MSRI Soergel bimodule workshop
June/July 2017

Week 1 Day 3 Afternoon: Supplementary/Advanced Exercises

Practice in two colors

1. Verify that the two-color relations (the "dot" relation and the "associativity" relation) imply the "idempotent decomposition" relation.

2. Let $m_{st} = m < \infty$. For $k > 0$, let $\underline{w} = stst \dots st$ of length $2(m+k)$. What is the dimension of $\mathrm{Hom}(BS(\underline{w}), R)$ in degree $-2k$? Draw a light leaf map in that degree. Now draw several different graphs realizing the same morphism.

Localization and diagrammatic presentation

After localization to Q, the fraction field of R, the Bott–Samelson bimodule $B_s \otimes_R Q$ splits as a direct sum of Q_s and Q (when using localization we ignore the grading). Therefore, for any subsequence $\mathbf{e} \subset \underline{w}$, there is a summand $Q_{\mathbf{e}} \overset{\oplus}{\subset} BS(\underline{w}) \otimes_R Q$, a tensor product of either $Q_{\underline{w}_i}$ or Q depending on whether $\mathbf{e}_i$ is 1 or 0. Obviously $Q_{\mathbf{e}} \cong Q_x$ when $\mathbf{e}$ expresses the element x.

3. Fix $\underline{w}$ arbitrary, and $\underline{x}$ reduced. Let $E(\underline{w}, x)$ denote the set of light leaves for subexpressions of $\underline{w}$ which terminate in x, living inside $\mathrm{Hom}(\underline{w}, \underline{x})$. Use localization and the Bruhat path dominance order to prove that the images in $\mathbb{BS}\mathrm{Bim}$ of the light leaves maps in $E(\underline{w}, x)$ are all linearly independent.

4. Show that the functor from $\mathcal{D}$ to $\mathbb{BS}\mathrm{Bim}$ is an equivalence of categories, assuming that double leaves form a basis for morphisms in $\mathcal{D}$.

For fun?

5. Find the appropriate notion of the Jones–Wenzl relation in type B_2, with the usual non-symmetric Cartan matrix. Find the orthogonal idempotents giving the direct sum decomposition $B_s B_t B_s B_t \cong B_{stst} \oplus B_{st} \oplus B_{st}$. (Warning: Computationally intensive.)