

# Grasshopper jumping

INDUCTION and integer inequalities are the topics of this note<sup>1</sup>.

## Problem

Let  $a_1, a_2, \dots, a_n$  be distinct positive integers and let  $M$  be a set of  $n - 1$  positive integers not containing  $s = a_1 + a_2 + \dots + a_n$ . A grasshopper is to jump along the real axis, starting at the point 0 and making  $n$  jumps to the right with lengths  $a_1, a_2, \dots, a_n$  in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in  $M$ .

We use induction on  $n$  and we use the problem as our induction hypothesis with one modification: set  $M$  has at most  $n - 1$  elements.

The base case  $n = 2$  is trivial.

Let  $A = \{a_i : 1 \leq i \leq n\}$  and  $M = \{m_i : 1 \leq i < n\}$ . Assume  $a_1 < a_2 < \dots < a_n$  and  $m_1 < m_2 < \dots < m_{n-1}$ . For the induction step we have several cases.

**Case:**  $a_n \in M$

There is an  $l : 1 \leq l < n : m_l = a_n$ .

If  $l = n - 1$ : there is an index  $k$  for which  $a_k \notin M$ . Then the order  $\{k, n, \dots\}$  never lands on any point in  $M$  because  $a_k + a_n > m_{n-1}$ .

If  $l < n - 1$ : Define  $M' = \{m_1, m_2, \dots, m_{l-1}\} \cup \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ . Use integers  $a_1, \dots, a_{n-1}$  and  $M'$  as induction step to get an order  $a_{\pi(1)}, \dots, a_{\pi(n-1)}$  with  $\pi \in S_{n-1}$ .

$a_{\pi(1)} \notin M'$  and  $a_{\pi(1)} < a_n$ , so  $a_{\pi(1)} \notin M$ .

$a_{\pi(1)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n \notin \{m_{l+1}, \dots, m_{n-1}\}$ .

Also  $a_{\pi(1)} + a_n > a_n$  so  $a_{\pi(1)} + a_n \notin \{m_1, m_2, \dots, m_{l-1}\}$ . That means  $a_{\pi(1)} + a_n \notin M$ .

We continue with similar reasoning with the rest:  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin M$  because  $a_{\pi(1)} + a_{\pi(2)} \notin \{m_{l+1} - a_n, \dots, m_{n-1} - a_n\}$ , so  $a_{\pi(1)} + a_n + a_{\pi(2)} \notin \{m_{l+1}, \dots, m_{n-1}\}$  and  $a_{\pi(1)} + a_n + a_{\pi(2)} > a_n$  etc.

This means  $\{\pi(1), n, \pi(2), \dots, \pi(n - 1)\}$  is a valid order.

<sup>1</sup> For an extension to signed jumps see

Géza Kós. On the grasshopper problem with signed jumps. *The American Mathematical Monthly*, 118:877–886, 2010. URL <https://arxiv.org/abs/1008.2936>

**Case:**  $a_n \notin M$

If there is an  $m_i < a_n$  then we can use the induction step with integers  $a_1, a_2, \dots, a_{n-1}$  and set  $M' = \{m_{i+1} - a_n, m_{i+2} - a_n, \dots, m_{n-1} - a_n\}$  to find an order and prepend  $a_n$  to that order.

If not, then  $\forall 1 \leq i < n : m_i > a_n$ .

$\sum_{j=1}^{n-1} a_j \geq m_1$  because otherwise we could have used order  $\{1, 2, \dots, n\}$ .

We have  $a_1 < a_n < m_1$  and  $\sum_{j=1}^{n-1} a_j \geq m_1$ , so there exists an  $1 \leq l < n - 1$  such that  $s' = \sum_{j=1}^l a_j < m_1$ .

Define  $M' = \{m_2 - a_n, m_3 - a_n, \dots, m_{n-1} - a_n\}$  and use  $M'$  with the integers  $a_1, a_2, \dots, a_{n-1}$  in an induction step which gives us an order  $\pi \in S_{n-1}$ .

Since  $a_{\pi(1)} < m_1$  and  $\sum_{j=1}^{n-1} a_{\pi(j)} \geq m_1$  there exists an  $1 < l \leq n - 1$  such that  $\sum_{j=1}^{l-1} a_{\pi(j)} < m_1$  and  $\sum_{j=1}^l a_{\pi(j)} \geq m_1$ .

We look at the order  $\{\pi(1), \dots, \pi(l-1), n, \pi(l), \dots, \pi(n-1)\}$  and claim it is a valid order.

Indeed  $\sum_{j=1}^{l-1} a_{\pi(j)} < m_1$ , so jumps  $\{\pi(1), \dots, \pi(l-1)\}$  won't encounter anything from  $M$ . We also have

$$\sum_{j=1}^{l-1} a_{\pi(j)} + a_n > \sum_{j=1}^l a_{\pi(j)} \geq m_1$$

which means  $\{\pi(1), \dots, \pi(l-1), a_n\}$  will avoid  $m_1$ . It will also avoid anything from  $M \setminus \{m_1\}$  because  $\{\pi(1), \dots, \pi(l-1)\}$  avoids anything from  $M'$ . The rest of the order is already bigger than  $m_1$  and avoids  $M \setminus \{m_1\}$  by induction.

## *Bibliography*

Géza Kós. On the grasshopper problem with signed jumps. *The American Mathematical Monthly*, 118:877–886, 2010. URL <https://arxiv.org/abs/1008.2936>.