MSRI Soergel bimodule workshop

June/July 2017

Week 1 Day 1 Afternoon: Basic Exercises

Miscellaneous exercises from lecture

1. a) Compute H_s^{-1}, and show that $\underline{H}_s$ is self-dual. Confirm that $\underline{H}_s^2 = \underline{H}_s(v+v^{-1})$.

b) Compute H_{st}^{-1} in terms of the standard basis. Given $w \in W$, for which $y \in W$ can there be a non-zero coefficient of H_y in the expression for H_w^{-1}? In the expression for $\overline{H_w}$? In the expression for $\omega(H_w)$?

c) Prove the uniqueness of the KL basis.

d) Find a formula for $H_w \underline{H}_s$.

e) Extrapolate the construction from lecture into a proof of the existence of the KL basis.

Kazhdan–Lusztig basis

2. Let $W = S_4$, with $S = \{s_1, s_2, s_3\}$ as above.

a) Consider the reduced expression $s_1s_2s_1s_3s_2s_1$ for the longest element w_0. Use the inductive algorithm to compute the KL basis element $\underline{H}_{w_0}$. Along the way, you will compute the KL basis elements $\underline{H}_1$, $\underline{H}_{s_1}$, $\underline{H}_{s_1s_2}$, $\underline{H}_{s_1s_2s_1}$, etcetera.

b) Repeat the calculation for the reduced expression $s_2s_3s_1s_2s_3s_1$. What is different this time? What non-trivial KL polynomials have you found?

c) Repeat the calculation for the reduced expression $s_1s_3s_2s_1s_3s_2$. You should now be able to deduce $\underline{H}_w$ for all $w \in W$.

3. Let (W,S) be a dihedral Coxeter group. That is

$$W = \langle s,t \mid s^2 = t^2 = (st)^{m_{st}} = e \rangle$$

where $e \in W$ is the identity, and $m_{st} \in \{2,3,4,\dots,\infty\}$. Given $0 \le m \le m_{st}$ write $st(m)$ for the product $stst\dots$ where m terms appear, and similarly for $ts(m)$. For example $st(0) = e$, $ts(1) = t$, $st(2) = st$, $ts(3) = tst$ etc.

a) Draw the Bruhat graph of W. Distinguish between the cases $m_{st} < \infty$ and $m_{st} = \infty$.

b) Use the inductive algorithm to compute $\underline{H}_{st(m)}$ for $m \le m_{st}$. Along the way, for $1 \le m < m_{st}$ find an explicit formula for the products

$$\underline{H}_s\underline{H}_{st(m)},\ \underline{H}_s\underline{H}_{ts(m)},\ \underline{H}_t\underline{H}_{ts(m)} \qquad \text{and} \qquad \underline{H}_t\underline{H}_{st(m)}$$

in terms of the Kazhdan–Lusztig basis. (*Hint:* Calculate the first few cases and then use induction. Use caution with small m.)

c) Conclude that $h_{x,y} = v^{\ell(y)-\ell(x)}$ for all $x \le y \in W$.

d) Using the formulas above, one can find an algebraic expression for $\underline{H}_{st(m)}$ in terms of $\underline{H}_s$ and $\underline{H}_t$, when $m \le m_{st}$. For example, when $m = 2$ one has $\underline{H}_{st} = \underline{H}_s\underline{H}_t$, and when $m = 3 \le m_{st}$ one has $\underline{H}_{sts} = \underline{H}_s\underline{H}_t\underline{H}_s - \underline{H}_s$. Find similar expressions when $m = 4,5,6$. Can you find a reasonable way to compute the coefficients which appear?

Remark. When m_{st} is finite, the longest element $w_0 = st(m_{st}) = ts(m_{st})$ has two reduced expressions, from which one will get two distinct algebraic expressions for $\underline{H}_{w_0}$ in terms of $\underline{H}_s$ and $\underline{H}_t$. Setting these expressions equal gives a relation in the Hecke algebra amongst Kazhdan–Lusztig generators, analogous to the braid relation for the standard generators.

4. a) Use the Deodhar defect formula to compute $\underline{H}_s\underline{H}_s\underline{H}_s$ in the standard basis.

b) Let s, t, u denote three distinct simple reflections. Use the Deodhar defect formula to compute $\underline{H}_s\underline{H}_t\underline{H}_u$. Is this product equal to $\underline{H}_{stu}$?

c) Let s and t be distinct simple reflections. What is $\varepsilon(\underline{H}_s\underline{H}_t\underline{H}_s)$?

d) Let $\{s, t, u, v\}$ be the simple reflections in type D_4, where s, u, v all commute. Compute the product $\underline{H}(\underline{w})$ for the reduced expression $\underline{w} = suvtsuv$. (Hint: there are 2^7 subexpressions, which is a lot. However, for each given element $x < w$, there are not many subexpressions for x. There is a lot of symmetry, so the number of x one must examine is relatively small.)

5. Compute the pairing $(\underline{H}_s\underline{H}_t\underline{H}_s, \underline{H}_s)$ in two different ways.

a) Use biadjunction and the quadratic relation to express this pairing in terms of $\varepsilon(\underline{H}_t\underline{H}_s)$.

b) Use the Deodhar defect formula on both sides, and the "false orthogonality" of the standard basis (see the supplemental exercises).