will obtain all solutions to the above linear system by adding a particular solution $\vec{x}_p$ to the solutions of the associated homogeneous system, $\vec{x}$. One particular solution is given above by

$$\vec{x}_p = \left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \\ 2 \\ 1 \end{array}\right] \tag{5.5}$$

Using this particular solution along with the solutions found in Example 5.60, we obtain the following solutions,

$$z\left[\begin{array}{c} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \\ 0 \end{array}\right] + w\left[\begin{array}{c} -\frac{4}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{array}\right] + \left[\begin{array}{c} 1 \\ 1 \\ 2 \\ 1 \end{array}\right]$$

Hence, any solution to the above linear system is of this form. ♠

5.9 The Coordinates of a Vector Relative to a Basis

Outcomes

A. *Find the coordinates of a vector relative to a given basis.*

B. *Use matrices to change the coordinates of a vector relative to one basis to coordinates relative to another basis.*

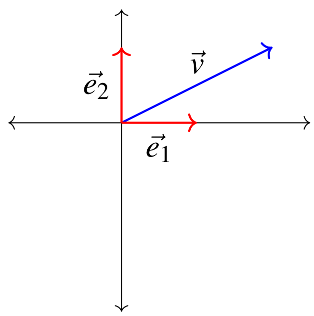
Coordinates of a Vector Relative to a Basis

In the diagram below, we see a vector $\vec{v}$ and the two vectors of the standard basis for $\mathbb{R}^2$. We are used to thinking of $\vec{v}$ algebraically, as $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. For our work to come, it will be important to realize that the algebraic notation is really using the fact that there is one, and only one, way to write $\vec{v}$ as a linear combination of the vectors in the standard basis:

$$\vec{v} = 2\vec{e}_1 + 1\vec{e}_2,$$

and we express this idea by talking about the *coordinates of* $\vec{v}$ *relative to the standard basis.*

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

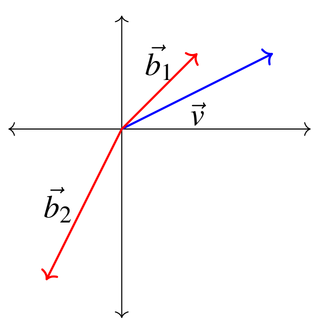


At this point we will introduce a bit of new notation:

$$[\vec{v}]_{Std} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

What this is trying to emphasize is that there is the vector (length and direction, remember?) $\vec{v}$, and we have associated with this geometric object some numbers, the coordinates of $\vec{v}$, but those coordinates depend on the fact that we can find a linear combination of the vectors in the *standard basis* that is equal to $\vec{v}$.

With that introduction, you won't be surprised to find out that now we will ask about expressing $\vec{v}$ as a linear combination of vectors in some other basis, B. Here's a picture:



Here we have the same vector $\vec{v}$, along with two other vectors $\vec{b}_1$ and $\vec{b}_2$. Since $\vec{b}_2$ is not a multiple of $\vec{b}_1$, the set $B = \{\vec{b}_1, \vec{b}_2\}$ is linearly independent and is therefore a basis for $\mathbb{R}^2$. This means that there is a unique way to write $\vec{v}$ as a linear combination of $\vec{b}_1$ and $\vec{b}_2$, and we should be able to use that linear combination to find the *coordinates of $\vec{v}$ relative to the basis B.*

For the picture above, it is the case that $[\vec{b}_1]_{Std} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $[\vec{b}_2]_{Std} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, and since $[\vec{v}]_{Std} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we have

$$\begin{aligned} \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} -1 \\ -2 \end{bmatrix} \\ \vec{v} &= 3\vec{b}_1 + 1\vec{b}_2 \end{aligned}$$

and so $[\vec{v}]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

So the vector $\vec{v}$ can be represented lots of different ways. But if we are given a basis, then there is only one way to write $\vec{v}$ as a linear combination of the vectors in that basis, and that linear combination generates the coordinates of $\vec{v}$ relative to that basis.

Let's formalize that a bit.

Definition 5.62: Coordinate Vector

Let $B = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$ and let $\vec{x}$ be an arbitrary vector in $\mathbb{R}^n$. Then $\vec{x}$ is uniquely represented as $\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ for scalars $a_1, \cdots, a_n$.
*The **coordinate vector of $\vec{x}$ with respect to the basis** B, written $[\vec{x}]_B$, is given by*

$$[\vec{x}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Consider the following example.

Example 5.63: Coordinate Vector

Let $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^2$ and let $\vec{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. Find $[\vec{x}]_B$.

Solution. First, note the order of the basis is important so label the vectors in the basis B as

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \{\vec{v}_1, \vec{v}_2\}$$

Now we need to find a_1 and a_2 such that $\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2$, that is:

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = a_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving this system gives $a_1 = 2, a_2 = -1$. Therefore the coordinate vector of $\vec{x}$ with respect to the basis B is

$$[\vec{x}]_B = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

♠

A couple of things to notice about the last example:

- When we were talking about the vector $\vec{x}$, we just said $\vec{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. We didn't say $[\vec{x}]_{Std} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. If we want to talk about the coordinates of a vector relative to any basis other than the standard basis, we will be explicit about the basis that we're using. Otherwise, just assume that we are talking about the standard basis.
- What we've managed to do, almost without thinking about it, is introduce a *function* that takes as input (the coordinates relative to the standard basis of) a vector and returns the coordinates of the same vector relative to the basis B. This function, the *change of coordinates function* deserves its own section.

The Change of Coordinates Function

Suppose you have a basis B of $\mathbb{R}^n$ and some vector $\vec{x}$. Since you know $\vec{x}$, you automatically know the coordinates of $\vec{x}$ relative to the standard basis, which we will denote $[\vec{x}]_{Std}$. (That is sort of complicated on a first read. Here's an example. Suppose $\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Then $[\vec{x}]_{Std} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. There. Not so bad after all.) We'd like to know the coordinates of $\vec{x}$ relative to the basis B, $[\vec{x}]_B$. We noted above that there is a function that does this. What can we say about that function? Can we easily compute $[\vec{x}]_B$?

Definition 5.64: Change of Coordinates Function

Fix an ordered basis B of $\mathbb{R}^n$. The ***change of coordinates function*** *C_B is the function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^n$ that computes the coordinates of a vector with respect to the basis B. So*

$$\begin{aligned} C_B : \mathbb{R}^n &\to \mathbb{R}^n \\ [\vec{x}]_{Std} &\mapsto [\vec{x}]_B \end{aligned}$$

So $C_B([\vec{x}]_{Std}) = [\vec{x}]_B$.

We think of C_B as changing the coordinates of the vector. Given the coordinates relative to the standard basis, C_B returns the coordinates of the same vector relative to the basis B.

Given any basis B, one can easily verify that the change of coordinates function is actually an isomorphism.

Theorem 5.65: C_B is a Linear Transformation

For any basis B of $\mathbb{R}^n$, the coordinate function

$$C_B : \mathbb{R}^n \to \mathbb{R}^n$$

is a linear transformation, and moreover an isomorphism.

Once we have established that the function C_B is a linear transformation, we know that there is a matrix, we will call it M_B, that represents that linear transformation relative to the standard basis. And finding the matrix M_B is easy: the columns of M_B are just the images of the standard basis vectors under the function C_B. In other words, the columns of M_B are nothing more or less than the coordinates (relative to the basis B) of the vectors in the standard basis.

Let's look at an example:

Example 5.66: Changing Coordinates

Let B be the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\}.$$

Find M_B, the matrix that changes coordinates from the standard coordinates to coordinates relative to B. So we need to find the matrix such that

$$M_B[\vec{x}]_{Std} = [\vec{x}]_B.$$

Solution. The first column of M_B should be the image of $\vec{e}_1$ under the linear transformation C_B. Thus we'd like to know the coordinates of $\vec{e}_1$ relative to the basis B. This means that we need to find the scalars a_1 and a_2 such that

$$a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In other words, we must solve

$$\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

and that gives us the first column of M_B.

By solving the equation

$$b_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

in the same fashion, we find that the second column of M_B is equal to $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$. So

$$M_B = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}.$$

♠

Now, let's look at that solution a little more closely. To find the two columns of M_B we multiplied the matrix $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$ by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and gathered up the solutions into the matrix ... $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$. And where did that 2×2 matrix come from? It was the inverse of the matrix whose columns are exactly the vectors in B. This gives us a recipe for finding the change of coordinates matrix M_B.

Proposition 5.67: Change of Coordinates Matrix

Given a basis B for $\mathbb{R}^n$, M_B, the change of coordinates matrix, is the matrix that has the property that

$$M_B[\vec{x}]_{Std} = [\vec{x}]_B.$$

To find M_B, do the following:

1. *Form A, the $n \times n$ matrix whose columns are the vectors in B.*
2. *Compute A^{-1}.*
3. *That's it. $M_B = A^{-1}$.*

But even better, look at the inverse function C_B^{-1} and its matrix M_B^{-1}. As C_B is an isomorphism, it has an inverse, and to find the matrix transformation associated with C_B^{-1}, we need to find the inverse of M_B. But by the algorithm above, that is simply the matrix A whose columns are the vectors in B.

Let us write things a little more generally. We've been working with two bases, the standard basis and the basis B. But there is no reason to restrict ourselves to working with the standard basis.

Definition 5.68: Change of Coordinates Function

Fix an ordered bases B_1 and B_2 of $\mathbb{R}^n$. The ***change of coordinates function*** *$C_{B_2B_1}$ is the function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^n$ that, given the coordinates of a vector with respect to B_1, computes the coordinates of the same vector with respect to the basis B_2. So*

$$\begin{aligned} C_{B_2B_1} : \mathbb{R}^n &\to \mathbb{R}^n \\ [\vec{x}]_{B_1} &\mapsto [\vec{x}]_{B_2} \end{aligned}$$

So $C_{B_2B_1}([\vec{x}]_{B_1}) = [\vec{x}]_{B_2}$.

Definition 5.69: Change of Coordinates Matrix

Let B_1 and B_2 be two bases for $\mathbb{R}^n$. The ***change of coordinates matrix*** *$M_{B_2B_1}$, is the matrix with the property that*

$$M_{B_2B_1}[\vec{x}]_{B_1} = [\vec{x}]_{B_2}.$$

So $M_{B_2B_1}$ changes coordinates from B_1 to B_2.
If B_1 is the standard basis, instead of writing M_{B_2Std} we will write M_{B_2}.

We will, of course, be interested in finding the matrix $M_{B_2B_1}$. By an argument that is similar to that preceding Proposition 5.67, we have

Proposition 5.70: Finding the Change of Coordinates Matrix

Given bases B_1 and B_2 of $\mathbb{R}^n$, to find $M_{B_2B_1}$, do the following:

1. *Form A_1, the $n \times n$ matrix whose columns are the vectors in B_1.*
2. *Form A_2, the $n \times n$ matrix whose columns are the vectors in B_2.*
3. *Compute $A_2^{-1}A_1$.*
4. *That's it. $M_{B_2B_1} = A_2^{-1}A_1$.*

Let us look at an example.

Example 5.71: Finding the Change of Coordinates Matrix

Suppose that the bases B_1 and B_2 are

$$B_1 = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \end{bmatrix} \right\} \quad B_2 = \{\vec{\beta}_1, \vec{\beta}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\}.$$

Let $\vec{v}$ be the vector $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

- *Use change of coordinates matrices to find $[\vec{v}]_{B_1}$ and $[\vec{v}]_{B_2}$.*
- *Find $M_{B_2B_1}$, the matrix that changes from B_1-coordinates to B_2-coordinates.*
- *Use $M_{B_2B_1}$ and $[\vec{v}]_{B_1}$ to compute $[\vec{v}]_{B_2}$.*

Solution.

- We know the coordinates of $\vec{v}$ with respect to the standard basis and we want the coordinates of $\vec{v}$ with respect to B_1. So we need the matrix M_{B_1Std}, also known as M_{B_1}. Using the algorithm of Proposition 5.67, let
$$A_1 = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}.$$
Then $M_{B_1Std} = A_1^{-1} = \begin{bmatrix} 3/5 & -1/5 \\ 1/5 & -2/5 \end{bmatrix}$ and
$$\begin{aligned} [\vec{v}]_{B_1} &= M_{B_1Std}[\vec{v}]_{Std} \\ &= \begin{bmatrix} 3/5 & -1/5 \\ 1/5 & -2/5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}.$$

To check whether this is correct, we see if the coordinates of $\vec{v}$ with respect to the basis B_1 really do give us $\vec{v}_1$ as a linear combination of the vectors in B_1:

$$\frac{3}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{-4}{5}\begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 6/5 \\ 3/5 \end{bmatrix} + \begin{bmatrix} 4/5 \\ 12/5 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

as needed.

To find $[\vec{v}]_{B_2}$ we argue similarly, using

$$A_2 = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}.$$

So $M_{B_2 Std} = A_2^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$ and

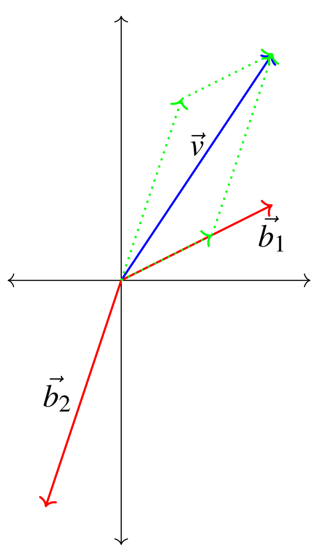
$$[\vec{v}]_{B_2} = M_{B_2 Std}[\vec{v}]_{Std} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

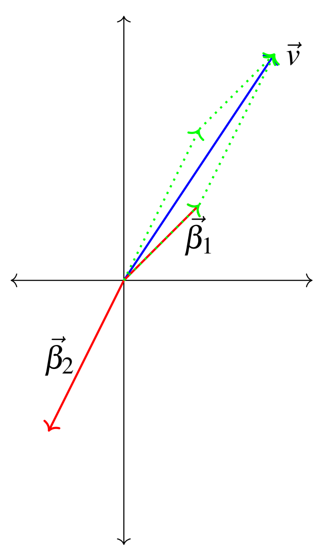
Again, we can check that this gives us the correct linear combination of the basis vectors in B_2 to create the vector $\vec{v}$:

$$1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

as needed.

Here are some pictures to showing that $\vec{v} = \frac{3}{5}\vec{b}_1 + \frac{-4}{5}\vec{b}_2$ and also $\vec{v} = 1\vec{\beta}_1 + (-1)\vec{\beta}_2$:





- We use the algorithm outlined in Proposition 5.70, and the matrices A_1 and A_2 that we found above. We're looking for $M_{B_2B_1}$, and we have

$$\begin{aligned} M_{B_2B_1} &= A_2^{-1}A_1 \\ &= \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}. \end{aligned}$$

- We are asked to use $M_{B_2B_1}$ and $[\vec{v}]_{B_1}$ to compute $[\vec{v}]_{B_2}$.

$$\begin{aligned} [\vec{v}]_{B_2} &= M_{B_2B_1}[\vec{v}]_{B_1} \\ &= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

which shows a pleasing agreement with our earlier calculation.

♠

5.10 The Matrix of a Linear Transformation II

Outcomes

A. *Find the matrix of a linear transformation with respect to general bases.*

The Matrix of a Linear Transformation with Respect to Arbitrary Bases

We know that, given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, there is a matrix A such that $T(\vec{x}) = A\vec{x}$. When we developed all of that machinery, we were working relative to the standard basis: given the coordinates of $\vec{x}$ relative to the standard basis, $A\vec{x}$ is the coordinate vector of $T(\vec{x})$ relative to the standard basis. Our goal now is to show how to represent that linear transformation relative to arbitrary bases.

"But why in the world might we want to do that?" you might reasonably ask. One reason is that there are linear transformations whose matrix (relative to the standard bases) is really complicated, while the matrix of the same transformation relative to other bases might be exceptionally easy, making it easy to analyze the behavior of the transformation. So it is worthwhile to be able to represent linear transformations relative to unusual bases.

But before we can talk about how to do that, we must establish an important lemma.

Lemma 5.72: Mapping of a Basis

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an isomorphism. Then T maps any basis of $\mathbb{R}^n$ to another basis for $\mathbb{R}^n$. Conversely, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation which maps a basis of $\mathbb{R}^n$ to another basis of $\mathbb{R}^n$, then it is an isomorphism.

Proof. First, suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation which is one to one and onto. Let $\left\{\vec{b}_1, \cdots, \vec{b}_n\right\}$ be a basis for $\mathbb{R}^n$. We wish to show that $\left\{T(\vec{b}_1), \cdots, T(\vec{b}_n)\right\}$ is also a basis for $\mathbb{R}^n$.

First consider why it is linearly independent. Suppose $\sum_{k=1}^n a_k T(\vec{b}_k) = \vec{0}$. Then by linearity we have $T\left(\sum_{k=1}^n a_k \vec{b}_k\right) = \vec{0}$ and since T is one to one, it follows that $\sum_{k=1}^n a_k \vec{b}_k = \vec{0}$. This requires that each $a_k = 0$ because $\left\{\vec{b}_1, \cdots, \vec{b}_n\right\}$ is independent, and it follows that $\left\{T(\vec{b}_1), \cdots, T(\vec{b}_n)\right\}$ is linearly independent.

Next take $\vec{w} \in \mathbb{R}^n$. Since T is onto, there exists $\vec{v} \in \mathbb{R}^n$ such that $T(\vec{v}) = \vec{w}$. Since $\left\{\vec{b}_1, \cdots, \vec{b}_n\right\}$ is a basis, in particular it is a spanning set and there are scalars c_k such that $T\left(\sum_{k=1}^n c_k \vec{b}_k\right) = T\left(\vec{b}\right) = \vec{w}$. Therefore $\vec{w} = \sum_{k=1}^n c_k T(\vec{b}_k)$ which is in the span $\left\{T(\vec{b}_1), \cdots, T(\vec{b}_n)\right\}$. Therefore, since $\left\{T(\vec{b}_1), \cdots, T(\vec{b}_n)\right\}$ is both linearly independent and spans $\mathbb{R}^n$, $\left\{T(\vec{b}_1), \cdots, T(\vec{b}_n)\right\}$ is a basis for $\mathbb{R}^n$, as claimed.

Suppose now that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation such that $T(\vec{b}_i) = \vec{w}_i$ where $\left\{\vec{b}_1, \cdots, \vec{b}_n\right\}$ and $\{\vec{w}_1, \cdots, \vec{w}_n\}$ are two bases for $\mathbb{R}^n$. We must show that T is an isomorphism, so we must show that T is both one to one and onto.

To show that T is one to one, let $T\left(\sum_{k=1}^n c_k \vec{b}_k\right) = \vec{0}$. Then $\sum_{k=1}^n c_k T(\vec{b}_k) = \sum_{k=1}^n c_k \vec{w}_k = \vec{0}$. It follows that each $c_k = 0$ because it is given that $\{\vec{w}_1, \cdots, \vec{w}_n\}$ is linearly independent. Hence $T\left(\sum_{k=1}^n c_k \vec{b}_k\right) = \vec{0}$ implies that $\sum_{k=1}^n c_k \vec{b}_k = \vec{0}$ and so T is one to one.

To show that T is onto, let $\vec{w}$ be an arbitrary vector in $\mathbb{R}^n$. This vector can be written as $\vec{w} = \sum_{k=1}^n d_k \vec{w}_k = \sum_{k=1}^n d_k T(\vec{b}_k) = T\left(\sum_{k=1}^n d_k \vec{b}_k\right)$. Therefore, T is also onto. ♠

We can now address the main goal of this section, which is how we can represent a linear transformation with respect to different bases.

We are comfortable with the fact that if T is a linear transformation with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$, then there is an $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$. So linear transformations can easily be computed using matrix multiplication. Furthermore, the columns of A are simply the images of the standard basis vectors $\vec{e}_i$ under the transformation T:

$$A = [T(\vec{e}_1)\ T(\vec{e}_2)\ \cdots\ T(\vec{e}_n)].$$

We are now going to start being careful about the fact that a vector $\vec{x}$ can have coordinates relative to different bases, and so let us rewrite the last paragraph emphasizing the fact that everything that we have done so far has been relative to the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$.

If T is a linear transformation with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$, then there is a matrix A_{StdStd} such that $[T(\vec{x})]_{Std} = A_{StdStd}[\vec{x}]_{Std}$. So linear transformations can easily be computed using matrix multiplication.

Furthermore, the columns of A_{StdStd} are simply the *coordinates relative to the standard basis for* $\mathbb{R}^m$ *of the* images of the *standard basis vectors* $\vec{e}_i$ *of* $\mathbb{R}^n$ under the transformation T:

$$A_{StdStd} = \left[\ [T(\vec{e}_1)]_{Std}\ [T(\vec{e}_2)]_{Std}\ \cdots\ [T(\vec{e}_n)]_{Std}\ \right].$$

Now we are going to think about representing that linear transformation with respect to arbitrary bases. So suppose that $B_1 = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ is a basis for $\mathbb{R}^n$ and B_2 is a basis for $\mathbb{R}^m$. We have a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, and we are looking for a matrix $A_{B_2B_1}$ such that the coordinates (relative to B_2) of the vector $T(\vec{x})$ can be found by multiplying the matrix times the coordinates (relative to B_1) of the vector $\vec{x}$. In other words, we want the matrix $A_{B_2B_1}$ such that

$$[T(\vec{x})]_{B_2} = A_{B_2B_1}[\vec{x}]_{B_1}.$$

Without justifying it yet, let us just state that we can find the matrix $A_{B_2B_1}$ in a fashion that is entirely analogous to the process we already know. The columns of $A_{B_2B_1}$ are simply the *coordinates relative to the basis* B_2 of the images of the *basis vectors* $\vec{b}_i$ *of* $\mathbb{R}^n$ under the transformation T:

$$A_{B_2B_1} = \left[\ [T(\vec{b}_1)]_{B_2}\ T(\vec{b}_2)]_{B_2}\ \cdots\ T(\vec{b}_n)]_{B_2}\ \right].$$

Here is the formal statement of the theorem:

Theorem 5.73: The Matrix of a Linear Transformation

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and suppose that the matrix A_{StdStd} represents the linear transformation T with respect to the standard bases. So

$$[T(\vec{x})]_{Std} = A_{StdStd}[\vec{x}]_{Std}.$$

Let B_1 and B_2 be bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Suppose that $B_1 = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$.
Define the $m \times n$ matrix $A_{B_2B_1}$ by letting the ith column of the matrix be the coordinates, relative to B_2, of the vector $T(\vec{b}_i)$, the image of the ith basis vector from B_1. In other words, let

$$A_{B_2B_1} = \left[\ [T(\vec{b}_1)]_{B_2}\ [T(\vec{b}_2)]_{B_2}\ \cdots\ [T(\vec{b}_n)]_{B_2}\ \right].$$

Finally, let M_{B_1} and M_{B_2} be the change of coordinate matrices from the standard basis to B_1 and B_2, respectively, representing the change of coordinate functions C_{B_1} and C_{B_2}.
Then the following holds:

$$A_{B_2B_1} = M_{B_2}A_{StdStd}M_{B_1}^{-1}. \tag{5.6}$$

and, furthermore, the matrix $A_{B_2B_1}$ represents the linear transformation T relative to the two bases B_1 and B_2. Thus

$$[T(\vec{x})]_{B_2} = A_{B_2B_1}[\vec{x}]_{B_1}.$$

<u>**Proof.**</u> The above equation 5.6 can be represented by the following diagram.

$$\begin{tikzcd}
\mathbb{R}^n_{Std} \arrow[r, "T\,/\,A_{StdStd}"] \arrow[d, swap, "C_{B_1}\,/\,M_{B_1}"] & \mathbb{R}^m_{Std} \arrow[d, "C_{B_2}\,/\,M_{B_2}"] \\
\mathbb{R}^n_{B_1} \arrow[r, swap, "T\,/\,A_{B_2B_1}"] & \mathbb{R}^m_{B_2}
\end{tikzcd}$$

In this diagram, the arrows are labeled with both the linear transformation (e.g., T) and the matrix that represents the linear transformation relative to the given bases (so A_{StdStd} is the matrix such that $[T(\vec{x})]_{Std} = A_{StdStd}[\vec{x}]_{Std}$). The subscripts on the $\mathbb{R}^n$ are suggesting the basis with which we should interpret the elements of $\mathbb{R}^n$. So, for example, in $\mathbb{R}^n_{Std}$, the coordinate vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is $\vec{e}_1$, while in $\mathbb{R}^n_{B_1}$, the same list of coordinates $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ represents the vector $\vec{b}_1$.

We are looking for the matrix of the linear transformation

$$C_{B_2} \circ T \circ C_{B_1}^{-1} : \mathbb{R}^n \to \mathbb{R}^m,$$

which we know exists as C_{B_1} is an isomorphism.

By Theorem 5.7, the columns are given by the images of the coordinate vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

under the function $C_{B_2} \circ T \circ C_{B_1}^{-1}$.

But since (for example)

$$C_{B_1}^{-1} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) = \vec{b}_1,$$

we readily obtain that

$$\begin{aligned} A_{B_2B_1} &= \left[C_{B_2} \circ T \circ C_{B_1}^{-1} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \; C_{B_2} \circ T \circ C_{B_1}^{-1} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \; \cdots \; C_{B_2} \circ T \circ C_{B_1}^{-1} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right) \right] \\ &= \left[C_{B_2}(T(\vec{b}_1)) \;\; C_{B_2}(T(\vec{b}_2)) \;\; \cdots \;\; C_{B_2}(T(\vec{b}_n)) \right] \\ &= \left[\; [T(\vec{b}_1)]_{B_2} \;\; [T(\vec{b}_2)]_{B_2} \;\; \cdots \;\; [T(\vec{b}_n)]_{B_2} \; \right] \end{aligned}$$

and this completes the proof. ♠

Consider the following example.

Example 5.74: Matrix of a Linear Transformation

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T\left(\left[\begin{array}{c} a \\ b \end{array}\right]\right) = \left[\begin{array}{c} b \\ a \end{array}\right]$.

Consider the two bases

$$B_1 = \left\{\vec{b}_1, \vec{b}_2\right\} = \left\{\left[\begin{array}{c} 1 \\ 0 \end{array}\right], \left[\begin{array}{r} -1 \\ 1 \end{array}\right]\right\}$$

and

$$B_2 = \left\{\left[\begin{array}{c} 1 \\ 1 \end{array}\right], \left[\begin{array}{r} 1 \\ -1 \end{array}\right]\right\}$$

Find the matrix A_{B_2,B_1} of T with respect to the bases B_1 and B_2.

Solution. By Theorem 5.73, the columns of $A_{B_2B_1}$ are the coordinate vectors of $T(\vec{b}_1)$ and $T(\vec{b}_2)$ with respect to the basis B_2.

Since

$$T\left(\left[\begin{array}{c} 1 \\ 0 \end{array}\right]\right) = \left[\begin{array}{c} 0 \\ 1 \end{array}\right],$$

a standard calculation yields

$$\left[\begin{array}{c} 0 \\ 1 \end{array}\right] = \left(\frac{1}{2}\right)\left[\begin{array}{c} 1 \\ 1 \end{array}\right] + \left(-\frac{1}{2}\right)\left[\begin{array}{r} 1 \\ -1 \end{array}\right],$$

the first column of $A_{B_2B_1}$ is $\left[\begin{array}{r} \frac{1}{2} \\ -\frac{1}{2} \end{array}\right]$.

The second column is found in a similar way. We have

$$T\left(\left[\begin{array}{r} -1 \\ 1 \end{array}\right]\right) = \left[\begin{array}{r} 1 \\ -1 \end{array}\right],$$

and with respect to B_2 calculate:

$$\left[\begin{array}{r} 1 \\ -1 \end{array}\right] = 0\left[\begin{array}{c} 1 \\ 1 \end{array}\right] + 1\left[\begin{array}{r} 1 \\ -1 \end{array}\right]$$

Hence the second column of $A_{B_2B_1}$ is given by $\left[\begin{array}{c} 0 \\ 1 \end{array}\right]$. We thus obtain

$$A_{B_2B_1} = \left[\begin{array}{rc} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{array}\right]$$

We can verify that this is the correct matrix $A_{B_2B_1}$ on the specific example

$$\vec{v} = \left[\begin{array}{r} 3 \\ -1 \end{array}\right]$$

First applying T gives

$$T(\vec{v}) = T\left(\left[\begin{array}{r} 3 \\ -1 \end{array}\right]\right) = \left[\begin{array}{r} -1 \\ 3 \end{array}\right]$$

and one can compute that

$$C_{B_2}\left(\left[\begin{array}{r} -1 \\ 3 \end{array}\right]\right) = \left[\begin{array}{r} 1 \\ -2 \end{array}\right].$$

On the other hand, one compute $C_{B_1}(\vec{v})$ as

$$C_{B_1}\left(\left[\begin{array}{r} 3 \\ -1 \end{array}\right]\right) = \left[\begin{array}{r} 2 \\ -1 \end{array}\right],$$

and finally applying $A_{B_1B_2}$ gives

$$\left[\begin{array}{rr} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{array}\right]\left[\begin{array}{r} 2 \\ -1 \end{array}\right] = \left[\begin{array}{r} 1 \\ -2 \end{array}\right]$$

as above.

We see that the same vector results from either method, as suggested by Theorem 5.73. ♠

If the bases B_1 and B_2 are equal, say B, then we write A_B instead of A_{BB}. The following example illustrates how to compute such a matrix. Note that this is what we did earlier when we considered only $B_1 = B_2$ to be the standard basis.

Example 5.75: Matrix of a Linear Transformation with respect to an Arbitrary Basis

Consider the basis B of $\mathbb{R}^3$ given by

$$B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{\left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right], \left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array}\right], \left[\begin{array}{r} -1 \\ 1 \\ 0 \end{array}\right]\right\}$$

And let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined on B as:

$$T\left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{r} 1 \\ -1 \\ 1 \end{array}\right], T\left[\begin{array}{r} 1 \\ 1 \\ 1 \end{array}\right] = \left[\begin{array}{r} 1 \\ 2 \\ -1 \end{array}\right], T\left[\begin{array}{r} -1 \\ 1 \\ 0 \end{array}\right] = \left[\begin{array}{r} 0 \\ 1 \\ 1 \end{array}\right]$$

1. *Find the matrix A_B of T relative to the basis B.*
2. *Then find the usual matrix A_{Std} that represents T with respect to the standard basis of $\mathbb{R}^3$.*

Solution.

Equation 5.6 from Theorem 5.73 tells us that $A_B = M_B A_{Std} M_B^{-1}$.

Now $C_B(\vec{b}_i) = \vec{e}_i$, so the matrix M_B^{-1} of the change of coordinates function C_B^{-1} is given by

$$M_B^{-1} = \left[C_B^{-1}(\vec{e}_1)\ C_B^{-1}(\vec{e}_2)\ C_B^{-1}(\vec{e}_2)\right] = \left[\begin{array}{rrr} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{array}\right]$$

Moreover the matrix product $A_{Std}M_B^{-1}$ of the transformation $T \circ C_B^{-1}$ is given by

$$\left[T \circ C_B^{-1}(\vec{e}_1)\ T \circ C_B^{-1}(\vec{e}_2)\ T \circ C_B^{-1}(\vec{e}_3)\right] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Thus

$$\begin{aligned} A_B &= M_B A_{Std} M_B^{-1} \\ &= [M_B^{-1}]^{-1}[A_{Std}M_B^{-1}] \\ &= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -5 & 1 \\ -1 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \end{aligned}$$

Consider how this works. Let $\vec{v}$ be an arbitrary vector in $\mathbb{R}^3$, and suppose that the coordinates of $\vec{v}$ relative to the basis B are $[\vec{v}]_B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Then the product $M_B^{-1}\begin{bmatrix} b_1 \\ b_2 \\ b_2 \end{bmatrix}$ gives us the coordinates of $\vec{v}$ relative to the standard basis:

$$[\vec{v}]_{Std} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Apply T to this linear combination to obtain

$$b_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 + b_2 \\ -b_1 + 2b_2 + b_3 \\ b_1 - b_2 + b_3 \end{bmatrix}$$

which are the coordinates of $T(\vec{v})$ relative to the standard basis.

If we take the product of the matrix A_B of the transformation (as found above) and multiply it by $\vec{b}$, we are supposed to find the coordinates of $T(\vec{v})$ relative to the basis B. So those coordinates should be given by

$$[T(\vec{v})]_B = \begin{bmatrix} 2 & -5 & 1 \\ -1 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2b_1 - 5b_2 + b_3 \\ -b_1 + 4b_2 \\ -2b_2 + b_3 \end{bmatrix}$$

Is this the coordinate vector of the above relative to the given basis B? We check as follows.

$$(2b_1 - 5b_2 + b_3) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-b_1 + 4b_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-2b_2 + b_3) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 + b_2 \\ -b_1 + 2b_2 + b_3 \\ b_1 - b_2 + b_3 \end{bmatrix}$$

and we get the coordinates of $T(\vec{v})$ relative to the standard basis that we found above.

Now we find the matrix of T with respect to the standard basis. Let A_{Std} be this needed matrix. Thus

$$A_{Std} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

as you can check by looking at each column of the product. For example $A_{Std} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

But this means that

$$A_{Std} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 3 & -3 \\ -3 & -2 & 4 \end{bmatrix}$$

Of course this is a very different matrix than the matrix of the linear transformation with respect to the non standard basis B. ♠

Chapter 6

Complex Numbers

6.1 Complex Numbers

Outcomes

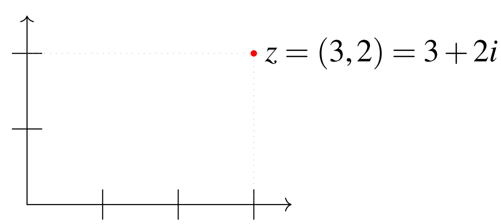
A. *Understand the geometric significance of a complex number as a point in the plane.*

B. *Prove algebraic properties of addition and multiplication of complex numbers, and apply these properties. Understand the action of taking the conjugate of a complex number.*

C. *Understand the absolute value of a complex number and how to find it as well as its geometric significance.*

Although very powerful, the real numbers are inadequate to solve equations such as $x^2 + 1 = 0$, and this is where complex numbers come in. We define the number i as the imaginary number such that $i^2 = -1$, and define complex numbers as those of the form $z = a + bi$ where a and b are real numbers. We call this the standard form, or Cartesian form, of the complex number z. Then, we refer to a as the *real* part of z, and b as the *imaginary* part of z. It turns out that such numbers not only solve the above equation, but in fact also solve any polynomial of degree at least 1 with complex coefficients. This property, called the Fundamental Theorem of Algebra, is sometimes referred to by saying $\mathbb{C}$ is algebraically closed. Gauss is usually credited with giving a proof of this theorem in 1797 but many others worked on it and the first completely correct proof was due to Argand in 1806.

Just as a real number can be considered as a point on the line, a complex number $z = a + bi$ can be considered as a point (a, b) in the plane whose x coordinate is a and whose y coordinate is b. For example, in the following picture, the point $z = 3 + 2i$ can be represented as the point in the plane with coordinates $(3, 2)$.



Addition of complex numbers is defined as follows.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

This addition obeys all the usual properties as the following theorem indicates.

Theorem 6.1: Properties of Addition of Complex Numbers

Let z,w, and v be complex numbers. Then the following properties hold.

- *Commutative Law for Addition*
$$z+w=w+z$$
- *Additive Identity*
$$z+0=z$$
- *Existence of Additive Inverse*
$$\text{For each } z\in\mathbb{C}, \text{there exists } -z\in\mathbb{C} \text{ such that } z+(-z)=0$$
$$\text{In fact if } z=a+bi, \text{ then } -z=-a-bi.$$
- *Associative Law for Addition*
$$(z+w)+v=z+(w+v)$$

The proof of this theorem is left as an exercise for the reader.

Now, multiplication of complex numbers is defined the way you would expect, recalling that $i^2=-1$.

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+i^2bd \\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

Consider the following examples.

Example 6.2: Multiplication of Complex Numbers

- $(2-3i)(-3+4i)=6+17i$
- $(4-7i)(6-2i)=10-50i$
- $(-3+6i)(5-i)=-9+33i$

The following are important properties of multiplication of complex numbers.

Theorem 6.3: Properties of Multiplication of Complex Numbers

Let z, w and v be complex numbers. Then, the following properties of multiplication hold.

- *Commutative Law for Multiplication*
$$zw = wz$$
- *Associative Law for Multiplication*
$$(zw)v = z(wv)$$
- *Multiplicative Identity*
$$1z = z$$
- *Existence of Multiplicative Inverse*
$$\text{For each } z \neq 0, \text{there exists } z^{-1} \text{ such that } zz^{-1} = 1$$
- *Distributive Law*
$$z(w+v) = zw + zv$$

You may wish to verify some of these statements. The real numbers also satisfy the above axioms, and in general any mathematical structure which satisfies these axioms is called a field. There are many other fields, in particular even finite ones particularly useful for cryptography, and the reason for specifying these axioms is that linear algebra is all about fields and we can do just about anything in this subject using any field. Although here, the fields of most interest will be the familiar field of real numbers, denoted as $\mathbb{R}$, and the field of complex numbers, denoted as $\mathbb{C}$.

An important construction regarding complex numbers is the complex conjugate denoted by a horizontal line above the number, $\overline{z}$. It is defined as follows.

Definition 6.4: Conjugate of a Complex Number

Let $z = a + bi$ be a complex number. Then the ***conjugate*** *of z, written $\overline{z}$ is given by*
$$\overline{a+bi} = a - bi$$

Geometrically, the action of the conjugate is to reflect a given complex number across the x axis. Algebraically, it changes the sign on the imaginary part of the complex number. Therefore, for a real number a, $\overline{a} = a$.

Example 6.5: Conjugate of a Complex Number

- *If* $z = 3+4i$*, then* $\overline{z} = 3-4i$*, i.e.,* $\overline{3+4i} = 3-4i$.
- $\overline{-2+5i} = -2-5i$.
- $\overline{i} = -i$.
- $\overline{7} = 7$.

Consider the following computation.

$$\begin{array}{rcl} \left(\overline{a+bi}\right)(a+bi) & = & (a-bi)(a+bi) \\ & = & a^2+b^2-(ab-ab)\,i = a^2+b^2 \end{array}$$

Notice that there is no imaginary part in the product, thus multiplying a complex number by its conjugate results in a real number.

Theorem 6.6: Properties of the Conjugate

Let z *and* w *be complex numbers. Then, the following properties of the conjugate hold.*

- $\overline{z \pm w} = \overline{z} \pm \overline{w}$.
- $\overline{(zw)} = \overline{z}\,\overline{w}$.
- $\overline{(\overline{z})} = z$.
- $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$.
- z *is real if and only if* $\overline{z} = z$.

Division of complex numbers is defined as follows. Let $z = a+bi$ and $w = c+di$ be complex numbers such that c,d are not both zero. Then the quotient z divided by w is

$$\begin{array}{rcl} \dfrac{z}{w} = \dfrac{a+bi}{c+di} & = & \dfrac{a+bi}{c+di} \times \dfrac{c-di}{c-di} \\ & = & \dfrac{(ac+bd)+(bc-ad)i}{c^2+d^2} \\ & = & \dfrac{ac+bd}{c^2+d^2} + \dfrac{bc-ad}{c^2+d^2}i. \end{array}$$

In other words, the quotient $\frac{z}{w}$ is obtained by multiplying both top and bottom of $\frac{z}{w}$ by $\overline{w}$ and then simplifying the expression.

Example 6.7: Division of Complex Numbers

- $$\frac{1}{i} = \frac{1}{i} \times \frac{-i}{-i} = \frac{-i}{-i^2} = -i$$

- $$\frac{2-i}{3+4i} = \frac{2-i}{3+4i} \times \frac{3-4i}{3-4i} = \frac{(6-4)+(-3-8)i}{3^2+4^2} = \frac{2-11i}{25} = \frac{2}{25} - \frac{11}{25}i$$

- $$\frac{1-2i}{-2+5i} = \frac{1-2i}{-2+5i} \times \frac{-2-5i}{-2-5i} = \frac{(-2-10)+(4-5)i}{2^2+5^2} = -\frac{12}{29} - \frac{1}{29}i$$

Interestingly every nonzero complex number $a+bi$ has a unique multiplicative inverse. In other words, for a nonzero complex number z, there exists a number z^{-1} (or $\frac{1}{z}$) so that $zz^{-1} = 1$. Note that $z = a+bi$ is nonzero exactly when $a^2+b^2 \neq 0$, and its inverse can be written in standard form as defined now.

Definition 6.8: Inverse of a Complex Number

Let $z = a+bi$ be a complex number. Then the multiplicative inverse of z, written z^{-1} exists if and only if $a^2+b^2 \neq 0$ and is given by

$$z^{-1} = \frac{1}{a+bi} = \frac{1}{a+bi} \times \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

Note that we may write z^{-1} as $\frac{1}{z}$. Both notations represent the multiplicative inverse of the complex number z. Consider now an example.

Example 6.9: Inverse of a Complex Number

Consider the complex number $z = 2+6i$. Then z^{-1} is defined, and

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{2+6i} \\
&= \frac{1}{2+6i} \times \frac{2-6i}{2-6i} \\
&= \frac{2-6i}{2^2+6^2} \\
&= \frac{2-6i}{40} \\
&= \frac{1}{20} - \frac{3}{20}i
\end{aligned}$$

You can always check your answer by computing zz^{-1}.

Another important construction of complex numbers is that of the absolute value, also called the modulus. Consider the following definition.

Definition 6.10: Absolute Value

The absolute value, or modulus, of a complex number, denoted $|z|$ *is defined as follows.*

$$|a+bi| = \sqrt{a^2+b^2}$$

Thus, if z is the complex number $z = a+bi$, it follows that

$$|z| = (z\bar{z})^{1/2}$$

Also from the definition, if $z = a+bi$ and $w = c+di$ are two complex numbers, then $|zw| = |z|\,|w|$. Take a moment to verify this.

The triangle inequality is an important property of the absolute value of complex numbers. There are two useful versions which we present here, although the first one is officially called the triangle inequality.

Proposition 6.11: Triangle Inequality

Let z, w *be complex numbers.*
The following two inequalities hold for any complex numbers z, w*:*

$$\begin{gathered} |z+w| \leq |z|+|w| \\ ||z|-|w|| \leq |z-w| \end{gathered}$$

The first one is called the Triangle Inequality.

Proof. Let $z = a+bi$ and $w = c+di$. First note that

$$z\overline{w} = (a+bi)(c-di) = ac+bd+(bc-ad)\,i$$

and so $|ac+bd| \leq |z\overline{w}| = |z|\,|w|$.

Then,

$$\begin{gathered} |z+w|^2 = (a+c+i(b+d))(a+c-i(b+d)) \\ = (a+c)^2+(b+d)^2 = a^2+c^2+2ac+2bd+b^2+d^2 \\ \leq |z|^2+|w|^2+2|z|\,|w| = (|z|+|w|)^2 \end{gathered}$$

Taking the square root, we have that

$$|z+w| \leq |z|+|w|$$

so this verifies the triangle inequality.

To get the second inequality, write

$$z = z-w+w, \; w = w-z+z$$

and so by the first form of the inequality we get both:

$$|z| \leq |z-w|+|w|, \; |w| \leq |z-w|+|z|$$

Hence, both $|z|-|w|$ and $|w|-|z|$ are no larger than $|z-w|$. This proves the second version because $||z|-|w||$ is one of $|z|-|w|$ or $|w|-|z|$. ♠

With this definition, it is important to note the following. You may wish to take the time to verify this remark.

Let $z=a+bi$ and $w=c+di$. Then $|z-w|=\sqrt{(a-c)^2+(b-d)^2}$. Thus the distance between the point in the plane determined by the ordered pair (a,b) and the ordered pair (c,d) equals $|z-w|$ where z and w are as just described.

For example, consider the distance between $(2,5)$ and $(1,8)$. Letting $z=2+5i$ and $w=1+8i$, $z-w=1-3i$, $(z-w)\overline{(z-w)}=(1-3i)(1+3i)=10$ so $|z-w|=\sqrt{10}$.

Recall that we refer to $z=a+bi$ as the standard form of the complex number. In the next section, we examine another form in which we can express the complex number.

6.2 Polar Form

Outcomes

A. *Convert a complex number from standard form to polar form, and from polar form to standard form.*

In the previous section, we identified a complex number $z=a+bi$ with a point (a,b) in the coordinate plane. There is another form in which we can express the same number, called the *polar form*. The polar form is the focus of this section. It will turn out to be very useful if not crucial for certain calculations as we shall soon see.

Suppose $z=a+bi$ is a complex number, and let $r=\sqrt{a^2+b^2}=|z|$. Recall that r is the **modulus** of z

. Note first that
$$\left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = \frac{a^2+b^2}{r^2} = 1$$
and so $\left(\frac{a}{r}, \frac{b}{r}\right)$ is a point on the unit circle. Therefore, there exists an angle θ (in radians) such that
$$\cos\theta = \frac{a}{r},\ \sin\theta = \frac{b}{r}$$
In other words θ is an angle such that $a = r\cos\theta$ and $b = r\sin\theta$, that is $\theta = \cos^{-1}(a/r)$ and $\theta = \sin^{-1}(b/r)$. We call this angle θ the **argument** of z.

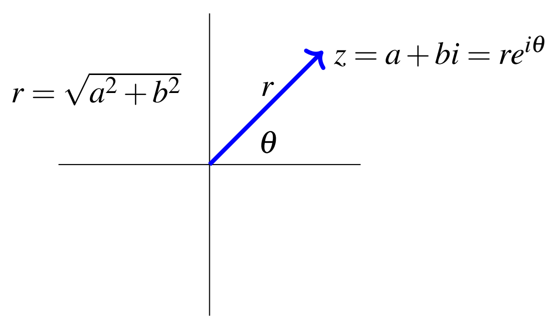
We often speak of the **principal argument** of z. This is the unique angle $\theta \in (-\pi, \pi]$ such that
$$\cos\theta = \frac{a}{r},\ \sin\theta = \frac{b}{r}$$

The polar form of the complex number $z = a + bi = r(\cos\theta + i\sin\theta)$ is for convenience written as:
$$z = re^{i\theta}$$
where θ is the argument of z.

Definition 6.12: Polar Form of a Complex Number

Let $z = a+bi$ be a complex number. Then the **polar form** *of z is written as*
$$z = re^{i\theta}$$
where $r = \sqrt{a^2+b^2}$ and θ is the argument of z.

When given $z = re^{i\theta}$, the identity $e^{i\theta} = \cos\theta + i\sin\theta$ will convert z back to standard form. Here we think of $e^{i\theta}$ as a short cut for $\cos\theta + i\sin\theta$. This is all we will need in this course, but in reality $e^{i\theta}$ can be considered as the complex equivalent of the exponential function where this turns out to be a true equality.



Thus we can convert any complex number in the standard (Cartesian) form $z = a+bi$ into its polar form. Consider the following example.

Example 6.13: Standard to Polar Form

Let $z = 2+2i$ be a complex number. Write z in the polar form

$$z = re^{i\theta}$$

Solution. First, find r. By the above discussion, $r = \sqrt{a^2+b^2} = |z|$. Therefore,

$$r = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$$

Now, to find θ, we plot the point $(2,2)$ and find the angle from the positive x axis to the line between this point and the origin. In this case, $\theta = 45^\circ = \frac{\pi}{4}$. That is we found the unique angle θ such that $\theta = \cos^{-1}(1/\sqrt{2})$ and $\theta = \sin^{-1}(1/\sqrt{2})$.

Note that in polar form, we always express angles in radians, not degrees.

Hence, we can write z as

$$z = 2\sqrt{2}e^{i\frac{\pi}{4}}$$

♠

Notice that the standard and polar forms are completely equivalent. That is not only can we transform a complex number from standard form to its polar form, we can also take a complex number in polar form and convert it back to standard form.

Example 6.14: Polar to Standard Form

Let $z = 2e^{2\pi i/3}$. Write z in the standard form

$$z = a+bi$$

Solution. Let $z = 2e^{2\pi i/3}$ be the polar form of a complex number. Recall that $e^{i\theta} = \cos\theta + i\sin\theta$. Therefore using standard values of sin and cos we get:

$$\begin{aligned} z = 2e^{i2\pi/3} &= 2(\cos(2\pi/3) + i\sin(2\pi/3)) \\ &= 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= -1+\sqrt{3}i \end{aligned}$$

which is the standard form of this complex number. ♠

You can always verify your answer by converting it back to polar form and ensuring you reach the original answer.

6.3 Roots of Complex Numbers

Outcomes

A. *Understand De Moivre's theorem and be able to use it to find the roots of a complex number.*

A fundamental identity is the formula of De Moivre with which we begin this section.

Theorem 6.15: De Moivre's Theorem

For any positive integer n, we have

$$\left(e^{i\theta}\right)^n = e^{in\theta}$$

Thus for any real number $r > 0$ and any positive integer n, we have:

$$\left(r\left(\cos\theta + i\sin\theta\right)\right)^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

Proof. The proof is by induction on n. It is clear the formula holds if $n = 1$. Suppose it is true for n. Then, consider $n+1$.

$$\left(r\left(\cos\theta + i\sin\theta\right)\right)^{n+1} = \left(r\left(\cos\theta + i\sin\theta\right)\right)^n\left(r\left(\cos\theta + i\sin\theta\right)\right)$$

which by induction equals

$$\begin{aligned}
&= r^{n+1}\left(\cos n\theta + i\sin n\theta\right)\left(\cos\theta + i\sin\theta\right) \\
&= r^{n+1}\left(\left(\cos n\theta\cos\theta - \sin n\theta\sin\theta\right) + i\left(\sin n\theta\cos\theta + \cos n\theta\sin\theta\right)\right) \\
&= r^{n+1}\left(\cos\left(n+1\right)\theta + i\sin\left(n+1\right)\theta\right)
\end{aligned}$$

by the formulas for the cosine and sine of the sum of two angles. ♠

The process used in the previous proof, called *mathematical induction* is very powerful in Mathematics and Computer Science and explored in more detail in the Appendix.

Now, consider a corollary of Theorem 6.15.

Corollary 6.16: Roots of Complex Numbers

Let z be a non zero complex number. Then there are always exactly k many k^{th} roots of z in $\mathbb{C}$.

Proof. Let $z = a + bi$ and let $z = |z|(\cos\theta + i\sin\theta)$ be the polar form of the complex number. By De Moivre's theorem, a complex number

$$w = re^{i\alpha} = r(\cos\alpha + i\sin\alpha)$$

is a k^{th} root of z if and only if

$$w^k = (re^{i\alpha})^k = r^k e^{ik\alpha} = r^k(\cos k\alpha + i\sin k\alpha) = |z|(\cos\theta + i\sin\theta)$$

This requires $r^k = |z|$ and so $r = |z|^{1/k}$. Also, both $\cos(k\alpha) = \cos\theta$ and $\sin(k\alpha) = \sin\theta$. This can only happen if

$$k\alpha = \theta + 2\ell\pi$$

for ℓ an integer. Thus

$$\alpha = \frac{\theta + 2\ell\pi}{k},\ \ell = 0, 1, 2, \cdots, k-1$$

and so the k^{th} roots of z are of the form

$$|z|^{1/k}\left(\cos\left(\frac{\theta + 2\ell\pi}{k}\right) + i\sin\left(\frac{\theta + 2\ell\pi}{k}\right)\right),\ \ell = 0, 1, 2, \cdots, k-1$$

Since the cosine and sine are periodic of period 2π, there are exactly k distinct numbers which result from this formula. ♠

The procedure for finding the k k^{th} roots of $z \in \mathbb{C}$ is as follows.

