```
inner2D = Assuming[{k > 0, r > 0}, Integrate[Exp[-2 Pi I k r Cos [t]], {t, 0, 2 Pi}]]
```

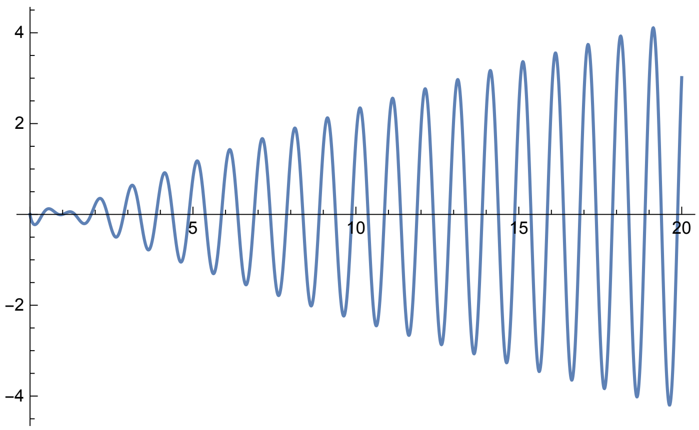
$2\pi\,\mathrm{BesselJ}[0, 2 k \pi r]$

```
Assuming [{a > 0, k > 0}, Integrate[Log[Sqrt[r^2 + a^2]] inner2D r, {r, 0, Infinity}]]
```

Integrate::idiv : Integral of r BesselJ[0, 2 k π r] Log[$a^2 + r^2$] does not converge on {0, ∞}. >>

$$\int_0^\infty 2\pi r\,\mathrm{BesselJ}[0, 2 k\pi r]\,\mathrm{Log}\left[\sqrt{a^2 + r^2}\right] dr$$

```
Plot[ Log[r] BesselJ[0, 2 Pi r] r, {r, 0, 20}]
```




```
inner3D := 2 Pi Integrate[Exp[-2 Pi I k r Cos[θ]] * Sin[θ], {θ, 0, Pi}]
```

```
inner3D
```

$$\frac{2\,\mathrm{Sin}[2 k \pi r]}{k r}$$

```
G = Limit[Assuming[{b > 0, k > 0},
    Integrate[Exp[-b r] / r inner3D r^2, {r, 0, Infinity}]], b → 0]
```

$$\frac{1}{k^2 \pi}$$

```
ibp = Simplify[D[r / Sqrt[r^2 + a^2], r]] * Integrate[r inner3D, r]
```

$$-\frac{a^2\,\mathrm{Cos}[2 k \pi r]}{k^2 \pi \left(a^2 + r^2\right)^{3/2}}$$

```
ft3d1 = Assuming[{a > 0, k > 0}, Integrate[-ibp, {r, 0, Infinity}]]
```

$$\frac{2\,a\,\mathrm{BesselK}[1, 2 a k \pi]}{k}$$

```
Assuming[{a > 0, r > 0}, Integrate[ft3d1 inner3D k^2, {k, 0, Infinity}]]
```

$$\frac{1}{\sqrt{a^2 + r^2}}$$

```
Limit[ft3d1, a → 0]
```

$$\frac{1}{k^2 \pi}$$

```
inner3Dexp = Expand[TrigToExp[inner3D] * Exp[-b r]]
```

$$\frac{i\, e^{-b r - 2 i k \pi r}}{k r} - \frac{i\, e^{-b r + 2 i k \pi r}}{k r}$$

```
ft3d2 = Limit[Assuming[{k > 0, a > 0, b > 0},
   Integrate[(1 / Sqrt[r^2 + a^2]) inner3Dexp r^2, {r, 0, Infinity}]], b → 0]
```

$$\frac{1}{2 k^2 \pi} \left(4 + i\, a\, k\, \pi^2 \, \text{BesselY}[1, -2\, i\, a\, k\, \pi] - i\, a\, k\, \pi^2 \, \text{BesselY}[1, 2\, i\, a\, k\, \pi]\right)$$

```
BesselY[1, -0.42 I]
```

$-0.214665 - 1.3108\, i$

```
-BesselY[1, 0.42 I]
```

$0.214665 - 1.3108\, i$

```
Limit[2 I a k Pi^2 Im[BesselY[1, I 2 Pi k a]] I, a → 0]
```

