Week 1: (Un)Natural Numbers

The problems are designed to develop your skills with formal definitions and provide some practice with inductive reasoning.

> **Collaboration Policy:** You should work on the problems yourself, before discussing with others, and with your cohorts are your cohort meeting. By the Assessed Cohort Meeting, you and all of your cohortmates, should be prepared to present and discuss solutions to all of the assigned problems. In addition to discussing with your cohortmates, you may discuss the problems with anyone you want, and use any resources you want **except for any materials from previous offerings of this course**, which are not permitted.

Problem 1 *Cohortroductions*

Everyone in your cohort should introduce themselves, and teach all the other cohort members how to pronounce your name correctly, and share something interesting about yourself with the group. This can be anything you want, but a few suggestions:

– What did you do over the summer?

– What are the best and worst things about living under the Covid lockdown?

– What is the most interesting thing about your hometown?

– Who is your spirit animal?

– If you could change anything about UVA, what would you change?

As the assessed cohort meeting, this will be treated like other questions: *one* of you will be selected, and will be expected to introduce yourself and the other members of your cohort, including pronouncing their names correctly and recounting something interesting you learned about each other cohort member.

Problem 2 *Cohort Namesake*

Each cohort has been named after someone who is (at least loosely) connected to theoretical computer science. You are not expected to do an extensive research project here, but find out something about the people your cohort is named after.

At your cohort preparation meeting, everyone should share what they learned, and the group should settle a few interesting things that you'll share with your Cohort Leader at the Assessed Cohort Meeting.

As an additional step for this problem, you should also agree on one interesting fact about, and the best URL you found, about your cohort namesake, and one of your cohortees should post a message in `#general` on discord to share this with the rest of the class.

Problem 3 *Higher Induction Practice*

Prove that any binary tree of height h has at most 2^{h-1} leaves.

Note: We haven't defined a *binary tree* (and the book doesn't). An adequate answer to this question will use the informal understanding of a binary tree which we expect you have entering this class (a tree where each node has 0, 1, or 2 children), but an excellent answer will include a definition of a binary tree and connect your proof to that definition.

Problem 4 *Addition is Commutative*

For this problem, we will use the successor definition of Natural Numbers from the *Constructing the Natural Numbers* video:

Definition 1 (Natural Numbers) We define the *Natural Numbers* as:

1. **0** is a Natural Number.
2. If n is a Natural Number, **S**(n) is a Natural Number.

We will use this definition of addition (from *Defining Addition*):

Definition 2 (Sum) The *sum* of two Natural Numbers a and b (denoted as $a + b$) is defined as:

1. If a is **0**, then $a + b$ is b.
2. Otherwise, a is **S**(p) for some Natural Number p, and $a + b$ is **S**$(p + b)$.

Prove that addition (as defined above) is *commutative* (that is, for all Natural Numbers a and b, $a + b$ is $b + a$). Note that what "is" means here is they are exactly the same representation (we are not using $=$, since we haven't defined it for our number representation). You can think of all the operations we have defined as just manipulating strings of symbols, and "x is y" meaning that x and y are exactly the same sequences of symbols.

Problem 5 *Countable Programs*

Prove that the set of all Python programs that you can execute on your laptop is *countable*.

Problem 6 *Uncountable Sets*

$(\star)$[1] Prove that the set of all undirected graphs (using the book's Definition 1.3, without the constraint that V is finite that was added for the previous problem) is not countable. (Note: be careful in any argument that you make that the graphs you are counting as actually *different*. The nodes on the graph have no labels, so the only way for two graphs to be different is if they have different structure.)

[1] When a problem is marked with a $\star$, it means we think this problem is challenging enough that students are expected to be able to solve it. We still hope everyone will attempt these problems and learn from trying to solve them, but you shouldn't get overly frustrated if you are not able to solve a $\star$ problem. Unlike the non-starred problems, you will not be "cold-called" to present a solution to a $\star$-level problem at an Assessed Cohort Meeting. Instead, if there is time in the meeting, students will have an opportunity to volunteer to discuss the problem (and receive potential bonus points for outstanding solutions).

