ENM 207

Lecture 6
Random Variable

Random Variable

- **Experiment (Physical Model)**

  → Compose of procedure & observation

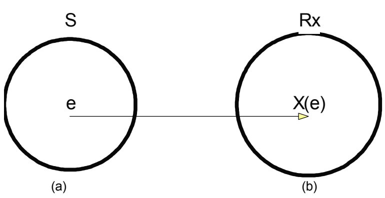
  → From observation, we get outcomes

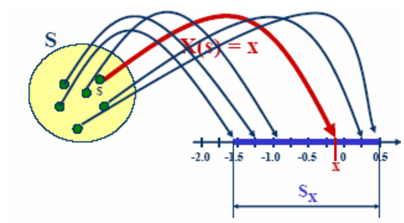
  → From all outcomes, we get a (mathematical) probability model called "Sample space"

  → From the model, we get $P[A]$, $A \subset S$

Random Variable

Definition: If ε is an experiment having sample space S, and X is a function that assigns a real number $X(e)$ to every outcome e; $e \in S$, then $X(e)$ is called a random variable





(a) The sample space of ε.

(b) Rx : The range of space X.

Example: Lets consider the coin tossing experiment that true coin was tossed three times and the sample space is

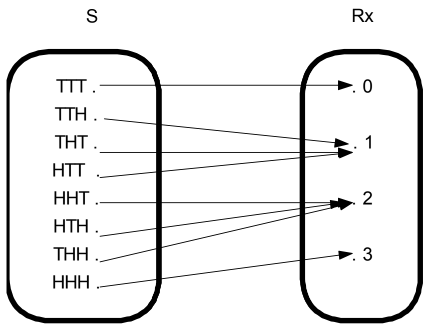
S={HHH,HHT,HTH,HTT,THH,THT,TTH,TTT}

If X is the number of heads showing ,then:

$$X(HHH)=3$$
$$X(HHT)=2$$
$$X(HTH)=2$$
$$X(HTT)=1$$
$$X(THH)=2$$
$$X(THT)=1$$
$$X(TTH)=1$$
$$X(TTT)=0$$

(They can be represented by 2 . Same random variables.)

The range space Rx={X:x=0,1,2,3} in this example



If the number of heads showing in any outcome is same in the different outcome, as you see in this figure , different values of e may lead to the same x.

The range space Rx , is made up of the possible values of x .

In coin tossing experiment ,

if the coin is true then there are eight equally likely outcomes, each having probability 1/8.

Lets

A is the event “exactly two heads”

x represent the number of heads.

The event that (x=2) relates to Rx , not S:

A={ HHT,HTH,THH } (therefore)

P(X=2)=P(A)=3/8 (eight equally likely outcomes in S)

Random Variable

Definition: If S is the sample space of an experiment ε and a random variable X with range space Rx is defined on S, and if event A is an event in S while event B is an event in Rx , than A and B are equivalent events if $A=\{e \in S: X(e) \in B\}$

Definition: If A is an event in the sample space , S and B is an event in the range space Rx of the random variable X, then we define the probability of B as $P(B)=P(A)$ where $A=\{e \in S: X(e) \in B\}$
The inverse of the function X:

$$X^{-1}(B) = \{e \in S : X(e) \in B\}$$

$$P(B) = P\left(x^{-1}(B)\right) = P(A)$$

Example: Consider the tossing of two true dice

Lets Y as the sum of the “up” faces. Then

$$Ry = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

probabilities are

$$\left(\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}\right)$$

There are 36 outcomes which , since the dice are true ,are equally likely

2: (1,1);
3: (1,2)-(2,1);
4: (2,2)-(1,3)-(3,1);
5: (2,3)-(3,2)-(1,4)-(4,1);
6: (3,3)-(2,4)-(4,2)-(1,5)-(5,1);
7: (3,4)-(4,3)-(2,5)-(5,2)-(1,6)-(6,1)
. . . .

