Natural Language Processing

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EE 645

Mar 25, 2023

This module

Latent Semantic Indexing

- SVD

Language models (Transformers)

- Low rank projections
- Transfer of information

Latent Semantic Indexing

Singular value decomposition

$$M = U\Sigma V^T$$

If M is $n \times p$,

U is $n \times n$

Σ is $n \times p$

V is $p \times p$

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U, V are both orthonormal

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Σ is diagonoal

all diagonal entries ≥ 0

(called singular values)

Singular value decomposition

M is $n \times p$,

$$M = U\Sigma V^T$$

cols of U: basis for cols of M

$U = \begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_n\end{bmatrix}$, each $\mathbf{u}_i \in \mathbb{R}^n$

$\mathbf{u}_i$ all have length 1, mutually perpendicular

cols of V: basis for rows of M

$V = \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_p\end{bmatrix}$, each $\mathbf{v}_i \in \mathbb{R}^n$

$\mathbf{v}_i$ all have length 1, mutually perpendicular

singular values: importance of basis vectors

$\sigma_1, \ldots, \sigma_{\min(n,p)}$

Multiplying out

M is $n \times p$,

$$M = \begin{bmatrix} \mathbf{u}_1 & \dots & \mathbf{u}_n \end{bmatrix} \mathrm{diag}\left(\sigma_1, \dots, \sigma_{\min(n,p)}\right) \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_p^T \end{bmatrix}$$

Instructive to multiply out:

$$M = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \ldots + \sigma_{\min(n,p)} \mathbf{u}_{\min(n,p)} \mathbf{v}_{min(n,p)}^T$$

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Number of non-zero singular values = rank of matrix
In fact, general definition of rank:

Rank of a matrix

M is defined rank-r if it can be written as a sum of r rank-1 matrices and no fewer.

Latent Semantic Indexing

p documents, total of n words in the documents

M is the $n \times p$ term-document matrix

Different ways to come up with M

  simplest $M_{ij} = 1$ if word $i \in$ doc j

Note: M loses information about relative ordering of words

  bag of words model

  formally equivalent to unigram language models

Latent Semantic Indexing

Singular value decomposition of M (assume $\sigma_1 \geq \sigma_2 \geq ...$)

$$M = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \ldots + \sigma_{\min(n,p)} \mathbf{u}_{\min(n,p)} \mathbf{v}_{min(n,p)}^T$$
$$\approx \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \ldots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T \qquad (r \ll \min(n,p)) \quad = U^{(r)} {V^{(r)}}^T$$

where $U^{(r)}$ ($V^{(r)}$) contains first r cols of U (V)

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Interpret the r vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ as choice of topics in each doc

Demo

Pros and cons

Pros
Simple and fast
Often used to optimize search

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Simple and fast
Often used to optimize search

Cons
Topics orthogonal?
Negative values
- signal words absent (ok!)
- docs similar using *absence* of words, (not ok!)

Non negative matrix factorization

LSI: $M \approx U^{(r)} {V^{(r)}}^T$
How about find best A, W such that

$$M \approx AW,$$

A has r cols, W has r rows, all entries ≥ 0
Lot harder than SVD, optimization NP-hard
Approximations exist (EM, algebraic)

Language Models

Statistical models of language

Unigram, Bigram, Trigram...

Little bit of information theory (offline)

- entropy
- representation in bits
- cross entropy

Perplexity (power of a language model)

- GPT-4 2.6
- GPT-3.5 4.5

Modern Language Models

Tokenizer (▸ OpenAI)

Modern Language Models

Tokenizer (OpenAI)

Brief history:

- Recurrent NN
- LSTMs
- Transformers

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Modern Language Models

Tokenizer (▸ OpenAI)

Brief history:

- Recurrent NN
- LSTMs
- Transformers
  - only focus on this!

Transformers

What is a transformer?

