MSRI Soergel bimodule workshop

June/July 2017

Week 1 Day 5 Morning: Basic Exercises

Lefschetz linear algebra

For a Soergel bimodule B, let $\overline{B}$ denote $B \otimes_R \mathbb{R}$ be the *right quotient*. For example, $\overline{BS(\underline{w})}$ has a basis over $\mathbb{R}$ given by 01-sequences. Just as $BS(\underline{w})$ has an intersection form valued in R, so too does $\overline{BS(\underline{w})}$ have an intersection form valued in $\mathbb{R}$.

Let $\rho \in \mathfrak{h}^*$ satisfy $\partial_s(\rho) = 1$ for all $s \in S$. For example, if $S = \{s\}$, then $\rho = \frac{\alpha_s}{2}$ will work. If $S = \{s, t\}$ and $m_{st} = 3$ then $\rho = \alpha_s + \alpha_t$ will work.

1. The endomorphism  of B_sB_s gives a degree 2 endomorphism L of the vector space $\overline{B_sB_s}$. What is $\langle c_{\text{bot}}, L^2(c_{\text{bot}})\rangle$? What is $\langle L(c_{\text{bot}}), L(c_{\text{bot}})\rangle$? Find an element b of degree zero which is perpendicular to $L(c_{\text{bot}})$. What is $\langle b, b\rangle$?

Now let L_0 be the degree 2 endomorphism of $\overline{B_sB_s}$ given by left multiplication by α_s. What is $L_0^2(c_{\text{bot}})$?

2. More generally, consider $\overline{(B_sB_s)}$, with the Lefschetz operator

$$L_{a,b} := \quad a\rho \,\Big|\quad \Big| \;+\; \Big|\, b\rho \,\Big|$$

for some $a, b \in \mathbb{R}$. For which a, b does the hard Lefschetz property hold? For which a, b do the Hodge–Riemann bilinear relations hold? For which a, b does (HR) hold with the opposite signatures?

3. Now we work with $\overline{BS(sts)}$ when $m_{st} = 3$. Let L be the degree 2 endomorphism of $\overline{B_sB_tB_s}$ given by left multiplication by ρ.

What is $L^3(c_{\text{bot}})$? What is $\langle c_{\text{bot}}, L^3(c_{\text{bot}})\rangle$? Find a basis for $\overline{B_sB_tB_s}^{-1}$ (i.e. the elements in degree -1) in the kernel of L^2. Are they orthogonal to $L^2(c_{\text{bot}})$ under the intersection form? Show that the form $(v, w) = \langle v, Lw\rangle$ on this orthogonal subspace of $\overline{B_sB_tB_s}^{-1}$ is negative definite.

Bonus problem: what does the picture look like when restricted to the the (images in $\overline{BS(sts)}$) of the summand $B_s \overset{\oplus}{\subset} B_sB_tB_s$? What does it look like when restricted to the summand $B_{sts} \overset{\oplus}{\subset} B_sB_tB_s$?

4. Let (V, L_V) and (W, L_W) be *Lefschetz spaces*, i.e. graded vector spaces equipped with a nondegenerate graded bilinear form and a Lefschetz operator. Suppose that $\sigma \colon V \to W(1)$ is a vector space map of degree $+1$, satisfying

- $\sigma L_V = L_W \sigma$,
- $\langle v, L_V v'\rangle_V = \langle \sigma v, \sigma v'\rangle_W$,
- σ is injective from negative degrees.

Suppose that (W, L_W) has (HR).

a) Prove that (V, L_V) has (hL). (Hint: There are two cases, for $v \in V$ of negative degree. Either σv is primitive, or σv is not primitive.)

b) Are there some reasonable extra conditions which would imply that σ sends primitives (of strictly negative degree) to primitives?

c) What extra conditions would guarantee that (V, L_V) has (HR), except in degree 0?

5. a) Let (V, L_V) and (W, L_W) be Lefschetz spaces, and suppose that $\sigma\colon V \to W(-d)$ is a vector space map of degree $-d$, satisfying $\sigma L_V = L_W \sigma$. When $d > 0$, prove that the Lefschetz form on W, restricted to the image of σ, is zero.

b) Deduce that the global intersection form on $\overline{B_s B_s}$, restricted to the (canonical) summand $\overline{B_s}(1)$, is zero.

c) By contrast, show that the global intersection form need not restrict to zero on a summand of the form $\overline{B_s}(-1)$. (This summand is non-canonical, so there are multiple choices of inclusion map.)

