MSRI Soergel bimodule workshop
June/July 2017

Week 2 Day 1 Afternoon: Basic Exercises

Perverse filtration

1. Compute the terms appearing in the minimal complex of $F_s^{\otimes m}$ for $m \geq 0$. Describe its perverse filtration explicitly.

2. Assuming Soergel's conjecture, show that the "perverse t-structure" on $K^b(\mathbb{SB}\mathrm{im})$ actually is a t-structure.

Rouquier complexes for dihedral groups

3. Read the Wikipedia article on Karoubi envelope. Let $\mathcal{A}$ be an additive category, $\overline{X}, \overline{Y} \in \mathcal{A}$, and $e_X : \overline{X} \to \overline{X}$, $e_Y : \overline{Y} \to \overline{Y}$ idempotents with corresponding direct summands X, Y in $\mathrm{Kar}(\mathcal{A})$. Show that there is an isomorphism

$$\mathrm{Hom}_{\mathrm{Kar}(\mathcal{A})}(X, Y) \cong e_Y \circ \mathrm{Hom}_{\mathcal{A}}(\overline{X}, \overline{Y}) \circ e_X,$$

compatible with composition in an appropriate sense.

4. Let $W = \langle s, t \rangle$ be dihedral.

a) Let $w \in W$, $u \in \{s, t\}$, and $\check{u}$ the simple reflection different from u. Show that

$$B_w B_u \cong \begin{cases} B_{wu} & \text{if } w \in \{1, \check{u}\}; \\ B_{wu} \oplus B_{w\check{u}} & \text{if } wu > w \text{ and } w \notin \{1, \check{u}\}; \\ B_w(-1) \oplus B_w(1) & \text{if } wu < w. \end{cases}$$

(Hint: Recall that B_w is the image of a Jones–Wenzl projector. Using the description of morphisms in the Karoubi envelope in the previous exercise, construct morphisms realizing each direct sum decomposition. In the case $wu < w$, these morphisms will generalize those used to show the decomposition $B_s B_s \cong B_s(-1) \oplus B_s(1)$.)

b) Recall the (perverse) minimal Rouquier complex

$$F_{st} = F_s F_t = (B_{st} \to B_s(1) \oplus B_t(1) \to R(2)).$$

For $m_{s,t} \geq 3$, use Gaussian elimination to show that $F_{st} F_s$ is homotopy equivalent to a (perverse) minimal complex of the form

$$B_{sts} \to B_{st}(1) \oplus B_{ts}(1) \to B_s(2) \oplus B_t(2) \to R(3),$$

which is therefore F_{sts}.

c) (Optional) Generalizing the previous part, show by induction that F_w is perverse for all $w \in W$.