MSRI Soergel bimodule workshop

June/July 2017

Week 1 Day 2 Morning: Basic Exercises

1. a) Let $f \in R$, and write $f = g + h\alpha_s$ for $g, h \in R^s$. What is $\partial_s(f)$? What is $\partial_s(f\alpha_s)$?

b) Let $c_s = \frac{\alpha_s}{2} \otimes 1 + 1 \otimes \frac{\alpha_s}{2}$ and $c_1 = 1 \otimes 1$ denote certain elements of B_s. Show that $\{c_s, c_1\}$ form a basis of B_s as a right R-module. For any $f \in R$, find nice formulas (involving the Demazure operator) for fc_s and fc_1 in terms of this basis. (Hint: if you're stuck, it may help to write $f = g + h\alpha_s$.)

Dihedral groups

2. Suppose (W, S) is a finite dihedral group, with $a_{s,t} = a_{t,s} = -2\cos(\frac{\pi}{m})$.

a) Find a formula for a quadratic polynomial $z \in R$ for which $s(z) = t(z) = z$.

b) Enumerate the *roots*, i.e. the W-orbit of α_s and α_t. Devise a reasonable notion of positive roots. (This was a previous exercise.)

c) Let $\mathbb{L}$ be the product of the positive roots. Show that $s(\mathbb{L}) = t(\mathbb{L}) = -\mathbb{L}$. If this is not familiar to you, just do the cases $m = 2, 3$.

d) Assume $m = 2$ and let w_0 be the longest element. What is $\partial_{w_0}(\mathbb{L})$? Repeat for $m = 3$. Care to generalize?

e) Let ω_s be the unique element satisfying $\partial_s(\omega_s) = 1$ and $\partial_t(\omega_s) = 0$. Let $Z := \prod_{x \in W/\langle t \rangle} x(\omega_s)$. Show that $s(Z) = t(Z) = Z$. (In fact, $R^{s,t} = \mathbb{R}[z, Z]$, though this is hard to prove without the Chevalley theorem.)

f) Suppose that $m = 2$. Find dual bases $\{a_i\}$ and $\{b_i\}$ for R over $R^{s,t}$, under the pairing $(f, g) \mapsto \partial_{w_0}(fg)$. Show that $\sum a_i b_i = \mathbb{L}$.

Bott–Samelson bimodules

3. Exercises from class:

a) Verify all the statements made in lecture about the Demazure operator ∂_s.

b) Why is $BS(\underline{w})$ free as a right R-module?

c) (This should be computationally very painful. Try it only to convince yourself that better technology would be useful!) When $m_{st} = 3$, verify that $B_s B_t B_s \cong B_s \oplus (R \otimes_{R^{s,t}} R(3))$.

4. Construct a map $B_s \otimes_R B_t \to B_{s,t}$ sending $1 \otimes 1 \otimes 1 \mapsto 1 \otimes 1$, when $m = 2$. Why is there no such map when $m > 2$?

Frobenius extensions

5. In class we stated that a (commutative) Frobenius extension $A \subset B$ (equipped with trace map $\partial \colon B \to A$) is the same data as a choice of biadjunction between Ind_A^B and Res_A^B. Prove this. En route you must prove the following: choose dual bases and consider $\Delta = \sum a_i \otimes b_i \in B \otimes_A B$. This element is independent of the choice of dual bases, and $f\Delta = \Delta f$ for all $f \in B$.

6. Let $H \subset G$ be an inclusion of finite abelian groups. Then $\mathbb{R}[H] \subset \mathbb{R}[G]$ is a Frobenius extension. Compute the units and counits of adjunction. What is ∂^2? Can this extension be graded in an interesting way? (For a challenge, look up non-commutative Frobenius extensions, and do not assume that the groups are abelian.)