MSRI Soergel bimodule workshop

June/July 2017

Week 2 Day 1 Morning: Supplementary/Advanced Exercises

The embedding theorem

1. Fix $x \in W$, and work in the category $\mathcal{D}/\mathcal{D}_{<x}$. The local intersection form on $BS(\underline{w})$ at x is a form on the space $\mathrm{Hom}^0(x, BS(\underline{w}))$, where x represents the object B_x in this quotient category. As we have seen, $\mathrm{Hom}^0(x, BS(\underline{w}))$ has a basis given by (upside-down) light leaves of degree 0.

a) Prove that the map $\iota\colon \mathrm{Hom}^0(x, BS(\underline{w})) \to \overline{BS(\underline{w})}$ which sends $\phi \mapsto \overline{\phi(c_{\mathrm{bot}})}$ is injective.

b) Suppose that $\overline{BS(\underline{w})}$ is equipped with a Lefschetz operator given by left multiplication by some $f \in \mathfrak{h}^* \subset R$. Prove that the image of ι consists of primitives in degree $-\ell(x)$.

2. This question tries to explain the results of previous exercises.

If you have not done it yet, do exercise 4 from 1-4a-basic. Do the signatures of the forms agree with the embedding theorem and the Hodge–Riemann bilinear relations?

If you have not done it yet, do exercise 1 from 1-4b-basic. How can you explain these signatures using the Hodge theory of Soergel bimodules?

Rouquier complexes

3. This exercise is very very computational! (Hint: If you're stuck, look at a paper by Elias–Krasner.) Suppose that $m_{st} = 3$. Find the most general chain map (of degree 0) from $F_sF_tF_s$ to $F_tF_sF_t$ and vice versa (i.e. you should get families of maps given by certain parameters). Compute their composition, an endomorphism of $F_sF_tF_s$. For certain parameters, find a homotopy map between this composition and the identity chain map.

4. This exercise is also very computational.

a) Compute the minimal complex FT of F_sF_s, including the differentials.

b) A *shift* is a complex of the form $R(n)[k]$, for some grading shift n and homological shift k. Compute all morphisms of complexes from each shift to FT. There should be only two nonzero maps, up to the action of R on morphisms.

c) One of the two maps, which we denote α, is a quasiisomorphism of complexes of R-bimodules. Show that α becomes an isomorphism (i.e. a homotopy equivalence) after tensoring with B_s on the right. (In this situation, we call α an *eigenmap*, and B_s an *eigencomplex.*)

d) Call the other map β. Let Λ_α and Λ_β denote the cones of α and β, respectively. Prove that the tensor product $\Lambda_\alpha\Lambda_\beta$ is nulhomotopic. (This categorifies the relation $(H_s^2 - v^2)(H_s^2 - v^{-2}) = 0$, and proves that FT is *categorically diagonalizable.*)

Positivity and quantum numbers

5. In this lecture series, we have been assuming that $a_{s,t} = -2\cos\frac{\pi}{m_{st}}$, or in other words, that $a_{s,t} = -(q + q^{-1})$ where $q = e^{\pm\frac{\pi i}{m}}$ is a primitive $2m$-th root of unity. In the very first exercise sheet, we have seen that one can set q to be other primitive $2m$-th roots of unity, and one will still obtain an action of W. (That is, the existence of a W action is an algebraic condition, and primitive roots of unity are algebraically equivalent.) However, our Hodge-theoretic results will fail.

What positivity considerations will fail if q is set to one of these other roots of unity? For example, what if $m = 53$ and $q = e^{\frac{3\pi i}{53}}$? (Hint: Try 1-4a-supplement, exercise 1c.)