Natural Language Processing

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This module

- Latent Semantic Indexing
  - SVD
- Language models (Transformers)
  - Low rank projections
  - Transfer of information

Latent Semantic Indexing

Singular value decomposition

$$M = U\Sigma V^T$$

If M is $n \times p$,

U is $n \times n$

Σ is $n \times p$

V is $p \times p$

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Σ is diagonoal

all diagonal entries ≥ 0

(called singular values)

Singular value decomposition

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$$M = U\Sigma V^T$$

cols of U: basis for cols of M

$U = \begin{bmatrix} \mathrm{u}_1 & \cdots & \mathrm{u}_n \end{bmatrix}$, each $\mathrm{u}_i \in \mathbb{R}^n$

u_i all have length 1, mutually perpendicular

cols of V: basis for rows of M

$V = \begin{bmatrix} \mathrm{v}_1 & \cdots & \mathrm{v}_p \end{bmatrix}$, each $\mathrm{v}_i \in \mathbb{R}^n$

v_i all have length 1, mutually perpendicular

singular values: importance of basis vectors

$\sigma_1, \ldots, \sigma_{\min(n,p)}$

Multiplying out

M is $n \times p$,

$$M = \begin{bmatrix} \mathrm{u}_1 & \dots & \mathrm{u}_n \end{bmatrix} \mathrm{diag}\left(\sigma_1, \dots, \sigma_{\min(n,p)}\right) \begin{bmatrix} \mathrm{v}_1^T \\ \vdots \\ \mathrm{v}_p^T \end{bmatrix}$$

Instructive to multiply out:

$$M = \sigma_1 \mathrm{u}_1 \mathrm{v}_1^T + \dots + \sigma_{\min(n,p)} \mathrm{u}_{\min(n,p)} \mathrm{v}_{min(n,p)}^T$$

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Number of non-zero singular values = rank of matrix

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Each of $u_i v_i^T$ is a rank-1 matrix
Number of non-zero singular values = rank of matrix
In fact, general definition of rank:

Rank of a matrix

M is defined rank-r if it can be written as a sum of r rank-1 matrices and no fewer.

Latent Semantic Indexing

p documents, total of n words in the documents

M is the $n \times p$ term-document matrix

Different ways to come up with M

simplest $M_{ij} = 1$ if word $i \in$ doc j

Note: M loses information about relative ordering of words

bag of words model

formally equivalent to unigram language models

Latent Semantic Indexing

Singular value decomposition of M (assume $\sigma_1 \geq \sigma_2 \geq ...$)

$$M = \sigma_1 \mathrm{u}_1 \mathrm{v}_1^T + \ldots + \sigma_{\min(n,p)} \mathrm{u}_{\min(n,p)} \mathrm{v}_{min(n,p)}^T$$

$$\approx \sigma_1 \mathrm{u}_1 \mathrm{v}_1^T + \ldots + \sigma_r \mathrm{u}_r \mathrm{v}_r^T \qquad (r \ll \min(n,p)) \quad = U^{(r)} {V^{(r)}}^T$$

where $U^{(r)}$ ($V^{(r)}$) contains first r cols of U (V)

Latent Semantic Indexing

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$$\begin{aligned} M &= \sigma_1 \mathrm{u}_1 \mathrm{v}_1^T + \ldots + \sigma_{\min(n,p)} \mathrm{u}_{\min(n,p)} \mathrm{v}_{min(n,p)}^T \\ &\approx \sigma_1 \mathrm{u}_1 \mathrm{v}_1^T + \ldots + \sigma_r \mathrm{u}_r \mathrm{v}_r^T \qquad (r \ll \min(n,p)) \quad = U^{(r)} {V^{(r)}}^T \end{aligned}$$

where $U^{(r)}$ ($V^{(r)}$) contains first r cols of U (V) Interpret the r vectors $\mathrm{u}_1, \ldots, \mathrm{u}_r$ as the r topics

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Latent Semantic Indexing

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Interpret the r vectors $\mathrm{v}_1, \ldots, \mathrm{v}_r$ as choice of topics in each doc

Demo

Pros and cons

Pros
Simple and fast
Often used to optimize search

Pros and cons

Pros

Simple and fast

Often used to optimize search

Cons

Topics orthogonal?

Negative values

- signal words absent (ok!)
- docs similar using *absence* of words, (not ok!)

