ENM 207

Lecture 4
Probability

Definitions in Probability

- **Outcome** is a possible observation in an experiment
- **Experiment** is any action or process to collect outcomes
- **Random experiment** is any action or process that gives different outcomes under the same conditions.
  - Tossing a coin, drawing cards from a deck, rolling a die, etc.
- **Sample Space** is the set of all possible outcomes of the experiment

Definitions in Probability

Example:

1. Flip a coin 3 times, Observe the sequence of heads/tails
   {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

2. Flip a coin 3 times, Observe # of heads
   {0, 1, 2, 3}

- **Event**
  - Set of outcomes (Must know all outcomes)
  - Event $\subset$ Sample Space
    - ***Simple (or elementary) event:*** consists of exactly one outcome
    - ***Compound event:*** consists of more than one outcome

Event Examples

For an experiment:
Roll a dice, observe the shown numbers

- **Outcomes:**
  - number = 1,2,3,4,5,6
- **Sample space:**
  - S = {1,2,3,4,5,6}
- **Event examples:**
  - Simple event: E = {4}
  - Compound event :
    - E1 = {number < 3} = {1,2}
    - E2 = {number is odd} = {1,3,5}

Set vs Probability

Set Algebra	Probability
Set	Event
Universal set	Sample space
Element	Outcome

Probability Concept

- Probability allows us to quantify the likelihood associated with uncertain events, that is, events that result from random experiments.
- Probabilities are reported as
  - proportions (between 0 and 1)
  - percentages (between 0% and 100%).
- Thus the statements
  - P(A)=.30, the probability of event A occurring is .30, and
  - the event A has a 30% chance of occurring

  are equivalent.

Probability of Event, P(☺)

P[☺]
is a function that maps event
in the sample space to real number

- From experiment: Roll a dice

Outcomes:
- number = 1,2,3,4,5,6

- **Sample space:**
  - S = {1,2,3,…,6}

$P[E_1] = 2/6 = 1/3$
$P[E_2] = 3/6 = 1/2$

- **Event examples**:
  - E1 = {number < 3} = {1,2}
  - E2 = {number is odd} = {1,3,5}

Probability of Event, P(☺)

- There are several ways to determine probability of an event:
  - As relative frequencies of occurrence;
    - Repeat the experiment
    - Calculate the relative frequency of the occurrence of the event of interest
  - By assuming that events are equal likely
    - Die, coin, etc.
  - From subjective estimates.
    - To find the probability that a horse will win a race,
      - Previous records of all the horses entered in the race
      - The records of the jockeys riding the horses

Assign probability value to an event

Example

- Purpose : To find the probability that a product can be defective
- Experiment : Each product in a production line is checked to determine whether it is defective or not.
  - There are two consequences into an experiment
    - DF: Defective
    - ND: Non defective
- Let
  - n : The number of experiments which were conducted
  - n(DF) : The number of defective products into *n* experiments
  - Using first approach, i.e. relative frequency of occurrence

$$P(DF) = n(DF)/n$$

  - As n grows large, n(DF)/n ratio converges to a steady number, called the limiting relative frequency, which is used to estimate P(DF).

Assign probability value to an event

Example :

- Check the 600 products (n)
- Classify each of them into two classes as defective and non defective
- Find the number of defective products
  - n(DF) = 60
- Estimate the probability that a product is a defective

$$\begin{aligned} P(DF) &= n(DF) / n \\ &= 60 / 600 \\ &= 0.10 \end{aligned}$$

- The product is defective with the probability of 0.10

Assign probability value to an event

- If the sample space for an experiment contains N elements, all of which are equally likely occur,
- Such as
  - rolling a die, $S = \{1,2,3,4,5,6\}$
  - tossing a coin, $S = \{H,T\}$
- the probability of each of the N points is equal and $1/N$.

