ENM 207

Lecture 13

Some Useful Continuous Distributions

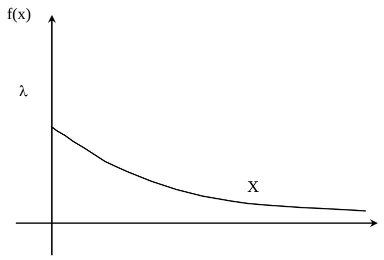
Exponential Distribution

Definition: A continuous random variable X assuming all nonnegative values is said to have an exponential distribution with parameter $\lambda>0$ if its pdf is given by

$$f(x)=\begin{cases}\lambda e^{-\lambda x} & x\geq 0\\ 0 & o.w.\end{cases}$$

$$\lambda = \frac{1}{\beta}$$

$$f(x)=\begin{cases}\dfrac{e^{\frac{-x}{\beta}}}{\beta} & 0\leq x<\infty\\ 0 & o.w.\end{cases}$$



Exponential Distribution

f(x) is a probability density function for

1) $\lambda > 0 \quad and \quad \lambda e^{-\lambda x} > 0 \quad thus \quad f(x) > 0$

2) $\int_0^\infty \lambda e^{-\lambda x} dx = \int f(x)dx = -e^{-\lambda x}\Big|_0^\infty = 0 - (-1) = 1$

Exponential Distribution

The expected value of X is obtained as follows

$$\mu = E(x) = \int_0^\infty x\lambda e^{-\lambda x} dx$$

Integrating by parts and letting $\lambda e^{-\lambda x} dx = dv$

X = u , we obtain $v = -e^{-\lambda x}$, du=dx. Thus

$$E(X) = \left[-x e^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx \right] = -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty = \frac{1}{\lambda}$$

Exponential Distribution

The variance of X is obtained by using partial integration.

$$E\left(x^2\right)=\frac{2}{\lambda^2}$$

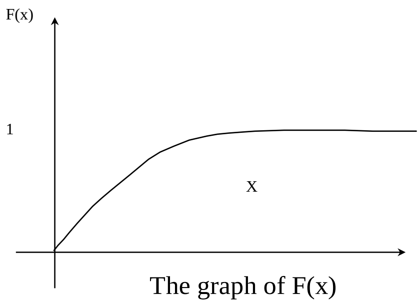
$$\sigma^2 = V(X) = E\left(x^2\right) - [E(x)]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Exponential Distribution

Cumulative Distribution Function

$$F(x) = P(X \le x) = \int_0^x \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^x = -e^{-\lambda x} - (-e^{-\lambda(0)}) = -e^{-\lambda x} + 1 = 1 - e^{-\lambda x}$$

$$F(x) = P(X \le x) = \begin{cases} 1 - e^{-\lambda x} & x \ge 0 \\ 0 & o.w. \end{cases}$$



The graph of F(x)

Exponential Distribution

Example:

The time X (in seconds) that it takes a librarian to locate a card in a file of records on checked-out books has an exponential distribution with expected time 20 seconds. Calculate the following probabilities:

a) $P(X \le 30)$

b) $P(X \ge 20)$

c) $P(20 \le X \le 30)$

d) For what value of t is $P(x \le t) = 0.5$

(t is the fifteenth percentile of the distribution)

Exponential Distribution

Example:

Suppose the length of time an electric bulb lasts, X , is a random variable with cumulative distribution.

$$F(x) = P(X \le x) = \begin{cases} 1 - e^{-\frac{x}{500}} & x \ge 0 \\ 0 & \text{o.w.} \end{cases}$$

find the probability that the bulb lasts

a) between 100 and 200 hours

b) beyond 300 hours

Exponential Distribution

Example:

Let X be the random variable representing the length of a telephone conversation. Let $f(x) = \lambda e^{-\lambda x} \quad 0 \le x \le \infty$.

a) Find the c.d.f , F(x),

b) Find $P(5 < x \le 10)$

Exponential Distribution

MEMORYLESS PROPERTY

- An important application of the exponential distribution is to model the distribution of component lifetime.
- The reason for the popularity on such applications is the "memoryless" property of the exponential distribution.
- Suppose component lifetime is exponentially distributed with parameter λ.
- After putting the component into service , we check it two hours later and find the component still working.
- What is the probability that it works at least an additional t hours?

Exponential Distribution

We can represent this event using this conditional probability

$$P\left(x \geq t + t_0 \middle| x \geq t_0\right)$$

By the definition of conditional probability

$$P\left(x \geq t + t_0 \middle| x \geq t_0\right) = \frac{P\left[\left(x \geq t + t_0\right) \cap \left(x \geq t_0\right)\right]}{P\left(x \geq t_0\right)}$$

But the event $x \geq t_0$ in the numerator is redundant , since both events can occur if and only if $x \geq t + t_0$.

