ENM 207

Lecture 8

The Expected Value and Variance of a Random Variable

The Expected Value for a Random Variable

- Since there is a range of possible values of a random variable, we would be interested in some central value such as the average.

- If X is a discrete random variable with probability function p(x), its weighted average value, denoted by E(x) or μ,is

$$\mu = E(x) = \sum_{all\ x_i} x_i \, p(x_i)$$

The Expected Value for a Random Variable

Similarly, for a continuous random variable X with p.d.f. f(x), the mean value is

$$E(x) = \int_{-\infty}^{\infty} x \; f(x)dx$$

The expected value of a function g(x):

$$E[g(x)] = \begin{cases} \sum_{all \;\; x_i} g(x_i)p(x_i) & , \; if \;\; x \;\; discrete \\ \int_{-\infty}^{\infty} g(x)f(x)dx & , \; if \;\; x \;\; continuous \end{cases}$$

Example

x_i	0	1	2	3
$p(x_i)$	1/27	10/27	8/27	8/27

$$E(x) = \sum_{\forall i} x_i \, p(x_i) = 0\frac{1}{27} + 1\frac{10}{27} + 2\frac{8}{27} + 3\frac{8}{27} = \frac{50}{27} = 1.85$$

Example

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample

Example

Let *X* be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \dfrac{20000}{x^3} & \mathrm{x} > 100 \\ 0 & \text{o.w.} \end{cases}$$

$$\mu = E(X) = \int_{100}^{\infty} x \frac{20000}{x^3} dx = \int_{100}^{\infty} \frac{20000}{x^2} dx = 200 \text{ hours}$$

SOME USEFUL EXPECTATION THEOREMS

Theorem

Let y be a discrete random variable with probability distribution p(y) and let c be a constant. Then the expected value or mean of c is

$$E(c) = c$$

Then the expected value of cy is

$$E(cy) = cE(y)$$

SOME USEFUL EXPECTATION THEOREMS

Theorem

Let y be a discrete random variable with probability distribution p(y) and let $g_1(y)$, $g_2(y)$,...., $g_k(y)$ be functions of y. Then

$$E[g_1(y)+g_2(y)+....+g_k(y)] = E[g_1(y)]+E[g_2(y)]+...+E[g_k(y)]$$

Example 4.5. p 92/ Walpole

Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0 & \text{o.w.} \end{cases}$$

Find the expected value of $g(X) = 4X + 3$

$$E(4X+3) = \int_{-1}^{2} \frac{(4x+3)x^2}{3} dx = \frac{1}{3}\int_{-1}^{2} (4x^3 + 3x^2) dx = 8$$

VARIANCE OF RANDOM VARIABLE

- Let Y be d.r.v. with probability distribution p(y) then the variance of y is

$$\sigma^2 = E\left[(x-\mu)^2\right] = E\left(x^2\right) - \mu^2 = E\left(x^2\right) - [E(x)]^2$$

$$\sigma^2 = E\left(x^2 - 2\mu x + \mu^2\right) = E\left(x^2\right) - 2\mu E(x) + \mu^2$$
$$= E\left(x^2\right) - 2\mu^2 + \mu^2 = E\left(x^2\right) - \mu^2$$

- The standard deviation of y is the positive square root of the variance of y:

$$\sigma = \sqrt{\sigma^2}$$

Example

Let Hi & Ti denote the observation of a head & a tail on the ith toss, for i=1,2.

Simple event	Description	P(Ei)	# of heads
E1	H1H2	1/4	2
E2	H1T2	1/4	1
E3	T1H2	1/4	1
E4	T1T2	1/4	0

Example

For the random variable

- Obtain probability mass function
- Calculate
  - Expected value of r.v.
  - Variance of r.v.

Example

The mathematical model for d.r.v. y is given below:

y_i	7	8	9	10	11	12	13
$p(y_i)$	0.10	0.10	0.15	0.20	0.20	0.15	0.10

a) Show that it is a probability mass function
b) P(y=9)=?
c) P(y<12)=?
d) Calculate E[X]
e) Calculate Var[x]

Example

Let the random variable X represents the number of automobiles that are used for official business purposes on any given workday. The probability distributions for company A and B are:

Company A

x	1	2	3
p(x)	0.3	0.4	0.3

Company B

x	0	1	2	3	4
p(x)	0.2	0.1	0.3	0.3	0.1

Show that the variance of the probability distribution of company B is greater than that for company A.

Example

y= is the # of firing pins tested in a sample of five selected from a large lot.

Suppose the cost of inspecting of a single pin is $300 if the pin is defective and $100 if not. Then the total cost c (in dollars) of the inspection is given by the equation c=200+100y .

Find the mean and the variance of c.

y	P(y)
1	0.6
2	0.3
3	0.1

Example

- The hospital period, in days, for patients following treatment for a certain type of kidney disorder is a random variable Y = X + 4, where X has the density function

$$f(x) = \begin{cases} \dfrac{32}{(x+4)^3}, & \text{x} > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

- Find the average number of days that a person is hospitalized following treatment for this disorder.

References

- Walpole, Myers, Myers, Ye, (2002),
  - *Probability & Statistics for Engineers & Scientists*
- Dengiz, B., (2004),
  - *Lecture Notes on Probability,* http://w3.gazi.edu.tr/web/bdengiz
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  - *Probability & Statistics in Engineering* & Management *Science*