MSRI Soergel bimodule workshop
June/July 2017

Week 1 Day 4 Morning: Basic Exercises

Note: the day 3 afternoon exercises had several exercises about light leaves.
Elements of Bott–Samelson bimodules and the global intersection form

1. a) Recall that B_s has a basis as a right R-module given by $c_1 = c_{bot} = 1 \otimes 1$ and $c_s = \frac{\alpha_s}{2} \otimes 1 + 1 \otimes \frac{\alpha_s}{2}$. Let $f \in R$ be arbitrary. Compute fc_1 and fc_s in this basis.

b) Recall that $BS(\underline{w})$ has a basis $\{c_{\underline{\varepsilon}}\}$ indexed by 01-sequences. For a linear polynomial $f \in \mathfrak{h}^* \in R$, express $fc_{\underline{\varepsilon}}$ in this basis.

2. Use the 01-basis of $BS(\underline{w})$ and an upper-triangularity argument to prove that the global intersection form is non-degenerate to degree 0.

Duality and invariant forms

3. Given an R-bimodule B, its dual $\mathbb{D}B$ is defined to be $\mathrm{Hom}_{(-,R)}(B, R)$, the right R-module maps. Clearly $\mathbb{D}^2 = \mathbb{1}$ on any bimodule which is free and finite rank as a right R-module.

a) Show that $\mathrm{Hom}^0_{(R,R)}(B, \mathbb{D}B)$ is isomorphic to the space of invariant forms on B. If a map $B \to \mathbb{D}B$ is an isomorphism, what does this say about the corresponding invariant form?

b) What is $\mathbb{D}B_s$? What about $\mathbb{D}BS(\underline{w})$?

c) Show (by the definition of B_w) that $\mathbb{D}B_w \cong B_w$ for all $w \in W$, and therefore there exists a nondegenerate invariant form on B_w.

d) If the Soergel conjecture holds, show that any non-zero invariant form on B_w is nondegenerate.

e) (Challenge) Show that the global intersection form on $BS(\underline{w})$ for a reduced expression restricts to a nonzero form on B_w. (Are c_{bot} and c_{top} in B_w?)

Local intersection form

4. a) In type B_2, compute the local intersection pairing of $BS(stst)$ at st, in all degrees.

b) In degree 0, make observations about the definiteness and signature of this form.

c) In type H_2, compute the local intersection pairings of $BS(ststs)$ at sts and s respectively. Any observations about definiteness and signature in degree 0?

d) In type D_4, compute the local intersection pairings of $BS(tuvstuv)$ at tuv, in all degrees. Definiteness and signature in degree 0?