# Stochastic Modelling in Fluid Dynamics: Itô vs Stratonovich

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**Abstract**

Suppose the observations of Lagrangian trajectories for fluid flow in some physical situation may be modelled sufficiently accurately by a spatially correlated Itô stochastic process obtained from data which is taken in fixed Eulerian space. Suppose we also want to apply Hamilton's principle to derive the stochastic fluid equations for this situation. Now, the variational calculus for applying Hamilton's principle requires the Stratonovich process, so we must transform from Itô noise in the *data frame* to the equivalent Stratonovich noise. However, the transformation from the Itô process in the data frame to the corresponding Stratonovich process shifts the drift velocity of the transformed Lagrangian fluid trajectory out of the data frame into a non-inertial frame obtained from the Itô correction. The question is, "Will fictitious forces arising from this transformation of reference frames make a difference in the interpretation of the solution behaviour of the resulting stochastic equations?" This issue will be resolved by elementary considerations.

## 1 Introduction

**The Kelvin circulation theorem.** The key element of fluid dynamics is the Kelvin circulation theorem, which is a statement of Newton's Force Law for distributions of mass on closed material loops $c(\boldsymbol{u}_t^L)$, where the subscript $t$ denotes explicit time dependence. By definition, such material loops move with the transport velocity $\boldsymbol{u}_t^L$ of the fluid flow. Newton's Force Law states that the time rate of change of the momentum $d\boldsymbol{P}/dt$ of such a loop of a given mass distribution is equal to the force $\boldsymbol{F}$ applied to it. For the fluid situation, this is written as

$$\frac{d\boldsymbol{P}}{dt} := \frac{d}{dt}\oint_{c(\boldsymbol{u}_t^L)} \boldsymbol{u}_t(\mathbf{x})\cdot d\mathbf{x} = \oint_{c(\boldsymbol{u}_t^L)} \boldsymbol{f}(\mathbf{x})\cdot d\mathbf{x} =: \boldsymbol{F}\,. \tag{1}$$

This Kelvin-Newton relation in (1) for loop momentum dynamics involves two kinds of velocity, both of which are defined in fixed Eulerian space with coordinates $\mathbf{x}$. The first velocity is $\boldsymbol{u}_t^L(\mathbf{x})$, which is the velocity at a given point $\mathbf{x}$ fixed in space along the path of the material masses distributed in the line elements along the moving loop. This is a Lagrangian quantity because its argument is the pullback of the tangents to the Lagrangian trajectories of fluid parcels moving through fixed Eulerian space under the smooth invertible flow map, $\mathbf{x}_t = \phi_t\mathbf{x}_0$. That is,

$$\frac{d}{dt}\phi_t(\mathbf{x}_0) = \phi_t^*\boldsymbol{u}^L(t,\mathbf{x}_0) = \boldsymbol{u}^L(t,\phi_t(\mathbf{x}_0)) = \boldsymbol{u}_t^L(\mathbf{x})\,. \tag{2}$$

The second velocity is $\boldsymbol{u}_t(\mathbf{x})$ in the integrand of the circulation integral. Physically, the velocity $\boldsymbol{u}_t(\mathbf{x})$ is the total momentum per unit mass of the fluid (called the *specific momentum*), and it is evaluated at *fixed points in space*. That is, the velocity $\boldsymbol{u}_t(\mathbf{x})$ in the integrand of (1) is an Eulerian quantity evaluated at fixed locations in an inertial frame, as is the force per unit mass, $\boldsymbol{f}_t(\mathbf{x})$. As is well known, the proof of the Kelvin circulation theorem involves transforming back and forth between the moving Lagrangian frame and the fixed Eulerian frame; see, e.g., [4, 16].

Notice that the Kelvin-Newton relation (1) is about the time rate of change of momentum distributed on closed loops. It is not about the *acceleration* of mass distributions on closed loops. That acceleration would be expressed, instead, as

$$\frac{d\boldsymbol{V}}{dt} := \frac{d}{dt}\oint_{c(\boldsymbol{u}_t^L)} \boldsymbol{u}_t^L(\mathbf{x})\cdot d\mathbf{x}\,. \tag{3}$$

The velocity $V$ only figures in the Kelvin-Newton relation in the special case that the specific momentum $\boldsymbol{u}_t(\mathbf{x})$ is linear in the fluid transport velocity $\boldsymbol{u}_t^L(\mathbf{x})$ at fixed points in Eulerian coordinates and with time-independent coefficients.