Procedure 6.17: Finding Roots of a Complex Number

Let w be a complex number. We wish to find the n^{th} roots of w, that is all z such that $z^n = w$. There are n distinct n^{th} roots and they can be found as follows:.

1. *Express both z and w in polar form $z = re^{i\theta}, w = se^{i\phi}$. Then $z^n = w$ becomes:*

$$(re^{i\theta})^n = r^n e^{in\theta} = se^{i\phi}$$

   We need to solve for r and θ.

2. *Solve the following two equations:*

$$\begin{aligned} r^n &= s \\ e^{in\theta} &= e^{i\phi} \end{aligned} \tag{6.1}$$

3. *The solutions to $r^n = s$ are given by $r = \sqrt[n]{s}$.*

4. *The solutions to $e^{in\theta} = e^{i\phi}$ are given by:*

$$n\theta = \phi + 2\pi\ell, \text{ for } \ell = 0,1,2,\cdots,n-1$$

   or

$$\theta = \frac{\phi}{n} + \frac{2}{n}\pi\ell, \text{ for } \ell = 0,1,2,\cdots,n-1$$

5. *Using the solutions r, θ to the equations given in (6.1) construct the n^{th} roots of the form $z = re^{i\theta}$.*

Notice that once the roots are obtained in the final step, they can then be converted to standard form if necessary. Let's consider an example of this concept. Note that according to Corollary 6.16, there are exactly 3 cube roots of a complex number.

Example 6.18: Finding Cube Roots

Find the three cube roots of i. In other words find all z such that $z^3 = i$.

Solution. First, convert each number to polar form: $z = re^{i\theta}$ and $i = 1e^{i\pi/2}$. The equation now becomes

$$(re^{i\theta})^3 = r^3 e^{3i\theta} = 1e^{i\pi/2}$$

Therefore, the two equations that we need to solve are $r^3 = 1$ and $3i\theta = i\pi/2$. Given that $r \in \mathbb{R}$ and $r^3 = 1$ it follows that $r = 1$.

Solving the second equation is as follows. First divide by i. Then, since the argument of i is not unique we write $3\theta = \pi/2 + 2\pi\ell$ for $\ell = 0,1,2$.

$$3\theta = \pi/2 + 2\pi\ell \text{ for } \ell = 0,1,2$$

$$\theta = \pi/6 + \frac{2}{3}\pi\ell \text{ for } \ell = 0,1,2$$

For $\ell = 0$:

$$\theta = \pi/6 + \frac{2}{3}\pi(0) = \pi/6$$

For $\ell = 1$:

$$\theta = \pi/6 + \frac{2}{3}\pi(1) = \frac{5}{6}\pi$$

For $\ell = 2$:

$$\theta = \pi/6 + \frac{2}{3}\pi(2) = \frac{3}{2}\pi$$

Therefore, the three roots are given by

$$1e^{i\pi/6}, 1e^{i\frac{5}{6}\pi}, 1e^{i\frac{3}{2}\pi}$$

Written in standard form, these roots are, respectively,

$$\frac{\sqrt{3}}{2} + i\frac{1}{2}, -\frac{\sqrt{3}}{2} + i\frac{1}{2}, -i$$

♠

The ability to find k^{th} roots can also be used to factor some polynomials.

Example 6.19: Solving a Polynomial Equation

Factor the polynomial $x^3 - 27$.

Solution. First find the cube roots of 27. By the above procedure , these cube roots are $3, 3\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)$, and $3\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)$. You may wish to verify this using the above steps.

Therefore, $x^3 - 27 =$

$$(x-3)\left(x - 3\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)\right)\left(x - 3\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)\right)$$

Note also $\left(x - 3\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)\right)\left(x - 3\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)\right) = x^2 + 3x + 9$ and so

$$x^3 - 27 = (x-3)\left(x^2 + 3x + 9\right)$$

where the quadratic polynomial $x^2 + 3x + 9$ cannot be factored without using complex numbers. ♠

Note that even though the polynomial $x^3 - 27$ has all real coefficients, it has some complex zeros, $3\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)$, and $3\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)$. These zeros are complex conjugates of each other. It is always the case that if a polynomial has real coefficients and a complex root, it will also have a root equal to the complex conjugate.

6.4 The Quadratic Formula

Outcomes

A. *Use the Quadratic Formula to find the complex roots of a quadratic equation.*

The roots (or solutions) of a quadratic equation $ax^2+bx+c=0$ where a,b,c are real numbers are obtained by solving the familiar quadratic formula given by

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

When working with real numbers, we cannot solve this formula if $b^2-4ac<0$. However, complex numbers allow us to find square roots of negative numbers, and the quadratic formula remains valid for finding roots of the corresponding quadratic equation. In this case there are exactly two distinct (complex) square roots of b^2-4ac, which are $i\sqrt{4ac-b^2}$ and $-i\sqrt{4ac-b^2}$.

Here is an example.

Example 6.20: Solutions to Quadratic Equation

Find the solutions to $x^2+2x+5=0$.

Solution. In terms of the quadratic equation above, $a=1$, $b=2$, and $c=5$. Therefore, we can use the quadratic formula with these values, which becomes

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)}$$

Solving this equation, we see that the solutions are given by

$$x = \frac{-2i \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

We can verify that these are solutions of the original equation. We will show $x = -1+2i$ and leave $x = -1-2i$ as an exercise.

$$\begin{aligned} x^2 + 2x + 5 &= (-1+2i)^2 + 2(-1+2i) + 5 \\ &= 1 - 4i - 4 - 2 + 4i + 5 \\ &= 0 \end{aligned}$$

Hence $x = -1+2i$ is a solution. ♠

What if the coefficients of the quadratic equation are actually complex numbers? Does the formula hold even in this case? The answer is yes. This is a hint on how to do Problem **??** below, a special case of the fundamental theorem of algebra, and an ingredient in the proof of some versions of this theorem.

Consider the following example.

Example 6.21: Solutions to Quadratic Equation

Find the solutions to $x^2 - 2ix - 5 = 0$.

Solution. In terms of the quadratic equation above, $a = 1$, $b = -2i$, and $c = -5$. Therefore, we can use the quadratic formula with these values, which becomes

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2i \pm \sqrt{(-2i)^2 - 4(1)(-5)}}{2(1)}$$

Solving this equation, we see that the solutions are given by

$$x = \frac{2i \pm \sqrt{-4+20}}{2} = \frac{2i \pm 4}{2} = i \pm 2$$

We can verify that these are solutions of the original equation. We will show $x = i+2$ and leave $x = i-2$ as an exercise.

$$\begin{aligned} x^2 - 2ix - 5 &= (i+2)^2 - 2i(i+2) - 5 \\ &= -1 + 4i + 4 + 2 - 4i - 5 \end{aligned}$$

$$= \quad 0$$

Hence $x = i + 2$ is a solution. ♠

We conclude this section by stating an essential theorem.

Theorem 6.22: The Fundamental Theorem of Algebra

Any polynomial of degree at least 1 *with complex coefficients has a root which is a complex number.*

Chapter 7

Spectral Theory

7.1 Eigenvalues and Eigenvectors of a Matrix

Outcomes

A. *Describe eigenvalues geometrically and algebraically.*

B. *Find eigenvalues and eigenvectors for a square matrix.*

Spectral Theory refers to the study of eigenvalues and eigenvectors of a matrix. It is of fundamental importance in many areas and is the subject of our study for this chapter.

Definition of Eigenvectors and Eigenvalues

In this section, we will work with the entire set of complex numbers, denoted by $\mathbb{C}$. Recall that the real numbers, $\mathbb{R}$ are contained in the complex numbers, so the discussions in this section apply to both real and complex numbers. For clarity, most of our examples and exposition will take place using real numbers but we will try to point out places where the fact that we are officially working with the complex numbers makes the mathematics cleaner.

To illustrate the idea behind what will be discussed, consider the following example.

Example 7.1: Eigenvectors and Eigenvalues

Let

$$A = \left[\begin{array}{rrr} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{array} \right]$$

Compute the product $A\vec{x}$ for

$$\vec{x} = \left[\begin{array}{r} 5 \\ -4 \\ 3 \end{array} \right] \text{ and } \vec{x} = \left[\begin{array}{r} 1 \\ 0 \\ 0 \end{array} \right].$$

What do you notice about $A\vec{x}$ in each of these products?

Solution. First, compute $A\vec{x}$ for

$$\vec{x} = \left[\begin{array}{r} 5 \\ -4 \\ 3 \end{array} \right]$$

This product is given by

$$A\vec{x} = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -50 \\ -40 \\ 30 \end{bmatrix} = 10 \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$$

In this case, the product $A\vec{x}$ resulted in a vector which is equal to 10 times the vector $\vec{x}$. In other words, $A\vec{x} = 10\vec{x}$.

Let's see what happens in the next product. Compute $A\vec{x}$ for the vector

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This product is given by

$$A\vec{x} = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

In this case, the product $A\vec{x}$ resulted in a vector equal to 0 times the vector $\vec{x}$, $A\vec{x} = 0\vec{x}$.

Perhaps this matrix is such that $A\vec{x}$ results in $k\vec{x}$, for every vector $\vec{x}$. However, consider

$$\begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 38 \\ -11 \end{bmatrix}$$

In this case, $A\vec{x}$ did not result in a vector of the form $k\vec{x}$ for some scalar k. ♠

There is something special about the first two products calculated in Example 7.1. Notice that for each, $A\vec{x} = k\vec{x}$ where k is some scalar. When this equation holds for some $\vec{x}$ and k, we call the scalar k an **eigenvalue** of A. Traditionally mathematicians use the special symbol λ (the Greek letter lambda) instead of k when referring to eigenvalues. In Example 7.1, the values 10 and 0 are eigenvalues for the matrix A and we can label these as $\lambda_1 = 10$ and $\lambda_2 = 0$.

When $A\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$, we call such an $\vec{x}$ an **eigenvector** of the matrix A. The eigenvectors of A are associated to an eigenvalue. Hence, if λ_1 is an eigenvalue of A and $A\vec{x} = \lambda_1\vec{x}$, we can label this eigenvector as $\vec{x}_1$. Note again that in order to be an eigenvector, $\vec{x}$ must be a nonzero vector.

There is also a geometric significance to eigenvectors. When you have a **nonzero** vector which, when multiplied by a matrix results in another vector which is parallel to the first or equal to $\vec{0}$, this vector is called an eigenvector of the matrix. This is the meaning when the vectors are in $\mathbb{R}^n$ and λ is a real number.

The formal definition of eigenvalues and eigenvectors is as follows.

Definition 7.2: Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix and let $\vec{x} \in \mathbb{C}^n$ be a ***nonzero vector*** *for which*

$$A\vec{x} = \lambda \vec{x} \tag{7.1}$$

for some scalar λ. Then λ is called an ***eigenvalue*** *of the matrix A and $\vec{x}$ is called an* ***eigenvector*** *of A associated with λ, or a λ-eigenvector of A.*
The set of all eigenvalues of an $n \times n$ matrix A is denoted by $\sigma(A)$ and is referred to as the ***spectrum*** *of A.*

The eigenvectors of a matrix A are those vectors $\vec{x}$ for which multiplication by A results in a vector in the same direction or opposite direction to $\vec{x}$. Since the zero vector $\vec{0}$ has no direction this would make no sense for the zero vector. As noted above, 0 is never allowed to be an eigenvector.

Let's look at eigenvectors in more detail. Suppose $\vec{x}$ satisfies 7.1. Then

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$
or
$$(A - \lambda I)\vec{x} = \vec{0}$$

for some $\vec{x} \neq \vec{0}$. Equivalently you could write $(\lambda I - A)\vec{x} = \vec{0}$, which is more commonly used. Hence, when we are looking for eigenvectors, we are looking for nontrivial solutions to this homogeneous system of equations!

Recall that the solutions to a homogeneous system of equations consist of the linear combinations of those basic solutions. In this context, we call the basic solutions of the homogeneous system of equations $(\lambda I - A)\vec{x} = \vec{0}$ the **basic eigenvectors** corresponding to λ. Note that these basic eigenvectors cannot be zero, and it follows that any (nonzero) linear combination of basic eigenvectors is again an eigenvector.

Definition 7.3: Basic Eigenvectors

The basic eigenvectors corresponding to an eigenvalue λ of a matrix A are the (nonzero) basic solutions of the homogeneous system of equations $(\lambda I - A)\vec{x} = \vec{0}$.

Suppose the matrix $(\lambda I - A)$ is invertible, so that $(\lambda I - A)^{-1}$ exists. Then the following equation would be true.

$$\begin{aligned}
\vec{x} &= I\vec{x} \\
&= \left((\lambda I - A)^{-1}(\lambda I - A)\right)\vec{x} \\
&= (\lambda I - A)^{-1}((\lambda I - A)\vec{x}) \\
&= (\lambda I - A)^{-1}\vec{0} \\
&= \vec{0}
\end{aligned}$$

This claims that $\vec{x} = \vec{0}$. However, we have required that $\vec{x} \neq 0$. Therefore $(\lambda I - A)$ cannot have an inverse!

Recall that if a matrix is not invertible, then its determinant is equal to 0. Therefore we can conclude that

$$\det(\lambda I - A) = 0 \tag{7.2}$$

Note that this is equivalent to $\det(A - \lambda I) = 0$.

The expression $\det(xI - A)$ is a polynomial (in the variable x) called the **characteristic polynomial of** A, and $\det(xI - A) = 0$ is called the **characteristic equation**. For this reason we may also refer to the eigenvalues of A as **characteristic values**, but the former is often used for historical reasons.

The following theorem claims that the roots of the characteristic polynomial are the eigenvalues of A. Thus when 7.2 holds, A has a nonzero eigenvector.

Theorem 7.4: The Existence of an Eigenvector

Let A be an $n \times n$ matrix and suppose $\det(\lambda I - A) = 0$ for some $\lambda \in \mathbb{C}$.
Then λ is an eigenvalue of A and thus there exists a nonzero vector $\vec{x} \in \mathbb{C}^n$ such that $A\vec{x} = \lambda \vec{x}$.

Proof. For A an $n \times n$ matrix, the method of Laplace Expansion demonstrates that $\det(\lambda I - A)$ is a polynomial of degree n. As such, the equation 7.2 has a solution $\lambda \in \mathbb{C}$ by the Fundamental Theorem of Algebra. The fact that λ is an eigenvalue is left as an exercise. ♠

Finding Eigenvectors and Eigenvalues

Now that eigenvalues and eigenvectors have been defined, we will study how to find them for a matrix A.

First, consider the following definition.

Definition 7.5: Algebraic Multiplicity of an Eigenvalue

Let A be an $n \times n$ matrix with characteristic polynomial given by $\det(xI - A)$. Suppose that λ is an eigenvalue for A. This means that the characteristic polynomial factors so that

$$\det(xI - A) = (x - \lambda)^n p(x),$$

where λ is not a root of the polynomial $p(x)$.
The ***algebraic multiplicity of*** λ *(or simply multiplicity if the context is clear) is defined to be the integer n. So the algebraic multiplicity of λ is the number of times that λ occurs as a root of the characteristic polynomial.*

For example, suppose the characteristic polynomial of A is given by $(x-2)^2$. Solving for the roots of this polynomial, we set $(x-2)^2 = 0$ and solve for x. We find that $\lambda = 2$ is a root that occurs twice. Hence, in this case, $\lambda = 2$ is an eigenvalue of A of algebraic multiplicity equal to 2.

We will now look at how to find the eigenvalues and eigenvectors for a matrix A in detail. The steps used are summarized in the following procedure.

Procedure 7.6: Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. *First, find the eigenvalues λ of A by solving the equation $\det(xI - A) = 0$.*
2. *For each λ, find the basic eigenvectors $\vec{x} \neq \vec{0}$ by finding the basic solutions to $(\lambda I - A)\vec{x} = \vec{0}$.*

To verify your work, make sure that $A\vec{x} = \lambda\vec{x}$ for each λ and associated eigenvector $\vec{x}$.

We will explore these steps further in the following example.

Example 7.7: Find the Eigenvalues and Eigenvectors

Let $A = \left[\begin{array}{rr} -5 & 2 \\ -7 & 4 \end{array}\right]$. Find its eigenvalues and eigenvectors.

Solution. We will use Procedure 7.6. First we find the eigenvalues of A by solving the equation

$$\det(xI - A) = 0$$

This gives

$$\begin{aligned}
\det\left(x\left[\begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array}\right] - \left[\begin{array}{rr} -5 & 2 \\ -7 & 4 \end{array}\right]\right) &= 0 \\
\det\left[\begin{array}{cc} x+5 & -2 \\ 7 & x-4 \end{array}\right] &= 0
\end{aligned}$$

Computing the determinant as usual, the result is

$$x^2 + x - 6 = 0$$

Solving this equation, we find that $\lambda_1 = 2$ and $\lambda_2 = -3$.

Now we need to find the basic eigenvectors for each λ. First we will find the eigenvectors for $\lambda_1 = 2$. We wish to find all vectors $\vec{x} \neq \vec{0}$ such that $A\vec{x} = 2\vec{x}$. These are the solutions to $(2I - A)\vec{x} = 0$.

$$\begin{aligned} \left(2\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] - \left[\begin{array}{cc} -5 & 2 \\ -7 & 4 \end{array}\right]\right)\left[\begin{array}{c} x \\ y \end{array}\right] &= \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \\ \left[\begin{array}{cc} 7 & -2 \\ 7 & -2 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] &= \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \end{aligned}$$

The augmented matrix for this system and corresponding reduced row-echelon form are given by

$$\left[\begin{array}{cc|c} 7 & -2 & 0 \\ 7 & -2 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{2}{7} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The solution is any vector of the form

$$\left[\begin{array}{c} \frac{2}{7}s \\ s \end{array}\right] = s\left[\begin{array}{c} \frac{2}{7} \\ 1 \end{array}\right]$$

Multiplying this vector by 7 we obtain a simpler description for the solution to this system, given by

$$t\left[\begin{array}{c} 2 \\ 7 \end{array}\right]$$

This gives the basic eigenvector for $\lambda_1 = 2$ as

$$\left[\begin{array}{c} 2 \\ 7 \end{array}\right]$$

To check, we verify that $A\vec{x} = 2\vec{x}$ for this basic eigenvector.

$$\left[\begin{array}{cc} -5 & 2 \\ -7 & 4 \end{array}\right]\left[\begin{array}{c} 2 \\ 7 \end{array}\right] = \left[\begin{array}{c} 4 \\ 14 \end{array}\right] = 2\left[\begin{array}{c} 2 \\ 7 \end{array}\right]$$

This is what we wanted, so we know this basic eigenvector is correct.

Next we will repeat this process to find the basic eigenvector for $\lambda_2 = -3$. We wish to find all vectors $\vec{x} \neq \vec{0}$ such that $A\vec{x} = -3\vec{x}$. These are the solutions to $((-3)I - A)\vec{x} = \vec{0}$.

$$\begin{aligned} \left((-3)\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] - \left[\begin{array}{cc} -5 & 2 \\ -7 & 4 \end{array}\right]\right)\left[\begin{array}{c} x \\ y \end{array}\right] &= \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \\ \left[\begin{array}{cc} 2 & -2 \\ 7 & -7 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right] &= \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \end{aligned}$$

The augmented matrix for this system and corresponding reduced row-echelon form are given by

$$\left[\begin{array}{rr|r} 2 & -2 & 0 \\ 7 & -7 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The solution is any vector of the form

$$\left[\begin{array}{c} s \\ s \end{array}\right] = s\left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

This gives the basic eigenvector for $\lambda_2 = -3$ as

$$\left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

To check, we verify that $A\vec{x} = -3\vec{x}$ for this basic eigenvector.

$$\left[\begin{array}{rr} -5 & 2 \\ -7 & 4 \end{array}\right]\left[\begin{array}{c} 1 \\ 1 \end{array}\right] = \left[\begin{array}{r} -3 \\ -3 \end{array}\right] = -3\left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

This is what we wanted, so we know this basic eigenvector is correct. ♠

The following is an example using Procedure 7.6 for a 3×3 matrix.

Example 7.8: Find the Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors for the matrix

$$A = \left[\begin{array}{rrr} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{array}\right]$$

Solution. We will use Procedure 7.6. First we need to find the eigenvalues of A. Recall that they are the solutions of the equation

$$\det(xI - A) = 0$$

In this case the equation is

$$\det\left(x\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] - \left[\begin{array}{rrr} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{array}\right]\right) = 0$$

which becomes

$$\det\left[\begin{array}{ccc} x-5 & 10 & 5 \\ -2 & x-14 & -2 \\ 4 & 8 & x-6 \end{array}\right] = 0$$

Using Laplace Expansion, compute this determinant and simplify. The result is the following equation.

$$(x-5)\left(x^2-20x+100\right)=0$$

Solving this equation, we find that the eigenvalues are $\lambda_1 = 5, \lambda_2 = 10$ and $\lambda_3 = 10$. Notice that 10 is a root of algebraic multiplicity two as

$$x^2-20x+100=(x-10)^2$$

Therefore, $\lambda_2 = 10$ is an eigenvalue of algebraic multiplicity two.

Now that we have found the eigenvalues for A, we can compute the eigenvectors.

First we will find the basic eigenvectors for $\lambda_1 = 5$. In other words, we want to find all non-zero vectors $\vec{x}$ so that $A\vec{x} = 5\vec{x}$. This requires that we solve the equation $(5I - A)\vec{x} = \vec{0}$ for $\vec{x}$ as follows.

$$\left(5\left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] - \left[\begin{array}{rrr} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{array}\right] \right) \left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

That is you need to find the solution to

$$\left[\begin{array}{rrr} 0 & 10 & 5 \\ -2 & -9 & -2 \\ 4 & 8 & -1 \end{array}\right] \left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

By now this is a familiar problem. You set up the augmented matrix and row reduce to get the solution. Thus the matrix you must row reduce is

$$\left[\begin{array}{rrr|r} 0 & 10 & 5 & 0 \\ -2 & -9 & -2 & 0 \\ 4 & 8 & -1 & 0 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{rrr|r} 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so the solution is any vector of the form

$$\left[\begin{array}{c} \frac{5}{4}s \\ -\frac{1}{2}s \\ s \end{array}\right] = s\left[\begin{array}{r} \frac{5}{4} \\ -\frac{1}{2} \\ 1 \end{array}\right]$$

where $s \in \mathbb{R}$. If we multiply this vector by 4, we obtain a simpler description for the solution to this system, as given by

$$t\left[\begin{array}{r} 5 \\ -2 \\ 4 \end{array}\right] \tag{7.3}$$

where $t \in \mathbb{R}$. Here, the basic eigenvector is given by

$$\vec{x}_1 = \left[\begin{array}{r} 5 \\ -2 \\ 4 \end{array} \right]$$

Notice that we cannot let $t = 0$ here, because this would result in the zero vector and eigenvectors are never equal to 0! Other than this value, every other choice of t in 7.3 results in an eigenvector.

It is a good idea to check your work! To do so, we will take the original matrix and multiply by the basic eigenvector $\vec{x}_1$. We check to see if we get $5\vec{x}_1$.

$$\left[\begin{array}{rrr} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{array} \right] \left[\begin{array}{r} 5 \\ -2 \\ 4 \end{array} \right] = \left[\begin{array}{r} 25 \\ -10 \\ 20 \end{array} \right] = 5 \left[\begin{array}{r} 5 \\ -2 \\ 4 \end{array} \right]$$

This is what we wanted, so we know that our calculations were correct.

Next we will find the basic eigenvectors for $\lambda_2, \lambda_3 = 10$. These vectors are the basic solutions to the equation,

$$\left(10 \left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] - \left[\begin{array}{rrr} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{array} \right] \right) \left[\begin{array}{c} x \\ y \\ z \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$$

That is you must find the solutions to

$$\left[\begin{array}{rrr} 5 & 10 & 5 \\ -2 & -4 & -2 \\ 4 & 8 & 4 \end{array} \right] \left[\begin{array}{c} x \\ y \\ z \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$$

Consider the augmented matrix

$$\left[\begin{array}{rrr|r} 5 & 10 & 5 & 0 \\ -2 & -4 & -2 & 0 \\ 4 & 8 & 4 & 0 \end{array} \right]$$

The reduced row-echelon form for this matrix is

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

and so the eigenvectors are of the form

$$\left[\begin{array}{c} -2s-t \\ s \\ t \end{array} \right] = s \left[\begin{array}{r} -2 \\ 1 \\ 0 \end{array} \right] + t \left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array} \right]$$

Note that you can't pick t and s both equal to zero because this would result in the zero vector and eigenvectors are never equal to zero.

Here, there are two basic eigenvectors, given by

$$\vec{x}_2 = \left[\begin{array}{r} -2 \\ 1 \\ 0 \end{array} \right] \text{ and } \vec{x}_3 = \left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array} \right].$$

Taking any (nonzero) linear combination of $\vec{x}_2$ and $\vec{x}_3$ will also result in an eigenvector for the eigenvalue $\lambda = 10$. As in the case for $\lambda = 5$, always check your work! For the first basic eigenvector, we can check $A\vec{x}_2 = 10\vec{x}_2$ as follows.

$$\left[\begin{array}{rrr} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{array}\right]\left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{r} -10 \\ 0 \\ 10 \end{array}\right] = 10\left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right]$$

This is what we wanted. Checking the second basic eigenvector, $\vec{x}_3$, is left as an exercise. ♠

It is important to remember that for any eigenvector $\vec{x}$, $\vec{x} \neq \vec{0}$. However, it is possible to have *eigenvalues* equal to zero. This is illustrated in the following example.

Example 7.9: A Zero Eigenvalue

Let

$$A = \left[\begin{array}{rrr} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{array}\right]$$

Find the eigenvalues and eigenvectors of A.

Solution. First we find the eigenvalues of A. We will do so using Definition 7.2.

In order to find the eigenvalues of A, we solve the following equation.

$$\det(xI - A) = \det\left[\begin{array}{ccc} x-2 & -2 & 2 \\ -1 & x-3 & 1 \\ 1 & -1 & x-1 \end{array}\right] = 0$$

This reduces to $x^3 - 6x^2 + 8x = 0$. You can verify that the solutions are $\lambda_1 = 0$, $\lambda_2 = 2$, and $\lambda_3 = 4$. Notice that while eigenvectors can never equal $\vec{0}$, it is possible to have an eigenvalue equal to 0.

Now we will find the basic eigenvectors. For $\lambda_1 = 0$, we need to solve the equation $(0I - A)\vec{x} = \vec{0}$. This equation becomes $-A\vec{x} = \vec{0}$, and so the augmented matrix for finding the solutions is given by

$$\left[\begin{array}{rrr|r} -2 & -2 & 2 & 0 \\ -1 & -3 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the eigenvectors are of the form $t\left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right]$ where $t \neq 0$ and the basic eigenvector is given by

$$\vec{x}_1 = \left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right]$$

We can verify that this eigenvector is correct by checking that the equation $A\vec{x}_1 = 0\vec{x}_1$ holds. The product $A\vec{x}_1$ is given by

$$A\vec{x}_1 = \left[\begin{array}{rrr} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{array}\right]\left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right] = \left[\begin{array}{r} 0 \\ 0 \\ 0 \end{array}\right]$$

This clearly equals $0\vec{x}_1$, so the equation holds. Hence, $A\vec{x}_1 = 0\vec{x}_1$ and so 0 is an eigenvalue of A.

Computing the other basic eigenvectors is left as an exercise. ♠

In the following sections, we examine ways to simplify this process of finding eigenvalues and eigenvectors by using properties of special types of matrices.

Eigenvalues and Eigenvectors for Special Types of Matrices

When trying to find the eigenvalues and eigenvectors of a matrix we'd like to work as little as possible. Sometimes we can trade a matrix A in for a simpler matrix B that has the same eigenvalues. We will show when this is possible by looking at what it means for two matrices to be *similar*. Then we will discuss using two special types of matrices that can help us find eigenvalues and eigenvectors more easily, our friends the elementary matrices and triangular matrices.

We start with the definition of what it means to say that two matrices are similar.

Definition 7.10: Similar Matrices

Let A and B be $n \times n$ matrices. Suppose there exists an invertible matrix P such that

$$A = P^{-1}BP$$

*Then A and B are called **similar matrices**.*

It turns out that we can use the concept of similar matrices to help us find the eigenvalues of matrices. Consider the following lemma.

Lemma 7.11: Similar Matrices and Eigenvalues

Let A and B be similar matrices, so that $A = P^{-1}BP$ where A, B are $n \times n$ matrices and P is invertible. Then A, B have the same eigenvalues.

Proof. We need to show that if $A = P^{-1}BP$, then A and B have the same eigenvalues.

Suppose $A = P^{-1}BP$ and λ is an eigenvalue of A, that is $A\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq 0$. Then

$$\begin{aligned} P^{-1}BP\vec{x} &= \lambda\vec{x} \\ BP\vec{x} &= P(\lambda\vec{x}) \\ B(P\vec{x}) &= \lambda(P\vec{x}). \end{aligned}$$

Since P is one to one and $\vec{x} \neq \vec{0}$, it follows that $P\vec{x} \neq \vec{0}$. Here, $P\vec{x}$ plays the role of the eigenvector in this equation. Thus λ is also an eigenvalue of B. One can similarly verify that any eigenvalue of B is also an eigenvalue of A, and thus both matrices have the same eigenvalues as desired.

♠

Note that this proof also demonstrates that the eigenvectors of A and B will (generally) be *different*. We see in the proof that $A\vec{x} = \lambda\vec{x}$, while $B(P\vec{x}) = \lambda(P\vec{x})$. Therefore, for an eigenvalue λ, A will have the eigenvector $\vec{x}$ while B will have the eigenvector $P\vec{x}$.

Now we will discuss how to use elementary matrices to simplify finding the eigenvectors and eigenvalues of a matrix A. Recall from Definition 2.46 that an elementary matrix E is obtained by applying one row operation to the identity matrix.

It is possible to use elementary matrices to simplify a matrix before searching for its eigenvalues and eigenvectors. This is illustrated in the following example.

Example 7.12: Simplify Using Elementary Matrices

Find the eigenvalues for the matrix

$$A = \left[\begin{array}{rrr} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{array} \right]$$

Solution. This matrix has big numbers and therefore we would like to simplify as much as possible before computing the eigenvalues.

We will do so using row operations. First, add 2 times the second row to the third row. To do so, left multiply A by $E(2,2)$. Then right multiply A by the inverse of $E(2,2)$ as illustrated.

$$\left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{array}\right]\left[\begin{array}{rrr} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{array}\right]\left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{array}\right]=\left[\begin{array}{rrr} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{array}\right]$$

By Lemma 7.11, the resulting matrix has the same eigenvalues as A where here, the matrix $E(2,2)$ plays the role of P.

We do this step again, as follows. In this step, we use the elementary matrix obtained by adding -3 times the second row to the first row.

$$\left[\begin{array}{rrr} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\left[\begin{array}{rrr} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{array}\right]\left[\begin{array}{rrr} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]=\left[\begin{array}{rrr} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{array}\right] \tag{7.4}$$

Again by Lemma 7.11, this resulting matrix has the same eigenvalues as A. At this point, we can easily find the eigenvalues. Let

$$B=\left[\begin{array}{rrr} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{array}\right]$$

Then, we find the eigenvalues of B (and therefore of A) by solving the equation $\det(xI-B)=0$. You should verify that this equation becomes

$$(x+2)(x+2)(x-3)=0$$

Solving this equation results in eigenvalues of $\lambda_1=-2, \lambda_2=-2$, and $\lambda_3=3$. Therefore, these are also the eigenvalues of A.

♠

Through using elementary matrices, we were able to create a matrix for which finding the eigenvalues was easier than for A. At this point, you could go back to the original matrix A and solve $(\lambda I-A)\vec{x}=0$ to obtain the eigenvectors of A.

Notice that when you multiply on the right by an elementary matrix, you are doing the column operation defined by the elementary matrix. In Equation 7.4 multiplication by the elementary matrix on the right merely involves taking three times the first column and adding to the second. Thus, without referring to the elementary matrices, the transition to the new matrix in 7.4 can be illustrated by

$$\left[\begin{array}{rrr} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{array}\right] \rightarrow \left[\begin{array}{rrr} 3 & -9 & 15 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{array}\right] \rightarrow \left[\begin{array}{rrr} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{array}\right]$$

The third special type of matrix we will consider in this section is the triangular matrix. Recall Definition 2.66 which states that an upper (lower) triangular matrix contains all zeros below (above) the main diagonal. Remember that finding the determinant of a triangular matrix is a simple procedure of taking the product of the entries on the main diagonal.. It turns out that there is also a simple way to find the eigenvalues of a triangular matrix.

In the next example we will demonstrate that the eigenvalues of a triangular matrix are the entries on the main diagonal.

Example 7.13: Eigenvalues for a Triangular Matrix

Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{bmatrix}$. *Find the eigenvalues of* A.

Solution. We need to solve the equation $\det(xI - A) = 0$ as follows

$$\det(xI - A) = \det \begin{bmatrix} x-1 & -2 & -4 \\ 0 & x-4 & -7 \\ 0 & 0 & x-6 \end{bmatrix} = (x-1)(x-4)(x-6) = 0$$

Solving the equation $(x-1)(x-4)(x-6) = 0$ for x results in the eigenvalues $\lambda_1 = 1, \lambda_2 = 4$ and $\lambda_3 = 6$. Thus the eigenvalues are the entries on the main diagonal of the original matrix. ♠

The same result is true for lower triangular matrices. For any triangular matrix, the eigenvalues are equal to the entries on the main diagonal. To find the eigenvectors of a triangular matrix, we use the usual procedure.

In the next section, we explore an important process involving the eigenvalues and eigenvectors of a matrix.

7.2 Diagonalization

Outcomes

A. *Determine when it is possible to diagonalize a matrix.*

B. *When possible, diagonalize a matrix.*

Similarity and Diagonalization

We begin this section by recalling the definition of similar matrices. Recall that if A,B are two $n \times n$ matrices, then they are **similar** if and only if there exists an invertible matrix P such that

$$A = P^{-1}BP$$

In this case we write $A \sim B$. The concept of similarity is an example of an **equivalence relation**.

Lemma 7.14: Similarity is an Equivalence Relation

Similarity is an equivalence relation, i.e. for $n \times n$ matrices A,B, and C,

1. *$A \sim A$ (reflexive)*
2. *If $A \sim B$, then $B \sim A$ (symmetric)*
3. *If $A \sim B$ and $B \sim C$, then $A \sim C$ (transitive)*

Proof. It is clear that $A \sim A$, taking $P = I$.

Now, if $A \sim B$, then for some P invertible,

$$A = P^{-1}BP$$

and so

$$PAP^{-1} = B$$

But then

$$\left(P^{-1}\right)^{-1}AP^{-1} = B$$

which shows that $B \sim A$.

Now suppose $A \sim B$ and $B \sim C$. Then there exist invertible matrices P, Q such that

$$A = P^{-1}BP, \; B = Q^{-1}CQ$$

Then,

$$A = P^{-1}\left(Q^{-1}CQ\right)P = (QP)^{-1}C(QP)$$

showing that A is similar to C. ♠

Another important concept necessary to this section is the trace of a matrix. Consider the definition.

Definition 7.15: Trace of a Matrix

If $A = [a_{ij}]$ is an $n \times n$ matrix, then the trace of A is

$$\text{trace}(A) = \sum_{i=1}^{n} a_{ii}.$$

In words, the trace of a matrix is the sum of the entries on the main diagonal.

Lemma 7.16: Properties of Trace

For $n \times n$ matrices A and B, and any $k \in \mathbb{R}$,

1. $\text{trace}(A + B) = \text{trace}(A) + \text{trace}(B)$
2. $\text{trace}(kA) = k \cdot \text{trace}(A)$
3. $\text{trace}(AB) = \text{trace}(BA)$

The following theorem includes a reference to the characteristic polynomial of a matrix. Recall that for any $n \times n$ matrix A, the characteristic polynomial of A is $c_A(x) = \det(xI - A)$.

Theorem 7.17: Properties of Similar Matrices

If A and B are $n \times n$ matrices and $A \sim B$, then

1. $\det(A) = \det(B)$
2. $rank(A) = rank(B)$
3. $trace(A) = trace(B)$
4. $c_A(x) = c_B(x)$
5. *A and B have the same eigenvalues*

We now proceed to the main concept of this section. When a matrix is similar to a diagonal matrix, the matrix is said to be diagonalizable. We define a diagonal matrix D as a matrix containing a zero in every entry except those on the main diagonal. More precisely, if d_{ij} is the ij^{th} entry of a diagonal matrix D, then $d_{ij} = 0$ unless $i = j$. Such matrices look like the following.

$$D = \left[\begin{array}{ccc} * & & 0 \\ & \ddots & \\ 0 & & * \end{array} \right]$$

where $*$ is a number which might not be zero.

The following is the formal definition of a diagonalizable matrix.

Definition 7.18: Diagonalizable

Let A be an $n \times n$ matrix. Then A is said to be ***diagonalizable*** *if there exists an invertible matrix P such that*

$$P^{-1}AP = D$$

where D is a diagonal matrix.

Notice that the above equation can be rearranged as $A = PDP^{-1}$. Suppose we wanted to compute A^{100}. By diagonalizing A first it suffices to then compute $\left(PDP^{-1}\right)^{100}$, which reduces to $PD^{100}P^{-1}$. This last computation is much simpler than A^{100}. While this process is described in detail later, it provides motivation for diagonalization.

Diagonalizing a Matrix

The most important theorem about diagonalizability is the following major result.

Theorem 7.19: Eigenvectors and Diagonalizable Matrices

An $n \times n$ matrix A is diagonalizable if and only if there is an invertible matrix P given by

$$P = \left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array} \right]$$

where the $\vec{x}_k$ are eigenvectors of A.
Moreover if A is diagonalizable, the corresponding eigenvalues of A are the diagonal entries of the diagonal matrix D.

Proof. Suppose P is given as above as an invertible matrix whose columns are eigenvectors of A. Then P^{-1} is of the form

$$P^{-1} = \left[\begin{array}{c} \vec{w}_1^T \\ \vec{w}_2^T \\ \vdots \\ \vec{w}_n^T \end{array} \right]$$

where $\vec{w}_k^T \vec{x}_j = \delta_{kj}$, which is the Kronecker's symbol defined by

$$\delta_{ij} = \left\{ \begin{array}{l} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{array} \right. .$$

Then

$$\begin{aligned}
P^{-1}AP &= \left[\begin{array}{c} \vec{w}_1^T \\ \vec{w}_2^T \\ \vdots \\ \vec{w}_n^T \end{array}\right] \left[\begin{array}{cccc} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{array}\right] \\
&= \left[\begin{array}{c} \vec{w}_1^T \\ \vec{w}_2^T \\ \vdots \\ \vec{w}_n^T \end{array}\right] \left[\begin{array}{cccc} \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 & \cdots & \lambda_n\vec{x}_n \end{array}\right] \\
&= \left[\begin{array}{ccc} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{array}\right].
\end{aligned}$$

Conversely, suppose A is diagonalizable so that $P^{-1}AP = D$. Let

$$P = \left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array}\right]$$

where the columns are the $\vec{x}_k$ and

$$D = \left[\begin{array}{ccc} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{array}\right].$$

Then

$$AP = PD = \left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array}\right] \left[\begin{array}{ccc} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{array}\right]$$

and so

$$\left[\begin{array}{cccc} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{array}\right] = \left[\begin{array}{cccc} \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 & \cdots & \lambda_n\vec{x}_n \end{array}\right]$$

showing the $\vec{x}_k$ are eigenvectors of A and the λ_k are eigenvectors. ♠

Notice that because the matrix P defined above is invertible it follows that the set of eigenvectors of A, $\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$, form a basis of $\mathbb{R}^n$.

We demonstrate the concept given in the above theorem in the next example. Note that not only are the columns of the matrix P formed by eigenvectors, but P must be invertible. We achieve this by using basic eigenvectors for the columns of P.

Example 7.20: Diagonalize a Matrix

Let

$$A = \left[\begin{array}{rrr} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{array}\right].$$

Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution. By Theorem 7.19 we use the eigenvectors of A as the columns of P, and the corresponding eigenvalues of A as the diagonal entries of D.

First, we will find the eigenvalues of A. To do so, we solve $\det(xI - A) = 0$ as follows.

$$\det\left(x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix} \right) = 0$$

This computation is left as an exercise, and you should verify that the eigenvalues are $\lambda_1 = 2, \lambda_2 = 2$, and $\lambda_3 = 6$.

Next, we need to find the eigenvectors. We first find the eigenvectors for $\lambda_1, \lambda_2 = 2$. Solving $(2I - A)\vec{x} = 0$ to find the eigenvectors, we find that the eigenvectors are

$$t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

where t, s are scalars. Hence there are two basic eigenvectors,

$$\vec{x}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

You can verify that the basic eigenvector for $\lambda_3 = 6$ is $\vec{x}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$.

Then, we construct the matrix P as follows.

$$P = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

That is, the columns of P are the basic eigenvectors of A. Then, you can verify that

$$P^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

Thus,

$$P^{-1}AP = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

You can see that the result here is a diagonal matrix where the entries on the main diagonal are the eigenvalues of A. We expected this based on Theorem 7.19. Notice that eigenvalues on the main diagonal *must* be in the same order as the corresponding eigenvectors in P. ♠

Consider the next important theorem.

Consider the next important theorem.

Theorem 7.21: Linearly Independent Eigenvectors

Let A be an $n \times n$ matrix, and suppose that A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. For each i, let $\vec{x}_i$ be a λ_i-eigenvector of A. Then $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ is linearly independent.

The corollary that follows from this theorem gives a useful tool in determining if A is diagonalizable.

Corollary 7.22: Distinct Eigenvalues

Let A be an $n \times n$ matrix and suppose A has n distinct eigenvalues. Then A is diagonalizable.

It is possible that a matrix A cannot be diagonalized. In other words, for some matrices A there is no invertible matrix P so that $P^{-1}AP$ is a diagonal matrix.

Consider the following example.

Example 7.23: A Matrix which cannot be Diagonalized

Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

If possible, find an invertible matrix P and diagonal matrix D so that $P^{-1}AP = D$.

Solution. Through the usual procedure, we find that the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 1$. To find the eigenvectors, we solve the equation $(\lambda I - A)\vec{x} = 0$. The matrix $(\lambda I - A)$ is given by

$$\begin{bmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{bmatrix}$$

Substituting in $\lambda = 1$, we have the matrix

$$\begin{bmatrix} 1 - 1 & -1 \\ 0 & 1 - 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Then, solving the equation $(\lambda I - A)\vec{x} = 0$ involves carrying the following augmented matrix to its reduced row-echelon form.

$$\left[\begin{array}{rr|r} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rr|r} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Then the eigenvectors are of the form

$$t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the basic eigenvector is

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In this case, the matrix A has one eigenvalue of algebraic multiplicity two, but only one basic eigenvector. In order to diagonalize A, we need to construct an invertible 2×2 matrix P. However, because A only has one basic eigenvector, we cannot construct this P. Notice that if we were to use $\vec{x}_1$ as both columns of P, P would not be invertible. For this reason, we cannot repeat eigenvectors in P.

Hence this matrix cannot be diagonalized. ♠

The idea that a matrix may not be diagonalizable suggests that conditions exist to determine when it is possible to diagonalize a matrix. We saw earlier in Corollary 7.22 that an $n\times n$ matrix with n distinct eigenvalues is diagonalizable. It turns out that there are other useful diagonalizability tests.

First we need the following definition.

Definition 7.24: Eigenspace, Geometric Multiplicity

Let A be an $n\times n$ matrix and $\lambda \in \mathbb{R}$. The eigenspace of A corresponding to λ, written $E_\lambda(A)$ is the set of all eigenvectors corresponding to λ, and its dimension is called the geometric multiplicity of λ. So

$$E_\lambda(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x}\}.$$

In other words, the eigenspace $E_\lambda(A)$ is all $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. Notice that this set can be written $E_\lambda(A) = \text{null}(\lambda I - A)$, showing that $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.

Recall that the algebraic multiplicity of an eigenvalue λ is the number of times that it occurs as a root of the characteristic polynomial.

Consider now the following lemma.

Lemma 7.25: Dimension of the Eigenspace, Geometric Multiplicity

If A is an $n\times n$ matrix and λ is an eigenvalue of A of algebraic multiplicity m, then

$$\dim(E_\lambda(A)) \leq m.$$

That is the geometric multiplicity of an eigenvalue is always at most its algebraic multiplicity.

Again in other words this result tells us that if λ is an eigenvalue of A, then the number of linearly independent λ-eigenvectors is never more than the alegbraic multiplicity of λ. This fact provides us with a useful test for diagonalizability:

Theorem 7.26: Diagonalizability Condition

Let A be an $n \times n$ matrix A. Then A is diagonalizable if and only if for each eigenvalue λ of A, its geometric multiplicity equals its algebraic multiplicity. That is $\dim(E_\lambda(A))$ is equal to the number of times λ occurs as a root of the characteristic polynomial.

Complex Eigenvalues

In some applications, a matrix may have eigenvalues which are complex numbers. For example, this often occurs in differential equations. These questions are approached in the same way as above.

Consider the following example.

Example 7.27: A Real Matrix with Complex Eigenvalues

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of A.

Solution. We will first find the eigenvalues as usual by solving the following equation.

$$\det \left(x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} \right) = 0$$

This reduces to $(x-1)\left(x^2-4x+5\right)=0$. The solutions are $\lambda_1 = 1, \lambda_2 = 2+i$ and $\lambda_3 = 2-i$.

There is nothing new about finding the eigenvectors for $\lambda_1 = 1$ so this is left as an exercise.

Consider now the eigenvalue $\lambda_2 = 2+i$. As usual, we solve the equation $(\lambda I - A)\vec{x} = 0$ as given by

$$\left((2+i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} \right) \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In other words, we need to solve the system represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 1+i & 0 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & -1 & i & 0 \end{array}\right]$$

We now use our row operations to solve the system. Divide the first row by $(1+i)$ and then take $-i$ times the second row and add to the third row. This yields

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now multiply the second row by $-i$ to obtain the reduced row-echelon form, given by

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the eigenvectors are of the form

$$t\left[\begin{array}{c} 0 \\ i \\ 1 \end{array}\right]$$

and the basic eigenvector is given by

$$\vec{x}_2 = \left[\begin{array}{c} 0 \\ i \\ 1 \end{array}\right]$$

As an exercise, verify that the eigenvectors for $\lambda_3 = 2-i$ are of the form

$$t\left[\begin{array}{r} 0 \\ -i \\ 1 \end{array}\right]$$

Hence, the basic eigenvector is given by

$$\vec{x}_3 = \left[\begin{array}{r} 0 \\ -i \\ 1 \end{array}\right]$$

As usual, be sure to check your answers! To verify, we check that $A\vec{x}_3 = (2-i)\vec{x}_3$ as follows.

$$\left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{array}\right]\left[\begin{array}{r} 0 \\ -i \\ 1 \end{array}\right] = \left[\begin{array}{c} 0 \\ -1-2i \\ 2-i \end{array}\right] = (2-i)\left[\begin{array}{r} 0 \\ -i \\ 1 \end{array}\right]$$

Therefore, we know that this eigenvector and eigenvalue are correct. ♠

Notice that in Example 7.27, two of the eigenvalues were given by $\lambda_2 = 2+i$ and $\lambda_3 = 2-i$. You may recall that these two complex numbers are **conjugates**. It turns out that whenever a matrix containing real entries has a complex eigenvalue λ, it also has an eigenvalue equal to $\overline{\lambda}$, the conjugate of λ.

Engage Active Learning App!

Lyryx Engage is an active learning app designed to increase student engagement in the study of linear algebra. The content is "chunked" into small blocks, each with an interactive assessment activity to promote comprehension.

Visit us at https://lyryx.com to register and access Lyryx Engage!

Lyryx Online Exercises!

Lyryx offers formative online exercises! The educational software captures instructors' experience using algorithms to analyze and evaluate students' work, identifying common errors and providing personalized feedback. Students are encouraged to try these randomized problems repeatedly, learning from their mistakes until mastery.

Visit us at https://lyryx.com to register and access Lyryx exercises!

7.3 Applications of Spectral Theory

Outcomes

A. *Use diagonalization to find a high power of a matrix.*

B. *Use diagonalization to solve dynamical systems.*

Raising a Matrix to a High Power

Suppose we have a matrix A and we want to find A^{50}. One could try to multiply A with itself 50 times, but this is computationally extremely intensive (try it!). However diagonalization allows us to compute high powers of a matrix relatively easily. Suppose A is diagonalizable, so that $P^{-1}AP = D$. We can rearrange this equation to write $A = PDP^{-1}$.

Now, consider A^2. Since $A = PDP^{-1}$, it follows that

$$A^2 = \left(PDP^{-1}\right)^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$$

Similarly,

$$A^3 = \left(PDP^{-1}\right)^3 = PDP^{-1}PDP^{-1}PDP^{-1} = PD^3P^{-1}$$

In general,

$$A^n = \left(PDP^{-1}\right)^n = PD^nP^{-1}$$

Therefore, we have reduced the problem to finding D^n. In order to compute D^n, then because D is diagonal we only need to raise every entry on the main diagonal of D to the power of n.

Through this method, we can compute large powers of matrices. Consider the following example.

Example 7.28: Raising a Matrix to a High Power

Let $A = \left[\begin{array}{rrr} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{array}\right]$. *Find* A^{50}.

Solution. We will first diagonalize A. The steps are left as an exercise and you may wish to verify that the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 1$, and $\lambda_3 = 2$.

The basic eigenvectors corresponding to $\lambda_1, \lambda_2 = 1$ are

$$\vec{x}_1 = \left[\begin{array}{r} 0 \\ 0 \\ 1 \end{array}\right] \text{ and } \vec{x}_2 = \left[\begin{array}{r} -1 \\ 1 \\ 0 \end{array}\right]$$

The basic eigenvector corresponding to $\lambda_3 = 2$ is

$$\vec{x}_3 = \left[\begin{array}{r} -1 \\ 0 \\ 1 \end{array}\right]$$

Now we construct P by using the basic eigenvectors of A as the columns of P. Thus

$$P = \left[\begin{array}{ccc} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{array} \right] = \left[\begin{array}{rrr} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right]$$

Then also

$$P^{-1} = \left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{array} \right]$$

which you may wish to verify.

Then,

$$\begin{aligned} P^{-1}AP &= \left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{array} \right] \left[\begin{array}{rrr} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{array} \right] \left[\begin{array}{rrr} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right] \\ &= \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array} \right] \\ &= D \end{aligned}$$

Now it follows by rearranging the equation that

$$A = PDP^{-1} = \left[\begin{array}{rrr} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array} \right] \left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{array} \right]$$

Therefore,

$$\begin{aligned} A^{50} &= PD^{50}P^{-1} \\ &= \left[\begin{array}{rrr} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array} \right]^{50} \left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{array} \right] \end{aligned}$$

By our discussion above, D^{50} is found as follows.

