MSRI Soergel bimodule workshop

June/July 2017

Week 1 Day 1 Afternoon: Supplementary/Advanced Exercises

Miscellaneous exercises from lecture

1. Some more questions from lecture, dealing with the standard trace and standard pairing.

a) Compute $\varepsilon(H_x H_y)$. When is it non-zero?

b) Show that $\varepsilon(ab) = \varepsilon(ba)$.

c) Show that the standard basis $\{H_x\}$ and the dual standard basis $\{\overline{H}_x\}$ are dual bases for the standard pairing.

d) Consequently, if $b = \overline{b}$ and $b' = \overline{b}'$ are self-dual, then you can pretend that the standard basis is self-dual for the standard pairing, i.e. if $b = \sum p_x H_x$ and $b' = \sum q_x H_x$ then $(b, b') = \sum p_x q_x$. We call this the "false orthogonality" of the standard basis.

e) Show that the KL basis is graded orthonormal for the standard pairing. That is, $(b_w, b_x) = \delta_{w,x} + v\mathbb{Z}[v]$.

f) Show that b_s is self-biadjoint.

Kazhdan–Lusztig polynomials

2. Let W be a Weyl group of type D_4 with generating reflections s, t, u, v such that s, u, v all commute. Consider the reduced expression $\underline{w} = suvtsuv$.

a) Use the Deodhar defect formula to write the element $b_{\underline{w}} = b_s b_u b_v b_t b_s b_u b_v$ in terms of the standard basis. (This was a previous exercise.)

b) Write the element $b_{\underline{w}}$ in terms of the KL basis.

c) Hence compute the KL polynomial $h_{suv,suvtsuv}$. (This is entry number 124 in Goresky's D_4 table.)

Deodhar defect formula

3. Prove the Deodhar defect formula.

4. Let $\underline{w} = s_1 \dots s_m$ be an expression. We write $x \leq \underline{w}$ if there exists a subexpression $\mathbf{e}$ of $\underline{w}$ with $x = \underline{w}^{\mathbf{e}}$. Given two subexpressions $\mathbf{e}, \mathbf{e}'$ of $\underline{w}$ let $x_0, x_1, \dots$ and $x'_0, x'_1, \dots$ be their Bruhat strolls (e.g. $x_i := s_1^{e_1} \dots s_i^{e_i}$). We define the *path dominance order* on subexpressions of $\underline{w}$ by saying that $\mathbf{e} \leq \mathbf{e}'$ if $x_i \leq x'_i$ for $1 \leq i \leq \ell(\underline{w})$.

Fix $x \leq \underline{w}$. Show that there is a unique subexpression $\mathbf{e}$ of $\underline{w}$ representing x, the *canonical subexpression* for x, which is characterized by the following equivalent conditions:

a) $\mathbf{e} \leq \mathbf{e}'$ for any subexpression $\mathbf{e}'$ of $\underline{w}$ with $\underline{w}^{\mathbf{e}'} = x$ (i.e. $\mathbf{e}$ is the unique minimal element representing x in the path dominance order);

b) $\mathbf{e}$ has no D's in its UD labelling;

c) $\mathbf{e}$ is of maximal defect among all subexpressions $\mathbf{e}'$ of $\underline{w}$ with $\underline{w}^{\mathbf{e}'} = x$.

(If you know about Bott–Samelson resolutions: What geometric fact does the existence of $\mathbf{e}$ correspond to? Do you think there is a unique maximal element in the path dominance order?)

Just for fun: unequal parameters

Hecke algebras can defined in more generality, using unequal parameters, which this sequence of exercises explores. One notable fact is that Kazhdan–Lusztig positivity can fail in the unequal parameters case! However, even in cases where it fails, it is possible to "categorify" the Hecke algebra. You can skip these exercises entirely if you want, though they make good practice!

5. Coxeter systems (W, S) are equipped with a standard length function ℓ, but can also be equipped with non-standard length functions, sometimes called *weights*. A weight L is a map $W \to \mathbb{Z}$ satisfying $L(uv) = L(u) + L(v)$ whenever $\ell(uv) = \ell(u) + \ell(v)$. Deduce the following elementary facts.

a) A weight function L is determined by the weights $L(s)$ of the simple reflections. Moreover, $L(s) = L(t)$ whenever m_{st} is odd.

b) Suppose one has an embedding of Coxeter groups $\iota\colon (W, S) \hookrightarrow (W', S')$ as in Q1 from the supplementary exercises this morning, where each simple reflection $s \in S$ is sent to a product Πt of commuting simple reflections $t \in S'$. This equips (W, S) with a weight L, given by $L(s) = \ell(\iota(s))$. For each possible value of m_{st}, what are the possible values of the ratio of $L(s)$ to $L(t)$?

6. Let (W, S, L) be a Coxeter system with a weight function. The *Hecke algebra with unequal parameters* $\mathbf{H}(W, S, L)$ is the $\mathbb{Z}[v^{\pm 1}]$-algebra generated by H_s, $s \in S$, subject to the usual braid relation and a new quadratic relation:

$$(H_s + v^{L(s)})(H_s - v^{-L(s)}) = 0.$$

Most features of the standard Hecke algebra extend to the Hecke algebra with unequal parameters, especially when the weight function is *positive* (i.e. $L(s) > 0$ for all $s \in S$).

a) Compute H_s^{-1}. Find the correct definition for a self-dual element b_s.

b) Find a formula for $H_w b_s$.

c) When L is positive, prove the existence and the uniqueness of the KL basis. What extra complication is required in the inductive construction?

d) Modify the Deodhar formula.

7. Continuing Q6: Let $\{s, t\}$ be the generators of a Coxeter group of type B, with $m_{st} = 4$, and let $L(s) = 1$ and $L(t) = 2$. Let $\mathbf{H}$ denote the Hecke algebra with unequal parameters. Compute the KL basis. Note that some KL polynomials have negative coefficients! This does not happen for usual Hecke algebras, as we will prove in this workshop.

8. Continuing Q6: When one Coxeter group (W, S, L) embeds inside another (W', S', ℓ) as the invariants under a diagram automorphism, one might expect there to be a corresponding relationship between their Hecke algebras (with unequal parameters). However, the relationship is quite subtle. We quickly explore this when (W', S') is $A_1^{\times n}$, and σ is the diagram automorphism which permutes the copies of A_1 cyclically. Therefore, (W, S, L) has type A_1, and $L(s) = n$.

a) Compute the self-dual generator of $\mathbf{H}(W', S', \ell)^\sigma$, and its square.

b) Compute the self-dual generator of $\mathbf{H}(W, S, L)$, and its square.

c) When $n = p$ is prime, show that these algebras are isomorphic modulo p.