Theoretical Neuroscience

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1 Integrate and Fire Model

$$CV = Q \tag{1}$$

$$C\frac{dV}{dt} = I \tag{2}$$

Recall Ohm's Law:

$$IR = V \Rightarrow I = \frac{V}{R} \tag{3}$$

At rest, the neuron is actually subject to discharge, hence the minus sign:

$$I = \frac{-V}{R} \tag{4}$$

But this is offset by the ATP pump:

$$I = I_{pump} - \frac{V}{R} \tag{5}$$

Where I_{pump} is the polarizing current at rest: $\frac{V_{rest}}{R}$. So:

$$I = I_{pump} - \frac{V}{R} = \frac{V_{rest}}{R} - \frac{V}{R} = \frac{1}{R}(V_{rest} - V) \tag{6}$$

Now let's add an external current, I_ext:

$$I = frac1R(V_{rest} - V) + I_ext \tag{7}$$

From 2:

$$C\frac{dV}{dt} = I = frac1R(V_{rest} - V) + I_ext \tag{8}$$

Multiply both sides by R:

$$RC\frac{dV}{dt} = V_{rest} - V + RI_ext \tag{9}$$

Define the membrane time constant τ:

$$RC \equiv \tau \tag{10}$$

Plug τ into equation 9:

$$\tau\frac{dV}{dt} = V_{rest} - V + RI_{ext} \tag{11}$$

In any given instant? V_{rest} and RI_{ext} are constants. IS THIS A DEFINITION:

$$V_\infty = V_{rest} + RI_{ext} \tag{12}$$

So, substituting 12 into 11:

$$\tau\frac{dV}{dt} = V_\infty - V \tag{13}$$

Note, the minus on the V is convenient for solving ODE.

Dividing both side by τ:

$$\frac{dV}{dt} = \frac{V_\infty - V}{\tau} \tag{14}$$

1.0.1 SOLVE ODE

Separate the variables (V and t), multiplying both sides by dt and dividing both by $V_\infty - V$:

$$\frac{1}{V_\infty - V}dV = \frac{1}{\tau}dt \tag{15}$$

Define:

$$z \equiv V_\infty - V \tag{16}$$

So WHERE DOES MINUS SIGN COME FROM:

$$\frac{1}{z}dz = -\frac{1}{\tau}dt \tag{17}$$

Then integrate both sides:

$$\int_{z(0)} z(t)\frac{1}{z}dz = \int_{\frac{1}{\tau}} dt \tag{18}$$

Take anti-derivatives:

$$\ln z(t) - \ln z(0) = -\frac{t}{\tau} \tag{19}$$

$$\ln z(t) - \ln z(0) = -\frac{t}{\tau} \tag{20}$$

Difference of the logs is the log of the quotient:

$$\ln(\frac{z(t)}{z(0)}) = -\frac{t}{\tau} \tag{21}$$

Take the exponent of both sides:

$$\frac{z(t)}{z(0)} = e^{\frac{t}{\tau}} \tag{22}$$

$$z(t) = z(0)e^{\frac{t}{\tau}} \tag{23}$$

The originals for z are:

$$z(t) = V_\infty - V(t) \tag{24}$$

$$z(t) = V_\infty - V(0) \tag{25}$$

Restoring these in the equation gives us:

$$V_\infty - V(t) = (V_\infty - V(0))e^{\frac{t}{\tau}} \tag{26}$$

Subtract V_∞ from both sides:

$$-V(t) = -V_\infty + (V_\infty - V(0))e^{\frac{t}{\tau}} \tag{27}$$

Reverse the sign on both sides:

$$V(t) = V_\infty + (V(0) - V_\infty)e^{\frac{t}{\tau}} \tag{28}$$

Modeling the evolution of voltage over time.

Let us inject a current $I(t)$ into a RC circuit with a battery of voltage V_{rest}. The total current can be divided into a resistor component and a capacitor component (Why?):

$$I(t) = I_C(t) + I_R(t) \tag{29}$$

Recall that $C = \frac{Q}{V} \to Q = VC$ and $I = \frac{d}{dt}Q$ (charge per unit of time) so:

$$I_C = \frac{d}{dt}VC = \frac{dV}{dt}C \tag{30}$$

Recall that $V = IR$ or, for this case, $V_R = I_R R$. Given the battery, with voltage V_{rest}, the total voltage in the circuit $V = V_{rest} + V_R$, so $V_R = V - V_{rest}$ and:

$$I_R = \frac{1}{R}(V - V_{rest}) \tag{31}$$

Summing the last two equations:

$$I(t) = I_C(t) + I_R(t) = \frac{1}{R}(V - V_{rest}) + \frac{dV}{dt}C \tag{32}$$

Multiply both sides by R:

$$RI(t) = (V - V_{rest}) + \frac{dV}{dt}RC \tag{33}$$

Isolate last term:

