MSRI Soergel bimodule workshop

June/July 2017

Week 1 Day 2 Afternoon: Basic Exercises

Diagrammatics

1. Let $A = \mathbb{R}[x]/(x^2)$ be an object in the monoidal category of $\mathbb{R}$-vector spaces. Let $\cap \colon A \otimes A \to \mathbb{R}$ denote the map which sends $f \otimes g$ to the coefficient of x in fg. Let $\cup \colon \mathbb{R} \to A \otimes A$ denote the map which sends 1 to $x \otimes 1 + 1 \otimes x$.

a) We wish to encode these maps diagrammatically, drawing $\cap$ as a cap and $\cup$ as a cup. Justify this diagrammatic notation, by checking the biadjointness/isotopy relations.

b) Draw a sequence of nested circles, as in an archery target. Evaluate this morphism.

2. This question is about the Temperley–Lieb category.

a) Confirm that the isotopy relation holds in vector spaces.

b) There is a map $V \otimes V \to V \otimes V$ which sends $x \otimes y \mapsto y \otimes x$. Draw this as an element of the Temperley–Lieb category (a linear combination of diagrams).

c) Find an endomorphism of 2 strands which is killed by placing a cap on top. Can you find one which is an idempotent? Also find an endomorphism killed by putting a cup on bottom.

d) (Harder) Find an idempotent endomorphism of 3 strands which is killed by any cap on top (there are two caps).

Diagrammatics for Frobenius extensions and Bott–Samelson bimodules

3. Look up the definition of a *Frobenius algebra object* in a monoidal category (on Wikipedia).

a) Express this definition diagrammatically.

b) Suppose that $A \subset B$ is a Frobenius extension. Using 1-manifold diagrams for induction and restriction bimodules, show that $B \otimes_A B$ is a Frobenius object in the category of B-bimodules. (Hint: The unit of the Frobenius object is normally portrayed as a dot, but should also be portrayed using some 1-manifold diagram. Which 1-manifold diagram? Your answer should involve a deformation retract.)