MSRI Soergel bimodule workshop

June/July 2017

Week 1 Day 3 Morning: Supplementary/Advanced Exercises

Standard filtration

1. Let $2c_s = \alpha_s \otimes 1 + 1 \otimes \alpha_s$ and $2d_s = \alpha_s \otimes 1 - 1 \otimes \alpha_s$ inside B_s. When s is understood, write $q_0 = c_s$ and $q_1 = d_s$. Let $\underline{w}$ have length d. For a subsequence $\underline{e} \subset \underline{w}$, write $q_{\underline{e}}$ for the tensor product $q_{e_1} q_{e_2} \cdots q_{e_d} \in B_{s_1} B_{s_2} \cdots B_{s_d}$. Prove that $\{q_{\underline{e}}\}_{\underline{e} \subset \underline{w}}$ forms a basis for the degree $+d$ part of $BS(\underline{w})$.

2. (Assumes knowledge of the support of a coherent sheaf.) For $w \in W$, let $\mathrm{Gr}_w = \{(w(v), v\} \subset \mathfrak{h} \times \mathfrak{h}$. Let $w_1, w_2, \ldots$ be an enumeration of the elements of W, and let B be an R-bimodule. Suppose there exists a filtration $0 \subset B^1 \subset \ldots \subset B^m = B$ such that $B^i / B^{i-1} \cong \oplus R_{w_i}^{\oplus n_i}$. Show that B^i is equal to the submodule of B consisting of sections with support on the subvariety $\cup_{j=1}^{i} \mathrm{Gr}_{w_j}$. Deduce that a standard filtration on a Soergel bimodule is unique and is preserved by all morphisms. (Hint: the support of any nonzero element of R_x is Gr_x.)

3. a) Suppose that $\underline{w}$ is a reduced expression. How many copies of Q_w appear in the localization of $BS(\underline{w})$? Does this depend on the reduced expression?

b) Suppose that $\underline{w}$ and $\underline{w'}$ are reduced expressions which differ by a single braid relation. Consider a $2m_{st}$ valent vertex, viewed as a map $BS(\underline{w}) \to BS(\underline{w'})$. Now apply the localization functor to this map. After restriction of this map to $Q_w \to Q_w$, do you get an isomorphism?

c) If you compose two $2m_{st}$-valent vertices $BS(\underline{w}) \to BS(\underline{w'}) \to BS(\underline{w})$, what can you say about the localized restriction to $Q_w \to Q_w$? Is it the identity map?

d) Now begin at $\underline{w}$, and apply an arbitrary chain of braid relations, viewed as $2m_{st}$-valent vertices, to get from $BS(\underline{w})$ to $BS(\underline{w})$. A priori, what can you say about the localized restriction to $Q_w \to Q_w$?

Diagrammatics for Soergel bimodules

4. Why does the general polynomial forcing relation follow from the case of linear polynomials? Give a quick and elegant argument.

5. Consider a universal diagram in two colors, whose boundary alternates $B_s B_t B_s B_t \ldots B_s B_t$ when read clockwise. Prove that the minimal degree of any nonzero universal diagram is 2, and that all such universal diagrams are deformation retracts of colored Temperley–Lieb diagrams. (Hint: what characterizes universal diagrams in the image of the functor from Temperley–Lieb? What happens when you try to connect two different connected components of the same color?)

Jones–Wenzl projectors

6. Inside TL_n, let T be the vector space of elements which are killed by all the $(n-1)$ caps on top, and let B be the space killed by cups on bottom. For an element $x \in TL_n$ let $\overline{x}$ denote the same element with each diagram flipped upside-down. Thus, for example, $x \in T$ if and only if $\overline{x} \in B$.

a) Show that any crossingless matching is either the identity diagram, or has both a cap on bottom and a cup on top.

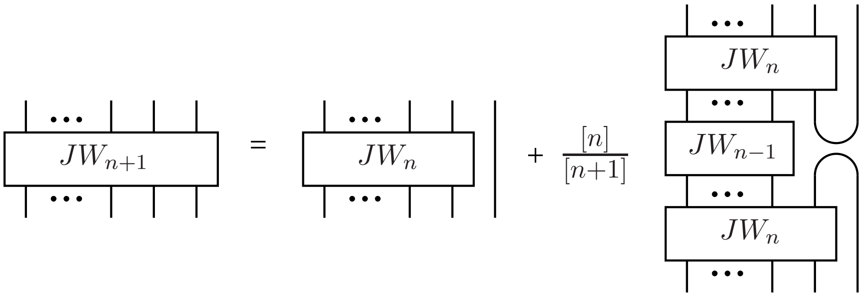
b) We now make the assumption (*): there exists some $f \in T$ for which the coefficient of the identity diagram is invertible. Why is this equivalent to the analogous assumption for B?

c) Let $f \in T$, with invertible coefficient c for the identity diagram. Let $g \in B$, with invertible coefficient d for the identity diagram. Compute the composition gf in two ways, and deduce that f and g are colinear.

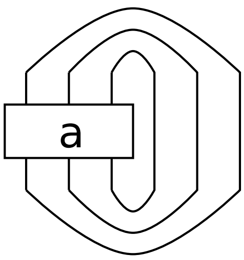
d) Assuming (*) deduce that $T = B$, and the space is one-dimensional, and that $f = \overline{f}$ for $f \in T$.

e) Thus, assuming (*), there is a unique element $JW_n \in T$ whose identity coefficient is 1. Prove that JW_n is idempotent. (If we construct JW_n in some other way, then this proves (*).)

7. Prove the following recursive formula. (Hint: there is an auxiliary computation that needs doing to prove this, similar in spirit to the next question. You will need to use induction for this computation. This is unlike the recursive formula in the basic exercises, which could be shown without induction.)



8. The *trace* of an element $a \in TL_n$ is the evaluation in $\mathbb{Z}[q, q^{-1}]$ of the closed diagram below. Calculate the trace of JW_n. (Hint: use induction.)



Suppose that q is a primitive $2(n+1)$-th root of unity. What is the trace of JW_n? What do you get when you rotate JW_n by one strand?