Non negative matrix factorization

LSI: $M \approx U^{(r)} {V^{(r)}}^T$

How about find best A, W such that

$$M \approx AW,$$

A has r cols, W has r rows, all entries ≥ 0

Lot harder than SVD, optimization NP-hard

Approximations exist (EM, algebraic)

Language Models

Statistical models of language

Unigram, Bigram, Trigram...

Little bit of information theory (offline)

- entropy
- representation in bits
- cross entropy

Perplexity (power of a language model)

- GPT-4 2.6
- GPT-3.5 4.5

Modern Language Models

Tokenizer (OpenAI)

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Brief history:

- Recurrent NN
- LSTMs
- Transformers

Modern Language Models

Tokenizer (OpenAI)

Brief history:

- Recurrent NN
- LSTMs
- Transformers
  - only focus on this!

Transformers

What is a transformer?

Central to Transformers is the notion of *attention*

Attention-like approaches in

- Linear Regression
- Kernels

Transformer core

Attention

Skip connections

Attention-like approaches

$n \times p$ design matrix X, target y

Each row is an example (key)

Each target is a number (value)

Given a test example z (query), output?

Recall

$$\hat{w} = (X^T X)^{-1} X^T \mathrm{y}, \qquad \text{Prediction: } \mathrm{z}^T \hat{\mathrm{w}}$$

If $\mathrm{x}_1, \ldots, \mathrm{x}_n$ are the n examples:

$$\mathrm{z}^T \hat{\mathrm{w}} = \sum_{i=1}^{n} (\mathrm{z}^T (X^T X)^{-1} \mathrm{x}_i) y_i$$

Attention

The term $\mathrm{z}^T (X^T X)^{-1} \mathrm{x}_i$ is the attention the key x_i gets from the query z. The output is a linear combination of values y_i, with y_i weighted by the attention placed x_i.

Other algorithms

Ridge Regression

$$z^T \hat{w} = \sum_{i=1}^{n} (z^T (X^T X + \lambda I)^{-1} x_i) y_i$$

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Support vector machines

Representer Theorem $w = \sum_{i=1}^{n} \beta_i x_i y_i$ (linear)

Soft prediction

$$z^T \hat{w} = \sum_{i=1}^{n} \beta_i (z^T x_i) y_i$$

β_i is obtained by solving the dual, most are 0

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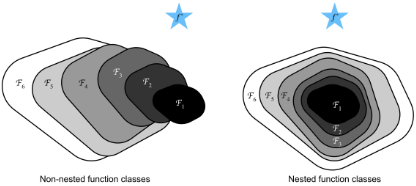
Attention

We specialize the observation in prior slides

Attention in Deep Learning: probability distribution over keys
- on any key must be ≥ 0
- must sum to 1 over all the keys
- in that sense, diff from OLS and kernel illustrations

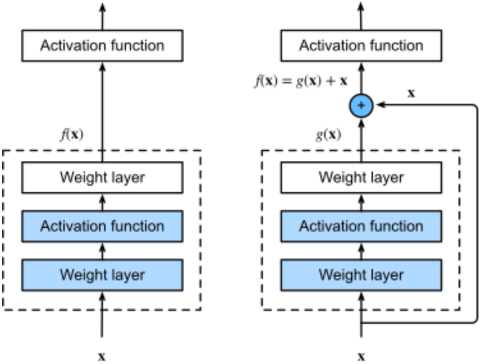
Arbitrary function and pass it through softmax

Skip connections



(Image source: Dive into deep learning)

Skip connections



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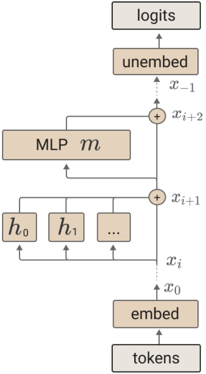
Putting them together



The final logits are produced by applying the unembedding.

$$T(t) = W_U x_{-1}$$

An MLP layer, m, is run and added to the residual stream.

$$x_{i+2} = x_{i+1} + m(x_{i+1})$$

Each attention head, h, is run and added to the residual stream.

$$x_{i+1} = x_i + \sum_{h \in H_i} h(x_i)$$

One residual block

Token embedding.

$$x_0 = W_E t$$

(Image source: A mathematical framework for transformer circuits, Anthropic)