Method of Characteristics for nozzle design.

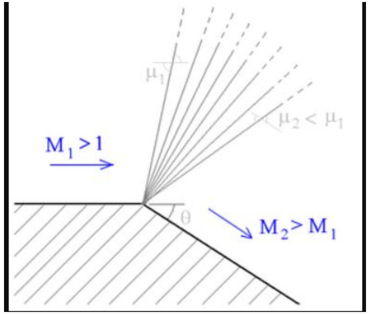
The project's aim was to computationally solve the potential flow equation describing compressible flows. This is a hyperbolic PDE which can be solved by using the Method of Characteristics, a numerical technique for the same has been developed and discussed in sections ahead.

https://github.com/v-Pranshu/MOC_NozzleDesign

Introduction

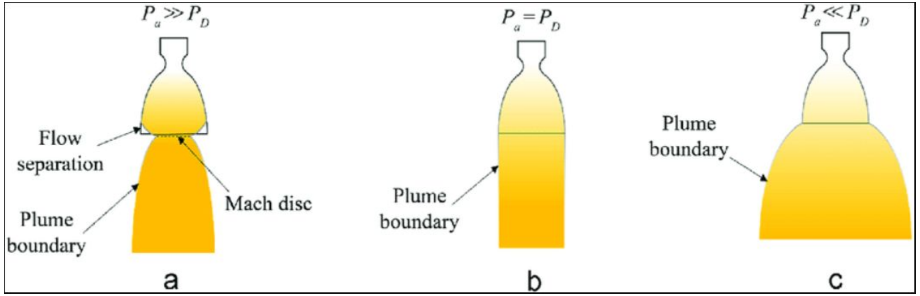
Nozzles have varying cross sectional areas and are used for accelerating an incoming flow. One application of nozzle is in rockets where they are used to provide a high exhaust velocity to the combusted products consequently getting optimal thrust.

For such a use case, the flow accelerates from subsonic conditions to supersonic at the exit, this is achieved by using a convergent divergent nozzle. Flow is Mach 1 for throat section after which the divergent section starts. This sudden increase in area causes the flow to turn around a convex corner leading to formation of shock waves known as Prandtl- Meyer expansion fan.



Design objectives

For obtaining optimum expansion at the exit of the nozzle, it is necessary to cancel out these shock waves produced by designing the wall contour such that it matches the angle of incident shock wave, consequently removing internal reflection and allowing for a uniform exit flow at the exit with a desired Mach number. Such a nozzle is also used for supersonic wind tunnels where uniform flow is desired at the inlet.



The potential equation

The potential equation is the primary equation that is solved to identify various parameters of the shock wave especially the angle of incidence of the shock which will decide the angles of the nozzle contour.

$$\left[1 - \frac{1}{a^2}\left(\frac{\partial \phi}{\partial x}\right)^2\right]\frac{\partial^2 \phi}{\partial x^2} + \left[1 - \frac{1}{a^2}\left(\frac{\partial \phi}{\partial y}\right)^2\right]\frac{\partial^2 \phi}{\partial y^2} - \frac{2}{a^2}\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x \, \partial y} = 0 \tag{11.12}$$

The above Partial Differential equation is hyperbolic in nature and therefore can be solved by method of characteristics. This basically converts the PDE to an ODE by using compatibility equation which is easily solvable.

$$\frac{\partial^2 \Phi}{\partial x \partial y} = \begin{vmatrix} 1-\frac{u^2}{a^2} & 0 & 1-\frac{v^2}{a^2} \\ dx & du & 0 \\ 0 & dv & dy \end{vmatrix} \begin{vmatrix} 1-\frac{u^2}{a^2} & -\frac{2uv}{a} & 1-\frac{v^2}{a^2} \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}^{-1} = \frac{N}{D}$$

Solving the equations and using Cramer's rule we get the above expression for Uxy which is indeterminate along the characteristics therefore N = 0 (compatibility condition) and D = 0

For D = 0, we get

$$\left(\frac{dy}{dx}\right)_{char} = \tan(\theta \mp \mu)$$

For N = 0, we get ,

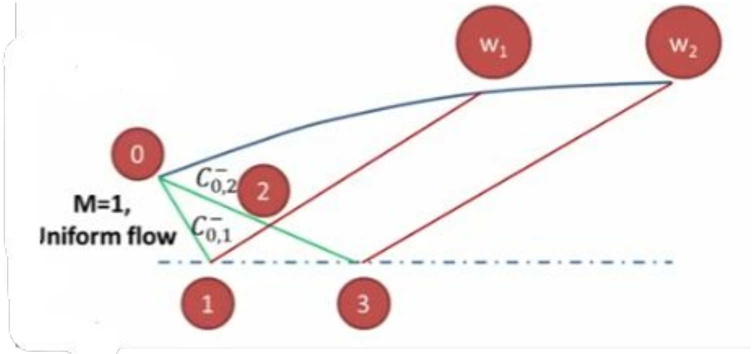
$$d\theta = -\sqrt{M^2 - 1}\frac{dV}{V} \qquad \text{along } C_-$$

