

## Schröder-Bernstein Theorem

BIJECTIONS from one-to-one functions are the topic<sup>1</sup> in this note. The problem statement is known as the Schröder-Bernstein Theorem.

<sup>1</sup> Exercise 1.5.11 on page 32 from Stephen Abbott. *Understanding Analysis*. Springer, 2 edition, 2015. ISBN 978-1-4939-2711-1.

### Problem

Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  be one-to-one functions. Then there exists a bijection  $h : X \rightarrow Y$ .

The given functions are one-to-one, so for subsets  $f(X)$  and  $g(Y)$  they are already bijections. This leads to the idea of partitioning  $X$  and  $Y$  such that we can compose a bijection  $h$  piece-wise from  $f$  and  $g^{-1}$  using the partitions. In particular given a subset  $A \subseteq X$ , we consider the sets  $A$ ,  $X \setminus A$ ,  $f(A)$ ,  $Y \setminus f(A)$  and  $g(Y \setminus f(A))$ . We want subsets  $A \subseteq X$ , such that  $A \cap g(Y \setminus f(A)) = \emptyset$ , as shown in figure 1.2. Let's define this as property  $P$ :

$$\forall A \subseteq X : P(A) \Leftrightarrow A \cap g(Y \setminus f(A)) = \emptyset$$

If we have a subset  $A \subseteq X$  that satisfies  $P(A)$ , then we can define the bijection  $h$ :

$$h(x) = \begin{cases} f(x) & : x \in A \\ g^{-1}(x) & : x \in g(Y \setminus f(A)) \end{cases}$$

The domain of  $h$  is  $A \cup g(Y \setminus f(A))$ , which is not necessarily equal to  $X$ , so we are not done yet. Our goal therefore is to find a subset  $A \subseteq X$  that satisfies  $P(A)$  and for which  $A \cup g(Y \setminus f(A)) = X$ .

Let

$$\Lambda = \{A \subseteq X : P(A)\}$$

be the set of all subsets of  $X$  that satisfy property  $P$  and let  $\bar{A}$  be the union of all such subsets

$$\bar{A} = \bigcup_{A \in \Lambda} A$$

**Lemma 1.1.**  $\bar{A}$  is the biggest subset of  $X$  that satisfies  $P$ .

*Proof.* First we show that  $\bar{A}$  satisfies  $P$ . Assume

$$\exists y \in Y \setminus f(\bar{A}) \text{ with } g(y) \in \bar{A}$$

Then there exists a set  $A \in \Lambda$  with  $g(y) \in A$ <sup>2</sup>.  $A \subseteq \bar{A}$ , so  $f(A) \subseteq f(\bar{A})$ . Therefore  $Y \setminus f(\bar{A}) \subseteq Y \setminus f(A)$ , so  $y \in Y \setminus f(A)$ . But this contradicts

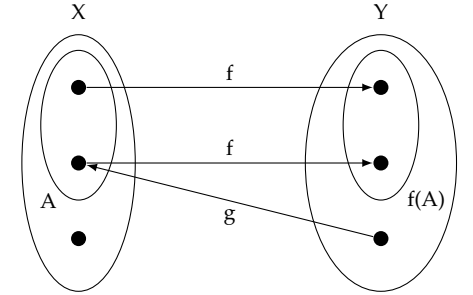


Figure 1.1:  $A$  violates  $P(A)$

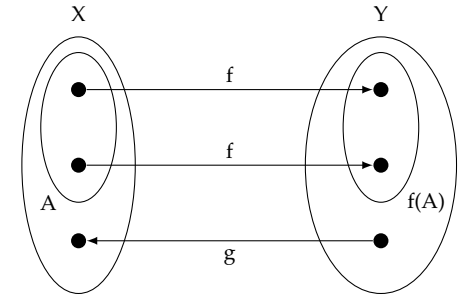


Figure 1.2:  $A$  satisfies  $P(A)$

<sup>2</sup> Because  $\bar{A} = \bigcup_{A \in \Lambda} A$ .

$A$  satisfying property  $P$ , so no such  $y$  exists. It follows that  $\bar{A}$  satisfies  $P$  too.

Assume there is a set  $A'$  that satisfies  $P$  and that is bigger than  $\bar{A}$ , so  $\bar{A} \subseteq A'$ . But  $A' \in \Lambda$  and  $\bar{A} = \bigcup_{A \in \Lambda} A$ , so  $A' \subseteq \bar{A}$ . That means  $A' = \bar{A}$ .  $\square$

With  $\bar{A}$  we can define the partitions  $X = \bar{A} \oplus (X \setminus \bar{A})$  and  $Y = f(\bar{A}) \oplus (Y \setminus f(\bar{A}))$ .

**Lemma 1.2.**

$$g(Y \setminus f(\bar{A})) = X \setminus \bar{A}$$

*Proof.* Because  $\bar{A}$  satisfies  $P$ , we already know that

$$g(Y \setminus f(\bar{A})) \subseteq X \setminus \bar{A}$$

Now assume

$$\exists x \in X \setminus \bar{A} \text{ such that } \forall y \in Y \setminus f(\bar{A}) : g(y) \neq x$$

But then  $\bar{A} \cup \{x\}$  satisfies  $P$ <sup>3</sup> and is bigger than  $\bar{A}$ . This contradicts lemma 1.1. So no such  $x$  exists and the lemma is proven.  $\square$

We can now define the bijection  $h : X \rightarrow Y$  with

$$h(x) = \begin{cases} f(x) & : x \in \bar{A} \\ g^{-1}(x) & : x \in X \setminus \bar{A} \end{cases}$$

which solves the problem in this section.<sup>4</sup>

<sup>3</sup> We have

$$Y \setminus f(\bar{A} \cup \{x\}) \subseteq Y \setminus f(\bar{A})$$

so

$$\forall y \in Y \setminus f(\bar{A} \cup \{x\}) : g(y) \notin \bar{A} \cup \{x\}$$

<sup>4</sup> The solution uses a nifty proof strategy: maximize a mathematical structure so that its “complement” has no choice but to satisfy a certain property, ie not satisfying the property would contradict the maximality.

## *Bibliography*

Stephen Abbott. *Understanding Analysis*. Springer, 2 edition, 2015.  
ISBN 978-1-4939-2711-1.