The form of the Kelvin circulation theorem in (1) persists for stochastic flow, provided the Lagrangian paths follow Stratonovich stochastic paths, as shown in [14] by using a Stratonovich stochastic version of Hamilton's principle for fluid dynamics. The observation of the persistence of the Kelvin form (1) for Stratonovich stochastic fluid trajectories has led to the SALT algorithm for uncertainty quantification and data assimilation for stochastic fluid models.[1] The SALT algorithm proceeds from data acquisition, to coarse graining, to uncertainty quantification by using stochastic fluid dynamical modelling, and finally to uncertainty reduction via data assimilation based on machine learning via particle filtering methods [5, 6].

**Problem statement.** The present note has a simple storyline. Suppose the Lagrangian trajectories for fluid flow in some physical situation are modelled sufficiently accurately by a spatially correlated Itô stochastic process obtained from data which is taken in fixed Eulerian space. For example, this could be drifter data on the surface of the Ocean as seen from a satellite. Suppose we also want to apply Hamilton's principle to derive the stochastic fluid equations for this situation. Now, the variational calculus for applying Hamilton's principle requires the Stratonovich process, so we must transform from Itô noise in the *data frame* to the equivalent Stratonovich noise. The question is, 'Will this transformation make a difference in the solution behaviour of the resulting stochastic equations?' The transformation from the Itô process in the data frame to the corresponding Stratonovich process shifts the drift velocity of the transformed Lagrangian fluid trajectory out of the data frame into a non-inertial frame obtained from the Itô correction. We do know the Itô correction perfectly well, though, since the spatial correlations of the Itô noise have been obtained from the observed data. So, perhaps all is well, even though the spatial correlations depend upon location.

Thus, the Itô correction shifts the Stratonovich drift velocity of the fluid into a spatially-dependent non-inertial frame relative to the *data frame*. (The data frame is the fixed Eulerian frame in which the Itô drift velocity was defined.) The shift of frame in Hamilton's principle introduces a fictitious force in the motion equation for $\boldsymbol{u}_t$, which is derived using variational calculus with the Stratonovich noise. This fictitious force generates circulation of the Stratonovich drift velocity. The question then arises, 'Is the circulation generated by the fictitious force important to the observed motion?'

To answer this question, we apply Hamilton's principle to derive the equations of motion in the example of the stochastic Euler-Boussinesq equations. In this case, including the fictitious force produces a "vortex force" analogous to the Coriolis force. Upon inspection, we will recognise the derived equations as a version of the Craik-Leibovich equations [9, 8], altered by the presence of stochastic advection by Lie transport.

The "vortex force" of the deterministic Craik-Leibovich (DCL) theory derived in [9, 8] was introduced to model the observed phenomenon of Langmuir circulations arising physically from wave–current interaction (WCI), [20]-[24]. The corresponding velocity shift due to WCI was called the "Stokes drift velocity" and was a *prescribed quantity* denoted as $\overline{\mathbf{u}}^S(\mathbf{x})$. The importance of including $\overline{\mathbf{u}}^S(\mathbf{x})$ in the DCL equations has been investigated for Kelvin-Helmholtz instability in [13] and for symmetric and geostrophic instabilities in the wave-forced ocean mixed layer in [12]. In fact, because of its effectiveness in generating Langmuir circulations, the DCL is a mainstay of the WCI literature.

[1] SALT is an acronym for stochastic advection by Lie Transport [5, 6].

The three-dimensional results of having transformed the stochastic Euler-Boussinesq (SEB) fluid equations into a stochastic version of Craik-Leibovich equations (SCL) have yet to be investigated. However, it would not be surprising if the SCL solutions were interpreted as possessing Langmuir circulations generated by the Itô correction to the stochastic drift velocity. Such an interpretation should be received with care, though, since they would be circulations of the *relative velocity*, $\boldsymbol{u}^L$, generated simply because the equations for $\boldsymbol{u}^L$ are not written in the inertial frame of the data. Thus, this note investigates how to deal with non-inertial fictitious forces in stochastic dynamics which arise from Itô corrections as changes of frame when applying mixed Itô and Stratonovich stochastic modelling in 3D SEB fluid dynamics. The answer to this question has already been given above in the comparison between equations (1) and (3). Namely, the Itô correction will generate no Langmuir circulations, as seen in the data frame with velocity $\boldsymbol{u}_t(\mathbf{x})$. However, Langmuir circulations could be generated by a fictitious force, which is felt in the relative drift frame of the Lagrangian particles with velocity $\boldsymbol{u}^L_t(\mathbf{x})$. Undergraduate physics students will recognise it when they see it again later. Nonetheless, we hope the explicit stochastic fluid dynamical calculations which demonstrate the answer for 3D SEB fluid dynamics below may be illuminating.

