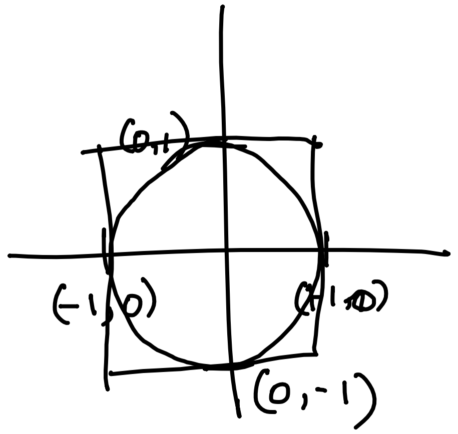
Applications :-

Needs to estimate π



Random experiment :- (X_1, X_2) is $\mathbb{R}^2$-random vector uniformly distributed over $[-1,1] \times [-1,1]$

Want to compute the prob. of $X_1^2 + X_2^2 \leq 1$.

Hence, that prob. will be $= \dfrac{\pi (1)^2}{2 \times 2} = \dfrac{\pi}{4}$.

$$P[X_1^2 + X_2^2 \leq 1] = \frac{\pi}{4} \quad \text{when } (X_1, X_2) \sim \text{Unif}\big[[-1,1] \times [-1,1]\big].$$

$$\Longleftrightarrow E\, \mathbb{1}_{\{X_1^2 + X_2^2 \leq 1\}} = \frac{\pi}{4}.$$

Algorithm :- i) Generate (x_{1i}, x_{2i}) from Unif $\big[[-1,1] \times [-1,1]\big]$, $i = 1 \cdots N$ ($N = 1000$).

ii) The proportion of times $x_{1i}^2 + x_{2i}^2 \leq 1$ will give you an estimate of $\frac{\pi}{4}$.

Another mode of convergence :-

Convergence in distⁿ :- (Weak convergence) :- Suppose $X_1, \dots, X_n$ is a sequence of random variables.

$\{X_n\} \xrightarrow{d} X$ (which is another random variable, but indp. of n) as $n \to \infty$

if $\{F_{X_n}(x)\} \longrightarrow F_X(x)$ ~~∀~~ for all $x \in C(F)$

F_{X_n}: CDF of X_n

F_X: CDF of X

$C(F)$ = {collection of continuity points of F}.

Examples :-

i) $Y_1, \dots, Y_n$ i.i.d. $N(0,1)$.

$$X_n := \frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow{d} ??$$

$$X_n \sim N\left(0, \frac{1}{n}\right)$$

p.d.f. of $N(0, \frac{1}{n})$

$$F_{X_n}(x) = \int_{-\infty}^{x} \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{nt^2}{2}} dt = \int_{-\infty}^{\sqrt{n}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

$$\xrightarrow{n\to\infty} \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

CDF of converging r.v. X

$$\text{Consider } F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Hence, $\{X_n\} \xrightarrow{d} X$, i.e, X is a degenerate random variable, which is degenerated at 0.

Example:- Suppose $Y_1, \ldots, Y_n$ i.i.d. $\text{Unif}(0, \theta)$, $\theta > 0$.

Want to know the limiting distn. of $\underset{Z_n}{\underset{\parallel}{n(\theta - Y_{(n)})}}$

Want to know $F_{Z_n}(x) \xrightarrow{d} ??$

Note that $F_{Z_n}(x) = P[Z_n \le x] = P[n(\theta - Y_{(n)}) \le x]$

$$= P\left[Y_{(n)} \ge \theta - \frac{x}{n}\right]$$

$$= 1 - \underbrace{P\left[Y_{(n)} < \theta - \frac{x}{n}\right]}_{\text{straightforward}}$$

Check:-

$$= \begin{cases} 1 & \text{if } x \ge n\theta. \\ 1 - \left(\frac{\theta - \frac{x}{n}}{\theta}\right)^n & \text{if } 0 < x < n\theta \\ 0 & \text{if } x < 0 \end{cases}$$

Further, observe that

$$F_{Z_n}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\frac{x}{\theta}} & \text{if } x \geq 0 \end{cases}$$

$\hookrightarrow$ This is the CDF of exp. distn. with mean θ.

Hence, $n(\theta - Y_{(n)}) \xrightarrow{d} L$, which is a random variable associated with exp. distn. with mean θ.

<u>Toolkit</u> :- (Proving convergence in distn. using MGF) :-

<u>Statement</u> :- $\{X_n\} \xrightarrow{d} X$ iff $M_{X_n}(t) \longrightarrow M_X(t) \;\forall t$

$M_{X_n}(t)$: MGF of X_n $\quad$ $M_X(t)$: MGF of X.

Example:- $X_n \sim \chi^2_n$

Want to know the limiting distⁿ of $Y_n = \frac{X_n - n}{\sqrt{2n}}$

Fact:- $M_{X_n}(t) = (1-2t)^{-\frac{n}{2}} \longrightarrow$ MGF of X_n

Consider $M_{Y_n}(t) = E\left[e^{t\left(\frac{\cancel{X}_n - n}{\sqrt{2n}}\right)}\right]$

$$= e^{-\frac{tn}{\sqrt{2n}}} E\left[e^{\frac{tX_n}{\sqrt{2n}}}\right]$$

$$= e^{-\frac{tn}{\sqrt{2n}}}\left(1 - \frac{2t}{\sqrt{2n}}\right)^{-\frac{n}{2}} \quad \left[\text{since } M_{X_n}(t) = (1-2t)^{-\frac{n}{2}}\right]$$

$$= \left(e^{t\sqrt{\frac{2}{n}}}\right)^{-\frac{n}{2}} \times \left(1 - \sqrt{\frac{2}{n}}\,t\right)^{-\frac{n}{2}}$$

$$= \left(e^{t\sqrt{\frac{2}{n}}} - t\sqrt{\frac{2}{n}}\, e^{t\sqrt{\frac{2}{n}}}\right)^{-\frac{n}{2}}$$

$$= \left[\left(1 + t\sqrt{\frac{2}{n}} + \frac{t^2}{2!}\frac{2}{n} \times \frac{t^3}{3!} \cancel{\times} \frac{2^{3/2}}{n^{3/2}} + \cdots \right) - t\sqrt{\frac{2}{n}} \left(1 + t\sqrt{\frac{2}{n}} + \frac{t^2 / \frac{2}{n}}{2!} + \cdots \right) \right]^{-\frac{n}{2}}$$

$$= \left[1 - \frac{t^2}{n} + \frac{K_1}{n^{3/2}} + \cdots \right]^{-\frac{n}{2}}.$$

$$\longrightarrow e^{-\frac{t^2}{2}} \quad \text{as } n \to \infty$$

$\hookrightarrow$ MGF of $N(0,1)$.

Hence, $\frac{X_n - n}{\sqrt{2n}} \xrightarrow{d} Z \sim N(0,1)$.