MSRI Soergel bimodule workshop

June/July 2017

Week 1 Day 4 Morning: Supplementary/Advanced Exercises

Note: the day 3 afternoon (supplementary) exercises had more exercises about light leaves.
Elements of Bott–Samelson bimodules and global intersection form

1. In lectures we saw that for any expression $\underline{w}$, $BS(\underline{w})$ has a basis as a right R-module given by 01-sequences. It contains two canonical elements c_{bot} and c_{top} which project to elements of minimal and maximal degree in $\overline{BS(\underline{w})}$. In this exercise we find a recursive formula for

$$N_{\underline{w}}(f) := \langle f^{\ell(\underline{w})} c_{\mathrm{bot}}, c_{\mathrm{bot}} \rangle.$$

for any degree two element $f \in R$, acting by left multiplication on $\overline{BS(\underline{w})}$.

a) Find a formula for $N_{\underline{w}}(f)$ in terms of $N_{\underline{w}'}(f)$, over all subexpressions $\underline{w}'$ obtained by omitting a simple reflection from $\underline{w}$.

b) Show that $N_{\underline{w}}(f) = 0$ unless $\underline{w}$ is reduced. (*Hint:* It might help to use the light leaves description of $BS(\underline{w})$ or the decomposition of $BS(\underline{w})$ into indecomposable Soergel bimodules.) Use this to simplify your formula in part (a).

c) Suppose that $\partial_s(f) > 0$ for all $s \in S$. Show that $N_{\underline{w}}(f) > 0$ for $\underline{w}$ reduced. (First prove that $sw > w$ if and only if $\partial_s(wf) > 0$.)

d) (Only if the question makes sense to you!) What is $N_{\underline{w}}(f)$ in terms of Schubert calculus?