### 1.1 Stochastic Kelvin circulation dynamics

Multi-time homogenisation for fluid dynamics in [7] was used to derive the following Itô representation of the stochastic vector field which generates a stochastic Lagrangian fluid trajectory in the Eulerian representation,

$$\mathrm{d}\mathbf{x}_t = \boldsymbol{u}_t(\mathbf{x}_t)\,dt + \boldsymbol{\xi}(\mathbf{x}_t)\,dB_t\,, \tag{4}$$

where subscript $t$ denotes explicit time dependence, i.e., not partial time derivative. In this notation, $dB_t$ denotes a Brownian motion in time, $t$, whose divergence-free vector amplitude $\boldsymbol{\xi}(\mathbf{x}_t)$ depends on the Eulerian spatial position $\mathbf{x} \in \mathbb{R}^3$ along the Lagrangian trajectory, $\mathbf{x}_t$ with initial condition $\mathbf{x}_0$. The differential notation (d) in equation (4) is short for

$$\mathbf{x}_t - \mathbf{x}_0 = \int_0^t \mathrm{d}\mathbf{x}_t = \int_0^t \boldsymbol{u}_t(\mathbf{x}_t)\,dt + \int_0^t \boldsymbol{\xi}(\mathbf{x}_t)\,dB_t\,, \tag{5}$$

where the first time integral in the sum on the right is a Lebesque integral and the second one is an Itô integral.

The Stratonovich representation (denoted with symbol $\circ$) of the Itô trajectory in (4) is given by

$$\mathrm{d}\mathbf{x}^L_t = \boldsymbol{u}^L_t(\mathbf{x}_t)\,dt + \boldsymbol{\xi}(\mathbf{x}_t)\circ dB_t\,. \tag{6}$$

The difference in drift velocities for the two equivalent representations (4) and (6) of the same Lagrangian trajectory $\mathrm{d}\mathbf{x}^L_t = \mathrm{d}\mathbf{x}_t$ is called the Itô correction [11]. It is given by,

$$\boldsymbol{u}^L_t(\mathbf{x}_t) - \boldsymbol{u}_t(\mathbf{x}_t) = -\frac{1}{2}\big(\boldsymbol{\xi}(\mathbf{x}_t)\cdot\nabla\big)\boldsymbol{\xi}(\mathbf{x}_t) =: \overline{\mathbf{u}}^S(\mathbf{x}_t)\,. \tag{7}$$

It may seem natural to identify the difference of velocities $\overline{\mathbf{u}}^S = \boldsymbol{u}^L_t - \boldsymbol{u}_t$ as the "Itô-Stokes drift velocity", as in [3]. This because one may regard this difference as the stochastic version of the classic Stokes drift velocity, which is traditionally written as $\overline{\mathbf{u}}^S = \overline{\mathbf{u}}^L - \overline{\mathbf{u}}^E$. Thus, $\overline{\mathbf{u}}^S$ is traditionally the difference between the Lagrangian mean fluid velocity $\overline{\mathbf{u}}^L$ and its Eulerian mean counterpart $\overline{\mathbf{u}}^E$. Likewise, in the present case, $\boldsymbol{u}_t$ is the Eulerian drift velocity (or, equivalently, $\boldsymbol{u}_t$ is the Eulerian momentum per unit mass in Newton's 2nd law) and $\boldsymbol{u}^L_t$ is the transport drift velocity for the corresponding equivalent Stratonovich representation of the Lagrangian trajectory.

