Counting and countable sets

Countable sets and counting schemes for infinite countable sets are the topics of the problem in this note.

Problem

Let $P = \{N \subset \mathbb{N} : N \text{ finite}\}$. Prove P is countable.

Let's revisit what it means for an infinite set to be countable: An infinite set M is countable if there is a bijection [1] from M to $\mathbb{N}$.

[1] A bijection is a function that is one-to-one and onto.

Given this definition, the problem statement is quite remarkable: the set of all the finite subsets of $\mathbb{N}$ is not "bigger" than $\mathbb{N}$.

Our strategy will be to start smaller and prove certain subsets of P are countable. We then expand it to P. We start by proving that the set of all subsets of $\mathbb{N}$ of size two is countable. We actually will prove something stronger, namely the set of ordered pairs of natural numbers is countable.

Theorem 1.1. *The set of ordered pairs $\mathbb{N} \times \mathbb{N}$ is countable.*

Mention puzzle 136 (Catching a Spy) from Levitin: Algorithmic Puzzles

Proof. We need a bijection from $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. There are many ways to do this [2].

[2] A very elegant way is described at http://www.math.upenn.edu/~wilf/website/recounting.pdf

The main idea we are going to use for our bijection is to order the pairs $(i, j) \in \mathbb{N} \times \mathbb{N}$ in rows, such that each pair in a row has the same value when summing the components of the pair. Figure 1.1 illustrates the idea. Row one has all pairs with components that sum up to two (in this case only one pair). Row two has all pairs with components that sum up to three, row three all pairs which sum to four, ... Notice also that in a row the pairs are sorted in increasing order of the first component.

We count the pairs from left to right in each row and go down the rows starting at the first row. For a given pair (i, j), how many pairs

$(1,1) \longrightarrow$ row 1

$(1,2), (2,1) \longrightarrow$ row 2

$(1,3), (2,2), (3,1) \longrightarrow$ row 3

$(1,4), (2,3), (3,2), (4,1) \longrightarrow$ row 4

...

Figure 1.1: Counting all pairs.

come before it in our counting scheme? It is in row $i+j-1$, so there are $k : 1 \leq k < i+j-1$ rows before it. Each row k has k pairs in it. This means there are

$$\sum_{k=1}^{i+j-2} k = \frac{(i+j-2)(i+j-1)}{2}$$

pairs in rows before our pair (i,j). There are $i-1$ pairs before (i,j) in the same row. Therefore, our counting function is

$$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad f(i,j) = i + \frac{(i+j-2)(i+j-1)}{2}$$

Suppose we have two pairs $(i_1, j_1) \neq (i_2, j_2)$. We have two cases:

- $i_1 + j_1 = i_2 + j_2$, same row, then $i_1 \neq i_2$, so $f(i_1, j_1) \neq f(i_2, j_2)$
- $i_1 + j_1 \neq i_2 + j_2$, different rows, so $f(i_1, j_1) \neq f(i_2, j_2)$

This means, f is one-to-one.

To prove that f is onto, we consider an arbitrary $n \in \mathbb{N}$ and find a pair (i,j) with $f(i,j) = n$. Working backwards and assuming we have a pair (i,j) with $f(i,j) = n$, it would fall on a row $r = i+j-1$. In each row k there are k pairs, n is on row r, so

$$\sum_{k=1}^{r-1} k = \frac{r(r-1)}{2} < n \leq \sum_{k=1}^{r} k = \frac{r(r+1)}{2}$$

Solving for r we have:[3]

[3] Note that $\frac{1+\sqrt{1+8n}}{2} - \frac{-1+\sqrt{1+8n}}{2} = 1$

$$r^2 - r - 2n < 0, \quad r^2 + r - 2n \geq 0, \quad r = \left\lceil \frac{-1+\sqrt{1+8n}}{2} \right\rceil$$

And then

$$i = n - \frac{r(r-1)}{2}, \quad j = r - i + 1$$

This means that given an arbitrary n, there exists a pair (i,j) with $f(i,j) = n$, so f is onto.

It follows that f is a bijection and $\mathbb{N} \times \mathbb{N}$ is countable. □

A corrollary to Theorem 1.1 let's us expand the countable subsets of P even more.