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{array} \right]^{50} = \left[\begin{array}{ccc} 1^{50} & 0 & 0 \\ 0 & 1^{50} & 0 \\ 0 & 0 & 2^{50} \end{array} \right]$$

It follows that

$$\begin{aligned} A^{50} &= \left[\begin{array}{rrr} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right] \left[\begin{array}{ccc} 1^{50} & 0 & 0 \\ 0 & 1^{50} & 0 \\ 0 & 0 & 2^{50} \end{array} \right] \left[\begin{array}{rrr} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{array} \right] \\ &= \left[\begin{array}{ccc} 2^{50} & -1+2^{50} & 0 \\ 0 & 1 & 0 \\ 1-2^{50} & 1-2^{50} & 1 \end{array} \right] \end{aligned}$$

♠

Through diagonalization, we can efficiently compute a high power of A. Once we have P, the only computation required is to use row reduction to find P^{-1}. But for some matrices finding the inverse is trivial, as we discuss in the next section.

Raising a Symmetric Matrix to a High Power

We already have seen how to use matrix diagonalization to compute powers of matrices. This requires computing eigenvalues of the matrix A, and finding an invertible matrix of eigenvectors P such that $P^{-1}AP$ is diagonal. In this section we will see that if the matrix A is symmetric (see Definition 2.30), then we can actually find such a matrix P that is an orthogonal matrix of eigenvectors. Thus P^{-1} is simply its transpose P^T, and P^TAP is diagonal. When this happens we say that A is **orthogonally diagonalizable**

In fact this happens if and only if A is a symmetric matrix as shown in the following important theorem.

Theorem 7.29: Principal Axis Theorem

The following conditions are equivalent for an $n \times n$ matrix A:

1. *A is symmetric.*
2. *A has an orthonormal set of n eigenvectors.*
3. *A is orthogonally diagonalizable.*

Proof. The complete proof is beyond this course, but to give an idea assume that A has an orthonormal set of eigenvectors, and let P consist of these eigenvectors as columns. Then $P^{-1} = P^T$, and $P^TAP = D$ a diagonal matrix. But then $A = PDP^T$, and

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$$

so A is symmetric.

Now given a symmetric matrix A, one shows that eigenvectors corresponding to different eigenvalues are always orthogonal. So it suffices to apply the Gram-Schmidt process on the set of basic eigenvectors of each eigenvalue to obtain an orthonormal set of eigenvectors. ♠

We demonstrate this in the following example.

Example 7.30: Orthogonal Diagonalization of a Symmetric Matrix

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$. *Find an orthogonal matrix P such that P^TAP is a diagonal matrix.*

Solution. In this case, verify that the eigenvalues are 2 and 1. First we will find an eigenvector for the eigenvalue 2. This involves row reducing the following augmented matrix.

$$\left[\begin{array}{ccc|c} 2-1 & 0 & 0 & 0 \\ 0 & 2-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 2-\frac{3}{2} & 0 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so an eigenvector is

$$\left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right]$$

Finally to obtain an eigenvector of length one (unit eigenvector) we simply divide this vector by its length to yield:

$$\left[\begin{array}{c} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array}\right]$$

Next consider the case of the eigenvalue 1. To obtain basic eigenvectors, the matrix which needs to be row reduced in this case is

$$\left[\begin{array}{ccc|c} 1-1 & 0 & 0 & 0 \\ 0 & 1-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1-\frac{3}{2} & 0 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the eigenvectors are of the form

$$\left[\begin{array}{c} s \\ -t \\ t \end{array}\right]$$

Note that all these vectors are automatically orthogonal to eigenvectors corresponding to the first eigenvalue. This follows from the fact that A is symmetric, as mentioned earlier.

We obtain basic eigenvectors

$$\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right] \text{ and } \left[\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right]$$

Since they are themselves orthogonal (by luck here) we do not need to use the Gram-Schmidt process and instead simply normalize these vectors to obtain

$$\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right] \text{ and } \left[\begin{array}{c} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array}\right]$$

An orthogonal matrix P that orthogonally diagonalizes A is then obtained by letting these basic vectors be the columns of P.

$$P = \left[\begin{array}{ccc} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{array} \right]$$

We verify this works. $P^T AP$ is of the form

$$\left[\begin{array}{ccc} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{array} \right] \left[\begin{array}{ccc} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{array} \right]$$

$$= \left[\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

which is the desired diagonal matrix. ♠

We can now apply this technique to efficiently compute high powers of a symmetric matrix.

Example 7.31: Powers of a Symmetric Matrix

Let $A = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{array} \right]$. *Compute* A^7.

Solution. We found in Example 7.30 that $P^T AP = D$ is diagonal, where

$$P = \left[\begin{array}{ccc} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{array} \right] \text{ and } D = \left[\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

Thus $A = PDP^T$ and $A^7 = PDP^T \; PDP^T \; \cdots PDP^T = PD^7P^T$ which gives:

$$
\begin{aligned}
A^7 &= \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^7 \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2^7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \frac{2^7}{\sqrt{2}} & \frac{2^7}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2^7+1}{2} & \frac{2^7-1}{2} \\ 0 & \frac{2^7-1}{2} & \frac{2^7+1}{2} \end{bmatrix}
\end{aligned}
$$

♠

Markov Matrices

There are applications of great importance which feature a special type of matrix. Matrices whose columns consist of non-negative numbers that sum to one are called **Markov matrices**.

Definition 7.32: Markov Matrix

An $n \times n$ matrix whose entries are nonnegative real numbers such that the sum of the entries in each column is equal to one is called a ***Markov matrix****.*

An important application of Markov matrices is in population migration, as illustrated in the following definition.

Definition 7.33: Migration Matrices

Let m locations be denoted by the numbers $1,2,\cdots,m$. Suppose it is the case that each year the proportion of residents in location j which move to location i is a_{ij}. Also suppose no one escapes or emigrates from without these m locations. This last assumption requires $\sum_i a_{ij} = 1$, and means that the matrix A, such that $A = \left[a_{ij}\right]$, is a Markov matrix. In this context, A is also called a ***migration matrix****.*

Consider the following example which demonstrates this situation.

Example 7.34: Migration Matrix

Let A be a Markov matrix given by

$$A = \left[\begin{array}{cc} .4 & .2 \\ .6 & .8 \end{array}\right]$$

Verify that A is a Markov matrix and describe the entries of A in terms of population migration.

Solution. The columns of A are comprised of non-negative numbers which sum to 1. Hence, A is a Markov matrix.

Now, consider the entries a_{ij} of A in terms of population. The entry $a_{11} = .4$ is the proportion of residents in location one which stay in location one in a given time period. Entry $a_{21} = .6$ is the proportion of residents in location 1 which move to location 2 in the same time period. Entry $a_{12} = .2$ is the proportion of residents in location 2 which move to location 1. Finally, entry $a_{22} = .8$ is the proportion of residents in location 2 which stay in location 2 in this time period.

Considered as a Markov matrix, these numbers are usually identified with probabilities. Hence, we can say that the probability that a resident of location one will stay in location one in the time period is .4. ♠

Observe that in Example 7.34 if there was initially, say, 15 thousand people in location 1 and 10 thousand in location 2, then after one year there would be $.4 \times 15 + .2 \times 10 = 8$ thousand people in location 1 the following year, and similarly there would be $.6 \times 15 + .8 \times 10 = 17$ thousand people in location 2 the following year.

More generally let $\vec{x}_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{mn} \end{bmatrix}$ where x_{in} is the population of location i at time period n. We call $\vec{x}_n$ the **state vector at period** n. In particular, we call $\vec{x}_0$ the initial state vector. If A is the migration matrix and $\vec{x}_n$ is the state vector at period n, we compute the population in each location i one time period later by $\vec{x}_{n+1} = A\vec{x}_n$. In order to find the population of location i after k years, we compute the i^{th} component of $A^k\vec{x}$. This discussion is summarized in the following theorem.

Theorem 7.35: State Vector

Let A be the migration matrix of a population and let $\vec{x}_n$ be the vector whose entries give the population of each location at time period n. Then $\vec{x}_n$ is the state vector at period n and it follows that

$$\vec{x}_{n+1} = A\vec{x}_n$$

The sum of the entries of $\vec{x}_n$ will equal the sum of the entries of the initial vector $\vec{x}_0$. Since the columns of A sum to 1, this sum is preserved for every multiplication by A as demonstrated below.

$$\sum_i \sum_j a_{ij} x_j = \sum_j x_j \left(\sum_i a_{ij} \right) = \sum_j x_j$$

Consider the following example.

Example 7.36: Using a Migration Matrix

Consider the migration matrix

$$A = \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix}$$

for locations $1, 2,$ *and* 3. *Suppose initially there are* 100 *residents in location* 1, 200 *in location* 2 *and* 400 *in location* 3. *Find the population in the three locations after* $1, 2,$ *and* 10 *units of time.*

Solution. Using Theorem 7.35 we can find the population in each location using the equation $\vec{x}_{n+1} = A\vec{x}_n$. For the population after 1 unit, we calculate $\vec{x}_1 = A\vec{x}_0$ as follows.

$$\begin{aligned} \vec{x}_1 &= A\vec{x}_0 \\ \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} &= \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \\ 400 \end{bmatrix} \\ &= \begin{bmatrix} 100 \\ 180 \\ 420 \end{bmatrix} \end{aligned}$$

Therefore after one time period, location 1 has 100 residents, location 2 has 180, and location 3 has 420. Notice that the **total** population is unchanged, it simply migrates within the given locations. We find the locations after two time periods in the same way.

$$\begin{aligned}
\vec{x}_2 &= A\vec{x}_1 \\
\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} &= \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix} \begin{bmatrix} 100 \\ 180 \\ 420 \end{bmatrix} \\
&= \begin{bmatrix} 102 \\ 164 \\ 434 \end{bmatrix}
\end{aligned}$$

We could progress in this manner to find the populations after 10 time periods. However from our above discussion, we can simply calculate $(A^n\vec{x}_0)_i$, where n denotes the number of time periods which have passed. Therefore, we compute the populations in each location after 10 units of time as follows.

$$\begin{aligned}
\vec{x}_{10} &= A^{10}\vec{x}_0 \\
\begin{bmatrix} x_{110} \\ x_{210} \\ x_{310} \end{bmatrix} &= \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix}^{10} \begin{bmatrix} 100 \\ 200 \\ 400 \end{bmatrix} \\
&= \begin{bmatrix} 115.08582922 \\ 120.13067244 \\ 464.78349834 \end{bmatrix}
\end{aligned}$$

Since we are speaking about populations, we would need to round these numbers to provide a logical answer. Therefore, we can say that after 10 units of time, there will be 115 residents in location one, 120 in location two, and 465 in location three. ♠

A second important application of Markov matrices is the concept of random walks. Suppose a walker has m locations to choose from, denoted $1,2,\cdots,m$. Let a_{ij} refer to the probability that the person will travel **to** location i **from** location j. Again, this requires that

$$\sum_{i=1}^{k} a_{ij} = 1$$

In this context, the vector $\vec{x}_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{mn} \end{bmatrix}$ contains the probabilities x_{in} that the walker ends up in location i at time n.

Example 7.37: Random Walks

Suppose three locations exist, referred to as locations $1,2$ and 3. The Markov matrix of probabilities $A = [a_{ij}]$ is given by

$$\begin{bmatrix} 0.4 & 0.1 & 0.5 \\ 0.4 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0.4 \end{bmatrix}$$

If the walker starts in location 1, calculate the probability that he ends up in location 3 at time $n = 2$.

Solution. Since the walker begins in location 1, we have

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The goal is to calculate x_{32}. To do this we calculate $\vec{x}_2$, using $\vec{x}_{n+1} = A\vec{x}_n$.

$$\begin{array}{rcl} \vec{x}_1 & = & A\vec{x}_0 \\ & = & \begin{bmatrix} 0.4 & 0.1 & 0.5 \\ 0.4 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ & = & \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix} \end{array}$$

$$\begin{array}{rcl} \vec{x}_2 & = & A\vec{x}_1 \\ & = & \begin{bmatrix} 0.4 & 0.1 & 0.5 \\ 0.4 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix} \\ & = & \begin{bmatrix} 0.3 \\ 0.42 \\ 0.28 \end{bmatrix} \end{array}$$

This gives the probabilities that our walker ends up in locations 1, 2, and 3. For this example we are interested in location 3, and the probability that our individual ends up at location 3 at time $n = 2$ is 0.28.

♠

Returning to the context of migration, suppose we wish to know how many residents will be in a certain location after a very long time. It turns out that if some power of the migration matrix has all positive entries, then there is a vector $\vec{x}_s$ such that $A^n\vec{x}_0$ approaches $\vec{x}_s$ as n becomes very large. Hence as more time passes and n increases, $A^n\vec{x}_0$ will become closer to the vector $\vec{x}_s$.

Consider Theorem 7.35. Let n increase so that $\vec{x}_n$ approaches $\vec{x}_s$. As $\vec{x}_n$ becomes closer to $\vec{x}_s$, so too does $\vec{x}_{n+1}$. For sufficiently large n, the statement $\vec{x}_{n+1} = A\vec{x}_n$ can be written as $\vec{x}_s = A\vec{x}_s$.

This discussion motivates the following theorem.

Theorem 7.38: Steady State Vector

*Let A be a migration matrix. Then there exists a **steady state vector** written $\vec{x}_s$ such that*

$$\vec{x}_s = A\vec{x}_s$$

where $\vec{x}_s$ has positive entries which have the same sum as the entries of $\vec{x}_0$.
As n increases, the state vectors $\vec{x}_n$ will approach $\vec{x}_s$.

Note that the condition in Theorem 7.38 can be written as $(I-A)\vec{x}_s = 0$, representing a homogeneous system of equations.

Consider the following example. Notice that it is the same example as the Example 7.36 but here it will involve a longer time frame.

Example 7.39: Populations over the Long Run

Consider the migration matrix

$$A = \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix}$$

for locations $1,2,$ *and* 3. *Suppose initially there are 100 residents in location 1, 200 in location 2 and 400 in location 4. Find the population in the three locations after a long time.*

Solution. By Theorem 7.38 the steady state vector $\vec{x}_s$ can be found by solving the system $(I-A)\vec{x}_s = 0$.

Thus we need to find a solution to

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix} \right) \begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix and the resulting reduced row-echelon form are given by

$$\left[\begin{array}{rrr|r} 0.4 & 0 & -0.1 & 0 \\ -0.2 & 0.2 & 0 & 0 \\ -0.2 & -0.2 & 0.1 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -0.25 & 0 \\ 0 & 1 & -0.25 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore, the eigenvectors are

$$t \begin{bmatrix} 0.25 \\ 0.25 \\ 1 \end{bmatrix}$$

The initial vector $\vec{x}_0$ is given by

$$\begin{bmatrix} 100 \\ 200 \\ 400 \end{bmatrix}$$

Now all that remains is to choose the value of t such that

$$0.25t + 0.25t + t = 100 + 200 + 400$$

Solving this equation for t yields $t = \frac{1400}{3}$. Therefore the population in the long run is given by

$$\frac{1400}{3} \begin{bmatrix} 0.25 \\ 0.25 \\ 1 \end{bmatrix} = \begin{bmatrix} 116.666\,666\,666\,666\,7 \\ 116.666\,666\,666\,666\,7 \\ 466.666\,666\,666\,666\,7 \end{bmatrix}$$

Again, because we are working with populations, these values need to be rounded. The steady state vector $\vec{x}_s$ is given by

$$\begin{bmatrix} 117 \\ 117 \\ 466 \end{bmatrix}$$

♠

We can see that the numbers we calculated in Example 7.36 for the populations after the 10^{th} unit of time are not far from the long term values.

Consider another example.

Example 7.40: Populations After a Long Time

Suppose a migration matrix is given by

$$A = \begin{bmatrix} \frac{1}{5} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{11}{20} & \frac{1}{4} & \frac{3}{10} \end{bmatrix}$$

Find the comparison between the populations in the three locations after a long time.

Solution. In order to compare the populations in the long term, we want to find the steady state vector $\vec{x}_s$. So we must solve the equation

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{5} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{11}{20} & \frac{1}{4} & \frac{3}{10} \end{bmatrix} \right) \begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix and the resulting reduced row-echelon form are given by

$$\left[\begin{array}{rrr|r} \frac{4}{5} & -\frac{1}{2} & -\frac{1}{5} & 0 \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & 0 \\ -\frac{11}{20} & -\frac{1}{4} & \frac{7}{10} & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -\frac{16}{19} & 0 \\ 0 & 1 & -\frac{18}{19} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

and so an eigenvector is

$$\begin{bmatrix} 16 \\ 18 \\ 19 \end{bmatrix}$$

Therefore, the proportion of population in location 2 to location 1 is given by $\frac{18}{16}$. The proportion of population 3 to location 2 is given by $\frac{19}{18}$. ♠

A Look Under the Hood: Eigenvalues of Markov Matrices

You may not have noticed it, but Theorem 7.38 and the discussion immediately following it foreshadow the following important proposition, the proof of which has a surprising and satisfying approach.

Proposition 7.41: Eigenvalues of a Markov Matrix

Let $A = \left[a_{ij}\right]$ be a Markov matrix. Then 1 is always an eigenvalue for A.

Proof. Remember that the determinant of a matrix always equals that of its transpose. Therefore,

$$\det(xI - A) = \det\left((xI - A)^T\right) = \det\left(xI - A^T\right)$$

because $I^T = I$. Thus the characteristic equation for A is the same as the characteristic equation for A^T. Consequently, A and A^T have the same eigenvalues. We will show that 1 is an eigenvalue for A^T and then it will follow that 1 is an eigenvalue for A.

Remember that for a Markov matrix, $\sum_i a_{ij} = 1$. Therefore, if $A^T = \left[b_{ij}\right]$ with $b_{ij} = a_{ji}$, it follows that

$$\sum_j b_{ij} = \sum_j a_{ji} = 1$$

Therefore, from matrix multiplication,

$$A^T \left[\begin{array}{c} 1 \\ \vdots \\ 1 \end{array}\right] = \left[\begin{array}{c} \sum_j b_{1j} \\ \vdots \\ \sum_j b_{nj} \end{array}\right] = \left[\begin{array}{c} 1 \\ \vdots \\ 1 \end{array}\right]$$

Notice that this shows that $\left[\begin{array}{c} 1 \\ \vdots \\ 1 \end{array}\right]$ is an eigenvector for A^T corresponding to the eigenvalue $\lambda = 1$. As explained above, this shows that $\lambda = 1$ is an eigenvalue for A because A and A^T have the same eigenvalues.

♠

Discrete Time Dynamical Systems

The migration matrices discussed above give an example of a discrete time dynamical system. We call them discrete because they involve discrete time values taken at a sequence of points rather than on a continuous interval of time.

Another example of a situation which can be studied in this way is a predator prey model. Consider the following model where x is the number of prey and y the number of predators in a certain area at a certain time. These are functions of $n \in \mathbb{N}$ where $n = 1, 2, \cdots$ are the ends of intervals of time which may be of interest in the problem. In other words, x_n is the number of prey at the end of the n^{th} interval of time. An example of this situation may be modeled by the following equation

$$\left[\begin{array}{c} x_{n+1} \\ y_{n+1} \end{array}\right] = \left[\begin{array}{rr} 2 & -3 \\ 1 & 4 \end{array}\right] \left[\begin{array}{c} x_n \\ y_n \end{array}\right]$$

This says that from time period n to $n+1$, x increases if there are more x and decreases as there are more y. In the context of this example, this means that as the number of predators increases, the number of prey decreases. As for y, it increases if there are more y and also if there are more x.

This is an example of a matrix recurrence, which we define now.

Definition 7.42: Matrix Recurrence

Suppose a dynamical system is given by

$$\begin{aligned} x_{n+1} &= ax_n + by_n \\ y_{n+1} &= cx_n + dy_n \end{aligned}$$

This system can be expressed as $V_{n+1} = AV_n$ *where* $V_n = \left[\begin{array}{c} x_n \\ y_n \end{array} \right]$ *and* $A = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right]$.

In this section, we will examine how to find solutions to a dynamical system given certain initial conditions. This process involves several concepts previously studied, including matrix diagonalization and Markov matrices. The procedure is given as follows.

Procedure 7.43: Solving a Dynamical System

Suppose a dynamical system is given by

$$\begin{aligned} x_{n+1} &= ax_n + by_n \\ y_{n+1} &= cx_n + dy_n \end{aligned}$$

Given initial conditions x_0 *and* y_0*, the solutions to the system are found as follows:*

1. *Express the dynamical system in the form* $V_{n+1} = AV_n$.
2. *Diagonalize* A *to be written as* $A = PDP^{-1}$.
3. *Then* $V_n = PD^nP^{-1}V_0$ *where* $V_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ *is the vector containing the initial conditions.*
4. *If given specific values for* n*, substitute into this equation. Otherwise, find a general solution for* V_n.

We will now consider an example in detail.

Example 7.44: Solutions of a Discrete Dynamical System

Suppose a dynamical system is given by

$$\begin{array}{rcl} x_{n+1} & = & 1.5x_n - 0.5y_n \\ y_{n+1} & = & 1.0x_n \end{array}$$

Express this system as a matrix recurrence and find solutions to the dynamical system for the initial conditions $x_0 = 20,\ y_0 = 10$.

Solution. First, we express the system as a matrix recurrence.

$$\begin{array}{rcl} V_{n+1} & = & AV_n \\ \left[\begin{array}{c} x_{n+1} \\ y_{n+1} \end{array} \right] & = & \left[\begin{array}{rr} 1.5 & -0.5 \\ 1.0 & 0 \end{array} \right] \left[\begin{array}{c} x_n \\ y_n \end{array} \right] \end{array}$$

Then

$$A = \left[\begin{array}{rr} 1.5 & -0.5 \\ 1.0 & 0 \end{array} \right]$$

You can verify that the eigenvalues of A are 1 and 0.5. By diagonalizing, we can write A in the form

$$P^{-1}DP = \left[\begin{array}{rr} 1 & 1 \\ 1 & 2 \end{array} \right] \left[\begin{array}{rr} 1 & 0 \\ 0 & 0.5 \end{array} \right] \left[\begin{array}{rr} 2 & -1 \\ -1 & 1 \end{array} \right]$$

Now given an initial condition

$$V_0 = \left[\begin{array}{c} x_0 \\ y_0 \end{array} \right]$$

the solution to the dynamical system is given by

$$\begin{array}{rcl} V_n & = & PD^nP^{-1}V_0 \\ \left[\begin{array}{c} x_n \\ y_n \end{array} \right] & = & \left[\begin{array}{rr} 1 & 1 \\ 1 & 2 \end{array} \right] \left[\begin{array}{rr} 1 & 0 \\ 0 & 0.5 \end{array} \right]^n \left[\begin{array}{rr} 2 & -1 \\ -1 & 1 \end{array} \right] \left[\begin{array}{c} x_0 \\ y_0 \end{array} \right] \\ & = & \left[\begin{array}{rr} 1 & 1 \\ 1 & 2 \end{array} \right] \left[\begin{array}{rr} 1 & 0 \\ 0 & (0.5)^n \end{array} \right] \left[\begin{array}{rr} 2 & -1 \\ -1 & 1 \end{array} \right] \left[\begin{array}{c} x_0 \\ y_0 \end{array} \right] \\ & = & \left[\begin{array}{c} y_0\left((0.5)^n - 1\right) - x_0\left((0.5)^n - 2\right) \\ y_0\left(2\left(0.5\right)^n - 1\right) - x_0\left(2\left(0.5\right)^n - 2\right) \end{array} \right] \end{array}$$

If we let n become arbitrarily large, this vector approaches

$$\left[\begin{array}{c} 2x_0 - y_0 \\ 2x_0 - y_0 \end{array} \right]$$

Thus for large n,

$$\left[\begin{array}{c} x_n \\ y_n \end{array} \right] \approx \left[\begin{array}{c} 2x_0 - y_0 \\ 2x_0 - y_0 \end{array} \right]$$

Now suppose the initial condition is given by

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

Then, we can find solutions for various values of n. Here are the solutions for values of n between 1 and 5

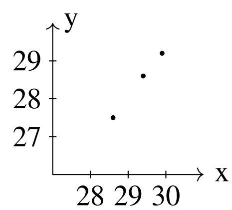
$$n = 1 : \begin{bmatrix} 25.0 \\ 20.0 \end{bmatrix}, n = 2 : \begin{bmatrix} 27.5 \\ 25.0 \end{bmatrix}, n = 3 : \begin{bmatrix} 28.75 \\ 27.5 \end{bmatrix}$$

$$n = 4 : \begin{bmatrix} 29.375 \\ 28.75 \end{bmatrix}, n = 5 : \begin{bmatrix} 29.688 \\ 29.375 \end{bmatrix}$$

Notice that as n increases, we approach the vector given by

$$\begin{bmatrix} 2x_0 - y_0 \\ 2x_0 - y_0 \end{bmatrix} = \begin{bmatrix} 2(20) - 10 \\ 2(20) - 10 \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \end{bmatrix}.$$

These solutions are graphed in the following figure.



♠

The following example demonstrates another system which exhibits some interesting behavior. When we graph the solutions, it is possible for the ordered pairs to spiral around the origin.

Example 7.45: Finding Solutions to a Dynamical System

Suppose a dynamical system is of the form

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

Find solutions to the dynamical system for given initial conditions.

Solution. Let

$$A = \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix}$$

To find solutions, we must diagonalize A. You can verify that the eigenvalues of A are complex and are given by $\lambda_1 = 0.7 + 0.7i$ and $\lambda_2 = 0.7 - 0.7i$. The eigenvector for $\lambda_1 = 0.7 + 0.7i$ is

$$\begin{bmatrix} 1 \\ i \end{bmatrix}$$

and the eigenvector for $\lambda_2 = 0.7 - 0.7i$ is

$$\left[\begin{array}{c} 1 \\ -i \end{array}\right]$$

Thus the matrix A can be written in the form

$$\left[\begin{array}{cc} 1 & 1 \\ i & -i \end{array}\right]\left[\begin{array}{cc} 0.7+0.7i & 0 \\ 0 & 0.7-0.7i \end{array}\right]\left[\begin{array}{cc} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{array}\right]$$

and so,

$$\begin{aligned} V_n &= PD^nP^{-1}V_0 \\ \left[\begin{array}{c} x_n \\ y_n \end{array}\right] &= \left[\begin{array}{cc} 1 & 1 \\ i & -i \end{array}\right]\left[\begin{array}{cc} (0.7+0.7i)^n & 0 \\ 0 & (0.7-0.7i)^n \end{array}\right]\left[\begin{array}{cc} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{array}\right]\left[\begin{array}{c} x_0 \\ y_0 \end{array}\right] \end{aligned}$$

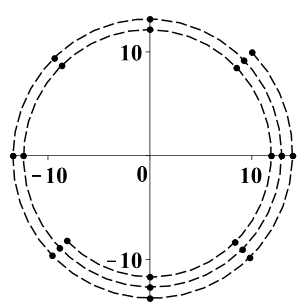
The explicit solution is given by

$$\left[\begin{array}{c} x_n \\ y_n \end{array}\right] = \left[\begin{array}{c} x_0\left(\frac{1}{2}(0.7-0.7i)^n + \frac{1}{2}(0.7+0.7i)^n\right) + y_0\left(\frac{1}{2}i(0.7-0.7i)^n - \frac{1}{2}i(0.7+0.7i)^n\right) \\ y_0\left(\frac{1}{2}(0.7-0.7i)^n + \frac{1}{2}(0.7+0.7i)^n\right) - x_0\left(\frac{1}{2}i(0.7-0.7i)^n - \frac{1}{2}i(0.7+0.7i)^n\right) \end{array}\right]$$

Suppose the initial condition is

$$\left[\begin{array}{c} x_0 \\ y_0 \end{array}\right] = \left[\begin{array}{c} 10 \\ 10 \end{array}\right]$$

Then one obtains the following sequence of values which are graphed below by letting $n = 1, 2, \cdots, 20$



In this picture, the dots are the values and the dashed line is to help to picture what is happening.

These points are getting gradually closer to the origin, but they are circling the origin in the clockwise direction as they do so. As n increases, the vector $\left[\begin{array}{c} x_n \\ y_n \end{array}\right]$ approaches $\left[\begin{array}{c} 0 \\ 0 \end{array}\right]$. ♠

Our discussion to this point has been focused on discrete time dynamical systems. However, matrix techniques can also be used to analyze the behavior of continuous time systems of differential equations.

One famous such model of predator-prey interactions is the Lotka Volterra system. This model is given by the system of two differential equations

$$\begin{aligned}\frac{dx}{dt} &= x(a-by)\\ \frac{dy}{dt} &= -y(c-dx)\end{aligned}$$

where a,b,c,d are positive constants. For example, you might have x be the population of moose and y the population of wolves on an island.

Note that these equations make logical sense. The top says that the rate at which the moose population increases would be ax if there were no predators y. However, this is modified by multiplying instead by $(a-by)$ because if there are predators, these will depress the rate of growth of the of moose. The more predators there are, the more pronounced is this effect. As to the predator equation, you can see that the equations predict that if there are many prey around, then the rate of growth of the predators would seem to be high. However, this is modified by the term $-cy$ because if there are many predators, there would be competition for the available food supply and this would tend to decrease $\frac{dy}{dt}$.

The behavior near an equilibrium point, which is a point where the right side of the differential equations equals zero, is of great interest. In this case, the equilibrium point is

$$x=\frac{c}{d},\ y=\frac{a}{b}$$

Then one defines new variables according to the formula

$$x+\frac{c}{d}=x,\ y+\frac{a}{b}=y$$

In terms of these new variables, the differential equations become

$$\begin{aligned}\frac{dx}{dt} &= \left(x+\frac{c}{d}\right)\left(a-b\left(y+\frac{a}{b}\right)\right)\\ \frac{dy}{dt} &= -\left(y+\frac{a}{b}\right)\left(c-d\left(x+\frac{c}{d}\right)\right)\end{aligned}$$

Multiplying out the right sides yields

$$\begin{aligned}\frac{dx}{dt} &= -bxy-b\frac{c}{d}y\\ \frac{dy}{dt} &= dxy+\frac{a}{b}dx\end{aligned}$$

The interest is for x,y small and so these equations are essentially equal to

$$\begin{aligned}\frac{dx}{dt} &= -b\frac{c}{d}y\\ \frac{dy}{dt} &= \frac{a}{b}dx\end{aligned}$$

Replace $\frac{dx}{dt}$ with the difference quotient $\frac{x(t+h)-x(t)}{h}$ where h is a small positive number and $\frac{dy}{dt}$ with a similar difference quotient. For example one could have h correspond to one day or even one hour. Thus, for h small enough, the following would seem to be a good approximation to the differential equations.

$$x(t+h) = x(t)-hb\frac{c}{d}y$$

$$y(t+h) \quad = \quad y(t) + h\frac{a}{b}dx$$

Let $1,2,3,\cdots$ denote the ends of discrete intervals of time having length h chosen above. Then the above equations take the form

$$\left[\begin{array}{c} x_{n+1} \\ y_{n+1} \end{array}\right] = \left[\begin{array}{cc} 1 & -\frac{hbc}{d} \\ \frac{had}{b} & 1 \end{array}\right]\left[\begin{array}{c} x_n \\ y_n \end{array}\right]$$

Note that the eigenvalues of this matrix are always complex.

We are not interested in time intervals of length h for h very small. Instead, we are interested in much longer lengths of time. Thus, replacing the time interval with mh,

$$\left[\begin{array}{c} x(n+m) \\ y(n+m) \end{array}\right] = \left[\begin{array}{cc} 1 & -\frac{hbc}{d} \\ \frac{had}{b} & 1 \end{array}\right]^m \left[\begin{array}{c} x_n \\ y_n \end{array}\right]$$

For example, if $m=2$, you would have

$$\left[\begin{array}{c} x(n+2) \\ y(n+2) \end{array}\right] = \left[\begin{array}{cc} 1-ach^2 & -2b\frac{c}{d}h \\ 2\frac{a}{b}dh & 1-ach^2 \end{array}\right]\left[\begin{array}{c} x_n \\ y_n \end{array}\right]$$

Note that most of the time, the eigenvalues of the new matrix will be complex.

You can also notice that the upper right corner will be negative by considering higher powers of the matrix. Thus letting $1,2,3,\cdots$ denote the ends of discrete intervals of time, the desired discrete dynamical system is of the form

$$\left[\begin{array}{c} x_{n+1} \\ y_{n+1} \end{array}\right] = \left[\begin{array}{cc} a & -b \\ c & d \end{array}\right]\left[\begin{array}{c} x_n \\ y_n \end{array}\right]$$

where a,b,c,d are positive constants and the matrix will likely have complex eigenvalues because it is a power of a matrix which has complex eigenvalues.

You can see from the above discussion that if the eigenvalues of the matrix used to define the dynamical system are less than 1 in absolute value, then the origin is stable in the sense that as $n \to \infty$, the solution converges to the origin. If either eigenvalue is larger than 1 in absolute value, then the solutions to the dynamical system will usually be unbounded, unless the initial condition is chosen very carefully. The next example exhibits the case where one eigenvalue is larger than 1 and the other is smaller than 1.

The following example demonstrates a familiar concept as a dynamical system.

Example 7.46: The Fibonacci Sequence

The Fibonacci sequence is the sequence given by

$$1,1,2,3,5,\cdots$$

which is defined recursively by the equations

$$\begin{aligned} x_0 &= 1 \\ x_1 &= 1 \\ x_{n+2} &= x_n + x_{n+1} \text{ for } n \geq 1 \end{aligned}$$

Show how the Fibonacci Sequence can be considered a dynamical system.

Solution. This sequence, important in both theoretical and applied mathematics, was first introduced to western mathematics by Leonardo of Pisa in 1202. His introductory problem involved keeping track of the number of reproducing rabbits on an island. The sequence can be found as the solution of a dynamical system as follows. Let $y_n = x_{n+1}$. Then the above recurrence relation can be written as

$$\left[\begin{array}{c} x_{n+1} \\ y_{n+1} \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right] \left[\begin{array}{c} x_n \\ y_n \end{array}\right], \left[\begin{array}{c} x_0 \\ y_0 \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

Let

$$A = \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right]$$

The eigenvalues of the matrix A are $\lambda_1 = \frac{1}{2} - \frac{1}{2}\sqrt{5}$ and $\lambda_2 = \frac{1}{2}\sqrt{5} + \frac{1}{2}$. The corresponding eigenvectors are, respectively,

$$\vec{x}_1 = \left[\begin{array}{c} -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{array}\right], \vec{x}_2 = \left[\begin{array}{c} \frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{array}\right]$$

You can see from a short computation (or a couple of seconds with a calculator) that one of the eigenvalues is smaller than 1 in absolute value while the other is larger than 1 in absolute value. Now, diagonalizing A gives us

$$\left[\begin{array}{cc} \frac{1}{2}\sqrt{5} - \frac{1}{2} & -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 & 1 \end{array}\right]^{-1} \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right] \left[\begin{array}{cc} \frac{1}{2}\sqrt{5} - \frac{1}{2} & -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc} \frac{1}{2}\sqrt{5} + \frac{1}{2} & 0 \\ 0 & \frac{1}{2} - \frac{1}{2}\sqrt{5} \end{array}\right]$$

Then it follows that for a given initial condition, the solution to this dynamical system is of the form

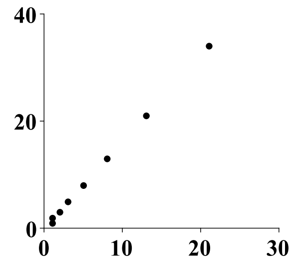
$$\begin{aligned} \left[\begin{array}{c} x_n \\ y_n \end{array}\right] = & \left[\begin{array}{cc} \frac{1}{2}\sqrt{5} - \frac{1}{2} & -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 & 1 \end{array}\right] \left[\begin{array}{cc} \left(\frac{1}{2}\sqrt{5} + \frac{1}{2}\right)^n & 0 \\ 0 & \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^n \end{array}\right] \cdot \\ & \left[\begin{array}{cc} \frac{1}{5}\sqrt{5} & \frac{1}{10}\sqrt{5} + \frac{1}{2} \\ -\frac{1}{5}\sqrt{5} & \frac{1}{5}\sqrt{5}\left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right) \end{array}\right] \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \end{aligned}$$

It follows that

$$x_n = \left(\frac{1}{2}\sqrt{5} + \frac{1}{2}\right)^n \left(\frac{1}{10}\sqrt{5} + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^n \left(\frac{1}{2} - \frac{1}{10}\sqrt{5}\right)$$

♠

Here is a picture of the ordered pairs (x_n, y_n) for $n = 0, 1, \cdots, n$.



There is so much more that can be said about dynamical systems. It is a major topic of study in differential equations and what is given above is just an introduction.

The Matrix Exponential

The goal of this section is to use the concept of the matrix exponential to solve first order linear differential equations. We begin by defining the matrix exponential.

Suppose A is a diagonalizable matrix. Then the **matrix exponential**, written e^A, can be easily defined. Recall that as A is diagonalizable, there is an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D$$

D is of the form

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \tag{7.5}$$

and it follows that

$$D^m = \begin{bmatrix} \lambda_1^m & & 0 \\ & \ddots & \\ 0 & & \lambda_n^m \end{bmatrix}$$

Since A is diagonalizable,

$$A = PDP^{-1}$$

and

$$A^m = PD^mP^{-1}$$

We now will examine what is meant by the matrix exponential e^A. Begin by formally writing the following power series for e^A:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{PD^kP^{-1}}{k!} = P\left(\sum_{k=0}^{\infty} \frac{D^k}{k!}\right)P^{-1}$$

If D is given above in 7.5, the above sum is of the form

$$P\left(\sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{k!}\lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \frac{1}{k!}\lambda_n^k \end{bmatrix}\right)P^{-1}$$

This can be rearranged as follows:

$$\begin{aligned} e^A &= P\begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_n^k \end{bmatrix}P^{-1} \\ &= P\begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}P^{-1} \end{aligned}$$

This justifies the following theorem.

Theorem 7.47: The Matrix Exponential

Let A be a diagonalizable matrix, with eigenvalues $\lambda_1, ..., \lambda_n$ and corresponding matrix of eigenvectors P. Then the matrix exponential, e^A, is given by

$$e^A = P\begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}P^{-1}$$

Example 7.48: Compute e^A for a Matrix A

Let

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Find e^A.

Solution. The eigenvalues work out to be $\lambda_1 = 1,\ \lambda_2 = 2$, and $\lambda_3 = 3$ and corresponding eigenvectors

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Then let

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } P = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

and so

$$P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Then the matrix exponential is

$$e^{At} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^1 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} e^2 & e^2 - e^3 & e^2 - e^3 \\ e^2 - e & e^2 & e^2 - e \\ -e^2 + e & -e^2 + e^3 & -e^2 + e + e^3 \end{bmatrix}$$

♠

The matrix exponential is a useful tool to solve autonomous systems of first order linear differential equations. These are equations which are of the form

$$\vec{x}' = A\vec{x},\ \vec{x}(0) = C$$

where A is a diagonalizable $n \times n$ matrix and C is a constant vector. $\vec{x}$ is a vector of functions in one variable, t:

$$\vec{x} = \vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Then $\vec{x}'$ refers to the first derivative of $\vec{x}$ and is given by

$$\vec{x}' = \vec{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix},\ x_i'(t) = \text{the derivative of } x_i(t)$$

Then it turns out that the solution to the above system of equations is $\vec{x}(t) = e^{At}C$. To see this, suppose A is diagonalizable so that

$$A = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1}$$

Then

$$e^{At} = P \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

$$e^{At}C = P \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}C$$

Differentiating $e^{At}C$ yields

$$\vec{x}' = \left(e^{At}C\right)' = P \begin{bmatrix} \lambda_1 e^{\lambda_1 t} & & \\ & \ddots & \\ & & \lambda_n e^{\lambda_n t} \end{bmatrix} P^{-1}C$$

$$= P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}C$$

$$\begin{aligned} &= P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1}P \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}C \\ &= A\left(e^{At}C\right) = A\vec{x} \end{aligned}$$

Therefore $\vec{x} = \vec{x}(t) = e^{At}C$ is a solution to $\vec{x}' = A\vec{x}$.

To prove that $\vec{x}(0) = C$ if $\vec{x}(t) = e^{At}C$:

$$\vec{x}(0) = e^{A0}C = P \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} P^{-1}C = C$$

Example 7.49: Solving an Initial Value Problem

Solve the initial value problem

$$\left[\begin{array}{c} x \\ y \end{array}\right]' = \left[\begin{array}{rr} 0 & -2 \\ 1 & 3 \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right], \left[\begin{array}{c} x(0) \\ y(0) \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

Solution. The matrix is diagonalizable and can be written as

$$\begin{array}{rcl} A & = & PDP^{-1} \\ \left[\begin{array}{rr} 0 & -2 \\ 1 & 3 \end{array}\right] & = & \left[\begin{array}{rr} 1 & 1 \\ -\frac{1}{2} & -1 \end{array}\right]\left[\begin{array}{rr} 1 & 0 \\ 0 & 2 \end{array}\right]\left[\begin{array}{rr} 2 & 2 \\ -1 & -2 \end{array}\right] \end{array}$$

Therefore, the matrix exponential is of the form

$$e^{At} = \left[\begin{array}{rr} 1 & 1 \\ -\frac{1}{2} & -1 \end{array}\right]\left[\begin{array}{cc} e^t & 0 \\ 0 & e^{2t} \end{array}\right]\left[\begin{array}{rr} 2 & 2 \\ -1 & -2 \end{array}\right]$$

The solution to the initial value problem is

$$\begin{array}{rcl} \vec{x}(t) & = & e^{At}C \\ \left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] & = & \left[\begin{array}{rr} 1 & 1 \\ -\frac{1}{2} & -1 \end{array}\right]\left[\begin{array}{cc} e^t & 0 \\ 0 & e^{2t} \end{array}\right]\left[\begin{array}{rr} 2 & 2 \\ -1 & -2 \end{array}\right]\left[\begin{array}{c} 1 \\ 1 \end{array}\right] \\ & = & \left[\begin{array}{c} 4e^t - 3e^{2t} \\ 3e^{2t} - 2e^t \end{array}\right] \end{array}$$

We can check that this works:

$$\begin{array}{rcl} \left[\begin{array}{c} x(0) \\ y(0) \end{array}\right] & = & \left[\begin{array}{c} 4e^0 - 3e^{2(0)} \\ 3e^{2(0)} - 2e^0 \end{array}\right] \\ & = & \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \end{array}$$

Lastly,

$$\vec{x}' = \left[\begin{array}{c} 4e^t - 3e^{2t} \\ 3e^{2t} - 2e^t \end{array}\right]' = \left[\begin{array}{c} 4e^t - 6e^{2t} \\ 6e^{2t} - 2e^t \end{array}\right]$$

and

$$A\vec{x} = \left[\begin{array}{rr} 0 & -2 \\ 1 & 3 \end{array}\right]\left[\begin{array}{c} 4e^t - 3e^{2t} \\ 3e^{2t} - 2e^t \end{array}\right] = \left[\begin{array}{c} 4e^t - 6e^{2t} \\ 6e^{2t} - 2e^t \end{array}\right]$$

which is the same thing. Thus this is the solution to the initial value problem. ♠

Engage Active Learning App!

Lyryx Engage is an active learning app designed to increase student engagement in the study of linear algebra. The content is "chunked" into small blocks, each with an interactive assessment activity to promote comprehension.

Visit us at https://lyryx.com to register and access Lyryx Engage!

Lyryx Online Exercises!

Lyryx offers formative online exercises! The educational software captures instructors' experience using algorithms to analyze and evaluate students' work, identifying common errors and providing personalized feedback. Students are encouraged to try these randomized problems repeatedly, learning from their mistakes until mastery.

Visit us at https://lyryx.com to register and access Lyryx exercises!

7.4 Orthogonality

Orthogonal Diagonalization

We begin this section by recalling some important definitions. Recall from Definition 4.126 that two non-zero vectors are called orthogonal if their dot product equals 0. A set of vectors is said to be orthonormal if every vector in the set has length one and any two vectors chosen from the set are orthogonal.

An orthogonal matrix U, from Definition 4.133, is one in which $UU^T = I$. In other words, the transpose of an orthogonal matrix is equal to its inverse. A key characteristic of orthogonal matrices, which will be essential in this section, is that the columns of an orthogonal matrix form an orthonormal set of vectors.

We now recall another important definition.

Definition 7.50: Symmetric and Skew Symmetric Matrices

A real $n \times n$ matrix A, is ***symmetric*** *if $A^T = A$. If $A = -A^T$, then A is called* ***skew symmetric.***

Before proving an essential theorem, we first examine the following lemma which will be used below.

Lemma 7.51: The Dot Product

Let $A = \left[a_{ij} \right]$ be a real symmetric $n \times n$ matrix, and let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then

$$(A\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A\vec{y})$$

Proof. This result follows from the definition of the dot product together with properties of matrix multiplication, as follows:

$$\begin{aligned} (A\vec{x}) \cdot \vec{y} &= (A\vec{x})^T \vec{y} \\ &= (\vec{x}^T A^T)\vec{y} \\ &= \vec{x}^T (A^T \vec{y}) \\ &= \vec{x} \cdot (A^T \vec{y}) \\ &= \vec{x} \cdot (A\vec{y}) \end{aligned}$$

The last step follows from $A^T = A$, since A is symmetric. ♠

We can now prove that the eigenvalues of a real symmetric matrix are real numbers and that eigenvectors corresponding to different eigenvalues are orthogonal.

Theorem 7.52: Orthogonal Eigenvectors

Let A be a real symmetric matrix. Then the eigenvalues of A are real numbers and eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof. Recall that for a complex number $a+ib$, the complex conjugate, denoted by $\overline{a+ib}$ is given by $\overline{a+ib} = a - ib$. The notation, $\overline{\vec{x}}$ will denote the vector which has every entry replaced by its complex conjugate.

Suppose A is a real symmetric matrix and $A\vec{x} = \lambda\vec{x}$ with $\vec{x} \neq \vec{0}$. We will first show that λ is a real number. As A is symmetric, $A = A^T$, and so

$$A^T\vec{x} = A\vec{x}.$$

Multiply on the left by $\overline{\vec{x}}^T$ to get

$$\overline{\vec{x}}^T(A^T\vec{x}) = \overline{\vec{x}}^T(A\vec{x}).$$

And then

$$\begin{aligned}
(\overline{\vec{x}}^T A^T)\vec{x} &= \overline{\vec{x}}^T(A\vec{x}) \\
(A\overline{\vec{x}})^T\vec{x} &= \overline{\vec{x}}^T(\lambda\vec{x}) \\
(\overline{A\vec{x}})^T\vec{x} &= \overline{\vec{x}}^T(\lambda\vec{x}) \text{ (as } A = \overline{A}) \\
(\overline{A\vec{x}})^T\vec{x} &= \lambda\overline{\vec{x}}^T\vec{x} \\
(\overline{\lambda\vec{x}})^T\vec{x} &= \lambda\overline{\vec{x}}^T\vec{x} \\
\overline{\lambda}\overline{\vec{x}}^T\vec{x} &= \lambda\overline{\vec{x}}^T\vec{x} \\
\overline{\lambda}\left(\overline{\vec{x}}^T\vec{x}\right) &= \lambda\left(\overline{\vec{x}}^T\vec{x}\right)
\end{aligned}$$

Dividing by $\overline{\vec{x}}^T\vec{x}$ on both sides yields $\overline{\lambda} = \lambda$ which says λ is real. To do this, we need to ensure that $\overline{\vec{x}}^T\vec{x} \neq 0$. Notice that $\overline{\vec{x}}^T\vec{x} = 0$ if and only if $\vec{x} = \vec{0}$. Since we chose $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$, $\vec{x}$ is an eigenvector and therefore must be nonzero.

To show that eigenvectors corresponding to distinct eigenvalues are orthogonal, suppose A is a real symmetric matrix, $A\vec{x} = \lambda\vec{x}$, and $A\vec{y} = \mu\vec{y}$ where $\mu \neq \lambda$. Then since A is symmetric, it follows from Lemma 7.51 about the dot product that

$$\lambda\vec{x}\cdot\vec{y} = A\vec{x}\cdot\vec{y} = \vec{x}\cdot A\vec{y} = \vec{x}\cdot\mu\vec{y} = \mu\vec{x}\cdot\vec{y}$$

Hence $(\lambda - \mu)\vec{x}\cdot\vec{y} = 0$. It follows that, since $\lambda - \mu \neq 0$, it must be that $\vec{x}\cdot\vec{y} = 0$, as claimed. ♠

The following theorem is proved in a similar manner.

Theorem 7.53: Eigenvalues of Skew Symmetric Matrix

The eigenvalues of a real skew symmetric matrix are either equal to 0 *or are pure imaginary numbers.*

Proof. First, note that if $A = 0$ is the zero matrix, then A is skew symmetric and has eigenvalues equal to 0.

Suppose $A = -A^T$ so A is skew symmetric and $A\vec{x} = \lambda\vec{x}$. Then

$$\overline{\lambda}\overline{\vec{x}}^T\vec{x} = \left(\overline{A\vec{x}}\right)^T\vec{x} = \overline{\vec{x}}^T A^T\vec{x} = -\overline{\vec{x}}^T A\vec{x} = -\lambda\overline{\vec{x}}^T\vec{x}$$

and so, dividing by $\overline{\vec{x}}^T\vec{x}$ as before, $\overline{\lambda} = -\lambda$. Letting $\lambda = a + ib$, this means $a - ib = -a - ib$ and so $a = 0$. Thus λ is not equal to zero, then λ is a pure imaginary number. ♠

Consider the following example.

Example 7.54: Eigenvalues of a Skew Symmetric Matrix

Let $A = \left[\begin{array}{rr} 0 & -1 \\ 1 & 0 \end{array} \right]$. *Find the eigenvalues of* A.

Solution. First notice that A is skew symmetric. By Theorem 7.53, the eigenvalues will either equal 0 or be pure imaginary. The eigenvalues of A are obtained by solving the usual equation

$$\det(xI - A) = \det \left[\begin{array}{rr} x & 1 \\ -1 & x \end{array} \right] = x^2 + 1 = 0$$

Hence the eigenvalues are $\pm i$, pure imaginary. ♠

Consider the following example.

Example 7.55: Eigenvalues of a Symmetric Matrix

Let $A = \left[\begin{array}{rr} 1 & 2 \\ 2 & 3 \end{array} \right]$. *Find its eigenvalues.*

Solution. First, notice that A is symmetric. By Theorem 7.52, the eigenvalues will all be real. The eigenvalues of A are obtained by solving the usual equation

$$\det(xI - A) = \det \left[\begin{array}{cc} x-1 & -2 \\ -2 & x-3 \end{array} \right] = x^2 - 4x - 1 = 0$$

The eigenvalues are given by $\lambda_1 = 2 + \sqrt{5}$ and $\lambda_2 = 2 - \sqrt{5}$ which are both real. ♠

Recall that a diagonal matrix $D = \left[d_{ij} \right]$ is one in which $d_{ij} = 0$ whenever $i \neq j$. In other words, all numbers not on the main diagonal are equal to zero.

Consider the following important theorem.

Theorem 7.56: Orthogonal Diagonalization

Let A *be a real symmetric matrix. Then there exists an orthogonal matrix* U *such that*

$$U^T A U = D$$

where D *is a diagonal matrix. Moreover, the diagonal entries of* D *are the eigenvalues of* A.

We can use this theorem to diagonalize a symmetric matrix, using orthogonal matrices. Consider the following corollary.

Corollary 7.57: Orthonormal Set of Eigenvectors

If A is a real $n \times n$ symmetric matrix, then there exists an orthonormal set of eigenvectors, $\{\vec{u}_1, \cdots, \vec{u}_n\}$.

