ENM 207

Lecture 10

Some Useful Discrete Distributions

Poisson Distribution

- The French Mathematician S.D. Poisson (1781-1840)

  provides a model for the relative frequency of the number of "rare events" that occur in a unit of time, unit of area, unit of volume, etc.

For example ;

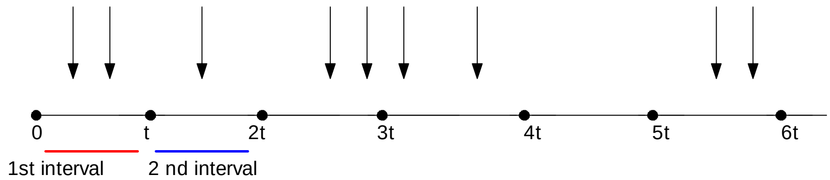
- The # of new jobs submitted to a computer in any one minute
- The # of fatal accidents per month in a manufacturing plant
- The # of visible defects in a diamond

are variables which are distributed by POISSON distribution.

The Characteristics of a Poisson Random Variable

- The random variable of interest , say X_t , is the # number of occurrence, say arrivals, that occur on the interval [0,t].
- The range space $Rx_t=\{0,1,2,\ldots\}$
- The experiment consists of counting the number of occurrence of a particular event during a given unit of time, or in a given area or volume (or weight, distance or any other unit of measurement)

Poisson Distribution



For example the r.v. of interest is the # of customer arrivals on the t interval .

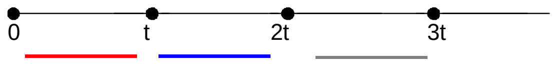
Lets X_t is the # of customers arrived in t.

X_t values are the 2, 1, 2, 2, 0, 2 for sequentially arrivals , respectively.

The probability that an event occurs in a given unit of time (area or volume) is the same for all the unit intervals.

The number of arrivals (or any events) during non-overlapping time intervals are independent r.v.s

Poisson Distribution



These intervals are non-overlapping intervals.

- The mean (or expected) number of events in each unit will be denoted by Greek letter , λ .

Poisson Distribution

One of the most useful discrete distribution is the poisson distribution.

While defining poisson process, we consider a collection of arbitrary time oriented occurrences often called “arrivals”.

The random variable of interest , say X_t , is the number of arrivals that occur on the interval [0,t] .

The range space $R_{X_t}=\{0,1,2,3\ldots\}$

The parameter λ is sometimes called the mean arrival rate or mean occurrence rate.

Poisson Distribution

We let

$$P(x) = P(X_t = x) = P_x(t) \quad x = 0,1,2\ldots$$
$$for \;\; x > 0$$
$$p(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & x = 0,1,2\ldots \\ 0 & o.w. \end{cases}$$

where λ= mean # of events, e= 2,71828

There are many real world phenomena for which the Poisson model is appropriate and where X is the r.v. distributed Poisson.

- Is this function a probability function?

i) $for\ \lambda > 0, x \geq 0, e^{-\lambda} > 0\ and\ \ x! > 0\ any\ p(x) > 0$

ii) Maclauren expansion of e^{λ}

$$\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!}\ldots$$

$$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty} e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\underbrace{\sum_{x=0}^{\infty}\frac{\lambda^x}{x!}}_{e^{\lambda}} = e^{-\lambda}e^{\lambda} = 1$$

Poisson Distribution

Mean of the Poisson distribution

$$\mu = E(x) = \sum_{x=0}^{\infty} x.e^{-\lambda} \frac{\lambda^x}{x!} \qquad \left(\lambda^x = \lambda^{x-1}.\lambda\right)$$

$$\mu = E(x) = \sum_{x=0}^{\infty} xe^{-\lambda} \frac{\lambda^{x-1}.\lambda}{x(x-1)!} = e^{-\lambda}\lambda \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = e^{-\lambda}\lambda e^{\lambda} = \lambda$$

Poisson Distribution

Variance of the Poisson distribution

$$E\left(x(x-1)\right) = E\left(x^2 - x\right) = \sum_{x=0}^{\infty} x(x-1)e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^2\lambda^{x-2}}{x(x-1)(x-2)!} = e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \underbrace{\frac{\lambda^{x-2}}{(x-2)!}}_{e^{\lambda}}$$

$$= e^{-\lambda}\lambda^2 e^{\lambda} = \lambda^2$$

Poisson Distribution

$$E(x(x-1)) = \lambda^2$$
$$E(x^2 - x) = \lambda^2$$
$$E(x^2) - E(x) = \lambda^2$$
$$E(x^2) = \lambda^2 + \lambda$$
$$Var(x) = E(x^2) - (E(x))^2$$
$$= \lambda^2 + \lambda - (\lambda)^2 = \lambda$$

Poisson Distribution

Example

During laboratory experiment the average number of radioactive particles passing through a counter in 1 milisecond is 4. What is the probability that 6 particles enter the counter in a given milisecond?

Poisson Distribution

On average a certain intersection results in 3 traffic accidents per month. What is the probability that for any given month at this intersection

a) Exactly 5 accidents will occur?
b) Less than 3 accidents will occur?
c) At least 2 accidents will occur?

References

- Walpole, Myers, Myers, Ye, (2002),
  - *Probability & Statistics for Engineers & Scientists*
- Dengiz, B., (2004),
  - *Lecture Notes on Probability,* http://w3.gazi.edu.tr/web/bdengiz
- Hines, Montgomery, (1990),
  - *Probability & Statistics in Engineering* & Management *Science*