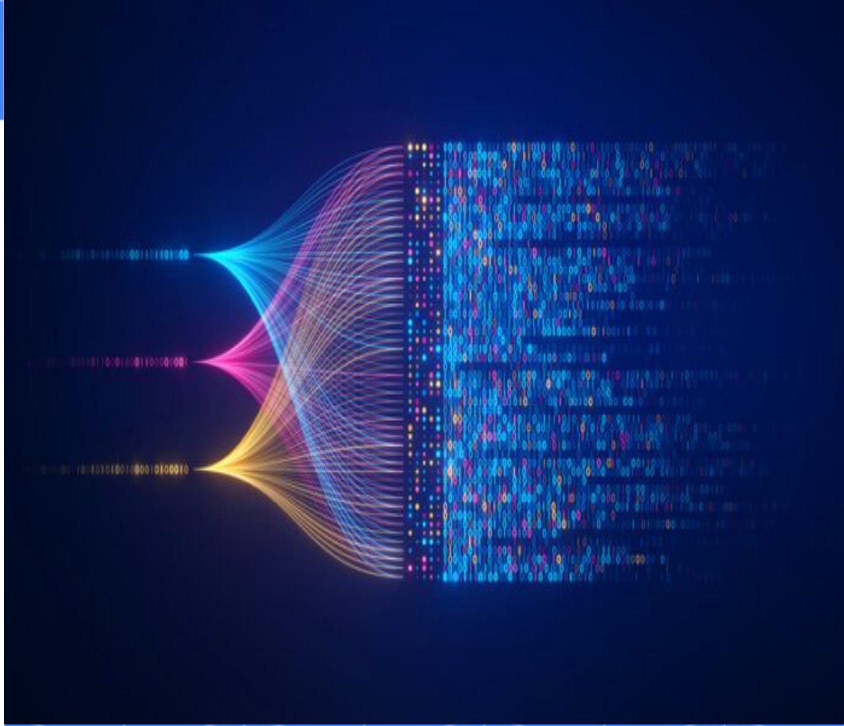
Starting At 6:30 PM

Google Developer Student Clubs
University of Toronto Mississauga

Foundations of Machine Learning: Linear Regression



Presented by;
Hamza & Rahul

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Introduction to the 4 part series on ML

```
lookup.KeyValue
f.constant(['em
=tf.constant([0
.lookup.StaticV

_buckets=5)
```

What, who, and when?

- Second of a planned 4 part series on ML
- The next topics are not set in stone so feel free to give us cool ideas
- The next two will also be presented by the two of us
- Dates not yet set in stone, but should be decided soon for workshop 3 so monitor the socials!

Important Notice

- This workshop is largely interactive with a large emphasis on group code along exercises
- So please scan the QR code to join the UTM GDSC discord where you can download the starter code
- Now you should've downloaded 'Linear_Regression_Workshop.ipynb' this is a jupyter notebook and can be opened without an IDE by googling UofT Jupyter Hub or going to jupyter.utoronto.ca/hub/login

SCAN ME

What really is Machine Learning...?

- Essentially just statistical methods that **learn** from a given data set
- Patterns in data **features** are captured in **parameters**
- **NOTE:** Parameters are measurable factors that define a system and determine its behaviour
- More technical definition: Given a simple mathematical objective (eg. the distance between your predictions and the label) we apply iterative mathematical optimization on the parameters to optimize for/minimize this objective

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Simple Linear Regression

```
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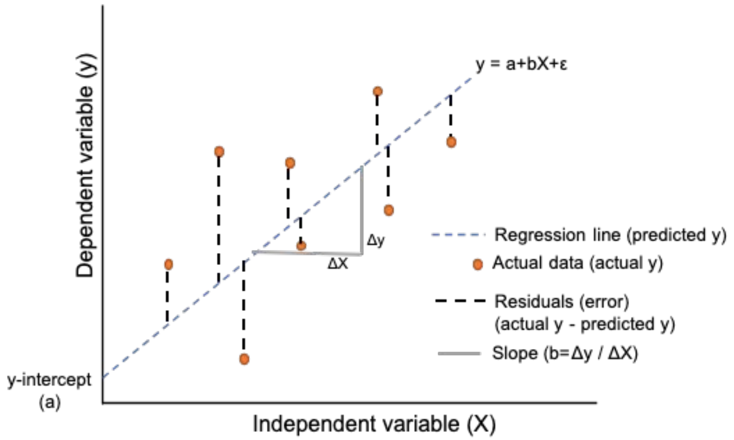
```

What is Linear Regression ?

- Essentially fitting a line through points
- More technically; Models the relationship between variables via a linear equation
- Goal; best line of fit means least deviations from line
- Minimizing the difference between observed and predicted values



Why Linear Regression over more powerful forms?