Proof. Since A is symmetric, then by Theorem 7.56, there exists an orthogonal matrix U such that $U^T AU = D$, a diagonal matrix whose diagonal entries are the eigenvalues of A. Therefore, since A is symmetric and all the matrices are real,

$$\overline{D} = \overline{D^T} = \overline{U^T A^T U} = U^T A^T U = U^T AU = D$$

showing D is real because each entry of D equals its complex conjugate.

Now let

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}$$

where the $\vec{u}_i$ denote the columns of U and

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

The equation, $U^T AU = D$ implies $AU = UD$ and

$$\begin{aligned} AU &= \begin{bmatrix} A\vec{u}_1 & A\vec{u}_2 & \cdots & A\vec{u}_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\vec{u}_1 & \lambda_2\vec{u}_2 & \cdots & \lambda_n\vec{u}_n \end{bmatrix} \\ &= UD \end{aligned}$$

where the entries denote the columns of AU and UD respectively. Therefore, $A\vec{u}_i = \lambda_i\vec{u}_i$. Since the matrix U is orthogonal, the ij^{th} entry of $U^T U$ equals δ_{ij} and so

$$\delta_{ij} = \vec{u}_i^T \vec{u}_j = \vec{u}_i \cdot \vec{u}_j$$

This proves the corollary because it shows the vectors $\{\vec{u}_i\}$ form an orthonormal set. ♠

Definition 7.58: Principal Axes

Let A be an $n \times n$ matrix. Then a principal axes of A is a set of orthonormal eigenvectors of A.

In the next example, we examine how to find such a set of orthonormal eigenvectors.

Example 7.59: Find an Orthonormal Set of Eigenvectors

Find an orthonormal set of eigenvectors for the symmetric matrix

$$A = \begin{bmatrix} 17 & -2 & -2 \\ -2 & 6 & 4 \\ -2 & 4 & 6 \end{bmatrix}$$

Solution. Recall Procedure 7.6 for finding the eigenvalues and eigenvectors of a matrix. You can verify that the eigenvalues are $18, 9, 2$. First find the eigenvector for 18 by solving the equation $(18I - A)\vec{x} = 0$. The appropriate augmented matrix is given by

$$\left[\begin{array}{ccc|c} 18-17 & 2 & 2 & 0 \\ 2 & 18-6 & -4 & 0 \\ 2 & -4 & 18-6 & 0 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore an eigenvector is

$$\left[\begin{array}{r} -4 \\ 1 \\ 1 \end{array}\right]$$

Next find the eigenvector for $\lambda = 9$. The augmented matrix and resulting reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 9-17 & 2 & 2 & 0 \\ 2 & 9-6 & -4 & 0 \\ 2 & -4 & 9-6 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus an eigenvector for $\lambda = 9$ is

$$\left[\begin{array}{r} 1 \\ 2 \\ 2 \end{array}\right]$$

Finally find an eigenvector for $\lambda = 2$. The appropriate augmented matrix and reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 2-17 & 2 & 2 & 0 \\ 2 & 2-6 & -4 & 0 \\ 2 & -4 & 2-6 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus an eigenvector for $\lambda = 2$ is

$$\left[\begin{array}{r} 0 \\ -1 \\ 1 \end{array}\right]$$

The set of eigenvectors for A is given by

$$\left\{ \left[\begin{array}{r} -4 \\ 1 \\ 1 \end{array}\right], \left[\begin{array}{r} 1 \\ 2 \\ 2 \end{array}\right], \left[\begin{array}{r} 0 \\ -1 \\ 1 \end{array}\right] \right\}$$

You can verify that these eigenvectors form an orthogonal set. By dividing each eigenvector by its magnitude, we obtain an orthonormal set:

$$\left\{ \frac{1}{\sqrt{18}}\left[\begin{array}{r} -4 \\ 1 \\ 1 \end{array}\right], \frac{1}{3}\left[\begin{array}{r} 1 \\ 2 \\ 2 \end{array}\right], \frac{1}{\sqrt{2}}\left[\begin{array}{r} 0 \\ -1 \\ 1 \end{array}\right] \right\}$$

♠

Consider the following example.

Example 7.60: Repeated Eigenvalues

Find an orthonormal set of three eigenvectors for the matrix

$$A = \begin{bmatrix} 10 & 2 & 2 \\ 2 & 13 & 4 \\ 2 & 4 & 13 \end{bmatrix}$$

Solution. You can verify that the eigenvalues of A are 9 (with algebraic multiplicity two) and 18 (with algebraic multiplicity one). Consider the eigenvectors corresponding to $\lambda = 9$. The appropriate augmented matrix and reduced row-echelon form are given by

$$\left[\begin{array}{ccc|c} 9-10 & -2 & -2 & 0 \\ -2 & 9-13 & -4 & 0 \\ -2 & -4 & 9-13 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so eigenvectors are of the form

$$\begin{bmatrix} -2y-2z \\ y \\ z \end{bmatrix}$$

We need to find two of these which are orthogonal. Let one be given by setting $z = 0$ and $y = 1$, giving $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

In order to find an eigenvector orthogonal to this one, we need to satisfy

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2y-2z \\ y \\ z \end{bmatrix} = 5y + 4z = 0$$

The values $y = -4$ and $z = 5$ satisfy this equation, giving another eigenvector corresponding to $\lambda = 9$ as

$$\begin{bmatrix} -2(-4) - 2(5) \\ (-4) \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix}$$

Next find the eigenvector for $\lambda = 18$. The augmented matrix and the resulting reduced row-echelon form are given by

$$\left[\begin{array}{ccc|c} 18-10 & -2 & -2 & 0 \\ -2 & 18-13 & -4 & 0 \\ -2 & -4 & 18-13 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so an eigenvector is

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Dividing each eigenvector by its length, the orthonormal set is

$$\left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \frac{\sqrt{5}}{15} \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

♠

In the above solution, the repeated eigenvalue implies that there would have been many other orthonormal bases which could have been obtained. While we chose to take $z = 0, y = 1$, we could just as easily have taken $y = 0$ or even $y = z = 1$. Any such change would have resulted in a different orthonormal set.

Recall the following definition.

Definition 7.61: Diagonalizable

An $n \times n$ matrix A is said to be ***non defective*** *or* ***diagonalizable*** *if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix.*

As indicated in Theorem 7.56 if A is a real symmetric matrix, there exists an orthogonal matrix U such that $U^T AU = D$ where D is a diagonal matrix. Therefore, every symmetric matrix is diagonalizable because if U is an orthogonal matrix, it is invertible and its inverse is U^T. In this case, we say that A is **orthogonally diagonalizable**. Therefore every symmetric matrix is in fact orthogonally diagonalizable. The next theorem provides another way to determine if a matrix is orthogonally diagonalizable.

Theorem 7.62: Orthogonally Diagonalizable

Let A be an $n \times n$ matrix. Then A is orthogonally diagonalizable if and only if A has an orthonormal set of eigenvectors.

Recall from Corollary 7.57 that every symmetric matrix has an orthonormal set of eigenvectors. In fact these three conditions are equivalent.

In the following example, the orthogonal matrix U will be found to orthogonally diagonalize a matrix.

Example 7.63: Diagonalize a Symmetric Matrix

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$. Find an orthogonal matrix U such that $U^T AU$ is a diagonal matrix.

Solution. In this case, the eigenvalues are 2 (with algebraic multiplicity one) and 1 (with algebraic multiplicity two). First we will find an eigenvector for the eigenvalue 2. The appropriate augmented matrix and resulting reduced row-echelon form are given by

$$\left[\begin{array}{ccc|c} 2-1 & 0 & 0 & 0 \\ 0 & 2-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 2-\frac{3}{2} & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so an eigenvector is

$$\left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right]$$

However, it is desired that the eigenvectors be unit vectors and so dividing this vector by its length gives

$$\left[\begin{array}{c} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array}\right]$$

Next find the eigenvectors corresponding to the eigenvalue equal to 1. The appropriate augmented matrix and resulting reduced row-echelon form are given by:

$$\left[\begin{array}{ccc|c} 1-1 & 0 & 0 & 0 \\ 0 & 1-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1-\frac{3}{2} & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the eigenvectors are of the form

$$\left[\begin{array}{r} s \\ -t \\ t \end{array}\right]$$

Two of these which are orthonormal are $\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$, choosing $s=1$ and $t=0$, and $\left[\begin{array}{c} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array}\right]$, letting $s=0$, $t=1$ and normalizing the resulting vector.

To obtain the desired orthogonal matrix, we let the orthonormal eigenvectors computed above be the columns.

$$\left[\begin{array}{rcc} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{array}\right]$$

To verify, compute U^TAU as follows:

$$U^TAU = \left[\begin{array}{ccc} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{array}\right] \left[\begin{array}{rcc} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{array}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

the desired diagonal matrix. Notice that the eigenvectors, which construct the columns of U, are in the same order as the eigenvalues in D. ♠

We conclude this section with a Theorem that generalizes earlier results.

Theorem 7.64: Triangulation of a Matrix

Let A be an $n \times n$ matrix. If A has n real eigenvalues, then an orthogonal matrix U can be found to result in the upper triangular matrix $U^T AU$.

This Theorem provides a useful Corollary.

Corollary 7.65: Determinant and Trace

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$. Then it follows that $\det(A)$ is equal to the product of the λ_1, while $\text{trace}(A)$ is equal to the sum of the λ_i.

Proof. By Theorem 7.64, there exists an orthogonal matrix U such that $U^T AU = P$, where P is an upper triangular matrix. Since P is similar to A, the eigenvalues of P are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Furthermore, since P is (upper) triangular, the entries on the main diagonal of P are its eigenvalues, so $\det(P) = \lambda_1 \lambda_2 \cdots \lambda_n$ and $\text{trace}(P) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Since P and A are similar, $\det(A) = \det(P)$ and $\text{trace}(A) = \text{trace}(P)$, and therefore the results follow. ♠

The Singular Value Decomposition

Singular Value Decomposition (SVD) can be thought of as a generalization of orthogonal diagonalization of a symmetric matrix to an arbitrary $m \times n$ matrix. This decomposition is the focus of this section.

Suppose that A is an $m \times n$ matrix. We will be interested in the eigenvalues and eigenvectors of the $n \times n$ matrix $A^T A$, and our first result concerns those eigenvalues.

Proposition 7.66: Eigenvalues of $A^T A$

For any real $m \times n$ matrix A, the eigenvalues of $A^T A$ are real and nonnegative.

Proof. As $A^T A$ is real and symmetric, Theorem 7.52 tells us that the eigenvalues of $A^T A$ are real. We must merely show that any such eigenvalue is nonnegative.

Suppose λ is a non-zero eigenvalue of $A^T A$ and let $\vec{x}$ be a corresponding eigenvector. We must show that λ is greater than zero. We will do this by examining the angle between $\vec{x}$ and $\vec{\lambda x}$, which is either 0 or π. Notice that $\vec{x}$ and $\vec{\lambda x}$ point in the same direction if and only if λ is greater than 0 if and only if the dot product $\vec{\lambda x} \cdot \vec{x}$ is greater than 0.

But we see that

$$\vec{\lambda x} \cdot \vec{x} = A^T A\vec{x} \cdot \vec{x} = A\vec{x} \cdot A\vec{x} > 0,$$

as $A\vec{x} \neq \vec{0}$. Thus we conclude that $\lambda\vec{x}$ and $\vec{x}$ point in the same direction, and so $\lambda > 0$. ♠

This tells us that the eigenvalues of $A^T A$ are either positive or zero. We will use the positive eigenvalues of $A^T A$ to define the **Singular Values** of A:

Definition 7.67: Singular Values

Let A be an $m \times n$ matrix. The singular values of A are the square roots of the positive eigenvalues of $A^T A$.

The following is a useful result that will help when computing the SVD of matrices.

Proposition 7.68

Let A be an $m \times n$ matrix. Then $A^T A$ and AA^T have the same **nonzero eigenvalues.**

Proof. Suppose A is an $m \times n$ matrix, and suppose that λ is a nonzero eigenvalue of $A^T A$. Then there exists a nonzero vector $\vec{x} \in \mathbb{R}^n$ such that

$$(A^T A)\vec{x} = \lambda\vec{x}. \tag{7.6}$$

Multiplying both sides of this equation by A yields:

$$\begin{aligned} A(A^T A)\vec{x} &= A\lambda\vec{x} \\ (AA^T)(A\vec{x}) &= \lambda(A\vec{x}). \end{aligned}$$

Since $\lambda \neq 0$ and $\vec{x} \neq \vec{0}_n$, $\lambda\vec{x} \neq \vec{0}_n$, and thus by equation (7.6), $(A^T A)\vec{x} \neq \vec{0}_m$; thus $A^T(A\vec{x}) \neq \vec{0}_m$, implying that $A\vec{x} \neq \vec{0}_m$.

Therefore $A\vec{x}$ is an eigenvector of AA^T corresponding to eigenvalue λ. An analogous argument can be used to show that every nonzero eigenvalue of AA^T is an eigenvalue of $A^T A$, thus completing the proof. ♠

Given an $m \times n$ matrix A, we will see how to express A as a product

$$A = U\Sigma V^T$$

where

- U is an $m \times m$ orthogonal matrix whose columns are eigenvectors of AA^T.
- V is an $n \times n$ orthogonal matrix whose columns are eigenvectors of $A^T A$.
- Σ is an $m \times n$ matrix whose only nonzero values lie on its main diagonal, and are the singular values of A.

This is called the **Singular Values Decomposition** of the matrix A.

How can we find such a decomposition? We are aiming to decompose A in the following form:

$$A = U \left[\begin{array}{cc} \sigma & 0 \\ 0 & 0 \end{array} \right] V^T$$

where σ is a $k \times k$ matrix of the form

$$\sigma = \left[\begin{array}{ccc} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{array} \right]$$

with $\sigma_1 \geq \sigma_2 \geq \cdot \geq \sigma_k$ being the singular values of A.

If we had sucha decomposition, then we would also have $A^T = V \left[\begin{array}{cc} \sigma & 0 \\ 0 & 0 \end{array} \right] U^T$ and it would follow that

$$A^T A = V \left[\begin{array}{cc} \sigma & 0 \\ 0 & 0 \end{array} \right] U^T U \left[\begin{array}{cc} \sigma & 0 \\ 0 & 0 \end{array} \right] V^T = V \left[\begin{array}{cc} \sigma^2 & 0 \\ 0 & 0 \end{array} \right] V^T$$

and so $A^T A V = V \left[\begin{array}{cc} \sigma^2 & 0 \\ 0 & 0 \end{array} \right]$. Similarly, $AA^T U = U \left[\begin{array}{cc} \sigma^2 & 0 \\ 0 & 0 \end{array} \right]$. Therefore, you would find an orthonormal basis of eigenvectors for AA^T make them the columns of a matrix so that the corresponding eigenvalues are decreasing. This gives U. You could then do the same for $A^T A$ to get V.

We formalize this discussion in the following theorem.

Theorem 7.69: Singular Value Decomposition

Let A be an $m \times n$ matrix. Then there exist orthogonal matrices U and V of the appropriate size such that $A = U \Sigma V^T$ where Σ is of the form

$$\Sigma = \left[\begin{array}{cc} \sigma & 0 \\ 0 & 0 \end{array} \right]$$

and σ is of the form

$$\sigma = \left[\begin{array}{ccc} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{array} \right]$$

for the σ_i the singular values of A.

Proof. By Theorem 7.29 and Proposition 7.66 we know that $A^T A$ has a set of n nonnegative eigenvalues. So there exist nonnegative numbers σ_i such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ and the eigenvalues of $A^T A$ are $\sigma_1^2 \geq \sigma_2^2, \ldots, \sigma_n^2$. We can assume that $\sigma_i > 0$ for $i \leq k$ and $\sigma_i = 0$ for $i > k$. As $A^T A$ is orthogonally diagonalizable, there exists an orthonormal basis, $\{\vec{v}_i\}_{i=1}^n$ such that $A^T A \vec{v}_i = \sigma_i^2 \vec{v}_i$. Thus for $i > k$, $A\vec{v}_i = \vec{0}$ because

$$A\vec{v}_i \cdot A\vec{v}_i = A^T A \vec{v}_i \cdot \vec{v}_i = \vec{0} \cdot \vec{v}_i = 0.$$

For $i = 1, \cdots, k$, define $\vec{u}_i \in \mathbb{R}^m$ by

$$\vec{u}_i = \sigma_i^{-1} A \vec{v}_i.$$

Thus $A\vec{v}_i = \sigma_i \vec{u}_i$. Now for any i and j that are less than or equal to k, we have

$$\begin{aligned}
\vec{u}_i \cdot \vec{u}_j &= \sigma_i^{-1} A\vec{v}_i \cdot \sigma_j^{-1} A\vec{v}_j \\
&= \sigma_i^{-1}\sigma_j^{-1}\left(A\vec{v}_i \cdot A\vec{v}_j\right) \\
&= \sigma_i^{-1}\sigma_j^{-1}\left(\vec{v}_i \cdot A^T A\vec{v}_j\right) \\
&= \sigma_i^{-1}\sigma_j^{-1}\left(\vec{v}_i \cdot \sigma_j^2 \vec{v}_j\right) \\
&= \frac{\sigma_j}{\sigma_i}\left(\vec{v}_i \cdot \vec{v}_j\right) \\
&= \delta_{ij}.
\end{aligned}$$

Thus $\{\vec{u}_i\}_{i=1}^k$ is an orthonormal set of vectors in $\mathbb{R}^m$. Also,

$$AA^T \vec{u}_i = AA^T \sigma_i^{-1} A\vec{v}_i = \sigma_i^{-1} AA^T A\vec{v}_i = \sigma_i^{-1} A\sigma_i^2 \vec{v}_i = \sigma_i^2 \vec{u}_i,$$

so our set $\{\vec{u}_i\}_{i=1}^k$ is an orthonormal set of eigenvectors corresponding, in order, to our eigenvalues $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$. Now extend $\{\vec{u}_i\}_{i=1}^k$ to an orthonormal basis for all of $\mathbb{R}^m$, $\{\vec{u}_i\}_{i=1}^m$ and let U be the matrix

$$U = \left[\begin{array}{ccc} \vec{u}_1 & \cdots & \vec{u}_m \end{array} \right]$$

while

$$V = \left[\begin{array}{ccc} \vec{v}_1 & \cdots & \vec{v}_n \end{array} \right].$$

Thus U is the matrix which has the $\vec{u}_i$ as columns and V is defined as the matrix which has the $\vec{v}_i$ as columns. Then

$$U^T AV = \left[\begin{array}{c} \vec{u}_1^T \\ \vdots \\ \vec{u}_k^T \\ \vdots \\ \vec{u}_m^T \end{array} \right] A \left[\vec{v}_1 \cdots \vec{v}_n\right]$$

$$= \left[\begin{array}{c} \vec{u}_1^T \\ \vdots \\ \vec{u}_k^T \\ \vdots \\ \vec{u}_m^T \end{array} \right] \left[\begin{array}{cccccc} \sigma_1 \vec{u}_1 & \cdots & \sigma_k \vec{u}_k & \vec{0} & \cdots & \vec{0} \end{array} \right] = \left[\begin{array}{cc} \sigma & 0 \\ 0 & 0 \end{array} \right]$$

where σ is given in the statement of the theorem. ♠

The singular value decomposition has as an immediate corollary which is given in the following interesting result.

Corollary 7.70: Rank and Singular Values

Let A be an $m \times n$ matrix. Then the rank of A and A^T equals the number of singular values.

Let's compute the Singular Value Decomposition of a simple matrix.

Example 7.71: Singular Value Decomposition

Let $A = \left[\begin{array}{rrr} 1 & -1 & 3 \\ 3 & 1 & 1 \end{array}\right]$. *Find the Singular Value Decomposition (SVD) of* A.

Solution. To begin, we compute AA^T and A^TA.

$$AA^T = \left[\begin{array}{rrr} 1 & -1 & 3 \\ 3 & 1 & 1 \end{array}\right]\left[\begin{array}{rr} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{array}\right] = \left[\begin{array}{rr} 11 & 5 \\ 5 & 11 \end{array}\right].$$

$$A^TA = \left[\begin{array}{rr} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{array}\right]\left[\begin{array}{rrr} 1 & -1 & 3 \\ 3 & 1 & 1 \end{array}\right] = \left[\begin{array}{rrr} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{array}\right].$$

Since AA^T is 2×2 while A^TA is 3×3, and AA^T and A^TA have the same *nonzero* eigenvalues (by Proposition 7.68), we compute the characteristic polynomial $c_{AA^T}(x)$ (because it's easier to compute than $c_{A^TA}(x)$).

$$\begin{array}{rcl} c_{AA^T}(x) & = & \det(xI - AA^T) = \left|\begin{array}{cc} x-11 & -5 \\ -5 & x-11 \end{array}\right| \\ & = & (x-11)^2 - 25 \\ & = & x^2 - 22x + 121 - 25 \\ & = & x^2 - 22x + 96 \\ & = & (x-16)(x-6) \end{array}$$

Therefore, the eigenvalues of AA^T are $\lambda_1 = 16$ and $\lambda_2 = 6$.

The eigenvalues of A^TA are $\lambda_1 = 16$, $\lambda_2 = 6$, and $\lambda_3 = 0$, and the singular values of A are $\sigma_1 = \sqrt{16} = 4$ and $\sigma_2 = \sqrt{6}$. By convention, we list the eigenvalues (and corresponding singular values) in non increasing order (i.e., from largest to smallest).

To find the matrix V:

To construct the matrix V we need to find eigenvectors for A^TA. Since the eigenvalues of AA^T are distinct, the corresponding eigenvectors are orthogonal, and we need only normalize them.

$\lambda_1 = 16$: solve $(16I - A^TA)Y = 0$.

$$\left[\begin{array}{rrr|r} 6 & -2 & -6 & 0 \\ -2 & 14 & 2 & 0 \\ -6 & 2 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } Y = \left[\begin{array}{c} t \\ 0 \\ t \end{array}\right] = t\left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right], \ t \in \mathbb{R}.$$

$\lambda_2 = 6$: solve $(6I - A^TA)Y = 0$.

$$\left[\begin{array}{rrr|r} -4 & -2 & -6 & 0 \\ -2 & 4 & 2 & 0 \\ -6 & 2 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } Y = \left[\begin{array}{r} -s \\ -s \\ s \end{array}\right] = s\left[\begin{array}{r} -1 \\ -1 \\ 1 \end{array}\right], \ s \in \mathbb{R}.$$

$\lambda_3 = 0$: solve $(-A^T A)Y = 0$.

$$\left[\begin{array}{rrr|r} -10 & -2 & -6 & 0 \\ -2 & -2 & 2 & 0 \\ -6 & 2 & -10 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } Y = \left[\begin{array}{r} -r \\ 2r \\ r \end{array}\right] = r\left[\begin{array}{r} -1 \\ 2 \\ 1 \end{array}\right], \ r \in \mathbb{R}.$$

Let

$$V_1 = \frac{1}{\sqrt{2}}\left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right], V_2 = \frac{1}{\sqrt{3}}\left[\begin{array}{r} -1 \\ -1 \\ 1 \end{array}\right], V_3 = \frac{1}{\sqrt{6}}\left[\begin{array}{r} -1 \\ 2 \\ 1 \end{array}\right].$$

Then

$$V = \frac{1}{\sqrt{6}}\left[\begin{array}{rrr} \sqrt{3} & -\sqrt{2} & -1 \\ 0 & -\sqrt{2} & 2 \\ \sqrt{3} & \sqrt{2} & 1 \end{array}\right].$$

Also,

$$\Sigma = \left[\begin{array}{rrr} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{array}\right],$$

and we use A, V^T, and Σ to find U.

Since V is orthogonal and $A = U\Sigma V^T$, it follows that $AV = U\Sigma$. Let $V = \left[\begin{array}{ccc} V_1 & V_2 & V_3 \end{array}\right]$, and let $U = \left[\begin{array}{cc} U_1 & U_2 \end{array}\right]$, where U_1 and U_2 are the two columns of U.

Then we have

$$\begin{array}{rcl} A\left[\begin{array}{ccc} V_1 & V_2 & V_3 \end{array}\right] & = & \left[\begin{array}{cc} U_1 & U_2 \end{array}\right]\Sigma \\ \left[\begin{array}{ccc} AV_1 & AV_2 & AV_3 \end{array}\right] & = & \left[\begin{array}{ccc} \sigma_1 U_1 + 0U_2 & 0U_1 + \sigma_2 U_2 & 0U_1 + 0U_2 \end{array}\right] \\ & = & \left[\begin{array}{ccc} \sigma_1 U_1 & \sigma_2 U_2 & 0 \end{array}\right] \end{array}$$

which implies that $AV_1 = \sigma_1 U_1 = 4U_1$ and $AV_2 = \sigma_2 U_2 = \sqrt{6}U_2$.

Thus,

$$U_1 = \frac{1}{4}AV_1 = \frac{1}{4}\left[\begin{array}{rrr} 1 & -1 & 3 \\ 3 & 1 & 1 \end{array}\right]\frac{1}{\sqrt{2}}\left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right] = \frac{1}{4\sqrt{2}}\left[\begin{array}{r} 4 \\ 4 \end{array}\right] = \frac{1}{\sqrt{2}}\left[\begin{array}{r} 1 \\ 1 \end{array}\right],$$

and

$$U_2 = \frac{1}{\sqrt{6}}AV_2 = \frac{1}{\sqrt{6}}\left[\begin{array}{rrr} 1 & -1 & 3 \\ 3 & 1 & 1 \end{array}\right]\frac{1}{\sqrt{3}}\left[\begin{array}{r} -1 \\ -1 \\ 1 \end{array}\right] = \frac{1}{3\sqrt{2}}\left[\begin{array}{r} 3 \\ -3 \end{array}\right] = \frac{1}{\sqrt{2}}\left[\begin{array}{r} 1 \\ -1 \end{array}\right].$$

Therefore,

$$U = \frac{1}{\sqrt{2}}\left[\begin{array}{rr} 1 & 1 \\ 1 & -1 \end{array}\right],$$

and

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \\ &= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix} \left(\frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{3} & 0 & \sqrt{3} \\ -\sqrt{2} & -\sqrt{2} & \sqrt{2} \\ -1 & 2 & 1 \end{bmatrix} \right). \end{aligned}$$

♠

Here is another example.

Example 7.72: Finding the SVD

Find an SVD for $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

Solution. Since A is 3×1, $A^T A$ is a 1×1 matrix whose eigenvalues are easier to find than the eigenvalues of the 3×3 matrix AA^T.

$$A^T A = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \end{bmatrix}.$$

Thus $A^T A$ has eigenvalue $\lambda_1 = 9$, and the eigenvalues of AA^T are $\lambda_1 = 9$, $\lambda_2 = 0$, and $\lambda_3 = 0$. Furthermore, A has only one singular value, $\sigma_1 = 3$.

To find the matrix V: To do so we find an eigenvector for $A^T A$ and normalize it. In this case, finding a unit eigenvector is trivial: $V_1 = \begin{bmatrix} 1 \end{bmatrix}$, and

$$V = \begin{bmatrix} 1 \end{bmatrix}.$$

Also, $\Sigma = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, and we use A, V^T, and Σ to find U.

Now $AV = U\Sigma$, with $V = \begin{bmatrix} V_1 \end{bmatrix}$, and $U = \begin{bmatrix} U_1 & U_2 & U_3 \end{bmatrix}$, where U_1, U_2, and U_3 are the columns of U. Thus

$$\begin{aligned} A \begin{bmatrix} V_1 \end{bmatrix} &= \begin{bmatrix} U_1 & U_2 & U_3 \end{bmatrix} \Sigma \\ \begin{bmatrix} AV_1 \end{bmatrix} &= \begin{bmatrix} \sigma_1 U_1 + 0U_2 + 0U_3 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1 U_1 \end{bmatrix} \end{aligned}$$

This gives us $AV_1 = \sigma_1 U_1 = 3U_1$, so

$$U_1 = \frac{1}{3} AV_1 = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

The vectors U_2 and U_3 are eigenvectors of AA^T corresponding to the eigenvalue $\lambda_2 = \lambda_3 = 0$. Instead of solving the system $(0I - AA^T)\vec{x} = 0$ and then using the Gram-Schmidt process on the resulting set of two basic eigenvectors, the following approach may be used.

Find vectors U_2 and U_3 by first extending $\{U_1\}$ to a basis of $\mathbb{R}^3$, then using the Gram-Schmidt algorithm to orthogonalize the basis, and finally normalizing the vectors.

Starting with $\{3U_1\}$ instead of $\{U_1\}$ makes the arithmetic a bit easier. It is easy to verify that

$$\left\{ \left[\begin{array}{r} -1 \\ 2 \\ 2 \end{array} \right], \left[\begin{array}{r} 1 \\ 0 \\ 0 \end{array} \right], \left[\begin{array}{r} 0 \\ 1 \\ 0 \end{array} \right] \right\}$$

is a basis of $\mathbb{R}^3$. Set

$$E_1 = \left[\begin{array}{r} -1 \\ 2 \\ 2 \end{array} \right], \vec{x}_2 = \left[\begin{array}{r} 1 \\ 0 \\ 0 \end{array} \right], \vec{x}_3 = \left[\begin{array}{r} 0 \\ 1 \\ 0 \end{array} \right],$$

and apply the Gram-Schmidt algorithm to $\{E_1, \vec{x}_2, \vec{x}_3\}$.

This gives us

$$E_2 = \left[\begin{array}{r} 4 \\ 1 \\ 1 \end{array} \right] \text{ and } E_3 = \left[\begin{array}{r} 0 \\ 1 \\ -1 \end{array} \right].$$

Therefore,

$$U_2 = \frac{1}{\sqrt{18}} \left[\begin{array}{r} 4 \\ 1 \\ 1 \end{array} \right], U_3 = \frac{1}{\sqrt{2}} \left[\begin{array}{r} 0 \\ 1 \\ -1 \end{array} \right],$$

and

$$U = \left[\begin{array}{rrr} -\frac{1}{3} & \frac{4}{\sqrt{18}} & 0 \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{2}} \end{array} \right].$$

Finally,

$$A = \left[\begin{array}{r} -1 \\ 2 \\ 2 \end{array} \right] = \left[\begin{array}{rrr} -\frac{1}{3} & \frac{4}{\sqrt{18}} & 0 \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{2}} \end{array} \right] \left[\begin{array}{r} 3 \\ 0 \\ 0 \end{array} \right] \left[\begin{array}{r} 1 \end{array} \right].$$

♠

Consider another example.

Example 7.73: Find the SVD

Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \end{bmatrix}$$

First consider $A^T A$

$$\begin{bmatrix} \frac{16}{5} & \frac{32}{5} & 0 \\ \frac{32}{5} & \frac{64}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

What are some eigenvalues and eigenvectors? Some computing shows the eigenvalues are 16 and 0 with

$$\begin{bmatrix} \frac{1}{5}\sqrt{5} \\ \frac{2}{5}\sqrt{5} \\ 0 \end{bmatrix}$$

being the unit eigenvector for $\lambda = 16$ and

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{2}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} \\ 0 \end{bmatrix}$$

being the two orthonormal eigenvectors for $\lambda = 0$.

Thus the matrix V is given by

$$V = \begin{bmatrix} \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next consider

$$AA^T = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

which has 16 as its only nonzero eigenvalue, with eigenvector $\begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$.

For the eigenvalue 0 you can compute the unit eigenvector is $\begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}$, and so we can let U be given by

$$U = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

To check this we compute $U^T AV$.

$$U^T AV = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This illustrates that if you have a good way to find the eigenvectors and eigenvalues for a symmetric matrix which has nonnegative eigenvalues, then you also have a good way to find the singular value decomposition of an arbitrary matrix.

Positive Definite Matrices

Positive definite matrices are often encountered in applications such mechanics and statistics.

We begin with a definition.

Definition 7.74: Positive Definite Matrix

Let A be an $n \times n$ symmetric matrix. Then A is positive definite if all of its eigenvalues are positive.

The relationship between a negative definite matrix and positive definite matrix is as follows.

Definition 7.75: Negative Definite Matrix

An $n \times n$ matrix A is negative definite if and only if $-A$ is positive definite.

Consider the following lemma.

Lemma 7.76: Positive Definite Matrix and Invertibility

If A is positive definite, then it is invertible.

Proof. If $A\vec{v} = \vec{0}$, then 0 is an eigenvalue if $\vec{v}$ is nonzero, which does not happen for a positive definite matrix. Hence $\vec{v} = \vec{0}$ and so A is one to one. This is sufficient to conclude that it is invertible. ♠

Notice that this lemma implies that if a matrix A is positive definite, then $\det(A) > 0$.

The following theorem provides another characterization of positive definite matrices. It gives a useful test for verifying if a matrix is positive definite.

Theorem 7.77: Positive Definite Matrix

Let A be a symmetric matrix. Then A is positive definite if and only if $\vec{x}^T A\vec{x}$ is positive for all nonzero $\vec{x} \in \mathbb{R}^n$.

Proof. Since A is symmetric, there exists an orthogonal matrix U so that

$$U^T AU = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = D,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of A. Let $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$, and define $\vec{y} = U^T\vec{x}$. Then

$$\vec{x}^T A\vec{x} = \vec{x}^T (UDU^T)\vec{x} = (\vec{x}^T U)D(U^T\vec{x}) = \vec{y}^T D\vec{y}.$$

Writing $\vec{y}^T = \left[\begin{array}{cccc} y_1 & y_2 & \cdots & y_n \end{array}\right]$,

$$\begin{aligned} \vec{x}^T A\vec{x} &= \left[\begin{array}{cccc} y_1 & y_2 & \cdots & y_n \end{array}\right] \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array}\right] \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots \lambda_n y_n^2. \end{aligned}$$

$(\Rightarrow)$ First we will assume that A is positive definite and prove that $\vec{x}^T A\vec{x}$ is positive.

Suppose A is positive definite, and $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$. Since U^T is invertible, $\vec{y} = U^T\vec{x} \neq \vec{0}$, and thus $y_j \neq 0$ for some j, implying $y_j^2 > 0$ for some j. Furthermore, since all eigenvalues of A are positive, $\lambda_i y_i^2 \geq 0$ for all i and $\lambda_j y_j^2 > 0$. Therefore, $\vec{x}^T A\vec{x} > 0$.

$(\Leftarrow)$ Now we will assume $\vec{x}^T A\vec{x}$ is positive and show that A is positive definite.

If $\vec{x}^T A\vec{x} > 0$ whenever $\vec{x} \neq \vec{0}$, choose $\vec{x} = U\vec{e}_j$, where $\vec{e}_j$ is the j^{th} column of I_n. Since U is invertible, $\vec{x} \neq \vec{0}$, and thus

$$\vec{y} = U^T\vec{x} = U^T(U\vec{e}_j) = \vec{e}_j.$$

Thus $y_j = 1$ and $y_i = 0$ when $i \neq j$, so

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots \lambda_n y_n^2 = \lambda_j,$$

i.e., $\lambda_j = \vec{x}^T A\vec{x} > 0$. Thus every eigenvalue of A is positive, and so A is a positive definite matrix. ♠

There are some other very interesting consequences which result from a matrix being positive definite. First one can note that the property of being positive definite is transferred to each of the principal submatrices which we will now define.

Definition 7.78: The Submatrix A_k

Let A be an $n \times n$ matrix. Denote by A_k the $k \times k$ matrix obtained by deleting the $k+1, \cdots, n$ columns and the $k+1, \cdots, n$ rows from A. Thus $A_n = A$ and A_k is the $k \times k$ submatrix of A which occupies the upper left corner of A.

Lemma 7.79: Positive Definite and Submatrices

Let A be an $n \times n$ positive definite matrix. Then each submatrix A_k is also positive definite.

Proof. This follows right away from the above definition. Let $\vec{x} \in \mathbb{R}^k$ be nonzero. Then

$$\vec{x}^T A_k \vec{x} = \left[\begin{array}{cc} \vec{x}^T & 0 \end{array} \right] A \left[\begin{array}{c} \vec{x} \\ 0 \end{array} \right] > 0$$

by the assumption that A is positive definite. ♠

There is yet another way to recognize whether a matrix is positive definite which is described in terms of these submatrices. We state the result, the proof of which can be found in more advanced texts.

Theorem 7.80: Positive Matrix and Determinant of A_k

Let A be a symmetric matrix. Then A is positive definite if and only if $\det(A_k)$ is greater than 0 for every submatrix A_k, $k = 1, \cdots, n$.

Proof. We prove the $\Leftarrow$ direction of the theorem by induction on n. It is clearly true if $n = 1$. Suppose then that it is true for $n-1$ where $n \geq 2$. Since $\det(A) = \det(A_n) > 0$, it follows that all the eigenvalues are nonzero. We need to show that they are all positive. Suppose not. Then there is some even number of them which are negative, even because the product of all the eigenvalues is known to be positive, equaling $\det(A)$. Pick two, λ_1 and λ_2 and let $A\vec{u}_i = \lambda_i \vec{u}_i$ where $\vec{u}_i \neq \vec{0}$ for $i = 1, 2$ and $\vec{u}_1 \cdot \vec{u}_2 = 0$. Now if $\vec{z} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2$ is an element of $\mathrm{span}\{\vec{u}_1, \vec{u}_2\}$, then since these are eigenvalues and $\vec{u}_1 \cdot \vec{u}_2 = 0$, a short computation shows

$$\begin{aligned} \vec{z}^T A \vec{z} &= (\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2)^T A (\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2) \\ &= |\alpha_1|^2 \lambda_1 \|\vec{u}_1\|^2 + |\alpha_2|^2 \lambda_2 \|\vec{u}_2\|^2 \\ &< 0. \end{aligned}$$

Also notice that if we let $\vec{x}$ be any vector in $\mathbb{R}^{n-1}$, we can use the induction hypothesis to write

$$\left[\begin{array}{cc} x^T & 0 \end{array} \right] A \left[\begin{array}{c} \vec{x} \\ 0 \end{array} \right] = \vec{x}^T A_{n-1} \vec{x} > 0.$$

Now the dimension of $\{\vec{z} \in \mathbb{R}^n : z_n = 0\}$ is $n-1$ and the dimension of $\mathrm{span}\{\vec{u}_1, \vec{u}_2\} = 2$ and so there must be some nonzero $\vec{z} \in \mathbb{R}^n$ which is in both of these subspaces of $\mathbb{R}^n$. However, the first computation above would require that $\vec{z}^T A \vec{z} < 0$ (as $z \in funcspan\{\vec{u}_1, \vec{u}_2\}$) while the second computation would require that $\vec{x}^T A \vec{x} > 0$. This contradiction shows that all the eigenvalues must be positive and A is a positive definite matrix. This proves the if part of the theorem.

The $\Rightarrow$ direction of the theorem can also be shown to be correct, but it is the direction which was just shown which is of most interest, so we omit the proof. ♠

Corollary 7.81: Symmetric and Negative Definite Matrix

Let A be symmetric. Then A is negative definite if and only if

$$(-1)^k \det(A_k) > 0$$

for every $k = 1, \cdots, n$.

Proof. This is immediate from the above theorem when we notice, that A is negative definite if and only if $-A$ is positive definite. Therefore, if $\det(-A_k) > 0$ for all $k = 1, \cdots, n$, it follows that A is negative definite. However, $\det(-A_k) = (-1)^k \det(A_k)$. ♠

The Cholesky Factorization

Another important theorem is the existence of a specific factorization of positive definite matrices. It is called the Cholesky Factorization and factors the matrix into the product of an upper triangular matrix and its transpose.

Theorem 7.82: Cholesky Factorization

Let A be a positive definite matrix. Then there exists an upper triangular matrix U whose main diagonal entries are positive, such that A can be written

$$A = U^T U$$

This factorization is unique.

The process for finding such a matrix U relies on simple row operations.

Procedure 7.83: Finding the Cholesky Factorization

Let A be a positive definite matrix. The matrix U that creates the Cholesky Factorization can be found through two steps.

1. *Using only type 3 elementary row operations (multiples of rows added to other rows) put A in upper triangular form. Call this matrix $\hat{U}$. Then $\hat{U}$ has positive entries on the main diagonal.*
2. *Divide each row of $\hat{U}$ by the square root of the diagonal entry in that row. The result is the matrix U.*

Of course you can always verify that your factorization is correct by multiplying U and U^T to ensure the result is the original matrix A.

Consider the following example.

Example 7.84: Cholesky Factorization

Show that $A = \begin{bmatrix} 9 & -6 & 3 \\ -6 & 5 & -3 \\ 3 & -3 & 6 \end{bmatrix}$ *is positive definite, and find the Cholesky factorization of* A.

Solution. First we show that A is positive definite. By Theorem 7.80 it suffices to show that the determinant of each submatrix is positive.

$$A_1 = \begin{bmatrix} 9 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 9 & -6 \\ -6 & 5 \end{bmatrix},$$

so $\det(A_1) = 9$ and $\det(A_2) = 9$. Since $\det(A) = 36$, it follows that A is positive definite.

Now we use Procedure 7.83 to find the Cholesky Factorization. Row reduce (using only type 3 row operations) until an upper triangular matrix is obtained.

$$\begin{bmatrix} 9 & -6 & 3 \\ -6 & 5 & -3 \\ 3 & -3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & -6 & 3 \\ 0 & 1 & -1 \\ 0 & -1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & -6 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

Now divide the entries in each row by the square root of the diagonal entry in that row, to give

$$U = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

You can verify that $U^T U = A$. ♠

Example 7.85: Cholesky Factorization

Let A *be a positive definite matrix given by*

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

Determine its Cholesky factorization.

Solution. You can verify that A is in fact positive definite.

To find the Cholesky factorization we first row reduce to an upper triangular matrix.

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{11}{3} & \frac{5}{3} \\ 0 & \frac{5}{3} & \frac{14}{5} \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{11}{3} & \frac{5}{3} \\ 0 & 0 & \frac{43}{11} \end{bmatrix}$$

Now divide the entries in each row by the square root of the diagonal entry in that row and simplify.

$$U = \left[\begin{array}{ccc} \sqrt{3} & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \\ 0 & \frac{1}{3}\sqrt{3}\sqrt{11} & \frac{5}{33}\sqrt{3}\sqrt{11} \\ 0 & 0 & \frac{1}{11}\sqrt{11}\sqrt{43} \end{array} \right]$$

$\spadesuit$

Finding Eigenvalues: QR Factorization and Power Methods

We know a method for finding the eigenvalues of a given matrix A: compute the characteristic polynomial of A and find all of its roots. Sadly, if A is large, then we have the problem of finding the roots of a polynomial of degree 420. This is not easily done algebraically, so we will resort to numerical methods to approximate the eigenvalues. This section describes one such approach, introducing QR factorization and power methods along the way, both of which have independent interest.

In this section we begin by describing a reliable way to factor a matrix. Called the QR factorization, it is guaranteed to *always* exist. While much can be said about the QR factorization, this section will be limited to real matrices. Therefore we assume the dot product used below is the usual dot product. We begin with a definition.

Definition 7.86: QR Factorization

Let A be a real $m \times n$ matrix. Then a QR factorization of A consists of an orthogonal matrix Q and an upper triangular matrix$R$ such that $A = QR$.

The following theorem claims that such a factorization exists.

Theorem 7.87: Existence of QR Factorization

Let A be any real $m \times n$ matrix with linearly independent columns. Then there exists an orthogonal matrix Q and an upper triangular matrix R having positive entries on the main diagonal such that

$$A = QR.$$

The procedure for obtaining the QR factorization for any matrix A is as follows.

Procedure 7.88: QR Factorization

Let A be an $m \times n$ matrix given by $A = \left[\begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \right]$ where the A_i are the linearly independent columns of A.

1. *Apply the Gram-Schmidt Process 4.139 to the columns of A, writing B_i for the resulting columns.*

2. *Normalize the B_i, to find $C_i = \frac{1}{\|B_i\|}B_i$.*

3. *Construct the orthogonal matrix Q as $Q = \left[\begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \right]$.*

4. *Construct the upper triangular matrix R as*

$$R = \left[\begin{array}{ccccc} \|B_1\| & A_2 \cdot C_1 & A_3 \cdot C_1 & \cdots & A_n \cdot C_1 \\ 0 & \|B_2\| & A_3 \cdot C_2 & \cdots & A_n \cdot C_2 \\ 0 & 0 & \|B_3\| & \cdots & A_n \cdot C_3 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \|B_n\| \end{array} \right]$$

5. *Finally, write $A = QR$ where Q is the orthogonal matrix and R is the upper triangular matrix obtained above.*

Notice that Q is an orthogonal matrix as the C_i form an orthonormal set. Since $\|B_i\| > 0$ for all i (since the length of a vector is always positive), it follows that R is an upper triangular matrix with positive entries on the main diagonal.

Consider the following example.

Example 7.89: Finding a QR Factorization

Let

$$A = \left[\begin{array}{cc} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{array} \right]$$

Find an orthogonal matrix Q and upper triangular matrix R such that $A = QR$.

Solution. First, observe that A_1, A_2, the columns of A, are linearly independent. Therefore we can use the Gram-Schmidt Process to create a corresponding orthogonal set $\{B_1, B_2\}$ as follows:

$$\begin{aligned}
B_1 &= A_1 = \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right] \\
B_2 &= A_2 - \frac{A_2 \cdot B_1}{\|B_1\|^2} B_1 \\
&= \left[\begin{array}{c} 2 \\ 1 \\ 0 \end{array}\right] - \frac{2}{2}\left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right] \\
&= \left[\begin{array}{r} 1 \\ 1 \\ -1 \end{array}\right]
\end{aligned}$$

Normalize each vector to create the set $\{C_1, C_2\}$ as follows:

$$\begin{aligned}
C_1 &= \frac{1}{\|B_1\|} B_1 = \frac{1}{\sqrt{2}}\left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right] \\
C_2 &= \frac{1}{\|B_2\|} B_2 = \frac{1}{\sqrt{3}}\left[\begin{array}{r} 1 \\ 1 \\ -1 \end{array}\right]
\end{aligned}$$

Now construct the orthogonal matrix Q as

$$\begin{aligned}
Q &= \left[\begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array}\right] \\
&= \left[\begin{array}{cc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{array}\right]
\end{aligned}$$

Finally, construct the upper triangular matrix R as

$$\begin{aligned}
R &= \left[\begin{array}{cc} \|B_1\| & A_2 \cdot C_1 \\ 0 & \|B_2\| \end{array}\right] \\
&= \left[\begin{array}{cc} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{array}\right]
\end{aligned}$$

It is left to the reader to verify that $A = QR$. ♠

The QR Factorization and Eigenvalues

The QR factorization of a matrix has a very useful application. It turns out that it can be used repeatedly to estimate the eigenvalues of a matrix. Consider the following procedure, which we present without proof.

Procedure 7.90: Using the QR Factorization to Estimate Eigenvalues

Let A be an invertible matrix. Define the matrices $A_1, A_2, \cdots$ as follows:

1. $A_1 = A$ *factored as* $A_1 = Q_1 R_1$
2. $A_2 = R_1 Q_1$ *factored as* $A_2 = Q_2 R_2$
3. $A_3 = R_2 Q_2$ *factored as* $A_3 = Q_3 R_3$

Continue in this manner, where in general $A_k = Q_k R_k$ and $A_{k+1} = R_k Q_k$.
Then it follows that this sequence of A_i converges to an upper triangular matrix which is similar to A. Therefore the eigenvalues of A can be approximated by the entries on the main diagonal of this upper triangular matrix.

Power Methods for Finding Eigenvalues

While the QR algorithm can be used to approximate all of the eigenvalues of a given matrix A, there are a couple of useful and fairly elementary techniques for finding both the largest eigenvalue of a matrix and the eigenvector and associated eigenvalue of A nearest to a given complex number. These are called *power methods*, as they work (not surprisingly) with powers of a matrix. Combining these power methods with the QR factorization will provide us a way to both improve the approximations to the values of the eigenvectors of A that are provided by Procedure 7.90 and to find eigenvectors associated with each eigenvalue.

First, we will discuss the Power Method, which finds the largest eigenvalue of A.

Suppose the $n\times n$ matrix A has a basis of eigenvectors $\{\vec{x}_1,\cdots,\vec{x}_n\}$ such that $A\vec{x}_n=\lambda_n\vec{x}_n$. Also assume that $|\lambda_1|\leq|\lambda_2|\leq\cdots\leq|\lambda_n|$. Now let $\vec{u}_1$ be some nonzero vector. Since $\{\vec{x}_1,\cdots,\vec{x}_n\}$ is a basis, there exists unique scalars, c_i such that

$$\vec{u}_1=\sum_{k=1}^{n}c_k\vec{x}_k$$

Assume you have not been so unlucky as to pick $\vec{u}_1$ in such a way that $c_n=0$. Then recursively define the sequence of vectors $\vec{u}_1,\vec{u}_2,\vec{u}_3\cdots$ by $\vec{u}_{k+1}=A\vec{u}_k$. Then, for any m we have

$$\vec{u}_m=A^m\vec{u}_1=\sum_{k=1}^{n-1}c_k\lambda_k^m\vec{x}_k+\lambda_n^m c_n\vec{x}_n. \tag{7.7}$$

For large m the last term, $\lambda_n^m c_n\vec{x}_n$, determines quite well the direction of the vector on the right. This is because $|\lambda_n|$ is larger than $|\lambda_k|$ for $k<n$ and so for a large m, the sum, $\sum_{k=1}^{n-1}c_k\lambda_k^m\vec{x}_k$, on the right is fairly insignificant. Therefore, for large m, $\vec{u}_m$ is essentially a multiple of the eigenvector $\vec{x}_n$, the one which goes with λ_n.

The only problem is that there is no control of the size of the vectors $\vec{u}_m$, which means that calculations can become impossible. But we can fix this by scaling. Let S_2 denote the entry of $A\vec{u}_1$ which is largest in absolute value. We call this a **scaling factor**. Then $\vec{u}_2$ will not be just $A\vec{u}_1$ but $A\vec{u}_1/S_2$. Next let S_3 denote the entry of $A\vec{u}_2$ which has largest absolute value and define $\vec{u}_3\equiv A\vec{u}_2/S_3$. Continue this way. The scaling just described does not destroy the relative insignificance of the term involving a sum in Equation 7.7. Indeed it amounts to nothing more than changing the units of length. Also note that from this scaling procedure, the absolute value of the largest element of $\vec{u}_k$ is always equal to 1. Therefore, for large m,

$$\vec{u}_m=\frac{\lambda_n^m c_n\vec{x}_n}{S_2S_3\cdots S_m}+(\text{relatively insignificant term}).$$

Therefore, the entry of $A\vec{u}_m$ which has the largest absolute value is essentially equal to the entry having largest absolute value of

$$A\left(\frac{\lambda_n^m c_n\vec{x}_n}{S_2S_3\cdots S_m}\right)=\frac{\lambda_n^{m+1}c_n\vec{x}_n}{S_2S_3\cdots S_m}\approx\lambda_n\vec{u}_m$$

and so for large m, it must be the case that $\lambda_n\approx S_{m+1}$. This suggests the following procedure.

Procedure 7.91: The Power Method: Finding the Largest Eigenvalue with its Eigenvector

1. *Start with a vector $\vec{u}_1$ which you hope has a component in the direction of $\vec{x}_n$. The vector $(1, \cdots, 1)^T$ is usually a pretty good choice.*

2. *If $\vec{u}_k$ is known, let*
$$\vec{u}_{k+1} = \frac{A\vec{u}_k}{S_{k+1}}$$
where S_{k+1} is the entry of $A\vec{u}_k$ which has largest absolute value.

