Groovy Numbers

Problem

$x \in \mathbb{R}$ is said to be a groovy number iff $\exists\ n \in \mathbb{N}$ such that $x = \sqrt{n} + \sqrt{n+1}$. Prove that if x is groovy, then $\forall r \in \mathbb{N} :\ x^r$ is groovy.

Binomial Expansion

In this section we explore a property of the binomial power expansion

$$(a+b)^r = \sum_{k=0}^{r} \binom{r}{k} a^{r-k} b^k$$

We define $\mathbb{N}_r = \{k \in \mathbb{N}_0 : 0 \leq k \leq r\}$ and its partition into two subsets $\mathbb{N}_r = \mathbb{E}_r \cup \mathbb{O}_r$, with $\mathbb{E}_r = \{k \in \mathbb{N}_r : k = 2u, u \in \mathbb{N}_0\}$ and $\mathbb{O}_r = \{k \in \mathbb{N}_r : k = 2u+1, u \in \mathbb{N}_0\}$. We then partition the binomial power expansion into two sums:

$$(a+b)^r = \sum_{k=0}^{r} \binom{r}{k} a^{r-k} b^k = \sum_{k \in \mathbb{E}_r} \binom{r}{k} a^{r-k} b^k + \sum_{k \in \mathbb{O}_r} \binom{r}{k} a^{r-k} b^k$$

Let

$$E(a,b,r) = \sum_{k \in \mathbb{E}_r} \binom{r}{k} a^{r-k} b^k \text{ and } O(a,b,r) = \sum_{k \in \mathbb{O}_r} \binom{r}{k} a^{r-k} b^k$$

Then

$$\begin{aligned}(a^2 - b^2)^r &= (a+b)^r (a-b)^r \\ &= (E(a,b,r) + O(a,b,r))(E(a,-b,r) + O(a,-b,r))\end{aligned}$$

But

$$E(a,-b,r) = E(a,b,r) \text{ and } O(a,-b,r) = -O(a,b,r)$$

so

$$\begin{aligned}
(a^2-b^2)^r &= (a+b)^r(a-b)^r \\
&= (E(a,b,r)+O(a,b,r))(E(a,-b,r)+O(a,-b,r)) \\
&= (E(a,b,r)+O(a,b,r))(E(a,b,r)-O(a,b,r)) \\
&= E(a,b,r)^2 - O(a,b,r)^2
\end{aligned}$$

We therefore proved

Lemma 1.1.

$$(a^2-b^2)^r = E(a,b,r)^2 - O(a,b,r)^2$$

Solution

Using lemma 1.1 with $a = \sqrt{n}$ and $b = \sqrt{n+1}$, we get

$$(-1)^r = E(\sqrt{n},\sqrt{n+1},r)^2 - O(\sqrt{n},\sqrt{n+1},r)^2 \tag{L}$$

Lemma 1.2.

$$\begin{aligned}
&E(\sqrt{n},\sqrt{n+1},r)^2 \in \mathbb{N}, \\
&O(\sqrt{n},\sqrt{n+1},r)^2 \in \mathbb{N}
\end{aligned}$$

Proof. We will look at two cases: r even and r odd.
Case 1. For $r = 2u$ even we have

$$\begin{aligned}
E(\sqrt{n},\sqrt{n+1},2u) &= \sum_{k=0}^{u} \binom{2u}{2k} (\sqrt{n})^{2u-2k}(\sqrt{n+1})^{2k} \\
&= \sum_{k=0}^{u} \binom{2u}{2k} (\sqrt{n})^{2(u-k)}(\sqrt{n+1})^{2k} \\
&= \sum_{k=0}^{u} \binom{2u}{2k} n^{u-k}(n+1)^{k}
\end{aligned}$$

so $E(\sqrt{n},\sqrt{n+1},r) \in \mathbb{N}$, and therefore $E(\sqrt{n},\sqrt{n+1},r)^2 \in \mathbb{N}$.

$$
\begin{aligned}
O(\sqrt{n}, \sqrt{n+1}, 2u) &= \sum_{k=0}^{u-1} \binom{2u}{2k+1} (\sqrt{n})^{2u-2k-1} (\sqrt{n+1})^{2k+1} \\
&= \frac{\sqrt{n+1}}{\sqrt{n}} \sum_{k=0}^{u-1} \binom{2u}{2k+1} (\sqrt{n})^{2(u-k)} (\sqrt{n+1})^{2k} \\
&= \frac{\sqrt{n+1}}{\sqrt{n}} \sum_{k=0}^{u-1} \binom{2u}{2k+1} n^{2(u-k)} (n+1)^{k} \\
&= \sqrt{n(n+1)} \sum_{k=0}^{u-1} \binom{2u}{2k+1} n^{2(u-k)-1} (n+1)^{k}
\end{aligned}
$$

so $O(\sqrt{n}, \sqrt{n+1}, r)^2 \in \mathbb{N}$.
Case 2. $r = 2u+1$ is handled in a similar fashion by factoring out $\sqrt{n}$ and $\sqrt{n+1}$ with the remainder $\in \mathbb{N}$.

□

From lemma 1.2 and equation (L) it follows that $E(\sqrt{n}, \sqrt{n+1}, r)^2$ and $O(\sqrt{n}, \sqrt{n+1}, r)^2$ are consecutive natural numbers. Let

$$m = min(E(\sqrt{n}, \sqrt{n+1}, r)^2, O(\sqrt{n}, \sqrt{n+1}, r)^2) \in \mathbb{N}$$

Then

$$x^r = (\sqrt{n} + \sqrt{n+1})^r = \sqrt{m} + \sqrt{m+1}$$