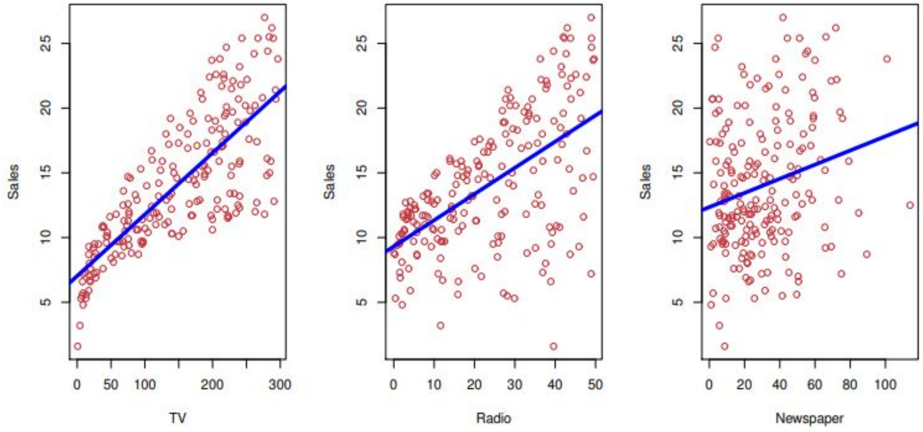
- Simplicity and Interpretability:
  a. Easy to understand and explain, even to non-technical audiences
  b. Clear relationship between independent and dependent variables
- Efficient Computation:
  a. Fast to compute, even with large datasets
  b. Less computationally expensive than many complex models
- Performance in Well-Structured Data:
  a. Highly effective with linear relationships
  b. Can outperform complex models when data structure is straightforward

Why Linear Regression over more powerful forms? - continued

- Less Prone to Overfitting:
    a. Simplicity reduces the risk of overfitting
    b. Easier to generalize to new data
- Good Starting Point for Analysis:
    a. Useful for initial analysis to understand data trends
    b. Provides a baseline for comparison with complex models
- Flexibility with Enhancements:
    a. Can be extended with polynomial and interaction terms
    b. Adaptable for various types of data through transformations

Linear Regression Theory - Form

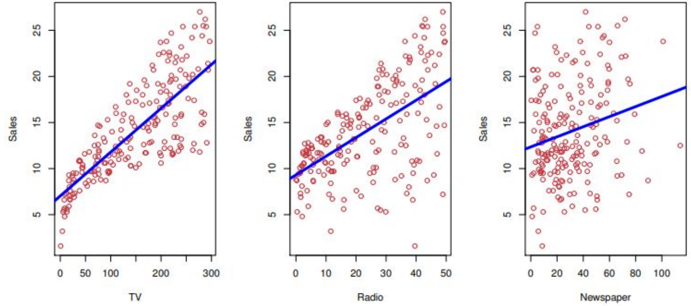
- $Y = \beta_0 + \beta_1 X + \epsilon$
- Y = independent variable
- X = dependant variable
- β_0 = Intercept
- β_1 = Slope
- ϵ = Error term
- Example: $Y = 5 + 7x + \epsilon$



Linear Regression Theory - Form

- Prediction Representation Per Datapoint : $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$
- Example:

TV (x)	Sales (y)
50	8
75	10



- If Y = 6 + 0.1x (Our prediction for best fit line)
- y-hat = 6 + 0.1*50 = 11 (not accurate, but this is what our equation gives us)

Linear Regression Theory - RSS

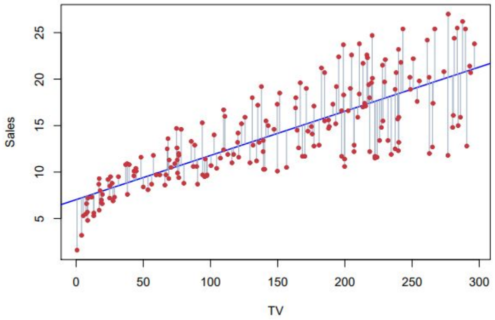
- Let $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$ be the prediction for Y based on the ith value of X. Then $e_i = y_i - \hat{y}_i$ represents the ith *residual*
- We define the *residual sum of squares* (RSS) as

$$\text{RSS} = e_1^2 + e_2^2 + \cdots + e_n^2,$$

or equivalently as

$$\text{RSS} = (y_1 - \hat{\beta}_0 - \hat{\beta}_1 x_1)^2 + (y_2 - \hat{\beta}_0 - \hat{\beta}_1 x_2)^2 + \ldots + (y_n - \hat{\beta}_0 - \hat{\beta}_1 x_n)^2.$$

TV (x)	Sales (y)
50	8
75	10



Lab Exercise

10 min

Collaboration is encouraged!

