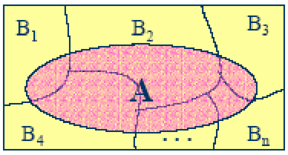
ENM 207

Lecture 5

Bayes Theorem

Total Probability Law



- Let $B_1, B_2, \ldots, B_n$ be mutual exclusive events whose union equals sample space S
- $P(B_i) > 0$

For any event A

$A = A \cap S = A \cap (B_1 \cup B_2 \cup \ldots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n)$

$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n)$

$$\text{Theorem:} \quad P(A) = \sum_{i=1}^{n} P(A \cap B_i)$$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + ... + P(A \cap B_n)$$

$$P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + ... + P(A \mid B_n)P(B_n)$$

$$\boxed{\text{Theorem:} \quad P[A] = \sum_{i=1}^{n} P[A|B_i]P[B_i]}$$

Example:

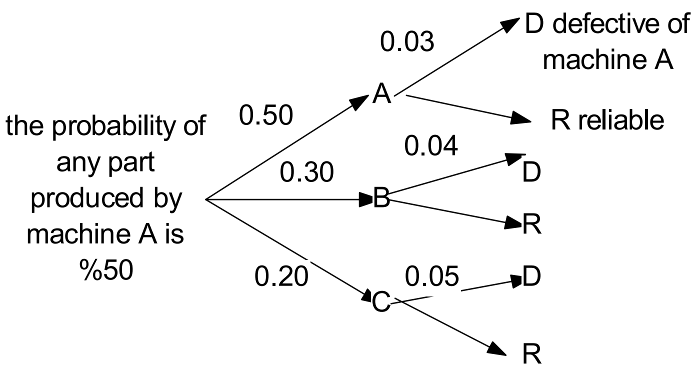
- Three machines A,B,C produce respectively 50%, 30%, 20% of the total number of items of a factory .
- The percentages of defective output of these machines are 3%, 4%, 5%, respectively. If an item is selected at random, find the probability that the item is defective.
- Let X be the event that an item is defective.
- Which theorem or which rule is used to solve this problem?

Solution:

Total probability law.

$$\begin{aligned} P(X) &= P(A)\times P(X \setminus A) + P(B)\times P(X \setminus B) + P(C)\times P(X \setminus C) \\ &= (0.50)\times(0.03) + (0.30)\times(0.04) + (0.20)\times(0.05) \\ &= 0.037 \end{aligned}$$

Another way to solve this problem: Tree Diagram



Bayes Theorem

- Another important result of the total probability law is known as "**Bayes Theorem**":

If $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S and A is an arbitrary event on S , then for r = 1,2, ,k

$$P(B_r \setminus A) = \frac{P(B_r) \times P(A \setminus B_r)}{\sum_{i=1}^{k} P(B_i) \times P(A \setminus B_i)} \quad \text{BAYES THEOREM}$$

$$P(B_r \setminus A) = \frac{P(B_r \cap A)}{P(A)} = \frac{P(B_r) \times P(A \setminus B_r)}{\sum_{i=1}^{k} P(B_i) \times P(A \setminus B_i)}$$

The numerator is a result of multiplication rule and the denumerator is a result of total probability law.

Example:

- Consider the factory in the preceeding example. Suppose an item is selected at random and is found to be defective. Find the probability that the item was produced by machine A;
- What is the P(A\X)?
- Which theorem or which law can be used to solve this problem?

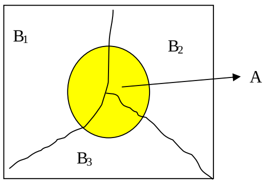
$$P(A \setminus X) = \frac{P(A) \times P(X \setminus A)}{P(A) \times P(X \setminus A) + P(B) \times P(X \setminus B) + P(C) \times P(X \setminus C)}$$

$$= \frac{(0.50) \times (0.03)}{(0.50) \times (0.03) + (0.30) \times (0.04) + (0.20) \times (0.05)} = \frac{15}{37}$$

Ex: 2.35 page 56 (Montgomery)

A manufacturer of telemetry equipment microprocessors is supplied from 3 different facilities. According to the historical data which are recorded by manufacturer company :

Supplying facility	Fraction defective	Fraction supplied by
1	0,02	0,15
2	0,01	0,80
3	0,03	0,05



Example: The director of manufacturing randomly selects a microprocessor , after testing procedure , finds that it is defective. Compute the probability that this defective product came from facility 3.

