A few results :-

Result 1 :- Let $Z = F(X)$, [X is continuous], then Z has uniform distⁿ over $[0, 1]$.

Proof :- $F_Z(z) = P[Z \le z] = P[F(X) \le z]$

CDF of Z

$= P[X \le F^{-1}(z)]$ because F is a strictly increasing on $\mathbb{R}$.

$= F(F^{-1}(z))$,

$= z$, $z \in [0, 1]$.

This implies that

$$Z \sim \text{Unif}\,[0, 1].$$

<u>Result 2:-</u> Let U be a random variable associated with Unif $[0,1]$ distn., and suppose that F is a contn. CDF. Then the CDF of $X := F^{-1}(U)$ will be F.

<u>Proof:-</u>
$$F_X(x) = P[X \leq x] = P[F^{-1}(U) \leq x]$$
$$= P[U \leq F(x)]$$
$$= F(x) \quad \forall x$$

F_X = CDF of X

since F is a strictly increasing fn.

Hence $X := F^{-1}(U)$ has the CDF F.

<u>Remark:-</u> If we want to generate the data from a contn. distn. F, it will be good enough to generate the data from $U[0,1]$, and then $F^{-1}(U)$ will be the data from F.

Explanation:- Suppose 0·53 is an obs. obtained from $U(0,1)$. Using this, we want to generate an obs. from exp. distn.

exp.

p.d.f. $f_X(x) = e^{-x}$, $x \geq 0$.

C.D.F. $F_X(x) = 1 - e^{-x}$

Hence, the obs. corr. to exp. distn. will be the soln. of $1 - e^{-x} = 0.53$

$\Leftrightarrow x = ??$

New Section

Joint dist^{n} :-

Real Example :- X : Weight, Y : Alcoholic, Z : Height.

Want to ~~see~~ the effect of (X, Y, Z) on the blood pressure. Here to study this problem, have to study the joint dist^{n} of (X, Y, Z).

Joint dist^{n} :- X & Y : Two random variables.

$F_{X,Y}$: Cumulative dist^{n} f^{n} of (X, Y).

Then $F_{X,Y}(x, y) = P[\omega : X(\omega) \leq x, Y(\omega) \leq y]$

$= P[X \leq x, Y \leq y]$.

$\quad \parallel$

$E\, 1_{\{X \leq x, Y \leq y\}}$

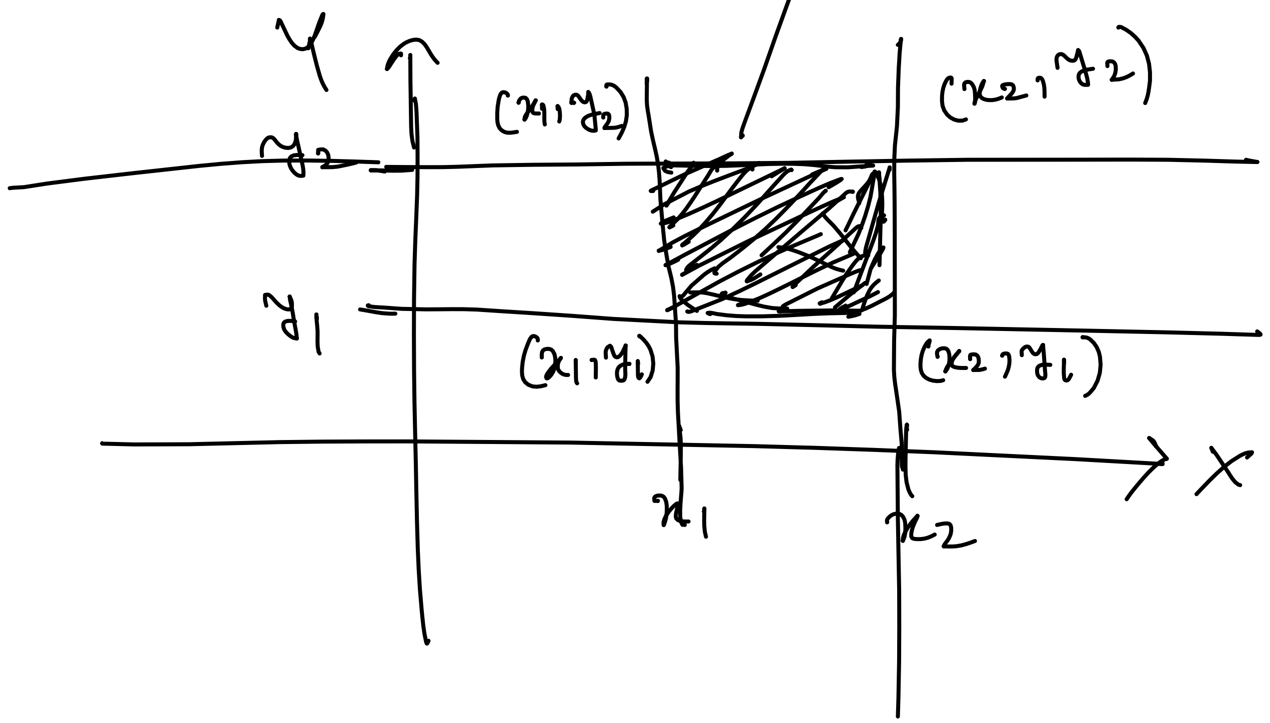
Remark :- This def^{n} can be extended for finitely many random variables.

