MSRI Soergel bimodule workshop
June/July 2017

Week 1 Day 4 Afternoon: Basic Exercises

1. In the Temperley–Lieb category with $q = 1$, compute the local intersection form on $\mathrm{Hom}(3, 5)$ and on $\mathrm{Hom}(1, 5)$. What are the signatures of these forms?

2. In type B_2, find a decomposition of the identity of $BS(stst)$ into primitive orthogonal idempotents. (You have computed the intersection form at st in a previous exercise.)

3. Fix w. Prove that the span of the light leaves factoring through elements $x < w$ is the same as the span of all morphisms factoring through Bott-Samelsons for rexes $\underline{x}$, with $x < w$. Hence this span is an ideal.

4. Prove that any semisimple abelian category can be given the structure of an object adapted cellular category. What is the poset Λ?

Recall that a *Krull–Schmidt category* is an additive category in which every object is isomorphic to a finite direct sum of indecomposable objects, and an object is indecomposable if and only if its endomorphism ring is local.

5. Let $\mathcal{C}$ be a Krull–Schmidt category over an algebraically closed field $\Bbbk$. Show that the multiplicity of B as summand of X is given by the rank of the pairing

$$\mathrm{Hom}(B, X) \times \mathrm{Hom}(X, B) \to \mathrm{End}(B)/\mathfrak{m}_B.$$

where $\mathfrak{m}_B$ denotes the maximal ideal of $\mathrm{End}(B)$. What is the correct statement for general fields or local rings $\Bbbk$?

6. Some exercises to get used to Krull–Schmidt categories:

a) Show that the Krull–Schmidt theorem holds in Krull–Schmidt categories: any object can be written as a direct sum of indecomposable objects, and this decomposition is unique up to permutation of the factors.

b) *(Idempotent lifting)* Let A be an algebra and $\mathfrak{m} \subset A$ an ideal such that $\mathfrak{m}^2 = 0$. Show that given an idempotent $e \in A/\mathfrak{m}$ there exists an idempotent $\widetilde{e} \in A$ such that $e = \widetilde{e}$ in $A/\mathfrak{m}$. Now prove the same statement assuming only that A is complete with respect to the topology defined by $\mathfrak{m}$.

c) Let $(\mathbb{O}, \mathfrak{m})$ be a complete local ring. Let $\mathcal{C}$ be a Karoubian $\mathbb{O}$-linear additive category such that all hom spaces are finitely generated. Show that $\mathcal{C}$ is Krull–Schmidt. (*Hint:* It might help to first consider the case when $\mathbb{O}$ is a field.)

d) Show that the category of graded modules over a polynomial ring is a Krull–Schmidt category. Conclude that the category of Soergel bimodules is Krull–Schmidt.

e) (*) Let X be an affine variety. When does the Krull–Schmidt theorem hold for vector bundles on X? (Answer: almost never.) Conclude that the Krull–Schmidt theorem fails for ungraded modules over a polynomial ring. (Optional: show that the Krull–Schmidt theorem holds for vector bundles on a projective algebraic variety.)