There is $s \in \mathbb{R}$ with $s^n = y$ for $y > 0$ and $n \geq 2$

Proof:

Consider set $A = \{ x \in \mathbb{R} : x^n < y \}$

$0 \in A \Rightarrow A \neq \emptyset$

$\forall x \in A : x < y \Rightarrow y$ upper bound

$\Rightarrow s := \sup A$ exists

Claim: $s^n = y$

We show this by eliminating the possibilities $s^n < y$, $s^n > y$

Assume $s^n < y$

We will work with expression $(s+\varepsilon)^n$

$$(s+\varepsilon)^n = \sum_{k=0}^{n} \binom{n}{k} s^{n-k} \varepsilon^k$$

$$= s^n + \sum_{k=1}^{n} \binom{n}{k} s^{n-k} \varepsilon^k$$

$$= s^n + \varepsilon \cdot \sum_{k=1}^{n} \binom{n}{k} s^{n-k} \varepsilon^{k-1}$$

we will make sure that

$0 < \varepsilon < 1$

$$\Rightarrow (s+\varepsilon)^n < s^n + \varepsilon \underbrace{\sum_{k=1}^{n} \binom{n}{k} s^{n-k}}_{B :=}$$

Now choose ε such that

$$0 < \varepsilon < \min\left(1, \frac{y - s^n}{B}\right)$$

$$\Rightarrow (s+\varepsilon)^n < s^n + \varepsilon \cdot B < s^n + \frac{y-s^n}{B} \cdot B = y$$

$$\Rightarrow s+\varepsilon \in A \quad \lightning \text{ contradicts } s = \sup A$$

Assume $s^n > y$

We will work with expression $(s-\varepsilon)^n$

$$(s-\varepsilon)^n = \sum_{k=0}^{n} \binom{n}{k} s^{n-k} (-\varepsilon)^k$$

$$= s^n + \sum_{k=1}^{n} \binom{n}{k} s^{n-k} (-\varepsilon)^k$$

$$= s^n + \sum_{\substack{k=1 \\ k \equiv 0 \bmod 2}}^{n} \binom{n}{k} s^{n-k} \varepsilon^k$$

$$- \sum_{\substack{k=1 \\ k \equiv 1 \bmod 2}}^{n} \binom{n}{k} s^{n-k} \varepsilon^k$$

$$> s^n - \sum_{\substack{k=1 \\ k \equiv 1 \bmod 2}}^{n} \binom{n}{k} s^{n-k} \varepsilon^k$$

$$= s^n - \varepsilon \cdot \sum_{\substack{k=1 \\ k \equiv 1 \bmod 2}}^{n} \binom{n}{k} s^{n-k} \varepsilon^{k-1}$$

Again, we make sure $0 < \varepsilon < 1$

$$\Rightarrow (s-\varepsilon)^n > s^n - \varepsilon \underbrace{\sum_{\substack{k=1 \\ k \equiv 1 \bmod 2}}^{n} \binom{n}{k} s^{n-k}}_{C :=}$$

Now we choose ε such that

$$0 < \varepsilon < \min\left(1, \frac{s^n - y}{c}\right)$$

$$\Rightarrow (s-\varepsilon)^n > s^n - \varepsilon \cdot c > s^n - \frac{s^n - y}{c} \cdot c = y$$

$\Rightarrow$ $s - \varepsilon$ is upper bound of A

$\lightning$ contradicts $s = \sup A$

So $s^n = y$

$\blacksquare$