Sequences and Series

Select exercises on sequences and series from Chapter 3 of the *Lectures on Real Analysis* textbook[1].

[1] F. Lárusson. *Lectures on Real Analysis*. Australian Mathematical Society Lecture Series. Cambridge University Press, 2012. ISBN 9781107026780. URL https://books.google.com/books?id=koj-IrXXwocC

Exercise 3.17, page 35

(a) Let $a \geq 0$ and $n \in \mathbb{N}$, $n \geq 2$. Show that

$$(1+a)^n \geq \frac{1}{2}n(n-1)a^2$$

(b) Show that $n^{\frac{1}{n}} \to 1$ as $n \to \infty$.

Solution. (a) Using the binomial expansion, we get

$$(1+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k = 1 + na + \frac{1}{2}n(n-1)a^2 + \ldots \geq \frac{1}{2}n(n-1)a^2$$

(b) Using the inequality from (a) with $a = n^{\frac{1}{n}} - 1$ we get

$$n = (n^{\frac{1}{n}} - 1 + 1)^n \geq \frac{1}{2}n(n-1)(n^{\frac{1}{n}} - 1)$$

So $\frac{2}{n-1} \geq (n^{\frac{1}{n}} - 1)$ and $n^{\frac{1}{n}} \to 1$. □

Exercise 3.18, page 35

Consider the recursively defined sequence (a_n) with $a_1 = 3$ and $a_{n+1} = \frac{a_n}{2} + \frac{3}{a_n}$. Show that (a_n) converges and find its limit.

Solution. Let's first prove by induction that $\forall n \in \mathbb{N} : 2 < a_n \leq 3$:

It's true for $a_1 = 3$. Assume it is true for a given n and let's do the induction step.

$$a_{n+1} = \frac{a_n}{2} + \frac{3}{a_n} > \frac{2}{2} + \frac{3}{3} = 2$$

Also

$$a_{n+1} = \frac{a_n}{2} + \frac{3}{a_n} \leq \frac{3}{2} + \frac{3}{2} = 3$$

At least we know (a_n) is bounded. Let us spy a little and assume (a_n) does converge, say to limit L. Then L must satisfy:

$$L = \frac{L}{2} + \frac{3}{L}$$

which works out to $L = \sqrt{6}$.

Let's try with a simpler sequence (b_n) such that $a_n = b_n\sqrt{6}$.

$$\begin{aligned} a_{n+1} = b_{n+1}\sqrt{6} &= \frac{a_n}{2} + \frac{3}{a_n} \\ &= \frac{b_n\sqrt{6}}{2} + \frac{3}{b_n\sqrt{6}} \\ &= \frac{b_n\sqrt{6}}{2} + \frac{\sqrt{6}}{2b_n} \end{aligned}$$

So (b_n) satisfies $b_{n+1} = \frac{1}{2}(b_n + \frac{1}{b_n})$. We prove that (b_n) is monoton decreasing:

$$\begin{aligned} b_{n+1} \leq b_n &\Leftrightarrow \\ \frac{1}{2}(b_n + \frac{1}{n}) \leq b_n &\Leftrightarrow \\ b_n^2 + 1 \leq 2b_n^2 &\Leftrightarrow \\ b_n^2 \geq 1 &\Leftrightarrow \\ b_n \geq 1 & \end{aligned}$$

We use the AGM inequality[2] and show:

$$b_{n+1} = \frac{1}{2}(b_n + \frac{1}{b_n}) \geq \sqrt{b_n \frac{1}{b_n}} = 1$$

So (b_n) is monoton decreasing and bounded below by 1, so (b_n) converges, and so does (a_n): $b_n \to 1$ and $a_n \to \sqrt{6}$.

□

[2] For positive x and y we have $(\sqrt{x} + \sqrt{y})^2 \geq 0$ which when expanded ends up at $\frac{x+y}{2} \geq \sqrt{xy}$.

Exercise 3.23, page 36

Let $\sum a_n$ be a series. Set $a_n^+ = max\{0, a_n\}$ and $a_n^- = min\{0, a_n\}$. Consider the series $\sum a_n^+$ and $\sum a_n^-$.
(a) Prove that $\sum a_n$ is absolutely convergent if and only if $\sum a_n^+$ and $\sum a_n^-$ both converge. Then $\sum a_n = \sum a_n^+ + \sum a_n^-$.
(b) Prove that if $\sum a_n$ is conditionally convergent, then $\sum a_n^+$ and $\sum a_n^-$ both diverge.

