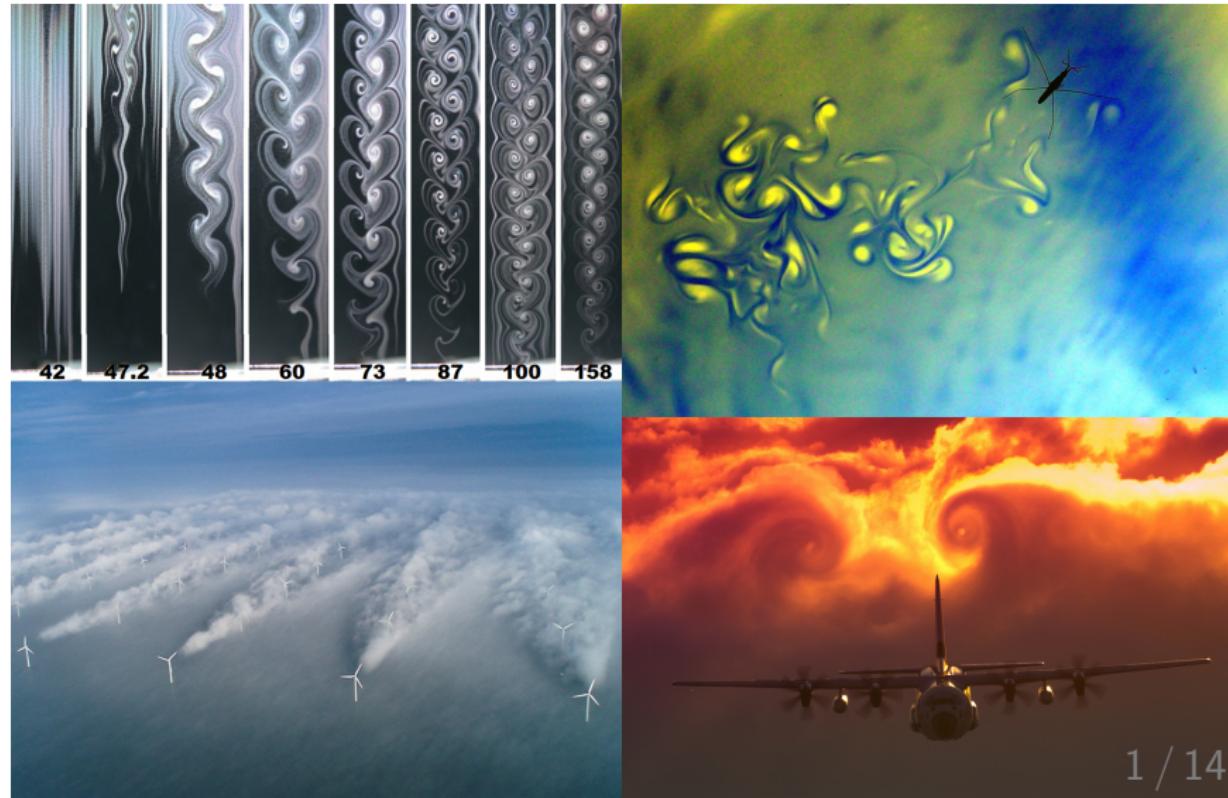


In Pursuit of a Fast High-order Poisson Solver: Volume Potential Evaluation

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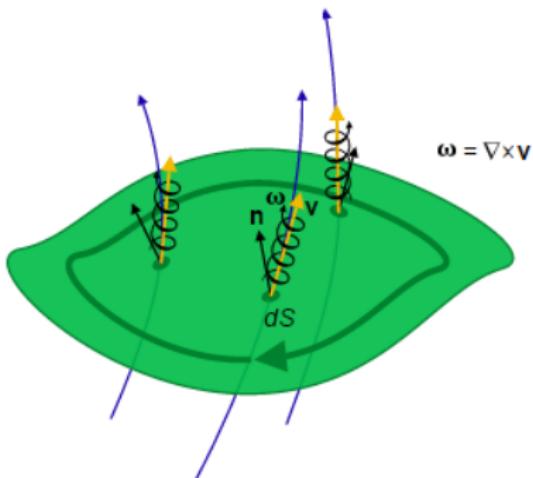
Introduction: Physical Examples and Motivating Problems



What is Vorticity?

$$\omega = \nabla \times \mathbf{u} \quad (1)$$

$$\Gamma = \oint_{\partial S} \mathbf{u} \cdot d\mathbf{l} = \iint_S \omega \cdot d\mathbf{S} \quad (2)$$



⁰<https://commons.wikimedia.org/wiki/File:Generalcirculation-vorticitydiagram.svg>