The Kelvin circulation integral for the Eulerian representation of the Lagrangian trajectory in (4) is defined as

$$I(t)=\oint_{c(\mathrm{d}\mathbf{x}_t^L)}\boldsymbol{u}_t\cdot d\mathbf{x}\,,\tag{8}$$

where $\boldsymbol{u}_t(\mathbf{x})$ is the Eulerian velocity at a fixed spatial position $\mathbf{x}\in\mathbb{R}^3$ and $\mathrm{d}\mathbf{x}_t^L$ is the Stratonovich representation of the transport velocity of the circulation loop moving along the Lagrangian trajectory determined by integrating the semimartingale relationship in the vector field (4) to find the path (5). Having transformied to the Stratonovich representation allows us to write the Kelvin theorem for the dynamics of the circulation loop using standard calculus operations.

In terms of Stratonovich vector field $\mathrm{d}\mathbf{x}_t^L$ in (6), we may use the ordinary rules of calculus to compute the evolution equation for the circulation in equation (8). For this calculation, we invoke the evolutionary version of the classic Kunita-Itô-Wentzell (KIW) formula [17, 18, 19] for a 1-form, as derived in [4]. The KIW formula produces the following dynamics,

$$\begin{aligned}\mathrm{d}\oint_{c(\mathrm{d}\mathbf{x}_t^L)}\boldsymbol{u}_t\cdot d\mathbf{x}&=\oint_{c(\mathrm{d}\mathbf{x}_t^L)}\big(\mathrm{d}+\mathcal{L}_{\mathrm{d}\mathbf{x}_t^L}\big)(\boldsymbol{u}_t\cdot d\mathbf{x})\\&=\oint_{c(\mathrm{d}\mathbf{x}_t^L)}\big(\mathrm{d}\boldsymbol{u}_t+(\mathrm{d}\mathbf{x}_t^L\cdot\nabla)\boldsymbol{u}_t+(\nabla\,\mathrm{d}\mathbf{x}_t^L)^T\cdot\boldsymbol{u}_t)\big)\cdot d\mathbf{x}\\&=\oint_{c(\mathrm{d}\mathbf{x}_t^L)}\big(\mathrm{d}\boldsymbol{u}_t-\mathrm{d}\mathbf{x}_t^L\times\mathrm{curl}\boldsymbol{u}_t+\nabla(\mathrm{d}\mathbf{x}_t^L\cdot\boldsymbol{u}_t)\big)\cdot d\mathbf{x}\,,\end{aligned}\tag{9}$$

where the operator $\mathcal{L}_{\mathrm{d}\mathbf{x}_t^L}$ denotes the Lie derivative with respect to the vector field $\mathrm{d}\mathbf{x}_t^L$. Equation (12) will play a role in deriving the Kelvin circulation theorem, itself, and thereby interpreting the solution behaviour of the fluid motion equation, derived below from Hamilton's principle.

In the next section, we will show how passing from the Itô representation of the Lagrangian trajectory in (4) to its equivalent Stratonovich representation in (6) enables the use of variational calculus to derive the equations of stochastic fluid motion via the approach of stochastic advection by Lie transport (SALT), based on Hamilton's variational principle using Stratonovich calculus, [14]. The resulting equations will raise the issue of fictitious forces and this issue will be resolved by elementary considerations.

## 2 SALT derivation of stochastic Euler-Boussinesq (SEB)

### 2.1 Hamilton's principle, motion equations and circulation theorems

Following [14] we apply Hamilton's principle $\delta S=0$ with the following action integral $S=\int_0^T\ell(\boldsymbol{u}_t^L,D,b)\,dt$ whose fluid Lagrangian $\ell(\boldsymbol{u}_t^L,D,b)$ depending on drift velocity $\boldsymbol{u}_t^L$, buoyancy function $b(\mathbf{x},t)$ and the density $D(\mathbf{x},t)d^3x$ for $(\mathbf{x},t)\in\mathbb{R}^3\times\mathbb{R}$. We constrain the variations to respect the *stochastic* advection equations with transport velocity $\mathrm{d}\mathbf{x}_t^L$ given in (6),

$$\mathrm{d}b+\mathrm{d}\mathbf{x}_t^L\cdot\nabla b=0\,,\quad\text{and}\quad\mathrm{d}D+\mathrm{div}(D\,\mathrm{d}\mathbf{x}_t^L)=0\,.\tag{10}$$

These relations ensure that the values of the advected quantities $b$ and $D(\mathbf{x},t)d^3x$ remain invariant along flow given by the stochastic Lagrangian trajectory in (5).

