MSRI Soergel bimodule workshop

June/July 2017

Week 2 Day 3 Afternoon: Basic and Supplemental Exercises

Supplemental is marked with (*).

1. Verify all the statements made about FT_2 in lecture. (Includes: the eigenmaps are chain maps, FT_2 is prediagonalizable, the Lagrange interpolation complexes are as described, etcetera). Verify that the Lagrange interpolation complexes descend to the appropriate idempotents in the Hecke algebra.

2. Let $A = \mathbb{Z}[x]/(x^n - 1)$, and consider the three-term complex

$$F = (\underline{A} \to A \to \mathbb{Z}).$$

Here, the underline indicates that the first term is in homological degree zero; $\mathbb{Z}$ represents the trivial module where x acts by 1; the first differential sends 1 to $x - 1$.

Prove that F is categorically diagonalizable: find the eigenmaps, prove that F is prediagonalizable, and construct the projections to eigencategories as bounded above complexes.

Remark. The $n = 2$ case of the above example is just a toy version of the full twist in type A_1.

3. (*) Let X denote the two term complex

$$(B_s \to B_s(2))$$

where the differential is multiplication by α_s on the left minus multiplication by α_s on the right, i.e. $d_X = \alpha_s \otimes 1 - 1 \otimes \alpha_s$. Let Y denote the analogous complex with $d_Y = \alpha_s \otimes 1 + 1 \otimes \alpha_s$.

a) Prove that X and Y are not homotopy equivalent.

b) Prove that $F_s \otimes X \cong Y(-1)$.

c) Conclude that the full subcategory of the homotopy category consisting of complexes Z where $F_s \otimes Z \cong Z(-1)$ is not triangulated (i.e. closed under mapping cones). (Hint: prove that B_s is in this subcategory.)

4. Compute the space of all chain maps from any shift of R into FT_2^{-1}, modulo homotopy. Deduce that FT_2^{-1} does not have enough eigenmaps to be diagonalizable. (Instead, it has "backward eigenmaps.")

5. (*) Compute the minimal complex of $J_3 = F_{s_2}F_{s_1}F_{s_1}F_{s_2}$. If J_3 had eigenmaps corresponding to the eigenvalues of j_3, which scalar functors would they come from? Does J_3 have enough eigenmaps to be categorically diagonalizable?

6. Assuming that the minimal complex of FT_3 is as given in class, and assuming that any pair of summands which could cancel (by Gaussian elimination) do actually cancel, verify that the cone of α_λ sends any indecomposable in cell λ to a complex in strictly lower cells. (Hint: Why is it sufficient to check this for R, B_s, and B_{sts}?)

7. (*) Suppose $\mathcal{A}$ is the (bounded above) homotopy category of an additive monoidal category, and suppose one can write the monoidal identity $\mathbb{1}$ as a finite convolution of complexes $\mathbf{P}_i$. Suppose that $\mathbf{P}_i \otimes \mathbf{P}_j \cong 0$ for $i \neq j$. Deduce that $\mathbf{P}_i \otimes \mathbf{P}_i \cong \mathbf{P}_i$.