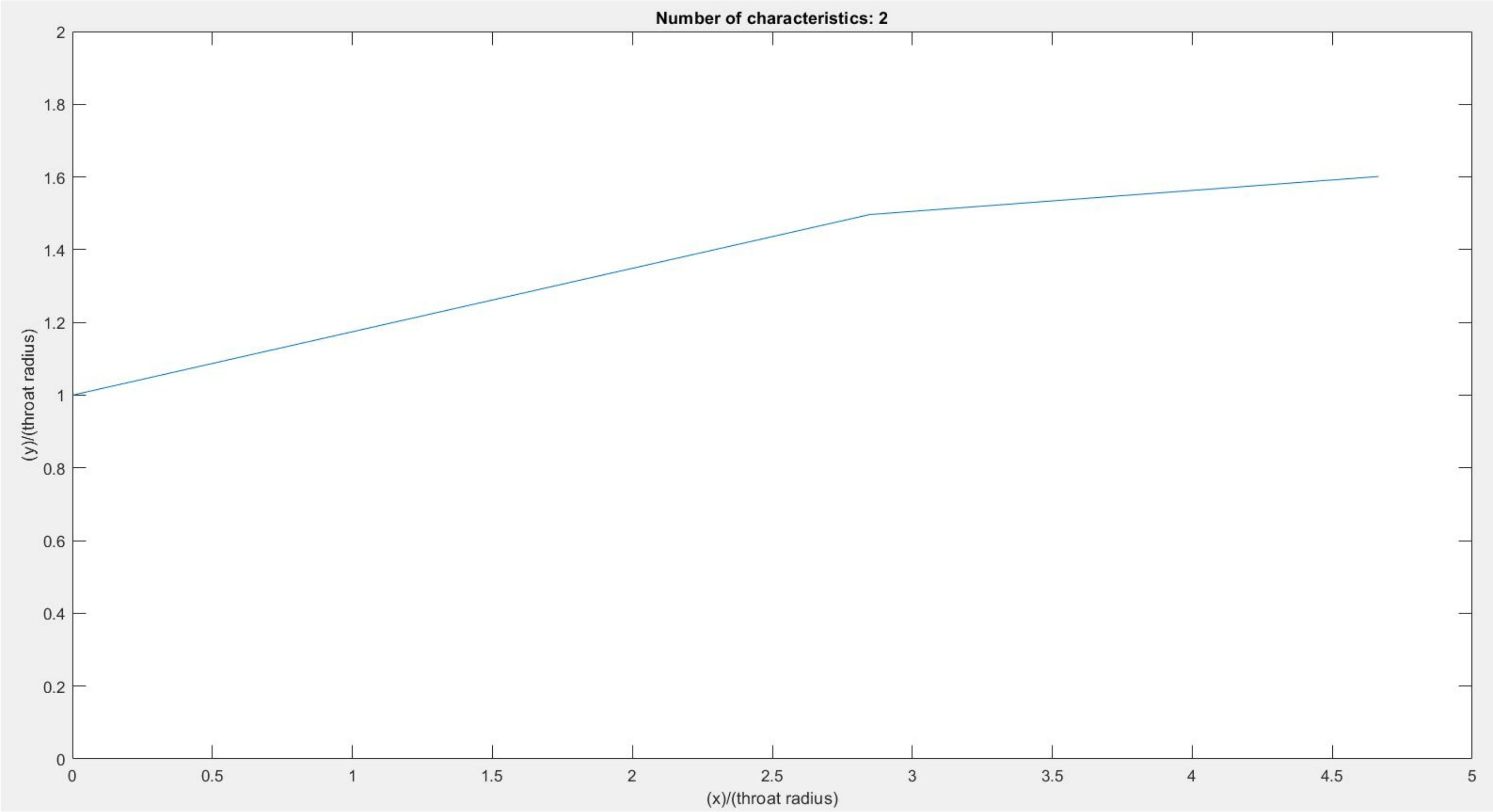
$$d\theta = \sqrt{M^2 - 1}\frac{dV}{V} \qquad \text{along } C_+$$