If an experiment can result in any one of *N* different equally likely outcomes, and if exactly *n* of these outcomes correspond to event A, then the probability of event A is

$$P(A) = \frac{n}{N}$$

Example, Walpole (p. 41)

A statistics class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students. If a person is randomly selected by the instructor to answer a question, find the probability that the student chosen is

a) An industrial engineering major
b) A civil engineering or an electrical engineering major

Solution

Probability Axioms

- For each element in the sample space (the set of outcomes of the experiment), we wish to assign a number P, called the probability of that outcome, such that:
- **Axiom 1:** For any event A, $0 \leq$ **P[A]** $\leq$ **1**
- **Axiom 2:** **P[S] = 1**
- **Axiom 3:** For events $A_1, A_2, \ldots, A_n$ of mutually exclusive events
  - $\mathbf{P[A_1 \cup A_2 \cup \ldots \cup A_n] = P[A_1]+P[A_2]+\ldots+P[A_n]}$

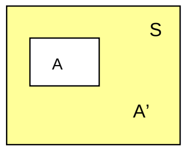
Consequences of Axioms

Theorem 1: if ∅ is the empty set , then P(∅) = 0

Proof:

Theorem 2 $P(\overline{A}) = 1 - P(A)$

Proof:



Consequences of Axioms

Theorem 3:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof:

Consequences of Axioms

Theorem 4

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)$$
$$-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$$

Homework

Proof $A\cup B\cup C = (A\cup B)\cup C$

(Hint: use theorem 3 since $A\cup B$ is an event)

Consequences of Axioms

Theorem 5

If $A\subset B$,then $P(A) \le P(B)$

Proof:

Example (Walpole, p. 40)

A die is loaded in such a way that an even number is twice as likely to occur as an odd number.

a) If E is the event that a number less than 4 occurs on a single toss of the die, P(E)=?

b) Let A be the event that an even number turns up and let B be the event that a number divisible by 3 occurs. $P(A \cup B) = ?$ and $P(A \cap B) = ?$

Solution

Example (Walpole, p. 42)

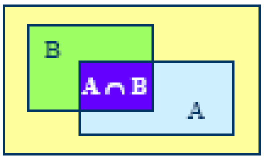
In a poker hand consists of 5 cards, find the probability of holding 2 aces and 3 jacks?

Conditional Probability

- Let A and B two events with P(B) > 0.

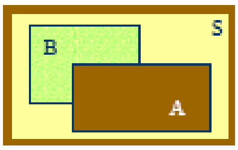
- **The conditional probability of A occurring given that event B has already occurred** is denoted by **P(A|B)** and can be calculated from the formula

$$P(A|B) = P(A \cap B)/P(B)$$

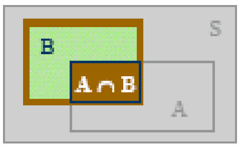


Conditional Probability

$$P(A\,|\,S) = \frac{P(A \cap S)}{P(S)} = \frac{\mathrm{P(A)}}{1} = \mathrm{P(A)}$$



$$P(A\,|\,B) = \frac{P(A \cap B)}{P(B)}$$



Conditional Probability

Ex: Lets consider a group of 100 persons of whom ;

- 40 are college graduates
- 20 are self employed
- 10 are both college graduates + self-employed

let

- B represents the set of college graduates
- A represents the set of self-employed
- A∩B is the set of college graduates who are self employed

From the group of 100 , one person is to be randomly selected.
(The chance of this person to come from college group). Then;

$$P(A)=\frac{20}{100}=0.2$$

$$P(B)=\frac{40}{100}=0.4$$

$$P(A\cap B)=\frac{10}{100}=0.10$$

Conditional Probability

Now suppose the following event is considered :

The probability of the event that selected person comes from self-employed given that the person is a college graduate **(In this case the reduced sample space will be A\B)**.

Obviously the sample space is reduced in that only college graduates are considered.

The probability , P(A\B) is thus given by

$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{0.1}{0.4}=0.25$$

The reduced sample space consists of the set of all subsets of S that belong to B

Conditional Probability

$$P(A \setminus B) = \frac{P(A \cap B)}{P(B)} \quad if \quad P(B) > 0$$

The conditional probability satisfies the properties required of probabilities. That is,

1) $0 \le P(A \setminus B) \le 1$
2) $P(S \setminus B) = 1$
3) $P(A_1 \cup A_2 \cup A_3 \cup \ldots \setminus B) = P(A_1 \setminus B) + P(A_2 \setminus B) + P(A_3 \setminus B) + \ldots$
   $\ldots\ldots\ldots for \quad A_1, A_2, A_3 \ldots\ldots,$

Example: 4.11 p 62/Schaums outline series

In a certain college, 25% of the students failed math, 15% of the students failed chemistry, 10% of the students failed both math and chemistry. A student is selected at random.

a) If he failed chemistry what is the probability that he failed math?

b) If he failed math, what is the probability that he failed chemistry?

c) What is the probability that he failed math or chemistry?

