Week 2: Fine Finite Computation

Authors: TODO: Cohort Name (names of all who contributed)

Collaborators and Resources: TODO: Replace with any additional collaborators and non-course resources you used

This is a template to help with your write-up for Week 2. The actual problem you will write up will be selected by your Cohort Leader at the Assessed Cohort Meeting.

Clone the Problem Set 2 Template Repository

See the Week 1 template for directions on Getting Started with LaTeX. Similarly to Week 1, one member of your cohort should create a copy of the Problem Set 2 repository, by following these steps (we recommend doing this together, with the one creating the repository sharing her screen for everyone to follow along):

1. Download the Problem Set 2 template from: https://uvatoc.github.io/ps/ps2.zip
2. In Overleaf, click on Create First Project or New Project in Overleaf and select Upload Project from the menu.
3. Click Select a .zip file and then select the ps2.zip file you downloaded in step 1.
4. Share the repository with your cohortmates by clicking the "Share" button at the top right of the overleaf window, and entering your cohortmates email addresses in the sharing form.

Click on ps2.tex to see the LaTeX source for this file, which is the file you will modify to prepare your solution. The first thing you should do in ps2.tex is set up your cohort name as the author of the submission by replacing the line, `\submitter{TODO: your name}`, with your the name of your cohort (e.g., `\submitter{Cohort Hopper (Ada Lovelace, Don Knuth)}`). For the list of cohort members, this should usually be everyone in your cohort, but if someone did not contribute during the week, they should not be included in your submission list (and should have informed us about their absence separately).

Before submitting your ps2.pdf file, also remember to:

– List your collaborators and resources, replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. You do not need to include

– Replace the second line in ps2.tex, `\usepackage{uvatoc}` with `\usepackage[response]{uvatoc}` so the directions do not appear in your final PDF. You can do this by using the LaTeX comment token, %. The rest of the line after a % is treated as a comment. You'll notice after you to this, when you Recompile the document, most of it will disappear (everything in \directions is left out, so only your solution will appear in the submitted document).

Problem 2 *Infinite Dominoes*

A domino is a tile with an unordered pair of numbers on it (e.g. $0, 5$ or $3, 3$). Dominoes come in sets containing all pairs of natural numbers less than or equal to some upper bound.

A pack of "double 6" dominoes will contain all unordered pairs of values from the set $\{0, 1, 2, 3, 4, 5, 6\}$ (there will be 28 total). A pack of "double 3" dominoes will contain all unordered pairs of values from the set $\{0, 1, 2, 3\}$ (there will be 10 total).

A *domino chain* is a sequence of dominoes ordered so that the second value of each domino matches the first value of the next. The domino sequence $(1, 2)(2, 5)(5, 5)(5, 0)$ is a valid domino chain, whereas $(1, 2)(2, 5)(5, 5)(0, 0)$ is not.

Consider a pack of "double $\mathbb{N}$" dominoes, which contains all of the infinitely-many unordered pairs of natural numbers. Show that there is an uncountable number of infinite-length domino chains that can be constructed from a pack of "double $\mathbb{N}$" dominoes.

Problem 5 *Compare 4 bit numbers* (Exercise 3.1 in TCS book)

Draw a Boolean circuit (using only *AND*, *OR*, and *NOT* gates) that computes the function $CMP_8 : \{0, 1\}^8 \rightarrow \{0, 1\}$ such that $CMP_8(a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3) = 1$ if and only if the number represented by $a_0a_1a_2a_3$ is larger than the number represented by $b_0b_1b_2b_3$. We will say that a_0, b_0 are the most significant bits and a_3, b_3 are least significant.

Problem 6 *Compare n bit numbers* (Exercise 3.2 in TCS book)

Prove that there exists a constant c such that for every n there is a Boolean circuit (using only *AND*, *OR*, and *NOT* gates) C of at most $c \cdot n$ gates that computes the function $CMP_{2n} : \{0, 1\}^{2n} \rightarrow \{0, 1\}$ such that $CMP_{2n}(a_0 \cdots a_{n-1}b_0 \cdots b_{n-1}) = 1$ if and only if the number represented by $a_0 \cdots a_{n-1}$ is larger than the number represented by $b_0 \cdots b_{n-1}$.

In other words, generalize the previous problem to describe how to compare n-bit numbers for any specific value n using *AND*, *OR*, and *NOT*. The total number of gates used should be upper bounded by some constant c times n (i.e. asymptotically linear).

Problem 7 NOR *equals* AON (based on Exercise 3.7 in TCS book)

Let *NOR* $: \{0, 1\}^2 \rightarrow \{0, 1\}$ defined as *NOR*(a, b) = *NOT*(*OR*(a, b)). Prove that $\{$*NOR*$\}$ is equivalent to *AND, OR, NOT*. In other words, show that any function that can be computed by *AND,OR,NOT* can also be computed using just *NOR*, and vice-versa.

Problem 8 *XOR does not equal* AON (based on Exercise 3.5 in TCS book)

Prove that the gates *XOR*$, 0, 1$ is *weaker* than *AND, OR, NOT*. (You can use any strategy you want to prove this; see the book for one hint of a possible strategy, but we think you may be able to find easier ways to prove this, and it is not necessary to follow the strategy given in the book.