$$\frac{dV}{dt}RC = -(V - V_{rest}) + RI(t) \tag{34}$$

1.1 Linear Differential Equation

Let's look at the units. The right hand side is in units of volts (recall that $V = IR$). The left hand side has one component $\frac{dV}{dt}$ with volts divided by units of time, so the other component must be a unit of time. We'll call this "time constant" τ: $RC = \tau$, so substituting τ for RC:

$$\tau \frac{dV}{dt} = \tau \frac{d}{dt} V = -(V - V_{rest}) + RI(t) \tag{35}$$

Note, if we set a $V_1 = V - V_{rest}$ then $\frac{d}{dT}(V - V_{rest}) = \frac{d}{dT}V - \frac{d}{dT}V_{rest}) = \frac{d}{dT}V$ (V_{rest} is a constant). And we cna say:

$$\tau \frac{dV_1}{dt} = -V_1 + RI(t) \tag{36}$$

Consider the solution for $I = 0$:

$$\tau \frac{dV_1}{dt} = -V_1 \tag{37}$$

So:

$$V = e^{\frac{-t}{\tau}} \text{ that is, exponential decay over time, back to } V_{rest} \tag{38}$$

2 Probability Distributions

2.1 Poisson

Parameter, λ, referred to as the "intensity" or "arrival rate." When rate is constant, referred to as homogeneous poisson process.

Derivation from Bernoulli

PDF

$$P(y|\lambda) = \frac{\lambda^{-y} e^{-\lambda}}{y!}$$

Table 1: Parameters in neuron model

Name	Symbol (Alt)	Formula	Units	Value	Definition
Capacitance			farad, F	.1-1nF	How much charge can be stored on a particular capacitor for a given potential difference across it: $C = \frac{V}{q}$
Charge	Q	$Q = CV$	coulomb		
Current	I	$I = C\frac{dV}{dt} = \frac{V}{R}$	amps, A		Flow of charge per time unit (Amount of positive charge flowing per unit of time past a point in a conductor)
Current density	I		μAcm^2		Flow of charge per time unit (Amount of positive charge flowing per unit of time past a point in a conductor)
Force	F		newton		
Resistance	R		ohm, Ω		
Voltage	V, E ?	u	volts, V (joules per coulomb)		Membrane (electric) potential. Work (joules) per charge (coulombs)
Reset voltage	V_{reset}		volts		Reset potential
Rest voltage	V_{rest}, E		volts	-70mV	Reset potential
Threshold voltage	V_{thresh}		volts (joules per coulumb)	-55 - -50mV	Threshold voltage, threshold potential
Work	W		joules (like all energy)		

First Moment

$$\frac{\lambda^{-y}e^{-\lambda}}{y!}\frac{dy}{d\lambda} = \frac{1}{y!}(y\lambda^{y-1}e^{-\lambda} - \lambda^{y}e^{-\lambda})$$

Set equal to 0 (and elminate constant $\frac{1}{y!}$):

$$y\lambda^{y-1}e^{-\lambda} = y\lambda^{y}e^{-\lambda}$$

Divide both sides by $e^{-\lambda}$:

$$y\lambda^{y-1} = \lambda^{y}$$

Divide both sides by λ^{y-1}:

$$y = \lambda$$

So:

$$E(y) =< y >= \lambda$$

Second Moment Same holds for second moment:

$$Var(y) =< (y- < y >)^2 >= \lambda$$

Fano Factor The Fano Factor of a distribution is defined as the ratio of the variance to the mean. For a poisson random variable:

$$\text{Fano factor } = fracVar(y)E(y) = \frac{\lambda}{\lambda} = 1$$

Fano factors in neural recordings range from $.3 - 1.8$ [REF?]

3 Physics

Table 2: Phyisics, forces

Law	Formula	Exposition
Capacitance	$C = \frac{Q}{V} = \frac{1}{4k\pi} \frac{A}{D}$	
Coulomb's Law	$\lvert F \rvert = k_e \frac{\lvert q_1 q_2 \rvert}{d^2}$	Force F = Coulomb's constant k_e times product of charge magnitudes $q_1 q_2$ divided by distance squared d^2 (aks r^2, radial distance)
Electric Field	$E = \frac{F}{q} = k_e \frac{Q}{d^2} = 2k\pi \frac{Q}{A} = 2k\pi\sigma$	Derived from Coulomb's law. Force exerted at a given distance. $\sigma = \frac{Q}{A}$ where A is area.
Ohm's Law	$I = \frac{V}{R}$, $IR = V$, $R = \frac{V}{I}$	I, the current through the conductor in units of amperes is a function of V,the potential difference measured across the conductor in units of volts, and R, the resistance of the conductor in units of ohm
Work	$W = Fd$	Work = force times distance

Electrical potential energy is a property of a charged particle.

By contrast, electrical potential is a property of a position.

Derivative rules (make table?)

$e^x = e^x$