MSRI Soergel bimodule workshop
June/July 2017

Week 1 Day 5 Afternoon: Basic Exercises

Category $\mathcal{O}$

1. These questions are about category $\mathcal{O}$ for $\mathfrak{sl}_2$.

a) Find a change of basis to check directly that $\Delta(5) \otimes L(1) \cong \Delta(4) \oplus \Delta(6)$.

b) Find projective resolutions of $\Delta(0)$ and $\Delta(-2)$.

c) Find a projective resolution of $L(0)$ and $L(-2)$.

d) Find a projective resolution of $\nabla(0)$ and $\nabla(-2)$ (if you know what these are).

e) After applying the Soergel functor, these resolutions are sent to complexes of Soergel modules. Write down these complexes. How can you deduce what the differentials are?

2. In this exercise we look at the effect of translation functors on category $\mathcal{O}$, and see that they are easily understood on Verma modules.

i) Let $\lambda \in \mathfrak{h}^*$ be an arbitrary weight, and let V be a finite dimensional representation of $\mathfrak{g}$. Show that $\Delta(\lambda) \otimes V$ has a Verma flag; that is, that there exists a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m = \Delta(\lambda) \otimes V$$

such that $F_i/F_{i-1} \cong \Delta(\mu_i)$ for some $\mu_i \in \mathfrak{h}^*$. What can you say about the multiset $\{\mu_i\}$?

ii) Now suppose that $\lambda, \mu \in \mathfrak{h}^*$ are such that $\lambda+\rho$ is strictly dominant, $\mu+\rho$ is dominant, and $\lambda - \mu \in \mathbb{Z}R$. Show that $T_\lambda^\mu(\Delta(w \cdot \lambda)) \cong \Delta(w \cdot \mu)$. Conclude that T_λ^μ gives an equivalence $\mathcal{O}_\lambda \to \mathcal{O}_\mu$ if $\lambda + \rho$ and $\mu + \rho$ are strictly dominant. Moreover, show that $T_\lambda^\mu \circ T_\nu^\lambda \cong T_\nu^\mu$ whenever $\mu + \rho, \lambda + \rho, \nu + \rho$ are all strictly dominant.

iii) Now suppose that λ is integral and that $\lambda + \rho$ is strictly dominant. Show we have an isomorphism

$$[\mathcal{O}_\lambda] \to \mathbb{Z}We_\lambda : [\Delta(w \cdot \lambda)] \mapsto e_\lambda \cdot w$$

where $e_\lambda = \sum_{x \in \mathrm{Stab}_W(\lambda+\rho)} x$.

iv) Let λ, μ be as above. In addition, assume that λ, μ are integral, that λ is regular (i.e. $\lambda + \rho$ is strictly dominant) and the μ is sub-regular (i.e. $e_\mu = (1 + s)$ for some $s \in S$). Show that we have a commutative diagram

$$\begin{tikzcd}
{[\mathcal{O}_\lambda]} \arrow[r, "T_\lambda^\mu"] \arrow[d, "\sim"] & {[\mathcal{O}_\mu]} \arrow[r, "T_\mu^\lambda"] \arrow[d, "\sim"] & {[\mathcal{O}_\lambda]} \arrow[d, "\sim"] \\
\mathbb{Z}W \arrow[r, "\cdot(1+s)"] & \mathbb{Z}W(1+s) \arrow[r, "\text{inclusion}"] & \mathbb{Z}W
\end{tikzcd}$$

(the vertical isomorphisms are those of the previous exercise).

v) (Optional) Can you give similar descriptions for more general weights? (I.e. non integral, or with e_λ more complicated?)