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Standard Error and Confidence Intervals

Standard Error

- The standard error of an estimator reflects how it varies under repeated sampling. We have

$$\mathrm{SE}(\hat{\beta}_1)^2 = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \quad \mathrm{SE}(\hat{\beta}_0)^2 = \sigma^2\left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right]$$

where $\sigma^2 = \mathrm{Var}(\epsilon)$

$$\sigma^2 = \frac{SSR}{n-p} = \frac{\sum_{i=1}^{n}(e_i)^2}{n-p}$$

Confidence Intervals

- These standard errors can be used to compute *confidence intervals.* A 95% confidence interval is defined as a range of values such that with 95% probability, the range will contain the true unknown value of the parameter. It has the form

$$\hat{\beta}_1 \pm 2 \cdot \mathrm{SE}(\hat{\beta}_1).$$

Confidence Intervals - continued

That is, there is approximately a 95% chance that the interval

$$\left[\hat{\beta}_1 - 2 \cdot \mathrm{SE}(\hat{\beta}_1),\ \hat{\beta}_1 + 2 \cdot \mathrm{SE}(\hat{\beta}_1)\right]$$

will contain the true value of β_1 (under a scenario where we got repeated samples like the present sample)

For the advertising data, the 95% confidence interval for β_1 is $[0.042, 0.053]$

Lab Exercise

5 min

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Hypothesis Testing

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```

What is Hypothesis Testing?

- Statistical tool used to make inferences about a population based on sample data
- Standard Errors are used to perform hypothesis tests on the coefficients
- The most common hypothesis test involves testing the null hypothesis...

Null Hypothesis

H_0 : There is no relationship between X and Y

versus the *alternative hypothesis*

H_A : There is some relationship between X and Y.

- Mathematically, this corresponds to testing

$$H_0 : \beta_1 = 0$$

versus

$$H_A : \beta_1 \neq 0,$$

since if $\beta_1 = 0$ then the model reduces to $Y = \beta_0 + \epsilon$, and X is not associated with Y.

t-statistic

- To test the null hypothesis, we compute a *t-statistic*, given by

$$t = \frac{\hat{\beta}_1 - 0}{\mathrm{SE}(\hat{\beta}_1)},$$

- This will have a t-distribution with $n - 2$ degrees of freedom, assuming $\beta_1 = 0$.

p-value

- Using statistical software, it is easy to compute the probability of observing any value equal to $|t|$ or larger. We call this probability the *p-value*.

	Coefficient	Std. Error	t-statistic	p-value
`Intercept`	7.0325	0.4578	15.36	< 0.0001
`TV`	0.0475	0.0027	17.67	< 0.0001

Lab Exercise

10 min

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RSE, R^2, F-statistic

RSE & R^2

- We compute the *Residual Standard Error* Used to calculate the lack of fit

$$\mathrm{RSE} = \sqrt{\frac{1}{n-2}\mathrm{RSS}} = \sqrt{\frac{1}{n-2}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2},$$

where the *residual sum-of-squares* is $RSS = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$.

- *R-squared* or fraction of variance explained is

$$R^2 = \frac{\mathrm{TSS} - \mathrm{RSS}}{\mathrm{TSS}} = 1 - \frac{\mathrm{RSS}}{\mathrm{TSS}}$$

where $\mathrm{TSS} = \sum_{i=1}^{n}(y_i - \bar{y})^2$ is the *total sum of squares*.

RSE = actual values deviate from the true regression line by approximately RSE units
TSS = measures the total variance in the response variable that we're trying to explain
RSS = measures the variance in the response variable that the model does not explain

F-statistic

- Definition: The F statistic is a ratio used in statistical testing to compare the model's fit against a model with no predictors.
- Purpose: It tests whether the group of variables in a model are jointly significant. Essentially, it asks, "Is the model better than nothing?"

F-statistic - continued

- Interpretation:
  a. A higher F statistic indicates that the model explains a significant amount of variance in the dependent variable.
  b. A lower F statistic suggests the model does not provide a better fit than one without independent variables.
- f_{value} = ((TSS - RSS) / (p-1)) / (RSS / (n - p))