-2

```
Limit[
 I a k Pi^2 Im[BesselY[1, -I 2 Pi k a]] I - I a k Pi^2 Im[BesselY[1, I 2 Pi k a]] I, a → 0]
```

2

```
Limit[ft3d2, a → 0]
```

$$\frac{3}{k^2 \pi}$$

```
Tt = Normal[Series[1 / Sqrt[r^2 + a^2], {a, d, 4}]] /. a → 0
```

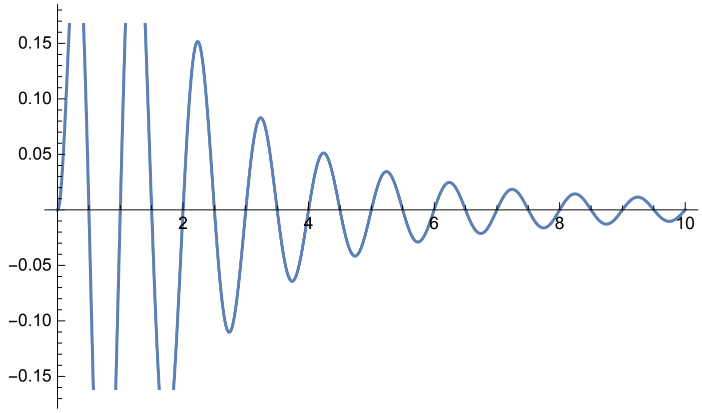
$$\frac{d^2 \left(2 d^2 - r^2\right)}{2 \left(d^2 + r^2\right)^{5/2}} + \frac{d^2}{\left(d^2 + r^2\right)^{3/2}} + \frac{1}{\sqrt{d^2 + r^2}} - \frac{d^3 \left(-2 d^3 + 3 d r^2\right)}{2 \left(d^2 + r^2\right)^{7/2}} + \frac{d^4 \left(8 d^4 - 24 d^2 r^2 + 3 r^4\right)}{8 \left(d^2 + r^2\right)^{9/2}}$$

```
s[n_] := Apart[(-d)^n D[1 / Sqrt[r^2 + a^2], {a, n}] / n! /. a → d, r]
```

```
s[1]
```

$$\frac{d^2}{\left(d^2 + r^2\right)^{3/2}}$$

```
Plot[(1 r/(r^2+1^2)^(3/2)) * Sin[2 Pi 1 r], {r, 0, 10}]
```




```
F[x_] := Assuming[{d > 0, k > 0}, Integrate[x inner3D r^2 , {r, 0, Infinity}]]
```

```
Fs[n_] := Total[F[Coefficient[s[n], d, Range[2 n]]] * d^Range[2 n]]
```

```
fs0 = ft3d1 /. a → d
```

$$\frac{2\, d\, \text{BesselK}[1,\ 2\, d\, k\, \pi]}{k}$$

```
fs1 = Fs[1]
```

$$4\, d^2\, \pi\, \text{BesselK}[0,\ 2\, d\, k\, \pi]$$

```
fs2 = Fs[2]
```

$$-2\, d^2\, \pi\, \text{BesselK}[0,\ 2\, d\, k\, \pi] + 4\, d^3\, k\, \pi^2\, \text{BesselK}[1,\ 2\, d\, k\, \pi]$$

```
fs3 = Fs[3]
```

$$-4\, d^3\, k\, \pi^2\, \text{BesselK}[1,\ 2\, d\, k\, \pi] + \frac{8}{3}\, d^4\, k^2\, \pi^3\, \text{BesselK}[2,\ 2\, d\, k\, \pi]$$

```
fs4 = Fs[4]
```

$$d^3\, k\, \pi^2\, \text{BesselK}[1,\ 2\, d\, k\, \pi] - 4\, d^4\, k^2\, \pi^3\, \text{BesselK}[2,\ 2\, d\, k\, \pi] + \frac{4}{3}\, d^5\, k^3\, \pi^4\, \text{BesselK}[3,\ 2\, d\, k\, \pi]$$

```
fs5 = Fs[5]
```

$$2\, d^4\, k^2\, \pi^3\, \text{BesselK}[2,\ 2\, d\, k\, \pi] - \frac{8}{3}\, d^5\, k^3\, \pi^4\, \text{BesselK}[3,\ 2\, d\, k\, \pi] + \frac{8}{15}\, d^6\, k^4\, \pi^5\, \text{BesselK}[4,\ 2\, d\, k\, \pi]$$

