Trigonometric Ratios of Compound Angles

An angle made up of the algebraic sum of two or more angles is called a compound angle. Some of the formulae and results regarding compound angles are:

sin(A + B) = sinA cosB + coA sinB

sin(A – B) = sinA cosB – cosA sinB

cos(A + B) = cosA cosB – sinA sinB

cos(A – B) = cosA cosB + sinA sinB

tan(A + B) = tanA + tan B/ 1 - tan A tanB

tan(A – B) = tanA - tan B/ 1 + tan A tan B

sin(A + B) sin(A – B) = $\sin^2$A – $\sin^2$B = $\cos^2$B – $\cos^2$A.

cos(A + B) = cos(A – B) = $\cos^2$A – $\sin^2$A – $\sin^2$B = $\cos^2$B – $\sin^2$A.

Illustration:

Prove that tan70^o = 2 tan50^o + tan20^o.

Solution:

tan70^o = tan(50^o + 20^o)

= tan 50^o + tan 20^o/1 - tan 50^o. tan 20^o

or, tan70^o (1 – tan 50^o tan20^o) = tan50^o + tan20^o

or, tan70^o = tan70^o tan50^o tan20^o + tan50^o + tan20^o

= cot20^o tan50^o tan20^o + tan50^o + tan20^o

= 2 tan50^o + tan20^o.

Illustration:

If A + B = 45^o, show that (1 + tanA) (1 + tanB) = 2.

Solution:

tan(A + B) = tan A + tan B / 1 - tan A tan B = 1

tanA + tanB + tanA tanB + 1 = 1 + 1

tanA (1 + tanB) + (1 + tanB) = 2

(1 + tanA) (1 + tanB) = 2

Trigonometric Ratios of Multiples of an Angle

sin2A = 2sinA cosA = 2tan A/ 1 + $\tan^2$ A

cos2A = cos2A – sin2A = 1 – 2sin2A = 2 cos2A – 1 = 1 - $\tan^2$ A/1 + $\tan^2$ A

tan2A = 2tan A/ 1 - $\tan^2$ A

sin3A = 3sinA – 4sin3A = 4sin(60^o – A) sinA sin(60^o + A)

cos3A = 4cos3A – 3cosA = 4cos(60^o A) cosA cos(60^o + A)

tan3A = 3tan A - $\tan^3$ A/1 - 3 $\tan^2$ A = tan(60^o – A) tan A tan (60^o + A)

Illustration:

Find the values of (i) sin 18^o (ii) tan 15^o

Solution:

(i) sin 18^o

Let θ = 18^o then 2θ = 36^o = 90^o – 3θ.

Now sin2θ = 2sinθ cosθ and

sin(90^o – 3θ) = $\cos^3\theta$ = 4$\cos^3\theta$ – 3cosθ

Hence we have 2sinθ cosθ = cosθ (4$\cos^2\theta$ – 3) = cosθ (1 – 4$\sin^2\theta$).

Hence, 2 sinθ = 1 – 4$\sin^2\theta$ (as cosθ # 0)

4$\sin^2\theta$ + 2sinθ – 1 = 0   sinθ = -2 $\pm$ √4 + 16 /2.4 = -1 $\pm$ √5/4

But as sinθ > 0 we have sinθ = √5 - 1/4 i.e. sin18^o = √5 + 1/4

(ii) tan 150^o

tan 150^o = tan (60^o – 45^o) = √3 - 1/ 1 + √3 = (√3 - 1)^2/3 -1 = 4 - 2 √3/2 = 2 - √3 .

Sum of sines/cosines in Terms of Products

sin A + sinB = 2sin A + B/2 cos A - B/2

sin A – sin B = 2sin A - B/2 cos A + B/2

cos A + cosB = 2cos A - B/2 cos A + B/2

cos A – cos B = –2sin A - B/2 sin A + B/2 (here notice (B – A))

tan A + tanB = sin (A + B)/ cos A. cos B

Conversely

2sinA cosB = sin(A + B) + sin (A – B)

2cosA sinB = sin(A + B) – sin (A – B)

2cosA cosB = cos(A + B) + cos (A – B)

2sinA sinB = cos(A – B) – cos (A + B)

Illustration:

If α, ß and g are in A.P., show that cotß = sin α - sinγ/ cosγ - cosα .

Solution:

Since α, ß and y are in A.P., 2ß = α + y cotß = α+y/2

= (cosα+y/2) / sinα+y/2 = (2cosα+y/2 sinα+y/2)/ 2sinα+y/2sinα+y/2 = sin α - siny/ cos y - cosα .

Illustration:

Show that sin 12^o.sin48^o.sin54^o = 1/8.

Solution:

L.H.S. = [cos 36^o – cos 60^o]sin 54^o = 1/2 [cos 36^o sin 54^o sin 54^o]

= 1/4 [2 cos 36^o sin 54^o – cos 54^o] = 1/4 [sin 90^o + sin 18^o – sin 54^o]

= 1/4 [1 – (sin 54^o– cos 18^o)] = 1/4 [1 – 2sin 18^o cos 36^o]

= 1/4 [1 - 2sin 18° cos36°] = 1/4 [1 - sin36° cos 36°/ cos18°]

= 1/4 [1 - 2sin 36° cos6°/2cos18°] = 1/4 [1 - sin 72°/2sin72°] = 1/4 [1 - 1/2] = 1/8 = R.H.S.

Alternative Method

Let θ = 12^o.

L.H.S. = 1/sin72° sin 12^o sin 48^o sin 72^o sin 54^o

= 1/ 4sin 72° 4 sin(60^o – 12^o) sin 12^o sin (60^o + 12^o) sin 54^o.

= 1/4 sin3 (12°) sin 54°/ sin72° = sin 36° sin 54°/ 8 sin 36° cos36° = 1/8 = R.H.S.