Sequences and Series

SELECT EXERCISES ON SEQUENCES AND SERIES from Chapter 3 of the *Lectures on Real Analysis* textbook[1].

[1] F. Lárusson. *Lectures on Real Analysis*. Australian Mathematical Society Lecture Series. Cambridge University Press, 2012. ISBN 9781107026780. URL https://books.google.com/books?id=koj-IrXXwocC

Exercise 3.17, page 35

(a) Let $a \geq 0$ and $n \in \mathbb{N}$, $n \geq 2$. Show that

$$(1+a)^n \geq \frac{1}{2}n(n-1)a^2$$

(b) Show that $n^{\frac{1}{n}} \to 1$ as $n \to \infty$.

Solution. (a) Using the binomial expansion, we get

$$(1+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k = 1 + na + \frac{1}{2}n(n-1)a^2 + \ldots \geq \frac{1}{2}n(n-1)a^2$$

(b) Using the inequality from (a) with $a = n^{\frac{1}{n}} - 1$ we get

$$n = (n^{\frac{1}{n}} - 1 + 1)^n \geq \frac{1}{2}n(n-1)(n^{\frac{1}{n}} - 1)$$

So $\frac{2}{n-1} \geq (n^{\frac{1}{n}} - 1)$ and $n^{\frac{1}{n}} \to 1$. □

Exercise 3.18, page 35

Consider the recursively defined sequence (a_n) with $a_1 = 3$ and $a_{n+1} = \frac{a_n}{2} + \frac{3}{a_n}$. Show that (a_n) converges and find its limit.

Solution. Let's first prove by induction that $\forall n \in \mathbb{N} : 2 < a_n \leq 3$:

It's true for $a_1 = 3$. Assume it is true for a given n and let's do the induction step.

$$a_{n+1} = \frac{a_n}{2} + \frac{3}{a_n} > \frac{2}{2} + \frac{3}{3} = 2$$

Also

$$a_{n+1} = \frac{a_n}{2} + \frac{3}{a_n} \leq \frac{3}{2} + \frac{3}{2} = 3$$

At least we know (a_n) is bounded. Let us spy a little and assume (a_n) does converge, say to limit L. Then L must satisfy:

$$L = \frac{L}{2} + \frac{3}{L}$$

which works out to $L = \sqrt{6}$.

Let's try with a simpler sequence (b_n) such that $a_n = b_n\sqrt{6}$.

$$\begin{aligned} a_{n+1} = b_{n+1}\sqrt{6} &= \frac{a_n}{2} + \frac{3}{a_n} \\ &= \frac{b_n\sqrt{6}}{2} + \frac{3}{b_n\sqrt{6}} \\ &= \frac{b_n\sqrt{6}}{2} + \frac{\sqrt{6}}{2b_n} \end{aligned}$$

So (b_n) satisfies $b_{n+1} = \frac{1}{2}(b_n + \frac{1}{b_n})$. We prove that (b_n) is monoton decreasing:

$$\begin{aligned} b_{n+1} \leq b_n &\Leftrightarrow \\ \frac{1}{2}(b_n + \frac{1}{n}) \leq b_n &\Leftrightarrow \\ b_n^2 + 1 \leq 2b_n^2 &\Leftrightarrow \\ b_n^2 \geq 1 &\Leftrightarrow \\ b_n \geq 1 & \end{aligned}$$

We use the AGM inequality[2] and show:

$$b_{n+1} = \frac{1}{2}(b_n + \frac{1}{b_n}) \geq \sqrt{b_n \frac{1}{b_n}} = 1$$

So (b_n) is monoton decreasing and bounded below by 1, so (b_n) converges, and so does (a_n): $b_n \to 1$ and $a_n \to \sqrt{6}$.

□

[2] For positive x and y we have $(\sqrt{x} + \sqrt{y})^2 \geq 0$ which when expanded ends up at $\frac{x+y}{2} \geq \sqrt{xy}$.

Exercise 3.23, page 36

Let $\sum a_n$ be a series. Set $a_n^+ = max\{0, a_n\}$ and $a_n^- = min\{0, a_n\}$. Consider the series $\sum a_n^+$ and $\sum a_n^-$.
(a) Prove that $\sum a_n$ is absolutely convergent if and only if $\sum a_n^+$ and $\sum a_n^-$ both converge. Then $\sum a_n = \sum a_n^+ + \sum a_n^-$.
(b) Prove that if $\sum a_n$ is conditionally convergent, then $\sum a_n^+$ and $\sum a_n^-$ both diverge.