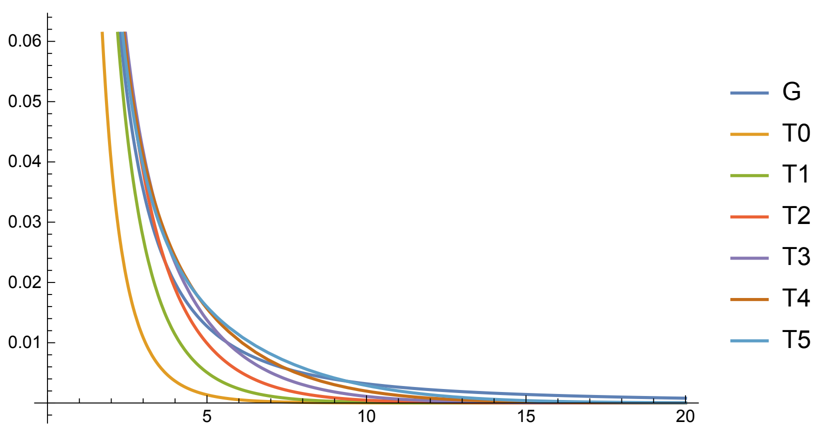
```
terms = {fs0, fs1, fs2, fs3, fs4, fs5}
```

$$\left\{\frac{2\,d\,\text{BesselK}[1,\,2\,d\,k\,\pi]}{k},\ 4\,d^2\,\pi\,\text{BesselK}[0,\,2\,d\,k\,\pi],\right.$$
$$-2\,d^2\,\pi\,\text{BesselK}[0,\,2\,d\,k\,\pi] + 4\,d^3\,k\,\pi^2\,\text{BesselK}[1,\,2\,d\,k\,\pi],$$
$$-4\,d^3\,k\,\pi^2\,\text{BesselK}[1,\,2\,d\,k\,\pi] + \frac{8}{3}\,d^4\,k^2\,\pi^3\,\text{BesselK}[2,\,2\,d\,k\,\pi],$$
$$d^3\,k\,\pi^2\,\text{BesselK}[1,\,2\,d\,k\,\pi] - 4\,d^4\,k^2\,\pi^3\,\text{BesselK}[2,\,2\,d\,k\,\pi] + \frac{4}{3}\,d^5\,k^3\,\pi^4\,\text{BesselK}[3,\,2\,d\,k\,\pi],$$
$$2\,d^4\,k^2\,\pi^3\,\text{BesselK}[2,\,2\,d\,k\,\pi] - \frac{8}{3}\,d^5\,k^3\,\pi^4\,\text{BesselK}[3,\,2\,d\,k\,\pi] + \left.\frac{8}{15}\,d^6\,k^4\,\pi^5\,\text{BesselK}[4,\,2\,d\,k\,\pi]\right\}$$