Some Brief Theory

Navier-Stokes momentum equation

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \frac{1}{3} \mu \nabla (\nabla \cdot \mathbf{u}) \quad (3)$$

where \mathbf{u} is the velocity field, p is the pressure field, and ρ is the density. Navier-Stokes can be recast as

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega - \omega \cdot \nabla \mathbf{u} = S(x, t) \quad (4)$$

viscous generation of vorticity, S . For incompressible flows velocity related to vorticity by

$$\nabla^2 \mathbf{u} = -\nabla \times \omega \quad (5)$$

Invert to obtain Biot-Savart integral

$$\mathbf{u}(x) = \int_{\Omega} K(x, y) \times \omega(y) dy \quad (6)$$

x is velocity eval point, y is non-zero vorticity domain, $K(x, y)$ singular Biot-Savart kernel.

Why Integral Equation Methods?

- ▶ Low-order solvers common (for both Lagrangian¹ and Eulerian² approaches)
- ▶ Some “high”-order work exists³, but is special purpose
- ▶ Ultimately, choice must be made between what form of Poisson equation is most useful
- ▶ Integral equations offer robust and flexible way, especially for complex geometries and for high-order

¹Moussa, C., Carley, M. J. (2008). A Lagrangian vortex method for unbounded flows. International journal for numerical methods in fluids, 58(2), 161-181.

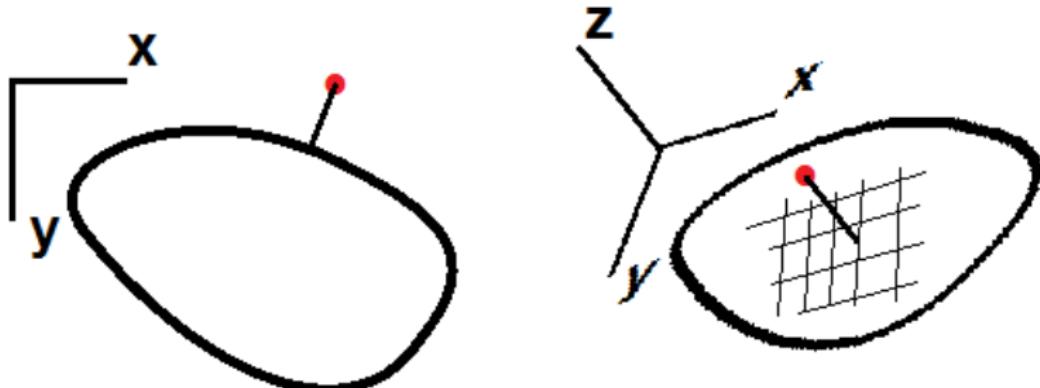
²R.E. Brown. Rotor Wake Modeling for Flight Dynamic Simulation of Helicopters. AIAA Journal, 2000. Vol. 38(No. 1): p. 57-63.

³J. Strain. Fast adaptive 2D vortex methods. Journal of computational physics 132.1 (1997): 108-122.

⁴Gholami, Amir, et al. "FFT, FMM, or Multigrid? A comparative Study of State-Of-the-Art Poisson Solvers for Uniform and Nonuniform Grids in the Unit Cube." SIAM Journal on Scientific Computing 38.3 (2016): C280-C306.

Methodology: Evaluation approach

- ▶ Volume potential share similarities to layer potentials
- ▶ Same main challenge: devising quadrature to handle singularity
- ▶ Take same approach: QBX
- ▶ But where do we put our expansion center, fictitious dimension?
- ▶ Off-surface: layer potential physically defined, off-volume has no requirements



Trial Scheme

- ▶ Absent any compelling choice for off-volume potential, choose obvious one:
- ▶ Consider 3D Poisson scheme: approximate $1/r$ kernel with $1/\sqrt{r^2 + a^2}$
- ▶ Effectively a parameter is the distance from expansion center to eval point in the fictitious dimension, and kernel is no longer singular
- ▶ Choose a “good” a so the kernel is smooth and take QBX approach of evaluating Taylor expansion of de-singularized kernel back at desired eval point

Is trial scheme high-order?

- ▶ No, in fact seems to be limited to second order regardless of expansion order.
- ▶ Consider example results in figure below for 5th order expansion.
- ▶ Why only second order?