In general, with the constraints in (10) Hamilton's principle will result in a motion equation in the Euler-Poincaré form [16]

$$\big(\mathrm{d}+\mathcal{L}_{\mathrm{d}\mathbf{x}_t^L}\big)\big(\boldsymbol{u}_t\cdot d\mathbf{x}\big)=\frac{1}{D}\frac{\delta\ell}{\delta b}db+d\frac{\delta\ell}{\delta D}\quad\text{with}\quad\boldsymbol{u}_t:=\frac{1}{D}\frac{\delta\ell}{\delta\boldsymbol{u}_t^L}\,.\tag{11}$$

This Euler-Poincaré equation will result in a Kelvin-Newton theorem of the form

$$
\mathrm{d}\oint_{c(\mathrm{d}\mathbf{x}_t^L)}(\boldsymbol{u}_t\cdot d\mathbf{x}) = \oint_{c(\mathrm{d}\mathbf{x}_t^L)}\frac{1}{D}\frac{\delta\ell}{\delta b}db + \oint_{c(\mathrm{d}\mathbf{x}_t^L)} d\frac{\delta\ell}{\delta D}\,, \tag{12}
$$

and the loop integral of an exact differential in the last termwill vanish. For more discussion of stochastic advection, see [4]. For discussion of other stochastic Kelvin theorems, see [10].

For the example of the stochastic Euler-Boussinesq (SEB) equations, pressure constraint in the well known deterministic action integral [13] must be altered to become,

$$
S = \int_0^T \ell(\boldsymbol{u}_t^L, D, b)\,dt = \int dt\int d^3x\left[\frac{1}{2}D|\boldsymbol{u}_t^L|^2 - D\boldsymbol{u}_t^L\cdot\overline{\mathbf{u}}^S(\mathbf{x}) - gDbz\right] - \int d^3x\int \mathrm{d}p(D-1), \tag{13}
$$

and again constrain the variations by requiring satisfaction of the stochastic advection relations in (10). Special care is required when imposing the incompressibility constraint, $\mathrm{div}(\mathrm{d}\mathbf{x}_t^L)=0$ by requiring that $(D=1)$, since the quantity $D$ is a stochastic quantity. As we shall see, this means we must determine the pressure Lagrange multiplier $(\mathrm{d}p)$ from a semimartingale equation. To finish the notation, $g$ in the Lagrangian (13) denotes the gravitational constant.

Hamilton's principle with the stochastic constraints (10) now yields a stochastic Kelvin-Newton theorem [16], expressible as, cf. (12),

$$
\mathrm{d}\oint_{c(\mathrm{d}\mathbf{x}_t^L)}\boldsymbol{u}_t\cdot d\mathbf{x} = -\,g\oint_{c(\mathrm{d}\mathbf{x}_t^L)} b\,dz\,dt - \oint_{c(\mathrm{d}\mathbf{x}_t^L)} d\Big(\mathrm{d}p - \frac{1}{2}|\boldsymbol{u}_t|^2 + \frac{1}{2}|\overline{\mathbf{u}}^S(\mathbf{x})|^2\Big)dt\,, \tag{14}
$$

in which $\boldsymbol{u}_t := \boldsymbol{u}_t^L - \overline{\mathbf{u}}^S$ and the closed loop $c(\mathrm{d}\mathbf{x}_t^L)$ moves with velocity $\mathrm{d}\mathbf{x}_t^L$ of the Lagrangian trajectory in (6). Again, the last term will vanish in the Kelvin-Newton theorem (14).

When $\overline{\mathbf{u}}^S$ vanishes, equation (14) yields Kelvin's circulation theorem for the stochastic Euler-Boussinesq (SEB) equations. Remarkably, though, when $\overline{\mathbf{u}}^S$ is finite, as given in (7), equation (14) yields Kelvin's circulation theorem for the stochastic Craik-Leibovich (SCL) equations, whose deterministic version (DCL) is used for modelling Langmuir circulations in the oceanic thermocline [8, 9].