Methodology

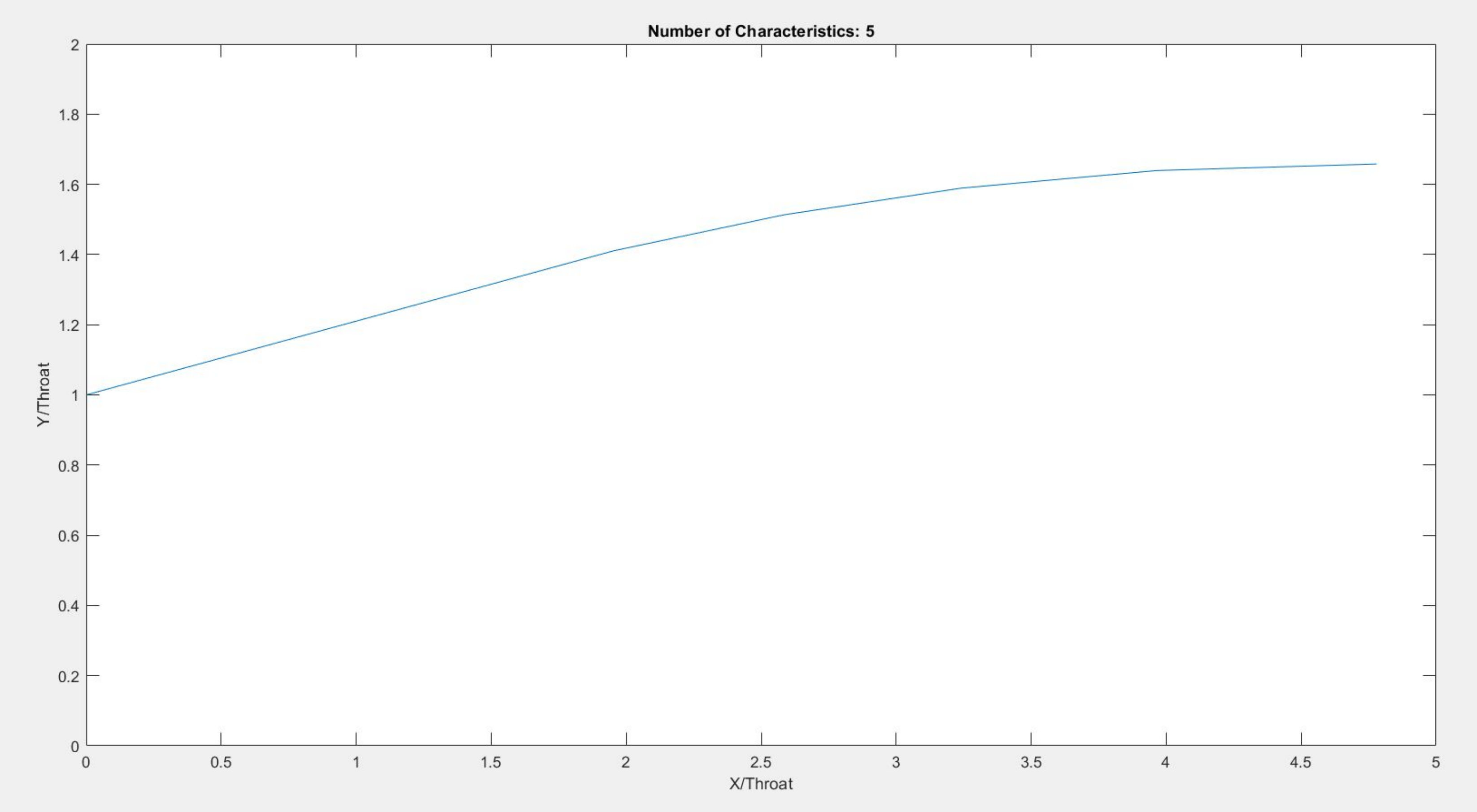
Consider the figure shown, here two characteristics/shock waves are shown that are reflected of the centre line of the nozzle to represent symmetry. Points 1 and 3 are centreline points, and w1 and w2 are wall points. The green characteristics lines represent the left running characteristics and the red lines represent the right running ones.

Using the equations described previously, the coordinates of the centre points are obtained and the corresponding theta values. Points like 2 are obtained by finding the intersection of the right running characteristic and C- 0,2. After finding point 2 and parameters for characteristic emanating from it, the wall point w1’s theta value can be calculated as it lies on the same wave. A similar procedure is repeated to find coordinates and theta values of all the wall points, these wall points are then joined by straight lines to form the nozzle contour.

