St. Petersburg Paradox :-

(prob. of head $= \frac{1}{2} =$ prob of tail)

Scheme of the game:- An unbiassed coin is tossing and will start the game with 2\$. Each time when head will appear, the money will be double & game will end when tail appears.

A few observations:-

1. Tail appears in the first toss, you will 2\$. (with prob. $\frac{1}{2}$.)
2. $\{H\,T\} \longrightarrow$ will gain 4 \$. (with prob $\frac{1}{4}$) $\left(\frac{1}{2}\right) \times \left(1-\frac{1}{2}\right)$
3. $\{H\,H\,T\} \longrightarrow$ " " 8 \$. (with prob $\frac{1}{8}$) $\frac{1}{2} \times \frac{1}{2} \times \left(1-\frac{1}{2}\right)$.

X : Amount of money you will gain.

$$E(X) = \left(2 \times \frac{1}{2}\right) + \left(4 \times \frac{1}{4}\right) + \left(8 \times \frac{1}{8}\right) + \cdots \quad = 1 + 1 + 1 \cdots$$

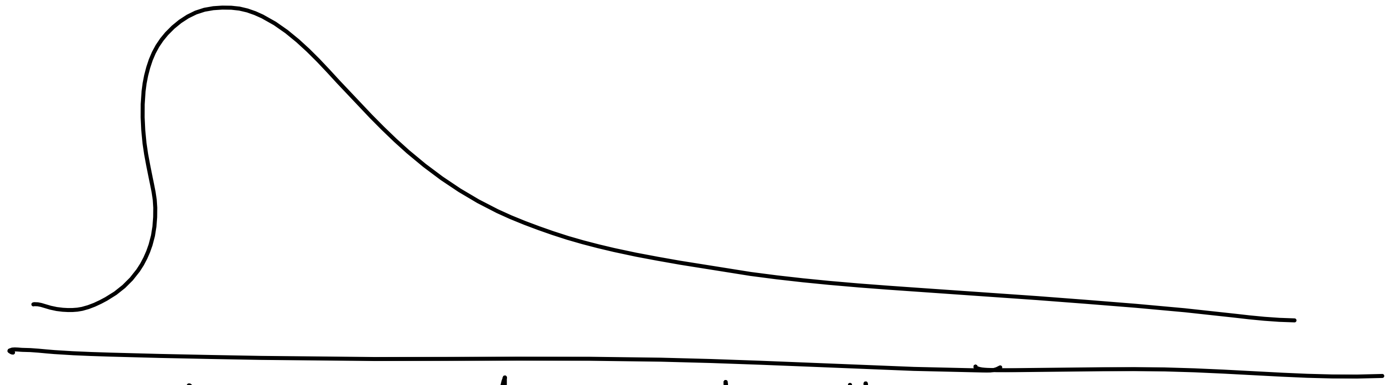
$$= \infty .$$

Some theoretical applications of moments :-

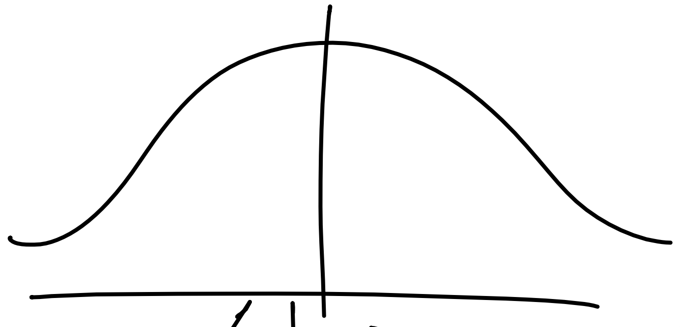
Skewness :- Mathematically, it is defined as

$$\text{skewness} = \frac{E\left[(X - E(X))\right]^3}{\left\{E(X - E(X))^2\right\}^{3/2}},$$

which can give us an idea how much the density f^{n} (pdf) is deviated from the symmetry.