3. *When the scaling factors, S_k are not changing much, S_{k+1} will be close to the eigenvalue and $\vec{u}_{k+1}$ will be close to an eigenvector.*

4. *Check your answer to see if it worked well.*

Now we turn to the *shifted inverse power method*, which finds the eigenvalue of A that is closest to a given complex (or real) number, along with the associated eigenvector. It tends to work extremely well, provided that you start with something which is fairly close to an eigenvalue.

Given an $n \times n$ matrix A, if μ is a complex number and you want to find the eigenvalue λ of A which is closest to μ, you could consider the eigenvalues and eigenvectors of the matrix $(A - \mu I)^{-1}$. Then $A\vec{x} = \lambda\vec{x}$ if and only if
$$(A - \mu I)\vec{x} = (\lambda - \mu)\vec{x}$$
If and only if
$$\frac{1}{\lambda - \mu}\vec{x} = (A - \mu I)^{-1}\vec{x}$$
Thus, if λ is the closest eigenvalue of A to μ then out of all eigenvalues of $(A - \mu I)^{-1}$, the eigenvalue given by $\frac{1}{\lambda - \mu}$ would be the largest, since if $\lambda - \mu$ is small, then $\frac{1}{\lambda - \mu}$ is large. But we just finished describing a procedure that produces the eigenvalue of a matrix with the largest absolute value!

So all we have to do is apply the power method to the matrix $(A - \mu I)^{-1}$. The eigenvector $\vec{u}$ that is you get from the power method will be the eigenvector which corresponds to the eigenvalue λ of A such that λ is the closest to μ of all eigenvalues of A. And once we have $\vec{u}$ in hand, we can find this closest value λ simply by computing $A\vec{u}$ and comparing the result to $\vec{u}$.

Example 7.92: Finding Eigenvalue and Eigenvector

Find the eigenvalue and eigenvector for
$$A = \left[\begin{array}{rrr} 3 & 2 & 1 \\ -2 & 0 & -1 \\ -2 & -2 & 0 \end{array}\right]$$
which is closest to $\mu = .9 + .9i$.

Solution. Form

$$(A-\mu I)^{-1} = \left(\begin{bmatrix} 3 & 2 & 1 \\ -2 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} - (.9+.9i) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} -0.61919 - 10.545i & -5.5249 - 4.9724i & -0.37057 - 5.8213i \\ 5.5249 + 4.9724i & 5.2762 + 0.24862i & 2.7624 + 2.4862i \\ 0.74114 + 11.643i & 5.5249 + 4.9724i & 0.49252 + 6.9189i \end{bmatrix}.$$

Then pick an initial guess and multiply by $(A-\mu I)^{-1}$ raised to a large power:

$$\left((A\mu I)^{-1}\right)^{15} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.61919 - 10.545i & -5.5249 - 4.9724i & -0.37057 - 5.8213i \\ 5.5249 + 4.9724i & 5.2762 + 0.24862i & 2.7624 + 2.4862i \\ 0.74114 + 11.643i & 5.5249 + 4.9724i & 0.49252 + 6.9189i \end{bmatrix}^{15} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5629 \times 10^{13} - 3.8993 \times 10^{12} i \\ -5.8645 \times 10^{12} + 9.7642 \times 10^{12} i \\ -1.5629 \times 10^{13} + 3.8999 \times 10^{12} i \end{bmatrix}$$

Now divide by an entry (try to pick the entry with largest absolute value) to make the vector have reasonable size. This yields

$$\begin{bmatrix} -0.99999 - 3.6140 \times 10^{-5} i \\ 0.49999 - 0.49999i \\ 1.0 \end{bmatrix}$$

which is close to

$$\vec{u} = \begin{bmatrix} -1.0 \\ 0.5 - 0.5i \\ 1.0 \end{bmatrix}$$

Then

$$A\vec{u} = \begin{bmatrix} 3 & 2 & 1 \\ -2 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} -1.0 \\ 0.5 - 0.5i \\ 1.0 \end{bmatrix} = \begin{bmatrix} -1.0 - 1.0i \\ 1.0 \\ 1.0 + 1.0i \end{bmatrix}$$

Now to determine the eigenvalue, you could just take the ratio of corresponding entries from $\vec{u}$ and $A\vec{u}$. Pick the two corresponding entries which have the largest absolute values. In this case, you would get the eigenvalue to be $\lambda = \frac{-1.0-1.0i}{-1.0} = 1+i$. Luckily, this happens to be the exact eigenvalue. Thus the eigenvalue closest to $\mu = 0.9 + 0.9i$ is $\lambda = 1+i$, and an eigenvector corresponding to this value of λ is $\vec{u} = \begin{bmatrix} -1.0 \\ 0.5 - 0.5i \\ 1.0 \end{bmatrix}$. ♠

Usually it won't work out quite this well but you can still find what is desired. Thus, once you have obtained approximate eigenvalues using the QR algorithm, you can approximate each eigenvalue more exactly, and produce eigenvectors associated with each eigenvalue, by using the shifted inverse power method.

Quadratic Forms

One of the applications of orthogonal diagonalization is that of quadratic forms and graphs of level curves of a quadratic form. This section has to do with rotation of axes so that with respect to the new axes, the graph of the level curve of a quadratic form is oriented parallel to the coordinate axes. This makes it much easier to understand. For example, we all know that $x_1^2 + x_2^2 = 1$ represents the equation in two variables whose graph in $\mathbb{R}^2$ is a circle of radius 1. But even if you remember that the graph of the equation $5x_1^2 + 4x_1x_2 + 3x_2^2 = 1$ is an ellipse, can you find the semi-major and semi-minor axes of that ellipse? We will use quadratic forms to simplify this problem.

We first formally define what is meant by a quadratic form. In this section we will work with only *real* quadratic forms, which means that the coefficients will all be real numbers.

Definition 7.93: Quadratic Form

A ***quadratic form*** *is a polynomial of degree two in n variables $x_1, x_2, \cdots, x_n$, written as a linear combination of x_i^2 terms and $x_i x_j$ terms.*

Consider the quadratic form $q = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + a_{12}x_1x_2 + \cdots$. We can write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ as the vector whose entries are the variables contained in the quadratic form.

Similarly, let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be a matrix whose entries are the coefficients of x_i^2 and $x_i x_j$ from q. It turns out that the matrix A is not unique, and we will discuss how to choose a unique such A in the example below. Using this matrix A, the quadratic form can be written as $q = \vec{x}^T A \vec{x}$.

$$\begin{aligned} q &= \vec{x}^T A \vec{x} \\ &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}x_n \\ \vdots \\ a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n \end{bmatrix} \\ &= a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + a_{12}x_1x_2 + \cdots \end{aligned}$$

Let's explore how to find our unique such matrix A. Consider the following example.

Example 7.94: Matrix of a Quadratic Form

Let a quadratic form q be given by

$$q = 6x_1^2 + 4x_1x_2 + 3x_2^2$$

Write q in the form $\vec{x}^T A \vec{x}$.

Solution. First, let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Then, writing $q = \vec{x}^T A \vec{x}$ gives

$$\begin{aligned} q &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= a_{11}x_1^2 + a_{21}x_1x_2 + a_{12}x_1x_2 + a_{22}x_2^2 \end{aligned}$$

Notice that we have an x_1x_2 term as well as an x_2x_1 term. Since multiplication is commutative, these terms can be combined. This means that q can be written

$$q = a_{11}x_1^2 + (a_{21} + a_{12})x_1x_2 + a_{22}x_2^2$$

Equating this to q as given in the example, we have

$$a_{11}x_1^2 + (a_{21} + a_{12})x_1x_2 + a_{22}x_2^2 = 6x_1^2 + 4x_1x_2 + 3x_2^2$$

Therefore,

$$\begin{aligned} a_{11} &= 6 \\ a_{22} &= 3 \\ a_{21} + a_{12} &= 4 \end{aligned}$$

This demonstrates that the matrix A is not unique, as there are several correct solutions to $a_{21} + a_{12} = 4$. However, we will *always* choose the coefficients such that $a_{21} = a_{12} = \frac{1}{2}(a_{21} + a_{12})$. This results in $a_{21} = a_{12} = 2$. This choice is key, as it will ensure that A turns out to be a symmetric matrix, and there is a unique symmetric matrix A such that $q = \vec{x}^T A\vec{x}$.

Hence for our example,

$$A = \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right] = \left[\begin{array}{cc} 6 & 2 \\ 2 & 3 \end{array}\right]$$

You can verify that $q = \vec{x}^T A\vec{x}$ holds for this choice of A. ♠

The above procedure for choosing A to be symmetric applies for any quadratic form q. We will *always* choose coefficients such that $a_{ij} = a_{ji}$.

We now turn our attention to the focus of this section. Our goal is to start with a quadratic form q as given above and find a way to rewrite it to eliminate the $x_i x_j$ terms. This is done through a change of variables. In other words, we wish to find y_i such that

$$q = d_{11}y_1^2 + d_{22}y_2^2 + \cdots + d_{nn}y_n^2$$

Letting $\vec{y} = \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array}\right]$ and $D = \left[d_{ij}\right]$, we can write $q = \vec{y}^T D\vec{y}$ where D is the matrix of coefficients from q. There is something special about this matrix D that is crucial. Since no $y_i y_j$ terms exist in q, it follows that $d_{ij} = 0$ for all $i \neq j$. Therefore, D is a diagonal matrix. Through this change of variables, we find the **principal axes** $y_1, y_2, \cdots, y_n$ of the quadratic form.

This discussion sets the stage for the following essential theorem.

Theorem 7.95: Diagonalizing a Quadratic Form

Let q be a quadratic form in the variables $x_1, \cdots, x_n$. It follows that q can be written in the form $q = \vec{x}^T A\vec{x}$ where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is the symmetric matrix of coefficients of q.
New variables $y_1, y_2, \cdots, y_n$ can be found such that $q = \vec{y}^T D\vec{y}$ where

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

and $D = \begin{bmatrix} d_{ij} \end{bmatrix}$ is a diagonal matrix. The matrix D contains the eigenvalues of A and is found by orthogonally diagonalizing A.

While not a formal proof, the following discussion should convince you that the above theorem holds. Let q be a quadratic form in the variables $x_1, \cdots, x_n$. Then, q can be written in the form $q = \vec{x}^T A\vec{x}$ for a symmetric matrix A. By Theorem 7.56 we can orthogonally diagonalize the matrix A such that $U^T AU = D$ for an orthogonal matrix U and diagonal matrix D.

Then, the vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ is found by $\vec{y} = U^T\vec{x}$. To see that this works, rewrite $\vec{y} = U^T\vec{x}$ as $\vec{x} = U\vec{y}$.
Since we know that $q = \vec{x}^T A\vec{x}$, proceed as follows:

$$\begin{aligned} q &= \vec{x}^T A\vec{x} \\ &= (U\vec{y})^T A(U\vec{y}) \\ &= \vec{y}^T (U^T AU)\vec{y} \\ &= \vec{y}^T D\vec{y} \end{aligned}$$

The following procedure details the steps for the change of variables given in the above theorem.

Procedure 7.96: Diagonalizing a Quadratic Form

Let q be a quadratic form in the variables $x_1, \cdots, x_n$ given by

$$q = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + a_{12}x_1x_2 + \cdots$$

Then, q can be written as $q = d_{11}y_1^2 + \cdots + d_{nn}y_n^2$ as follows:

1. *Find the symmetric matrix A such that $q = \vec{x}^T A\vec{x}$.*
2. *Orthogonally diagonalize A. So find an orthogonal matrix U such that $U^T AU = D$ for a diagonal matrix D.*
3. *Let $\vec{y} = U^T\vec{x}$ and write $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$.*
4. *The quadratic form q will now be given by*

$$q = d_{11}y_1^2 + \cdots + d_{nn}y_n^2 = \vec{y}^T D\vec{y}$$

where $D = \left[d_{ij}\right]$ is the diagonal matrix found by orthogonally diagonalizing A.

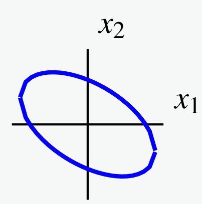
Consider the following example.

Example 7.97: Choosing New Axes to Simplify a Quadratic Form

Consider the following level curve

$$6x_1^2 + 4x_1x_2 + 3x_2^2 = 7$$

shown in the following graph.



Use a change of variables to choose new axes such that the ellipse is oriented parallel to the new coordinate axes. In other words, use a change of variables to rewrite q to eliminate the x_1x_2 term.

Solution. Notice that the level curve is given by $q = 7$ for $q = 6x_1^2 + 4x_1x_2 + 3x_2^2$. This is the same quadratic form that we examined earlier in Example 7.94. Therefore we know that we can write $q = \vec{x}^T A\vec{x}$ for the matrix

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

Now we want to orthogonally diagonalize A to write $U^TAU = D$ for an orthogonal matrix U and diagonal matrix D. The details are left to the reader, and you can verify that the resulting matrices are

$$\begin{aligned} U &= \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \\ D &= \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

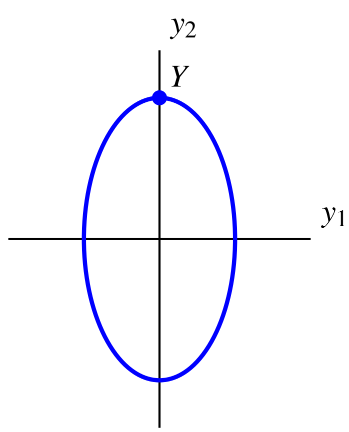
Now we change variables. Let the new variables $\vec{y}$ be defined by

$$\begin{aligned} \vec{y} &= U^T\vec{x} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} \frac{2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_2 \\ -\frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_2 \end{bmatrix} \end{aligned}$$

We can now express the quadratic form q in terms of y, using the entries from D as coefficients as follows:

$$\begin{aligned} q &= d_{11}y_1^2 + d_{22}y_2^2 \\ &= 7y_1^2 + 2y_2^2. \end{aligned}$$

Hence the level curve can be written $7y_1^2 + 2y_2^2 = 7$. The graph of this equation is given by:



The change of variables results in new axes such that with respect to the new axes, the ellipse is oriented parallel to the coordinate axes. These are called the **principal axes** of the quadratic form.

We can, of course, use simple algebra to check that our change of variables worked in the way that it was supposed to. Recall that we changed variables so that $y_1 = \frac{2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_2$ and $y_2 = -\frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_2$. So we have

$$\begin{aligned}
q &= 7y_1^2 + 2y_2^2 \\
&= 7\left(\frac{2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_2\right)^2 + 2\left(-\frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_2\right)^2 \\
&= 7\left(\frac{4}{5}x_1 + \frac{4}{5}x_1x_2 + \frac{1}{5}x_2^2\right) + 2\left(\frac{1}{5}x_1^2 - \frac{4}{5}x_1x_2 + \frac{4}{5}x_2^2\right) \\
&= 6x_1^2 - 4x_1x_2 + 3x_2^2 \\
&= q
\end{aligned}$$

which is comforting.

To answer the question suggested at the beginning of this subsection, notice that the point $Y = \left(0, \sqrt{\frac{7}{2}}\right)$ in the graph above is a point on the ellipse that is farthest from the origin, the center of the ellipse. If we let

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{\frac{7}{2}} \end{bmatrix}$$

be the position vector of Y, then

$$\vec{x} = U\vec{y} = \begin{bmatrix} -\sqrt{\frac{7}{10}} \\ \sqrt{\frac{28}{10}} \end{bmatrix}$$

is the position vector of the point on the original level curve in the third quadrant that is furthest from the origin. Thus the semi-major axis of the original ellipse is simply $\|\vec{x}\| = \sqrt{\frac{35}{10}}$. Finding the semi-minor axis is left as an exercise for you to complete. ♠

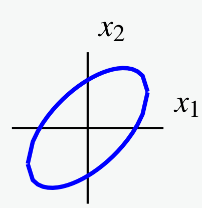
The following is another example of diagonalizing a quadratic form.

Example 7.98: Choosing New Axes to Simplify a Quadratic Form

Consider the level curve

$$5x_1^2 - 6x_1x_2 + 5x_2^2 = 8$$

shown in the following graph.



Use a change of variables to choose new axes such that the ellipse is oriented parallel to the new coordinate axes. In other words, use a change of variables to rewrite q to eliminate the x_1x_2 term.

Solution. First, express the level curve as $\vec{x}^T A\vec{x}$ where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and A is symmetric. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Then $q = \vec{x}^T A\vec{x}$ is given by

$$\begin{aligned} q &= \left[\begin{array}{cc} x_1 & x_2 \end{array}\right] \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] \\ &= a_{11}x_1^2 + (a_{12}+a_{21})x_1x_2 + a_{22}x_2^2 \end{aligned}$$

Equating this to the given description for q, we have

$$5x_1^2 - 6x_1x_2 + 5x_2^2 = a_{11}x_1^2 + (a_{12}+a_{21})x_1x_2 + a_{22}x_2^2$$

This implies that $a_{11} = 5, a_{22} = 5$ and in order for A to be symmetric, $a_{12} = a_{21} = \frac{1}{2}(a_{12}+a_{21}) = -3$. The result is $A = \left[\begin{array}{rr} 5 & -3 \\ -3 & 5 \end{array}\right]$. We can write $q = \vec{x}^T A\vec{x}$ as

$$\left[\begin{array}{cc} x_1 & x_2 \end{array}\right] \left[\begin{array}{rr} 5 & -3 \\ -3 & 5 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = 8$$

Next, orthogonally diagonalize the matrix A to write $U^T AU = D$. The details are left to the reader and the necessary matrices are given by

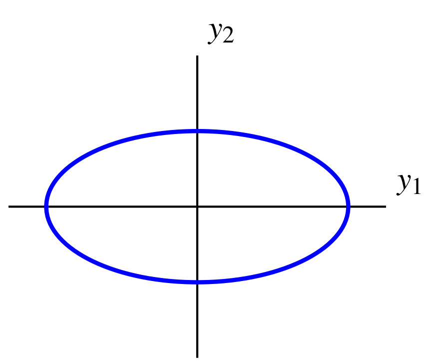
$$\begin{aligned} U &= \left[\begin{array}{cc} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{array}\right] \\ D &= \left[\begin{array}{cc} 2 & 0 \\ 0 & 8 \end{array}\right] \end{aligned}$$

Write $\vec{y} = \left[\begin{array}{c} y_1 \\ y_2 \end{array}\right]$, such that $\vec{x} = U\vec{y}$. Then it follows that q is given by

$$\begin{aligned} q &= d_{11}y_1^2 + d_{22}y_2^2 \\ &= 2y_1^2 + 8y_2^2 \end{aligned}$$

Therefore the level curve can be written as $2y_1^2 + 8y_2^2 = 8$.

This is an ellipse which is parallel to the coordinate axes. Its graph is of the form



Thus this change of variables chooses new axes such that with respect to these new axes, the ellipse is oriented parallel to the coordinate axes. ♠

Chapter 8

Some Curvilinear Coordinate Systems

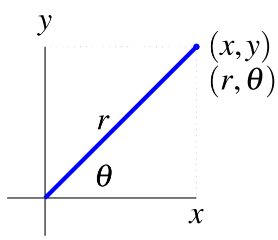
8.1 Polar Coordinates and Polar Graphs

Outcomes

A. *Understand polar coordinates.*

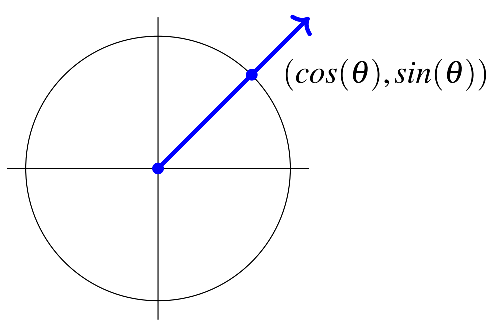
B. *Convert points between Cartesian and polar coordinates.*

You have likely encountered the Cartesian coordinate system in many aspects of mathematics. There is an alternative way to represent points in space, called **polar coordinates**. The idea is suggested in the following picture.



Consider the point above, which would be specified as (x,y) in Cartesian coordinates. We can also specify this point using polar coordinates, which we write as (r,θ). The number r is the distance from the origin$(0,0)$ to the point, while θ is the angle shown between the positive x axis and the line from the origin to the point. In this way, the point can be specified in polar coordinates as (r,θ).

Now suppose we are given an ordered pair (r,θ) where r and θ are real numbers. We want to determine the point specified by this ordered pair. We can use θ to identify a ray from the origin as follows. Let the ray pass from $(0,0)$ through the point $(\cos\theta,\sin\theta)$ as shown.



The ray is identified on the graph as the line from the origin, through the point $(\cos(\theta),\sin(\theta))$. Now if $r>0$, go a distance equal to r in the direction of the displayed arrow starting at $(0,0)$. If $r<0$, move in the opposite direction a distance of $|r|$. This is the point determined by (r,θ).

It is common to assume that θ is in the interval $[0, 2\pi)$ and $r > 0$. In this case, there is a very simple relationship between the Cartesian and polar coordinates, given by

$$x = r\cos(\theta), \quad y = r\sin(\theta) \tag{8.1}$$

These equations demonstrate how to find the Cartesian coordinates when we are given the polar coordinates of a point. They can also be used to find the polar coordinates when we know (x, y). A simpler way to do this is the following equations:

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \tan(\theta) &= \tfrac{y}{x} \end{aligned} \tag{8.2}$$

In the next example, we look at how to find the Cartesian coordinates of a point specified by polar coordinates.

Example 8.1: Finding Cartesian Coordinates

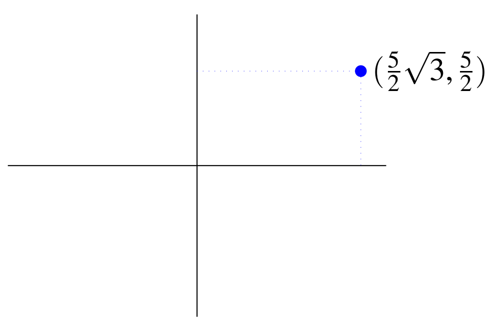
The polar coordinates of a point in the plane are $(5, \pi/6)$. Find the Cartesian coordinates of this point.

Solution. The point is specified by the polar coordinates $(5, \pi/6)$. Therefore $r = 5$ and $\theta = \pi/6$. From 8.1

$$x = r\cos(\theta) = 5\cos\left(\frac{\pi}{6}\right) = \frac{5}{2}\sqrt{3}$$

$$y = r\sin(\theta) = 5\sin\left(\frac{\pi}{6}\right) = \frac{5}{2}$$

Thus the Cartesian coordinates are $\left(\frac{5}{2}\sqrt{3}, \frac{5}{2}\right)$. The point is shown in the below graph.



♠

Consider the following example of the case where $r < 0$.

Example 8.2: Finding Cartesian Coordinates

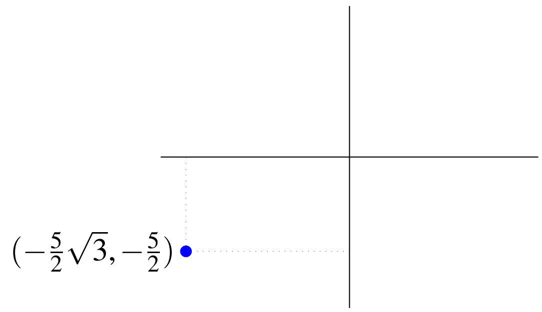
The polar coordinates of a point in the plane are $(-5, \pi/6)$. Find the Cartesian coordinates.

Solution. For the point specified by the polar coordinates $(-5, \pi/6)$, $r = -5$, and $x\theta = \pi/6$. From 8.1

$$x = r\cos(\theta) = -5\cos\left(\frac{\pi}{6}\right) = -\frac{5}{2}\sqrt{3}$$

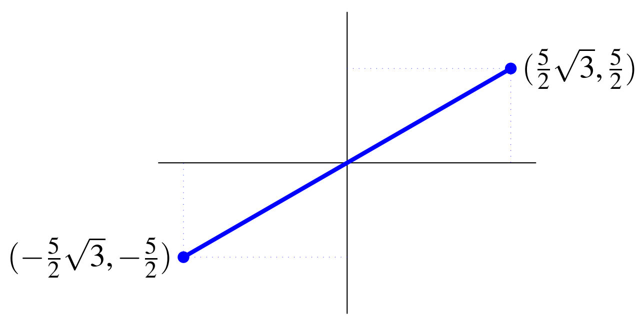
$$y = r\sin(\theta) = -5\sin\left(\frac{\pi}{6}\right) = -\frac{5}{2}$$

Thus the Cartesian coordinates are $\left(-\frac{5}{2}\sqrt{3}, -\frac{5}{2}\right)$. The point is shown in the following graph.



Recall from the previous example that for the point specified by $(5, \pi/6)$, the Cartesian coordinates are $\left(\frac{5}{2}\sqrt{3}, \frac{5}{2}\right)$. Notice that in this example, by multiplying r by -1, the resulting Cartesian coordinates are also multiplied by -1. ♠

The following picture exhibits both points in the above two examples to emphasize how they are just on opposite sides of $(0,0)$ but at the same distance from $(0,0)$.



In the next two examples, we look at how to convert Cartesian coordinates to polar coordinates.

Example 8.3: Finding Polar Coordinates

Suppose the Cartesian coordinates of a point are $(3,4)$. Find a pair of polar coordinates which correspond to this point.

Solution. Using equation 8.2, we can find r and θ. Hence $r = \sqrt{3^2+4^2} = 5$. It remains to identify the angle θ between the positive x axis and the line from the origin to the point. Since both the x and y values are positive, the point is in the first quadrant. Therefore, θ is between 0 and $\pi/2$. Using this and 8.2, we have to solve:

$$\tan(\theta) = \frac{4}{3}$$

Conversely, we can use equation 8.1 as follows:

$$3 = 5\cos(\theta)$$

$$4 = 5\sin(\theta)$$

Solving these equations, we find that, approximately, $\theta = 0.927295$ radians. ♠

Consider the following example.

Example 8.4: Finding Polar Coordinates

Suppose the Cartesian coordinates of a point are $\left(-\sqrt{3}, 1\right)$ *. Find the polar coordinates which correspond to this point.*

Solution. Given the point $\left(-\sqrt{3}, 1\right)$,

$$\begin{aligned} r &= \sqrt{1^2 + (-\sqrt{3})^2} \\ &= \sqrt{1+3} \\ &= 2 \end{aligned}$$

In this case, the point is in the second quadrant since the x value is negative and the y value is positive. Therefore, θ will be between $\pi/2$ and π. Solving the equations

$$-\sqrt{3} = 2\cos(\theta)$$

$$1 = 2\sin(\theta)$$

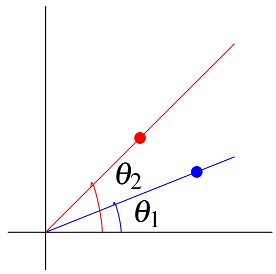
we find that $\theta = 5\pi/6$. Hence the polar coordinates for this point are $(2, 5\pi/6)$. ♠

Consider this example. Suppose we used $r = -2$ and $\theta = 2\pi - (\pi/6) = 11\pi/6$. These coordinates specify the same point as above. Observe that there are infinitely many ways to identify this particular point with polar coordinates. In fact, every point can be represented with polar coordinates in infinitely many ways. Because of this, it will usually be the case that θ is confined to lie in some interval of length 2π and $r > 0$, for real numbers r and θ.

Just as with Cartesian coordinates, it is possible to use relations between the polar coordinates to specify points in the plane. The process of sketching the graphs of these relations is very similar to that used to sketch graphs of functions in Cartesian coordinates. Consider a relation between polar coordinates of the form, $r = f(\theta)$. To graph such a relation, first make a table of the form

θ	r
θ_1	$f(\theta_1)$
θ_2	$f(\theta_2)$
$\vdots$	$\vdots$

Graph the resulting points and connect them with a curve. The following picture illustrates how to begin this process.



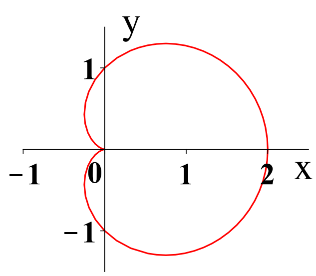
To find the point in the plane corresponding to the ordered pair $(f(\theta), \theta)$, we follow the same process as when finding the point corresponding to (r, θ).

Consider the following example of this procedure, incorporating computer software.

Example 8.5: Graphing a Polar Equation

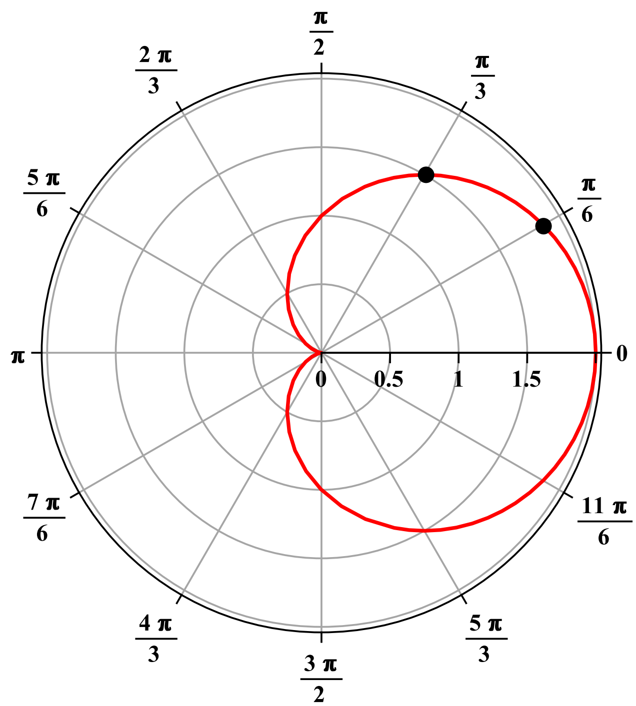
Graph the polar equation $r = 1 + \cos\theta$.

Solution. We will use the computer software *Maple* to complete this example. The command which produces the polar graph of the above equation is: $>$ plot(1+cos(t),t= 0..2*Pi,coords=polar). Here we use t to represent the variable θ for convenience. The command tells Maple that r is given by $1 + \cos(t)$ and that $t \in [0, 2\pi]$.



The above graph makes sense when considered in terms of trigonometric functions. Suppose $\theta = 0, r = 2$ and let θ increase to $\pi/2$. As θ increases, $\cos\theta$ decreases to 0. Thus the line from the origin to the point on the curve should get shorter as θ goes from 0 to $\pi/2$. As θ goes from $\pi/2$ to π, $\cos\theta$ decreases, eventually equaling -1 at $\theta = \pi$. Thus $r = 0$ at this point. This scenario is depicted in the above graph, which shows a function called a **cardioid**.

The following picture illustrates the above procedure for obtaining the polar graph of $r = 1 + \cos(\theta)$. In this picture, the concentric circles correspond to values of r while the rays from the origin correspond to the angles which are shown on the picture. The dot on the ray corresponding to the angle $\pi/6$ is located at a distance of $r = 1 + \cos(\pi/6)$ from the origin. The dot on the ray corresponding to the angle $\pi/3$ is located at a distance of $r = 1 + \cos(\pi/3)$ from the origin and so forth. The polar graph is obtained by connecting such points with a smooth curve, with the result being the figure shown above.



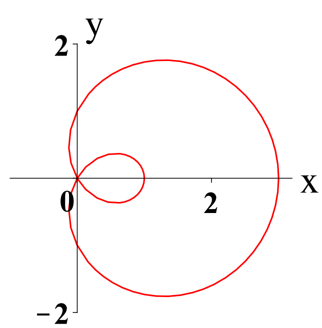
♠

Consider another example of constructing a polar graph.

Example 8.6: A Polar Graph

Graph $r = 1 + 2\cos\theta$ *for* $\theta \in [0, 2\pi]$.

Solution. The graph of the polar equation $r = 1 + 2\cos\theta$ for $\theta \in [0, 2\pi]$ is given as follows.

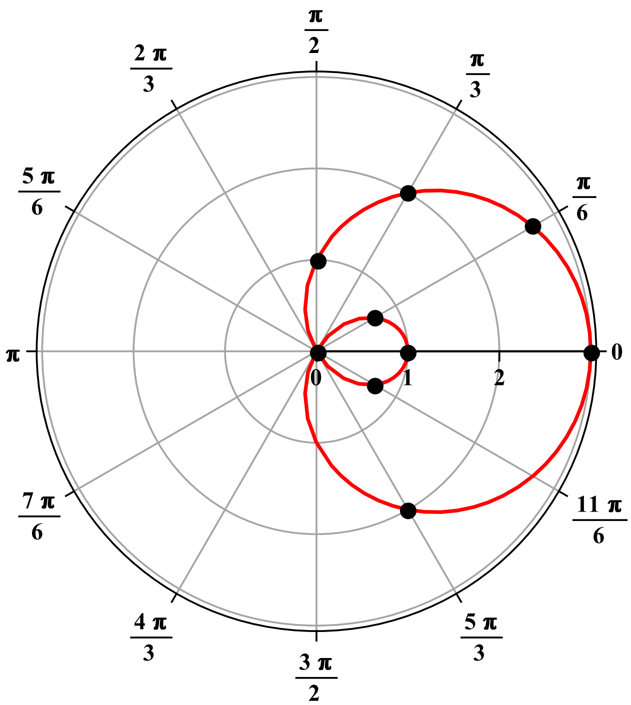


To see the way this is graphed, consider the following picture. First the indicated points were graphed and then the curve was drawn to connect the points. When done by a computer, many more points are used to create a more accurate picture.

Consider first the following table of points.

θ	$\pi/6$	$\pi/3$	$\pi/2$	$5\pi/6$	π	$4\pi/3$	$7\pi/6$	$5\pi/3$
r	$\sqrt{3}+1$	2	1	$1-\sqrt{3}$	-1	0	$1-\sqrt{3}$	2

Note how some entries in the table have $r < 0$. To graph these points, simply move in the opposite direction. These types of points are responsible for the small loop on the inside of the larger loop in the graph.



♠

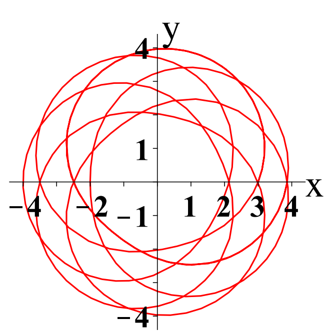
The process of constructing these graphs can be greatly facilitated by computer software. However, the use of such software should not replace understanding the steps involved.

The next example shows the graph for the equation $r = 3 + \sin\left(\frac{7\theta}{6}\right)$. For complicated polar graphs, computer software is used to facilitate the process.

Example 8.7: A Polar Graph

Graph $r = 3 + \sin\left(\frac{7\theta}{6}\right)$ *for* $\theta \in [0, 14\pi]$.

Solution.



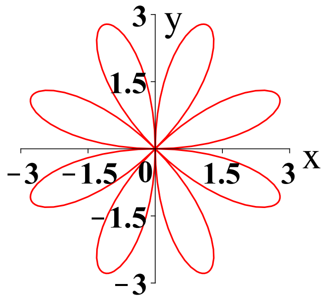
♠

The next example shows another situation in which r can be negative.

Example 8.8: A Polar Graph: Negative r

Graph $r = 3sin(4\theta)$ *for* $\theta \in [0, 2\pi]$.

Solution.



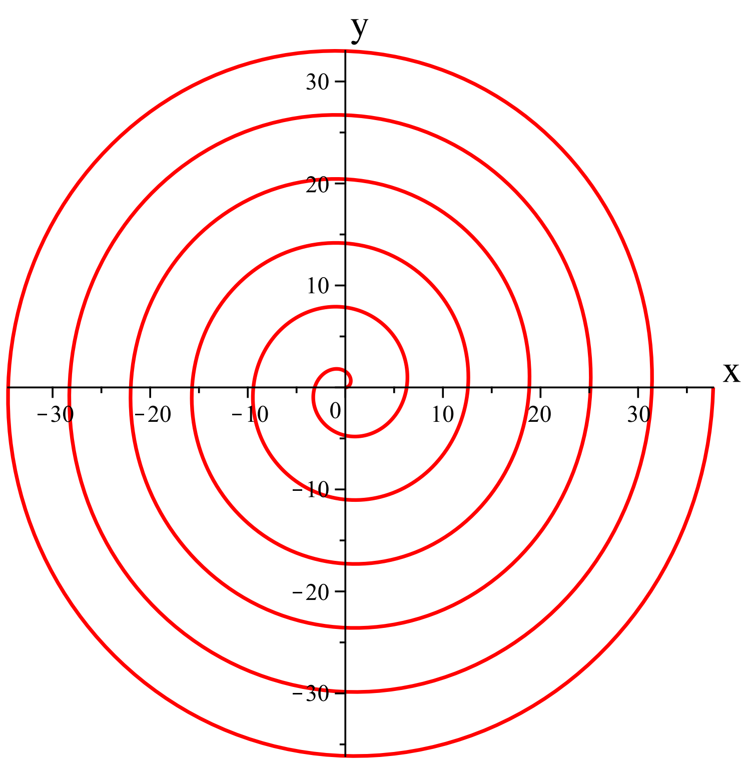
♠

We conclude this section with an interesting graph of a simple polar equation.

Example 8.9: The Graph of a Spiral

Graph $r = \theta$ *for* $\theta \in [0, 2\pi]$.

Solution. The graph of this polar equation is a spiral. This is the case because as θ increases, so does r.



♠

In the next section, we will look at two ways of generalizing polar coordinates to three dimensions.

8.2 Spherical and Cylindrical Coordinates

Outcomes

A. *Understand cylindrical and spherical coordinates.*

B. *Convert points between Cartesian, cylindrical, and spherical coordinates.*

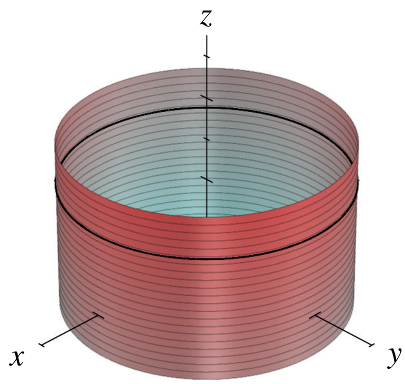
Spherical and cylindrical coordinates are two generalizations of polar coordinates to three dimensions. We will first look at **cylindrical coordinates** .

When moving from polar coordinates in two dimensions to cylindrical coordinates in three dimensions, we use the polar coordinates in the xy plane and add a z coordinate. For this reason, we use the notation (r,θ,z) to express cylindrical coordinates. The relationship between Cartesian coordinates (x,y,z) and cylindrical coordinates (r,θ,z) is given by

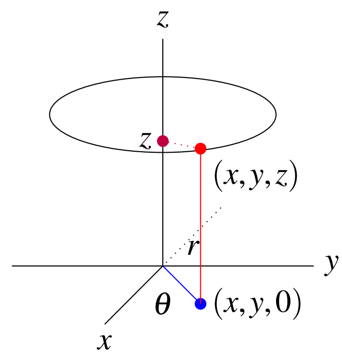
$$\begin{aligned}
x &= r\cos(\theta) \\
y &= r\sin(\theta) \\
z &= z
\end{aligned}$$

where $r \geq 0$, $\theta \in [0,2\pi)$, and z is simply the Cartesian coordinate. Notice that x and y are defined as the usual polar coordinates in the xy-plane. Recall that r is defined as the length of the ray from the origin to the point $(x,y,0)$, while θ is the angle between the positive x-axis and this same ray.

To illustrate this coordinate system, consider the following two pictures. In the first of these, both r and z are known. The cylinder corresponds to a given value for r. A useful way to think of r is as the distance between a point in three dimensions and the z-axis. Every point on the cylinder shown is at the same distance from the z-axis. Giving a value for z results in a horizontal circle, or cross section of the cylinder at the given height on the z axis (shown below as a black line on the cylinder). In the second picture, the point is specified completely by also knowing θ as shown.



r and z are known



r, θ and z are known

Every point of three dimensional space other than the z axis has unique cylindrical coordinates. Of course there are infinitely many cylindrical coordinates for the origin and for the z-axis. Any θ will work if $r = 0$ and z is given.

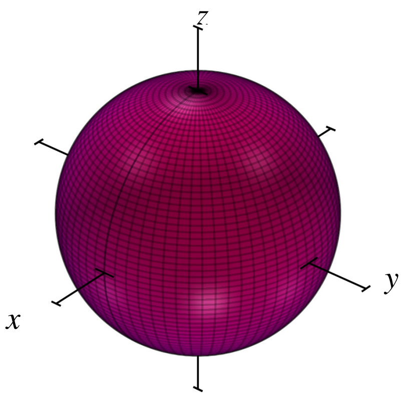
Consider now **spherical coordinates**, the second generalization of polar form in three dimensions. For a point (x, y, z) in three dimensional space, the spherical coordinates are defined as follows.

ρ : the length of the ray from the origin to the point
θ : the angle between the positive x-axis and the ray from the origin to the point $(x, y, 0)$
ϕ : the angle between the positive z-axis and the ray from the origin to the point of interest

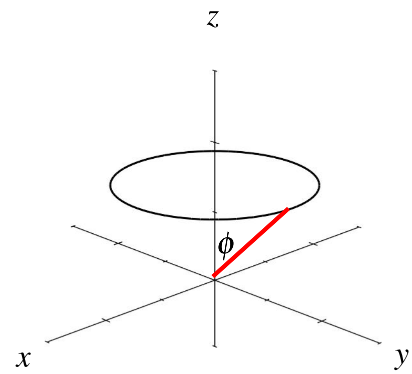
The spherical coordinates are determined by (ρ, ϕ, θ). The relation between these and the Cartesian coordinates (x, y, z) for a point are as follows.

$$\begin{aligned} x &= \rho \sin(\phi) \cos(\theta),\ \phi \in [0, \pi] \\ y &= \rho \sin(\phi) \sin(\theta),\ \theta \in [0, 2\pi) \\ z &= \rho \cos\phi,\ \rho \geq 0. \end{aligned}$$

Consider the pictures below. The first illustrates the surface when ρ is known, which is a sphere of radius ρ. The second picture corresponds to knowing both ρ and ϕ, which results in a circle about the z-axis. Suppose the first picture demonstrates a graph of the Earth. Then the circle in the second picture would correspond to a particular latitude.

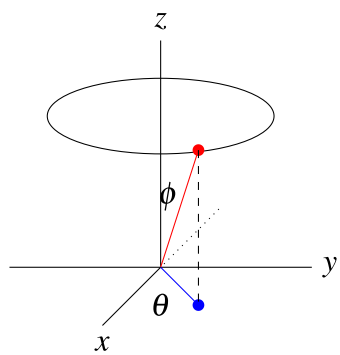


ρ is known



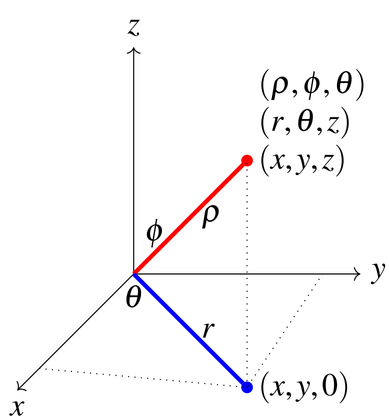
ρ and ϕ are known

Giving the third coordinate, θ completely specifies the point of interest. This is demonstrated in the following picture. If the latitude corresponds to ϕ, then we can think of θ as the longitude.



ρ, ϕ and θ are known

The following picture summarizes the geometric meaning of the three coordinate systems.



Therefore, we can represent the same point in three ways, using Cartesian coordinates, (x,y,z), cylindrical coordinates, (r,θ,z), and spherical coordinates (ρ,ϕ,θ).

Using this picture to review, call the point of interest P for convenience. The Cartesian coordinates for P are (x,y,z). Then ρ is the distance between the origin and the point P. The angle between the positive z axis and the line between the origin and P is denoted by ϕ. Then θ is the angle between the positive x axis and the line joining the origin to the point $(x,y,0)$ as shown. This gives the spherical coordinates, (ρ,ϕ,θ). Given the line from the origin to $(x,y,0)$, $r=\rho\sin(\phi)$ is the length of this line. Thus r and θ determine a point in the xy-plane. In other words, r and θ are the usual polar coordinates and $r\geq 0$ and $\theta\in[0,2\pi)$. Letting z denote the usual z coordinate of a point in three dimensions, (r,θ,z) are the cylindrical coordinates of P.

The relation between spherical and cylindrical coordinates is that $r=\rho\sin(\phi)$ and the θ is the same as the θ of cylindrical and polar coordinates.

We will now consider some examples.

Example 8.10: Describing a Surface in Spherical Coordinates

Express the surface $z=\frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$ *in spherical coordinates.*

Solution. We will use the equations from above:

$$\begin{aligned}
&x=\rho\sin(\phi)\cos(\theta),\phi\in[0,\pi]\\
&y=\rho\sin(\phi)\sin(\theta),\ \theta\in[0,2\pi)\\
&z=\rho\cos\phi,\ \rho\geq 0
\end{aligned}$$

To express the surface in spherical coordinates, we substitute these expressions into the equation. This is done as follows:

$$\rho\cos(\phi)=\frac{1}{\sqrt{3}}\sqrt{(\rho\sin(\phi)\cos(\theta))^2+(\rho\sin(\phi)\sin(\theta))^2}=\frac{1}{3}\sqrt{3}\rho\sin(\phi).$$

This reduces to

$$\tan(\phi)=\sqrt{3}$$

and so $\phi=\pi/3$. ♠

Example 8.11: Describing a Surface in Spherical Coordinates

Express the surface $y=x$ *in terms of spherical coordinates.*

Solution. Using the same procedure as the previous example, this says $\rho\sin(\phi)\sin(\theta)=\rho\sin(\phi)\cos(\theta)$. Simplifying, $\sin(\theta)=\cos(\theta)$, which you could also write $\tan(\theta)=1$. ♠

We conclude this section with an example of how to describe a surface using cylindrical coordinates.

Example 8.12: Describing a Surface in Cylindrical Coordinates

Express the surface $x^2+y^2=4$ *in cylindrical coordinates.*

Solution. Recall that to convert from Cartesian to cylindrical coordinates, we can use the following equations:

$$x = r\cos(\theta), y = r\sin(\theta), z = z$$

Substituting these equations in for x, y, z in the equation for the surface, we have

$$r^2\cos^2(\theta) + r^2\sin^2(\theta) = 4$$

This can be written as $r^2(\cos^2(\theta) + \sin^2(\theta)) = 4$. Recall that $\cos^2(\theta) + \sin^2(\theta) = 1$. Thus $r^2 = 4$ or $r = 2$. ♠

Chapter 9

Vector Spaces

9.1 Algebraic Considerations

Outcomes

A. *Develop the abstract concept of a vector space through axioms.*

B. *Deduce basic properties of vector spaces.*

C. *Use the vector space axioms to determine if a set and its operations constitute a vector space.*

We have been working a lot with the set of vectors in $\mathbb{R}^n$. Some of the great power of linear algebra comes from generalizing the ideas, techniques and results that we have developed so that they can be used in other settings. So in this section we build the idea of an abstract *vector space*.

To this point we have had vectors and scalars, and we have been very careful to think of a vector $\vec{u}$ as an element of $\mathbb{R}^n$ or $\mathbb{C}^n$. We have also, almost without thinking about it, used real numbers, or occasionally complex numbers, as scalars. For this chapter we will be allowing ourselves to use different objects as vectors, but our scalars will still be either the real numbers (almost all of the time) or the complex numbers (for an example or two). If the set of scalars is $\mathbb{R}$, then we will be working with a *real vector space*. If the set of scalars is $\mathbb{C}$, then we will have a *complex vector space*. So most of the time we will be looking at real vector spaces in this chapter. And of course we will only be able to give a brief introduction to this rich and interesting field.

The definition of a vector space is focused on the two basic operations with which we are familiar, vector addition and scalar multiplication, which are nothing more than functions. We will denote vector addition by the symbol "$+$", while scalar multiplication will be denoted (at least for the official definition, but not long thereafter) by the symbol "$\cdot$". The needed properties of those functions and how we want them to interact with each other are what we specify in the definition of a vector space. For the following definition, remember that $V \times V$ is the set of ordered pairs $(\vec{u}, \vec{v})$, where $\vec{u}, \vec{v} \in V$ while $\mathbb{R} \times V$ is the set of ordered pairs $(r, \vec{v})$, where $r \in \mathbb{R}$ and $\vec{v} \in V$.

Definition 9.1: Vector Space

A nonempty set V, together with two functions vector addition ($+ : V \times V \to V$) and scalar multiplication ($\cdot : \mathbb{R} \times V \to V$), is called a ***real vector space*** *if the following conditions hold.*

- *V is Closed under Addition:*

$$\text{If } \vec{v}, \vec{w} \text{ are elements of } V, \text{ then } \vec{v} + \vec{w} \text{ is also an element of } V.$$

- *The Commutative Law of Addition*

$$\text{For any } \vec{v}, \vec{w} \in V, \vec{v} + \vec{w} = \vec{w} + \vec{v}.$$

- *The Associative Law of Addition*

$$\text{For any } \vec{u}, \vec{v}, \vec{w} \in V, (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}).$$

- *The Existence of an Additive Identity*

$$\text{There is an element of } V, \text{ called } \vec{0}, \text{ such that for any } \vec{v} \in V, \vec{v} + \vec{0} = \vec{v}.$$

- *The Existence of an Additive Inverse*

$$\text{For any } \vec{v} \in V \text{ there is an element of } V, \text{ called } -\vec{v}, \text{ such that } \vec{v} + (-\vec{v}) = \vec{0}.$$

- *Closed under Scalar Multiplication*

$$\text{If } \vec{v} \text{ is an element of } V, \text{ and } r \text{ is an element of } \mathbb{R}, \text{ then } r \cdot \vec{v} \text{ is also an element of } V.$$

- *Distributive Law of Scalar Multiplication over Vector Addition*

$$\text{For any } r \in \mathbb{R} \text{ and any } \vec{v}, \vec{w} \in V, \ r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}.$$

- *Distributive Law of Scalar Multiplication over Scalar Addition*

$$\text{For any } r, s \in \mathbb{R} \text{ and any } \vec{v} \in V, \ (r + s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}.$$

- *Associative Law of Scalar Multiplication*

$$\text{For any } r, s \in \mathbb{R} \text{ and any } \vec{v}, \in V, r \cdot (s \cdot \vec{v}) = (rs) \cdot \vec{v}.$$

- *Existence of a Multiplicative Identity*

$$\text{For any } \vec{v}, \in V, 1 \cdot \vec{v} = \vec{v}.$$

If, in the above axioms, scalars can be chosen from the set $\mathbb{C}$ of complex numbers, we will say that V is a ***complex vector space.***

As mentioned above, for reading simplicity the symbol "·" for scalar multiplication will almost never be used, so we will write $r\vec{v}$ rather than the officially correct $r \cdot \vec{v}$.

It is important to note that we have seen much of this content before, in terms of $\mathbb{R}^n$. In particular, you should look back at Theorem 4.9 and Theorem 4.12, Just to get a feel for how the arguments go, the first thing that we will prove in this section is that $\mathbb{R}^n$ is an example of a vector space. This means that all discussions in this chapter will pertain to $\mathbb{R}^n$. While it may be useful to consider all concepts of this chapter in terms of $\mathbb{R}^n$, it is also important to understand that these concepts apply to *all* vector spaces.

