Range of U & V:-

$U = X_1 + X_2$, $\quad V = X_1 - X_2$, $\quad \Rightarrow \quad X_1 = \frac{U+V}{2}$ & $X_2 = \frac{U-V}{2}$.

$X_1 \geq 0$
$X_2 \geq 0$,

Unconditionally, $\quad U \geq 0$ & $-\infty < V < \infty$.

Conditional Range:-

$$0 < X_1 < \infty$$

$$\Leftrightarrow 0 < \frac{U+V}{2} < \infty$$

$$\Leftrightarrow -V < U < \infty \,/\, \underbrace{-U < V < \infty}_{(1)}$$

Also, $\quad 0 < X_2 < \infty$

$$\Rightarrow 0 < \frac{U-V}{2} < \infty \Leftrightarrow V < U < \infty \,/\, \underbrace{-\infty < V < U}_{(2)}.$$

$$(1) \,\&\, (2) \Rightarrow \max(V, -V) < U < \infty \quad \text{——— } (3)$$

$(3) \Rightarrow$ if $-\infty < V < 0 \Rightarrow -V < U < \infty$

& if $0 < V < \infty \Rightarrow V < U < \infty$

Hence, the joint density of U & V is

$f_{U,V}(u,v) = \frac{1}{2} e^{-u}$ if $-\infty < v < 0$ & $-v < u < \infty$

AND $0 < v < \infty$ & $v < u < \infty$.

Let f_V be the p.d.f. of V.

Then $$f_V(v) = \frac{1}{2}\int_{-v}^{\infty} e^{-u}\,du \quad \text{if} \quad -\infty < v < 0$$

$$= \frac{1}{2} e^{v} \quad \text{when} \quad -\infty < v < 0 \quad \text{—— (4)}$$

And $$f_V(v) = \frac{1}{2}\int_{v}^{\infty} e^{-u}\,du \quad \text{if} \quad 0 < v < \infty$$

$$= \frac{1}{2} e^{-v} \quad \text{when} \quad -\infty < v < 0. \quad \text{—— (5).}$$

Combining (4) & (5), we can write

$$f_V(v) = \frac{1}{2} e^{-|v|}, \quad v \in \mathbb{R}.$$

<u>Remark</u> :- i) If $\mathcal{U} = \{u_1, \ldots, u_n\}$ are r.n. from std. exp.

& $\mathcal{V} = \{v_1, \ldots, v_n\}$ " " " std exp

(moreover, they are indep)

$\Rightarrow \{u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n\}$ will be the r.n. from std. Laplace dist^n.

ii) Note that $f_X(x) = \frac{1}{2} e^{-|x|}$ is ~~no~~ NOT diff. at $x = 0$. Based on earlier result, we can avoid this issue ~~many~~ often.

<u>Example</u> :- If $X_1, \ldots, X_n$ are i.i.d. $N(0,1)$, then

$$Y = \sum_{i=1}^{n} X_i^2 \sim \chi_n^2$$ [Chi-square dist^n. with n degrees of freedom].

Q :- What is the p.d.f. of Y??

$\underline{\text{Sol}^n}$:- $X_i \sim N(0,1) \quad \forall \quad i=1,\dots,n.$

Consider the tranformation :- (Multivariate extension of polar tranformation)

$$X_1 = \sqrt{Y} \cos\theta_1$$

$$X_2 = \sqrt{Y} \sin\theta_1 \cos\theta_2$$

$$\vdots$$

$$X_{n-1} = \sqrt{Y} \sin\theta_1 \cdots \sin\theta_{n-2} \cos\theta_{n-1}$$

$$X_n = \sqrt{Y} \sin\theta_1 \cdots \sin\theta_{n-2} \sin\theta_{n-1}$$

Tranformation is $(X_1,\dots,X_n) \longrightarrow (Y, \theta_1, \theta_2, \dots, \theta_{n-1})$.

Note that $X_1^2 + \dots + X_n^2 = \sum_{i=1}^{n} X_i^2 = Y.$

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = c \times e^{-\sum_{i=1}^{n} x_i^2}.$$

Hence, using Jacobian method, $f_{Y,\theta_1,\dots,\theta_{n-1}}(y, \theta_1,\dots,\theta_{n-1}) = c \times e^{-\sum_{i=1}^{n} y} |J|.$

It is a know fact that

$|J| = y^{n/2 - 1} \; K(\underset{\sim}{\theta})$, when $\underset{\sim}{\theta} = (\theta_1, \dots, \theta_{n-1})$

$\longmapsto$ some $f^{\underline{n}}$ of θ.

