MSRI Soergel bimodule workshop
June/July 2017

Week 1 Day 1 Morning: Basic Exercises

Coxeter groups

1. a) Let (W, S) be a Coxeter system and $w \in W$. Using the exchange condition, prove the *descent property*, that

$$\{s \in S \mid \ell(ws) < \ell(w)\} = \{s \in S \mid w \text{ has a rex ending in } s\}.$$

This set is called the *right descent set* of w.

b) Let $W = S_4$, the symmetric group on $\{1, 2, 3, 4\}$. Then W has the structure of a Coxeter group with $S = \{s_1, s_2, s_3\}$ where s_i denotes the transposition $(i, i + 1)$. Show that $w = s_1 s_2 s_1 s_3 s_2 s_1$ is a reduced expression. What is the right descent set of w? For each element s of the right descent set, find the reflection which should be removed from this expression to obtain a reduced expression for ws (cf. the exchange condition).

Remark. Using Matsumoto's theorem and the descent property, if $t_1 t_2 \cdots t_d s$ is **not** reduced, but $t_1 t_2 \cdots t_d$ is reduced, then one can apply braid relations to $t_1 t_2 \cdots t_d$ to get the simple reflection s at the end. In practice, finding a sequence of braid relations which bring s to the end will help you figure out which reflection to remove for the exchange condition.

2. Let $W = S_4$, with notation as above.

a) To warm up, draw the (weak right) Bruhat graph. This is a graded graph whose vertices are the (twenty-four) elements $w \in W$, each assigned a height equal to $\ell(w)$. There is an edge between w and v if $w = vs$ for some $s \in S$.

b) For each $w \in W$, draw its *reduced expression graph* or *rex graph* Γ_w. The vertices of Γ_w are reduced expressions for W. There is an edge between two reduced expressions if they differ by a single application of a braid relation (it helps to label the edge with the number m_{st} associated to this braid relation).

c) What cycles appear in these rex graphs?

d) W is known as the Coxeter group of type A_3; you should familiarize yourself with the classification of finite Coxeter groups into types A through I. Draw the rex graph for the longest element in types $A_1 \times A_1 \times A_1$, $A_1 \times A_2$, and B_3. Do it for H_3 if you're feeling ambitious.

3. Let (W, S) be a Coxeter group of rank n. Its *Coxeter complex* is a simplicial complex constructed as follows:

- There is an $(n-1)$-simplex labeled by w for each $w \in W$. The n faces of this $(n-1)$-simplex are labeled by the simple reflections s.
- Whenever $w = sv$, glue the simplices w and v along the face s. (Technically, one should fix the orientations when gluing faces. If $\ell(w) = \ell(v) + 1$, then glue the outward face of s in v to the inward face of s in w.)

Draw the Coxeter complex for the following Coxeter groups: $I_2(m)$ for m finite, $I_2(\infty)$, A_3 (the barycentric subdivision of a tetrahedron), B_3, $\widetilde{A_2}$, $\widetilde{B_2}$.

Quantum numbers

4. Suppose that W is a dihedral group, with $S = \{s,t\}$ and $m = m_{s,t}$. Instead of writing $a_{s,t} = -2\cos(\frac{\pi}{m_{s,t}})$, let us just write $a_{s,t} = -(q+q^{-1})$. After all, when $q = e^{\frac{\pi i}{m_{s,t}}}$, the two formula agree. This will allow us to write formulae which work simultaneously for all dihedral groups, using quantum numbers.

a) Consider the quantum number

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \ldots + q^{3-n} + q^{1-n}.$$

One has $[1] = 1$ and $[0] = 0$. Find a formula for $[2][n]$ in terms of quantum numbers.

b) The statement that q^2 is a primitive m-th root of unity is equivalent to what statement about quantum numbers? The statement that q is a primitive $2m$-th root of unity is equivalent to what statement about quantum numbers? What about when q is a primitive m-th root of unity for m odd? Compare $[m-k]$ and $[k]$. Compare $[m+k]$ and $[m-k]$.

c) Compute the matrix for the action of $(st)^k$ on the 2-dimensional space spanned by α_s and α_t, in terms of quantum numbers. When does (st) have finite order m? When $m = 2k+1$ is the order of (st), what is $(st)^k(\alpha_s)$?

d) Assume that q is a primitive $2m$-th root of unity. The *positive roots* for the dihedral group are the elements in the W-orbit of the simple roots $\{\alpha_s, \alpha_t\}$, which have the form $a\alpha_s + b\alpha_t$ for $a, b \geq 0$. Find a simple enumeration of these roots as linear combinations of α_s and α_t.