Central to Transformers is the notion of *attention*

Attention-like approaches in

- Linear Regression
- Kernels

Transformer core

Attention

Skip connections

Attention-like approaches

$n \times p$ design matrix X, target $\mathbf{y}$
Each row is an example (key)
Each target is a number (value)
Given a test example $\mathbf{z}$ (query), output?
Recall

$$\hat{w} = (X^T X)^{-1} X^T \mathbf{y}, \qquad \text{Prediction: } \mathbf{z}^T \hat{\mathbf{w}}$$

If $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are the n examples:

$$\mathbf{z}^T \hat{\mathbf{w}} = \sum_{i=1}^{n} (\mathbf{z}^T (X^T X)^{-1} \mathbf{x}_i) y_i$$

Attention

The term $\mathbf{z}^T (X^T X)^{-1} \mathbf{x}_i$ is the attention the key $\mathbf{x}_i$ gets from the query $\mathbf{z}$. The output is a linear combination of values y_i, with $\mathbf{y}_i$ weighted by the attention placed $\mathbf{x}_i$.

Other algorithms

Ridge Regression

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Support vector machines

Representer Theorem $\mathbf{w} = \sum_{i=1}^{n}\beta_i\mathbf{x}_i y_i$ (linear)

Soft prediction

$$\mathbf{z}^T\hat{\mathbf{w}} = \sum_{i=1}^{n}\beta_i(\mathbf{z}^T\mathbf{x}_i)\mathbf{y}_i$$

β_i is obtained by solving the dual, most are 0

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Attention

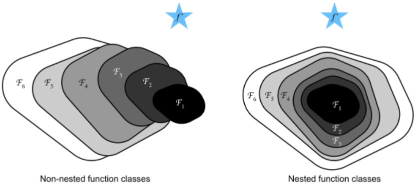
We specialize the observation in prior slides

Attention in Deep Learning: probability distribution over keys

- on any key must be ≥ 0
- must sum to 1 over all the keys
- in that sense, diff from OLS and kernel illustrations

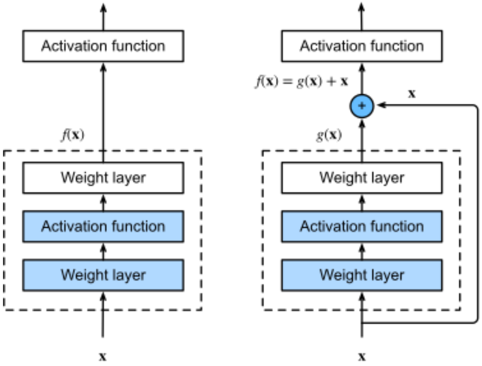
Arbitrary function and pass it through softmax

Skip connections



(Image source: Dive into deep learning)

Skip connections



(Image source: Dive into deep learning)

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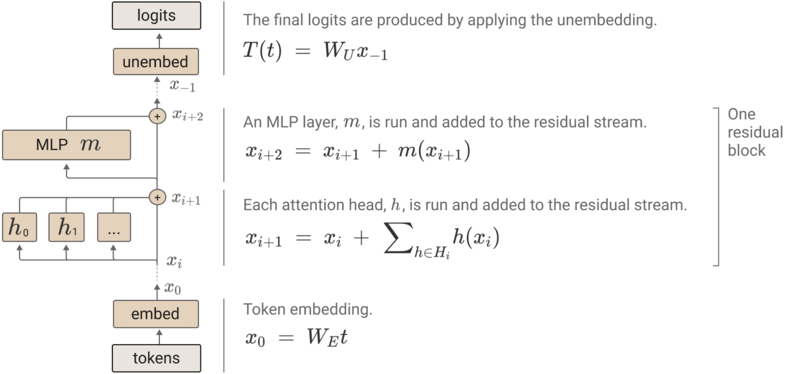
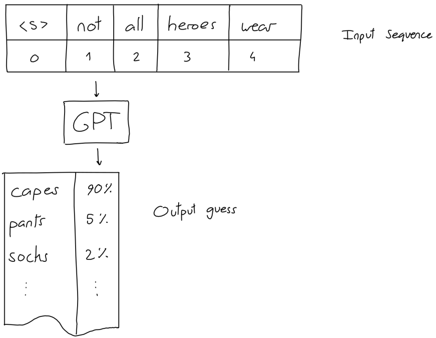
Putting them together