Let A be the event that an item is defective

B_1 be the event that the item comes from facility 1

B_2 be the event that the item comes from facility 2

B_3 be the event that the item comes from facility 3

We can find this probability using BAYES THEOREM.

$$P(B_3 \setminus A) = \frac{P(B_3) \times P(A \setminus B_3)}{P(B_1) \times P(A \setminus B_1) + P(B_2) \times P(A \setminus B_2) + P(B_3) \times P(A \setminus B_3)}$$

$$\frac{P(B_3 \cap A)}{P(A)} = \frac{0.05 \times 0.03}{0.15 \times 0.02 + 0.80 \times 0.01 + 0.05 \times 0.03} = \frac{3}{25}$$

Example: 6 p 61 (Walpole et al.)

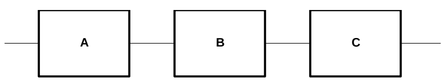
A regional telephone company operates three identical relay stations at different locations. During a one year period, the number of malfunctions reported by each station and causes are shown below:

Stations	A	B	C
Problem with electricity supplied	2	1	1
Computer malfunction	4	3	2
Malfunctioning electrical equipment	5	4	2
Caused by other human errors	7	7	5

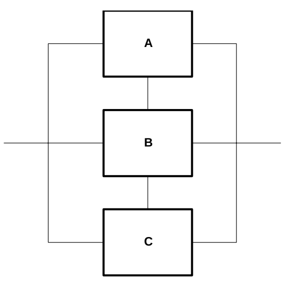
Suppose that a malfunction was reported and it was found to be caused by other human errors. What is the probability that it came from station C?

SYSTEM RELIABILITY

- Systems - electronic, mechanical or a combination of both –are composed of components.
- A component of a system is represented by a capital letter.
- Two systems each composed of tree components A; B; C are shown below.
- Systems according to their components connections can be classified in two groups. Such as series and parallel.



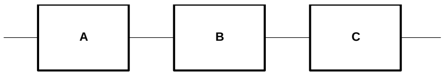
A) Series system



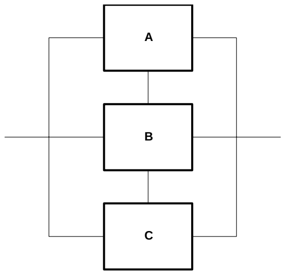
B) Parallel system

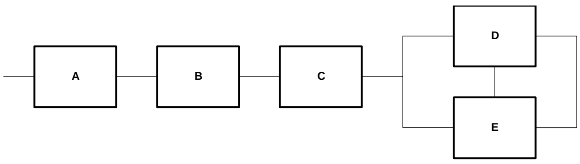
Definition:

- If the system fails when any of the components fails, it is called a series system.

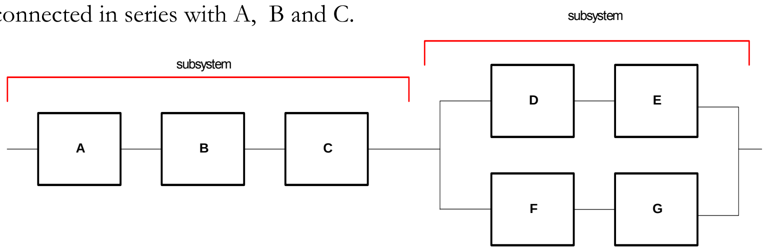


•If the system fails only when all of its components fail, it is called a parallel system.





This system is composed of five components A, B, C, D and E as shown above. Components D and E are form a two-component parallel system. This subsystem is connected in series with A, B and C.



Two subsystems are connected in series. The first subsystem contains three components such as A, B and C.

The second contains two series subsystems: the first composed of D and E, the second composed of F and G.

The reliability of a series system is

P(system functions) = P(all componenets function)

The multiplication rule can be applied because the components operate independently of each other in a series system. If there is k components in a series system

P(series system functions) = P(A functions) . P(B functions). . P(K functions)

$$P_A \times P_B \times P_C \ldots P_K$$

where p_I is the probability that i^{th} component functions, I=A, B, ..., K

Reliability of Series System

1. If the system is in series form that all components are connected as a series.

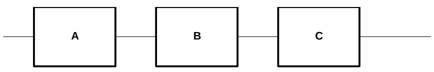
$$P(\text{series system functions}) = \prod_{i}^{n} P(\text{each component functions})$$

If there are k components like $A_1, A_2,, , A_k$

$$P(\text{series system functions}) = \prod_{i}^{k} P(A_i)$$

Because a series system will fail if any one of its components fail.

