MSRI Soergel bimodule workshop

June/July 2017

Week 1 Day 4 Afternoon: Supplementary/Advanced Exercises

1. Prove that a rex move $\underline{w} \to \underline{w'}$ (obviously with $w = w'$) induces an isomorphism between the top summands B_w and $B_{w'}$. Prove that this isomorphism does not depend on the choice of rex move.

Light leaves and indecomposables

2. Let $\underline{w}$ and $\underline{x}$ be rexes. We work modulo terms lower than x. We have seen that the coefficient of H_x inside $\underline{H}(\underline{w})$ describes the graded rank of $\mathrm{Hom}(BS(\underline{w}), B_x)$ modulo lower terms, and that light leaves for $\underline{w}$ with terminus x give a basis for this space (as a right R-module). Let e_w denote the idempotent in $\mathrm{End}(BS(\underline{w}))$ which picks out the indecomposable B_w. Let the *x-kernel of $\underline{w}$* be those linear combinations of light leaves with terminus x which vanish after precomposition with e_w. Then $\mathrm{Hom}(B_w, B_x)$ is precisely $\mathrm{Hom}(BS(\underline{w}), B_x)$ modulo the x-kernel, modulo lower terms.

a) Justify that the graded rank of $\mathrm{Hom}(BS(\underline{w}), B_x)$ modulo lower terms should agree with the coefficient of H_x in the character of $BS(\underline{w})$.

b) Assuming that $[B_w] = \underline{H}_w$, justify that the graded rank of $\mathrm{Hom}(B_w, B_x)$ modulo lower terms should agree with the coefficient $h_{x,w}$ of H_x in $\underline{H}_w$.

c) Let $m_{st} = 3$. Compute the x-kernel of the rex $\underline{sts}$, for each $x \le sts$. Do the graded ranks agree with your expectations?

3. Recall that $\underline{H}_w \underline{H}_s = \underline{H}_{ws} \sum_y \mu(y, w, s) \underline{H}_y$ for various integers $\mu(y, w, s)$.

a) Using the inductive algorithm, prove that $\mu(y, w, s)$ is zero unless $ys < y$. When $ys < y$, prove that $\mu(y, w, s)$ is equal to the coefficient of v^1 in $h_{y,w}$.

b) Assuming that one knows the y-kernel of $\underline{w}$, construct a diagrammatic basis of $\mathrm{Hom}^0(B_w B_s, B_y)$. Use symbols to denote e_w and e_y. (Hint: How does the light leaf construction connect degree $+1$ maps from $\underline{w}$ and degree $+0$ maps from $\underline{w}s$?)

Constructing idempotents

4. In this exercise, we work in type B_2, so that $m_{st} = 4$, and we use a non-symmetric Cartan matrix where $a_{s,t} = -1$ and $a_{t,s} = -2$.

a) In type B_2, write $\underline{H}_s \underline{H}_t \underline{H}_s \underline{H}_t$ as a sum of KL basis elements. How do you expect $B_s B_t B_s B_t$ to decompose?

b) Calculate the graded rank of $\mathrm{Hom}(B_s B_t, B_s B_t B_s B_t)$. Compute a diagrammatic basis of maps in degree 0 (you should have found it to be a 2-dimensional space).

c) Calculate the graded rank of $\mathrm{Hom}(B_s B_t B_s B_t, B_s B_t)$. Compute a diagrammatic basis of maps in degree 0. Why is this really easy, given the last part?

d) Calculate the graded rank of $\mathrm{End}(B_s B_t)$ and deduce that the only degree zero map is the identity.

e) Therefore, one can construct a 2×2 matrix given by composing a map $B_s B_t \to B_s B_t B_s B_t$ of degree 0 with a map $B_s B_t B_s B_t \to B_s B_t$ of degree 0, and computing the coefficient of the identity. This is called a *local intersection form*; one thinks of it as a bilinear form on $\mathrm{Hom}(B_s B_t B_s B_t, B_s B_t)$... How? Compute this matrix.

f) Whenever two maps pair under the local intersection form to the value 1, one can construct an idempotent in $\mathrm{End}(B_sB_tB_sB_t)$ which factors through B_sB_t. Whenever one has dual bases under the local intersection form, the corresponding idempotents will be orthogonal. Find dual bases and compute these orthogonal idempotents.

g) You have just proven that B_sB_t occurs as a summand inside $B_sB_tB_sB_t$ precisely 2 times. Can there be any other summands besides B_{stst}? Why or why not?

h) Suppose that we work in characteristic 2. How many times does B_sB_t occur as a summand inside $B_sB_tB_sB_t$?

5. What happens if you repeat the previous exercise in type H_2? One has $m_{st} = 5$, and $a_{s,t} = a_{t,s} = -\phi$, the (negative) golden ratio.

6. If you want more exercise, repeat again in type H_2, except with the goal of decomposing $B_sB_tB_sB_tB_s$.

7. Let V be the standard representation of $\mathfrak{sl}_2$. Compute the decomposition of $V \otimes V \otimes V$ into direct summands, by constructing an idempotent decomposition of the identity. Does this remind you of any previous exercises? What happens when q is an 8-th root of unity?