Solution. We will use the partial sums:

$$s_n = \sum_{k=1}^{n} a_k, \quad s_n^a = \sum_{k=1}^{n} |a_k|$$
$$s_n^+ = \sum_{k=1}^{n} a_k^+, \quad s_n^- = \sum_{k=1}^{n} a_k^-$$

(a) $(\Rightarrow)$

We have $\forall n \in \mathbb{N} : |a_n| \geq a_n^+$ and $|a_n| \geq (-1)a_n^-$. Using the comparison test we find $\sum a_n^+$ and $\sum a_n^-$ converge.

$(\Leftarrow)$ $\sum a_n^+$ and $\sum a_n^-$ converge, so then also $\sum a_n^+ + (-1)\sum a_n^-$ converges. But $s_n^a = s_n^+ + (-1)s_n^-$, so $\sum |a_n|$ converges too.

(b) $\sum a_n$ converges conditionally. If both $\sum a_n^+$ and $\sum a_n^-$ converge, then from (a) we would have $\sum a_n$ converges absolutely, contradicting the premise. So at least one of $\sum a_n^+$ or $\sum a_n^-$ must diverge.

Assume $\sum a_n^+$ diverges (the other case is similar). s_n^+ is monotonically increasing and divergent, so it is unbounded. We have $s_n^+ = s_n - s_n^-$ and s_n is bounded. It follows that s_n^- has to be unbounded, so $\sum a_n^-$ diverges also.

$\square$

Exercise 3.24, page 36

Let $\sum a_n$ be a conditionally convergent series. Prove that for every $\sigma \in \mathbb{R}$ there is a rearrangement of $\sum a_n$ that converges to σ.

Solution. We will construct this rearrangement.

We know from the previous exercise that both $\sum a_n^+$ and $\sum a_n^-$ diverge and both s_n^+ and s_n^- are unbounded.

Assume first that $\sigma > 0$ (the other case is similar). Since s_n^+ is unbounded, there exists[3] a $N_1 \in \mathbb{N}$ such that

$$\sum_{k=1}^{N_1-1} a_k^+ \leq \sigma$$
$$\sum_{k=1}^{N_1} a_k^+ > \sigma$$

Let $d_1 = |\sum_{k=1}^{N_1} a_k^+ - \sigma|$. We see that $0 < d_1 \leq |a_{N_1}^+|$. Our rearrangement will start with the first N_1 terms from $\sum a_n^+$. For the next terms we turn to $\sum a_n^-$. s_n^- is also unbounded, so there exists a $M_1 \in \mathbb{N}$ such that

[3] This N_1 has to exist because s_n^+ is unbounded. If it was only zeros it would converge and be bounded.

$$\sum_{k=1}^{M_1-1} a_k^- \geq d_1$$
$$\sum_{k=1}^{M_1} a_k^- < d_1$$

We add the first M_1 terms from $\sum a_n^-$ to the rearrangement. Let $d_2 = |\sum_{k=1}^{N_1} a_k^+ + \sum_{k=1}^{M_1} a_k^- - \sigma|$. We see that $0 < d_2 \leq |a_{M_1}^-|$.

Next we go back to $\sum a_n^+$ for more terms. The tail of $\sum a_n^+$ starting at $N_1 + 1$ is also unbounded, so there must exist a N_2 such that

$$\sum_{k=N_1+1}^{N_2-1} a_k^+ \leq d_2$$
$$\sum_{k=N_1+1}^{N_2} a_k^+ > d_2$$

We add the terms $\sum_{k=N_1+1}^{N_2} a_k^+$ to the rearrangement and define

$$d_3 = |\sum_{k=1}^{N_1} a_k^+ + \sum_{k=1}^{M_1} a_k^- + \sum_{k=N_1+1}^{N_2} a_k^+ - \sigma|$$

We see that $0 < d_3 \leq |a_{N_2}^+|$.

We go back down with the help of terms from the tail of $\sum a_n^-$ starting at M_1, a tail that is also unbounded. There must exist a M_2 such that

$$\sum_{k=M_1+1}^{M_2-1} a_k^+ \geq d_3$$
$$\sum_{k=M_1+1}^{M_2} a_k^+ < d_3$$

We add the terms $\sum_{k=M_1+1}^{M_2} a_k^-$ to the rearrangement and define

$$d_4 = |\sum_{k=1}^{N_1} a_k^+ + \sum_{k=1}^{M_1} a_k^- + \sum_{k=N_1+1}^{N_2} a_k^+ + \sum_{k=M_1+1}^{M_2} a_k^- - \sigma|$$

We see that $0 < d_4 \leq |a_{M_2}^-|$.

We continue in this way, switching between terms in $\sum a_n^+$ and $\sum a_n^-$, constructing a rearrangement of $\sum a_n$ that has partial sums that have distance d_n from σ.

The sequence (d_n) of distances is bounded by $(|a_n|)$ and $\sum a_n$ is a conditionally convergent series, so $a_n \to 0$. That means that $d_n \to 0$ and the rearrangement converges to σ.

$\square$

Bibliography

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