MSRI Soergel bimodule workshop

June/July 2017

Week 2 Day 1 Afternoon: Supplementary/Advanced Exercises

Lefschetz calculations in Soergel bimodules

1. In this exercise we prove an "easy" case of hard Lefschetz. Assume that B_x is a Soergel bimodule such that hard Lefschetz holds on $\overline{B_x}$. (You can equip B_x with an invariant form if you wish, but it won't be important for this exercise.) We fix $s \in S$ such that $xs < x$, and consider B_xB_s.

One has an operator

$$L_\zeta := (\rho \cdot -)\,\mathrm{id}_{B_s} + \mathrm{id}_{B_x}(\zeta\rho \cdot -)$$

on B_xB_s, for each $\zeta \in \mathbb{R}$. That is, L_ζ is left multiplication by ρ, plus middle multiplication by $\zeta\rho$. (Compare this with 1-5a-basic, exercise 2.) It induces a Lefschetz operator L_ζ on $\overline{B_xB_s}$.

a) Show that $B_xB_s = B_x(1) \oplus B_x(-1)$. (You should be able to give an abstract argument, but in part b) the following fact is useful (see "Singular Soergel bimodules"): there exists an (R, R^s)-bimodule $B_{\overline{x}}$ such that $B_{\overline{x}} \otimes_{R^s} R \cong B_x$.)

b) Rewrite the Lefschetz operator L_z on B_xB_s using a fixed choice of isomorphism $B_xB_s = B_x(1) \oplus B_x(-1)$. Conclude that in the right quotient $\overline{B_xB_s}$, L_ζ has the form

$$\begin{pmatrix} \rho \cdot - & 0 \\ \zeta\gamma & \rho \cdot - \end{pmatrix}.$$

for some non-zero scalar γ. (As above, $\rho \cdot -$ denotes the degree two endomorphism of left multiplication by ρ.) For a hint, see question 2 from the morning basic exercises.

c) Conclude that L_ζ satisfies hard Lefschetz on $\overline{B_xB_s}$ if and only if $\zeta \neq 0$.

2. We continue the notation of the previous exercise. Assume that B is equipped with a Lefschetz operator L given by left multiplication, so that $\overline{B}$ has hard Lefschetz. Let $\{e_i\}$ be a collection of elements of B^{-k-1} which project to an orthonormal basis of $\overline{B}^{-k-1}$ (with respect to the Lefschetz form). Let $\{f_i\}$ be a collection of elements projecting to an orthonormal basis of $P^{-k+1} \subset \overline{B}^{-k+1}$.

a) Find a basis for $\overline{B}^{-k+1}$. Find a basis for $\overline{BB_s}^{-k}$, using the maps α and β.

Let L continue to denote the same operator of left multiplication, now considered to act on BB_s. Let M denote multiplication by some linear polynomial ρ immediately to the left of B_s in BB_s, and suppose that $\partial_s(\rho) = 1$. For $v, w \in \overline{BB_s}^{-k}$, one can pair them by the formula $\langle v, L^{k-1}Mw\rangle$. As discussed in lecture, this pairing is the limit of the Lefschetz pairing induced by $L + \zeta M$ on $\overline{BB_s}$.

b) Compute the matrix of this pairing in the basis you found above.

c) Deduce that the signature of this pairing on $\overline{BB_s}^{-k}$ is equal to the signature of the Lefschetz form on $P^{-k+1} \subset \overline{B}^{-k+1}$.