Example 9.2: $\mathbb{R}^n$

$\mathbb{R}^n$, under the usual operations of vector addition and scalar multiplication, is a vector space.

Solution. To show that $\mathbb{R}^n$ is a vector space, we need to show that the above axioms hold. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^n$. We first prove the axioms for vector addition.

- To show that $\mathbb{R}^n$ is closed under addition, we must show that for two vectors in $\mathbb{R}^n$ their sum is also in $\mathbb{R}^n$. The sum $\vec{u}+\vec{v}$ is given by:

$$\left[\begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_n \end{array}\right] + \left[\begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_n \end{array}\right] = \left[\begin{array}{c} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{array}\right]$$

  The sum is a vector with n entries, showing that it is in $\mathbb{R}^n$. Hence $\mathbb{R}^n$ is closed under vector addition.

- To show that addition is commutative, consider the following:

$$\begin{aligned}
\vec{u}+\vec{v} &= \left[\begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_n \end{array}\right] + \left[\begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_n \end{array}\right] \\
&= \left[\begin{array}{c} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{array}\right] \\
&= \left[\begin{array}{c} v_1+u_1 \\ v_2+u_2 \\ \vdots \\ v_n+u_n \end{array}\right] \\
&= \left[\begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_n \end{array}\right] + \left[\begin{array}{c} u_1 \\ u_2 \\ \vdots \\ u_n \end{array}\right] \\
&= \vec{v}+\vec{u}
\end{aligned}$$

Hence addition of vectors in $\mathbb{R}^n$ is commutative.

- We will show that addition of vectors in $\mathbb{R}^n$ is associative in a similar way.

$$\begin{aligned}
(\vec{u}+\vec{v})+\vec{w} &= \left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right) + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \\
&= \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \\
&= \begin{bmatrix} (u_1+v_1)+w_1 \\ (u_2+v_2)+w_2 \\ \vdots \\ (u_n+v_n)+w_n \end{bmatrix} \\
&= \begin{bmatrix} u_1+(v_1+w_1) \\ u_2+(v_2+w_2) \\ \vdots \\ u_n+(v_n+w_n) \end{bmatrix} \\
&= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{bmatrix} \\
&= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \right) \\
&= \vec{u}+(\vec{v}+\vec{w})
\end{aligned}$$

Hence addition of vectors is associative.

- Next, we show the existence of an additive identity. Let $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

$$\vec{u}+\vec{0} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{aligned} &= \begin{bmatrix} u_1+0 \\ u_2+0 \\ \vdots \\ u_n+0 \end{bmatrix} \\ &= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \vec{u} \end{aligned}$$

Hence the zero vector $\vec{0}$ is an additive identity.

- Next, we prove the existence of an additive inverse. Let $-\vec{u} = \begin{bmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{bmatrix}$.

$$\begin{aligned} \vec{u}+(-\vec{u}) &= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{bmatrix} \\ &= \begin{bmatrix} u_1-u_1 \\ u_2-u_2 \\ \vdots \\ u_n-u_n \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= \vec{0} \end{aligned}$$

Hence $-\vec{u}$ is an additive inverse.

We now need to prove the axioms related to scalar multiplication. Let r,s be real numbers and let $\vec{u},\vec{v}$ be vectors in $\mathbb{R}^n$.

- We first show that $\mathbb{R}^n$ is closed under scalar multiplication. To do so, we show that $r\vec{u}$ is also a vector with n entries.

$$r\vec{u} = r\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ru_1 \\ ru_2 \\ \vdots \\ ru_n \end{bmatrix}$$

The vector $r\vec{u}$ is again a vector with n entries, showing that $\mathbb{R}^n$ is closed under scalar multiplication.

- We wish to show that $r(\vec{u}+\vec{v}) = r\vec{u}+r\vec{v}$.

$$\begin{aligned}
r(\vec{u}+\vec{v}) &= r\left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\right) \\
&= r\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} \\
&= \begin{bmatrix} r(u_1+v_1) \\ r(u_2+v_2) \\ \vdots \\ r(u_n+v_n) \end{bmatrix} \\
&= \begin{bmatrix} ru_1+rv_1 \\ ru_2+rv_2 \\ \vdots \\ ru_n+rv_n \end{bmatrix} \\
&= \begin{bmatrix} ru_1 \\ ru_2 \\ \vdots \\ ru_n \end{bmatrix} + \begin{bmatrix} rv_1 \\ rv_2 \\ \vdots \\ rv_n \end{bmatrix} \\
&= r\vec{u}+r\vec{v}
\end{aligned}$$

- Next, we wish to show that $(r+s)\vec{u} = r\vec{u}+s\vec{u}$.

$$\begin{aligned}
(r+s)\vec{u} &= (r+s)\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \\
&= \begin{bmatrix} (r+s)u_1 \\ (r+s)u_2 \\ \vdots \\ (r+s)u_n \end{bmatrix} \\
&= \begin{bmatrix} ru_1+su_1 \\ ru_2+su_2 \\ \vdots \\ ru_n+su_n \end{bmatrix}
\end{aligned}$$

$$\begin{aligned} &= \begin{bmatrix} ru_1 \\ ru_2 \\ \vdots \\ ru_n \end{bmatrix} + \begin{bmatrix} su_1 \\ su_2 \\ \vdots \\ su_n \end{bmatrix} \\ &= r\vec{u} + s\vec{u} \end{aligned}$$

- We wish to show that $r(s\vec{u}) = (rs)\vec{u}$.

$$\begin{aligned} r(s\vec{u}) &= r\left(s \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \right) \\ &= r\left(\begin{bmatrix} su_1 \\ su_2 \\ \vdots \\ su_n \end{bmatrix} \right) \\ &= \begin{bmatrix} r(su_1) \\ r(su_2) \\ \vdots \\ r(su_n) \end{bmatrix} \\ &= \begin{bmatrix} (rs)u_1 \\ (rs)u_2 \\ \vdots \\ (rs)u_n \end{bmatrix} \\ &= (rs) \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \\ &= (rs)\vec{u} \end{aligned}$$

- Finally, we need to show that $1\vec{u} = \vec{u}$.

$$\begin{aligned} 1\vec{u} &= 1 \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \\ &= \begin{bmatrix} 1u_1 \\ 1u_2 \\ \vdots \\ 1u_n \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$
$$= \vec{u}$$

By the above proofs, it is clear that $\mathbb{R}^n$ satisfies the vector space axioms. Hence, $\mathbb{R}^n$ is a vector space under the usual operations of vector addition and scalar multiplication. ♠

We now consider some other examples of vector spaces.

Example 9.3: Vector Space of Polynomials

Let $\mathbb{P}_2$ be

$$\mathbb{P}_2 = \left\{ a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R} \text{ for all } i \right\}. \tag{9.1}$$

So $\mathbb{P}_2$ is the set of all polynomials of degree at most 2 as well as the zero polynomial. Define addition to be the standard addition of polynomials, and scalar multiplication the usual multiplication of a polynomial by a number. Then $\mathbb{P}_2$ is a vector space.

Although it seems unnatural to mention the zero polynomial separately in the discussion above, it is necessary. Officially, the degree of the zero polynomial is undefined, so we cannot say that its degree is less than or equal to 2. But we will want the zero polynomial as part of our vector space (do you see why?), so we add it into the set $\mathbb{P}_2$ separately.

Solution.

To show that $\mathbb{P}_2$ is a vector space, we verify the axioms. Let $p(x), q(x), r(x)$ be polynomials in $\mathbb{P}_2$ and let r, s be real numbers. Write $p(x) = p_2x^2 + p_1x + p_0$, $q(x) = q_2x^2 + q_1x + q_0$, and $r(x) = r_2x^2 + r_1x + r_0$.

- We first prove that $\mathbb{P}_2$ is closed under addition. For two polynomials in $\mathbb{P}_2$ we need to show that their sum is also a polynomial in $\mathbb{P}_2$. Notice that

$$\begin{aligned} p(x) + q(x) &= p_2x^2 + p_1x + p_0 + q_2x^2 + q_1x + q_0 \\ &= (p_2 + q_2)x^2 + (p_1 + q_1)x + (p_0 + q_0) \end{aligned}$$

  The sum is a polynomial of the form described in Equation 9.1, and so is an element of $\mathbb{P}_2$. Thus $\mathbb{P}_2$ is closed under addition.

- We need to show that addition is commutative, that is $p(x) + q(x) = q(x) + p(x)$.

$$\begin{aligned} p(x) + q(x) &= p_2x^2 + p_1x + p_0 + q_2x^2 + q_1x + q_0 \\ &= (p_2 + q_2)x^2 + (p_1 + q_1)x + (p_0 + q_0) \\ &= (q_2 + p_2)x^2 + (q_1 + p_1)x + (q_0 + p_0) \\ &= q_2x^2 + q_1x + q_0 + p_2x^2 + p_1x + p_0 \\ &= q(x) + p(x) \end{aligned}$$

- Next, we need to show that addition is associative. That is, that $(p(x)+q(x))+r(x) = p(x)+(q(x)+r(x))$.

$$\begin{aligned}
(p(x)+q(x))+r(x) &= \left(p_2x^2+p_1x+p_0+q_2x^2+q_1x+q_0\right)+r_2x^2+r_1x+r_0 \\
&= (p_2+q_2)x^2+(p_1+q_1)x+(p_0+q_0)+r_2x^2+r_1x+r_0 \\
&= (p_2+q_2+r_2)x^2+(p_1+q_1+r_1)x+(p_0+q_0+r_0) \\
&= p_2x^2+p_1x+p_0+(q_2+r_2)x^2+(q_1+r_1)x+(q_0+r_0) \\
&= p_2x^2+p_1x+p_0+\left(q_2x^2+q_1x+q_0+r_2x^2+r_1x+r_0\right) \\
&= p(x)+(q(x)+r(x))
\end{aligned}$$

- Next, we must prove that there exists an additive identity. Let $0(x) = 0x^2+0x+0$, which is an element of $\mathbb{P}_2$ by Equation 9.1.

$$\begin{aligned}
p(x)+0(x) &= p_2x^2+p_1x+p_0+0x^2+0x+0 \\
&= (p_2+0)x^2+(p_1+0)x+(p_0+0) \\
&= p_2x^2+p_1x+p_0 \\
&= p(x)
\end{aligned}$$

  Hence an additive identity exists, specifically the zero polynomial. Which explains why we needed to make sure that the zero polynomial is an element of $\mathbb{P}_2$.

- Next we must prove that there exists an additive inverse. Let $-p(x) = -p_2x^2-p_1x-p_0$ and consider the following:

$$\begin{aligned}
p(x)+(-p(x)) &= p_2x^2+p_1x+p_0+\left(-p_2x^2-p_1x-p_0\right) \\
&= (p_2-p_2)x^2+(p_1-p_1)x+(p_0-p_0) \\
&= 0x^2+0x+0 \\
&= 0(x)
\end{aligned}$$

  Hence an additive inverse $-p(x)$ exists such that $p(x)+(-p(x)) = 0(x)$.

We now need to verify the axioms related to scalar multiplication.

- First we prove that $\mathbb{P}_2$ is closed under scalar multiplication. That is, we show that if $p(x) \in \mathbb{P}_2$ and $r \in \mathbb{R}$, then $rp(x)$ is also an element of $\mathbb{P}_2$.

$$rp(x) = r\left(p_2x^2+p_1x+p_0\right) = rp_2x^2+rp_1x+rp_0 \in \mathbb{P}_2.$$

  Therefore $\mathbb{P}_2$ is closed under scalar multiplication.

- We need to show that $r(p(x)+q(x)) = rp(x)+rq(x)$.

$$\begin{aligned}
r(p(x)+q(x)) &= r\left(p_2x^2+p_1x+p_0+q_2x^2+q_1x+q_0\right) \\
&= r\left((p_2+q_2)x^2+(p_1+q_1)x+(p_0+q_0)\right) \\
&= r(p_2+q_2)x^2+r(p_1+q_1)x+r(p_0+q_0) \\
&= (rp_2+rq_2)x^2+(rp_1+rq_1)x+(rp_0+rq_0) \\
&= rp_2x^2+rp_1x+rp_0+rq_2x^2+rq_1x+rq_0 \\
&= rp(x)+rq(x)
\end{aligned}$$

- Next we show that $(r+s)p(x) = rp(x) + sp(x)$.

$$\begin{aligned}
(r+s)p(x) &= (r+s)(p_2x^2 + p_1x + p_0) \\
&= (r+s)p_2x^2 + (r+s)p_1x + (r+s)p_0 \\
&= rp_2x^2 + rp_1x + rp_0 + sp_2x^2 + sp_1x + sp_0 \\
&= rp(x) + sp(x)
\end{aligned}$$

- The next axiom which needs to be verified is $r(sp(x)) = (rs)p(x)$.

$$\begin{aligned}
r(sp(x)) &= r\left(s\left(p_2x^2 + p_1x + p_0\right)\right) \\
&= r\left(sp_2x^2 + sp_1x + sp_0\right) \\
&= rsp_2x^2 + rsp_1x + rsp_0 \\
&= (rs)\left(p_2x^2 + p_1x + p_0\right) \\
&= (rs)p(x)
\end{aligned}$$

- Finally, we show that $1p(x) = p(x)$.

$$\begin{aligned}
1p(x) &= 1\left(p_2x^2 + p_1x + p_0\right) \\
&= 1p_2x^2 + 1p_1x + 1p_0 \\
&= p_2x^2 + p_1x + p_0 \\
&= p(x)
\end{aligned}$$

Since the above axioms hold, we know that $\mathbb{P}_2$ as described above is a vector space. ♠

In fact there is nothing particularly special about the fact that we were working with polynomials of degree at most two in the example above. The obvious modifications show that $\mathbb{P}_n$ is a vector space for any natural number n, and in fact the set $\mathbb{P}$ of all polynomials is also a vector space.

Another important example of a vector space is the set of all matrices of the same size.

Example 9.4: Vector Space of Matrices

Let $\mathbb{M}_{2,3}$ be the set of all 2×3 matrices. Using the usual operations of matrix addition and scalar multiplication, show that $\mathbb{M}_{2,3}$ is a vector space.

Solution. Let A, B be 2×3 matrices in $\mathbb{M}_{2,3}$. We first prove the axioms for addition.

- In order to prove that $\mathbb{M}_{2,3}$ is closed under matrix addition, we show that the sum $A+B$ is in $\mathbb{M}_{2,3}$. This means showing that $A+B$ is a 2×3 matrix.

$$A + B = \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right] + \left[\begin{array}{ccc} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{array}\right]$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$$

You can see that the sum is a 2×3 matrix, so it is in $\mathbb{M}_{2,3}$. It follows that $\mathbb{M}_{2,3}$ is closed under matrix addition.

- The remaining axioms regarding matrix addition follow from properties of matrix addition. Therefore $\mathbb{M}_{2,3}$ satisfies the axioms of matrix addition.

We now turn our attention to the axioms regarding scalar multiplication. Let A,B be matrices in $\mathbb{M}_{2,3}$ and let r be a real number.

- We first show that $\mathbb{M}_{2,3}$ is closed under scalar multiplication. That is, we show that rA a 2×3 matrix.

$$\begin{aligned} rA &= r\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \\ &= \begin{bmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \end{bmatrix} \end{aligned}$$

This is a 2×3 matrix in $\mathbb{M}_{2,3}$ which proves that the set is closed under scalar multiplication.

- The remaining axioms of scalar multiplication follow from properties of scalar multiplication of matrices. Therefore $\mathbb{M}_{2,3}$ satisfies the axioms of scalar multiplication.

In conclusion, $\mathbb{M}_{2,3}$ satisfies the required axioms and is a vector space. ♠

While here we proved that the set of all 2×3 matrices is a vector space, there is nothing special about this choice of matrix size. In fact if we instead consider $\mathbb{M}_{m,n}$, the set of all $m\times n$ matrices, then $\mathbb{M}_{m,n}$ is a vector space under the operations of matrix addition and scalar multiplication.

We now examine an example of a set that does not satisfy all of the above axioms, and is therefore *not* a vector space.

Example 9.5: Not a Vector Space

Let V denote the set of 2×3 matrices. Let addition in V be defined by $A+B=A$ for matrices A,B in V. Let scalar multiplication in V be the usual scalar multiplication of matrices. Show that V is not a vector space.

Solution. In order to show that V is not a vector space, it suffices to find only one axiom which is not satisfied. We will begin by examining the axioms for addition until one is found which does not hold. Let A,B be matrices in V.

- We first want to check if addition is closed. Consider $A+B$. By the definition of addition in the example, we have that $A+B=A$. Since A is a 2×3 matrix, it follows that the sum $A+B$ is in V, and V is closed under addition.

- We now wish to check if addition is commutative. That is, we want to check if $A+B=B+A$ **for all** choices of A and B in V. From the definition of addition, we have that $A+B=A$ and $B+A=B$. Therefore, we can find A, B in V such that these sums are not equal. One example is

$$A=\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 0 & 0\end{array}\right], B=\left[\begin{array}{ccc}0 & 0 & 0\\ 1 & 0 & 0\end{array}\right]$$

  Using the operation defined by $A+B=A$, we have

$$\begin{aligned}A+B &= A\\ &= \left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 0 & 0\end{array}\right]\\ B+A &= B\\ &= \left[\begin{array}{ccc}0 & 0 & 0\\ 1 & 0 & 0\end{array}\right]\end{aligned}$$

  It follows that $A+B \neq B+A$. Therefore addition as defined for V is not commutative and V fails this axiom. Hence V is not a vector space.

♠

Consider another example of a vector space.

Example 9.6: Vector Space of Functions

Let S be a nonempty set and define $\mathbb{F}_S$ to be the set of real functions defined on S. In other words, we write $\mathbb{F}_S : S \mapsto \mathbb{R}$. Letting r be a scalar and f,g functions with domain S, the vector operations are defined as

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (rf)(x) &= r(f(x))\end{aligned}$$

Show that $\mathbb{F}_S$ is a vector space.

Solution. To verify that $\mathbb{F}_S$ is a vector space, we must prove the axioms beginning with those for addition. Let f,g,h be functions in $\mathbb{F}_S$.

- First we check that addition is closed. For functions f,g defined on the set S, their sum given by

$$(f+g)(x)=f(x)+g(x)$$

  is again a function defined on S. Hence this sum is in $\mathbb{F}_S$ and $\mathbb{F}_S$ is closed under addition.

- Secondly, we check the commutative law of addition:

$$(f+g)(x)=f(x)+g(x)=g(x)+f(x)=(g+f)(x)$$

  Since x is arbitrary, $f+g=g+f$.

- Next we check the associative law of addition:

$$((f+g)+h)(x) = (f+g)(x)+h(x) = (f(x)+g(x))+h(x)$$

$$= f(x)+(g(x)+h(x)) = (f(x)+(g+h)(x)) = (f+(g+h))(x)$$

  and so $(f+g)+h = f+(g+h)$.

- Next we check for an additive identity. Let 0 denote the function which is given by $0(x) = 0$. Then this is an additive identity because

$$(f+0)(x) = f(x)+0(x) = f(x)+0 = f(x) = f(x)$$

  and so $f+0 = f$.

- Finally, check for an additive inverse. Let $-f$ be the function which satisfies $(-f)(x) = -(f(x))$. Then

$$(f+(-f))(x) = f(x)+(-f)(x) = f(x)-(f(x)) = 0$$

  Hence $f+(-f) = 0$.

Now, check the axioms for scalar multiplication.

- We first need to check that $\mathbb{F}_S$ is closed under scalar multiplication. For a function $f(x)$ in $\mathbb{F}_S$ and real number r, the function $(rf)(x) = r(f(x))$ is again a function defined on the set S. Hence $r(f(x))$ is in $\mathbb{F}_S$ and $\mathbb{F}_S$ is closed under scalar multiplication.

- Fix scalars r and s. To check the first distributive property,

$$((r+s)f)(x) = (r+s)f(x) = rf(x)+sf(x) = (rf+sf)(x)$$

  and so $(r+s)f = rf+sf$.

- And for the second distributive property

$$(r(f+g))(x) = r(f+g)(x) = r(f(x)+g(x))$$

$$= rf(x)+rg(x) = (rf+rg)(x)$$

  and so $r(f+g) = rf+rg$.

- For the penultimate axiom, again let r and s be scalars. Then

$$((rs)f)(x) = (rs)f(x) = r(sf(x)) = (r(sf))(x)$$

  so $(rs)f = r(sf)$.

- Finally $(1f)(x) = 1f(x) = f(x)$ so $1f = f$.

It follows that V satisfies all the required axioms and is a vector space. ♠

Having defined what a vector space *is*, and having seen several examples of vector spaces, now we turn our attention to describing what we can say about vector spaces in general. Several useful properties follow logically from the axioms that define a vector space, For example, consider the following important theorem.

Theorem 9.7: Uniqueness

In any vector space V, the following are true:

1. *$\vec{0}$, the additive identity, is unique.*
2. *For any vector $\vec{u} \in V$, the additive inverse of $\vec{u}$, $-\vec{u}$, is unique.*
3. *$0\vec{u} = \vec{0}$ for all vectors $\vec{u}$.*
4. *$(-1)\vec{u} = -\vec{u}$ for all vectors $\vec{u}$.*

Proof.

1. When we say that the additive identity is unique, we mean that if two vectors acts like the additive identity, then they are equal. To prove this uniqueness, we will assume that $\vec{u}$ and $\vec{v}$ both act like the additive identity, then $\vec{u} = \vec{v}$.

   Since $\vec{v}$ is an additive identity, when we add it to $\vec{u}$, we should get $\vec{u}$. Thus,
   $$\vec{u} + \vec{v} = \vec{u}$$
   Since $\vec{u}$ is an additive identity, when we add it to $\vec{v}$, we should get $\vec{v}$:
   $$\vec{v} + \vec{u} = \vec{v}.$$
   So by the commutative property:
   $$\vec{u} = \vec{u} + \vec{v} = \vec{v} + \vec{u} = \vec{v}$$
   and so $\vec{u} = \vec{v}$, which is what was claimed.

   At this point we are justified in talking about *the* additive identity in the vector space V, and giving it a name, $\vec{0}$.

2. When we say that the additive inverse of $\vec{u}$ is unique, we mean that if $\vec{v}$ and $\vec{w}$ both act like additive inverses of $\vec{u}$, then $\vec{v} = \vec{w}$. So, assume that $\vec{v}$ and $\vec{w}$ both act like additive inverses of $\vec{u}$. We will argue that $\vec{v} = \vec{w}$.

   Since $\vec{v}$ is an additive inverse of $\vec{u}$:
   $$\vec{u} + \vec{v} = \vec{0}.$$
   As $\vec{w}$ is also an additive inverse of $\vec{u}$,
   $$\vec{w} + \vec{u} = \vec{0}.$$

Then the following holds:

$$\vec{v} = \vec{0} + \vec{v} = (\vec{w} + \vec{u}) + \vec{v} = \vec{w} + (\vec{u} + \vec{v}) = \vec{w} + \vec{0} = \vec{w}$$

Thus if $\vec{v} = \vec{w}$, as claimed.

At this point, for any vector $\vec{u}$, we are justified in talking about *the* additive inverse of $\vec{u}$ and giving it a name: $-\vec{u}$.

3. This statement claims that for all vectors $\vec{u}$, scalar multiplication by 0 equals the zero vector $\vec{0}$. Consider the following, using the fact that we can write $0 = 0 + 0$:

$$0\vec{u} = (0+0)\vec{u} = 0\vec{u} + 0\vec{u}$$

We use a small trick here: add $-0\vec{u}$ to both sides. This gives

$$\begin{array}{rcl} 0\vec{u} + (-0\vec{u}) & = & 0\vec{u} + 0\vec{u} + (-0\vec{u}) \\ \vec{0} & = & 0\vec{u} + 0 \\ \vec{0} & = & 0\vec{u} \end{array}$$

This proves that scalar multiplication of any vector by 0 results in the zero vector $\vec{0}$.

4. Finally, we wish to show that scalar multiplication of -1 and any vector $\vec{u}$ results in the additive inverse of that vector, $-\vec{u}$. Recall from 2. above that the additive inverse is unique. Consider the following:

$$\begin{array}{rcl} (-1)\vec{u} + \vec{u} & = & (-1)\vec{u} + 1\vec{u} \\ & = & (-1+1)\vec{u} \\ & = & 0\vec{u} \\ & = & \vec{0} \end{array}$$

By the uniqueness of the additive inverse shown earlier, any vector which acts like the additive inverse must be equal to the additive inverse. It follows that $(-1)\vec{u} = -\vec{u}$.

♠

An important use of the additive inverse is the following theorem.

Theorem 9.8

Let V be a vector space. Then $\vec{v} + \vec{w} = \vec{v} + \vec{w}$ implies that $\vec{w} = \vec{w}$ for all $\vec{v}, \vec{w}, \vec{w} \in V$

The proof follows from the vector space axioms, in particular the existence of an additive inverse $(-\vec{v})$. The proof is left as an exercise to the reader.

9.2 Spanning Sets

Outcomes

A. *Given a vector space V and a set of vectors $S \subseteq V$, determine if S is a spanning set for V.*

Having defined what a vector space is in the previous section, we now want to investigate what we can say about them. Most of what we develop in the rest of the chapter will look very familiar, since we have been spending our time talking about $\mathbb{R}^n$, and (since $\mathbb{R}^n$ *is* a vector space) everything that we can say about vector spaces in general must be true about the vector space $\mathbb{R}^n$. So, for the rest of this chapter, you should expect lots of statements that say something like "If V is a vector space, then $\langle$*something*$\rangle$," and that *something* will be a statement or definition that echos a statement or definition from earlier in the book. So the ideas won't be surprising, but the fact that the ideas are applicable to a wide variety of different vector spaces is new and worthwhile.

In this section we will focus on the concept of the span of a set of vectors.

Consider the following definition.

Definition 9.9: Subset

Let X and Y be two sets. If all elements of X are also elements of Y then we say that X is a ***subset*** *of Y and we write*

$$X \subseteq Y$$

In particular, we often speak of subsets of a vector space, such as $X \subseteq V$. By this we mean that every element in the set X is an element of the vector space V.

Definition 9.10: Linear Combination

Let V be a vector space and let $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\} \subseteq V$. A vector $\vec{v} \in V$ is called a ***linear combination*** *of the $\vec{v}_i$ if there exist scalars $r_i \in \mathbb{R}$ such that*

$$\vec{v} = r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_n\vec{v}_n$$

This definition leads to define the span of a set of vectors.

Definition 9.11: Span of Vectors

Let $S = \{\vec{v}_1, \cdots, \vec{v}_n\} \subseteq V$. Then the ***span of*** *S is defined to be*

$$\mathrm{span}(S) = \mathrm{span}\{\vec{v}_1, \cdots, \vec{v}_n\} = \left\{ \sum_{i=1}^{n} r_i\vec{v}_i \mid r_i \in \mathbb{R} \right\}$$

When we say that a vector $\vec{v}$ is in $\mathrm{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$ we mean that $\vec{v}$ can be written as a linear combination of the $\vec{v}_i$. We say that a collection of vectors $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a **spanning set** for V if $V = \mathrm{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$.

Consider the following example.

Example 9.12: Matrix Span

Let $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Determine if A and B are elements of $\mathrm{span}\{M_1, M_2\}$.

Solution.

First consider A. We want to see if scalars r_1, r_2 can be found such that $A = r_1M_1 + r_2M_2$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = r_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The solution to this equation is given by

$$\begin{aligned} 1 &= r_1 \\ 2 &= r_2 \end{aligned}$$

and it follows that A is in $\mathrm{span}\{M_1, M_2\}$.

Now consider B. Again we write $B = r_1M_1 + r_2M_2$ and see if a solution can be found for r_1, r_2.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = r_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Clearly no values of r_1 and r_2 can be found such that this equation holds. Therefore B is not in $\mathrm{span}\{M_1, M_2\}$.

♠

Consider another example.

Example 9.13: Polynomial Span

Show that $p(x) = 7x^2 + 4x - 3$ is in span $\left\{4x^2 + x, x^2 - 2x + 3\right\}$.

Solution. To show that $p(x)$ is in the given span, we need to show that it can be written as a linear combination of polynomials in the span. Suppose scalars r_1, r_2 existed such that

$$7x^2 + 4x - 3 = r_1(4x^2 + x) + r_2(x^2 - 2x + 3)$$

If this linear combination were to hold, the following would be true:

$$\begin{array}{rcl} 4r_1 + r_2 & = & 7 \\ r_1 - 2r_2 & = & 4 \\ 3r_2 & = & -3 \end{array}$$

You can verify that $r_1 = 2, r_2 = -1$ satisfies this system of equations. This means that we can write $p(x)$ as follows:

$$7x^2 + 4x - 3 = 2(4x^2 + x) - (x^2 - 2x + 3)$$

Hence $p(x)$ is in the given span. ♠

Consider the following example.

Example 9.14: Spanning Set

Let $S = \left\{x^2 + 1, x - 2, 2x^2 - x\right\}$. Show that S is a spanning set for $\mathbb{P}_2$, the set of all polynomials of degree at most 2.

Solution. Let $p(x) = ax^2 + bx + c$ be an arbitrary polynomial in $\mathbb{P}_2$. To show that S is a spanning set, it suffices to show that $p(x)$ can be written as a linear combination of the elements of S. In other words, we wish to find scalars r, s, t such that:

$$p(x) = ax^2 + bx + c = r(x^2 + 1) + s(x - 2) + t(2x^2 - x).$$

If a solution r, s, t can be found, then this shows that any such polynomial $p(x)$ can be written as a linear combination of the polynomials in S and thus S spans $\mathbb{P}_2$.

$$\begin{array}{rcl} ax^2 + bx + c & = & r(x^2 + 1) + s(x - 2) + t(2x^2 - x) \\ & = & rx^2 + r + sx - 2s + 2tx^2 - tx \\ & = & (r + 2t)x^2 + (s - t)x + (r - 2s) \end{array}$$

For this to be true, the following must hold:

$$a = r + 2t$$

$$\begin{aligned} b &= s - t \\ c &= r - 2s \end{aligned}$$

To check that a solution exists, set up the augmented matrix and row reduce:

$$\left[\begin{array}{rrr|r} 1 & 0 & 2 & a \\ 0 & 1 & -1 & b \\ 1 & -2 & 0 & c \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|c} 1 & 0 & 0 & \frac{1}{2}a + 2b + \frac{1}{2}c \\ 0 & 1 & 0 & \frac{1}{4}a - \frac{1}{4}c \\ 0 & 0 & 1 & \frac{1}{4}a - b - \frac{1}{4}c \end{array}\right]$$

Clearly a solution exists for any choice of a, b, c. Hence S is a spanning set for $\mathbb{P}_2$. ♠

9.3 Linear Independence

Outcomes

A. *Determine if a set of vectors is linearly independent.*

In this section, we will again explore concepts introduced earlier in terms of $\mathbb{R}^n$ and extend them to apply to abstract vector spaces.

Definition 9.15: Linear Independence

Let V be a vector space. We say that a set $\{\vec{v}_1, \cdots, \vec{v}_n\} \subseteq V$, is ***linearly independent*** *if*

$$\sum_{i=1}^{n} a_i \vec{v}_i = \vec{0} \text{ implies } a_1 = \cdots = a_n = 0$$

where the a_i are real numbers.

The set of vectors is called ***linearly dependent*** *if it is not linearly independent.*

We have already seen, for *any* set of vectors $\{\vec{v}_1, \cdots, \vec{v}_n\}$, that

$$0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_n = \vec{0}.$$

If our set is linearly independent, this is just saying that the *only* way a linear combination of the vectors can add up to the zero vector is if all of the coefficients are equal to 0.

Of course, we start with an example:

Example 9.16: Linear Independence

Let $S \subseteq \mathbb{P}_2$ be

$$S = \left\{x^2 + 2x - 1, 2x^2 - x + 3\right\}$$

Determine if S is linearly independent.

Solution. To determine if this set S is linearly independent, we assume that a linear combination of the vectors in S is equal to $\vec{0}$, and prove that all of the coefficients in the sum must be equal to 0. So assume that there are real numbers r and s such that

$$r(x^2 + 2x - 1) + s(2x^2 - x + 3) = \vec{0} = 0x^2 + 0x + 0$$

If S is linearly independent, then $r = s = 0$ will be the only solution. We proceed as follows.

$$\begin{array}{rcl} r(x^2 + 2x - 1) + s(2x^2 - x + 3) & = & 0x^2 + 0x + 0 \\ rx^2 + 2rx - r + 2sx^2 - sx + 3s & = & 0x^2 + 0x + 0 \\ (r + 2s)x^2 + (2r - s)x - r + 3s & = & 0x^2 + 0x + 0 \end{array}$$

It follows that

$$\begin{array}{rcl} r + 2s & = & 0 \\ 2r - s & = & 0 \\ -r + 3s & = & 0 \end{array}$$

The augmented matrix and resulting reduced row-echelon form are given by

$$\left[\begin{array}{rr|r} 1 & 2 & 0 \\ 2 & -1 & 0 \\ -1 & 3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence the only solution to our system of equations is $r = s = 0$ and thus the set S is linearly independent. ♠

The next example shows us what it means for a set to be dependent.

Example 9.17: Dependent Set

Determine if the set $S \subseteq \mathbb{R}^3$ given below is independent.

$$S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$$

Solution. To determine if S is linearly independent, we look for solutions to

$$r \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Notice that this equation has nontrivial solutions, for example $r = 2$, $s = 3$ and $t = -1$. Therefore S is linearly dependent. ♠

The following is an important result regarding linearly dependent sets.

Lemma 9.18: Dependent Sets

Let V be a vector space and suppose $W = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ is a subset of V. Then W is linearly dependent if and only if there is some $i \leq k$ such that $\vec{v}_i$ can be written as a linear combination of $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{i-1}, \vec{v}_{i+1}, \cdots, \vec{v}_k\}$.

Revisit Example 9.17 with this in mind. Notice that we can write one of the three vectors as a combination of the others.

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

By Lemma 9.18 this set is dependent.

If we know that one particular set is linearly independent, we can use this information to determine if a related set is linearly independent. Consider the following example.

Example 9.19: Related Independent Sets

Let V be a vector space and suppose $S \subseteq V$ is a set of linearly independent vectors given by $S = \{\vec{u}, \vec{v}, \vec{w}\}$. Let $R \subseteq V$ be given by $R = \left\{2\vec{u} - \vec{w}, \vec{w} + \vec{v}, 3\vec{v} + \frac{1}{2}\vec{u}\right\}$. Show that R is also linearly independent.

Solution. To determine if R is linearly independent, we write

$$r(2\vec{u}-\vec{w})+s(\vec{w}+\vec{v})+t\left(3\vec{v}+\frac{1}{2}\vec{u}\right)=\vec{0}$$

To show that$R$ is a linearly independent set, we must show that the only solution to this equation will be $r=s=t=0$. We proceed as follows.

$$\begin{aligned}
r(2\vec{u}-\vec{w})+s(\vec{w}+\vec{v})+t\left(3\vec{v}+\frac{1}{2}\vec{u}\right) &= \vec{0} \\
2r\vec{u}-a\vec{w}+s\vec{w}+s\vec{v}+3t\vec{v}+\frac{1}{2}t\vec{u} &= \vec{0} \\
\left(2r+\frac{1}{2}t\right)\vec{u}+(s+3t)\vec{v}+(-r+s)\vec{w} &= \vec{0}
\end{aligned}$$

We know that the set $S=\{\vec{u},\vec{v},\vec{w}\}$ is linearly independent. Since our last equation is a linear combination of the vectors in S which is equal to the zero vector, all of the coefficients in that equation, $\left(2r+\frac{1}{2}t\right)$, $(s+3t)$, and $(-r+s)$, must be equal to 0.

In other words:

$$\begin{aligned}
2r+\frac{1}{2}t &= 0 \\
s+3t &= 0 \\
-r+s &= 0
\end{aligned}$$

The augmented matrix and resulting reduced row-echelon form are given by:

$$\left[\begin{array}{rrr|r} 2 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 3 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Hence the only solution is $r=s=t=0$ and the set is linearly independent. ♠

We know that if a set of vectors U is linearly independent, then there is only one way to write $\vec{0}$ as a linear combination of the vectors in U: $0\vec{u}_1+0\vec{u}_2+\cdots+0\vec{u}_n$. This property of being uniquely representable by the vectors in a linearly independent U extends to every vector that is in the span of U. The following theorem in the setting of $\mathbb{R}^n$ was seen earlier as Theorem 4.81. The proof given there generalizes quite directly to prove this general statement:

Theorem 9.20: Unique Representation

Let V be a vector space and let $U=\{\vec{v}_1,\cdots,\vec{v}_k\}\subseteq V$ be an independent set. If $\vec{v}\in \operatorname{span} U$, then $\vec{v}$ can be written uniquely as a linear combination of the vectors in U.

Consider the span of a linearly independent set of vectors. Suppose we take a vector which is not in this span and add it to the set. The following lemma claims that the resulting set is still linearly independent. We will use this result to expand a linearly independent set of vectors to a larger set that is still linearly independent.

Lemma 9.21: Adding to a Linearly Independent Set

Suppose $\vec{v} \notin \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$ and $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent. Then the set

$$\{\vec{u}_1, \cdots, \vec{u}_k, \vec{v}\}$$

is also linearly independent.

Proof. Suppose $\sum_{i=1}^{k} c_i\vec{u}_i + d\vec{v} = \vec{0}$. It is required to verify that each $c_i = 0$ and that $d = 0$. But if $d \neq 0$, then you can solve for $\vec{v}$ as a linear combination of the vectors, $\{\vec{u}_1, \cdots, \vec{u}_k\}$,

$$\vec{v} = -\sum_{i=1}^{k} \left(\frac{c_i}{d}\right) \vec{u}_i$$

contrary to the assumption that $\vec{v}$ is not in the span of the $\vec{u}_i$. Therefore, $d = 0$. But then $\sum_{i=1}^{k} c_i\vec{u}_i = \vec{0}$ and the linear independence of $\{\vec{u}_1, \cdots, \vec{u}_k\}$ implies each $c_i = 0$ also. ♠

Consider the following example.

Example 9.22: Adding to a Linearly Independent Set

Let $S \subseteq \mathbb{M}_{2,2}$ be the linearly independent set

$$S = \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right] \right\}$$

Show that the set $R \subseteq \mathbb{M}_{2,2}$ given by

$$R = \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right] \right\}$$

is also linearly independent.

Solution. Instead of writing a linear combination of the matrices which equals 0 and showing that the coefficients must equal 0, we can instead use Lemma 9.21.

To do so, it suffices to show that

$$\left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right] \notin \text{span} \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right] \right\}$$

Write

$$\begin{aligned}
\left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right] &= a \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] + b \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right] \\
&= \left[\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right] + \left[\begin{array}{cc} 0 & b \\ 0 & 0 \end{array}\right] \\
&= \left[\begin{array}{cc} a & b \\ 0 & 0 \end{array}\right]
\end{aligned}$$

Clearly there are no possible a,b to make this equation true. Hence the new matrix does not lie in the span of the matrices in S. By Lemma 9.21, R is also linearly independent. ♠

9.4 Subspaces and Bases

Outcomes

A. *Utilize the subspace test to determine if a set is a subspace of a given vector space.*

B. *Extend a linearly independent set and shrink a spanning set to a basis of a given vector space.*

In this section we will examine the concept of subspaces introduced earlier in terms of $\mathbb{R}^n$. Here, we will discuss these concepts in terms of abstract vector spaces.

Consider the definition of a subspace.

Definition 9.23: Subspace

Let V be a vector space. A nonempty subset $W \subseteq V$ is said to be a **subspace** *of V if $r\vec{u}+s\vec{v} \in W$ whenever $r,s \in \mathbb{R}$ and $\vec{u},\vec{v} \in W$.*

Take a moment to compare the definition above with Definition 4.84. Although not stated in the same terms, it is easy to see that the definition of a subspace of $\mathbb{R}^n$ is equivalent to the definition of a subspace of a vector space V given above. So everything you thought was a subspace is still a subspace, but our definition works in a more general setting, too. That is a pattern that will continue as we work through this chapter.

The span of a set of vectors as described in Definition 9.11 is an example of a subspace. The following fundamental result says that subspaces are subsets of a vector space which are themselves vector spaces.

Theorem 9.24: Subspaces are Vector Spaces

Let W be a nonempty collection of vectors in a vector space V. Then W is a subspace if and only if W satisfies the vector space axioms, using the same operations as those defined on V.

Proof. Suppose first that W is a subspace. It is obvious that all the algebraic laws hold on W because W is a subset of V and the algebraic laws hold on V. Thus $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ along with the other axioms. Does W contain $\vec{0}$? Yes because it contains $0\vec{u}=\vec{0}$. See Theorem 9.7.

Is W closed under the operations of vector addition and scalar multiplication? That is, when you add vectors of W do you get a vector in W? When you multiply a vector in W by a scalar, do you get a vector in W? Yes. This is contained in the definition of what it means for W to be a subspace. Does every vector in W have an additive inverse that is an element of W? Yes by Theorem 9.7 because $-\vec{v}=(-1)\vec{v}$ which is given to be an element of W provided $\vec{v}\in W$.

Next suppose W is a vector space. Then by definition, it is closed with respect to linear combinations. Hence it is a subspace. ♠

Consider the following useful Corollary.

Corollary 9.25: Span is a Subspace

Let V be a vector space $S=\{\vec{v}_1,\cdots,\vec{v}_n\}\subseteq V$. If $W=span(S)$ then W is a subspace of V.

When determining spanning sets the following theorem proves useful.

Theorem 9.26: Spanning Set

Let V be a vector space, let U be a subspace of V, and let $S=\{\vec{v}_1,\cdots,\vec{v}_n\}\subseteq V$. If $S\subseteq U$, then $span(S)\subseteq U$.

In other words, this theorem claims that any subspace that contains a set of vectors must also contain the span of these vectors.

The following example will show that two spans, described differently, can in fact be equal.

Example 9.27: Equal Span

Let $S=\{p(x),q(x)\}\subseteq\mathbb{P}_n$ be polynomials and suppose $U=span\{2p(x)-q(x),p(x)+3q(x)\}$ and $W=span(S)$. Show that $U=W$.

Solution. We will use Theorem 9.26 to show that $U\subseteq W$ and $W\subseteq U$. It will then follow that $U=W$.

1. $U\subseteq W$

   Notice that $2p(x)-q(x)$ and $p(x)+3q(x)$ are both elements of $W=\text{span}(S)$. Since $\text{span}(S)$ is a subspace of $\mathbb{P}_n$, by Theorem 9.26 W is a superset of the span of these polynomials and so $U\subseteq W$.

2. $W \subseteq U$

   Notice that

$$\begin{aligned} p(x) &= \frac{3}{7}(2p(x)-q(x))+\frac{1}{7}(p(x)+3q(x)) \\ q(x) &= -\frac{1}{7}(2p(x)-q(x))+\frac{2}{7}(p(x)+3q(x)) \end{aligned}$$

   Hence $p(x), q(x)$ are elements of span $\{2p(x)-q(x), p(x)+3q(x)\}$. By Theorem 9.26, U must contain the span of these polynomials and so $W \subseteq U$.

♠

To prove that a set is a vector space, one must verify each of the axioms given in Definition 9.1. This may be a cumbersome task, and so here is a shorter procedure to verify a set of vectors is a subspace of a vector space V:

Procedure 9.28: Subspace Test

Suppose W is a subset of a vector space V. To determine if W is a subspace of V, it is sufficient to determine if the following three conditions hold, using the operations of V:

1. *The additive identity $\vec{0}$ of V is an element of W.*
2. *For any vectors $\vec{w}_1, \vec{w}_2 \in W$, the vector $\vec{w}_1 + \vec{w}_2$ is also an element of W.*
3. *For any vector $\vec{w} \in W$ and any scalar r, the product $r\vec{w}$ is also an element of W.*

If a set $W \subseteq V$ satisfies these three conditions, then W is nonempty by (1) and conditions (2) and (3) guarantee that W satisfies the requirements of Definition 9.23. Similarly, if W is a subspace and satisfies Definition 9.23, then W immediately satisfies conditions (2) and (3) above. The fact that $\vec{0} \in W$ follows from the fact that W is nonempty. If $\vec{w} \in W$, then by (3) $0\vec{w} \in W$ and by Theorem 9.7 $0\vec{w} = \vec{0} \in W$, so W satisfies (1). Therefore to check if some $W \subseteq V$ is a subspace of the vector space V, it suffices to check these three conditions.

Consider the following example.

Example 9.29: Improper Subspaces

Let V be an arbitrary vector space. Then V is a subspace of itself. Similarly, the set $\left\{\vec{0}\right\}$ containing only the zero vector is also a subspace.

*These two subspaces described above are called **improper subspaces**. of V. Any subspace of a vector space V which is not equal to V or $\left\{\vec{0}\right\}$ is called a **proper subspace**.*

*The subspace $\left\{\vec{0}\right\}$ is called the **zero subspace**.*

Solution. Using the subspace test in Procedure 9.28 we can show that V and $\left\{\vec{0}\right\}$ are subspaces of V.

Since V satisfies the vector space axioms it also satisfies the three steps of the subspace test. Therefore V is a subspace.

Let us consider the set $\left\{\vec{0}\right\}$.

1. The vector $\vec{0}$ is clearly an element of $\left\{\vec{0}\right\}$, so the first condition is satisfied.

2. Let $\vec{w}_1, \vec{w}_2$ be elements of $\left\{\vec{0}\right\}$. Then $\vec{w}_1 = \vec{0}$ and $\vec{w}_2 = \vec{0}$ and so
$$\vec{w}_1 + \vec{w}_2 = \vec{0} + \vec{0} = \vec{0}$$
It follows that the sum is an element of $\left\{\vec{0}\right\}$ and the second condition is satisfied.

3. Let $\vec{w}_1$ bean element of $\left\{\vec{0}\right\}$ and let r be an arbitrary scalar. Then
$$r\vec{w}_1 = r\vec{0} = \vec{0}$$
Hence the product is an element of $\left\{\vec{0}\right\}$ and the third condition is satisfied.

It follows that $\left\{\vec{0}\right\}$ is a subspace of V. ♠

Consider another example.

Example 9.30: Subspace of Polynomials

Let $\mathbb{P}_2$ be the vector space of polynomials of degree two or less. Let $W \subseteq \mathbb{P}_2$ be the set of all polynomials of degree two or less which have 1 as a root. Show that W is a subspace of $\mathbb{P}_2$.

Solution. First, express W as follows:
$$W = \left\{p(x) = ax^2 + bx + c, a, b, c, \in \mathbb{R} \mid p(1) = 0\right\}$$

We need to show that W satisfies the three conditions of Procedure 9.28.

1. The zero polynomial of $\mathbb{P}_2$ is given by $0(x) = 0x^2 + 0x + 0 = 0$. Clearly $0(1) = 0$ so $0(x)$ is an element of W.

2. Let $p(x), q(x)$ be polynomials in W. It follows that $p(1) = 0$ and $q(1) = 0$. Now consider $r(x) = p(x) + q(x)$. Notice that
$$\begin{aligned} r(1) &= p(1) + q(1) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$
Therefore the sum $p(x) + q(x)$is also an element of W and the second condition is satisfied.

3. Let $p(x)$ be a polynomial in W and let r be a scalar. It follows that $p(1) = 0$. Consider the product $rp(x)$.

$$\begin{aligned} rp(1) &= r(0) \\ &= 0 \end{aligned}$$

Therefore the product is an element of W and the third condition is satisfied.

It follows that W is a subspace of $\mathbb{P}_2$. ♠

Recall the definition of basis of a subspace of $\mathbb{R}^n$, Definition 4.89. Now we consider this concept in the context of general vector spaces.

Definition 9.31: Basis

*Let V be a vector space. Then a set of vectors $B = \{\vec{v}_1, \cdots, \vec{v}_n\}$ is called a **basis** for V if $\text{span}(B) = V$, and B is a linearly independent set of vectors.*

The plural of basis is bases, which is pronounced base-ees. (If we pronounced it like "bases" we'd never be able to tell if we were talking about one basis or many bases.)

Consider the following example.

Example 9.32: Polynomials of Degree at Most Two

Let $\mathbb{P}_2$ be the set polynomials of degree no more than 2. Is $\left\{x^2, x, 1\right\}$ a basis for $\mathbb{P}_2$?

Solution. We know that $\mathbb{P}_2$ is a vector space defined under the usual addition and scalar multiplication of polynomials.

Now, since clearly $\mathbb{P}_2 = \text{span}\left\{x^2, x, 1\right\}$, the set $\left\{x^2, x, 1\right\}$ is a basis for $\mathbb{P}$ if it is a linearly independent set. Suppose then that

$$ax^2 + bx + c = 0x^2 + 0x + 0$$

where a, b, c are real numbers. This means that $ax^2 + bx + c = 0$ for all real numbers x. But as a nonzero quadratic polynomial has no more than two roots, It is clear that this can only occur if $a = b = c = 0$. Hence the set $\left\{x^2, x, 1\right\}$ is linearly independent and forms a basis of $\mathbb{P}_2$. ♠

We have seen, in some sense, that a linearly independent set of vectors is large enough to get the job done, but no larger. For example, if $L = \{\vec{u}_1, \cdots, \vec{u}_r\}$ is linearly independent and $\vec{v} \in \text{span}(L)$, then we can write $\vec{v}$ as a linear combination of the $\vec{u}_i$'s, and we can do it in only one way. The next theorem, the Exchange Theorem, says that a linearly independent spanning set is a minimal spanning set. No set with fewer vectors than the linearly independent set can span the same subspace. This is an essential result and a key to understanding the structure of finite dimensional vector spaces. The proof is rather technical, so either give it a pass on a first reading, or grab a cup of coffee and some paper and prepare to work through the details. But really, everything just hinges on the fact that scalar addition is commutative.

Theorem 9.33: Exchange Theorem

Let $L = \{\vec{u}_1, \cdots, \vec{u}_r\}$ be a linearly independent set of vectors and let $S = \{\vec{v}_1, \cdots, \vec{v}_s\}$, with both L and S being subsets of a vector space V. If $L \subseteq \text{span}(S)$, then $r \leq s$.

In particular, if $\text{span}(L) = \text{span}(S)$, then $r \leq s$. So for a given subspace, a linearly independent spanning set is going to be of the smallest possible size.

Proof. The proof of this theorem is exactly the same as the proof of the Exchange Theorem in $\mathbb{R}^n$, Theorem 4.91. We reproduce it, with a couple of additional comments, here.

Assume that L and S are as described in the statement of the theorem, and assume that $L \subseteq \text{span}(S)$. We must show that r, the number of vectors in the linearly independent set L, is less than or equal to s, the number of vectors in the spanning set S.

Suppose, by way of contradiction, that $s < r$.