Random Variable

- Discrete Random Variables
  - Examples:
    - X = # of customers in a bank
    - Y = sum of the up faces in a rolling two dice
- Continuous Random Variables
  - Examples:
    - X = processing time for a product
    - Y= height of a person

Discrete Random Variable

Definition:

- **X** is a **discrete random variable** if the range of **X** is countable

  $$Sx = \{x1,x2,\ldots\}$$

- **X** is a **finite random variable** if all values with nonzero probability are in the finite set

  $$Sx = \{x1,x2,\ldots,xn\}$$

Discrete Probability Distributions

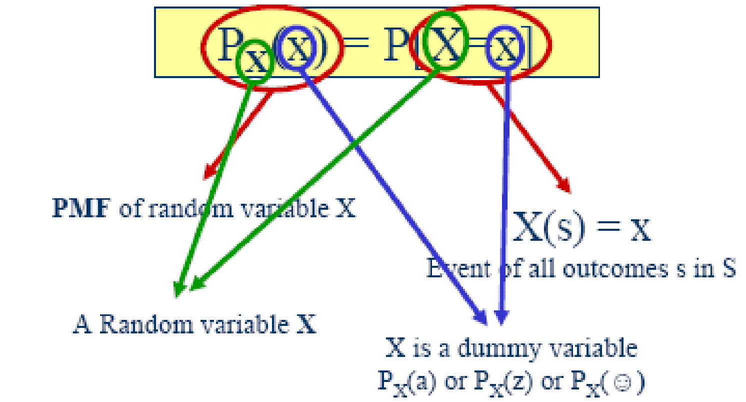
- For a (discrete) probability model,

  P[A] = [0,1]

- For a discrete random variable, the probability model is called a

  "Probability Mass Function (PMF) " or
  "Probability Distribution"

Discrete Probability Distributions



Discrete Probability Distributions

Definition: If X is a discrete random variable , it can be associated a number $P(X = x_i) = p(x_i)$ with each outcome x_i in Rx for $i = 1,2,\ldots,n$ where

$$1.\text{For any } x,\quad p(x_i) \geq 0 \; for\; all\; i$$

$$2.\sum_{x \in R_x} p(x) = 1$$

$$3.\text{For any event } B \subset R_x, P(B), \text{the probability that X is in the set B is}$$

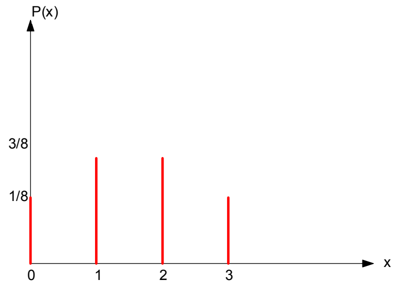
$$\text{P(B)} = \sum_{x \in B} p(x)$$

Example: In the coin tossing experiment , where X is the number of heads. We represent the probability distribution as a tabular or graphical form.

i) Tabular Representation

X	p(x)
0	1/8
1	3/8
2	3/8
3	1/8

ii) Graphical Representation



0 r.v. represents the outcome that it doesn't contain head , 1 represents the outcome that it contains one head, 2 represents two heads, 3 represents three heads.

Example: 3.3 (Walpole et al. p 66)

A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Example

There are 5 firing pins of which 3 are defective.

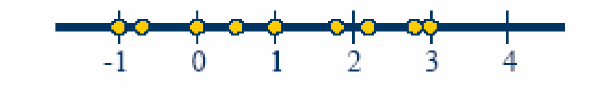
Let Y = number of firing pins tested until the first defective is found.

Let Di represents defective firing pin on the ith trial and

Let Gi represents a good firing pin on the ith trial.

Define probability distribution function for Y

Continuous Random Variables



- In Discrete: countable set of numbers
  - R_X= {-1,0,1,3,4}

- In Continuous: uncountable set of numbers
  - R_X = Interval between 2 limits
  - R_X = (x1,x2) = (-1,3)

Continuous Random Variables

- Measuring T, the download time

  R_T= {t | 0 < t < 12}

- Guess the download time is (0, 10] minutes
- Guess the download time is [5, 8] minutes
- Guess the download time is [5, 5.5] minutes

- **Chance that our guess is correct is decreasing**

- Guess the download time is exactly 5.25 min.

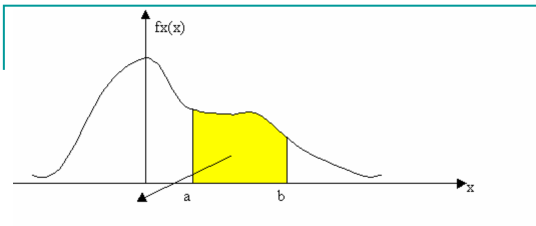
Probability of each individual outcome is zero. The interesting probability is an interval.