Being loop integrals of exact differentials, the last terms in equations (12) and (14) both vanish. However, including the last term allows us to envision the SCL equations in full. Namely, for the Lagrangian trajectory $\mathrm{d}\mathbf{x}_t^L$ in equation (6), applying the KIW formula (12) to the Kelvin circulation integral on the left side of equation (14) yields the stochastic motion equation, as

$$
\mathrm{d}\boldsymbol{u}_t - \mathrm{d}\mathbf{x}_t^L\times\mathrm{curl}\boldsymbol{u}_t + \nabla\big(\mathrm{d}\mathbf{x}_t^L\cdot\boldsymbol{u}_t\big) = -\,gb\,\nabla z\,dt - \nabla\mathrm{d}p - \nabla\Big(-\frac{1}{2}|\boldsymbol{u}_t|^2 + \frac{1}{2}|\overline{\mathbf{u}}^S(\mathbf{x})|^2\Big)dt\,. \tag{15}
$$

The SCL motion equation (15) includes all three of the velocities $\boldsymbol{u}_t$, $\boldsymbol{u}_t^L$ and $\boldsymbol{u}_t^S$. Although the velocities are mixed in this equation, it implies a compact version of the Kelvin circulation theorem,

$$
\mathrm{d}\oint_{c(\mathrm{d}\mathbf{x}_t^L)}\boldsymbol{u}_t\cdot d\mathbf{x} = -\,g\oint_{c(\mathrm{d}\mathbf{x}_t^L)} b\,dz\,dt\,, \tag{16}
$$

where the closed loop $c(\mathrm{d}\mathbf{x}_t^L)$ is transported by the stochastic vector field $\mathrm{d}\mathbf{x}_t^L$ in (6) and the integrals of the gradients are the closed loop have vanished. As we have discussed, in the physical understanding of the Kelvin circulation theorem, one may regard the velocity $\boldsymbol{u}_t$ in the integrand as an Eulerian quantity and the flow velocity $\mathrm{d}\mathbf{x}_t^L$ of the material loop as a Lagrangian quantity.

**Remark 2.1 (Determining the pressure semimartingale)** *To determine the pressure semimartingale* $(\mathrm{d}p)$ *one imposes* $\mathrm{div}\boldsymbol{u}_t=0$ *on the divergence of the motion equation* (15) *to find a semimartingale Poisson equation*

$$
\Delta\Big(\mathrm{d}p + \mathrm{d}\mathbf{x}_t^L\cdot\boldsymbol{u}_t + \Big(-\frac{1}{2}|\boldsymbol{u}_t|^2 + \frac{1}{2}|\overline{\mathbf{u}}^S(\mathbf{x})|^2\Big)dt\Big) = \mathrm{div}\Big(\mathrm{d}\mathbf{x}_t^L\times\mathrm{curl}\boldsymbol{u}_t - \ gb\,\nabla z\,dt\Big)\,, \tag{17}
$$

*with Neumann boundary conditions obtained by preservation of the condition that $\boldsymbol{u}_t$ have no normal component on the fixed boundary of the flow domain.*

**Remark 2.2 (Completing the stochastic dynamical system)** *The SCL motion equation (15) is completed by the auxiliary stochastic advection equations for $b$ and $D$ in equation (10). The constraint $D-1=0$ imposed by the Lagrange multiplier $\mathrm{d}p$ (the pressure semimartingale) in (13) ensures that the velocity $\boldsymbol{u}_t^L$ is divergence free, provided the drift velocity $\overline{\mathbf{u}}^S(\mathbf{x})$ in (7) also has no divergence.*

Equation (15) may be equivalently written in terms of only $\boldsymbol{u}_t^L$ and $\boldsymbol{u}_t^S$ as

$$
\begin{aligned}
\mathrm{d}\boldsymbol{u}_t^L - \mathrm{d}\mathbf{x}_t^L \times \mathrm{curl}\boldsymbol{u}_t^L + \nabla\big(\mathrm{d}\mathbf{x}_t^L\cdot\boldsymbol{u}_t^L\big) = &-\ gb\,\nabla z\,dt + \mathrm{d}\mathbf{x}_t^L \times \mathrm{curl}\overline{\mathbf{u}}^S(\mathbf{x}) \\
&- \nabla\Big(\mathrm{d}p + \mathrm{d}\mathbf{x}_t^L\cdot\boldsymbol{u}^S(\mathbf{x})\Big) + \nabla\Big(\frac{1}{2}|\boldsymbol{u}_t^L - \overline{\mathbf{u}}^S(\mathbf{x})|^2 - \frac{1}{2}|\overline{\mathbf{u}}^S(\mathbf{x})|^2\Big)dt\,,
\end{aligned}
\tag{18}
$$

