MSRI Soergel bimodule workshop

June/July 2017

Week 1 Day 2 Morning: Supplementary/Advanced Exercises

1. Let F be a functor, and E_1 and E_2 be two right adjoints of F (equipped with their respective units and counits of adjunction). Then there is a unique isomorphism $E_1 \to E_2$ that respects the units and counits of adjunction. Thus right adjoints are unique up to unique isomorphism. Similarly for left adjoints.

Remark. For a ring extension, the left and right adjoints of the restriction functor are each canonical. For a Frobenius extension, they are also isomorphic. However, there is no canonical isomorphism from a left to a right adjoint. The space of such isomorphisms is equivalent to the space of possible Frobenius traces.

2. Continuing Q2 from basic. (These exercises are a bit more computational.)

a) Show that $R^{s,t} = \mathbb{R}[z, Z]$. Either use the Chevalley theorem, or do it explicitly for $m = 2, 3$.

b) Suppose that $m = 3$. Find dual bases for R over $R^{s,t}$. Show that $\sum a_i b_i = \mathbb{L}$.

c) Suppose that $m = 3$. Find dual bases for R^s over $R^{s,t}$, under the pairing using $\partial_s \partial_t$. Show that $\sum a_i b_i = \frac{\mathbb{L}}{\alpha_s}$.

3. Suppose that W is a finite group acting faithfully on a euclidean vector space V of dimension n. Let R be the coordinate ring of V, and R^W the invariant subring. Chevalley's theorem states that when W is generated by reflections, then R^W is generated by n algebraically independent homogeneous polynomials, known as a "basic set" of invariants. The basic set itself is not unique, but the multiset of degrees of the polynomials in the basic set is determined by the group W. (Proving algebraic independence is a theorem! These exercises are just to get you to explore invariant polynomials, not to prove theorems.)

a) (Type A) Find a basic set for the symmetric group S_n acting on its standard n-dimensional representation. Recall that this action is generated by the reflections which flip x_i with x_j for two standard basis elements, and keep the rest of the basis fixed.

b) (Type B) Find a basic set for the signed symmetric group SS_n acting on its standard n-dimensional representation. Recall that this action is generated by the reflections above, as well as the reflection which sends x_i to $-x_i$ and keeps the rest of the basis fixed.

c) (Type D) Find a basic set for the even signed symmetric group ESS_n acting on its standard n-dimensional representation. Recall that this action is generated by the symmetric group and by the reflection which sends x_i to $-x_j$ and x_j to $-x_i$, and keeps the rest of the basis fixed.

4. Some "counterexamples" to the Chevalley theorem:

a) Find an example where W is not generated by reflections, and R^W is **not** a polynomial ring, i.e. it is not generated by algebraically independent elements. (Minimal example: type A_1.)

b) Find an example where W is infinite, and R^W is a polynomial ring, but with $n - 1$ generators rather than n. (Minimal example: type $I_2(\infty)$.)

5. Let us examine the set of degrees $\{d_i\}$.

a) Show that the trace of w on the symmetric tensor S^kV is given by the coefficient of t^k in

$$\frac{1}{\det(1-tw)}.$$

b) Show that the dimension of the invariant subspace V^W is given by the trace of

$$\frac{1}{|W|}\sum_{w\in W} w.$$

c) By computing the dimension of each graded piece of R^W, show that

$$\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det(1-tw)} = \prod_{i=1}^{n}\frac{1}{1-t^{d_i}}.$$

d) Recall that an element of W is a reflection if all but one eigenvalue is 1, and the remaining eigenvalue is -1. Let N denote the number of reflections in W (also the number of positive roots). Show that $\prod d_i = |W|$ and $\sum(d_i - 1) = N$.

e) Verify that the basic sets you found in Q3 have the correct degrees. What must the degrees of a finite dihedral group be?