## Grasshopper jumping

INDUCTION and integer inequalities are the topics of this note<sup>1</sup>.

## Problem

Let  $a_1, a_2, \ldots, a_n$  be distinct positive integers and let M be a set of n-1 positive integers not containing  $s=a_1+a_2+\ldots+a_n$ . A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths  $a_1, a_2, \ldots, a_n$  in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

We use induction on n and we use the problem as our induction hypothesis with one modification: set M has at most n-1 elements.

The base case n = 2 is trivial.

Let  $A = \{a_i : 1 \le i \le n\}$  and  $M = \{m_i : 1 \le i < n\}$ . Assume  $a_1 < a_2 < \ldots < a_n$  and  $m_1 < m_2 < \ldots < m_{n-1}$ . For the induction step we have several cases.

**Case:**  $a_n \in M$ 

There is an  $l : 1 \le l < n : m_1 = a_n$ .

If l = n - 1: there is an index k for which  $a_k \notin M$ . Then the order  $\{k, n, \ldots\}$  never lands on any point in M because  $a_k + a_n > m_{n-1}$ .

If l < n-1: Define  $M' = \{m_1, m_2, \dots, m_{l-1}\} \cup \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ . Use integers  $a_1, \dots, a_{n-1}$  and M' as induction step to get an order  $a_{\pi(1)}, \dots, a_{\pi(n-1)}$  with  $\pi \in S_{n-1}$ .

 $a_{\pi(1)} \notin M'$  and  $a_{\pi(1)} < a_n$ , so  $a_{\pi(1)} \notin M$ .

 $a_{\pi(1)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n \notin \{m_{l+1}, \dots, m_{n-1}\}$ . Also  $a_{\pi(1)} + a_n > a_n$  so  $a_{\pi(1)} + a_n \notin \{m_1, m_2, \dots, m_{l-1}\}$ . That means  $a_{\pi(1)} + a_n \notin M$ .

We continue with similar reasoning with the rest:  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin M$  because  $a_{\pi(1)} + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \in \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \in \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \in \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \in \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \in \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \in \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ 

<sup>1</sup> For an extension to signed jumps see Géza Kós. On the grasshopper problem with signed jumps. *The Ameri-*

can Mathematical Monthly, 118:877–886, 2010. URL https://arxiv.org/abs/1008.2936

 $a_{\pi(2)} \notin \{m_{l+1}, \dots, m_{n-1}\}$  and  $a_{\pi(1)} + a_n + a_{\pi(2)} > a_n$  etc.

This means  $\{\pi(1), n, \pi(2), \dots, \pi(n-1)\}$  is a valid order.

Case:  $a_n \notin M$ 

If there is an  $m_i < a_n$  then we can use the induction step with integers  $a_1, a_2, \ldots, a_{n-1}$  and set  $M' = \{m_{i+1} - a_n, m_{i+2} - a_n, \ldots, m_{n-1} - a_n\}$ to find an order and prepend  $a_n$  to that order.

If not, then  $\forall 1 \leq i < n : m_i > a_n$ .

 $\sum_{i=1}^{n-1} a_i \ge m_1$  because otherwise we could have used order  $\{1, 2, \dots, n\}$ .

We have  $a_1 < a_n < m_1$  and  $\sum_{j=1}^{n-1} a_j \ge m_1$ , so there exists an  $1 \le l < m_1$ n - 1 such that  $s' = \sum_{i=1}^{l} a_i < m_1$ .

Define  $M' = \{m_2 - a_n, m_3 - a_n, \dots, m_{n-1} - a_n\}$  and use M' with the integers  $a_1, a_2, \dots, a_{n-1}$  in an induction step which gives us an order  $\pi \in S_{n-1}$ .

Since  $a_{\pi(1)} < m_1$  and  $\sum_{j=1}^{n-1} a_{\pi(j)} \ge m_1$  there exists an  $1 < l \le n-1$ such that  $\sum_{j=1}^{l-1} a_{\pi(j)} < m_1$  and  $\sum_{j=1}^{l} a_{\pi(j)} \ge m_1$ . We look at the order  $\{\pi(1),\ldots,\pi(l-1),n,\pi(l),\ldots,\pi(n-1)\}$  and

claim it is a valid order.

Indeed  $\sum_{j=1}^{l-1} a_{\pi(j)} < m_1$ , so jumps  $\{\pi(1), \ldots, \pi(l-1)\}$  won't encounter anything from M. We also have

$$\sum_{j=1}^{l-1} a_{\pi(j)} + a_n > \sum_{j=1}^{l} a_{\pi(j)} \ge m_1$$

which means  $\{\pi(1), \ldots, \pi(l-1), a_n\}$  will avoid  $m_1$ . It will also avoids anything from  $M \setminus \{m_1\}$  because  $\{\pi(1), \dots, \pi(l-1)\}$  avoids anything from M'. The rest of the order is already bigger than  $m_1$  and avoids  $M \setminus \{m_1\}$  by induction.

## Bibliography

Géza Kós. On the grasshopper problem with signed jumps. *The American Mathematical Monthly*, 118:877–886, 2010. URL https://arxiv.org/abs/1008.2936.