Since each vector $\vec{u}_j \in L$ is an element of $\text{span}\{\vec{v}_1, \cdots, \vec{v}_s\}$, there exist scalars a_{ij} such that

$$\vec{u}_j = \sum_{i=1}^{s} a_{ij}\vec{v}_i, \ j = 1, 2, \ldots, r.$$

As we have assumed that $s < r$, the matrix $A = \left[a_{ij}\right]$ has fewer rows, s, than columns, r. Then the homogeneous system of linear equations $A\vec{x} = 0$ has, as we saw back in Chapter 1, a non trivial solution $\vec{d}$. So there is a vector $\vec{d} \in \mathbb{R}^r$ with $\vec{d} \neq \vec{0}$ such that $A\vec{d} = \vec{0}$. In other words,

$$\sum_{j=1}^{r} a_{ij} d_j = 0, \ i = 1, 2, \cdots, s.$$

Now we use these scalars d_j to construct a linear combination of the vectors in L:

$$\begin{aligned} \sum_{j=1}^{r} d_j \vec{u}_j &= \sum_{j=1}^{r} d_j \sum_{i=1}^{s} a_{ij}\vec{v}_i \\ &= \sum_{i=1}^{s} \left(\sum_{j=1}^{r} a_{ij} d_j \right) \vec{v}_i = \sum_{i=1}^{s} 0\vec{v}_i = \vec{0}. \end{aligned}$$

But this contradicts the assumption that $L = \{\vec{u}_1, \cdots, \vec{u}_r\}$ is linearly independent, because not all the d_j are zero.

Our assumption that $s < r$ led to a contradiction, so we conclude that $r \leq s$, as needed.

♠

The following corollary follows from the Exchange Theorem.

Corollary 9.34: All Finite Bases of V are of the Same Size

Let B_1, B_2 be two bases of a vector space V. Suppose B_1 contains m vectors and B_2 contains n vectors. Then $m = n$.

Proof. Notice that $\text{span}(B_1) = \text{span}(B_2)$. Since B_1 is linearly independent and has the same span as B_2, by Theorem 9.33, $m \leq n$. As B_2 is linearly independent and has the same span as B_1, $n \leq m$. Therefore $m = n$. ♠

Given the result of the previous corollary, we know that if a vector space V has a finite basis, then every basis of V has exactly the same number of vectors. Thus we get to define the dimension of such a vector space.

Definition 9.35: Dimension

A vector space V is of ***dimension*** *n if it has a basis consisting of n vectors. Such a vector space is said to be* ***finite dimensional****.*

Not every vector space is finite dimensional; $\mathbb{P}$, the collection of all polynomials, is an example of an infinite dimensional vector space. But our discussion for now will concentrate on finite dimensional vector spaces.

Example 9.36: Dimension of a Vector Space

Let $\mathbb{P}_2$ be the set of all polynomials of degree at most 2. Find the dimension of $\mathbb{P}_2$.

Solution. If we can find a basis of $\mathbb{P}_2$ then the number of vectors in the basis will give the dimension. Recall from Example 9.32 that a basis of $\mathbb{P}_2$ is given by

$$S = \left\{x^2, x, 1\right\}$$

There are three polynomials in S and hence the dimension of $\mathbb{P}_2$ is three. ♠

It is important to note that a basis for a vector space is not unique. A vector space can have many bases. Consider the following example.

Example 9.37: A Different Basis for Polynomials of Degree Two

Let $\mathbb{P}_2$ be the polynomials of degree no more than 2. Is $\left\{x^2 + x + 1, 2x + 1, 3x^2 + 1\right\}$ a basis for $\mathbb{P}_2$?

Solution. Suppose these vectors are linearly independent but do not form a spanning set for $\mathbb{P}_2$. Then by Lemma 9.21, we could find a fourth polynomial in $\mathbb{P}_2$ to create a new linearly independent set containing four polynomials. However this would imply that we could find a basis of $\mathbb{P}_2$ of more than three polynomials. This contradicts the result of Example 9.36 in which we determined the dimension of $\mathbb{P}_2$ is three. Therefore if these vectors are linearly independent they must also form a spanning set and thus a basis for $\mathbb{P}_2$.

Suppose then that

$$\begin{aligned} r\left(x^2 + x + 1\right) + s\left(2x + 1\right) + t\left(3x^2 + 1\right) &= 0 \\ (r + 3t)x^2 + (r + 2s)x + (r + s + t) &= 0 \end{aligned}$$

We know that $\{x^2, x, 1\}$ is linearly independent, and so it follows that

$$\begin{aligned} r+3t &= 0 \\ r+2s &= 0 \\ r+s+t &= 0 \end{aligned}$$

and there is only one solution to this system of equations, $r = s = t = 0$. Therefore, these vectors are linearly independent and form a basis for $\mathbb{P}_2$. ♠

Consider the following theorem.

Theorem 9.38: Every Subspace has a Basis

Let W be a nonzero subspace of a finite dimensional vector space V. Suppose V has dimension n. Then W has a basis with no more than n vectors.

Proof. Let $\vec{v}_1 \in V$ where $\vec{v}_1 \neq 0$. If $\text{span}\{\vec{v}_1\} = V$, then it follows that $\{\vec{v}_1\}$ is a basis for V. Otherwise, there exists $\vec{v}_2 \in V$ which is not an element of $\text{span}\{\vec{v}_1\}$. By Lemma 9.21 $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set of vectors. Then $\{\vec{v}_1, \vec{v}_2\}$ is a basis for V and we are done. If $\text{span}\{\vec{v}_1, \vec{v}_2\} \neq V$, then there exists $\vec{v}_3 \notin \text{span}\{\vec{v}_1, \vec{v}_2\}$ and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a larger linearly independent set of vectors. Continuing this way, the process must stop before $n+1$ steps because if not, it would be possible to obtain $n+1$ linearly independent vectors contrary to the Exchange Theorem, Theorem 9.33. ♠

If in fact W is an n-dimensional subspace of an n-dimensional vector space V, then $W = V$.

Theorem 9.39: Subspace of Same Dimension

Let V be a vector space of dimension n and let W be a subspace. Then $W = V$ if and only if the dimension of W is also n.

Proof. First suppose $W = V$. Then obviously the dimension of $W = n$.

Now suppose that the dimension of W is n. Let a basis for W be $\{\vec{w}_1, \cdots, \vec{w}_n\}$. If W is not equal to V, then let $\vec{v}$ be a vector of V which is not an element of W. Thus $\vec{v}$ is not an element of $\text{span}\{\vec{w}_1, \cdots, \vec{w}_n\}$ and by Lemma 9.73, $\{\vec{w}_1, \cdots, \vec{w}_n, \vec{v}\}$ is linearly independent which contradicts Theorem 9.33 because it would be an independent set of $n+1$ vectors even though each of these vectors is in a spanning set of n vectors, a basis of V. ♠

Consider the following example.

Example 9.40: Basis of a Subspace

Let $U = \left\{ A \in \mathbb{M}_{22} \;\middle|\; A \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} A \right\}$. Then U is a subspace of $\mathbb{M}_{22}$. Find a basis of U, and hence $\dim(U)$.

Solution. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_{22}$. Then

$$A \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a+b & -b \\ c+d & -d \end{bmatrix}$$

and
$$\left[\begin{array}{rr} 1 & 1 \\ 0 & -1 \end{array}\right] A = \left[\begin{array}{rr} 1 & 1 \\ 0 & -1 \end{array}\right]\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = \left[\begin{array}{cc} a+c & b+d \\ -c & -d \end{array}\right].$$

If $A \in U$, then $\left[\begin{array}{cc} a+b & -b \\ c+d & -d \end{array}\right] = \left[\begin{array}{cc} a+c & b+d \\ -c & -d \end{array}\right]$.

Equating entries leads to a system of four equations in the four variables a, b, c and d.

$$\begin{array}{rcl} a+b & = & a+c \\ -b & = & b+d \\ c+d & = & -c \\ -d & = & -d \end{array} \quad \text{or} \quad \begin{array}{rcl} b-c & = & 0 \\ -2b-d & = & 0 \\ 2c+d & = & 0 \end{array}.$$

The solution to this system is $a = s$, $b = -\frac{1}{2}t$, $c = -\frac{1}{2}t$, $d = t$ for any $s, t \in \mathbb{R}$, and thus

$$A = \left[\begin{array}{cc} s & \frac{t}{2} \\ -\frac{t}{2} & t \end{array}\right] = s\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] + t\left[\begin{array}{cc} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{array}\right].$$

Let
$$B = \left\{\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{array}\right]\right\}.$$

Then $\text{span}(B) = U$, and it is routine to verify that B is an independent subset of $\mathbb{M}_{22}$. Therefore B is a basis of U, and $\dim(U) = 2$. ♠

The following theorem claims that a spanning set of a vector space V can be shrunk down to a basis of V. Similarly, a linearly independent set within V can be enlarged to create a basis of V.

Theorem 9.41: Basis of V

If $V = \text{span}\{\vec{u}_1, \cdots, \vec{u}_n\}$ is a vector space, then some subset of $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is a basis for V. Also, if $\{\vec{u}_1, \cdots, \vec{u}_k\} \subseteq V$ is linearly independent and the vector space is finite dimensional, then the set $\{\vec{u}_1, \cdots, \vec{u}_k\}$ can be enlarged to obtain a basis of V.

Proof. Let
$$S = \{E \subseteq \{\vec{u}_1, \cdots, \vec{u}_n\} \mid \text{span}\{E\} = V\}.$$

For $E \in S$, let $|E|$ denote the number of elements of E. Let
$$m = \min\{|E| \text{ such that } E \in S\}.$$

Thus there exist vectors
$$\{\vec{v}_1, \cdots, \vec{v}_m\} \subseteq \{\vec{u}_1, \cdots, \vec{u}_n\}$$
such that
$$\text{span}\{\vec{v}_1, \cdots, \vec{v}_m\} = V$$
and m is as small as possible for this to happen. If this set is linearly independent, it follows it is a basis for V and the theorem is proved. On the other hand, if the set is not linearly independent, then there exist scalars, $c_1, \cdots, c_m$ such that
$$\vec{0} = \sum_{i=1}^{m} c_i \vec{v}_i$$

and not all the c_i are equal to zero. Suppose $c_k \neq 0$. Then solve for the vector $\vec{v}_k$ in terms of the other vectors, say $\vec{v}_k = \sum_{i \neq k} r_i \vec{v}_i$. Then we can show that

$$V = \text{span}\left\{\vec{v}_1, \cdots, \vec{v}_{k-1}, \vec{v}_{k+1}, \cdots, \vec{v}_m\right\}.$$

For if $\vec{v} \in V$, then there exist scalars s_i such that

$$\vec{v} = \sum_{i=1}^{n} s_i \vec{v}_i = \sum_{i \neq k} s_i \vec{v}_i + s_k \vec{v}_k = \sum_{i \neq k} s_i \vec{v}_i + s_k \left(\sum_{i \neq k} r_i \vec{v}_i \right).$$

This contradicts the definition of m as the size of the smallest spanning set and proves the first part of the theorem.

To obtain the second part, begin with $\{\vec{u}_1, \cdots, \vec{u}_k\}$ and suppose a basis for V is

$$\{\vec{v}_1, \cdots, \vec{v}_m\}$$

If

$$\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\} = V,$$

then $k = m$. If not, there exists a vector

$$\vec{u}_{k+1} \notin \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$$

Then from Lemma 9.21, $\{\vec{u}_1, \cdots, \vec{u}_k, \vec{u}_{k+1}\}$ is also linearly independent. Continue adding vectors in this way until m linearly independent vectors have been obtained. Then

$$\text{span}\{\vec{u}_1, \cdots, \vec{u}_m\} = V$$

because if it did not do so, there would exist $\vec{u}_{m+1}$ as just described and $\{\vec{u}_1, \cdots, \vec{u}_{m+1}\}$ would be a linearly independent set of vectors having $m+1$ elements. This contradicts the fact that $\{\vec{v}_1, \cdots, \vec{v}_m\}$ is a basis. In turn this would contradict Theorem 9.33. Therefore, this list is a basis. ♠

Recall Example 9.22 in which we added a matrix to a linearly independent set to create a larger linearly independent set. By Theorem 9.41 we can extend a linearly independent set to a basis.

Example 9.42: Adding to a Linearly Independent Set

Let $S \subseteq \mathbb{M}_{22}$ be the linearly independent set given by

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Enlarge S to a basis of $\mathbb{M}_{22}$.

Solution. Recall from the solution of Example 9.22 that the set $R \subseteq \mathbb{M}_{22}$ given by

$$R = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

is also linearly independent. However this set is still not a basis for $\mathbb{M}_{22}$ as it is not a spanning set. In particular, $\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right]$ is not an element of $\text{span}(R)$. Therefore, this matrix can be added to R by Lemma 9.21 to obtain a new linearly independent set given by

$$T = \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right] \right\}$$

This set is linearly independent and now spans $\mathbb{M}_{22}$. Hence T is a basis, called the standard basis of $\mathbb{M}_{22}$. ♠

Next we consider the case where you have a spanning set and you want a subset which is a basis. The above discussion involved adding vectors to a set. The next example involves removing vectors.

Example 9.43: Basis from a Spanning Set

Let $V = \mathbb{P}_3$, the vector space of polynomials of degree no more than 3. Consider the following vectors in V:

$$\begin{aligned} \vec{v}_1 &= 2x^2 + x + 1 \\ \vec{v}_2 &= x^3 + 4x^2 + 2x + 2 \\ \vec{v}_3 &= 2x^3 + 2x^2 + 2x + 1 \\ \vec{v}_4 &= x^3 + 4x^2 - 3x + 2 \\ \vec{v}_5 &= x^3 + 3x^2 + 2x + 1 \end{aligned}$$

As $\{x^3, x^2, x, 1\}$ is a basis for $\mathbb{P}_3$, we know that V has dimension 4, so the set of vectors displayed above is not linearly independent. Determine a linearly independent subset of these vectors that has the same span. Determine whether this subset is a basis for V.

Solution.

We will build a maximal linearly independent subset of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ by repeatedly using Lemma 9.21. We will start with the linearly independent set of vectors $\{\vec{v}_1\}$ and just check, one by one, whether we can add subsequent vectors to our linearly independent set and maintain our linear independence.

- Is $\{\vec{v}_1, \vec{v}_2\}$ linearly independent? By Lemma 9.21, the answer to this question is "yes" if and only if $\vec{v}_2$ is not an element of $\text{span}\{\vec{v}_1\}$. Since $\vec{v}_2$ is not a multiple of $\vec{v}_1$ (look at the x^3 term), we know that $\vec{v}_2 \notin \text{span}\{\vec{v}_1\}$, and so $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.

- Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent? We must check whether $\vec{v}_3$ is an element of $\text{span}\{\vec{v}_1, \vec{v}_2\}$. Suppose it is, so suppose $\vec{v}_3$ can be written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$. This means that there are scalars a and b such that $\vec{v}_3 = a\vec{v}_1 + b\vec{v}_2$. But then b must equal 2 (from the x^3 term) and $a = -3$ (from the x^2 term, given that $b = 2$). But these choices of a and b don't work (look at the x term). Thus $\vec{v}_3 \notin \text{span}\{\vec{v}_1, \vec{v}_2\}$, and so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

- Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ linearly independent? So we want to see if $\vec{v}_4$ is an element of $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. This means that we seek scalars a, b, and c such that $\vec{v}_4 = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$. By equating the coefficients on each term, that means that we are looking for a solution to this set of equations:

$$\begin{aligned} b+2c &= 1 \\ 2a+4b+2c &= 4 \\ a+2b+2c &= -3 \\ a+2b+\ \ c &= 2 \end{aligned}.$$

Looking at the augmented matrix

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 2 & 4 & 2 & 4 \\ 1 & 2 & 2 & -3 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

we find its reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and so $\vec{v}_4 = -15\vec{v}_1 + 11\vec{v}_2 - 5\vec{v}_3$. Since we can write $\vec{v}_4$ as a linear combination of the previous three vectors, adding it to our set would ruin its linear independence and not increase the span, so we will have to leave $\vec{v}_4$ out.

- Is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_5\}$ linearly independent? Now we have to see if $\vec{v}_5$ is an element of $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Using the same procedure as in the previous step, we set up the linear equations to write $\vec{v}_5$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. The augmented matrix we obtain is

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 2 & 4 & 2 & 3 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so there is no solution to our system of equations and thus $\vec{v}_5$ is not a linear combination of the vectors up to this point. So we can add $\vec{v}_5$ to our linearly independent set, yielding a maximal linearly independent subset: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_5\}$, and the span of this subset is the same as the span of the collection of five vectors with which we began.

Since our set of four linearly independent vectors spans a four-dimensional subspace of the four-dimensional vector space $\mathbb{P}_3$, we must have $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_5\} = \mathbb{P}_3$ by Theorem 9.39, and so we have built a basis for V.

♠

Consider the following example.

Example 9.44: Shrinking a Spanning Set

Consider the set $S \subseteq \mathbb{P}_2$ given by

$$S = \left\{1, x, x^2, x^2 + 1\right\}$$

Show that S spans $\mathbb{P}_2$, then remove vectors from S until a basis is created.

Solution. First we need to show that S spans $\mathbb{P}_2$. Let $ax^2 + bx + c$ be an arbitrary polynomial in $\mathbb{P}_2$. Write

$$ax^2 + bx + c = r(1) + s(x) + t(x^2) + u(x^2 + 1)$$

Then,

$$\begin{aligned} ax^2 + bx + c &= r(1) + s(x) + t(x^2) + u(x^2 + 1) \\ &= (t + u)x^2 + s(x) + (r + u) \end{aligned}$$

It follows that

$$\begin{aligned} a &= t + u \\ b &= s \\ c &= r + u \end{aligned}$$

Clearly a solution exists for all a, b, c and so S is a spanning set for $\mathbb{P}_2$. By Theorem 9.41, some subset of S is a basis for $\mathbb{P}_2$.

Recall that a basis must be both a spanning set and a linearly independent set. Therefore we must remove a vector from S keeping this in mind. Suppose we remove x from S. The resulting set would be $\left\{1, x^2, x^2 + 1\right\}$. This set is clearly linearly dependent (and also does not span $\mathbb{P}_2$) and so is not a basis.

Suppose we remove $x^2 + 1$ from S. The resulting set is $\left\{1, x, x^2\right\}$ which is both linearly independent and spans $\mathbb{P}_2$. Hence this is a basis for $\mathbb{P}_2$. Note that removing any one of $1, x^2$, or $x^2 + 1$ will result in a basis. ♠

Now the following is a fundamental result about subspaces.

Theorem 9.45: Basis from a Linearly Independent Set

Let V be a finite dimensional vector space and let W be a non-zero subspace of V. Suppose that $L = \{\vec{w}_1, \cdots, \vec{w}_s\}$ is a linearly independent subset of W. Then L can be extended to a basis of W.

Proof. Since L is a linearly independent subset of W and W is finite dimensional, this is an immediate corollary of Theorem 9.41. ♠

This also proves the following corollary. Let V play the role of W in the above theorem and begin with a basis for W, enlarging it to form a basis for V as discussed above.

Corollary 9.46: Basis Extension

Let W be any non-zero subspace of a finite dimensional vector space V. Then every basis of W can be extended to a basis for V.

Consider the following example.

Example 9.47: Basis Extension

Let $V = \mathbb{R}^4$ and let

$$W = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Extend this basis of W to a basis of V.

Solution. An easy way to do this is to take the reduced row-echelon form of the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{9.2}$$

Note how the given vectors were placed as the first two columns in the matrix and then the matrix was extended by adding the standard basis vectors of $\mathbb{R}^4$. So it is clear that the span of the columns of this matrix yield all of $\mathbb{R}^4$. Now determine the pivot columns. The reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} \tag{9.3}$$

The pivot columns are

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and now this is an extension of the given basis for W to a basis for $\mathbb{R}^4$.

Why does this work? The columns of 9.2 obviously span $\mathbb{R}^4$, so the column space of the matrix is equal to $\mathbb{R}^4$. And the first four columns of the matrix are a basis for the column space, and hence they span $\mathbb{R}^4$. ♠

9.5 Sums and Intersections

Outcomes

A. *Show that the sum of two subspaces is a subspace.*

B. *Show that the intersection of two subspaces is a subspace.*

We begin this section with a definition.

Definition 9.48: Sum and Intersection

Let V be a vector space, and let U and W be subspaces of V. Then

1. $U+W=\{\vec{u}+\vec{w} \mid \vec{u}\in U \text{ and } \vec{w}\in W\}$ *and is called the* ***sum*** *of U and W.*
2. $U\cap W=\{\vec{v} \mid \vec{v}\in U \text{ and } \vec{v}\in W\}$ *and is called the* ***intersection*** *of U and W.*

Therefore the intersection of two subspaces is all the vectors shared by both. If there are no nonzero vectors shared by both subspaces, meaning that $U\cap W=\left\{\vec{0}\right\}$, the sum $U+W$ takes on a special name.

Definition 9.49: Direct Sum

Let V be a vector space and suppose U and W are subspaces of V such that $U\cap W=\left\{\vec{0}\right\}$. Then the sum of U and W is called the ***direct sum*** *and is denoted $U\oplus W$.*

An interesting result is that both the sum $U+W$ and the intersection $U\cap W$ are subspaces of V.

Proposition 9.50: Intersection is a Subspace

Let V be a vector space and suppose U and W are subspaces. Then both $U+W$ and $U\cap W$ are subspaces of V.

Proof. We will show that $U\cap W$ is a subspace of V. The proof that $U+W$ is also a subspace of V is similar and is left as an exercise.

To establish that $U\cap W$ is a subspace of V using the subspace test, we must show three things:

1. $\vec{0}\in U\cap W$
2. For vectors $\vec{v}_1,\vec{v}_2\in U\cap W, \vec{v}_1+\vec{v}_2\in U\cap W$
3. For scalar a and vector $\vec{v}\in U\cap W, a\vec{v}\in U\cap W$

We proceed to show each of these three conditions hold.

1. Since U and W are subspaces of V, they each contain $\vec{0}$. By definition of the intersection, $\vec{0}\in U\cap W$.
2. Let $\vec{v}_1,\vec{v}_2\in U\cap W$. Then in particular, $\vec{v}_1,\vec{v}_2\in U$. Since U is a subspace, it follows that $\vec{v}_1+\vec{v}_2\in U$. The same argument holds for W. Therefore $\vec{v}_1+\vec{v}_2$ is in both U and W and by definition is also in $U\cap W$.
3. Let a be a scalar and $\vec{v}\in U\cap W$. Then in particular, $\vec{v}\in U$. Since U is a subspace, it follows that $a\vec{v}\in U$. The same argument holds for W so $a\vec{v}$ is in both U and W. By definition, it is in $U\cap W$.

Therefore $U\cap W$ is a subspace of V. ♠

The following theorem relates the dimensions of the various subspaces that we have been discussing.

Theorem 9.51: Dimension of Sum

Let V be a vector space with subspaces U and W. Suppose U and W each have finite dimension. Then $U+W$ also has finite dimension which is given by

$$dim(U+W)=dim(U)+dim(W)-dim(U\cap W)$$

Notice that when $U\cap W=\left\{\vec{0}\right\}$, the sum becomes the direct sum and the above equation becomes

$$\dim(U\oplus W)=\dim(U)+\dim(W)$$

9.6 Linear Transformations

Outcomes

A. *Understand the definition of a linear transformation in the context of vector spaces.*

Recall that a function is simply a transformation of a vector to result in a new vector. Consider the following definition.

Definition 9.52: Linear Transformation

Let V and W be vector spaces. Suppose $T : V \to W$ is a function, where for each $\vec{v} \in V, T(\vec{v}) \in W$. Then T is a ***linear transformation*** *if whenever r, s are scalars and $\vec{v}_1$ and $\vec{v}_2$ are vectors in V*

$$T(r\vec{v}_1 + s\vec{v}_2) = rT(\vec{v}_1) + sT(\vec{v}_2)$$

Several important examples of linear transformations include the zero transformation, the identity transformation, and the scalar transformation.

Example 9.53: Linear Transformations

Let V and W be vector spaces.

1. ***The zero transformation***
   $0 : V \to W$ *is defined by* $0(\vec{v}) = \vec{0}$ *for all* $\vec{v} \in V$.
2. ***The identity transformation***
   $1_V : V \to V$ *is defined by* $1_V(\vec{v}) = \vec{v}$ *for all* $\vec{v} \in V$.
3. ***The scalar transformation*** *Let* $r \in \mathbb{R}$.
   $s_r : V \to V$ *is defined by* $s_r(\vec{v}) = r\vec{v}$ *for all* $\vec{v} \in V$.

Solution. We will show that the scalar transformation s_r is linear, the rest are left as an exercise.

By Definition 9.52 we must show that for all scalars k, p and vectors $\vec{v}_1$ and $\vec{v}_2$ in V, $s_r(k\vec{v}_1 + p\vec{v}_2) = ks_r(\vec{v}_1) + ps_r(\vec{v}_2)$.

$$\begin{aligned} s_r(k\vec{v}_1 + p\vec{v}_2) &= r(k\vec{v}_1 + p\vec{v}_2) \\ &= rk\vec{v}_1 + rp\vec{v}_2 \\ &= k(r\vec{v}_1) + p(r\vec{v}_2) \\ &= ks_r(\vec{v}_1) + ps_r(\vec{v}_2) \end{aligned}$$

Therefore s_r is a linear transformation. ♠

Consider the following important theorem.

Theorem 9.54: Properties of Linear Transformations

Let V and W be vector spaces, and $T : V \to W$ a linear transformation. Then

1. *T preserves the zero vector.*
$$T(\vec{0}) = \vec{0}$$

2. *T preserves additive inverses. For all $\vec{v} \in V$,*
$$T(-\vec{v}) = -T(\vec{v})$$

3. *T preserves linear combinations. For all $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in V$ and all $k_1, k_2, \ldots, k_m \in \mathbb{R}$,*
$$T(k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_m\vec{v}_m) = k_1T(\vec{v}_1) + k_2T(\vec{v}_2) + \cdots + k_mT(\vec{v}_m).$$

Proof.

1. Let $\vec{0}_V$ denote the zero vector of V and let $\vec{0}_W$ denote the zero vector of W. We want to prove that $T(\vec{0}_V) = \vec{0}_W$. Let $\vec{v} \in V$. Then $0\vec{v} = \vec{0}_V$ and
$$T(\vec{0}_V) = T(0\vec{v}) = 0T(\vec{v}) = \vec{0}_W.$$

2. Let $\vec{v} \in V$; then $-\vec{v} \in V$ is the additive inverse of $\vec{v}$, so $\vec{v} + (-\vec{v}) = \vec{0}_V$. Thus
$$\begin{array}{rcl} T(\vec{v} + (-\vec{v})) & = & T(\vec{0}_V) \\ T(\vec{v}) + T(-\vec{v})) & = & \vec{0}_W \\ T(-\vec{v}) & = & \vec{0}_W - T(\vec{v}) = -T(\vec{v}). \end{array}$$

3. This result follows from preservation of addition and preservation of scalar multiplication. A formal proof would be by induction on m.

♠

Consider the following example using the above theorem.

Example 9.55: Linear Combination

Let $T : \mathbb{P}_2 \to \mathbb{R}$ be a linear transformation such that
$$T(x^2 + x) = -1; T(x^2 - x) = 1; T(x^2 + 1) = 3.$$
Find $T(4x^2 + 5x - 3)$.

Solution. We provide two solutions to this problem.

Solution 1: Suppose $a(x^2 + x) + b(x^2 - x) + c(x^2 + 1) = 4x^2 + 5x - 3$. Then
$$(a + b + c)x^2 + (a - b)x + c = 4x^2 + 5x - 3.$$

Solving for a, b, and c results in the unique solution $a = 6$, $b = 1$, $c = -3$.

Thus

$$\begin{array}{rcl} T(4x^2+5x-3) & = & T\left(6(x^2+x)+(x^2-x)-3(x^2+1)\right) \\ & = & 6T(x^2+x)+T(x^2-x)-3T(x^2+1) \\ & = & 6(-1)+1-3(3) = -14. \end{array}$$

Solution 2: Notice that $S = \{x^2+x, x^2-x, x^2+1\}$ is a basis of $\mathbb{P}_2$, and thus x^2, x, and 1 can each be written as a linear combination of elements of S. In fact we have

$$\begin{array}{rcl} x^2 & = & \frac{1}{2}(x^2+x)+\frac{1}{2}(x^2-x) \\ x & = & \frac{1}{2}(x^2+x)-\frac{1}{2}(x^2-x) \\ 1 & = & (x^2+1)-\frac{1}{2}(x^2+x)-\frac{1}{2}(x^2-x). \end{array}$$

Then

$$\begin{array}{rcl} T(x^2) & = & T\left(\frac{1}{2}(x^2+x)+\frac{1}{2}(x^2-x)\right) = \frac{1}{2}T(x^2+x)+\frac{1}{2}T(x^2-x) \\ & = & \frac{1}{2}(-1)+\frac{1}{2}(1) = 0. \\ T(x) & = & T\left(\frac{1}{2}(x^2+x)-\frac{1}{2}(x^2-x)\right) = \frac{1}{2}T(x^2+x)-\frac{1}{2}T(x^2-x) \\ & = & \frac{1}{2}(-1)-\frac{1}{2}(1) = -1. \\ T(1) & = & T\left((x^2+1)-\frac{1}{2}(x^2+x)-\frac{1}{2}(x^2-x)\right) \\ & = & T(x^2+1)-\frac{1}{2}T(x^2+x)-\frac{1}{2}T(x^2-x) \\ & = & 3-\frac{1}{2}(-1)-\frac{1}{2}(1) = 3. \end{array}$$

Therefore,

$$\begin{array}{rcl} T(4x^2+5x-3) & = & 4T(x^2)+5T(x)-3T(1) \\ & = & 4(0)+5(-1)-3(3) = -14. \end{array}$$

The advantage of **Solution 2** over **Solution 1** is that if you were now asked to find $T(-6x^2-13x+9)$, it is easy to use $T(x^2) = 0$, $T(x) = -1$ and $T(1) = 3$:

$$\begin{array}{rcl} T(-6x^2-13x+9) & = & -6T(x^2)-13T(x)+9T(1) \\ & = & -6(0)-13(-1)+9(3) = 13+27 = 40. \end{array}$$

More generally,

$$\begin{array}{rcl} T(ax^2+bx+c) & = & aT(x^2)+bT(x)+cT(1) \\ & = & a(0)+b(-1)+c(3) = -b+3c. \end{array}$$

♠

Suppose two linear transformations act in the same way on $\vec{v}$ for all vectors. Then we say that these transformations are equal.

Definition 9.56: Equal Transformations

Let S and T be linear transformations from V to W. Then we say that S and T are equal, and write $S = T$, when for every $\vec{v} \in V$,

$$S(\vec{v}) = T(\vec{v})$$

The definition above requires that two transformations have the same action on every vector in order for them to be equal. The next theorem argues that it is only necessary to check the action of the transformations on basis vectors, or even just any spanning set of vectors.

Theorem 9.57: Transformation of a Spanning Set

Let V and W be vector spaces and suppose that S and T are linear transformations from V to W. Then in order for S and T to be equal, it suffices that $S(\vec{v}_i) = T(\vec{v}_i)$ where $V = span\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

This theorem tells us that a linear transformation is completely determined by its actions on a spanning set. We can also examine the effect of a linear transformation on a basis.

Theorem 9.58: Transformation of a Basis

Suppose V and W are vector spaces with V being n-dimensional. Let $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ be any vectors in W that may or may not be distinct. Then for any basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ of V there is a unique linear transformation $T : V \to W$ with $T(\vec{v}_i) = \vec{w}_i$.
Furthermore, if

$$\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n$$

is a vector of V, then

$$T(\vec{v}) = k_1\vec{w}_1 + k_2\vec{w}_2 + \cdots + k_n\vec{w}_n.$$

Engage Active Learning App!

Lyryx Engage is an active learning app designed to increase student engagement in the study of linear algebra. The content is "chunked" into small blocks, each with an interactive assessment activity to promote comprehension.

Visit us at https://lyryx.com to register and access Lyryx Engage!

Lyryx Online Exercises!

Lyryx offers formative online exercises! The educational software captures instructors' experience using algorithms to analyze and evaluate students' work, identifying common errors and providing personalized feedback. Students are encouraged to try these randomized problems repeatedly, learning from their mistakes until mastery.

Visit us at https://lyryx.com to register and access Lyryx exercises!

9.7 Isomorphisms

Outcomes

A. *Apply the concepts of one to one and onto to transformations of vector spaces.*

B. *Determine if a linear transformation of vector spaces is an isomorphism.*

C. *Determine if two vector spaces are isomorphic.*

One to One and Onto Transformations

Recall the following definitions, given here in terms of vector spaces.

Definition 9.59: One to One

Let V and W be vector spaces. Then a linear transformation $T : V \to W$ is called one to one if whenever $\vec{v}_1 \in V$ and $\vec{v}_2 \in V$ and $\vec{v}_1 \neq \vec{v}_2$, then

$$T(\vec{v}_1) \neq T(\vec{v}_2)$$

Definition 9.60: Onto

Let V and W be vector spaces. Then a linear transformation $T : V \to W$ is called onto if for all $\vec{w} \in \vec{W}$ there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

Recall that every linear transformation T has the property that $T(\vec{0}) = \vec{0}$. This will be necessary to prove the following useful lemma.

Lemma 9.61: One to One

The assertion that a linear transformation T is one to one is equivalent to saying that if $T(\vec{v}) = \vec{0}$, then $\vec{v} = 0$.

Proof. Suppose first that T is one to one. We already know that $T(\vec{0}) = \vec{0}$. If $T(\vec{v}) = \vec{0}$, then the fact that T is one to one lets us conclude that $\vec{v} = \vec{0}$, as needed.

To prove the converse, suppose that if $T(\vec{v}) = \vec{0}$, then $\vec{v} = 0$. We must show that T is one to one. So assume that $T(\vec{v}) = T(\vec{u})$. Then $T(\vec{v}) - T(\vec{u}) = T(\vec{v} - \vec{u}) = \vec{0}$ which shows that $\vec{v} - \vec{u} = 0$ or in other words, $\vec{v} = \vec{u}$. ♠

Consider the following example.

Example 9.62: One to One Transformation

Let $S : \mathbb{P}_2 \to \mathbb{M}_{2,2}$ be a linear transformation defined by

$$S(ax^2+bx+c) = \left[\begin{array}{cc} a+b & a+c \\ b-c & b+c \end{array}\right] \text{ for all } ax^2+bx+c \in \mathbb{P}_2.$$

Prove that S is one to one but not onto.

Solution.

Suppose $p(x) = ax^2+bx+c$ and $S(p(x)) = \left[\begin{array}{cc} a+b & a+c \\ b-c & b+c \end{array}\right] = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$. This leads to a homogeneous system of four equations in three variables. Putting the augmented matrix in reduced row-echelon form:

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The solution is $a = b = c = 0$. This tells us that if $S(p(x)) = 0$, then $p(x) = ax^2+bx+c = 0x^2+0x+0 = 0$. Therefore it is one to one.

To show that S is **not** onto, we must show that there is a matrix $A \in \mathbb{M}_{2,2}$ such that for every $p(x) \in \mathbb{P}_2$, $S(p(x)) \neq A$. The easiest way to show that such a matrix exists is to exhibit one, so consider

$$A = \left[\begin{array}{cc} 0 & 1 \\ 0 & 2 \end{array}\right],$$

and suppose $p(x) = ax^2+bx+c \in \mathbb{P}_2$ is such that $S(p(x)) = A$. Then

$$\begin{array}{ll} a+b=0 & a+c=1 \\ b-c=0 & b+c=2 \end{array}$$

Solving this system

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{array}\right].$$

Since the system is inconsistent, there is no $p(x) \in \mathbb{P}_2$ so that $S(p(x)) = A$, and therefore S is not onto.

♠

Example 9.63: An Onto Transformation

Let $T : \mathbb{M}_{2,2} \to \mathbb{R}^2$ be a linear transformation defined by

$$T\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = \left[\begin{array}{c} a+d \\ b+c \end{array}\right] \text{ for all } \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \in \mathbb{M}_{2,2}.$$

Prove that T is onto but not one to one.

Solution. To show that T is onto, we will show that any vector in $\mathbb{R}^2$ is the image of some 2×2 matrix under the transformation T. To that end, let $\left[\begin{array}{c} x \\ y \end{array}\right]$ be an arbitrary vector in $\mathbb{R}^2$. Notice that $T\left[\begin{array}{cc} x & y \\ 0 & 0 \end{array}\right] = \left[\begin{array}{c} x \\ y \end{array}\right]$, so $\left[\begin{array}{c} x \\ y \end{array}\right]$ is the image of some matrix under the transforemation T, and hence T is onto.

By Lemma 9.61 T is one to one if and only if $T(A)=\vec{0}$ implies that $A=0$ the zero matrix. Observe that

$$T\left(\left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right]\right) = \left[\begin{array}{c} 1+-1 \\ 0+0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

There exists a nonzero matrix A such that $T(A)=\vec{0}$. It follows that T is not one to one. ♠

The following theorem demonstrates that a one to one transformation preserves linear independence.

Theorem 9.64: One to One and Independence

Let V and W be vector spaces and let $T:V\to W$ be a linear transformation. If T is one to one and $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\}$ is a linearly independent subset of V, then $\{T(\vec{v}_1),T(\vec{v}_2),\ldots,T(\vec{v}_k)\}$ is a linearly independent subset of W.

Proof. Let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. Suppose that

$$a_1T(\vec{v}_1)+a_2T(\vec{v}_2)+\cdots+a_kT(\vec{v}_k)=\vec{0}_W$$

for some $a_1,a_2,\ldots,a_k\in\mathbb{R}$. We seek to prove that $a_1=a_2=\cdots=a_k=0$. Since linear transformations preserve linear combinations (addition and scalar multiplication), our assumption tells us that

$$T(a_1\vec{v}_1+a_2\vec{v}_2+\cdots+a_k\vec{v}_k)=\vec{0}_W.$$

Now, since T is one to one, the only vector that T can map to $\vec{0}_W$ is $\vec{0}_V$, and thus

$$a_1\vec{v}_1+a_2\vec{v}_2+\cdots+a_k\vec{v}_k=\vec{0}_V.$$

However, $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\}$ is independent so $a_1=a_2=\cdots=a_k=0$, as needed. Thus by the definition of linear independence, $\{T(\vec{v}_1),T(\vec{v}_2),\ldots,T(\vec{v}_k)\}$ is linearly independent. ♠

A similar claim can be made regarding onto transformations. In this case, an onto transformation preserves a spanning set.

Theorem 9.65: Onto and Spanning

Let V and W be vector spaces and let $T:V\to W$ be a linear transformation. If T is onto and $V=span\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\}$, then

$$W=span\{T(\vec{v}_1),T(\vec{v}_2),\ldots,T(\vec{v}_k)\}.$$

Proof. Suppose that T is onto and let $\vec{w} \in W$. Then there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. Since $V = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, there exist $a_1, a_2, \ldots a_k \in \mathbb{R}$ such that $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$. Using the fact that T is a linear transformation,

$$\begin{aligned}\vec{w} = T(\vec{v}) &= T(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k) \\ &= a_1T(\vec{v}_1) + a_2T(\vec{v}_2) + \cdots + a_kT(\vec{v}_k),\end{aligned}$$

i.e., $\vec{w} \in \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$, and thus

$$W \subseteq \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}.$$

Since $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k) \in W$, it follows from that $\text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\} \subseteq W$, and therefore $W = \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$. ♠

Isomorphisms

The focus of this section is on linear transformations which are both one to one and onto. When this is the case, we call the transformation an isomorphism.

Definition 9.66: Isomorphism

*Let V and W be two vector spaces and let $T : V \to W$ be a linear transformation. Then T is called an **isomorphism** if T is both one to one and onto.*

Definition 9.67: Isomorphic

*Let V and W be two vector spaces. We say that V and W are **isomorphic.** if there is an isomorphism $T : V \to W$.*

Consider the following example of an isomorphism.

Example 9.68: Isomorphism

Let $T : \mathbb{M}_{2,2} \to \mathbb{R}^4$ be defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_{2,2}.$$

Show that T is an isomorphism.

Solution. Notice that if we can prove T is an isomorphism, it will mean that $\mathbb{M}_{2,2}$ and $\mathbb{R}^4$ are isomorphic. It remains to prove that

1. T is a linear transformation;
2. T is one-to-one;
3. T is onto.

T **is linear:** Let k, p be scalars.

$$\begin{aligned}
T\left(k\left[\begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array}\right]+p\left[\begin{array}{cc} a_2 & b_2 \\ c_2 & d_2 \end{array}\right]\right) &= T\left(\left[\begin{array}{cc} ka_1 & kb_1 \\ kc_1 & kd_1 \end{array}\right]+\left[\begin{array}{cc} pa_2 & pb_2 \\ pc_2 & pd_2 \end{array}\right]\right) \\
&= T\left(\left[\begin{array}{cc} ka_1+pa_2 & kb_1+pb_2 \\ kc_1+pc_2 & kd_1+pd_2 \end{array}\right]\right) \\
&= \left[\begin{array}{c} ka_1+pa_2 \\ kb_1+pb_2 \\ kc_1+pc_2 \\ kd_1+pd_2 \end{array}\right] \\
&= \left[\begin{array}{c} ka_1 \\ kb_1 \\ kc_1 \\ kd_1 \end{array}\right]+\left[\begin{array}{c} pa_2 \\ pb_2 \\ pc_2 \\ pd_2 \end{array}\right] \\
&= k\left[\begin{array}{c} a_1 \\ b_1 \\ c_1 \\ d_1 \end{array}\right]+p\left[\begin{array}{c} a_2 \\ b_2 \\ c_2 \\ d_2 \end{array}\right] \\
&= kT\left(\left[\begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array}\right]\right)+pT\left(\left[\begin{array}{cc} a_2 & b_2 \\ c_2 & d_2 \end{array}\right]\right)
\end{aligned}$$

Therefore T is linear.

T **is one-to-one:** By Lemma 9.61 we need to show that if $T(A)=0$ then $A=0$ for some matrix $A \in \mathbb{M}_{2,2}$.

$$T\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]=\left[\begin{array}{c} a \\ b \\ c \\ d \end{array}\right]=\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]$$

This clearly only occurs when $a=b=c=d=0$ which means that

$$A=\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]=\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]=0$$

Hence T is one-to-one.

T **is onto:** Let

$$\vec{x}=\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right] \in \mathbb{R}^4,$$

and define matrix $A \in \mathbb{M}_{2,2}$ as follows:

$$A = \left[\begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array} \right].$$

Then $T(A) = \vec{x}$, and therefore T is onto.

Since T is a linear transformation which is one-to-one and onto, T is an isomorphism. Hence $\mathbb{M}_{2,2}$ and $\mathbb{R}^4$ are isomorphic. ♠

An important property of isomorphisms is that the inverse of an isomorphism is itself an isomorphism and the composition of isomorphisms is an isomorphism. We begin with inverses.

If $T : V \to W$ is an isomorphism, then we can define the inverse function T^{-1} as follows:

Definition 9.69: Inverse Function

Suppose that $T : V \to W$ is an isomorphism from the vector space V to the vector space W. Then we define the ***inverse*** *of T, denoted T^{-1}, as the function*

$$\begin{aligned} T^{-1} &: W \to V \\ \vec{w} &\mapsto \text{the unique vector } \vec{v} \in V \text{ such that } T(\vec{v}) = \vec{w}. \end{aligned}$$

Noice that if $T : V \to W$ is an isomorphism, for any $\vec{w} \in W$, the $\vec{v} \in V$ required by the definition of T^{-1} exists because T is onto and is unique as T is one to one.

Proposition 9.70: The Inverse of an Isomorphism is an Isomorphism

Suppose $T : V \to W$ is an isomorphism. Then $T^{-1} : W \to V$ is also an isomorphism.

Proof. We are given that T is an isomorphism, and we must show that T^{-1} is an isomorphism. So we must show that T^{-1} is a linear transformation that is one to one and onto. We discuss each in turn:

- $T^{-1} : W \to V$ *is a linear transformation:* Let $\vec{w}_1$ and $\vec{w}_2$ be vectors in W, and let a and b be scalars. We must show that $T^{-1}(a\vec{w}_1 + b\vec{w}_2) = aT^{-1}(\vec{w}_1) + bT^{-1}(\vec{w}_2)$.

  Since $\vec{w}_1$ and $\vec{w}_2$ are each elements of W and the linear transformation T is onto, we know that there are vectors $\vec{v}_1$ and $\vec{v}_2$, each elements of V, such that $T(\vec{v}_1) = \vec{w}_1$ and $T(\vec{v}_2) = \vec{w}_2$. This means that $T^{-1}(\vec{w}_1) = \vec{v}_1$ and $T^{-1}(\vec{w}_2) = \vec{v}_2$. Substituting, we now will be finished if we can show

  $$T^{-1}(aT(\vec{v}_1) + bT(\vec{v}_2)) = a\vec{v}_1 + b\vec{v}_2.$$

  This isn't hard to see. All we need to do is show that when we apply the function T to the stuff on the right of the equals sign, we get the stuff inside the parentheses on the left of the equals sign. So we must show

  $$T(a\vec{v}_1 + b\vec{v}_2) = aT(\vec{v}_1) + bT(\vec{v}_2).$$

  But since we know that T is a linear transformation, this equation is known to be true, and we are finished. So we know that T^{-1} is a linear transformation.

- T^{-1} *is one to one:* Suppose that $T^{-1}(\vec{w}_1) = T^{-1}(\vec{w}_2) = \vec{v}$. We must show that $\vec{w}_1 = \vec{w}_2$. By the definition of T^{-1}, we know that $T(\vec{v}) = \vec{w}_1$ and $T(\vec{v}) = \vec{w}_2$. But as T is a function, we can conclude that $\vec{w}_1 = \vec{w}_2$, as needed

- T^{-1} *is onto:* Fix $\vec{v} \in V$. We must show there is some $\vec{w} \in W$ such that $T^{-1}(\vec{w}) = \vec{v}$. Consider $\vec{w} = T(\vec{v})$. Then
$$T^{-1}(\vec{w}) = T^{-1}(T(\vec{v})) = \vec{v},$$
and we have found the needed vector $\vec{w}$, so we can conclude that T^{-1} is onto.

♠

A quick aside: Give yourself extra points if you noticed that we only used the fact that T is a linear transformation in the first part of the above proof. In fact, the inverse of *any* one to one and onto function is a one to one and onto function, whether the function in question is a linear transformation or not.

Now a reminder of how we define the composition of functions:

Definition 9.71: Composition of Transformations

Let V, W, and Z be vector spaces and suppose $T : V \to W$ and $S : W \to Z$ are linear transformations. Then the composition of S and T is the function
$$S \circ T : V \to Z$$
defined by
$$(S \circ T)(\vec{v}) = S(T(\vec{v})) \text{ for all } \vec{v} \in V$$

We start with linear transformations $T : V \to W$ and $S : W \to Z$. Then we define a new *function* $S \circ T : V \to Z$. There is no *a priori* reason to think that this new function is nice in any way, but it turns out that it is not only nice, it very nice. An isomorphism, in fact:

Proposition 9.72: The Composition of Isomorphisms is an Isomorphism

If $T : V \to W$ is an isomorphism and $S : W \to Z$ is an isomorphism for the vector spaces V, W, and Z, then $S \circ T$ defined by $(S \circ T)(v) = S(T(v))$ is also an isomorphism.

Proof.

Suppose that T and S are as described.

- *$S \circ T$ is a linear transformation:* Let a and b be scalars, and consider
$$\begin{aligned} S \circ T(a\vec{v}_1 + b\vec{v}_2) &\equiv S(T(a\vec{v}_1 + b\vec{v}_2)) = S(aT(\vec{v}_1) + bT(\vec{v}_2)) \\ &= aS(T(\vec{v}_1)) + bS(T(\vec{v}_2)) \equiv a(S \circ T)(\vec{v}_1) + b(S \circ T)(\vec{v}_2) \end{aligned}$$
Hence $S \circ T$ is a linear map.

- *$S \circ T$ is one to one:* If $(S \circ T)(\vec{v}) = 0$, then $S(T(\vec{v})) = \vec{0}$ and since S is a one to one linear transformation, it follows from Lemma 9.61 that $T(\vec{v}) = \vec{0}$ and hence by the lemma again, this time using the fact that T is one to one, $\vec{v} = \vec{0}$. Thus $S \circ T$ is one to one, once again using Lemma 9.61.

- *$S \circ T$ is onto:* To show that $S \circ T$ is onto, let $\vec{z} \in Z$. Then since S is onto, there exists $\vec{w} \in W$ such that $S(\vec{w}) = \vec{z}$. Also, since T is onto, there exists $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. It follows that $S(T(\vec{v})) = \vec{z}$ and so $S \circ T$ is also onto.

Having shown that the function $S \circ T$ is a one to one, onto, linear transformation, we can conclude that $S \circ T$ is an isomorphism, as claimed. ♠

Suppose we say that two vector spaces V and W are related if there exists an isomorphism of one to the other, written as $V \sim W$. Then the above propositions suggest that $\sim$ is an equivalence relation. That is: $\sim$ satisfies the following conditions:

- $V \sim V$
- If $V \sim W$, it follows that $W \sim V$
- If $V \sim W$ and $W \sim Z$, then $V \sim Z$

We leave the proof of these to the reader.

The following fundamental lemma describes the relation between bases and isomorphisms.

Lemma 9.73: Bases and Isomorphisms

Let V and W be finite dimensional vector spaces and let $T : V \to W$ be a linear transformation. T is an isomorphism if and only if whenever $B = \{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, it follows that $T(B) = \{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis for W.

Proof.

First, assume that T is an isomorphism and that $B = \{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V. We must show that $T(B) = \{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis for W.

Since T is one-to-one and B is linearly independent, Theorem 9.64 tell us that $T(B)$ is linearly independent. And as T is onto and B spans V, Theorem 9.65 guarantees that $T(B)$ spans W. So by definition, $T(B)$ is a basis for W and the transformation T preserves bases, as claimed.

For the converse, suppose that $T : V \to W$ preserves bases. We must show that T is an isomorphism. Since V is finite dimensional, there is a basis $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for V. As T preserves bases, we know that $T(B) = \{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)\}$ is a basis for W, and hence the dimension of W is no more than n.

To show that T is onto, fix $\vec{w} \in W$. We must find some $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. As $T(B)$ is a basis for W, we know that there are scalars r_i such that

$$r_1 T(\vec{v}_1) + r_2 T(\vec{v}_2) + \cdots + r_n T(\vec{v}_n) = \vec{w}.$$

But then, as T is linear,

$$T(r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_n\vec{v}_n) = \vec{w}.$$

and so $\vec{w}$ is the image of some vector in V, as needed.

We show that T is one to one with an argument by contradiction. Suppose that T is not one to one. Then there is a non-zero vector $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$. Extend the linearly independent set $\{\vec{v}\}$ to a basis B for V. By assumption, the image of B under the transformation T is a basis for W. But $T(B)$ includes the vector $T(\vec{v}) = \vec{0}$, and no set that includes the zero vector can be linearly independent. So $T(B)$ is not a basis for W, which is a contradiction. So we conclude that T must be an injection, as claimed.

♠

The next theorem characterizes exactly when two finite-dimensional vector spaces are isomorphic.

Theorem 9.74: Isomorphic Vector Spaces

Suppose V and W are two finite dimensional vector spaces. Then the two vector spaces are isomorphic if and only if they have the same dimension.
In the case that V and W have the same dimension, then for a linear transformation $T : V \to W$, the following are equivalent.