Corollary. *Set of all finite sequences of length k, $\mathbb{N}^k$ is countable.*[4]

[4] $\mathbb{N}^k$ is the set of sequences of length k, or the cartesian product $\mathbb{N} \times \mathbb{N} \times \ldots \mathbb{N}$. The set of pairs is $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$.

Proof. Follows by induction on k: Assuming $\mathbb{N}^{k-1}$ is countable, then

$$\mathbb{N}^k = \mathbb{N}^{k-1} \times \mathbb{N}$$

is also countable according to Theorem 1.1. □

From the corrollary we now know[5] that the set of all subsets of $\mathbb{N}$ of size k is countable (it's a subset of $\mathbb{N}^k$). The problem in this section asks us to prove that P is countable, which means the union of all these countable sets is countable. The next theorem will prove just that.

[5] We keep using the fact that the set of all finite subsets of $\mathbb{N}$ of size k is a subset of the set of all sequences of size k. To see this impose an order on a set of size k and you get a sequence.

Theorem 1.2. *Let $A_n,\ n \in \mathbb{N}$ be countable sets. Then*

$$\bigcup_{n=1}^{\infty} A_n$$

is countable[6].

[6] Exercise 1.5.3 on page 30 from Stephen Abbott. *Understanding Analysis*. Springer, 2 edition, 2015. ISBN 978-1-4939-2711-1.

Proof. A_n is countable, so there exists a bijection $f_n : \mathbb{N} \to A_n$. We already know that $\mathbb{N} \times \mathbb{N}$ is countable, so there exists a bijection $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

We define $F : \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$

$$F(n) = f_i(j), \text{ where } (i,j) = g(n)$$

We claim that F is a bijection.

Take $n_1 \neq n_2$. Then $g(n_1) \neq g(n_2)$ and $(i_1, j_2) \neq (i_2, j_2)$, so

$$f_{i_1}(j_1) \neq f_{i_2}(j_2)$$

It means $F(n_1) \neq F(n_2)$ and F is one-to-one.

Now pick an arbitrary $a \in \bigcup_{n=1}^{\infty} A_n$. Then there exists $i \in \mathbb{N}$ with $a \in A_i$. [7] There also exists $j \in \mathbb{N}$ with $f_i(j) = a$. The pair (i,j) is in $\mathbb{N} \times \mathbb{N}$, so there exists $n \in \mathbb{N}$ with $g(n) = (i,j)$. It follows that $F(n) = a$ and F is onto. □

[7] We assume here the A_i are disjoint, if not we make them disjoint and their union stays the same.

Let us use this last theorem to prove the following statement[8]:

[8] Exercise 0.0.1 on page xiii from T. Tao. *An Introduction to Measure Theory*. Graduate Studies in Mathematics. American Mathematical Society, 2021. ISBN 9781470466404. URL https://books.google.com/books?id=k0lDEAAAQBAJ.

Theorem 1.3. *If $(x_\alpha)_{\alpha \in A}$ is a collection of numbers $(x_\alpha) \in [0, +\infty]$ such that $\sum_{\alpha \in A} x_\alpha < \infty$, then $x_\alpha = 0$ for all but at most countably many $\alpha \in A$, even if A itself is uncountable.*

Proof. We adopt the same definition of sum over the collection of numbers as in Terence Tao's book:

$$\sum_{\alpha \in A} x_\alpha = \sup\{\sum_{\alpha \in F} x_\alpha : F \subset A, F \text{ finite}\}$$

For each $n \in \mathbb{N}$ we define the subset $A_n \subset A$:

$$A_n = \{\alpha : \alpha \in A, x_\alpha \geq \frac{1}{n}\}$$

The sets A_n have to be finite because otherwise the sum $\sum_{\alpha \in A_n} x_\alpha$ would be an infinite sum of numbers not converging to zero, therefore it would diverge which is a contradiction to $\sum_{\alpha \in A} x_\alpha < \infty$.

We also know that $\bigcup_{n=1}^{\infty} A_n$ collect all the non-zero elements x_α and according to the previous theorem this union is countable. □

Bibliography

Stephen Abbott. *Understanding Analysis*. Springer, 2 edition, 2015. ISBN 978-1-4939-2711-1.

T. Tao. *An Introduction to Measure Theory*. Graduate Studies in Mathematics. American Mathematical Society, 2021. ISBN 9781470466404. URL https://books.google.com/books?id=k0lDEAAAQBAJ.