Solution

Example: The probability that a regularly scheduled flight departs on time is $P(D)=0.83$; the probability that it arrives on time is $P(A)=0.82$; and the probability that it departs and arrives on time is $P(D \cap A)= 0.78$. Find the probability that a plane

a) Arrives on time given that it departed on time
b) Departed on time given that it has arrived on time.
c) Arrives on time given that it did not depart on time.

Solution:

a)

b)

Solution:

c)

Example:

There are three columns entitled "Art" (A) ,"Books" (B) and "Cinema" (C) in a new magazine. Reading habits of a randomly selected reader with respect to choosen columns are:

read regularly	A	B	C	$A\cap B$	$A\cap C$	$B\cap C$	$A\cap B\cap C$
probability	.14	.23	.37	.08	.09	.13	.05

(The probability of reading only A column is 14%)

1) What is the probability of column A given that they read column B?
2) What is the probability of reading A given that they are reading B or C columns?
3) What is the probability of reading column A given that they are reading at least one column?
4) What is the probability of reading A or B columns given that they read C columns?

Solution:

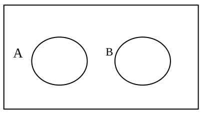
The Multiplication Rule

From conditional probability definition we know

$$P(A \cap B) = P(B) \times P(A \setminus B) \quad ; P(B) > 0$$
$$P(A \cap B) = P(A) \times P(B \setminus A) \quad ; P(A) > 0$$

This statement is the obvious consequence of conditional probability.

It should be noted that if A and B are mutually exclusive events than $A \cap B = \varnothing$ So that $P(B \setminus A) = 0$ and $P(A \setminus B) = 0$



A and B are mutually exclusive events

Example: One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball drawn from the second bag is black?

The Multiplication Rule

- If, in an experiment, the events A_1, A_2, …, A_k can occur, then

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_k)$$

$$= P(A_1)P(A_2 \backslash A_1)P(A_3 \backslash A_1 \cap A_2)\ldots P(A_k \backslash A_1 \cap A_2 \cap \ldots \cap A_{k-1})$$

Example: Three cards are drawn in succession, without replacement, from an ordinary deck of playing card. Find the probability that event $A_1 \cap A_2 \cap A_3$ occurs, where

A_1 is the event that the first card is red ace,

A_2 is the event that the second card is 10 or a jack, and

A_3 is the event that the third card is greater than 3 but less than 7.

Independent Events

Definition : Event A and B are independent iff

$P(A \cap B) = P(A)P(B)$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$
$$= \frac{P(A)P(B)}{P(B)}$$
$$P(A \mid B) = P(A)$$
$$P(B \mid A) = P(B)$$

Two events, A and B, are **independent events** if the probability that either one occurs is not affected by the occurrence of the other.

Independent Interpretation

$P(A) = 0.3$

$P(A|B) = 0.3$

No matter event B occurs or not, event A is not affected

Independent Events

From this definition we can say if A and B are independent events

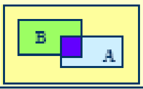
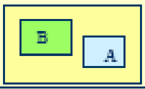
$$P(A \cap B) = P(A)P(B)$$

$$P(A \setminus B) = P(A)$$

$$P(B \setminus A) = P(B)$$

$$\underbrace{P(B \setminus A)}_{P(B)} = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(A)P(B)$$

Independent vs Disjoint

Independent	Disjoint
	
P[AB] ≠ 0	P[AB] = 0
P[A∩B] = P[A]∗P[B]	P[A∪B] = P[A]+P[B]

Note : Independent = Disjoint iff P(A) = 0 or P(B) = 0

Independent Events

- Several events, $A_1, A_2, A_3, \ldots, A_k$, are **independent** if the probability of each event is unaltered by the occurrence of any subset of the remaining events.
- In this case, the product rule can be applied to any subset of the **k** events.
- That is, the probability that all the events in any *subset* occur equals the product of their individual probabilities of occurring. In particular, for all k events,

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots\ldots \cap A_k) = P(A_1)P(A_2)P(A_3)\ldots.P(A_k)$$

Example: A coin is biased so that a head is twice likely to occur as a tail. If the coin is tossed 3 times, what is the probability of getting 2 tails and 1 head?

References

- Walpole, Myers, Myers, Ye, (2002),
  - *Probability & Statistics for Engineers & Scientists*
- Dengiz, B., (2004),
  - *Lecture Notes on Probability,* http://w3.gazi.edu.tr/web/bdengiz
- Hines, Montgomery, (1990),
  - *Probability & Statistics in Engineering* & Management *Science*