6. a) Let C be a finite dimensional graded algebra, and P a (non-graded) projective (resp. simple) module. Show that P admits a graded lift.

b) Show that $\overline{B_x}$ is indecomposable as a graded R-module if and only if it is indecomposable as an ungraded R-module.

Category $\mathcal{O}$

These questions are entirely about category $\mathcal{O}$ for $U(\mathfrak{sl}_2)$. The algebra $U(\mathfrak{sl}_2)$ is generated by operators e, h, f with $ef - fe = h$, $he - eh = 2e$, $hf - fh = -2f$. Every representation in $\mathcal{O}$ will be *weight*, i.e. $V = \oplus V_k$, $k \in \mathbb{C}$, where $v \in V_k$ means that $hv = kv$. Such a vector v is called a *weight vector* of weight k.

7. Fix $\lambda \in \mathbb{C}$. Let v_+ satisfy $hv_+ = \lambda v_+$ and $ev_+ = 0$. Then the Verma module $\Delta(\lambda)$ is freely generated by v_+ under this condition. In other words, $\Delta(\lambda)$ has a basis $\{f^k v_+\}$ for $k \geq 0$.

a) Is this a basis of weight vectors? What are the weights?

b) Why should e send V_k to V_{k+2} for any weight representation V?

c) Compute $e(f^k v_+)$ in this basis. Compute $f(f^k v_+)$ in this basis.

d) Repeat this computation for the basis $\{f^{(k)} v_+\}$, where $f^{(k)} = \frac{f^k}{k!}$.

8. Continuing the previous question.

a) Prove that there is a map $\Delta(\mu) \to \Delta(\lambda)$ if and only if $\Delta(\lambda)$ has a weight vector x of weight μ such that $ex = 0$.

b) For which k is $f^k v_+$ such a weight vector?

c) Conclude that there is a map $\Delta(\mu) \to \Delta(\lambda)$ if and only if either $\mu = \lambda$, or $\lambda \in \mathbb{Z}_{\geq 0}$ and $\mu = -\lambda - 2$.

9. The operator $c = fe + h^2 + 2h$ is called the Casimir element. It lives in the center of $U(\mathfrak{sl}_2)$.

a) Compute cv_+ inside $\Delta(\lambda)$.

b) When $\mu = -\lambda - 2$, deduce from this computation that the central characters of $\Delta(\lambda)$ and $\Delta(\mu)$ agree on c.

c) When $\mu = -\lambda - 2$ and $\lambda \in \mathbb{Z}_{\geq 0}$, deduce using the previous exercise that the central characters of $\Delta(\lambda)$ and $\Delta(\mu)$ agree.

d) (Extra credit) Using only the knowledge that $\chi_\lambda(z) = \chi_\mu(z)$ when $\mu = -\lambda - 2$ and $\lambda \in \mathbb{Z}_{\geq 0}$, can you deduce that $\chi_\lambda(z) = \chi_\mu(z)$ when $\mu = -\lambda - 2$ for all $\lambda \in \mathbb{C}$?

10. The goal of this exercise is to study an object of category $\mathcal{O}$ on which the center only acts by generalized eigenvalues. Let $V = \Delta(-1) \otimes \mathbb{C}^2$. More precisely, let V denote the vector space with basis $\{(f^k v_+) \otimes \uparrow\} \cup \{(f^k v_+) \otimes \downarrow\}$. On the first tensor factor, $\mathfrak{sl}_2$ acts as on $\Delta(-1)$. On the second tensor factor, $\uparrow$ has weight $+1$, $\downarrow$ has weight -1, and $e(\downarrow) = \uparrow$, $e(\uparrow) = 0$, $f(\uparrow) = \downarrow$, $f(\downarrow) = 0$. Finally, on a pure tensor we have $e(a \otimes b) = ea \otimes b + a \otimes eb$, and similarly for h and f.

a) Prove that if a and b are weight vectors, then so is $a \otimes b$. What is its weight?

b) Compute how e and f and h act on the basis of V.

c) Find a vector x of weight 0 with $ex = 0$, giving a map $\Delta(0) \to V$. Show the map is injective.

d) Find a vector $\bar{y}$ in the quotient $V/\Delta(0)$, of weight -2, with $e\bar{y} = 0$. Show that this gives an isomorphism from $\Delta(-2)$ to the quotient. Thus V has a filtration by Verma modules.

e) Does $\bar{y}$ lift to an element $y \in V$ with $ey = 0$?

f) Consider the action of c on the -2 weight space of V. Prove that c acts by a nontrivial Jordan block. (Why did you know it acted by a Jordan block in the first place?)