MSRI Soergel bimodule workshop

June/July 2017

Week 1 Day 5 Morning: Supplementary/Advanced Exercises

Lefschetz linear algebra

1. Let $H = \bigoplus H^i$ be a finite dimensional graded $\mathbb{R}$-vector space and $L : H^\bullet \to H^{\bullet+2}$ an operator of degree 2. Show that H admits a representation of $\mathfrak{sl}_2(\mathbb{R}) = \mathbb{R}f \oplus \mathbb{R}h \oplus \mathbb{R}e$ with $e = L$ and $hx = mx$ for all $x \in H^m$ if and only if L satisfies the hard Lefschetz theorem (i.e. $L^m : H^{-m} \to H^m$ is an isomorphism for all $m \geq 0$).

2. Prove that the Lefschetz decomposition is orthogonal for the Lefschetz form.

3. Suppose that $H = \oplus H^i$ and $W = \oplus W^j$ are finite dimensional graded real vector spaces with forms $\langle -, - \rangle$ and Lefschetz operators L_H and L_W. Suppose that $H^{\mathrm{odd}} = 0$ or $H^{\mathrm{even}} = 0$, that L satisfies the hard Lefschetz theorem on H and that

$$\underline{\dim} W := \sum \dim W^i v^i = (v + v^{-1}) \underline{\dim} H.$$

Show that W satisfies (HR) if and only if the signature of the Lefschetz form $(-,-)^{-i}_{L_W}$ on W^{-i} is equal to the dimension of the primitive subspace $P^{-i+1}_{L_H} \subset H^{-i+1}$ (by convention $P^1_{L_H} = 0$).

4. (If you know a little Hodge theory). Show that the hard Lefschetz theorem for complex algebraic varieties of (complex) dimension n is a formal consequence of the Hodge–Riemann bilinear relations for varieties of dimension $n-1$ and the weak Lefschetz theorem. Why is this not a proof of the hard Lefschetz theorem? (This exercise is intended to explain where the terminology "weak Lefschetz substitute" comes from. We will see next week that although one does not have an analogue of the weak Lefschetz theorem for Soergel bimodules, the first differential on a Rouquier complex provides a substitute.)

5. This question explores Hodge theory for the Grassmannian $H^*(Gr(3,6))$, using a combinatorial model. Let $P(3,6)$ denote the set of partitions which fit inside a 3×3 rectangle (I will describe elements of $P(3,6)$ using Young tableaux). The *degree* of a partition will be -9 plus twice the number of boxes; for example, the partition $(3,1,1)$ has degree $+1$. We say that two partitions are *complimentary* if one can be glued to the 180 degree rotation of the other to obtain the full 3×3 rectangle; for example, $(3,2,0)$ and $(3,1,0)$ are complimentary.

Let H denote the graded vector space with basis $\{v_\lambda\}_{\lambda \in P(3,6)}$. Place a symmetric bilinear form on H, where $\langle v_\lambda, v_\mu \rangle = 1$ when λ and μ are complimentary, and it equals zero otherwise. Place an operator $L \colon H \to H(2)$ on this space, where $Lv_\lambda = \sum_\mu v_\mu$ is the sum over partitions $\mu \in P(3,6)$ obtained from λ by adding a single box.

(If you want a shorter exercise, replace $P(3,6)$ with $P(n, n+m)$ for smaller values of n and m.)

a) Prove that L is a Lefschetz operator.

b) Prove that L has the hard Lefschetz property. Compute a basis of each primitive subspace.

c) Prove that L has the Hodge-Riemann bilinear relations.

(If you had instead done the exercise for $P(n, n+m)$, how many summands would there be in the Lefschetz decomposition?)