where we have dropped the term $\mathrm{d}\overline{\mathbf{u}}^S(\mathbf{x})$ because $\overline{\mathbf{u}}^S(\mathbf{x})$ in equation (7) is time-independent. The remaining terms involving $\overline{\mathbf{u}}^S(\mathbf{x})$ comprise a stochastic version of the 'vortex force' in DCL and an added stochastic contribution to the pressure. This vortex force appears in the corresponding Kelvin theorem as a source of circulation of the velocity $\boldsymbol{u}_t^L$, viz.,

$$
\mathrm{d}\oint_{c(\mathrm{d}\mathbf{x}_t^L)}(\boldsymbol{u}_t^L - \overline{\mathbf{u}}^S(\mathbf{x}))\cdot d\mathbf{x} = -\ g\oint_{c(\mathrm{d}\mathbf{x}_t^L)} b\,dz\,dt\,. \tag{19}
$$

The "vortex force" of the Deterministic Craik-Leibovich (DCL) theory was introduced to model the observed phenomenon of Langmuir circulations arising physically from wave–current interaction (WCI), [20]-[24]. The importance of including $\overline{\mathbf{u}}^S$ in the DCL equations is investigated for Kelvin-Helmholtz instability in [13] and for symmetric and geostrophic instabilities in the wave-forced ocean mixed layer in [12]. The results of having made the "vortex force" of the SCL theory stochastic have yet to be investigated in solutiopns of the 3D SEB equations.

Equation (18) with $\boldsymbol{u}_t := \boldsymbol{u}_t^L - \overline{\mathbf{u}}^S$ is an example of our earlier discussion after equation (3) in which the acceleration $V$ figures in the Kelvin-Newton relation, because the specific momentum $\boldsymbol{u}_t(\mathbf{x})$ is linear in the fluid transport velocity $\boldsymbol{u}_t^L(\mathbf{x})$ at fixed points in Eulerian coordinates and with time-independent coefficients. In this case, equations (18) and (19) exemplify the $a = F/m$ version of Newton's law which arises in this special case. Thus, the stochastic "vortex force" in equation (19) is a *fictitious force* which arises from insisting on writing the acceleration instead of the rate of change of momentum in Newton's Force Law. The stochastic motion equation (15) has no fictitious "vortex force", because it is written entirely in the Eulerian data frame. The fictitious "vortex force" only arises in equation (18) upon replacing rate of change of Eulerian specific momentum $\boldsymbol{u}_t$ in (15) with rate of change of the Lagrangian transport velocity (Lagrangian acceleration) $\boldsymbol{u}_t^L$ in equation (18).

### 2.2 Vorticity and PV dynamics

The curl of the SCL motion equation (18) yields the dynamics for the *total vorticity*

$$
\boldsymbol{\omega}_t := \mathrm{curl}(\boldsymbol{u}_t^L - \overline{\mathbf{u}}^S) = \mathrm{curl}\,\boldsymbol{u}_t\,, \tag{20}
$$

which is given by

$$
\mathrm{d}\boldsymbol{\omega}_t - \mathrm{curl}\big(\mathrm{d}\mathbf{x}_t^L \times \boldsymbol{\omega}_t\big) = -g\nabla b\times\nabla z\,. \tag{21}
$$

The total vorticity dynamics (21), in turn, yields a stochastic advection law for the *total potential vorticity*, defined by $q := \boldsymbol{\omega}_t\cdot\nabla b$; namely,

$$
\mathrm{d}q + \mathrm{d}\mathbf{x}_t^L\cdot\nabla q = 0\,. \tag{22}
$$

In turn, this implies preservation of spatial integrals

$$C_\Phi = \int_{\mathcal{D}} \Phi(q,b)\, d^3x \,, \tag{23}$$

for arbitrary differentiable functions $\Phi$, provided ${\rm d}\mathbf{x}_t^L$ has no normal component at the boundary $\partial\mathcal{D}$ of the flow domain $\mathcal{D}$.