Exponential Distribution

$$P\left(x \geq t + t_0 \middle| x \geq t_0\right) = \frac{P\left(x \geq t + t_0\right)}{P\left(x \geq t_0\right)} = \frac{1 - F\left(t + t_0; \lambda\right)}{1 - F\left(t_0; \lambda\right)}$$

$$= \frac{1 - \left[1 - e^{-\lambda(t + t_0)}\right]}{1 - \left[1 - e^{-\lambda t_0}\right]} = \frac{e^{-\lambda t} e^{-\lambda t_0}}{e^{-\lambda t_0}} = e^{-\lambda t}$$

This conditional probability is identical to the original probability $P(x \geq t)$ that the component lasted t hours.

Thus the distr. of additional lifetime is exactly the same as the original dist. of life time. We can say that the dist. of remaining life time is independent of current age.

Relationship Between Exponential and Poisson Distribution

- The exponential distribution is closely related to the poisson distribution.
- If the number of occurrences has a poisson distribution then the time between occurrences has an exponential distribution.
- For example if the number of orders for a certain item received per week has a poisson distribution, then the time between orders would have an exponential distribution.
- One variable is discrete (the countable one) , and the other(time) is continuous.

Relationship Between Exponential and Poisson Distribution

- In developing the poisson distribution from the poisson process, we fixed time at some value *t* (we consider unit time as *t*) and we developed the distribution of the number of occurrences in the interval [0,t].
- We noted this random variable by X and the distribution is

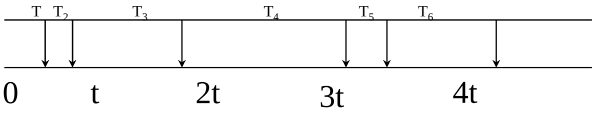
$$P(x) = \begin{cases} \dfrac{e^{-\lambda t}(\lambda t)^x}{x!} & x = 0,1,2\ldots \\ 0 & o.w. \end{cases}$$

Relationship Between Exponential and Poisson Distribution

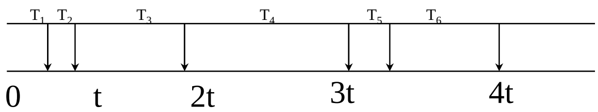
- Now consider p(0), which is the probability of no occurrences on [0, *t*]. This is given by

$$P(0) = e^{-\lambda t}$$

- Recall that we originally fixed time at t.
- Another interpretation of $P(0) = e^{-\lambda t}$ is that this the probability that the time between two occurrence is greater than t.



Relationship Between Exponential and Poisson Distribution



The elapsed time between two occurrences is greater than t, means that T > t.

T is interarrival time of these occurences. If T > t is shown means that

there is no occurrence on this interval t.

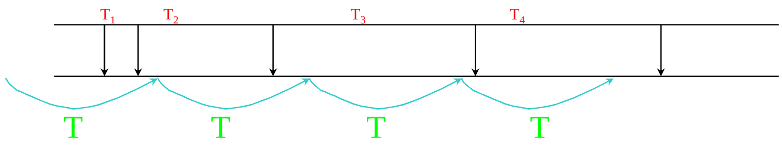
Consider this time as a random variable T.

We note that $P(0) = P(T > t) = e^{-\lambda t} \qquad t \geq 0$

Because $P(0) = e^{-\lambda t}$

Relationship Between Exponential and Poisson Distribution

$$P(T \le t) + P(T > t) = 1$$
$$P(T \le t) = 1 - P(T > t)$$
$$F(t) = 1 - e^{-\lambda t}$$
$$P(T \le t) = F(t)$$



$$f(t) = F'(t) = \frac{\partial F(t)}{\partial t} = -\left(-\lambda e^{-\lambda t}\right) = \lambda e^{-\lambda t}$$

$$f(t) = \begin{Bmatrix} \lambda e^{-\lambda t} & t \ge 0 \\ 0 & o.w. \end{Bmatrix}$$ This is the exponential density function.

Exponential Distribution

Example:

Calls are received at a 24-hour "successive hotline" according to a Poisson process with rate α=0.5 calls per day. The # of days between successive calls has an exponential distribution with parameter value 0.5.

a) Find the probability that more than 2 days elapsed between calls.

$$P(x > 2) = 1 - P(x \le 2) = 1 - \left(1 - e^{(-0.5)2}\right) = e^{-1} = 0.368$$

b) The expected time between successive calls

$$E(x) = \frac{1}{\lambda} = \frac{1}{0,5} = 2 \ \ days$$

Exponential Distribution

Example:

Let X=the time between two successive arrivals at the drive-up window of a local bank. If X has an exponential distribution with $\lambda=1$ compute

a) The Expected time between two successive arrivals.

b) The standart deviation of time between successive arrivals.

c) $P(X\leq 4)=?$ d) $P(2\leq X\leq 5)=?$

References

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