MSRI Soergel bimodule workshop

June/July 2017

Week 1 Day 5 Afternoon: Supplementary/Advanced Exercises

1. In the following exercise we examine the first example discovered by Bernstein-Gelfand-Gelfand of a submodule of a Verma module which is not generated by highest weight vectors. This implies that simple modules occur with multiplicity $\neq 0, 1$ in Verma modules in general. (*Challenge:* why?) This example was the first that suggested that this is a challenging question, and was one of the key steps towards the Kazhdan-Lusztig conjecture.

We consider $\mathfrak{g} = \mathfrak{sl}_4$. It has a basis consisting of E_{ij} for $1 \leq i \neq j \leq j$ and $E_{ii} - E_{i+1,i+1}$ for $1 \leq i < 4$, where E_{ij} denote matrix units. Positive roots correspond to E_{ij} with $i < j$. Consider the weight χ defined by

$$\chi(E_{11} - E_{22}) = \chi(E_{33} - E_{44}) = 0$$
$$\chi(E_{22} - E_{33}) = 1$$

and consider the Verma module $\Delta(\chi - \rho)$ of highest weight $\chi - \rho$. Denote the highest weight vector by $v_{\chi-\rho}$.

Suppose $f_1 = E_{32} v_{\chi-\rho}$ and let $\widetilde{M} = U(\mathfrak{g}) \cdot f_1$.

a) By considering central characters, write down a list of possible highest weight vectors in $\Delta(\chi-\rho)$. (Remember that finding a non-zero highest weight vector in $\Delta(\chi-\rho)$ of highest weight λ is the same thing as giving a non-zero map $\Delta(\lambda) \to \Delta(\chi - \rho)$).

b) We will use the following fact, which is non-trivial, and proved in BGG*: all non-trivial highest weight vectors (i.e. $\neq v_{\chi-\rho}$) in $\Delta(\chi - \rho)$ are contained in $\widetilde{M}$. Assume this from now on.

c) Consider $x = E_{42}E_{21}v_{\chi-\rho} + E_{43}E_{31}v_{\chi-\rho}$. Then $x \notin \widetilde{M}$ and $E_{ik}x \in \widetilde{M}$ for $i < k$.

d) Deduce that the submodule generated by x and $\widetilde{M}$ is a proper submodule of $\Delta(\chi - \rho)$ which is not generated by vectors of highest weight.

(*) Bernstein I.N., Gelfand I.M., Gelfand S.I., *Structure of Representations that are generated by vectors of highest weight,* Functional. Anal. Appl. 5 (1971)