Continuous Probability Distributions

- Suppose that *X* is a random variable, its sample space is *S*, whose random variable function set *X*(*S*) is a continuum of numbers such as an interval.
- The set B={ a $\le X \le$ b } is an event in *S* and therefore the probability P(B)=P($a \le X \le b$) is defined.
- There is a piecewise continuous function.
- f : R→R such that P(a ≤ X ≤ b) is equal to the area under the graph of x=a and x=b.
- In the language of calculus,

$$P(a \le x \le b) = \int_a^b f(x)dx$$

In this case X said to be a continuous random variable.



P(a ≤ X ≤ b) = the area of yellow shaded region

The function f is called the distribution or the density function of X.

It satisfies the conditions

$$i)\ f(x) \ge 0$$

$$ii)\ \int_R f(x)dx = 1$$ Where R represents the real numbers set.

That is , f is nonnegative and the total area under its graph is 1. This area is equal to the probability of S.

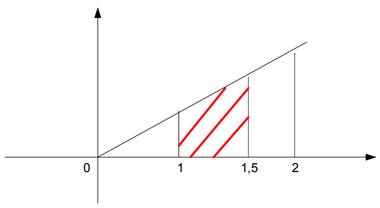
Continuous Probability Distributions

Example: Let X be a continuous random variable with the following distribution :

$$f(x)=\begin{cases}\frac{1}{2}x & if \quad 0\le x\le 2\\ 0 & o.w.\end{cases}$$

$$P(1\le x\le 1{,}5)=\int_{1}^{1{,}5}\frac{1}{2}xdx=\frac{1}{2}\frac{x^2}{2}\Big|_{1}^{1{,}5}=\frac{(1{,}5)^2}{4}-\frac{(1)^2}{4}=\frac{5}{16}$$

As you see the density function f(x) is the simple linear function.



The area of shaded region in this diagram is a trapezoid. You can use geometrical relation to find the same probability value.

$$P(1\le x\le 1{,}5)=\frac{1}{2}\times\frac{1}{2}\left(\frac{1}{2}+\frac{3}{4}\right)=\frac{5}{16}$$

Continuous Probability Distributions

Theorem: The function *f*(*x*) is a probability density function for continuous random variable *X*, defined over the set of real numbers *Rx*, if

1. $f(x)\ge 0 \quad for\ all\ x\in Rx$
2. $\int_R f(x)dx=1$
3. $It\ is\ piecewise\ continuous$
4. $f(x)=0\ \ if\ \ x\ is\ not\ in\ the\ range\ Rx$

Example: 3-13/p.74 Montgomery

$$f(x)=\begin{cases}\lambda e^{-\lambda t} & t\geq 0\\ 0 & o.w.\end{cases}$$

$$P(T\geq 100)=\int_{100}^{\infty}\lambda e^{-\lambda t}dt=e^{-100\lambda}$$

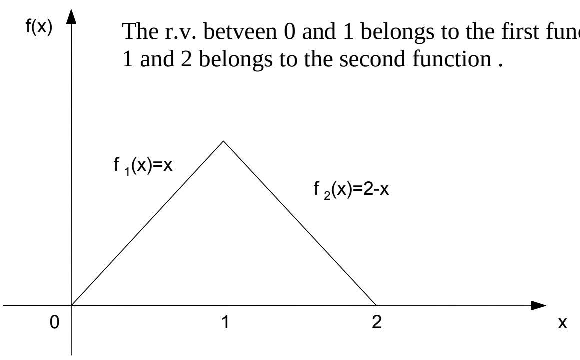
$$P(T\geq 100 \setminus T>99)=\frac{P(T\geq 100 \quad and \quad T>99)}{T(T>99)}=\frac{P(T\geq 100)}{P(T>99)}$$

$$=\frac{\int_{100}^{\infty}\lambda e^{-\lambda t}dt}{\int_{99}^{\infty}\lambda e^{-\lambda t}dt}=\frac{-e^{-100\lambda}}{-e^{-99\lambda}}=e^{-\lambda}$$

Example: 3-14/p74. Montgomery

$$f(x)=\begin{cases}x & 0\leq x<1\\ 2-x & 1\leq x<2\\ 0 & o.w.\end{cases}$$

The r.v. betveen 0 and 1 belongs to the first function of x, and r.v. between 1 and 2 belongs to the second function .



$$a)P\left(-1 < x < \frac{1}{2}\right) = ?$$

$$b)P\left(x \le \frac{3}{2}\right) = ?$$

$$c)P(x \le 3) = ?$$

$$d)P(x \ge 2{,}5) = ?$$

$$e)P\left(\frac{1}{4} < x < \frac{3}{2}\right) = ?$$

References

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  - *Probability & Statistics for Engineers & Scientists*
- Dengiz, B., (2004),
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