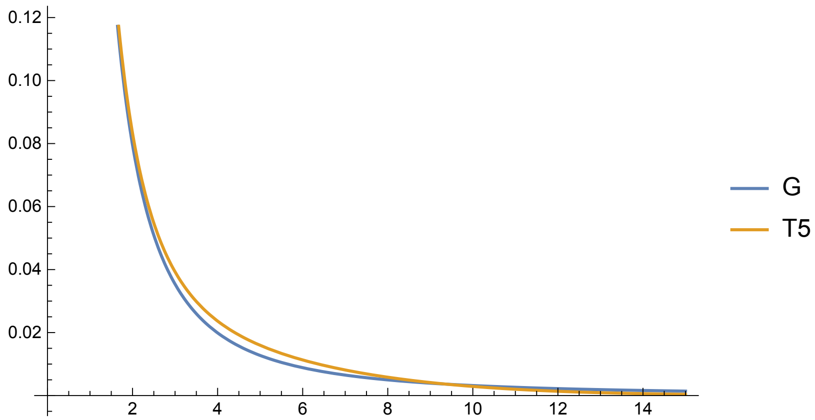
```
T = Accumulate[terms]
```

$$\left\{\frac{2\,d\,\text{BesselK}[1,\,2\,d\,k\,\pi]}{k},\ 4\,d^2\,\pi\,\text{BesselK}[0,\,2\,d\,k\,\pi] + \frac{2\,d\,\text{BesselK}[1,\,2\,d\,k\,\pi]}{k},\right.$$
$$2\,d^2\,\pi\,\text{BesselK}[0,\,2\,d\,k\,\pi] + \frac{2\,d\,\text{BesselK}[1,\,2\,d\,k\,\pi]}{k} + 4\,d^3\,k\,\pi^2\,\text{BesselK}[1,\,2\,d\,k\,\pi],$$
$$2\,d^2\,\pi\,\text{BesselK}[0,\,2\,d\,k\,\pi] + \frac{2\,d\,\text{BesselK}[1,\,2\,d\,k\,\pi]}{k} + \frac{8}{3}\,d^4\,k^2\,\pi^3\,\text{BesselK}[2,\,2\,d\,k\,\pi],$$
$$2\,d^2\,\pi\,\text{BesselK}[0,\,2\,d\,k\,\pi] + \frac{2\,d\,\text{BesselK}[1,\,2\,d\,k\,\pi]}{k} + d^3\,k\,\pi^2\,\text{BesselK}[1,\,2\,d\,k\,\pi] - \frac{4}{3}\,d^4\,k^2\,\pi^3\,\text{BesselK}[2,\,2\,d\,k\,\pi] + \frac{4}{3}\,d^5\,k^3\,\pi^4\,\text{BesselK}[3,\,2\,d\,k\,\pi],$$
$$2\,d^2\,\pi\,\text{BesselK}[0,\,2\,d\,k\,\pi] + \frac{2\,d\,\text{BesselK}[1,\,2\,d\,k\,\pi]}{k} + d^3\,k\,\pi^2\,\text{BesselK}[1,\,2\,d\,k\,\pi] + \frac{2}{3}\,d^4\,k^2\,\pi^3\,\text{BesselK}[2,\,2\,d\,k\,\pi] - \frac{4}{3}\,d^5\,k^3\,\pi^4\,\text{BesselK}[3,\,2\,d\,k\,\pi] + \left.\frac{8}{15}\,d^6\,k^4\,\pi^5\,\text{BesselK}[4,\,2\,d\,k\,\pi]\right\}$$

```
Plot[Evaluate@({G, T} /. d → 0.1), {k, 0, 20},
 PlotLegends → {"G", "T0", "T1", "T2", "T3", "T4", "T5"}]
```



```
Plot[{G, T[[6]] /. d → 0.1}, {k, 0, 15}, PlotLegends → {"G", "T5"}]
```




```
Assuming[d > 0, Integrate[(G - T[[4]])^2, {k, 0, Infinity}]]
```

$$\frac{3\,d^3\,\pi^3}{256}$$

```
Expand[(G - T[[4]])^2]
```

$$\frac{1}{k^4\,\pi^2} - \frac{4\,d^2\,\mathrm{BesselK}[0,\,2\,d\,k\,\pi]}{k^2} + 4\,d^4\,\pi^2\,\mathrm{BesselK}[0,\,2\,d\,k\,\pi]^2 - \frac{4\,d\,\mathrm{BesselK}[1,\,2\,d\,k\,\pi]}{k^3\,\pi} + \frac{8\,d^3\,\pi\,\mathrm{BesselK}[0,\,2\,d\,k\,\pi]\,\mathrm{BesselK}[1,\,2\,d\,k\,\pi]}{k} + \frac{4\,d^2\,\mathrm{BesselK}[1,\,2\,d\,k\,\pi]^2}{k^2} - \frac{16}{3}\,d^4\,\pi^2\,\mathrm{BesselK}[2,\,2\,d\,k\,\pi] + \frac{32}{3}\,d^6\,k^2\,\pi^4\,\mathrm{BesselK}[0,\,2\,d\,k\,\pi]\,\mathrm{BesselK}[2,\,2\,d\,k\,\pi] + \frac{32}{3}\,d^5\,k\,\pi^3\,\mathrm{BesselK}[1,\,2\,d\,k\,\pi]\,\mathrm{BesselK}[2,\,2\,d\,k\,\pi] + \frac{64}{9}\,d^8\,k^4\,\pi^6\,\mathrm{BesselK}[2,\,2\,d\,k\,\pi]^2$$

```
Assuming[d > 0, Integrate[
  Total[Coefficient[Expand[(G - T[[4]])^2], k, Range[-4, 1]] * k^Range[-4, 1]],
  {k, 0, Infinity}]]
```

