Bivariate Normal Distribution:-

Real life application:- Heights of parents and their next generation jointly follow multivariate normal distn.

(X, Y) : Bivariate Random vector.

Joint p.d.f. $\leftarrow f_{X,Y}(x,y)$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{-\left\{\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]\right\}},$$

where $x \in \mathbb{R}$, $y \in \mathbb{R}$,

$\mu_x \in \mathbb{R}$, $\mu_y \in \mathbb{R}$, $\sigma_x \in \mathbb{R}^+$, $\sigma_y \in \mathbb{R}^+$, $\rho \in (-1, 1)$.

Fact:- If $f_{X,Y}(x,y) = c \ (>0)$, then we have

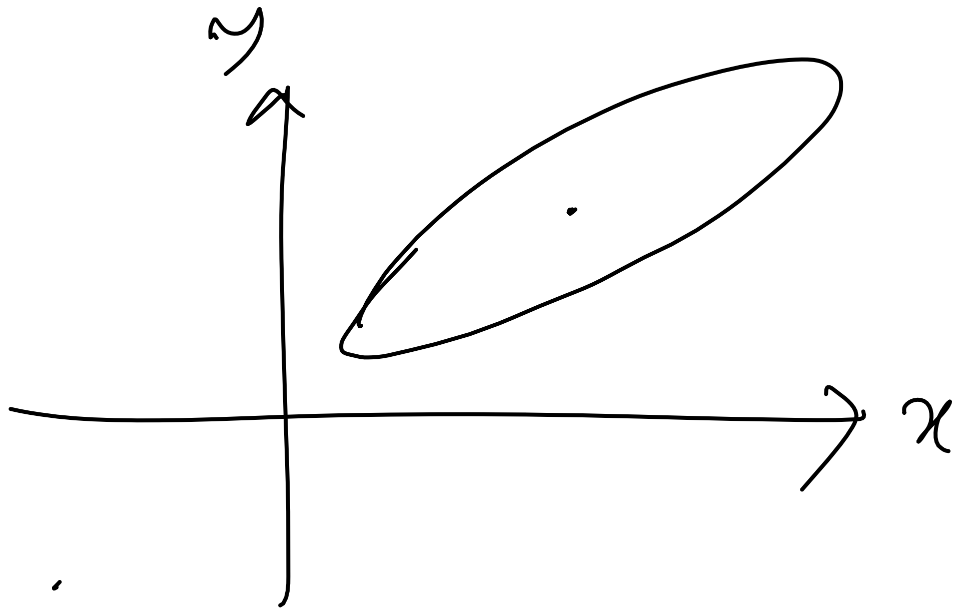
$$f_{X,Y}(x,y) = c$$

$$\Leftrightarrow \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\left[\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right\}\right]} = c.$$

$$\Leftrightarrow \left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) = K$$

some constant

The locus of (x,y) is the equation of ellipse.



Marginal of X:-

$$f_{X,Y}(x,y) = c \times e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\times\left(\frac{y-\mu_y}{\sigma_y}\right)\right]}.$$

$$C = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

$f_X(x)$: Marginal density of X.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy.$$

Consider $\frac{x-\mu_x}{\sigma_x} = u$ & $\frac{y-\mu_y}{\sigma_y} = v$. —— (**)

Check:-

$$f_X(x) = \frac{1}{2\pi\sigma_x\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-\left[\frac{1}{2(1-\rho^2)}\left(u^2+v^2-2\rho uv\right)\right]}dv$$

Observe that $u^2+v^2-2\rho uv = (v-\rho u)^2 + u^2(1-\rho^2)$ —— (*)

Now, we have, using (*) & (**)

$$f_X(x) = \frac{1}{2\pi\sigma_x\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-\left[\frac{1}{2(1-\rho^2)}\left\{(v-\rho u)^2+u^2(1-\rho^2)\right\}\right]}dv$$

$$= \frac{1}{2\pi\sigma_x\sqrt{1-\rho^2}}e^{-\frac{u^2}{2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\{v-\rho u\}^2}dv.$$

$N(\rho u, 1-\rho^2)$

$$= \frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}\times\underbrace{\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}(v-\rho u)^2}dv}_{1}$$

Hence, $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}$ since

$$\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[v-\rho u\right]^2} dv = 1$$ since

the integrand is the normal density with

mean $= \rho u$ & variance $= (1-\rho^2)$.

