How many trailing zeros in $n!$

GREATEST DIVIDING EXPONENT and its properties is the topic of the problem in this note.

Problem

Write a program that calculates for an arbitrary positive integer n how many trailing zeros there are in $n!$.

Let's first try to figure out for any natural number n what the number of trailing zeros is. A useful concept here is the greatest dividing exponent [1]:

[1] Eric W. Weisstein. Greatest dividing exponent. From MathWorld—A Wolfram Web Resource. URL http://mathworld.wolfram.com/GreatestDividingExponent.html

Definition 1.1. The greatest dividing exponent $gde(n, b)$ of a base b with respect to a number n is the largest integer value of k such that $b^k \mid n$, where $b^k \leq n$.

Lemma 1.2.

$$gde(n, ab) = min(gde(n, a), gde(n, b)), \text{ with } (a, b) = 1$$

Proof. Assume $gde(n, a) \leq gde(n, b)$. Then $a^{gde(n,a)} \mid n$ and $b^{gde(n,a)} \mid n$, with $(a^{gde(n,a)}, b^{gde(n,a)}) = 1$, so $(ab)^{gde(n,a)} \mid n$. By definition of gde we then have $gde(n, a) \leq gde(n, ab)$.

We also have $(ab)^{gde(n,ab)} \mid n$, so $a^{gde(n,ab)} \mid n$. By definition of gde we then have $gde(n, a) \geq gde(n, ab)$.

It follows that $gde(n, a) = gde(n, ab)$. □

It's clear that the number of trailing zeros of n equals $gde(n, 10)$. From lemma 1.2 we are looking for $min(gde(n!, 2), gde(n!, 5))$.

Lemma 1.3.

$$gde(n!, p) = \sum_{k=1}^{\lfloor log_p n \rfloor} \left\lfloor \frac{n}{p^k} \right\rfloor, \text{ for a prime } p \leq n$$

Proof. We define the following subsets of $\{1, \ldots, n\}$:

$$M_p^k = \{i : 1 \leq i \leq n : p^k \mid i\}$$

For $k > \lfloor log_p n \rfloor$ the sets M_p^k are empty, so we only consider $k \leq \lfloor log_p n \rfloor$. Each member of one set M_p^k contributes k to $gde(n!, p)$, so the whole set contributes $k|M_p^k|$. From $p^k \mid i$ it follows that also $p^{k-1} \mid i$, so $M_p^k \subseteq M_p^{k-1}$ for $k = 2, \ldots, \lfloor log_p n \rfloor$. Being careful not to count the contributions more than once we get:

$$gde(n!, p) = \sum_{k=1}^{\lfloor log_p n \rfloor} |M_p^k|$$

With $|M_p^k| = \left\lfloor \frac{n}{p^k} \right\rfloor$ we conclude the proof. $\square$

Lemma 1.4.

$$gde(n!, 2) \geq gde(n!, 5) \textit{ for any } n \geq 1$$

Proof. Plugging in the expression of *gde* from lemma 1.3 into the claim of this lemma we get:

$$gde(n!, 2) \geq gde(n!, 5) \Leftrightarrow \sum_{k=1}^{\lfloor log_2 n \rfloor} \left\lfloor \frac{n}{2^k} \right\rfloor \geq \sum_{k=1}^{\lfloor log_5 n \rfloor} \left\lfloor \frac{n}{5^k} \right\rfloor$$

We establish:

$$\begin{aligned} log_2 n \geq log_5 n &\Leftrightarrow log_2 n \geq log_2 n \; log_5 2 \\ &\Leftrightarrow 1 \geq log_5 2, \text{ which is true} \end{aligned}$$

For each $1 \leq k \leq \lfloor log_5 n \rfloor$ we have:

$$\left\lfloor \frac{n}{2^k} \right\rfloor \geq \left\lfloor \frac{n}{5^k} \right\rfloor$$

and for $\lfloor log_5 n \rfloor + 1 \leq k \leq \lfloor log_2 n \rfloor$ we have:

$$\left\lfloor \frac{n}{2^k} \right\rfloor > 0$$

Adding up the inequalities establishes the claim. $\square$

From the three lemmas we found that:

$$\begin{aligned} \text{(number of trailing zeros in } n!) &= gde(n!, 10) \\ &= min(gde(n!, 2), gde(n!, 5)) \\ &= gde(n!, 5) \\ &= \sum_{k=1}^{\lfloor log_5 n \rfloor} \left\lfloor \frac{n}{5^k} \right\rfloor \end{aligned}$$

so our program needs to calculate the expression:

$$\sum_{k=1}^{\lfloor log_5 n \rfloor} \left\lfloor \frac{n}{5^k} \right\rfloor$$

The following small Haskell function does it:

Listing 1.1: Haskell code

```
gdefac :: Int -> Int

gdefac n = fst (until (\(x, y) -> y == 0)
                (\(x, y) -> let
                                y' = div y 5
                              in (x + y', y'))
                (0, n))
```

It works on tuples of numbers. It keeps dividing the second number in the tuple by 5 until zero and adding the division results together into the first number of the tuple. In the end it returns the first number in the tuple.

Bibliography

Eric W. Weisstein. Greatest dividing exponent. From MathWorld—A Wolfram Web Resource. URL http://mathworld.wolfram.com/GreatestDividingExponent.html.