Solution. We will use the partial sums:

$$
\begin{aligned}
s_n &= \sum_{k=1}^{n} a_k, \quad s_n^a = \sum_{k=1}^{n} |a_k| \\
s_n^+ &= \sum_{k=1}^{n} a_k^+, \quad s_n^- = \sum_{k=1}^{n} a_k^-
\end{aligned}
$$

(a) $(\Rightarrow)$

We have $\forall n \in \mathbb{N} : |a_n| \geq a_n^+$ and $|a_n| \geq (-1)a_n^-$. Using the comparison test we find $\sum a_n^+$ and $\sum a_n^-$ converge.

$(\Leftarrow)$ $\sum a_n^+$ and $\sum a_n^-$ converge, so then also $\sum a_n^+ + (-1)\sum a_n^-$ converges. But $s_n^a = s_n^+ + (-1)s_n^-$, so $\sum |a_n|$ converges too.

(b) $\sum a_n$ converges conditionally. If both $\sum a_n^+$ and $\sum a_n^-$ converge, then from (a) we would have $\sum a_n$ converges absolutely, contradicting the premise. So at least one of $\sum a_n^+$ or $\sum a_n^-$ must diverge.

Assume $\sum a_n^+$ diverges (the other case is similar). s_n^+ is monotonically increasing and divergent, so it is unbounded. We have $s_n^+ = s_n - s_n^-$ and s_n is bounded. It follows that s_n^- has to be unbounded, so $\sum a_n^-$ diverges also.

□

Exercise 3.24, page 36

Let $\sum a_n$ be a conditionally convergent series. Prove that for every $\sigma \in \mathbb{R}$ there is a rearrangement of $\sum a_n$ that converges to σ.

Solution. We will construct this rearrangement.

We know from the previous exercise that both $\sum a_n^+$ and $\sum a_n^-$ diverge and both s_n^+ and s_n^- are unbounded.

Assume first that $\sigma > 0$ (the other case is similar). Since s_n^+ is unbounded, there exists[3] a $N_1 \in \mathbb{N}$ such that

$$
\begin{aligned}
\sum_{k=1}^{N_1-1} a_k^+ &\leq \sigma \\
\sum_{k=1}^{N_1} a_k^+ &> \sigma
\end{aligned}
$$

Let $d_1 = |\sum_{k=1}^{N_1} a_k^+ - \sigma|$. We see that $0 < d_1 \leq |a_{N_1}^+|$. Our rearrangement will start with the first N_1 terms from $\sum a_n^+$. For the next terms we turn to $\sum a_n^-$. s_n^- is also unbounded, so there exists a $M_1 \in \mathbb{N}$ such that

[3] This N_1 has to exist because s_n^+ is unbounded. If it was only zeros it would converge and be bounded.

$$\sum_{k=1}^{M_1-1} a_k^- \geq d_1$$
$$\sum_{k=1}^{M_1} a_k^- < d_1$$

We add the first M_1 terms from $\sum a_n^-$ to the rearrangement. Let $d_2 = |\sum_{k=1}^{N_1} a_k^+ + \sum_{k=1}^{M_1} a_k^- - \sigma|$. We see that $0 < d_2 \leq |a_{M_1}^-|$.

Next we go back to $\sum a_n^+$ for more terms. The tail of $\sum a_n^+$ starting at $N_1 + 1$ is also unbounded, so there must exist a N_2 such that

$$\sum_{k=N_1+1}^{N_2-1} a_k^+ \leq d_2$$
$$\sum_{k=N_1+1}^{N_2} a_k^+ > d_2$$

We add the terms $\sum_{k=N_1+1}^{N_2} a_k^+$ to the rearrangement and define

$$d_3 = |\sum_{k=1}^{N_1} a_k^+ + \sum_{k=1}^{M_1} a_k^- + \sum_{k=N_1+1}^{N_2} a_k^+ - \sigma|$$

We see that $0 < d_3 \leq |a_{N_2}^+|$.