Therefore, $X \sim N(\mu_x, \sigma_x^2)$.

lly, $Y \sim N(\mu_y, \sigma_y^2)$.

Remark:- If $(X,Y) \sim N_2(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$

Then $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$ & $Y \sim \mathcal{N}(\mu_y, \sigma_y^2)$.

However, $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$ & $Y \sim \mathcal{N}(\mu_y, \sigma_y^2)$

may NOT imply $(X,Y) \sim N_2(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$.

Counter example:- (X,Y) with p.d.f. $f_{X,Y}(x,y)$.

Let us consider

$$f_{X,Y}(x,y) = \{x - 2\Phi(x) - 2\Phi(y) + 4\Phi(x)\Phi(y)\}\, \phi(x)\,\phi(y)$$

Here Φ: CDF of $N(0,1)$ & ϕ: PDF of $\mathcal{N}(0,1)$.

Although $f_{X,Y}$ is NOT $N_2(\cdots)$ but $f_X(x)$ & $f_Y(y)$ are normal density.

Characterization of $N_2(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$:-

Suppose $(X, Y) \sim N_2(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$

$$\iff a_1 X + a_2 Y \sim N(\cdot, \cdot) \quad \forall\ a_1, a_2.$$

$$\overset{\parallel}{\langle \underset{\sim}{a}, \underset{\sim}{Z} \rangle} \sim N(\cdot, \cdot), \text{ where } \underset{\sim}{a} = (a_1\ a_2)$$

$$\underset{\sim}{Z} = (X, Y).$$

<u>Remark</u>:- This result can be extended for any finite dimensional dist^n (or even Hilbert space). In fact, that gives us an idea how to define Gaussian dist^n in Hilbert space)

<u>Example</u>:- Suppose,

$$(X, Y) \sim N_2(0, 0, 1, 1, \rho).$$

Want to show $(5X + 3Y, 10X + 7Y) \sim N_2(\cdot, \cdot)$.

Take for any a_1 & a_2,

$$a_1(5X + 3Y) + a_2(10X + 7Y)$$

$$= (5a_1 + 10a_2)X + (3a_1 + 7a_2)Y = KX + K^* Y \sim N_1(\cdot, \cdot). \quad \forall\ a_1, a_2$$

$$\Rightarrow (5X + 3Y, 10X + 7Y) \sim N_2(\cdot\ \cdot\ \cdot\ \cdot\ \cdot).$$

Independent Random variables :-

$\underset{\sim}{X} = (X_1, \dots, X_d) \sim F_{X_1, \dots, X_d}$.

Joint CDF

F_{X_i} : CDF of X_i, $i = 1, \dots, d$.

Then $X_1, \dots, X_d$ will be independent if

Statement $\leftarrow$ $F_{X_1, \dots, X_d}(x_1, \dots, x_d) = F_{X_1}(x_1) \times F_{X_2}(x_2) \times \dots \times F_{X_d}(x_d)$.

If $f_{X_1, \dots, X_d}$: Joint PDF of $(X_1, \dots, X_d)$

f_{X_i} : Marginal PDF of X_i

Statement :- $(X_1, \dots, X_d)$ will be independent if

$$f_{X_1, \dots, X_d}(x_1, \dots, x_d) = 1 \times f_{X_1}(x_1) \times f_{X_2}(x_2) \dots \times f_{X_d}(x_d).$$

Remark:- $(X,Y) \sim N_2(0,0,1,1,\rho)$

$$\propto e^{-\frac{1}{2(1-\rho^2)}\left[x^2+y^2-2\rho xy\right]} \longrightarrow f_{X,Y}$$

If $\rho = 0$, then $f_{X,Y} = k_1 e^{-\frac{x^2}{2}} \times \underline{k_2 e^{-\frac{y^2}{2}}}$

$N(0,1) \times N(0,1)$.

Hence, $\rho = 0 \Rightarrow X$ is indep of Y.