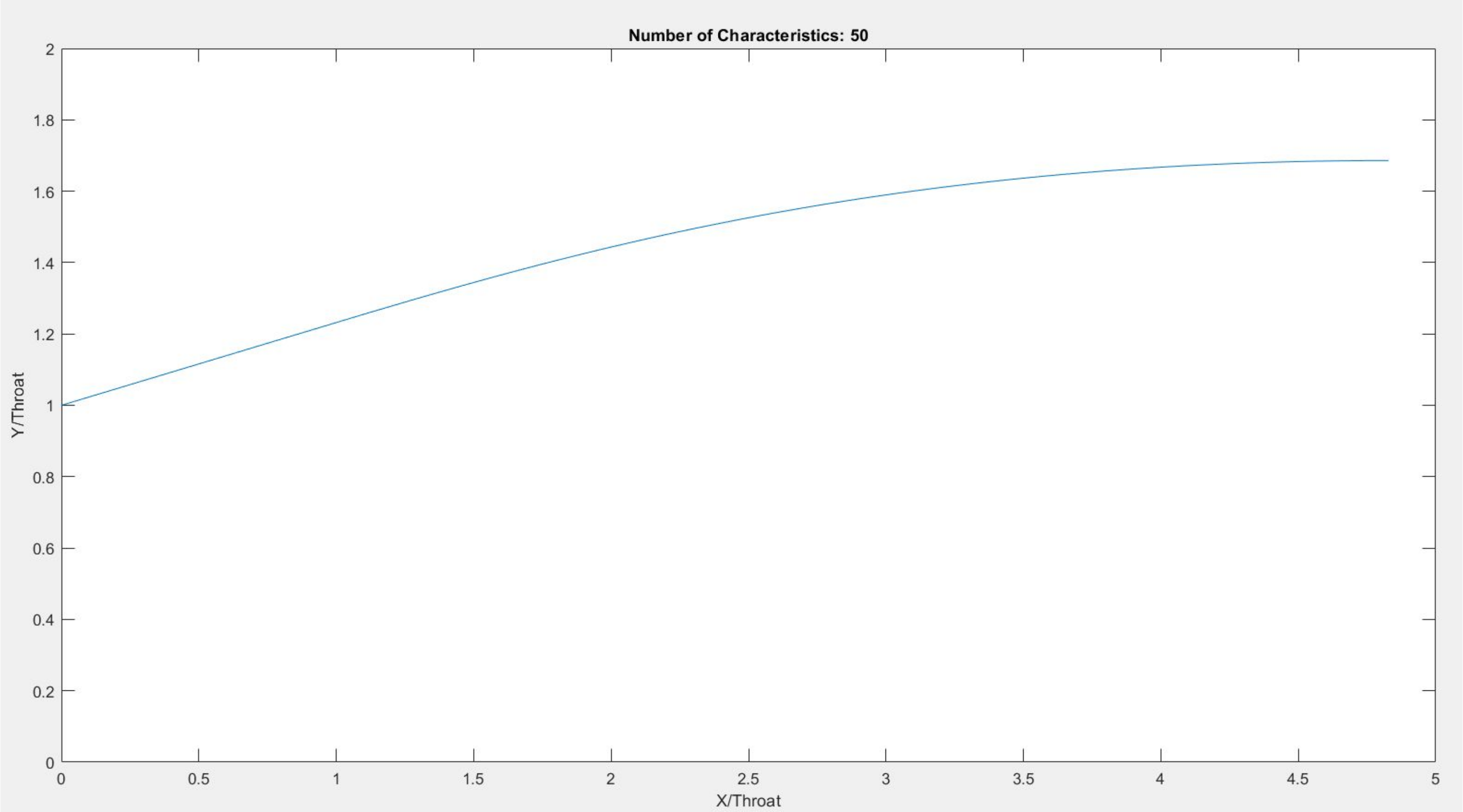




Nozzle contour obtained for desired exit mach of 2 with 2 characteristics considered.



Nozzle contour obtained for desired exit mach of 2 with 5 characteristics considered.



Nozzle contour obtained for desired exit mach of 2 with 50 characteristics considered.

Conclusions

- The potential equation with the assumptions of inviscid, irrotational compressible flow is solved using MOC numerically to parameters related to various characteristics involved.
- The Nozzle contour is designed such that the wall angles match the angle of incidence of shock waves to prevent reflection allowing for uniform exit flow and optimum expansion.
- Increasing the number of characteristics considered for the numerical computation enhances the smoothness of the nozzle wall contour and also its accuracy.
- Although the equation being solved has various assumptions that make it vary from real life situations, results obtained from these MOC computations can be a great starting point for CFD simulations and optimizations to reach the final design.
- Code with in detail algorithm explanation can be found in the github repository:

https://github.com/v-Pranshu/MOC_NozzleDesign

References

- Gasdynamics: Fundamentals and Applications, NPTEL Course IISC Bangalore
- Notes on MOC by IIST
- Ansys MOC course
- NASA Resources on compressible flows
- Fundamentals of Aerodynamics, Textbook by J.D Anderson