MSRI Soergel bimodule workshop

June/July 2017

Week 1 Day 1 Morning: Supplementary/Advanced Exercises

Coxeter groups

1. To practice with Coxeter groups, we play with some embeddings and foldings.

a) Let $\{s, t, u\}$ be the simple reflections inside the Coxeter group of type A_3. Show that the subgroup generated by (su) and t is a Coxeter group of type $B_2 = I_2(4)$, with simple reflections $\{su, t\}$, by checking the braid relation. This implies that B_2 embeds inside A_3 as the invariants under a certain automorphism σ, induced by a diagram automorphism.

b) Let $\{s, t, u, v\}$ be the simple reflections inside the Coxeter group of type A_4. Show that the subgroup generated by (su) and (tv) is a Coxeter group of type $H_2 = I_2(5)$, with simple reflections $\{su, tv\}$. However, this subgroup is not the invariants of any diagram automorphism.

c) Embed the Coxeter group of type $I_2(m)$ inside the Coxeter group of type A_{m-1} for $m \geq 3$, using products of distinct simple reflections.

2. More embedding exercises.

a) Embed H_3 inside D_6. Embed H_4 inside E_8. Generalize this.

b) Look at star-shaped Coxeter groups: A_2, A_3, D_4, $\widetilde{D}_4$, and so forth. Consider the subgroup generated by the hub and by the product of the spokes. What subgroups do you get?

3. Now we do the previous exercises "in reverse." Let (W, S) be a Coxeter group, and fix $s \in S$. Consider the set Γ_s of elements of W which have a unique reduced expression, and which have s in their right descent set. Γ_s has the structure of a labeled graph, where each element $w \in \Gamma_s$ is labeled by the (unique!) element $t \in S$ in its left descent set, and where w, v are connected by an edge if and only if $w = uv$ for some $u \in S$.

a) Let $\{s, t\}$ be the simple reflections in type B_2. Compute that Γ_s is A_3, with the labelings corresponding to the embedding of B_2 inside A_3 from Q1.

b) Do the same for $I_2(m)$ and A_{m-1}.

c) Let (W, S) be the Coxeter group of type H_4. For $s \in S$, compute the labeled graph Γ_s.

d) Repeat the exercise for $I_2(\infty)$. What labeled graph do you obtain?

(If you know about such things, Γ_s is the W-graph of the left cell containing s. See Lusztig "Some examples of square integrable functions on a p-adic group".)

4. Here is a non-Coxeter presentation of S_4, which Ben is quite interested in. This is a very optional exercise. The generators are $s = (12)$, $t = (13)$, $u = (14)$.

a) Which braid relations do these satisfy? If there were only braid relations, what Coxeter group would it be?

b) What additional relations are satisfied?

c) Count the number of elements of each length with respect to this presentation.

5. The even signed symmetric group ESS_n was defined as the subgroup of the signed symmetric group SS_n where the number of sign changes was a multiple of 2. Let $m \in \mathbb{Z}$, $m > 2$. Prove that (unless n is small) the subset of SS_n where the number of sign changes is a multiple of m is not a subgroup.

6. Coxeter systems (W, S) are equipped with a standard length function ℓ, but can also be equipped with non-standard length functions, sometimes called *weights*. A weight L is a map $W \to \mathbb{Z}$ satisfying $L(uv) = L(u) + L(v)$ whenever $\ell(uv) = \ell(u) + \ell(v)$. Deduce the following elementary facts.

a) A weight function L is determined by the weights $L(s)$ of the simple reflections. Moreover, $L(s) = L(t)$ whenever m_{st} is odd.

b) Suppose one has an embedding of Coxeter groups $\iota\colon (W, S) \hookrightarrow (W', S')$ as in Q1, where each simple reflection $s \in S$ is sent to a product Πt of commuting simple reflections $t \in S'$. This equips (W, S) with a weight L, given by $L(s) = \ell(\iota(s))$. For each possible value of m_{st}, what are the possible values of the ratio of $L(s)$ to $L(t)$? It will help to remind oneself of the classification of finite Coxeter groups.

7. Continuing Q3 from basic: The *dual Coxeter complex* is a CW complex obtained by dualizing the Coxeter complex. In other words, there is a 0-cell for each simplex, a 1-cell connecting 0-cells if the simplices meet in a (codimension 1) face, a 2-cell glued along 1-cells if the faces all meet in a codimension 2 face, etc. Show that the dual Coxeter complex can be constructed directly as follows:

- There is a 0-cell for each $w \in W$. Said another way, there is a 0-cell for each coset of the trivial subgroup.
- There is a 1-cell for each pair $\{w, ws\}$ with s a simple reflection. Said another way, there is a 1-cell for each coset of each rank 1 parabolic subgroup.
- There is a 2-cell for each coset of each **finite** rank 2 parabolic subgroup.
- ...
- There is a k-cell for each coset of each finite rank k parabolic subgroup.
- However, if the rank of W is n, then the process ends at $k = n - 1$.

Also, draw the dual Coxeter complex for the same list of Coxeter groups.

Remark. In fact, one can also construct the *completed dual Coxeter complex* by also including the step $k = n$. This makes no difference when W is infinite, but glues in a single n-cell when W is finite. The resulting complex is contractible. This is shown in Ronan, "Lectures on buildings."

Generalizing the reflection representation

8. We will now use the term "Cartan matrix" to refer to any matrix indexed by S, satisfying $a_{s,s} = 2$ and $a_{s,t} = 0 \iff a_{t,s} = 0$, with coefficients in a base ring $\Bbbk$ (not necessarily integers). A Cartan matrix need not be symmetric, or even symmetrizable (i.e. conjugate by a diagonal matrix to a symmetric matrix).

a) Given a Cartan matrix, one can still construct a vector space $\mathfrak{h}^*$ with involutions $s \in S$ acting upon it. Show that (st) has order m if and only if $a_{s,t}a_{t,s}$ is algebraically equivalent to $[2]^2$ for q a primitive $2m$-th root of unity.

b) Show that any Cartan matrix admitting a representation of a Weyl group, and satisfying $a_{s,t} = 0 \iff a_{t,s} = 0$, is symmetrizable.

c) Show that the following matrix admits a representation of the affine Weyl group $\tilde{A}_4$, for any $q \in \mathbb{C}^*$. When is it symmetrizable? When is it conjugate, by a diagonal matrix, to a representation defined over $\mathbb{R}$?

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & -q^{-1} \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -q \\ -q & 0 & 0 & -q^{-1} & 2 \end{pmatrix}.$$