MSRI Soergel bimodule workshop

June/July 2017

Week 2 Day 3 Morning: Basic Exercises

Jones and HOMFLY polynomials

1. Compute the trace of $s_1 s_2 \ldots s_{n-1}$ acting on $V^{\otimes n}$, where $V = \mathbb{C}^2$, in two ways: directly, and using the Temperley-Lieb algebra.

2. Compute the Jones polynomial and the HOMFLY polynomial of the trefoil. The trefoil is the closure of σ^3 in the braid group on two strands.

Cell theory

3. This question explores the cell theory of various algebras with fixed basis. For each algebra, find the left cells, right cells, and two-sided cells. What do the cell modules look like?

a) A matrix algebra, with its basis of matrix entries.

b) A product of matrix algebras, with its basis of matrix entries for each term in the product.

c) A polynomial ring $\mathbb{C}[x]$, with the basis $\{x^k\}$.

d) A polynomial ring $\mathbb{C}[x]$, with the basis $\{1, x, x^2 - 1, x^3 - 2x, \ldots\}$. To interpret this basis, consider the ring $\mathbb{C}[q, q^{-1}]$, and the subring of invariants under $q \mapsto q^{-1}$. This subring is a polynomial ring generated by $x = [2]$, and the basis described is $\{[n]\}$.

4. Compute the left cells, right cells, and two-sided cells for the Hecke algebra $\mathbf{H}(S_4)$ with its Kazhdan-Lusztig basis. Do the same for every dihedral group.

Robinson-Schensted

5. Pick a few random elements of S_{10} and apply the Schensted algorithm to find the corresponding triple (P, Q, λ).

Schur-Weyl duality

6. a) Why does $(1 - s_i - s_{i+1} + s_i s_{i+1} + s_{i+1} s_i - s_i s_{i+1} s_i)$ act trivially on $V^{\otimes n}$, for $V = \mathbb{C}^2$?

b) What is the kernel of the action of $\mathbb{C}[S_4]$ on $V^{\otimes 4}$, when $V = \mathbb{C}^3$?

Jucys-Murphy elements

7.

Clearly $H_{s_i} j_i H_{s_i} = j_{i+1}$, by definition. Deduce that

$$H_{s_i} j_i = j_{i+1} H_{s_i} + X \tag{1}$$

for some X. Find X.

8. Let A denote the subalgebra of $\mathbf{H}(S_n)$ generated by j_i, j_{i+1}, and $H = H_{s_i}$. Let x be an eigenvector for both j_i and j_{i+1} in some $\mathbf{H}$ module, with eigenvalues λ_i and λ_{i+1} respectively. The relation (1) implies that the subspace $A \cdot x$ is at most 2-dimensional, spanned by x and Hx.

a) Suppose that x and Hx are colinear. What does this imply about the eigenvalues $(\lambda_i, \lambda_{i+1})$? (There are two cases.)

b) Suppose that x and Hx are not colinear. Find another eigenvector y for j_i and j_{i+1} (linearly independent from v), and compute its eigenvalues.

c) (Optional) Find the matrix for H in the basis $\{x, y\}$, assuming that $\lambda_i = v^a$ and $\lambda_j = v^b$. This is called *Young seminormal form*.