

How many trailing zeros in $n!$

GREATEST DIVIDING EXPONENT and its properties is the topic of the problem in this note.

Problem

Write a program that calculates for an arbitrary positive integer n how many trailing zeros there are in $n!$.

Let's first try to figure out for any natural number n what the number of trailing zeros is. A useful concept here is the greatest dividing exponent ¹:

Definition 1.1. The greatest dividing exponent $gde(n, b)$ of a base b with respect to a number n is the largest integer value of k such that $b^k \mid n$, where $b^k \leq n$.

Lemma 1.2.

$$gde(n, ab) = \min(gde(n, a), gde(n, b)), \text{ with } (a, b) = 1$$

Proof. Assume $gde(n, a) \leq gde(n, b)$. Then $a^{gde(n, a)} \mid n$ and $b^{gde(n, a)} \mid n$, with $(a^{gde(n, a)}, b^{gde(n, a)}) = 1$, so $(ab)^{gde(n, a)} \mid n$. By definition of gde we then have $gde(n, a) \leq gde(n, ab)$.

We also have $(ab)^{gde(n, ab)} \mid n$, so $a^{gde(n, ab)} \mid n$. By definition of gde we then have $gde(n, a) \geq gde(n, ab)$.

It follows that $gde(n, a) = gde(n, ab)$. \square

It's clear that the number of trailing zeros of n equals $gde(n, 10)$. From lemma 1.2 we are looking for $\min(gde(n!, 2), gde(n!, 5))$.

Lemma 1.3.

$$gde(n!, p) = \sum_{k=1}^{\lfloor \log_p n \rfloor} \left\lfloor \frac{n}{p^k} \right\rfloor, \text{ for a prime } p \leq n$$

¹ Eric W. Weisstein. Greatest dividing exponent. From MathWorld—A Wolfram Web Resource. URL <http://mathworld.wolfram.com/GreatestDividingExponent.html>

Proof. We define the following subsets of $\{1, \dots, n\}$:

$$M_p^k = \{i : 1 \leq i \leq n : p^k \mid i\}$$

For $k > \lfloor \log_p n \rfloor$ the sets M_p^k are empty, so we only consider $k \leq \lfloor \log_p n \rfloor$. Each member of one set M_p^k contributes k to $gde(n!, p)$, so the whole set contributes $k|M_p^k|$. From $p^k \mid i$ it follows that also $p^{k-1} \mid i$, so $M_p^k \subseteq M_p^{k-1}$ for $k = 2, \dots, \lfloor \log_p n \rfloor$. Being careful not to count the contributions more than once we get:

$$gde(n!, p) = \sum_{k=1}^{\lfloor \log_p n \rfloor} |M_p^k|$$

With $|M_p^k| = \left\lfloor \frac{n}{p^k} \right\rfloor$ we conclude the proof. \square

Lemma 1.4.

$$gde(n!, 2) \geq gde(n!, 5) \text{ for any } n \geq 1$$

Proof. Plugging in the expression of gde from lemma 1.3 into the claim of this lemma we get:

$$gde(n!, 2) \geq gde(n!, 5) \Leftrightarrow \sum_{k=1}^{\lfloor \log_2 n \rfloor} \left\lfloor \frac{n}{2^k} \right\rfloor \geq \sum_{k=1}^{\lfloor \log_5 n \rfloor} \left\lfloor \frac{n}{5^k} \right\rfloor$$

We establish:

$$\begin{aligned} \log_2 n \geq \log_5 n &\Leftrightarrow \log_2 n \geq \log_2 n \log_5 2 \\ &\Leftrightarrow 1 \geq \log_5 2, \text{ which is true} \end{aligned}$$

For each $1 \leq k \leq \lfloor \log_5 n \rfloor$ we have:

$$\left\lfloor \frac{n}{2^k} \right\rfloor \geq \left\lfloor \frac{n}{5^k} \right\rfloor$$

and for $\lfloor \log_5 n \rfloor + 1 \leq k \leq \lfloor \log_2 n \rfloor$ we have:

$$\left\lfloor \frac{n}{2^k} \right\rfloor > 0$$

Adding up the inequalities establishes the claim. \square

From the three lemmas we found that:

$$\begin{aligned} (\text{number of trailing zeros in } n!) &= gde(n!, 10) \\ &= \min(gde(n!, 2), gde(n!, 5)) \\ &= gde(n!, 5) \\ &= \sum_{k=1}^{\lfloor \log_5 n \rfloor} \left\lfloor \frac{n}{5^k} \right\rfloor \end{aligned}$$

so our program needs to calculate the expression:

$$\sum_{k=1}^{\lfloor \log_5 n \rfloor} \left\lfloor \frac{n}{5^k} \right\rfloor$$

The following small Haskell function does it:

Listing 1.1: Haskell code

```
gdefac :: Int -> Int

gdefac n = fst (until (\(x, y) -> y == 0)
                  (\(x, y) -> let
                                y' = div y 5
                                in (x + y', y'))
                (0, n))
```

It works on tuples of numbers. It keeps dividing the second number in the tuple by 5 until zero and adding the division results together into the first number of the tuple. In the end it returns the first number in the tuple.

Bibliography

Eric W. Weisstein. Greatest dividing exponent. From MathWorld—
A Wolfram Web Resource. URL [http://mathworld.wolfram.com/
GreatestDividingExponent.html](http://mathworld.wolfram.com/GreatestDividingExponent.html).