Efficiency :- Let $U_n(\underline{X})$ be an unbiased estimator of $g(\theta)$.

The efficiency of $U_n(X)$ is defined by

$$\text{Efficiency}\left[U_n(x)\right] = \frac{\text{CRLB of } g(\theta)}{\text{Var}\left[U_n(x)\right]}$$

Exercise :- Derive the efficiency of sample median when data are generated from $N(\mu, 1)$.

How to find an estimator :-

i) Method of Maximum likelihood estimation. (MLE)

ii) Method of Moment estimation (MME).

What is likelihood :-

<u>Set up :-</u> $X_1, \dots, X_n \overset{i.i.d.}{\sim} f(x; \theta)$

The joint p.d.f. of $X_1, \dots, X_n$ is $f_{X_1, \dots, X_n}(x_1, \dots, x_n; \theta)$

$$= \prod_{i=1}^{n} f_{X_i}(x_i; \theta).$$

$$= L(\theta; x_1, \dots, x_n)$$

likelihood is a f^{n} of θ given the observations $(x_1, \dots, x_n)$.

Defn. of MLE :-

$\hat{\theta}_n$ is the MLE of θ if

$$L(\hat{\theta}_n; x_1, \dots, x_n) > L(\theta; x_1, \dots, x_n) \quad \forall \ \theta \in \Theta .$$

In other words, $\hat{\theta}_n = \underset{\theta \in \Theta}{\operatorname{argmax}} \ L(\theta; x_1, \dots, x_n)$.

Result :- MLE is always a fn. of sufficient stat.

Proof :-

$$L(\theta; x_1, \dots, x_n) = f(x_1, \dots, x_n; \theta)$$

$$\overset{NFT}{=} g(\theta, T_n(x_1, \dots, x_n))\, h(x_1 \dots x_n)$$

$\circledast$

Maximizing $L(\theta; x_1 \dots, x_n)$ w.r.t. θ is equivalent to maximize $g(\theta; T_n(x_1 \dots x_n))$ w.r.t. θ (follows from NFT). Hence, MLE will be a fn of $T_n(x_1, \dots, x_n)$. Proved.

A few examples:–

i) $X_1, \ldots, X_n \overset{\text{i.i.d}}{\sim} \text{Bin}(1, \theta)$, $\theta \in (0,1)$,

Want to find MLE of θ.

$$L(\theta; x_1, \ldots, x_n) = \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n - \sum_{i=1}^{n} x_i}$$

$$\underbrace{\log L}_{\text{log Likelihood}} (\theta; x_1 \ldots x_n) = \sum_{i=1}^{n} x_i \log \theta + \left(n - \sum_{i=1}^{n} x_i\right) \log(1-\theta).$$

$$\frac{\partial \log L(\theta; x_1, \ldots, x_n)}{\partial \theta} = 0 \iff \theta_n^* = \frac{\sum_{i=1}^{n} x_i}{n}.$$

And
$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{\sum x_i}{\theta^2} - \frac{n - \sum_{i=1}^{n} x_i}{(1-\theta)^2} \Bigg|_{\theta = \theta_n^*} < 0$$

Hence, $\theta_n^* = \hat{\theta}_n = \frac{\sum x_i}{n}$ is the MLE of θ.

2) Let $X_1, \dots, X_n \sim \text{Unif}\,[0, \theta]$. $\rightarrow$ Want to find MLE of θ.

$\hookrightarrow$ $x_1 \dots x_n \in [0, \theta]$.

$$L(\theta; x_1 \dots x_n) = \frac{1}{\theta^n} \quad \text{if} \quad 0 \le x_{(1)} \le x_{(2)} \dots \le x_{(n)} \le \theta$$

$$= 0 \quad , \quad \text{o.w.}$$

$\theta \in [X_{(n)}, \infty)$

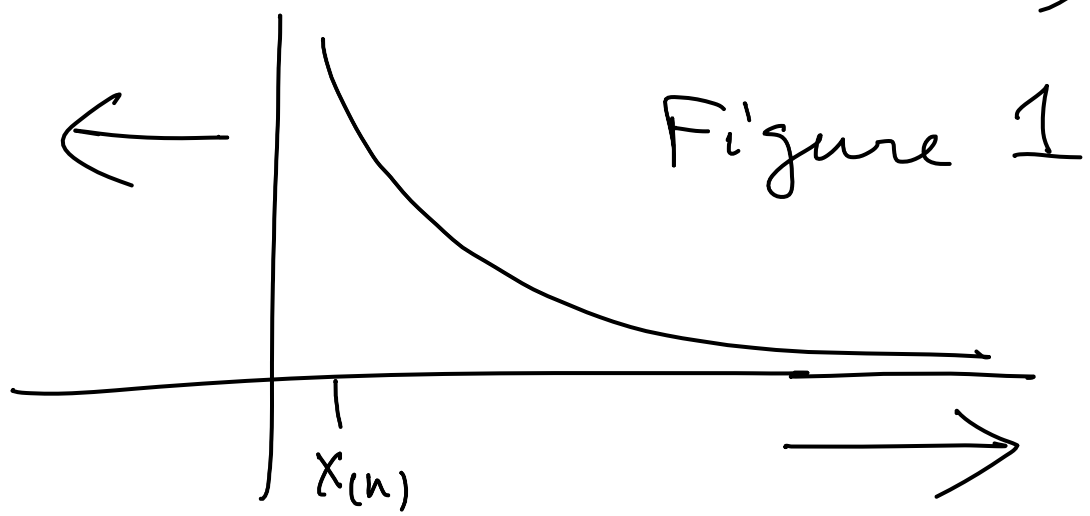


Figure 1.

