Calculate the spin-orbit splitting of the L-shell hole. With a reference state $|0\rangle$ the 2p-hole states span a six-dimensional subspace corresponding to the removal of a $2p$-electron ($2p_x$, $2p_y$, $2p_z$) of spin α or β

$$|p_{i\sigma}^{-1}\rangle = a_{i\sigma}|0\rangle \qquad i = x, y, z \qquad \sigma = \alpha, \beta \tag{1}$$

and the diagonalization of the spin-orbit operator in this basis leads for the atomic case to two degenerate sets of $J = 1/2$ and $J = 3/2$

$$V_{ij}^{\sigma\tau} = \langle p_{i\sigma}^{-1}|V|p_{j\tau}^{-1}\rangle = \langle 0|a_{i\sigma}^{\dagger} V a_{j\tau}|0\rangle \tag{2}$$

Explicit blocking of the spin part yields the matrix

$$\mathbf{V} = \begin{pmatrix} V_{ij}^{\alpha\alpha} & V_{ij}^{\alpha\beta} \\ V_{ij}^{\beta\alpha} & V_{ij}^{\beta\beta} \end{pmatrix} \tag{3}$$

where each subblock is a 3x3 matrix labeld by the p orbitals. Spin-orbit selectron roules give

$$\begin{aligned} V_{ij}^{\alpha\alpha} &= \langle p_{i\alpha}^{-1}|l_z s_z|p_{j\alpha}^{-1}\rangle && (4)\\ V_{ij}^{\beta\beta} &= \langle p_{i\beta}^{-1}|l_z s_z|p_{j\beta}^{-1}\rangle && (5)\\ V_{ij}^{\alpha\beta} &= \langle p_{i\alpha}^{-1}|-l_1 s_{-1}|p_{j\beta}^{-1}\rangle && (6)\\ V_{ij}^{\beta\alpha} &= \langle p_{i\beta}^{-1}|-l_{-1} s_1|p_{j\alpha}^{-1}\rangle && (7)\\ & && (8) \end{aligned}$$

and the off-diagonal blocks can be transformed to a common spin component with the Wigner-Eckart theorem (notation $C_{m_1,m_2,m}^{j_1,j_2,j}$)

$$V_{ij}^{\alpha\beta} = \langle p_{i\beta}^{-1}|-l_1 s_0|p_{j\beta}^{-1}\rangle C_{\frac{1}{2},-1,-\frac{1}{2}}^{\frac{1}{2},1,\frac{1}{2}}/C_{\frac{1}{2},0,\frac{1}{2}}^{\frac{1}{2},1,\frac{1}{2}} = \langle p_{i\beta}^{-1}|-l_1 s_0|p_{j\beta}^{-1}\rangle\sqrt{2/3}/\sqrt{1/3} = \langle p_{i\beta}^{-1}|(l_x+il_y)s_z|p_{j\beta}^{-1}\rangle \tag{9}$$

$$V_{ij}^{\beta\alpha} = \langle p_{i\beta}^{-1}|-l_{-1} s_0|p_{j\beta}^{-1}\rangle C_{-\frac{1}{2},1,\frac{1}{2}}^{\frac{1}{2},1,\frac{1}{2}}/C_{\frac{1}{2},0,\frac{1}{2}}^{\frac{1}{2},1,\frac{1}{2}} = \langle p_{i\beta}^{-1}|-l_{-1} s_0|p_{j\beta}^{-1}\rangle(-\sqrt{2/3})/\sqrt{1/3} = \langle p_{i\beta}^{-1}|(l_x-il_y)s_z|p_{j\beta}^{-1}\rangle \tag{10}$$

and finally

$$V_{ij}^{\alpha\alpha} = \langle p_{i\beta}^{-1}|l_z s_z|p_{j\beta}^{-1}\rangle C_{-\frac{1}{2},0,-\frac{1}{2}}^{\frac{1}{2},1,\frac{1}{2}}/C_{\frac{1}{2},0,\frac{1}{2}}^{\frac{1}{2},1,\frac{1}{2}} = -\langle p_{i\beta}^{-1}|l_z s_z|p_{j\beta}^{-1}\rangle \tag{11}$$

i.e.

$$\mathbf{V} = \langle p_{\imath\beta}^{-1}| \begin{pmatrix} -l_z s_z & (l_x + il_y)s_z \\ (l_x - il_y)s_z & l_z s_z \end{pmatrix} |p_{j\beta}^{-1}\rangle \tag{12}$$