MSRI Soergel bimodule workshop

June/July 2017

Week 1 Day 3 Morning: Basic Exercises

Diagrammatics for Soergel bimodules

1. Check that the one color relations hold in Soergel bimodules.

Remark. (For several problems below) In an additive category, in order to demonstrate morphism-theoretically that $X \cong M \oplus N$, one must provide morphisms

$$p_M \colon X \to M, \qquad i_M \colon M \to X, \qquad p_N \colon X \to N, \qquad i_N \colon N \to X,$$

which satisfy the following relations:

$$\begin{aligned} p_M i_M &= \mathbb{1}_M, \\ p_N i_N &= \mathbb{1}_N, \\ p_M i_N &= 0, \\ p_N i_M &= 0, \\ \mathbb{1}_X &= i_M p_M + i_N p_N. \end{aligned}$$

Comprehend this fact. The final equation decomposes the identity of X into orthogonal idempotents.

2. Show that $B_s B_s \cong B_s(1) \oplus B_s(-1)$ by following the rubric of the remark above.

3. a) Consider a one-color Soergel diagram without polynomials, viewed as a graph (with boundary) having only trivalent and univalent vertices. Prove that any two trees with the same boundary are equal. Prove that any graph which is not a tree evaluates to zero.

b) Prove that any universal morphism (in many colors) with empty boundary is equal to a polynomial. (Hint: use induction on the number of connected components.)

4. Use the Soergel Hom formula to compute the size (i.e. graded rank) of the following Hom spaces:

$$\mathrm{Hom}(B_s, B_t), \quad \mathrm{Hom}(B_s B_s, B_s), \quad \text{and } \mathrm{Hom}(B_s, B_s B_t B_s) \quad (\text{assuming } m_{st} > 2).$$

Construct a diagrammatic basis for each space. (Hint: You only need universal diagrams.)

5. a) Write down the two-color relations when $m_{st} = 2$. Prove that $B_s B_t \cong B_t B_s$ by constructing inverse isomorphisms.

b) Write down the two-color relations when $m_{st} = 3$. Prove that $B_s B_t B_s \cong X \oplus B_s$, where X is the image of an idempotent constructed using two 6-valent vertices, by following the rubric of the remark above.

c) (Still with $m_{st} = 3$.) One similarly has $B_t B_s B_t \cong Y \oplus B_t$. Prove that X is isomorphic to Y. (Extra credit: Extend the remark above to a rubric which describes when two summands of different objects are isomorphic.)

Jones–Wenzl projectors

6. Let TL_n be the Temperley–Lieb algebra with n strands, where a circle evaluates to $-[2] = -(q+q^{-1}) \in \mathbb{Q}(q)$. The Jones-Wenzl projector JW_n is the unique element of TL_n for which the coefficient of the identity diagram is 1, and which is killed by all caps above. (In a supplementary exercise you prove that JW_n is unique, and is an idempotent.) Clearly JW_1 is just the identity element, where the condition of being killed by caps is vacuous. In previous exercises you computed JW_2 and maybe JW_3.

Verify the following recursive formula.