One real example (Theorem of Total prob) :-

(It is study happened in England in 1954).

U: Upper income group.

M: Middle income group.

L: Lower " ".

U_1, M_1, L_1: Income of parent's generation.

U_2, M_2, L_2: " " present "

It is also given that $P(U_1) = 0.1$, $P(M_1) = 0.4$, $P(L_1) = 0.5$.

	U_2	M_2	L_2	
U_1	0.45	0.48	0.07	1
M_1	0.05	0.70	0.25	1
L_1	0.01	0.50	0.49	1

The figures in the table indicate that

$P(U_2|U_1) = 0.45$, $P(M_2|U_1) = 0.48$

$P(L_2|U_1) = 0.07$ ---- so on.

Suppose, want to compute $P(U_2)$

$$P(U_2) = P(U_2 \cap U_1) + P(U_2 \cap M_1) + P(U_2 \cap L_1).$$

// Multiplication law

$$= P(U_2|U_1)\,P(U_1) + P(U_2|M_1)\,P(M_1) + P(U_2|L_1)\,P(L_1)$$

Theorem of total prob.

$$= 0.45 \times 0.10 + 0.05 \times 0.40 + 0.01 \times 0.50 = 0.07$$

Similarly, one can compute $P(M_2)$ & $P(L_2)$ (Try by yourself).

<u>Next issue:-</u> Suppose, we know that an individual belongs to upper income group. Now, want to know that what is the prob. that the parents will belong to upper income group??

So, we want to compute

$$P(U_1 | U_2) = \frac{P(U_1 \cap U_2)}{P(U_2)}$$

Multiplication law

$$= \frac{P(U_2|U_1)\,P(U_1)}{P(U_2|U_1)\,P(U_1) + P(U_2|M_1)\,P(M_1) + P(U_2|L_1)\,P(L_1)}$$

$$= \frac{0.45 \times 0.10}{0.07} \approx 0.64$$

$$\approx (64\%).$$

REMARK:- This compution is essentially based on Bayes' theorem.

Bayes' theorem:-

Let A and $B_1, \ldots, B_n$ be events, where $\bigcup_{i=1}^{n} B_i = \Omega$ and $B_i \cap B_j = \phi \ \forall i \neq j$, and $P(B_i) > 0 \ \forall i$.

Then
$$P(B_j \mid A) = \frac{P(A \mid B_j)\, P(B_j)}{\sum_{i=1}^{n} P(A \mid B_i)\, P(B_i)}.$$

Proof:-

Complete the proof (follow the last derivation).

Another example:- (Breast Cancer) It was studied in UK in 1982.

+ : Having breast cancer.

− : Not having breast cancer.

D : Test is positive.

It is given that

$$P(+) = 0.01 \Rightarrow P(-) = 1 - 0.01 = 0.99$$

$$P(D|+) = 0.8 \quad \& \quad P(D|-) = 0.1.$$

a Suppose we want to know what is the prob. that person is suffering ~~fro~~ by breast cancer when the test is positive.

$$P(+|D) = \frac{P(D|+)\,P(+)}{P(D|+)\,P(+) + P(D|-)\,P(-)}$$

$$= \frac{0.8 \times 0.01}{0.8 \times 0.01 + 0.1 \times 0.99} \approx 0.0747$$

$$\approx 7.5\% !!$$

This gives you an insight about the unconditional & conditional prob. !!

Next topic:- **Independent events:-**

A and B are two indep. events if
$P(A|B) = P(A)$ (i.e, conditional prob = unconditional prob).

$$\iff \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\iff P(A \cap B) = P(A)\,P(B) \quad \text{———} \quad (*)$$

Remark:- (*) can be extented for finitly many events also. Notationally speaking, if $A_1, \dots, A_n$ are indep. then $P(A_1 \cap \dots A_n) = P(A_1)\,P(A_2) \dots P(A_n)$.

A few facts:- i) If $A \underset{\text{indep}}{\perp\!\!\!\perp} B$, then $A^c \underset{\text{indep}}{\perp\!\!\!\perp} B$.

(Proof:- $P(A^c \cap B) = \underbrace{P(B) - P(A \cap B)}_{\text{try to convince}} = P(B) - P(A)P(B)$
$= P(B)[1 - P(A)] = P(B)P(A^c)$)

ii) If $A \perp\!\!\!\perp B$, then $A \perp\!\!\!\perp B^C$.

iii) If $A \perp\!\!\!\perp B$, then $A^C \perp\!\!\!\perp B^C$.

Try to write the proof.

<u>Further concepts:-</u> <u>Pairwise & Mutual independence</u>

<u>Pairwise indp:-</u> Let $A_1, \ldots, A_n$ be events. They are pairwise indp if

$$P(A_i \cap A_j) = P(A_i)\, P(A_j) \quad \forall\, i \neq j.$$

Mutual independence:-

Let $A_1, \dots, A_n$ be n many events. They are mutually indep if

$$P(A_i \cap A_j) = P(A_i)\, P(A_j) \quad \forall\, i \neq j.$$

$$P(A_i \cap A_j \cap A_K) = P(A_i)\, P(A_j)\, P(A_K) \quad \forall\, i \neq j \neq K$$

$$\vdots$$

$$P(A_1 \cap \dots A_n) = P(A_1)\, P(A_2) \cdots P(A_n).$$

Remark:- Mutual indepence $\Rightarrow$ Pairwise indep.
but other way may not true alway.

One counter example:—

Suppose $\Omega = \{1, 2, 3, 4\}$, $A_1 = \{1, 2\}$, $A_2 = \{1, 3\}$ &
$A_3 = \{1, 4\}$.

Note that $P(A_1 \cap A_2) = P(\{1\}) = \frac{1}{4}$.

$$P(A_1)\, P(A_2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

Check that A_1, A_2 & A_3 are pairwise indep.

However, $P(A_1 \cap A_2 \cap A_3) = P(\{1\}) = \frac{1}{4}$.

but $P(A_1)\, P(A_2)\, P(A_3) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

so, $P(A_1 \cap A_2 \cap A_3) \neq P(A_1)\, P(A_2)\, P(A_3)$

Hence, A_1, A_2 & A_3 are NOT mutually indep.