$$-\frac{1}{2}\,d^3\,\pi^3$$

```
Assuming[d > 0, Integrate[
  Coefficient[Expand[(G - T[[4]])^2], k, {2, 3, 4}] * k^{2, 3, 4}, {k, 0, Infinity}]]
```

$$\left\{\frac{3\,d^3\,\pi^3}{8},\ 0,\ \frac{35\,d^3\,\pi^3}{256}\right\}$$

```
Assuming[d > 0, Integrate[
  Coefficient[Expand[(G - T[[4]])^2], k, {2, 3, 4}] * k^{2, 3, 4}, {k, 0, Infinity}]]
```

$$\left\{\frac{3\,d^3\,\pi^3}{8},\ 0,\ \frac{35\,d^3\,\pi^3}{256}\right\}$$

```
Coefficient[Expand[(G - T[[4]])^2], k, 4] k^4
```

$\frac{64}{9} d^8 k^4 \pi^6 \text{BesselK}[2, 2 d k \pi]^2$

```
Assuming[d > 0, Integrate[64 / 9 d^8 k^4 π^6 BesselK[2, 2 d k π]^2, {k, 0, Infinity}]]
```

$\frac{35 d^3 \pi^3}{256}$

```
Assuming[d > 0,
 Integrate[Coefficient[Expand[(G - T[[4]])^2], k, 4] k^4, {k, 0, Infinity}]]
```

$\frac{35 d^3 \pi^3}{256}$

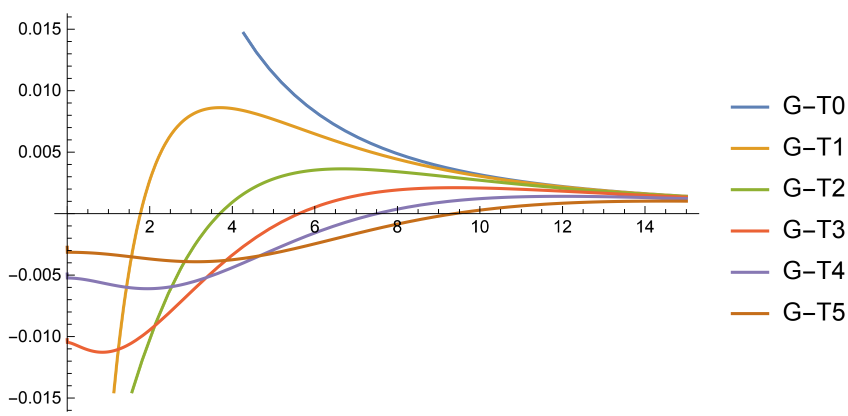
```
Assuming[d > 0, Integrate[(G - T[[5]])^2, {k, 0, Infinity}]]
```

$\frac{175 d^3 \pi^3}{32\,768}$

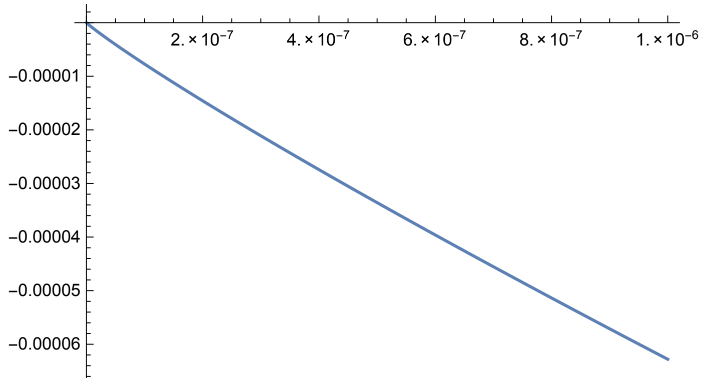
```
Assuming[d > 0, Integrate[(G - T[[6]])^2, {k, 0, Infinity}]]
```