Preliminary Error Analysis

- ▶ We would like to examine the error $\epsilon = |\text{Exact potential} - \text{QBX computed potential}|$ and its dependence on a
- ▶ Call $G(r) = \frac{1}{r}$, $f(r, a) = \frac{1}{\sqrt{r^2 + a^2}}$, and the k-th order Taylor series expansion about d and evaluated at $a = 0$:

$$T_k(r, d) = \sum_{n=0}^k \frac{(-d)^n}{n!} f^{(n)}(r, d)$$

- ▶ So our error is:

$$\epsilon = \int_{\Omega} G(r) \sigma(r) dr - \int_{\Omega} T(r, d) \sigma(r) dr$$

where $\sigma(r)$ is the density (vorticity in our physical example).

- ▶ This form seems complicated to inspect, is there a way to avoid the integrals and factor out the density?

Error in Fourier Space

- ▶ Consider the action of the Fourier transform on the error:

$$\mathcal{F}[\epsilon] = \mathcal{F} \left[\int G \sigma dr \right] - \mathcal{F} \left[\int T \sigma dr \right]$$

and by the convolution theorem:

$$= \mathcal{F}[G] \mathcal{F}[\sigma] - \mathcal{F}[T] \mathcal{F}[\sigma] = \mathcal{F}[\sigma] (\mathcal{F}[G] - \mathcal{F}[T])$$

$$\mathcal{F}[T_k] = \sum_{n=0}^k \frac{(-d)^n}{n!} \mathcal{F}[f^{(n)}(r, d)]$$

- ▶ This looks more reasonable, let's examine the behavior of $\mathcal{F}[G] - \mathcal{F}[T]$ with respect to d .

Fourier Transform Particulars

- ▶ Need 3D Fourier transform; both G and T are radially symmetric, so simplifications can be made: transforms can be given in terms of the scalar k in Fourier space.
- ▶ It is known that $\mathcal{F}[1/r] = 1/\pi k^2$
- ▶ With some work one can show:

$$\mathcal{F}\left[\frac{1}{\sqrt{r^2 + a^2}}\right] = \frac{2a}{k} K_1(2\pi ak)$$

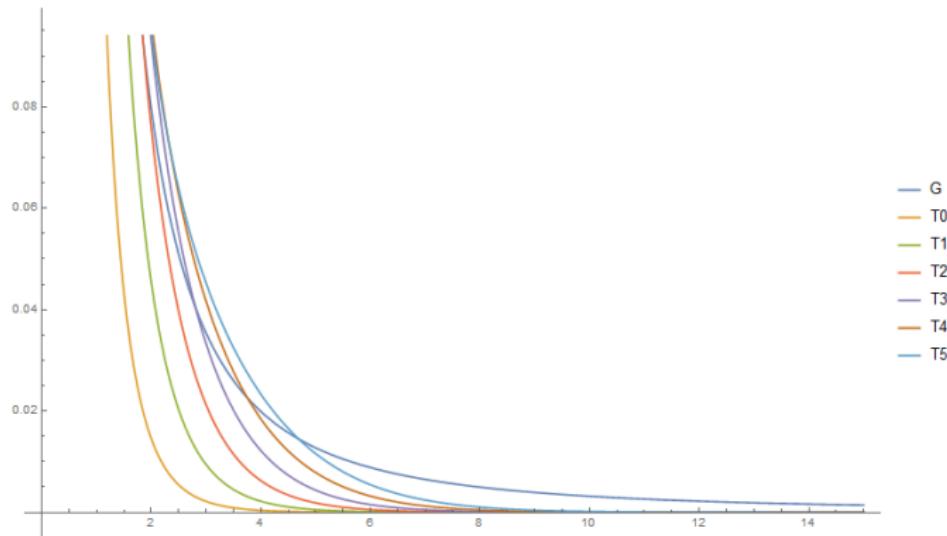
where $K_1(x)$ is the modified Bessel function of second kind

- ▶ Reduces to expected form for $\lim_{a \rightarrow 0} \frac{2a}{k} K_1(2\pi ak) = 1/\pi k^2$
- ▶ Without concerning ourselves with details, in general we find:

