MSRI Soergel bimodule workshop

June/July 2017

Week 1 Day 1 Afternoon: Basic Exercises

Miscellaneous exercises from lecture

1. a) Compute H_s^{-1}, and show that b_s is self-dual. Confirm that $b_s^2 = b_s(v + v^{-1})$.

b) Compute H_{st}^{-1} in terms of the standard basis. Given $w \in W$, for which $y \in W$ can there be a non-zero coefficient of H_y in the expression for H_w^{-1}? In the expression for $\overline{H_w}$? In the expression for $\omega(H_w)$?

c) Prove the uniqueness of the KL basis.

d) Find a formula for $H_w b_s$.

e) Extrapolate the construction from lecture into a proof of the existence of the KL basis.

Kazhdan–Lusztig basis

2. Let $W = S_4$, with $S = \{s, t, u\}$.

a) Consider the reduced expression $stsuts$ for the longest element w_0. Use the inductive algorithm to compute the KL basis element b_{w_0}. Several things you compute along the way are recorded in the next part of the question.

b) What are the KL basis elements b_s, b_{st}, b_{sts}, b_{stsu}, and b_{stsut}, in terms of the standard basis? (Hint: if you did this right, there are no non-trivial KL polynomials!) What are the product $b_s b_t$, $b_{st} b_s$, $b_{sts} b_u$, $b_{stsu} b_t$ and $b_{stsut} b_s$, in terms of the Kazhdan-Lusztig basis.

c) Which other KL basis elements can you deduce, using the symmetries of S_4? Which KL basis elements are missing? Which products $b_w b_x$ do you know, for $x \in S$ with $wx > w$, and which are missing?

d) Compute the missing KL basis elements and products $b_w b_x$.

e) Which non-trivial KL polynomials have you found?

3. Let (W, S) be a dihedral Coxeter group. That is

$$W = \langle s, t \mid s^2 = t^2 = (st)^{m_{st}} = e \rangle$$

where $e \in W$ is the identity, and $m_{st} \in \{2, 3, 4, \ldots, \infty\}$. Given $0 \leq m \leq m_{st}$ write $st(m)$ for the product $stst\ldots$ where m terms appear, and similarly for $ts(m)$. For example $st(0) = e$, $ts(1) = t$, $st(2) = st$, $ts(3) = tst$ etc.

a) Draw the Bruhat graph of W. Distinguish between the cases $m_{st} < \infty$ and $m_{st} = \infty$.

b) Use the inductive algorithm to compute $b_{st(m)}$ for $m \leq m_{st}$. Along the way, for $1 \leq m < m_{st}$ find an explicit formula for the products

$$b_s b_{st(m)}, \; b_s b_{ts(m)}, \; b_t b_{ts(m)} \qquad \text{and} \qquad b_t b_{st(m)}$$

in terms of the Kazhdan–Lusztig basis. (*Hint:* Calculate the first few cases and then use induction. Use caution with small m.)

c) Conclude that $h_{x,y} = v^{\ell(y) - \ell(x)}$ for all $x \leq y \in W$.

d) Using the formulas above, one can find an algebraic expression for $b_{st(m)}$ in terms of b_s and b_t, when $m \leq m_{st}$. For example, when $m = 2$ one has $b_{st} = b_s b_t$, and when $m = 3 \leq m_{st}$ one has $b_{sts} = b_s b_t b_s - b_s$. Find similar expressions when $m = 4, 5, 6$. Can you find a reasonable way to compute the coefficients which appear?

Remark. When m_{st} is finite, the longest element $w_0 = st(m_{st}) = ts(m_{st})$ has two reduced expressions, from which one will get two distinct algebraic expressions for b_{w_0} in terms of b_s and b_t. Setting these expressions equal gives a relation in the Hecke algebra amongst Kazhdan–Lusztig generators, analogous to the braid relation for the standard generators.

4. a) Use the Deodhar defect formula to compute $b_s b_s b_s$ in the standard basis.

b) Let s, t, u denote three distinct simple reflections. Use the Deodhar defect formula to compute $b_s b_t b_u$. Is this product equal to b_{stu}?

c) Let s and t be distinct simple reflections. What is $\varepsilon(b_s b_t b_s)$?

d) Let $\{s, t, u, v\}$ be the simple reflections in type D_4, where s, u, v all commute. Compute the product $b_s b_u b_v b_t b_s b_u b_v$, associated with the reduced expression $\underline{w} = suvtsuv$. (Hint: there are 2^7 subexpressions, which is a lot. However, for each given element $x < w$, there are not many subexpressions for x. There is a lot of symmetry, so the number of x one must examine is relatively small.)

5. Compute the pairing $(b_s b_t b_s, b_s)$ in two different ways.

a) Use biadjunction and the quadratic relation to express this pairing in terms of $\varepsilon(b_t b_s)$.

b) Use the Deodhar defect formula on both sides, and the "false orthogonality" of the standard basis (see the supplemental exercises).