Hence, the joint density of $Y, \theta_1, \dots, \theta_{n-1}$ is

$$f_{Y, \theta_1, \dots, \theta_{n-1}}(y, \theta_1, \dots, \theta_{n-1}) = c \times e^{-\frac{1}{2} y} \times y^{n/2 - 1} \; K(\underset{\sim}{\theta}).$$

$\longmapsto$ It is clear that $Y, (\theta_1, \dots, \theta_{n-1})$ are indp dist^{n}.

Let f_Y be the p.d.f. of Y.

Then
$$f_Y(y) = \underset{\theta_1 \cdots \theta_{n-1}}{\int\int \cdots \int} c \cdot e^{-\frac{1}{2} y} y^{n/2 - 1} \; K(\underset{\sim}{\theta}) \, d\underset{\sim}{\theta}$$

$$= c_* e^{-\frac{1}{2} y} y^{n/2 - 1}, \quad y \geq 0.$$

This is the p.d.f. of $Y = \sum_{i=1}^{n} x_i^2 \sim \chi^2_n$.

Remark:- If $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$,

then $\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_n$. (directly follows from the earlier result)

$X_i \sim N(\mu, \sigma^2)$

$\Rightarrow \frac{X_i - \mu}{\sigma} \sim N(0,1)$

Approach 3:- $\underset{\sim}{X} = (X_1, \ldots, X_n)$

Q:- Interested to know the dist$^{\underline{n}}$. of $U = U(X_1, \ldots, X_n)$.

Procedure:- i) Derive the MGF of U.

ii) Check that $M_U(t)$ is coinciding with the MGF of any other dist$^{\underline{n}}$. (Random variable) or not.

Example:-

Suppose $X_1, \dots, X_n \sim$ Poin (λ_i).

Want to know the distn of $\sum_{i=1}^{n} X_i$.

Let $Y = \sum_{i=1}^{n} X_i$

MGF of Y

$$M_Y(t) = E\left[e^{tY}\right] = E\left[e^{t\sum_{i=1}^{n} X_i}\right] = \prod_{i=1}^{n} M_{X_i}(t)$$

since X_i's are indep.

$$= \prod_{i=1}^{n} \left\{ e^{\lambda_i (e^t - 1)} \right\}$$

$$= e^{\sum_{i=1}^{n} \lambda_i (e^t - 1)}.$$

$\hookrightarrow$ MGF of Poisson $\left(\sum_{i=1}^{n} \lambda_i\right)$.

Hence, $Y \sim$ Poisso $\left(\sum_{i=1}^{n} \lambda_i\right)$.

Example:-

$X_i \sim N(\mu_i, \sigma_i^2)$, X_i's are indep., $i = 1, \ldots, n$.

Want to know the distⁿ of $Y = a + \sum_{i=1}^{n} k_i X_i$.

$$M_Y(t) = E[e^{tY}] = E\left[e^{t\left(\sum_{i=1}^{n} k_i X_i + a\right)}\right]$$

$$= e^{at} E\left[e^{t\sum_{i=1}^{n} k_i X_i}\right]$$

[Now, $e^{t\sum_{i=1}^{n} k_i x_i} = \prod_{i=1}^{n} e^{t k_i x_i} = \prod_{i=1}^{n} e^{l_i x_i}$, where $l_i = t k_i$.]

$$= e^{at} E\left[e^{t\sum_{i=1}^{n} \cancel{k_i} x_i}\right]$$

$$= e^{at} \prod_{i=1}^{n} E\left[e^{l_i X_i}\right]$$ because X_i's are indep.

$$= e^{at} \prod_{i=1}^{n} \left(e^{l_i \mu_i + \frac{1}{2} l_i^2 \sigma_i^2}\right)$$ (Using the MGF of $N(\mu, \sigma^2)$. $X \sim N(\mu, \sigma^2)$, $M_X(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}$.)

$$= e^{at} \times \left[e^{\sum_{i=1}^{n} l_i \mu_i + \frac{1}{2}\sum_{i=1}^{n} l_i^2 \sigma_i^2}\right]$$

Put $u_i = k_i t_i$, we will have

$$M_Y(t) = e^{t\left(a + \sum_{i=1}^{n} k_i \mu_i\right) + \frac{1}{2} t^2 \sum_{i=1}^{n} k_i \sigma_i^2\Big)}.$$

$\checkmark$

This is the MGF of

$$N\left(a + \sum_{i=1}^{n} k_i \mu_i, \sum_{i=1}^{n} k_i^2 \sigma_i^2\right).$$