MSRI Soergel bimodule workshop

June/July 2017

Week 2 Day 1 Morning: Basic Exercises

Hodge theory of Soergel bimodules

1. Prove that the summand $B_x \subset BS(\underline{x})$ (for a reduced expression) contains both c_{bot} and c_{top}.

2. Fix a Soergel bimodule B and consider the two maps $\alpha, \beta : B \to BB_s = B \otimes_R B_s$ given by

$$\alpha(b) := bc_{\mathrm{id}} \quad \text{and} \quad \beta(b) := bc_s.$$

Together, $\alpha(B)$ and $\beta(B)$ span BB_s. Show that β is a morphism of bimodules, whilst α is a morphism of left modules. Find a formula for $\alpha(br)$ for $b \in B$ and $r \in R$.

Suppose that B is equipped with an invariant form $\langle -, - \rangle_B$. Prove that there is a unique invariant form $\langle -, - \rangle_{BB_s}$ on BB_s, which we call the *induced form*, satisfying

$$\langle \alpha(b), \alpha(b') \rangle_{BB_s} = \partial_s(\langle b, b' \rangle_B) \tag{1}$$

$$\langle \alpha(b), \beta(b') \rangle_{BB_s} = \langle b, b' \rangle_B \text{ and } \langle \beta(b), \alpha(b') \rangle_{BB_s} = \langle b, b' \rangle_B \tag{2}$$

$$\langle \beta(b), \beta(b') \rangle_{BB_s} = \langle b, b' \rangle_B \alpha_s \tag{3}$$

for all $b, b' \in B$. Show that the intersection form on a Bott–Samelson bimodule agrees with the form induced many times from the canonical form on R.

Now consider $\overline{BB_s}$, with its induced form valued in $\mathbb{R}$. Calculate a matrix for this form in some basis. Prove that the induced form is non-degenerate whenever the original form on $\overline{B}$ is non-degenerate.

Rouquier complexes

3. ("Gaussian elimination") Let $\mathcal{A}$ be an additive category. Consider a complex $A^\bullet \in C^b(\mathcal{A})$ of the form

$$\cdots \to A^{i-1} \xrightarrow{d^{i-1}} \begin{matrix} B^i \\ \oplus \\ C^i \end{matrix} \xrightarrow{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}} \begin{matrix} B^{i+1} \\ \oplus \\ C^{i+1} \end{matrix} \xrightarrow{d^{i+1}} A^{i+2} \to \cdots,$$

where $\delta : C^i \to C^{i+1}$ is an isomorphism. Show that $A^\bullet$ is homotopy equivalent to a complex of the form

$$\cdots \to A^{i-1} \xrightarrow{d^{i-1}} B^i \xrightarrow{\alpha'} B^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \to \cdots$$

with some new differential α'. Find α'.

4. Let F_s and F_s^{-1} denote the Rouquier complexes introduced in lectures. Check that $F_s F_s^{-1} \cong R$ in $K^b(R\text{-Bim})$ as sketched in the lectures.

5. Let $W = S_4$ with $s = (1\ 2)$, $t = (2\ 3)$, $u = (3\ 4)$. Write down the summands appearing in the minimal complex of $F_s F_u F_t F_s F_u$.

6. Suppose that $m_{st} = 2$. Find explicitly a chain map from $F_s F_t$ to $F_t F_s$ and back. Renormalize your maps such that the composition is the identity chain map.