We go back down with the help of terms from the tail of $\sum a_n^-$ starting at M_1, a tail that is also unbounded. There must exist a M_2 such that

$$\sum_{k=M_1+1}^{M_2-1} a_k^+ \geq d_3$$
$$\sum_{k=M_1+1}^{M_2} a_k^+ < d_3$$

We add the terms $\sum_{k=M_1+1}^{M_2} a_k^-$ to the rearrangement and define

$$d_4 = |\sum_{k=1}^{N_1} a_k^+ + \sum_{k=1}^{M_1} a_k^- + \sum_{k=N_1+1}^{N_2} a_k^+ + \sum_{k=M_1+1}^{M_2} a_k^- - \sigma|$$

We see that $0 < d_4 \leq |a_{M_2}^-|$.

We continue in this way, switching between terms in $\sum a_n^+$ and $\sum a_n^-$, constructing a rearrangement of $\sum a_n$ that has partial sums that have distance d_n from σ.

The sequence (d_n) of distances is bounded by $(|a_n|)$ and $\sum a_n$ is a conditionally convergent series, so $a_n \to 0$. That means that $d_n \to 0$ and the rearrangement converges to σ.

$\square$

Exercise 3.30, page 37

Show that there is a sequence (a_n) such that for every real number x, there is a subsequence of (a_n) converging to x.

Solution. At first glance this seems quite a fantastical premise. How can there be a sequence that for every real number contains a subsequence converging to that number? Isn't $\mathbb{R}$ uncountable? Well, the best way to prove the existence of such a sequence is to construct it.

First we want to make our life easier: we use the fact that there exists a bijection between the interval $(0,1)$ and $\mathbb{R}$. There are many bijections between these two sets to choose from and we will choose a continuous one:

$$
\begin{aligned}
F &: \mathbb{R} \to (0,1) \\
F(x) &= \frac{1}{1+e^x}
\end{aligned}
$$

and its inverse

$$
\begin{aligned}
F^{-1} &: (0,1) \to \mathbb{R} \\
F^{-1}(x) &= \ln\left(\frac{1-x}{x}\right)
\end{aligned}
$$

If we can construct subsequences that converge to $x \in (0,1)$ then we can use F^{-1} to map them over to $y \in \mathbb{R}$ and because of continuity the mapping of the subsequence will converge to y. The construction idea is to map a given $n \in \mathbb{N}$ to a pair $(i,j) \in \mathbb{N} \times \mathbb{N}$. This (i,j) pair will have the following meaning: j subdivides $(0,1)$ into j subintervals of length $\frac{1}{j}$ and i will select which of those j subintervals we mean. A given $x \in (0,1)$ will fall into one of them and its corresponding (i,j) pair will determine the n we use in the subsequence. Increasing the j and then choosing the corresponding i subinterval containing x will get us closer and closer to x.

This is the construction idea. We still have to deal with the technicalities.

First we want a bijection from $\mathbb{N}$ to a subset of $\mathbb{N} \times \mathbb{N}$ where the pairs (i,j) satisfy $i \leq j$. We use a similar approach to the one we used in a previous note: `https://sagenhaft.space/posts/math_notes/counting/counting.pdf`.

We order the pairs $(i,j) \in \mathbb{N} \times \mathbb{N}$ satisfying $i \leq j$ in rows, such that row r has pairs $(1,r),(2,r),\ldots,(r,r)$. Figure 1.1 illustrates the idea. Our bijection will count going down the rows and going left to right in each row. So the order is $(1,1),(1,2),(2,2),(1,3),(2,3),(3,3),\ldots$

$(1,1) \longrightarrow$ row 1

$(1,2),(2,2) \longrightarrow$ row 2

$(1,3),(2,3),(3,3) \longrightarrow$ row 3

$(1,4),(2,4),(3,4),(4,4) \longrightarrow$ row 4

$\cdots$

Figure 1.1: Going from n to (i,j) with $i \leq j$.