Example



Given that P_A=0.90 P_B=0.95 P_C=0.90 find the reliability of the series system shown in this figure.

P(System functions)= (0.90)x(0.95)x(0.90)=0.7695

The reliability of a parallel system containing k components can be calculated in a similar manner. Since a parallel system will fail only if all components fail,

$$P(parallel\ system\ \ fails) = (1 - P_A) \times (1 - P_B) \ldots (1 - P_C)$$

Where p_I is the probability that the i^{th} component functions, i=A,B,...,K.

2. If the system is parallel and also if it contains k components,

$$P(parallel \quad system \quad fails) = \left(\prod_{i}^{k} (1 - P(A_i)) \right)$$

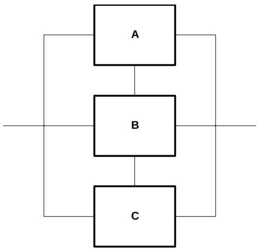
Because as you know a parallel system will fail only if all components will fail.

From this equation

$$P(parallel \quad system \quad functions) = \left(1 - \prod_{i}^{k} (1 - P(A_i)) \right)$$

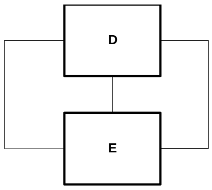
Note: These formulations can be used to calculate the reliabilities of series systems , parallel systems , or any combinations of them. These systems must satisfy the assumption that the components operate independently.

Example:



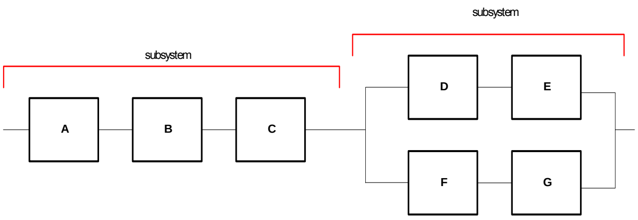
P(first subcircuits)=1-(1-P_A)(1-P_B)(1-P_C)

P(A)=0.95, P(B)=0.95 , P(C)=0.90, P(D)=0.90, P(E)=0.98



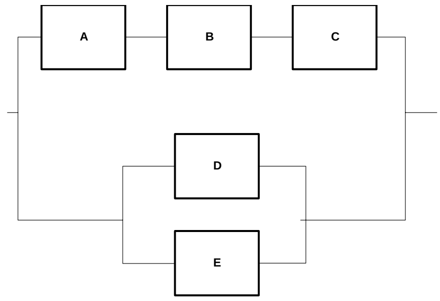
P(second subcircuits)=1-P(D)P(E)

If the subcircuits are connected in series



P(two subcircuits are connected in series)=P(first subcircuits)P(second subcircuits)

- If two subcircuits are connected in parallel



two subcircuits are connected in parallel = 1-(1-P(first scir.))(1-P(sec.scir.))

Example

3 identical component A, B, C means that ($P_A = P_B = P_C$)

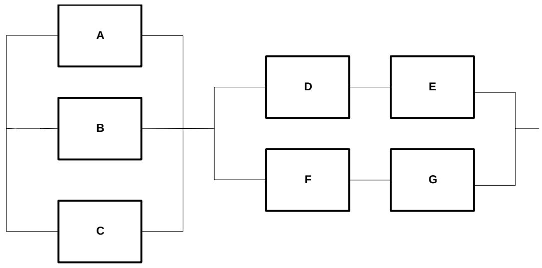
If P(System functions) = 0.95, then find $P_A = P_B = P_C$

a) when the components are connected in series

b) when the components are connected in parallel

Example: Find the reliability of the system shown in this figure given that

$P_A = 0.90 \quad P_B = 0.95 \quad P_C = 0.95 \quad P_D = 0.92 \quad P_E = 0.97$
$P_F = 0.92 \quad P_G = 0.97$



This figure represents that the system is a series of two parallel subsystems.

References

- Walpole, Myers, Myers, Ye, (2002),
  - *Probability & Statistics for Engineers & Scientists*
- Dengiz, B., (2004),
  - *Lecture Notes on Probability,* http://w3.gazi.edu.tr/web/bdengiz
- Hines, Montgomery, (1990),
  - *Probability & Statistics in Engineering* & Management *Science*