Lab Exercise

10 min

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Multiple Linear Regression

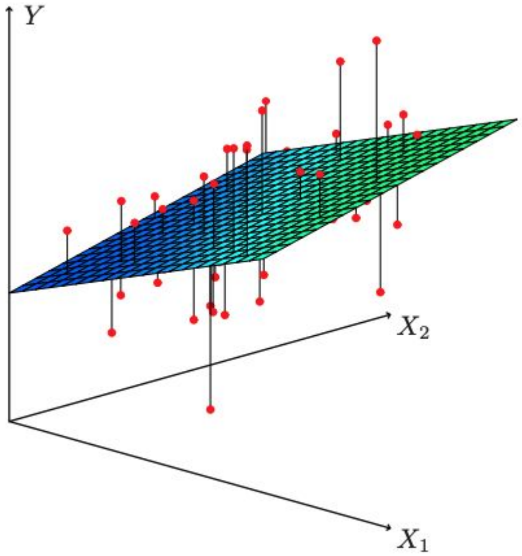
Multiple Linear Regression

- Here our model is

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p + \epsilon,$$

- We interpret β_j as the *average* effect on Y of a one unit increase in X_j, *holding all other predictors fixed*. In the advertising example, the model becomes

$$\texttt{sales} = \beta_0 + \beta_1 \times \texttt{TV} + \beta_2 \times \texttt{radio} + \beta_3 \times \texttt{newspaper} + \epsilon.$$



Interpreting Regression Coefficients

- The ideal scenario is when the predictors are uncorrelated — a *balanced design*:
  - Each coefficient can be estimated and tested separately.
  - Interpretations such as *"a unit change in X_j is associated with a β_j change in Y, while all the other variables stay fixed"*, are possible.
- Correlations amongst predictors cause problems:
  - The variance of all coefficients tends to increase, sometimes dramatically
  - Interpretations become hazardous — when X_j changes, everything else changes.

Estimation and Prediction

- Given estimates $\hat\beta_0, \hat\beta_1, \ldots \hat\beta_p$, we can make predictions using the formula

$$\hat{y} = \hat\beta_0 + \hat\beta_1 x_1 + \hat\beta_2 x_2 + \cdots + \hat\beta_p x_p.$$

- We estimate $\beta_0, \beta_1, \ldots, \beta_p$ as the values that minimize the sum of squared residuals

$$\begin{aligned} \mathrm{RSS} &= \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \\ &= \sum_{i=1}^{n} (y_i - \hat\beta_0 - \hat\beta_1 x_{i1} - \hat\beta_2 x_{i2} - \cdots - \hat\beta_p x_{ip})^2. \end{aligned}$$

This is done using standard statistical software.
The values $\hat\beta_0, \hat\beta_1, \ldots, \hat\beta_p$ that minimize RSS are the multiple least squares regression coefficient estimates.

Lab Exercise

10 min

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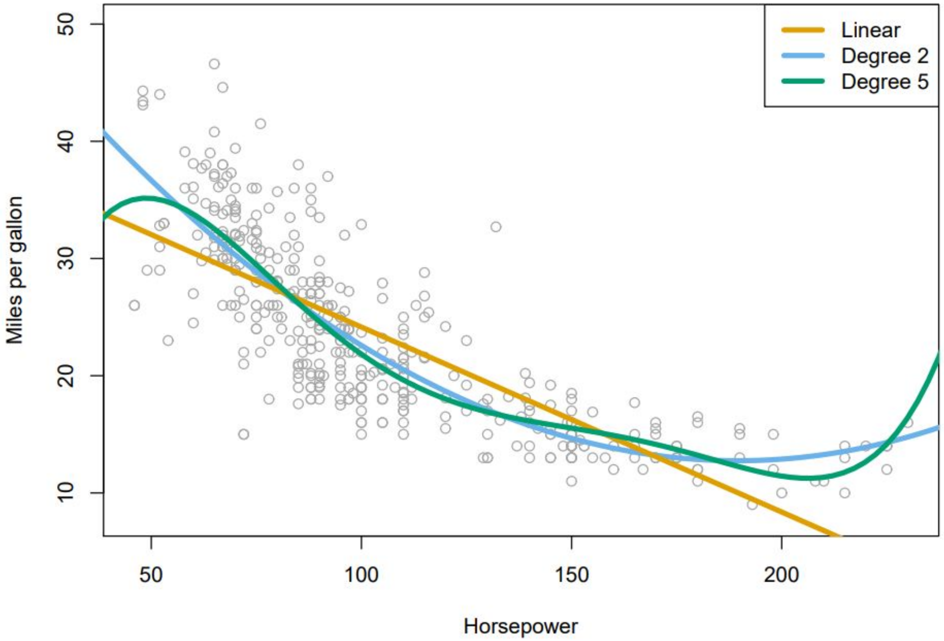
Polynomial Regression

Polynomial Regression Figure



Polynomial Regression

The figure suggests that

$$\texttt{mpg} = \beta_0 + \beta_1 \times \texttt{horsepower} + \beta_2 \times \texttt{horsepower}^2 + \epsilon$$

may provide a better fit.

	Coefficient	Std. Error	t-statistic	p-value
Intercept	56.9001	1.8004	31.6	< 0.0001
horsepower	-0.4662	0.0311	-15.0	< 0.0001
$\texttt{horsepower}^2$	0.0012	0.0001	10.1	< 0.0001

Lab Exercise (HW)

10 min

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Thank you!

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