$$\mathcal{F}[T_k] = \sum_{n=-1}^k C_n d^{n+2} k^n K_n(2\pi kd)$$

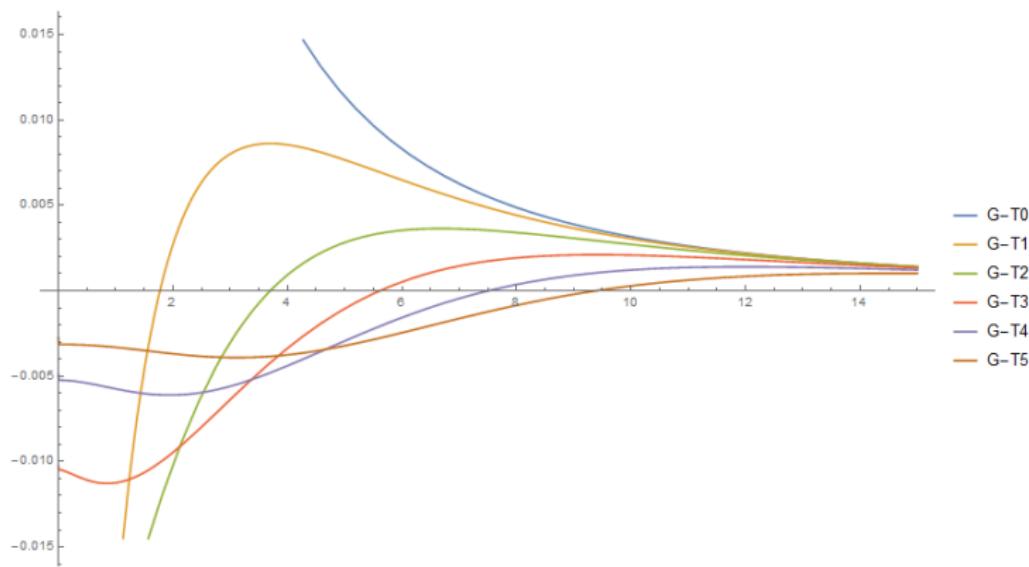
Fourier Space Behavior

- ▶ How well does T_k approximate G in Fourier space?
- ▶ Example figure shows G vs T_k for $d = 0.2$, higher order expansions do reasonably well qualitatively
- ▶ One issue: modified Bessel function of second kind have $\log(k)$ -type singularities at 0, while G has a k^{-2} singularity



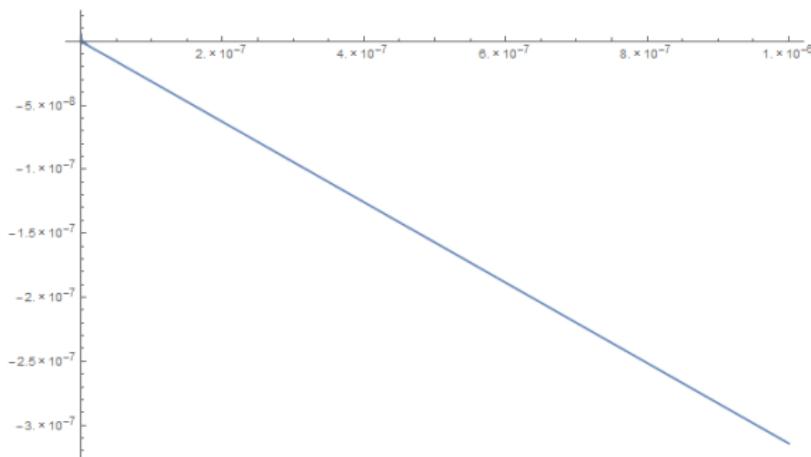
Examination of error: k dependence

- ▶ k dependence tells us how well the expansion preserves low vs high modes in real space
- ▶ Example figure shows k dependence for $d = 0.2$
- ▶ One way of thinking about the error quantitatively would be $\int (\mathcal{F}[G] - \mathcal{F}[T])^2 dk$, we would like to minimize this.
- ▶ Spoiler: closed form expression 2 slides away



Examination of error: d dependence

- ▶ Ultimately, a k-th order method should have the error be proportional to d^k
- ▶ However examine example figure for $|G - T_5|/d$ for $k = 5$ (we saw that a moderate order expansion only weakly depended on k , holds for other choice of k)
- ▶ Looks linear! Add back in factor of d , error seems to go as d^2 . Looks linear at any zoom range of d .



Closed form expression for error

- ▶ While $|\mathcal{F}[G] - \mathcal{F}[T]|$ is messy, as it turns out $\int (\mathcal{F}[G] - \mathcal{F}[T])^2 dk$ reduces concisely.
- ▶ For $T_3 : \frac{3\pi^3 d^3}{256}, T_4 : \frac{175\pi^3 d^3}{32768}, T_5 : \frac{3059\pi^3 d^3}{1048576}$
- ▶ Pick up extra power of d due to integration across all k compared to at a particular k
- ▶ Alternately, consider Taylor series expansion of T_5 in Fourier space with respect to d :

$$\frac{1}{\pi k^2} + \frac{\pi d^2}{10} + \frac{1}{20}\pi^3 d^4 k^2 + \mathcal{O}(d^6)$$

Future effort

- ▶ Suggests need for alternate basis in Fourier space more able to represent k^{-2} singularity
- ▶ Alternate basis in turn would suggest appropriate de-singularized kernel in real space
- ▶ Caveat: If an inverse Fourier transform exists and the result is smooth enough!