1. *T is one to one.*
2. *T is onto.*
3. *T is an isomorphism.*

Proof. Suppose first these two vector spaces have the same dimension. Let a basis for V be $\{\vec{v}_1, \cdots, \vec{v}_n\}$ and let a basis for W be $\{\vec{w}_1, \cdots, \vec{w}_n\}$. Now define T as follows.

$$T(\vec{v}_i) = \vec{w}_i$$

for $\sum_{i=1}^{n} c_i \vec{v}_i$ an arbitrary vector of V,

$$T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i T(\vec{v}_i) = \sum_{i=1}^{n} c_i \vec{w}_i.$$

It is necessary to verify that this is well defined. Suppose then that

$$\sum_{i=1}^{n} c_i \vec{v}_i = \sum_{i=1}^{n} \hat{c}_i \vec{v}_i$$

Then

$$\sum_{i=1}^{n} (c_i - \hat{c}_i) \vec{v}_i = 0$$

and since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis, $c_i = \hat{c}_i$ for each i. Hence

$$\sum_{i=1}^{n} c_i \vec{w}_i = \sum_{i=1}^{n} \hat{c}_i \vec{w}_i$$

and so the mapping is well defined. Also if a, b are scalars,

$$T\left(a \sum_{i=1}^{n} c_i \vec{v}_i + b \sum_{i=1}^{n} \hat{c}_i \vec{v}_i\right) \quad = \quad T\left(\sum_{i=1}^{n} (ac_i + b\hat{c}_i) \vec{v}_i\right) = \sum_{i=1}^{n} (ac_i + b\hat{c}_i) \vec{w}_i$$

$$\begin{aligned} &= a\sum_{i=1}^{n} c_i\vec{w}_i + b\sum_{i=1}^{n} \hat{c}_i\vec{w}_i \\ &= aT\left(\sum_{i=1}^{n} c_i\vec{v}_i\right) + bT\left(\sum_{i=1}^{n} \hat{c}_i\vec{v}_i\right) \end{aligned}$$

Thus T is a linear map.

Now if

$$T\left(\sum_{i=1}^{n} c_i\vec{v}_i\right) = \sum_{i=1}^{n} c_i\vec{w}_i = \vec{0},$$

then since the $\{\vec{w}_1, \cdots, \vec{w}_n\}$ are independent, each $c_i = 0$ and so $\sum_{i=1}^{n} c_i\vec{v}_i = \vec{0}$ also. Hence T is one to one. If $\sum_{i=1}^{n} c_i\vec{w}_i$ is a vector in W, then it equals

$$\sum_{i=1}^{n} c_i T\vec{v}_i = T\left(\sum_{i=1}^{n} c_i\vec{v}_i\right)$$

showing that T is also onto. Hence T is an isomorphism and so V and W are isomorphic.

Next suppose these two vector spaces are isomorphic. Let T be the name of the isomorphism. Then for $\{\vec{v}_1, \cdots, \vec{v}_n\}$ a basis for V, it follows that a basis for W is $\{T\vec{v}_1, \cdots, T\vec{v}_n\}$ showing that the two vector spaces have the same dimension.

Now suppose the two vector spaces have the same dimension.

First consider the claim that 1. $\Rightarrow$ 2. If T is one to one, then if $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V, then $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent. If it is not a basis, then it must fail to span W. But then there would exist $\vec{w} \notin \operatorname{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ and it follows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n), \vec{w}\}$ would be linearly independent which is impossible because there exists a basis for W of n vectors. Hence

$$\operatorname{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\} = W$$

and so $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis. Hence, if $\vec{w} \in W$, there exist scalars c_i such that

$$\vec{w} = \sum_{i=1}^{n} c_i T(\vec{v}_i) = T\left(\sum_{i=1}^{n} c_i\vec{v}_i\right)$$

showing that T is onto. This shows that 1. $\Rightarrow$ 2.

Next consider the claim that 2. $\Rightarrow$ 3. Since 2. holds, it follows that T is onto. It remains to verify that T is one to one, since then T will be both onto and one to one, i.e., an isomorphism. Since T is onto, there exists a basis of the form $\{T(\vec{v}_i), \cdots, T(\vec{v}_n)\}$. If $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is linearly independent, then this set of vectors must also be a basis for V because if not, there would exist $\vec{u} \notin \operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$ so $\{\vec{v}_1, \cdots, \vec{v}_n, \vec{u}\}$ would be a linearly independent set which is impossible because by assumption, there exists a basis which has n vectors. So why is$\{\vec{v}_1, \cdots, \vec{v}_n\}$ linearly independent? Suppose

$$\sum_{i=1}^{n} c_i\vec{v}_i = \vec{0}$$

Then

$$\sum_{i=1}^{n} c_i T\vec{v}_i = \vec{0}$$

Hence each $c_i = 0$ and so, as just discussed, $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis for V. Now it follows that a typical vector in V is of the form $\sum_{i=1}^n c_i \vec{v}_i$. If $T\left(\sum_{i=1}^n c_i \vec{v}_i\right) = \vec{0}$, it follows that

$$\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$$

and so, since $\{T(\vec{v}_i), \cdots, T(\vec{v}_n)\}$ is independent, it follows each $c_i = 0$ and hence $\sum_{i=1}^n c_i \vec{v}_i = \vec{0}$. Thus T is one to one as well as onto and so it is an isomorphism.

If T is an isomorphism, it is both one to one and onto by definition so 3. implies both 1. and 2. ♠

Note the interesting way of defining a linear transformation in the first part of the argument by describing what it does to a basis and then "extending it linearly".

Consider the following example.

Example 9.75

Show that $\mathbb{R}^3$ is isomorphic to $\mathbb{P}_2$.

Solution. First, observe that a basis for $\mathbb{P}_2$ is $\{1, x, x^2\}$ and a basis for $\mathbb{R}^3$ is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Since these two vector spaces have the same dimension, they are isomorphic. An example of an isomorphism is this:

$$T(\vec{e}_1) = 1, T(\vec{e}_2) = x, T(\vec{e}_3) = x^2$$

and extend T linearly as in the above proof. Thus

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = T(a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3) = aT(\vec{e}_1) + bT(\vec{e}_2) + cT(\vec{e}_3) = a + bx + cx^2.$$

♠

9.8 The Kernel And Image Of A Linear Map

Outcomes

A. *Describe the kernel and image of a linear transformation.*

B. *Use the kernel and image to determine if a linear transformation is one to one or onto.*

Here we consider the case where the linear map is not necessarily an isomorphism. First here is a definition of what is meant by the image and kernel of a linear transformation.

Definition 9.76: Kernel and Image

Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Then the **image** *of T denoted as $\operatorname{im}(T)$ is defined to be the set*

$$\operatorname{im}(T) = \{T(\vec{v}) \mid \vec{v} \in V\}.$$

In words, it consists of all vectors in W which equal $T(\vec{v})$ for some $\vec{v} \in V$.
The **kernel**, $\ker(T)$, *consists of all $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$. That is,*

$$\ker(T) = \left\{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\right\}.$$

Then in fact, both $\operatorname{im}(T)$ and $\ker(T)$ are subspaces of W and V respectively:

Proposition 9.77: Kernel and Image as Subspaces

Let V, W be vector spaces and let $T : V \to W$ be a linear transformation. Then $\ker(T)$ is a subspace of V and $\operatorname{im}(T)$ is a subspace of W.

Proof. Notice that $\ker(T) \subseteq V$ and $\operatorname{im}(T) \subseteq W$ by definition. To show that $\ker(T)$ is a subspace of V, It is necessary to show that if $\vec{v}_1, \vec{v}_2$ are vectors in $\ker(T)$ and if r, s are scalars, then $r\vec{v}_1 + s\vec{v}_2$ is also in $\ker(T)$. But

$$T(r\vec{v}_1 + s\vec{v}_2) = rT(\vec{v}_1) + sT(\vec{v}_2) = r\vec{0} + s\vec{0} = \vec{0}$$

Thus $\ker(T)$ is a subspace of V.

Next suppose $T(\vec{v}_1), T(\vec{v}_2)$ are two vectors in $\operatorname{im}(T)$. Again to show that $\operatorname{im}(T)$ is a subspace of W, we must show that $rT(\vec{v}_2) + sT(\vec{v}_2)$ is an element of $\operatorname{im}(T)$. But

$$rT(\vec{v}_2) + sT(\vec{v}_2) = T(r\vec{v}_1 + s\vec{v}_2)$$

as T is a linear transformation, and this last vector is in $\operatorname{im}(T)$ by definition, showing that $\operatorname{im}(T)$ is a subspace of W. ♠

Consider the following example.

Example 9.78: Kernel and Image of a Transformation

Let $T : \mathbb{P}_1 \to \mathbb{R}$ be the linear transformation defined by

$$T(p(x)) = p(1) \text{ for all } p(x) \in \mathbb{P}_1.$$

Find the kernel and image of T.

Solution. We will first find the kernel of T. It consists of all polynomials in $\mathbb{P}_1$ that have 1 for a root.

$$\begin{aligned} \ker(T) &= \{p(x) \in \mathbb{P}_1 \mid p(1) = 0\} \\ &= \{ax+b \mid a,b \in \mathbb{R} \text{ and } a+b=0\} \\ &= \{ax-a \mid a \in \mathbb{R}\} \end{aligned}$$

Therefore a basis for $\ker(T)$ is

$$\{x-1\}$$

Notice that $\ker(T)$ is a subspace of $\mathbb{P}_1$.

Now consider the image. It consists of all numbers which can be obtained by evaluating all polynomials in $\mathbb{P}_1$ at 1.

$$\begin{aligned} \operatorname{im}(T) &= \{p(1) \mid p(x) \in \mathbb{P}_1\} \\ &= \{a+b \mid ax+b \in \mathbb{P}_1\} \\ &= \{a+b \mid a,b \in \mathbb{R}\} \\ &= \mathbb{R} \end{aligned}$$

Therefore a basis for $\operatorname{im}(T)$ is

$$\{1\}$$

Notice that $\operatorname{im}(T)$ is a subspace of $\mathbb{R}$, and in fact is the space $\mathbb{R}$ itself. ♠

Example 9.79: Kernel and Image of a Linear Transformation

Let $T : \mathbb{M}_{2,2} \mapsto \mathbb{R}^2$ be defined by

$$T \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] = \left[\begin{array}{c} a-b \\ c+d \end{array} \right]$$

Then T is a linear transformation. Find a basis for $\ker(T)$ and $\operatorname{im}(T)$.

Solution. You can verify that T represents a linear transformation.

Now we want to find a way to describe all matrices A such that $T(A) = \vec{0}$, that is the matrices in $\ker(T)$. Suppose $A = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right]$ is such a matrix. Then

$$T \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] = \left[\begin{array}{c} a-b \\ c+d \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \end{array} \right]$$

The values of a,b,c,d that make this true are given by solutions to the system

$$\begin{array}{rcl} a-b & = & 0 \\ c+d & = & 0 \end{array}$$

The solution is $a=s, b=s, c=t, d=-t$ where s,t are scalars. We can describe $\ker(T)$ as follows.

$$\ker(T)=\left\{\left[\begin{array}{cc} s & s \\ t & -t \end{array}\right]\right\}=\operatorname{span}\left\{\left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right],\left[\begin{array}{cc} 0 & 0 \\ 1 & -1 \end{array}\right]\right\}$$

It is clear that this set is linearly independent and therefore forms a basis for $\ker(T)$.

We now wish to find a basis for $\operatorname{im}(T)$. We can write the image of T as

$$\operatorname{im}(T)=\left\{\left[\begin{array}{c} a-b \\ c+d \end{array}\right]\right\}$$

Notice that this can be written as

$$\operatorname{span}\left\{\left[\begin{array}{c} 1 \\ 0 \end{array}\right],\left[\begin{array}{c} -1 \\ 0 \end{array}\right],\left[\begin{array}{c} 0 \\ 1 \end{array}\right],\left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right\}$$

However this is clearly not linearly independent. By removing vectors from the set to create an independent set gives a basis of $\operatorname{im}(T)$.

$$\left\{\left[\begin{array}{c} 1 \\ 0 \end{array}\right],\left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right\}$$

Notice that these vectors have the same span as the set above but are now linearly independent and span $\operatorname{im}(T)$, which is equal to $\mathbb{R}^2$.

A quicker path to the question of finding a basis for $\operatorname{im}(T)$ would be to notice that $T\left(\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right]\right)=\left[\begin{array}{c} 1 \\ 0 \end{array}\right]$ and $T\left(\left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right]\right)=\left[\begin{array}{c} 0 \\ 1 \end{array}\right]$. This means that both $\left[\begin{array}{c} 1 \\ 0 \end{array}\right]$ and $\left[\begin{array}{c} 0 \\ 1 \end{array}\right]$ are elements of $\operatorname{im}(T)$. Since these two linearly independent vectors span $\mathbb{R}^2$, they show that $\operatorname{im}(T)=\mathbb{R}^2$ and form a basis for $\operatorname{im}(T)$. ♠

A major result is the relation between the dimension of the kernel and dimension of the image of a linear transformation. A special case was done earlier in the context of matrices. Recall that for an $m\times n$ matrix A, it was the case that the dimension of the kernel of A added to the rank of A equals n.

Theorem 9.80: Dimension of Kernel + Image

Let $T: V\to W$ be a linear transformation where V,W are vector spaces. Suppose the dimension of V is n. Then $n=\dim(\ker(T))+\dim(\operatorname{im}(T))$.

Proof. From Proposition 9.77, $\operatorname{im}(T)$ is a subspace of W. By Theorem 9.45, there exists a basis for $\operatorname{im}(T), \{T(\vec{v}_1),\cdots,T(\vec{v}_r)\}$. Similarly, there is a basis for $\ker(T), \{\vec{u}_1,\cdots,\vec{u}_s\}$. Then if $\vec{v}\in V$, there exist scalars c_i such that

$$T(\vec{v})=\sum_{i=1}^{r} c_i T(\vec{v}_i)$$

Hence $T\left(\vec{v}-\sum_{i=1}^r c_i\vec{v}_i\right)=0$. It follows that $\vec{v}-\sum_{i=1}^r c_i\vec{v}_i$ is in $\ker(T)$. Hence there are scalars a_i such that

$$\vec{v}-\sum_{i=1}^r c_i\vec{v}_i=\sum_{j=1}^s a_j\vec{u}_j$$

Hence $\vec{v}=\sum_{i=1}^r c_i\vec{v}_i+\sum_{j=1}^s a_j\vec{u}_j$. Since $\vec{v}$ is arbitrary, it follows that

$$V=\operatorname{span}\{\vec{u}_1,\cdots,\vec{u}_s,\vec{v}_1,\cdots,\vec{v}_r\}$$

If the vectors $\{\vec{u}_1,\cdots,\vec{u}_s,\vec{v}_1,\cdots,\vec{v}_r\}$ are linearly independent, then it will follow that this set is a basis. Suppose then that

$$\sum_{i=1}^r c_i\vec{v}_i+\sum_{j=1}^s a_j\vec{u}_j=0$$

Apply T to both sides to obtain

$$\sum_{i=1}^r c_iT(\vec{v}_i)+\sum_{j=1}^s a_jT(\vec{u}_j)=\sum_{i=1}^r c_iT(\vec{v}_i)=\vec{0}$$

Since $\{T(\vec{v}_1),\cdots,T(\vec{v}_r)\}$ is linearly independent, it follows that each $c_i=0$. Hence $\sum_{j=1}^s a_j\vec{u}_j=0$ and so, since the $\{\vec{u}_1,\cdots,\vec{u}_s\}$ are linearly independent, it follows that each $a_j=0$ also. It follows that $\{\vec{u}_1,\cdots,\vec{u}_s,\vec{v}_1,\cdots,\vec{v}_r\}$ is a basis for V and so

$$n=s+r=\dim(\ker(T))+\dim(\operatorname{im}(T))$$

♠

Consider the following definition.

Definition 9.81: Rank of Linear Transformation

Let $T:V\to W$ be a linear transformation and suppose V,W are finite dimensional vector spaces. Then the rank of T denoted as $\operatorname{rank}(T)$ is defined as the dimension of $\operatorname{im}(T)$. The nullity of T is the dimension of $\ker(T)$. Thus the above theorem says that $\operatorname{rank}(T)+\dim(\ker(T))=\dim(V)$.

Recall the following important result.

Theorem 9.82: Subspace of Same Dimension

Let V be a vector space of dimension n and let W be a subspace. Then $W=V$ if and only if the dimension of W is also n.

From this theorem follows the next corollary.

Corollary 9.83: One to One and Onto Characterization

Let $T:V\to W$ be a linear map where the dimension of V is n and the dimension of W is m. Then T is one to one if and only if $\ker(T)=\left\{\vec{0}\right\}$ and T is onto if and only if $\operatorname{rank}(T)=m$.

Proof. The statement $\ker(T) = \left\{\vec{0}\right\}$ is equivalent to saying if $T(\vec{v}) = \vec{0}$, it follows that $\vec{v} = \vec{0}$. Thus by Lemma 9.61 T is one to one. If T is onto, then $\mathrm{im}(T) = W$ and so $\mathrm{rank}(T)$ which is defined as the dimension of $\mathrm{im}(T)$ is m. If $\mathrm{rank}(T) = m$, then by Theorem 9.39, since $\mathrm{im}(T)$ is a subspace of W, it follows that $\mathrm{im}(T) = W$. ♠

Example 9.84: One to One Transformation

Let $S : \mathbb{P}_2 \to \mathbb{M}_{2,2}$ be a linear transformation defined by

$$S(ax^2+bx+c) = \left[\begin{array}{cc} a+b & a+c \\ b-c & b+c \end{array}\right] \text{ for all } ax^2+bx+c \in \mathbb{P}_2.$$

Prove that S is one to one but not onto.

Solution. You may recall this example from earlier—it is Example 9.62. Here we will determine that S is one to one, but not onto, using the method provided in Corollary 9.83.

By definition,

$$\ker(S) = \{ax^2+bx+c \in \mathbb{P}_2 \mid a+b=0, a+c=0, b-c=0, b+c=0\}.$$

Suppose $p(x) = ax^2+bx+c \in \ker(S)$. This leads to a homogeneous system of four equations in three variables. Putting the augmented matrix in reduced row-echelon form:

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since the unique solution is $a=b=c=0$, $\ker(S) = \{\vec{0}\}$, and thus S is one-to-one by Corollary 9.83.

Similarly, by Corollary 9.83, if S is onto it will have $\mathrm{rank}(S) = \dim(\mathbb{M}_{2,2}) = 4$. The image of S is given by

$$\mathrm{im}(S) = \left\{\left[\begin{array}{cc} a+b & a+c \\ b-c & b+c \end{array}\right]\right\} = \mathrm{span}\left\{\left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right], \left[\begin{array}{rr} 0 & 1 \\ -1 & 1 \end{array}\right]\right\}$$

These matrices are linearly independent which means this set forms a basis for $\mathrm{im}(S)$. Therefore the dimension of $\mathrm{im}(S)$, also called $\mathrm{rank}(S)$, is equal to 3. It follows that S is not onto. ♠

Engage Active Learning App!

Lyryx Engage is an active learning app designed to increase student engagement in the study of linear algebra. The content is "chunked" into small blocks, each with an interactive assessment activity to promote comprehension.

Visit us at https://lyryx.com to register and access Lyryx Engage!

Lyryx Online Exercises!

Lyryx offers formative online exercises! The educational software captures instructors' experience using algorithms to analyze and evaluate students' work, identifying common errors and providing personalized feedback. Students are encouraged to try these randomized problems repeatedly, learning from their mistakes until mastery.

Visit us at https://lyryx.com to register and access Lyryx exercises!

9.9 The Matrix of a Linear Transformation

Outcomes

A. *Find the matrix of a linear transformation with respect to general bases in vector spaces.*

You may recall from $\mathbb{R}^n$ that the matrix of a linear transformation depends on the bases chosen. This concept is explored in this section, where the linear transformation now maps from one arbitrary vector space to another.

Let $T : V \to W$ be an isomorphism where V and W are vector spaces. Recall from Lemma 9.73 that T maps a basis in V to a basis in W. When discussing this Lemma, we were not specific on what these bases looked like. In this section we will make such a distinction.

Consider now an important definition.

Definition 9.85: Coordinate Isomorphism

Let V be a vector space with $\dim(V) = n$, and let $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be an ordered basis of V (meaning that the order that the vectors are listed is taken into account). Let $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ denote the standard basis of $\mathbb{R}^n$. We define a function $C_B : V \to \mathbb{R}^n$ by

$$C_B(a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n) = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n = \left[\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_n \end{array}\right].$$

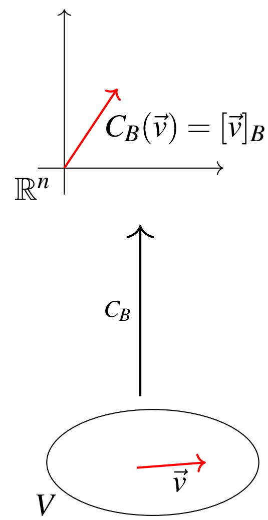
Then C_B is a linear transformation such that $C_B(\vec{b}_i) = \vec{e}_i$, $1 \leq i \leq n$.
C_B is an isomorphism, called the **coordinate isomorphism** *corresponding to B.*

We continue with another related definition.

Definition 9.86: Coordinate Vector

Let V be a finite dimensional vector space with $\dim(V) = n$, and let $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be an ordered basis of V. The ***coordinate vector of $\vec{v}$ with respect to*** *B is $C_B(\vec{v})$, which will also be denoted $[\vec{v}]_B$*

We have defined a function mapping vectors in V (at the bottom of the diagram below) to vectors in $\mathbb{R}^n$. The goal is to identify a random vector $\vec{v}$ in this random vector space with its coordinates, which is a familiar looking vector in $\mathbb{R}^n$, i.e. just an n-tuple of numbers. The notation is supposed to be reminiscent of the notation of Section 5.10, where we were discussing different bases for $\mathbb{R}^n$



This example should make it clear both how this function works, and its crucial dependence on the basis B that is chosen for the vector space V:

Example 9.87: Coordinate Vector

Let $V = \mathbb{P}_2$ and $\vec{v} = -x^2 - 2x + 4$. Find $[\vec{v}]_B$ for the following bases of $\mathbb{P}_2$:

1. $B_1 = \left\{1, x, x^2\right\}$
2. $B_2 = \left\{x^2, x, 1\right\}$
3. $B_3 = \left\{x + x^2, x, 4\right\}$

Solution.

1. First, note the order of the vectors in each basis is important. Now we need to find a_1, a_2, a_3 such that $\vec{v} = a_1(1) + a_2(x) + a_3(x^2)$, that is:
$$-x^2 - 2x + 4 = a_1(1) + a_2(x) + a_3(x^2)$$
Clearly the solution is
$$\begin{array}{rcl} a_1 & = & 4 \\ a_2 & = & -2 \\ a_3 & = & -1 \end{array}$$
Therefore the coordinate vector is
$$[\vec{v}]_{B_1} = \left[\begin{array}{r} 4 \\ -2 \\ -1 \end{array}\right],$$
and we have identified the polynomial $\vec{v} = -x^2 - 2x + 4$ with the coordinate vector $\left[\begin{array}{r} 4 \\ -2 \\ -1 \end{array}\right]$.

2. Again remember that the order of the vectors in the basis is important. We proceed as above. We need to find a_1, a_2, a_3 such that $\vec{v} = a_1(x^2) + a_2(x) + a_3(1)$, that is:

$$-x^2 - 2x + 4 = a_1(x^2) + a_2(x) + a_3(1)$$

Here the solution is

$$\begin{array}{rcl} a_1 & = & -1 \\ a_2 & = & -2 \\ a_3 & = & 4 \end{array}$$

Therefore the coordinate vector is

$$[\vec{v}]_{B_2} = \left[\begin{array}{r} -1 \\ -2 \\ 4 \end{array} \right].$$

This time, because the order of the vectors in B_2 is not the same as the order of the vectors in B_1, we have identified the polynomial $\vec{v} = -x^2 - 2x + 4$ with the coordinate vector $\left[\begin{array}{r} -1 \\ -2 \\ 4 \end{array} \right]$.

3. Now we need to find a_1, a_2, a_3 such that $\vec{v} = a_1(x + x^2) + a_2(x) + a_3(4)$, that is:

$$\begin{array}{rcl} -x^2 - 2x + 4 & = & a_1(x + x^2) + a_2(x) + a_3(4) \\ & = & a_1(x^2) + (a_1 + a_2)(x) + a_3(4) \end{array}$$

The solution is

$$\begin{array}{rcl} a_1 & = & -1 \\ a_2 & = & -1 \\ a_3 & = & 1 \end{array}$$

and the coordinate vector is

$$[\vec{v}]_{B_3} = \left[\begin{array}{r} -1 \\ -1 \\ 1 \end{array} \right],$$

identifying the same polynomial with an entirely different coordinate vector. Again, everything depends on the basis B with which you are working.

♠

Given that the coordinate transformation $C_B : V \to \mathbb{R}^n$ is an isomorphism, its inverse exists.

Theorem 9.88: Inverse of the Coordinate Isomorphism

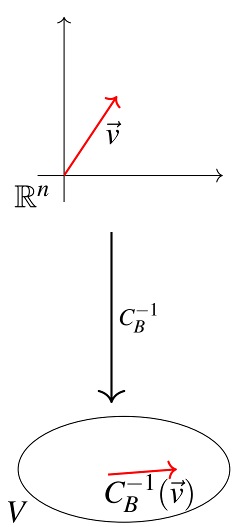
Let V be a finite dimensional vector space with dimension n and ordered basis $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$. Then $C_B : V \to \mathbb{R}^n$ is an isomorphism whose inverse,

$$C_B^{-1} : \mathbb{R}^n \to V$$

is given by

$$C_B^{-1}(\vec{v}) = a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n \text{ for all } \vec{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n.$$

This inverse of the coordinate isomorphism is actually easier to work with than the coordinate isomorphism itself. The picture looks like this, where again we have placed V at the bottom of the picture and $\mathbb{R}^n$ at the top, to match the diagram from a couple of pages back:



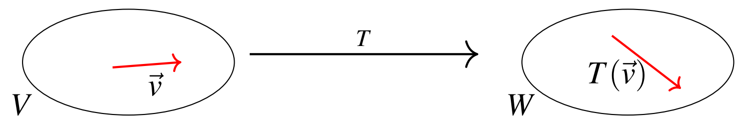
Suppose we are given, for the vector space V, the ordered basis $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$. Then if $\vec{v}$ is an element of $\mathbb{R}^3$ and $\vec{v} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$, the value of $C_B^{-1}\left(\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}\right)$ is simply $2\vec{b}_1 + 5\vec{b}_2 + 3\vec{b}_3$.

We now ready to discuss the main result of this section, which is how to represent a linear transformation from one arbitrary vector space to another with respect to different bases of the vector spaces.

Let V and W be finite dimensional vector spaces, and suppose

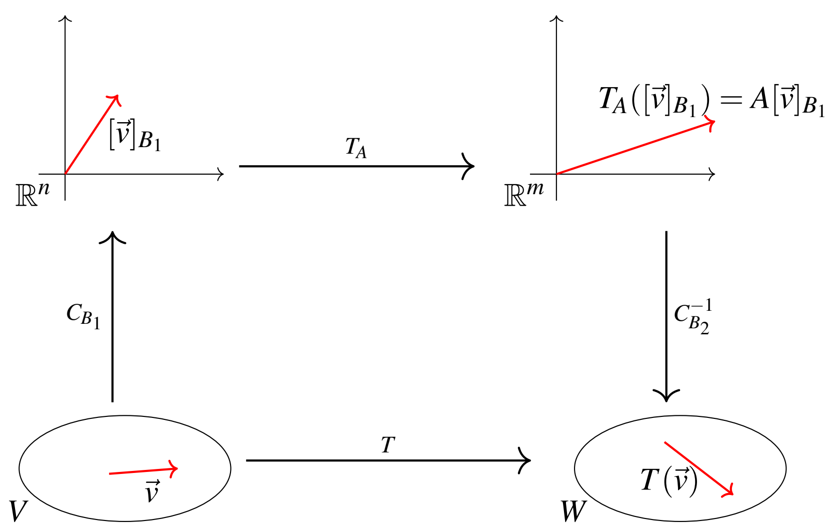
- $\dim(V) = n$ and $B_1 = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ is an ordered basis of V;
- $\dim(W) = m$ and $B_2 = \{\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_m\}$ is an ordered basis of W.

If $T : V \to W$ be a linear transformation, we have this picture:



The problem is that, given $\vec{v}$, it may be difficult to compute $T(\vec{v})$. But we can work around this by using the identification of T with $\mathbb{R}^n$ and the identification of W with $\mathbb{R}^m$ through their respective coordinate isomorphisms to find a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ that represents the linear transformation T. And we are experts at computing linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$; we just use multiplication by a particular matrix A.

Here is the way to picture what is going on:



We know that an $m \times n$ matrix A can be used to define a linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by $T_A(\vec{v}) = A\vec{v}$. Our goal now is to find the special matrix A such that $C_{B_2}^{-1} \circ T_A \circ C_{B_1} = T$.

To find the matrix A, notice that

$$C_{B_2}^{-1} \circ T_A \circ C_{B_1} = T \text{ implies that } T_A \circ C_{B_1} = C_{B_2} \circ T,$$

and thus for any $\vec{v} \in V$,

$$C_{B_2}(T(\vec{v})) = T_A(C_{B_1}(\vec{v})) = AC_{B_1}(\vec{v}).$$

In other words,

$$[T(\vec{v})]_{B_2} = A[\vec{v}]_{B_1}.$$

Since $[\vec{b}_i]_{B_1} = \vec{e}_i$ for each $\vec{b}_i \in B_1$, $A[\vec{b}_i]_{B_1} = A\vec{e}_i$, which is simply the i^{th} column of A. Therefore, the i^{th} column of A is equal to $[T(\vec{b}_i)]_{B_2}$, the coordinate vector (relative to B_2) of the image of the i^{th} basis vector from B_1.

So our needed matrix A corresponding to the ordered bases B_1 and B_2, which we denote by $A_{B_2B_1}(T)$, is given by

$$A_{B_2B_1}(T) = \left[\begin{array}{cccc} [T(\vec{b}_1)]_{B_2} & [T(\vec{b}_2)]_{B_2} & \cdots & [T(\vec{b}_n)]_{B_2} \end{array}\right].$$

This result is given in the following theorem.

Theorem 9.89

Let V and W be vector spaces of dimension n and m respectively, with $B_1 = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ an ordered basis of V and $B_2 = \{\vec{\beta}_1, \vec{\beta}_2, \ldots, \vec{\beta}_m\}$ an ordered basis of W. Suppose $T : V \to W$ is a linear transformation. Then the unique matrix $A_{B_2B_1}(T)$ of T corresponding to B_1 and B_2 is given by

$$A_{B_2B_1}(T) = \left[\begin{array}{cccc} [T(\vec{b}_1)]_{B_2} & [T(\vec{b}_2)]_{B_2} & \cdots & [T(\vec{b}_n)]_{B_2} \end{array}\right].$$

This matrix satisfies $[T(\vec{v})]_{B_2} = A_{B_2B_1}(T)[\vec{v}]_{B_1}$ for all $\vec{v} \in V$.

Please take a moment and see how closely this theorem parallels both Theorem 5.7 from Section 5.2 and Theorem 5.73 from Section 5.10. In each case, to find the ith column of the matrix that represents a linear transformation, all you do is apply the transformation to the ith basis vector and write down the coordinates of the resulting vector. We really aren't doing anything particularly new or surprising here, we are just doing the same old thing in a setting where the bases involved are different. The fact that our vector spaces have bases consisting of a finite number of vectors is all we need to get this to work.

We demonstrate this content in the following examples.

Example 9.90: Matrix of a Linear Transformation

Let $T : \mathbb{P}_3 \to \mathbb{R}^4$ be an isomorphism defined by

$$T(ax^3 + bx^2 + cx + d) = \left[\begin{array}{c} a+b \\ b-c \\ c+d \\ d+a \end{array}\right]$$

Suppose $B_1 = \left\{x^3, x^2, x, 1\right\}$ is an ordered basis of $\mathbb{P}_3$ and

$$B_2 = \left\{ \left[\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{r} 0 \\ 0 \\ -1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}\right] \right\}$$

be an ordered basis of $\mathbb{R}^4$. Find the matrix $A_{B_2B_1}(T)$.

Solution. To find $A_{B_2B_1}(T)$, we use the following definition.

$$A_{B_2B_1}(T) = \left[\begin{array}{cccc} [T(x^3)]_{B_2} & [T(x^2)]_{B_2} & [T(x)]_{B_2} & [T(x^2)]_{B_2} \end{array}\right]$$

First we find the result of applying T to the basis B_1.

$$T(x^3) = \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array}\right], T(x^2) = \left[\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array}\right], T(x) = \left[\begin{array}{r} 0 \\ -1 \\ 1 \\ 0 \end{array}\right], T(1) = \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ 1 \end{array}\right].$$

Next we apply the coordinate isomorphism C_{B_2} to each of these vectors. We will show the first in detail.

$$C_{B_2}\left(\left[\begin{array}{c}1\\0\\0\\1\end{array}\right]\right) = a_1\left[\begin{array}{c}1\\0\\1\\0\end{array}\right] + a_2\left[\begin{array}{c}0\\1\\0\\0\end{array}\right] + a_3\left[\begin{array}{r}0\\0\\-1\\0\end{array}\right] + a_4\left[\begin{array}{c}0\\0\\0\\1\end{array}\right]$$

This implies that

$$\begin{array}{rcl} a_1 & = & 1 \\ a_2 & = & 0 \\ a_1 - a_3 & = & 0 \\ a_4 & = & 1 \end{array}$$

which has a solution given by

$$\begin{array}{rcl} a_1 & = & 1 \\ a_2 & = & 0 \\ a_3 & = & 1 \\ a_4 & = & 1 \end{array}$$

Therefore $[T(x^3)]_{B_2} = \left[\begin{array}{c}1\\0\\1\\1\end{array}\right]$.

You can verify that the following are true.

$$[T(x^2)]_{B_2} = \left[\begin{array}{c}1\\1\\1\\0\end{array}\right], \ [T(x)]_{B_2} = \left[\begin{array}{r}0\\-1\\-1\\0\end{array}\right], \ [T(1)]_{B_2} = \left[\begin{array}{r}0\\0\\-1\\1\end{array}\right]$$

Using these vectors as the columns of $A_{B_2B_1}(T)$ we have

$$A_{B_2B_1}(T) = \left[\begin{array}{rrrr} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

♠

The next example demonstrates that this method can be used to solve different types of problems. We will examine the above example and see if we can work backwards to determine the action of T from the matrix $A_{B_2B_1}(T)$.

Example 9.91: Finding the Action of a Linear Transformation

Let $T : \mathbb{P}_3 \to \mathbb{R}^4$ be an isomorphism with

$$A_{B_2B_1}(T) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

where $B_1 = \left\{x^3, x^2, x, 1\right\}$ is an ordered basis of $\mathbb{P}_3$ and

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is an ordered basis of $\mathbb{R}^4$. If $p(x) = ax^3 + bx^2 + cx + d$, find $T(p(x))$.

Solution. Recall that $[T(p(x))]_{B_2} = A_{B_2B_1}(T)[p(x)]_{B_1}$. Then we have

$$\begin{aligned} C_{B_2}(T(p(x))) &= A_{B_2B_1}(T)C_{B_1}(p(x)) \\ &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\ &= \begin{bmatrix} a+b \\ b-c \\ a+b-c-d \\ a+d \end{bmatrix} \end{aligned}$$

Therefore

$$\begin{aligned} T(p(x)) &= C_{B_2}^{-1} \begin{bmatrix} a+b \\ b-c \\ a+b-c-d \\ a+d \end{bmatrix} \\ &= (a+b) \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (b-c) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (a+b-c-d) \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} + (a+d) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} a+b \\ b-c \\ c+d \\ a+d \end{bmatrix} \end{aligned}$$

You can verify that this was the definition of $T(p(x))$ given in the previous example. ♠

We can also find the matrix of the composition of multiple transformations.

Theorem 9.92: Matrix of Composition

Let V, W and U be finite dimensional vector spaces, and suppose $T : V \to W$, $S : W \to U$ are linear transformations. Suppose V, W and U have ordered bases of B_1, B_2 and B_3 respectively. Then the matrix of the transformation $S \circ T : V \to U$ is given by

$$A_{B_3B_1}(S \circ T) = A_{B_3B_2}(S)A_{B_2B_1}(T).$$

The next important theorem gives a condition on when T is an isomorphism.

Theorem 9.93: Isomorphism

Let V and W be vector spaces such that both have dimension n and let $T : V \to W$ be a linear transformation. Suppose B_1 is an ordered basis of V and B_2 is an ordered basis of W.
*Then the conditions that $A_{B_2B_1}(T)$ is invertible for **all** B_1 and B_2, and that $A_{B_2B_1}(T)$ is invertible for **some** B_1 and B_2 are equivalent. In fact, these occur if and only if T is an isomorphism.*
If T is an isomorphism, the matrix $A_{B_2B_1}(T)$ is invertible and its inverse is given by $[A_{B_2B_1}(T)]^{-1} = A_{B_1B_2}(T^{-1})$.

Consider the following example.

Example 9.94

Suppose $T : \mathbb{P}_3 \to \mathbb{M}_{22}$ is a linear transformation defined by

$$T(ax^3 + bx^2 + cx + d) = \begin{bmatrix} a+d & b-c \\ b+c & a-d \end{bmatrix}$$

for all $ax^3 + bx^2 + cx + d \in \mathbb{P}_3$. Let $B_1 = \{x^3, x^2, x, 1\}$ and

$$B_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

be ordered bases of $\mathbb{P}_3$ and $\mathbb{M}_{22}$, respectively.

1. *Find $A_{B_2B_1}(T)$.*
2. *Verify that T is an isomorphism by proving that $A_{B_2B_1}(T)$ is invertible.*
3. *Find $A_{B_1B_2}(T^{-1})$, and verify that $A_{B_1B_2}(T^{-1}) = [A_{B_2B_1}(T)]^{-1}$.*
4. *Use $A_{B_1B_2}(T^{-1})$ to find T^{-1}.*

Solution.

1.

$$\begin{aligned}
A_{B_2B_1}(T) &= \left[\ [T(x^3)]_{B_2} \quad [T(x^2)]_{B_2} \quad [T(x)]_{B_2} \quad [T(1)]_{B_2}\ \right] \\
&= \left[\ [\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}]_{B_2} \quad [\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}]_{B_2} \quad [\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}]_{B_2} \quad [\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}]_{B_2}\ \right] \\
&= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}
\end{aligned}$$

2. $\det(A_{B_2B_1}(T)) = 4$, so the matrix is invertible, and hence T is an isomorphism.

3. We know that

$$T^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = x^3,\ T^{-1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = x^2,\ T^{-1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = x, \text{ and } T^{-1}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 1,$$

so

$$\begin{aligned}
T^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} &= \frac{1+x^3}{2},\ T^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{x^2-x}{2}, \\
T^{-1}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} &= \frac{x+x^2}{2},\ T^{-1}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{x^3-1}{2}.
\end{aligned}$$

Therefore,

$$M_{B_1B_2}(T^{-1}) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

You should verify that $A_{B_2B_1}(T)A_{B_1B_2}(T^{-1}) = I_4$. From this it follows that $A_{B_2B_1}(T)^{-1} = A_{B_1B_2}(T^{-1})$.

4.

$$\begin{aligned}
\left[T^{-1}\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right]_{B_1} &= A_{B_1B_2}(T^{-1})\left[\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right]_{B_2} \\
T^{-1}\begin{bmatrix} p & q \\ r & s \end{bmatrix} &= C_{B_1}^{-1}\left(A_{B_1B_2}(T^{-1})\left[\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right]_{B_2}\right) \\
&= C_{B_1}^{-1}\left(\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}\right) \\
&= C_{B_1}^{-1}\left(\frac{1}{2}\begin{bmatrix} p+s \\ q+r \\ r-q \\ p-s \end{bmatrix}\right) \\
&= \frac{1}{2}(p+s)x^3 + \frac{1}{2}(q+r)x^2 + \frac{1}{2}(r-q)x + \frac{1}{2}(p-s).
\end{aligned}$$

Engage Active Learning App!

Lyryx Engage is an active learning app designed to increase student engagement in the study of linear algebra. The content is "chunked" into small blocks, each with an interactive assessment activity to promote comprehension.

Visit us at https://lyryx.com to register and access Lyryx Engage!

Lyryx Online Exercises!

Lyryx offers formative online exercises! The educational software captures instructors' experience using algorithms to analyze and evaluate students' work, identifying common errors and providing personalized feedback. Students are encouraged to try these randomized problems repeatedly, learning from their mistakes until mastery.

Visit us at https://lyryx.com to register and access Lyryx exercises!

Appendix A

Some Prerequisite Topics

The topics presented in this section are important concepts in mathematics and therefore should be examined.

A.1 Sets and Set Notation

A set is a collection of things called elements. For example $\{1,2,3,8\}$ would be a set consisting of the elements 1,2,3, and 8. To indicate that 3 is an element of $\{1,2,3,8\}$, it is customary to write $3 \in \{1,2,3,8\}$. We can also indicate when an element is not in a set, by writing $9 \notin \{1,2,3,8\}$ which says that 9 is not an element of $\{1,2,3,8\}$. Sometimes a rule specifies a set. For example you could specify a set as all integers larger than 2. This would be written as $S = \{x \in \mathbb{Z} : x > 2\}$. This notation says: S is the set of all integers, x, such that $x > 2$.

Suppose A and B are sets with the property that every element of A is an element of B. Then we say that A is a subset of B. For example, $\{1,2,3,8\}$ is a subset of $\{1,2,3,4,5,8\}$. In symbols, we write $\{1,2,3,8\} \subseteq \{1,2,3,4,5,8\}$. It is sometimes said that “A is contained in B" or even “B contains A". The same statement about the two sets may also be written as $\{1,2,3,4,5,8\} \supseteq \{1,2,3,8\}$.

We can also talk about the *union* of two sets, which we write as $A \cup B$. This is the set consisting of everything which is an element of at least one of the sets, A or B. As an example of the union of two sets, consider $\{1,2,3,8\} \cup \{3,4,7,8\} = \{1,2,3,4,7,8\}$. This set is made up of the numbers which are in at least one of the two sets.

In general
$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$
Notice that an element which is in *both* A and B is also in the union, as well as elements which are in only one of A or B.

Another important set is the intersection of two sets A and B, written $A \cap B$. This set consists of everything which is in *both* of the sets. Thus $\{1,2,3,8\} \cap \{3,4,7,8\} = \{3,8\}$ because 3 and 8 are those elements the two sets have in common. In general,
$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

If A and B are two sets, $A \setminus B$ denotes the set of things which are in A but not in B. Thus
$$A \setminus B = \{x \in A : x \notin B\}$$
For example, if $A = \{1,2,3,8\}$ and $B = \{3,4,7,8\}$, then $A \setminus B = \{1,2,3,8\} \setminus \{3,4,7,8\} = \{1,2\}$.

A special set which is very important in mathematics is the empty set denoted by $\emptyset$. The empty set, $\emptyset$, is defined as the set which has no elements in it. It follows that the empty set is a subset of every set. This is true because if it were not so, there would have to exist a set A, such that $\emptyset$ has something in it which is not in A. However, $\emptyset$ has nothing in it and so it must be that $\emptyset \subseteq A$.

We can also use brackets to denote sets which are intervals of numbers. Let a and b be real numbers. Then

- $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- $[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $(a,b) = \{x \in \mathbb{R} : a < x < b\}$
- $(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$
- $[a,\infty) = \{x \in \mathbb{R} : x \geq a\}$
- $(-\infty,a] = \{x \in \mathbb{R} : x \leq a\}$

These sorts of sets of real numbers are called intervals. The two points a and b are called endpoints, or bounds, of the interval. In particular, a is the *lower bound* while b is the *upper bound* of the above intervals, where applicable. Other intervals such as $(-\infty,b)$ are defined by analogy to what was just explained. In general, the curved parenthesis, (, indicates the end point is not included in the interval, while the square parenthesis, [, indicates this end point is included. The reason that there will always be a curved parenthesis next to ∞ or $-\infty$ is that these are not real numbers and cannot be included in the interval in the way a real number can.

To illustrate the use of this notation relative to intervals consider three examples of inequalities. Their solutions will be written in the interval notation just described.

Example A.1: Solving an Inequality

Solve the inequality $2x+4 \leq x-8$.

Solution. We need to find x such that $2x+4 \leq x-8$. Solving for x, we see that $x \leq -12$ is the answer. This is written in terms of an interval as $(-\infty,-12]$. ♠

Consider the following example.

Example A.2: Solving an Inequality

Solve the inequality $(x+1)(2x-3) \geq 0$.

Solution. We need to find x such that $(x+1)(2x-3) \geq 0$. The solution is given by $x \leq -1$ or $x \geq \frac{3}{2}$. Therefore, x which fit into either of these intervals gives a solution. In terms of set notation this is denoted by $(-\infty,-1] \cup [\frac{3}{2},\infty)$. ♠

Consider one last example.

Example A.3: Solving an Inequality

Solve the inequality $x(x+2) \geq -4$.

Solution. This inequality is true for any value of x where x is a real number. We can write the solution as $\mathbb{R}$ or $(-\infty,\infty)$. ♠

In the next section, we examine another important mathematical concept.

A.2 Well Ordering and Induction

We begin this section with some important notation. Summation notation, written $\sum_{i=1}^{j} i$, represents a sum. Here, i is called the index of the sum, and we add iterations until $i = j$. For example,

$$\sum_{i=1}^{j} i = 1+2+\cdots+j$$

Another example:

$$a_{11}+a_{12}+a_{13} = \sum_{i=1}^{3} a_{1i}$$

The following notation is a specific use of summation notation.

Notation A.4: Summation Notation

Let a_{ij} be real numbers, and suppose $1 \leq i \leq r$ while $1 \leq j \leq s$. These numbers can be listed in a rectangular array as given by

$$\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rs} \end{array}$$

Then $\sum_{j=1}^{s}\sum_{i=1}^{r} a_{ij}$ means to first sum the numbers in each column (using i as the index) and then to add the sums which result (using j as the index). Similarly, $\sum_{i=1}^{r}\sum_{j=1}^{s} a_{ij}$ means to sum the vectors in each row (using j as the index) and then to add the sums which result (using i as the index).

Notice that since addition is commutative, $\sum_{j=1}^{s}\sum_{i=1}^{r} a_{ij} = \sum_{i=1}^{r}\sum_{j=1}^{s} a_{ij}$.

We now consider the main concept of this section. Mathematical induction and well ordering are two extremely important principles in math. They are often used to prove significant things which would be hard to prove otherwise.

Definition A.5: Well Ordered

A set is well ordered if every nonempty subset S, contains a smallest element z having the property that $z \leq x$ for all $x \in S$.

In particular, the set of natural numbers defined as

$$\mathbb{N} = \{1,2,\cdots\}$$

is well ordered.

Consider the following proposition.

Proposition A.6: Well Ordered Sets

Any set of integers larger than a given number is well ordered.

This proposition claims that if a set has a lower bound which is a real number, then this set is well ordered.

Further, this proposition implies the principle of mathematical induction. The symbol $\mathbb{Z}$ denotes the set of all integers. Note that if a is an integer, then there are no integers between a and $a+1$.

Theorem A.7: Mathematical Induction

A set $S \subseteq \mathbb{Z}$, having the property that $a \in S$ and $n+1 \in S$ whenever $n \in S$, contains all integers $x \in \mathbb{Z}$ such that $x \geq a$.

Proof. Let T consist of all integers larger than or equal to a which are not in S. The theorem will be proved if $T = \emptyset$. If $T \neq \emptyset$ then by the well ordering principle, there would have to exist a smallest element of T, denoted as b. It must be the case that $b > a$ since by definition, $a \notin T$. Thus $b \geq a+1$, and so $b-1 \geq a$ and $b-1 \notin S$ because if $b-1 \in S$, then $b-1+1 = b \in S$ by the assumed property of S. Therefore, $b-1 \in T$ which contradicts the choice of b as the smallest element of T. ($b-1$ is smaller.) Since a contradiction is obtained by assuming $T \neq \emptyset$, it must be the case that $T = \emptyset$ and this says that every integer at least as large as a is also in S. ♠

Mathematical induction is a very useful device for proving theorems about the integers. The procedure is as follows.

Procedure A.8: Proof by Mathematical Induction

Suppose S_n is a statement which is a function of the number n, for $n = 1, 2, \cdots$, and we wish to show that S_n is true for all $n \geq 1$. To do so using mathematical induction, use the following steps.

1. ***Base Case:*** *Show S_1 is true.*
2. *Assume S_n is true for some n, which is the **induction hypothesis**. Then, using this assumption, show that S_{n+1} is true.*

Proving these two steps shows that S_n is true for all $n = 1, 2, \cdots$.

We can use this procedure to solve the following examples.

Example A.9: Proving by Induction

Prove by induction that $\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Solution. By Procedure A.8, we first need to show that this statement is true for $n = 1$. When $n = 1$, the statement says that

$$\begin{aligned} \sum_{k=1}^{1} k^2 &= \frac{1(1+1)(2(1)+1)}{6} \\ &= \frac{6}{6} \end{aligned}$$

$$= \quad 1$$

The sum on the left hand side also equals 1, so this equation is true for $n = 1$.

Now suppose this formula is valid for some $n \geq 1$ where n is an integer. Hence, the following equation is true.

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{1.1}$$

We want to show that this is true for $n+1$.

Suppose we add $(n+1)^2$ to both sides of equation 1.1.

$$\begin{aligned} \sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \end{aligned}$$

The step going from the first to the second line is based on the assumption that the formula is true for n. Now simplify the expression in the second line,

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

This equals

$$(n+1)\left(\frac{n(2n+1)}{6} + (n+1)\right)$$

and

$$\frac{n(2n+1)}{6} + (n+1) = \frac{6(n+1)+2n^2+n}{6} = \frac{(n+2)(2n+3)}{6}$$

Therefore,

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6} = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

showing the formula holds for $n+1$ whenever it holds for n. This proves the formula by mathematical induction. In other words, this formula is true for all $n = 1, 2, \cdots$. ♠

Consider another example.

Example A.10: Proving an Inequality by Induction

Show that for all $n \in \mathbb{N}$, $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$.

Solution. Again we will use the procedure given in Procedure A.8 to prove that this statement is true for all n. Suppose $n = 1$. Then the statement says

$$\frac{1}{2} < \frac{1}{\sqrt{3}}$$

which is true.

Suppose then that the inequality holds for n. In other words,

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}<\frac{1}{\sqrt{2n+1}}$$

is true.

Now multiply both sides of this inequality by $\frac{2n+1}{2n+2}$. This yields

$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\cdot\frac{2n+1}{2n+2}<\frac{1}{\sqrt{2n+1}}\frac{2n+1}{2n+2}=\frac{\sqrt{2n+1}}{2n+2}$$

The theorem will be proved if this last expression is less than $\dfrac{1}{\sqrt{2n+3}}$. This happens if and only if

$$\left(\frac{1}{\sqrt{2n+3}}\right)^2=\frac{1}{2n+3}>\frac{2n+1}{(2n+2)^2}$$

which occurs if and only if $(2n+2)^2>(2n+3)(2n+1)$ and this is clearly true which may be seen from expanding both sides. This proves the inequality. ♠

Let's review the process just used. If S is the set of integers at least as large as 1 for which the formula holds, the first step was to show $1\in S$ and then that whenever $n\in S$, it follows $n+1\in S$. Therefore, by the principle of mathematical induction, S contains $[1,\infty)\cap\mathbb{Z}$, all positive integers. In doing an inductive proof of this sort, the set S is normally not mentioned. One just verifies the steps above.

Index

www.lyryx.com