(Image source: A mathematical framework for transformer circuits, Anthropic)

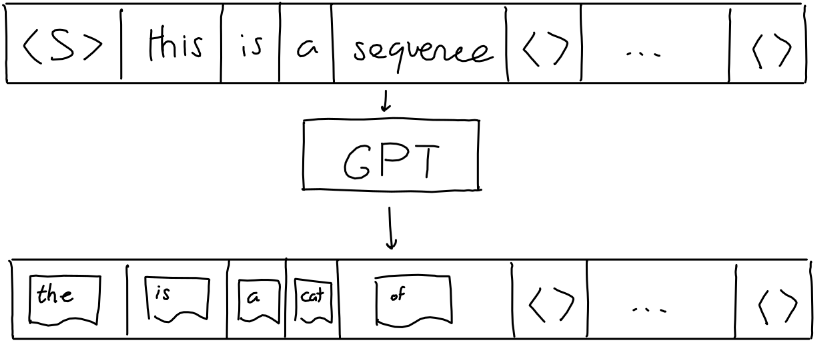
What is a Language Model?



(Image source: GPT architecture on a napkin)

What does a Transformer output?

Context has 2048 tokens (though pic shows words)



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Representation of tokens

GPT has a vocabulary of 50,257 tokens

$$\boxed{\text{The}} \longrightarrow [0\ 0\ 0\ 0\ 1\ 0\ 0\ \dots]$$

For every token in context

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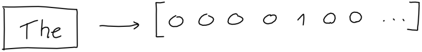

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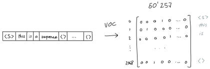

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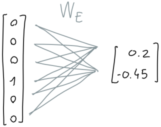
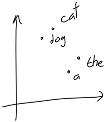
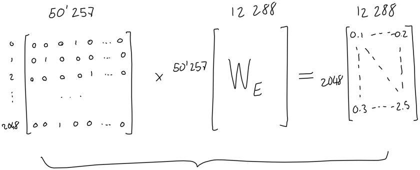
Embeding tokens

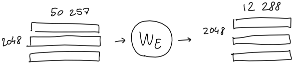




Linear map

In actuality, each token $\to \mathbb{R}^{12288}$





Positional Encoding

Each position (0-2047) $\to \mathbb{R}^{12288}$
P: position matrix ($2048\times$ 12288)

$$p_{i,2j} = \sin\left(\frac{i}{M^{2j/d}}\right)$$
$$p_{i,2j+1} = \cos\left(\frac{i}{M^{2j/d}}\right)$$

M is a large number (not important)

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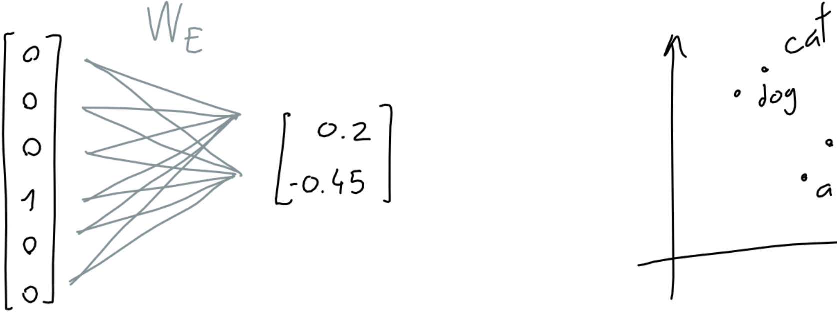
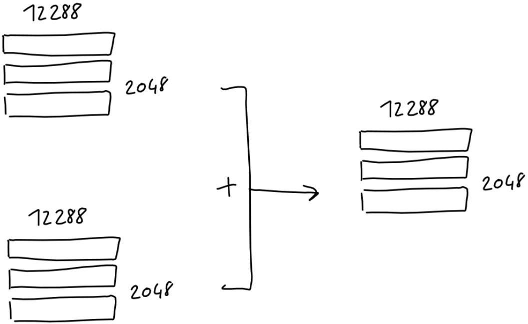
Idea: mimic binary representation of numbers
relative location is a linear transform