It is amply indicated by Figure 1 that $\hat{\theta}_{n, MLE} = X_{(n)}$.

3) $X_1, \ldots, X_n \sim \text{Cauchy}(\mu, 1)$.

$$f_X(x) = \frac{1}{\pi\{1+(x-\mu)^2\}}, \quad x \in \mathbb{R}, \quad \mu \in \mathbb{R}.$$

$$L(\mu; x_1, \ldots, x_n) = \frac{1}{\pi^n \left\{\prod_{i=1}^{n} \{1+(x_i-\mu)^2\}\right\}}$$

$$\log L(\mu; x_1, \ldots, x_n) = -n \log \pi - \sum_{i=1}^{n} \log \{1+(x_i-\mu)^2\}.$$

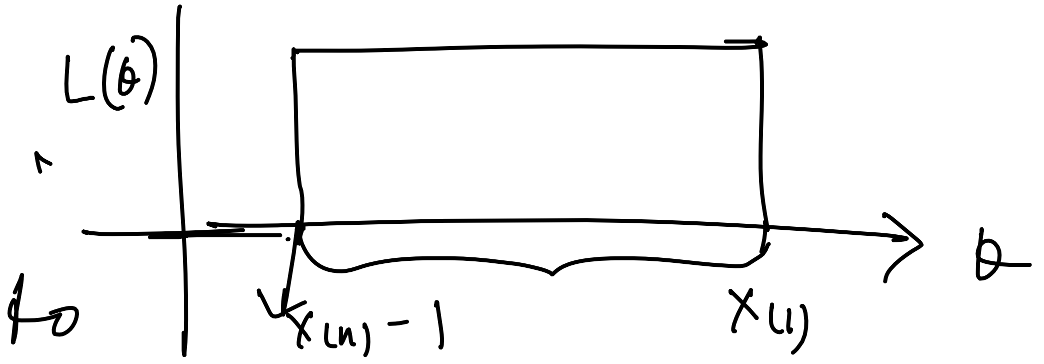
$$\frac{\partial \log L(\mu; x_1, \ldots, x_n)}{\partial \mu} = -\sum_{i=1}^{n} \frac{2(x_i-\mu)(-1)}{\{1+(x_i-\mu)^2\}}$$

<u>Remark:-</u> In principle, MLE exists here; however, the closed form of MLE (of μ) is not available. One needs to apply Numerical method to obtain the MLE of μ!

Example 4:- (MLE may not be unique)

$X_1, \ldots, X_n \underset{\sim}{i.i.d}$ Unif $[\theta, \theta+1]$.

↳ Want to find MLE of θ.



$L(\theta; x_1, \ldots, x_n)$

$= 1$ if $\underbrace{\theta \le x_{(1)}}_{\theta \le X_{(1)}} \le x_{(2)} \cdots \le \underbrace{x_{(n)} \le \theta + 1}_{\Downarrow}$

$\theta \ge X_{(n)} - 1$

$= 0$ o.w.

$\theta \in [X_{(n)} - 1, X_{(1)}]$

Hence, MLE of θ will be any value taken from the interval $[X_{(n)} - 1, X_{(1)}]$.

Hence, MLE is NOT unique here.

Example 5:- Let $X_1, \ldots, X_n \overset{i.i.d}{\sim}$ Laplace $(\mu, 1)$.

$\frac{1}{2} e^{-|x-\mu|}$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$

$$L(\cancel{\mu}; x_1, \ldots, x_n) = \frac{1}{2^n} e^{-\sum_{i=1}^{n} |x_i - \mu|}$$

$$\log L(\mu; x_1, \ldots, x_n) = -n \log 2 - \sum_{i=1}^{n} |x_i - \mu|.$$

Maximizing $\log L(\mu; x_1, \ldots, x_n)$ w.r.t. μ is equivalent to minimizing $\sum_{i=1}^{n} |x_i - \mu|$ w.r.t. μ.

Hence, the median will be the MLE of μ.

Result:- Invariance (Continuous Mapping) property of MLE.

Statement:- If $\hat{\theta}_n$ is the MLE of θ.

$\Rightarrow g(\hat{\theta}_n)$ will be the MLE of $g(\theta)$

as long as g is one-to-one $f^{\underline{n}}$

(strictly increasing $f^{\underline{n}}$)

PF. Θ does NOT include $X_{(n)}$

$L(\theta) = \frac{1}{\theta^n} \underbrace{X_{(n)} < \theta}$ ....

Hence, in that case MLE does NOT exist