Additive property of χ^2 distⁿ :-

Statement :- If $X \sim \chi^2_m$ & $Y \sim \chi^2_n$, and $X \perp\!\!\!\perp Y$, then $X+Y \sim \chi^2_{m+n}$.

Proof :- Derive $M_{X+Y}(t)$, and try to show that it is same as that of χ^2_{m+n}.

MGF of $(X+Y)$

t- distribution :- Suppose $X \sim N(0,1)$ & $Y \sim \chi^2_n$. Moreover, $X \perp\!\!\!\perp Y$. Then $\frac{X}{\sqrt{\frac{Y}{n}}} \sim t$ distⁿ with n degrees of freedom.

The p.d.f. of the t- distⁿ with n degrees of freedom is

$$f_T(t) = \frac{\Gamma\frac{n+1}{2}}{\sqrt{n\pi}\,\Gamma\frac{n}{2}}\left(1+\frac{t^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}, \quad t \in \mathbb{R}.$$

Remark :- If $n=1$, then $f_T(t)$ is nothing but the standard Cauchy density. If $n \to \infty$, $f_T(t)$ will coincide with the standard normal density (Try to prove it algebraically by yourself).

No moment exists (≥ 1 order)

all moments exist.

A few facts :- i) Standard t- density is symmetric about o.

ii) $n = 1$, $f_T(t) = c \times \frac{1}{(1+t^2)}$, $t \in \mathbb{R}$.

$E(T)$ does NOT exist.

iii) $n=2$, $f_T(t) = c \times \frac{1}{(1+\frac{t^2}{2})^{3/2}}$, $t \in \mathbb{R}$.

Note that here $E(T)$ exists. but $E(T^2)$ does NOT exist.

iv) $n = m$,

$$f_T(t) = c \times \frac{1}{\left(1+\frac{t^2}{m}\right)^{\frac{m+1}{2}}}, \quad t \in \mathbb{R}.$$

$k \geq 1$

$$E(T^k) = c \times \int \frac{t^k}{\left(1+\frac{t^2}{m}\right)^{\frac{m+1}{2}}} \, dt$$

$$= c \times \int \frac{t^k}{\text{polynomial of order } (m+1)} \, dt$$

$$< \infty \quad \text{if} \quad k < m.$$

so, $E(T^k) < \infty$ if $k < m$.

Significance :- In case of testing of hypothesis, suppose you want to test $H_0: \mu = \mu_0$ ag. $H_1: \mu \neq \mu_0$ (when σ is unknown), the test-statistic will follow t-distⁿ..
(data is obtained from $N(\mu, \sigma^2)$)

F - distribution :-

Def^n :- If $U \sim \chi^2_n$ & $V \sim \chi^2_m$, and $U \perp\!\!\!\perp V$.

Then $\dfrac{U/n}{V/m} \sim$ F dist^n with m & n degrees of freedom.

The density f^n of F dist^n is

$$f_W(w) = \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}\right)^{m/2} w^{\frac{m}{2}-1} \left(1+\frac{m}{n}w\right)^{-\frac{m+n}{2}}, \quad w \geq 0.$$

W is a random variable associated with $F_{m,n}$.

Significance:- F dist^n is useful in ANOVA / ANCOVA.

Fact:- $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$.

$$\left.\begin{aligned} \overline{X_n} &\sim N\left(\mu, \frac{\sigma^2}{n}\right) \\ \frac{(n-1) S_n^2}{\sigma^2} &\sim \chi^2_{n-1} \end{aligned}\right\} \text{ indep.}$$

So, $$\frac{\overline{X_n}}{\sqrt{\frac{(n-1) S_n^2}{\sigma^2} \Big/ n-1}} \sim t_{n-1}.$$

(by definition).

Modes of convergence in probability theory :-

i) Convergence in probability (included).

ii) Convergence in distn / weak convergence (included).

iii) Convergence almost surely (excluded).

iv) Convergence in r-th mean (excluded).

Convergence in prob :- Suppose $\{X_n^{(\omega)}\}_{n \geq 1}$ is a sequence of random variables. Then $\{X_n\} \xrightarrow[\text{converges in prob.}]{P} X$ (is a another random variable; in various applications, X can be a constant, i.e., degenerate random variable).

if for every $\varepsilon > 0$,

$$P\left[\omega : |X_n^{(\omega)} - X^{(\omega)}| > \varepsilon\right] \longrightarrow 0 \text{ as } n \to \infty$$

In other words, $\forall\ \varepsilon > 0$,

$$P[w: |X_n(w) - X(w)| \leq \varepsilon] \longrightarrow 1 \text{ as } n \to \infty, \quad \forall \varepsilon > 0.$$

$$\left[\lim_{n\to\infty} P[w: |X_n(w) - X(w)| < \varepsilon] = 1 \right]$$

[Convergence in almost surely :-

$\{X_n\} \xrightarrow{a.s.} X$ if $\forall\ \varepsilon > 0$,

$$P[w: \lim_{n\to\infty} |X_n(w) - X(w)| \leq \varepsilon] = 1.$$]

Remark :- Convergence in almost surely

$\Rightarrow$ Convergence in prob.

But other way ~~is~~ may NOT hold.

A Mathematical ~~sent~~ result :- (a sufficient condition to check convergence in prob.)

Result:- $\{X_n\} \xrightarrow{P} X$ if $E(X_n) \to E(X)$ as $n \to \infty$

and $Var(X_n) \longrightarrow Var(X)$ as $n \to \infty$.

Proof:- $\{X_n\} \xrightarrow{P} X$. Take any $\varepsilon > 0$,

$$P[|X_n - X| > \varepsilon]$$

$$= P[(X_n - X)^2 > \varepsilon^2]$$

$$\leq \frac{E(X_n - X)^2}{\varepsilon^2} \quad [\text{Using Markov's inequality}].$$

a few steps. algebra

$$= \frac{\{E(X_n) - E(X)\}^2 + E[X_n - E(X_n)]^2 + E[X - E(X)]^2}{\varepsilon^2} \quad \circledast$$

(2) $\longrightarrow 0$ as $n \to \infty$

if $E(X_n) - E(X) \longrightarrow 0$ as $n \to \infty$

& $Var(X_n) - Var(X) \longrightarrow 0$ as $n \to \infty$.