Positional encoding matrix P



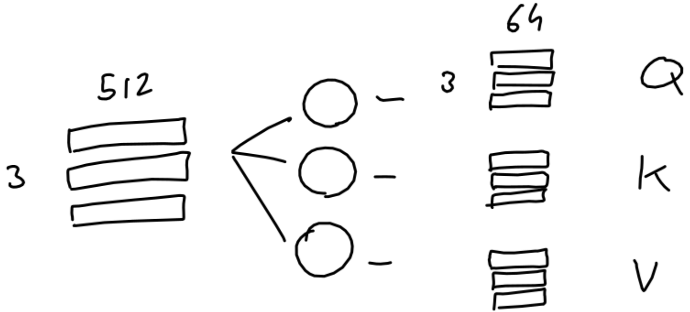
(Image source: Dive into Deep Learning)

Embedding all 2048 tokens



(Image source: GPT architecture on a napkin)

Transformer core: attention



In GPT-3: query, key, values are 128-long vectors

(Image source: GPT architecture on a napkin)

Transformer core: attention

Compute softmax$((QK^T)V)$
For query $\mathbf{q}_i$ from token i, compute

$$\sum_{j=1}^{n} \alpha(\mathbf{q}_i, \mathbf{k_j})\mathbf{v_j}$$

for every key $\mathbf{k}_j$ and value $\mathbf{v}_j$ from token j,

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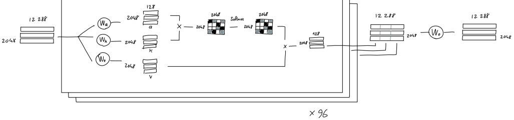
$$\alpha(\mathbf{q}_i, \mathbf{k_j}) = \text{softmax}_j(\mathbf{x}_i^T W_q W_k \mathbf{x}_j / \sqrt{128})$$

and $\mathbf{x}_i$ and $\mathbf{x}_j$ are the embeddings of tokens i and j from prior layer

Multiheaded attention

96 parallel attention heads
Think of each computing a different representation
Followed by a Feedforward (1 hidden layer)



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GPT-3

GPT-3 has 96 layers as above
  layers also have dropouts

Parameters (estimate)
Embedding: 50527×12288

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Attention

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    Not counting dropouts, biases, layer norm scalings

    Each attention head: $12288 \times 128 \times 3$

  Layer pooling $128 \times 96 \times 12288 = 12288 \times 12288$

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$96\times(\text{Attention}+\text{MLP})$

$= 96 \times (12288\times128\times3\times96+12288\times12288\times9)$

Total: 174.6 billion parameters, (reported 175 billion)

What happens at each layer

Think of each layer as a representation of token
First layer: direct embedding
Subsequent layers: contextualized embeddings
    Richer representation that includes context

What can we do with these rich representations?

Downstream tasks

We have been talking about:
Contextual representation $\rightarrow$ Language model

But in fact, lot lot more

- Translation
- Summarization
- General Knowledge Q&A
- Chatbots
- Programming... and the list goes on

LLMs are few-shot learners

Two general ways to build

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Fine tuning:

- Uses 1000s/100,000 more examples
- Gradient updates are performed on model
- Original LLMs or subset or (likely) add-on

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Two general ways to build

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Gradient updates are performed on model

Original LLMs or subset or (likely) add-on

Few shot learning: no parameter updates

Few examples, 10s

(whatever fits into 2048 tokens)

No gradient updates

Use off the shelf predictions