$$F_{(X_1,\cdots,X_n)}(x_1,\cdots,x_n) = P[w: X_1(w) \le x_1, X_2(w) \le x_2, \cdots, X_n(w) \le x_n].$$

One crucial observation:- (X,Y): Random vector

$$P[x_1 \le X \le x_2,\ y_1 \le Y \le y_2] = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1)$$

Area of the shaded region.



Note that the shaded area

= Area of the rectangle with the upper corner point (x_2,y_2) − " (x_1,y_2) − " ... (x_2,y_1) + " ... (x_1,y_1)

$$= F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1) + F_{X,Y}(x_1,y_1)$$

Now, Note that since $P(\cdot) \geq 0$, we have for any $x_1 < x_2$ & $y_1 < y_2$, we have

$$F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1) + F_{X,Y}(x_1, y_1) \geq 0. \quad (*)$$

Exercise :- Derive (*) for d-dimensional case ($d > 1$).

Discrete Bivariate (Multivariate) Random vectors :-

(X, Y) : Random vectors.

X : Taking values $x_1, \ldots, x_n$

Y : Taking values $y_1, \ldots, y_m$.

Then joint p.m.f. of (X, Y) is defined as $P_{X,Y}(x_i, y_j) = P[X = x_i, Y = y_j]$.

Remark :- The defn. can be extended for d-dimensional case as well.

Example:-

A fair coin is tossed thrice.

Sample space $\leftarrow \Omega = \{HHH, HHT, HHT, HTT, THH, THT, TTH, TTT\}$.

X: No of heads in the first toss.

Y: No. of heads in three tosses.

X can take either 0 or 1; $X \in \{0, 1\}$,

$Y \in \{0, 1, 2, 3\}$

$x \backslash y$	0	1	2	3
0	1/8	2/8	1/8	0
1	0	1/8	2/8	1/8

check carefully.

P_X: Marginal p.m.f. of X

P_Y: Marginal p.m.f. of Y.

Want to compute

$$P[Y=0] = P[Y=0, X=0] + P[Y=0, X=1]$$

$$= \frac{1}{8} + 0 = \frac{1}{8}.$$

Ily, $P[Y=1] = P[Y=1, X=1] + P[Y=1, X=0]$

$$= \frac{1}{8} + \frac{2}{8} = \frac{3}{8}.$$

Ily, can compute $P[Y=2]$ & $P[Y=3]$.