$\frac{3059 d^3 \pi^3}{1\,048\,576}$

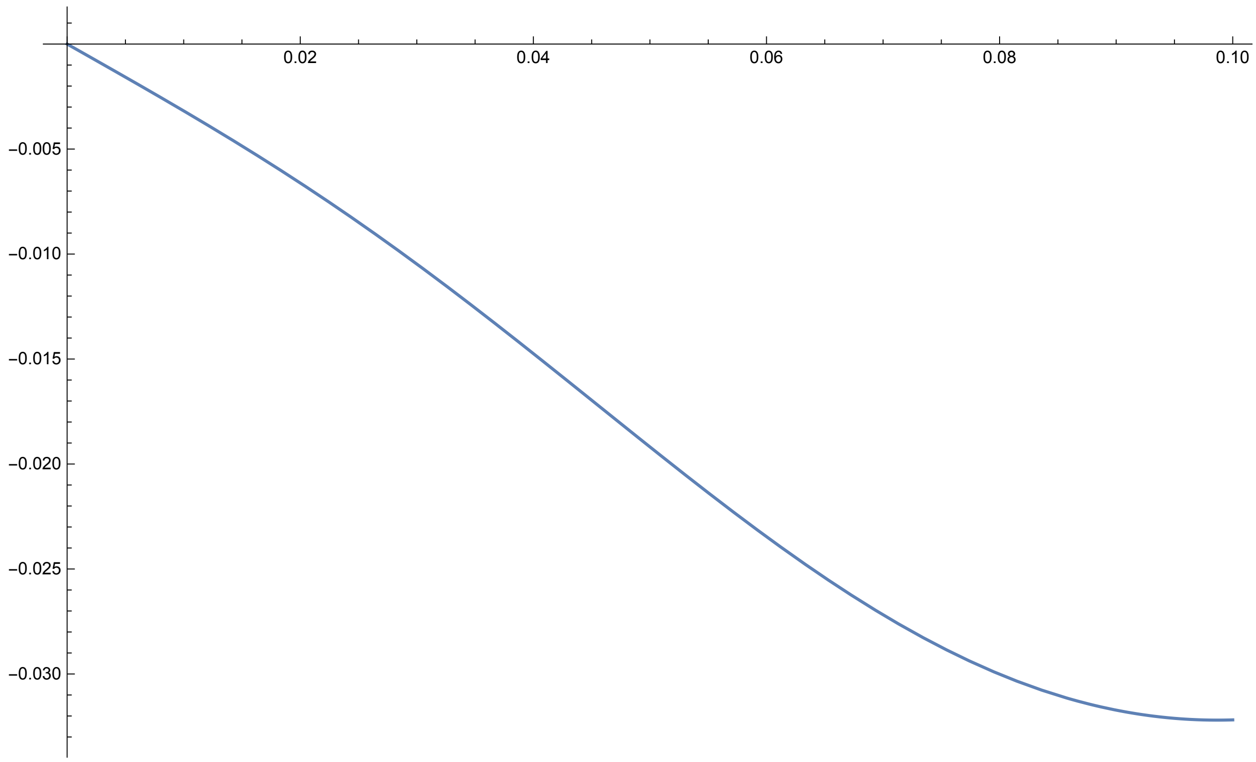
```
Plot[Evaluate@(G - T /. d → 0.1), {k, 0, 15},
 PlotLegends → {"G-T0", "G-T1", "G-T2", "G-T3", "G-T4", "G-T5"}]
```



```
Plot[fs2 / d /. k → 3, {d, 0, 0.000001}]
```




```
Plot[(G - T[[6]]) / d /. k → 5, {d, 0, 0.1}]
```




```
Assuming[{k >= 0}, Series[T[[6]], {d, 0, 5}]]
```

$$\frac{1}{k^2 \pi} + \frac{\pi d^2}{10} + \frac{1}{20} k^2 \pi^3 d^4 + O[d]^6$$