Lets first deduce the inverse, going from (i, j) to n in that order. For a given (i, j) we know we are in row j at pair i in that row. Each row k before row j has k pairs in it, therefore the corresponding position n in the counting order is:

$$\begin{aligned} n &= \sum_{k=1}^{j-1} k + i \\ &= \frac{j(j-1)}{2} + i \end{aligned}$$

We can test this: in the Figure 1.1, pair $(2, 4)$ should be the eighth pair. $\frac{4 \times 3}{2} + 2 = 8$, so it checks out. We denote $M = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i \leq j\}$ and define the function f:

$$\begin{aligned} &f : M \to \mathbb{N} \\ &f(i, j) = \frac{j(j-1)}{2} + i \end{aligned}$$

It is easy to prove that f is a bijection. Suppose we have two pairs $(i_1, j_1) \neq (i_2, j_2)$. If $j_1 \neq j_2$ then they are in different rows. If $j_1 = j_2$ then we must have $i_1 \neq i_2$, so again their mapping is different. It follows that f is injective.

Given $n \in \mathbb{N}$, can we find (i, j) such that $f(i, j) = n$? The nth pair falls on some row r. There are $\frac{r(r-1)}{2}$ pairs in the rows before row r and $\frac{r(r+1)}{2}$ pairs in the first r rows. Therefore:

$$\frac{r(r-1)}{2} < n \leq \frac{r(r+1)}{2}$$

The two relevant values for these two quadratic inequalities are $\frac{1+\sqrt{1+8n}}{2}$ and $\frac{-1+\sqrt{1+8n}}{2}$ because we have to stay positive. Notice that their difference is $\frac{1+\sqrt{1+8n}}{2} - \frac{-1+\sqrt{1+8n}}{2} = 1$, so there is only one positive integer satisfying both inequalities (as we hoped) and that positive integer is our sought after row r:

$$r = \left\lceil \frac{-1 + \sqrt{1+8n}}{2} \right\rceil$$

Lets verify this for fun again, making sure that the eighth pair is on row four:

$$\left\lceil \frac{-1 + \sqrt{1 + 8 \times 8}}{2} \right\rceil = \left\lceil \frac{-1 + \sqrt{65}}{2} \right\rceil = \lceil 3.53113 \rceil = 4$$

We know that $j = r$ and then $i = n - \frac{j(j-1)}{2}$. This means that f is surjective and therefore a bijection.

The inverse $f^{-1}(n)$ is:

$f^{-1} : \mathbb{N} \to M$

$f^{-1}(n) = (i, j)$, where $j = \left\lceil \dfrac{-1+\sqrt{1+8n}}{2} \right\rceil$ and $i = n - \dfrac{j(j-1)}{2}$

For a given pair (i, j) lets divide interval $(0, 1)$ into j non-overlapping intervals:

$$(0, \frac{1}{j}], (\frac{1}{j}, \frac{2}{j}], \ldots, (\frac{j-2}{j}, \frac{j-1}{j}], (\frac{j-1}{j}, 1)$$

Except for the last subinterval, all other subintervals are left-exclusive and right-inclusive. The last one is open on both ends. This is just a technicality, but we now have a set of intervals that don't intersect and their union is $(0, 1)$.

A given $x \in (0, 1)$ will fall into one of these subintervals. We will use this fact shortly.

We are ready to define our sequence (a_n):

$$a_n = \ln(\frac{j-i}{i}), \text{ where } j = \left\lceil \frac{-1+\sqrt{1+8n}}{2} \right\rceil \text{ and } i = n - \frac{j(j-1)}{2}$$

For any $x \in \mathbb{R}$ we first get $y = F(x) = \frac{1}{1+e^x}$ which places us in interval $(0, 1)$. We choose the following subsequence of (a_{n_k}): choose the n_k so that the corresponding (i, j) pair according to our bijection f^{-1} is the ith interval of the division of $(0, 1)$ into j non-overlapping intervals that contains y. Keep increasing j and selecting the corresponding (a_{n_k}) according to this criteria. This subsequence converges to x.

This construction is not unique. We made pretty arbitrary choices along the way. There are more than one sequence (a_n) with the desired property.

$\square$

Bibliography

F. Lárusson. *Lectures on Real Analysis*. Australian Mathematical Society Lecture Series. Cambridge University Press, 2012. ISBN 9781107026780. URL https://books.google.com/books?id=koj-IrXXwocC.