# 3 Conclusion

The central theorem for fluid dynamics (the Kelvin theorem) involves two frames in which velocities are measured. The integrand is in a fixed frame and the circulation loop is in the moving frame of the Lagrangian fluid parcels. The frame of the specific momentum in the integrand is Eulerian and the frame of the moving loop is Lagrangian. Likewise the data observation frame and the fluid motion frame will differ, if one is modelled as Itô and the other as Stratonovich. Thus, it makes sense that the shifts between frames which occur in transforming a Lagrangian trajectory from Itô to Stratonovich form would introduce non-inertial fictitious forces in the motion equations. This was already clear from the Coriolis force and the Craik-Leibovich vortex force in deterministic modelling of fluid dynamics. In particular, the Coriolis force arises because the fluid is moving in a reference with coordinates $\mathbf{x}$ on the surface of the rotating Earth. The Coriolis parameter is ${\rm curl}\boldsymbol{R}(\mathbf{x}) = 2\boldsymbol{\Omega}(\mathbf{x})$ where $\boldsymbol{\Omega}(\mathbf{x})$ is the angular velocity of the Earth, relative to the fixed stars. Newton's momentum force law in (1) becomes

$$\frac{d\boldsymbol{P}}{dt} := \frac{d}{dt}\oint_{c(\boldsymbol{u}_t^L)} \boldsymbol{u}_t(\mathbf{x})\cdot d\mathbf{x} = \frac{d}{dt}\oint_{c(\boldsymbol{u}_t^L)} \big(\boldsymbol{u}_t^L(\mathbf{x}) + \boldsymbol{R}(\mathbf{x})\big)\cdot d\mathbf{x} = \oint_{c(\boldsymbol{u}_t^L)} \boldsymbol{f}(\mathbf{x})\cdot d\mathbf{x} =: \boldsymbol{F}\,, \tag{24}$$

and Newton's acceleration force law in (3) becomes

$$\frac{d\boldsymbol{V}}{dt} := \frac{d}{dt}\oint_{c(\boldsymbol{u}_t^L)} \boldsymbol{u}_t^L(\mathbf{x})\cdot d\mathbf{x} = \oint_{c(\boldsymbol{u}_t^L)} \big(\boldsymbol{f}(\mathbf{x}) + \boldsymbol{u}_t^L \times 2\boldsymbol{\Omega}\big)\cdot d\mathbf{x} =: \boldsymbol{F} + \boldsymbol{F}_{Coriolis}\,. \tag{25}$$

Similarly, waves are Eulerian while fluid motion is Lagrangian: waves move relative to fixed space through the moving fluid, while the motion of the fluid Doppler shifts the wave frequency. In the Craik-Leibovich model, the Eulerian velocity (defined as the total specific momentum) is posited as $\overline{\mathbf{u}}_t = \overline{\mathbf{u}}_t^L - \overline{\mathbf{u}}^S(\mathbf{x})$. This is the difference between the Lagrangian fluid transport velocity $\overline{\mathbf{u}}_t^L$ and another velocity $\overline{\mathbf{u}}^S(\mathbf{x})$ called the Stokes drift velocity due to the waves, which must be prescribed from observed wave conditions. The Craik-Leibovich fictitious vortex force arises as in (18) for the same reason as for the Coriolis force in equations (24) and (25), except that one replaces $\boldsymbol{R}(\mathbf{x}) \to -\overline{\mathbf{u}}^S(\mathbf{x})$.

What does all this mean for the original problem of comparing Itô data with Stratonovich equations of motion derived from Hamilton's principle for stochastic fluid equations in the Euler-Poincaré form (11)? It means that no fictitious forces due to changes of frame by the Itô correction need to be considered as long as one evolves the total specific momentum, $\boldsymbol{u}_t = \boldsymbol{u}_t^L + \frac{1}{2}\big(\boldsymbol{\xi}(\mathbf{x}_t)\cdot\nabla\big)\boldsymbol{\xi}(\mathbf{x}_t)$, which lives naturally in the Eulerian data frame. However, if one decides to evolve the Lagrangian transport velocity, $\boldsymbol{u}_t^L$, instead of the Eulerian specific momentum then fictitious forces will arise due to the Itô correction, $-\frac{1}{2}\big(\boldsymbol{\xi}(\mathbf{x}_t)\cdot\nabla\big)\boldsymbol{\xi}(\mathbf{x}_t)$.

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