Asymmetric density
(here skewness > 0)

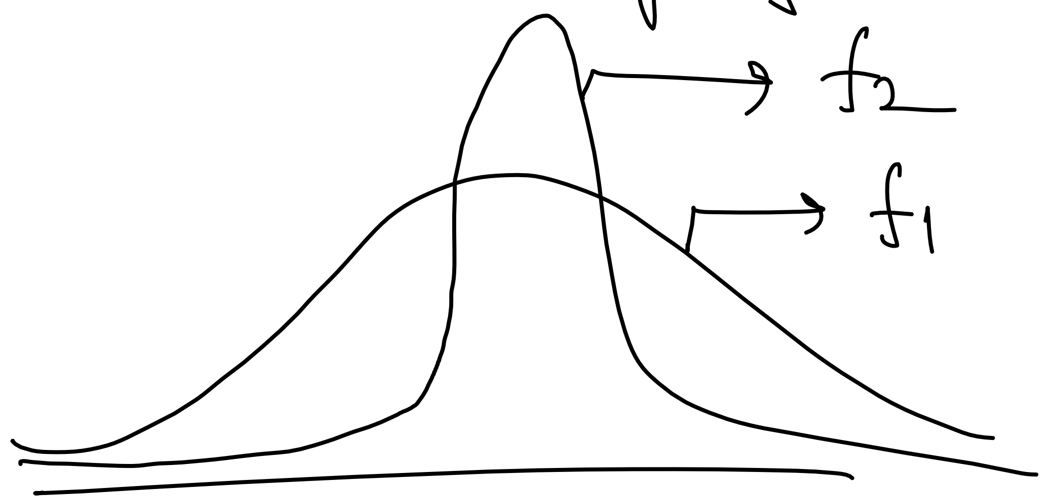


symmetric density

skewness = 0

Asymmetric density
(here skewness < 0).

Kurtosis:- It measures the peakedness of the density f^n:-



It is defined mathematically as

$$= \frac{E\left[(X - E(X))\right]^4}{\left[\underset{\substack{\parallel \\ \text{Variance}}}{E\,(X - E(X))^2}\right]^2}$$

Remark:- In case of real data, $\mathcal{X} = \{x_1, \dots, x_n\}$, you can think about the sample version of skewness

$$= \frac{\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \frac{1}{n}\sum_{i=1}^{n} x_i\right)^3}{\left[\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \frac{1}{n}\sum_{i=1}^{n} x_i\right)^2\right]^{3/2}}$$

and the sample version of kurtosis

$$= \frac{\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \frac{1}{n}\sum_{i=1}^{n} x_i\right)^4}{\left[\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \frac{1}{n}\sum_{i=1}^{n} x_i\right)^2\right]^2}.$$

Another theoretical applications of moments :-

If moments exist, then moments characterize the distribut-ions.

i.e, X & Y : 2 r.vs.

$X \to$ CDF :- F $\qquad$ $Y \to$ CDF : G.

$$\text{If } \quad E(X^s) = E(Y^s) \quad \forall\, s$$

$$\Leftrightarrow \quad X \overset{d}{=} Y$$

$\hookrightarrow$ (Why it is important

$X \sim N(0,1) \to \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \; x \in \mathbb{R}$

$Y \sim \text{Laplace}(0,1) \hookrightarrow \frac{1}{2} e^{-|x|} \; x \in \mathbb{R}.$

$E(X) = E(Y) = 0$

leaves $E(X^2) \neq E(Y^2)$

Real life significance of this result :-

In testing of hypothesis problem (2-sample test), such result enables us to formulate appropriate test statistic

$$\int \left[\widehat{E}(X^s) - \widehat{E}(Y^s)\right]^2 ds$$

$\hookrightarrow$ One of the probable test statistic

A few mathematical results related to Expectation:-

Result 1:- Let $g_1, \ldots, g_r$ be r real valued f^n, and X be an r.v. Then if $E(g_i(X))$ exists for $i = 1, \ldots, r$, then $E\left(\sum_{i=1}^{r} g_i(X)\right)$ exists, and moreover,

$$E\left(\sum_{i=1}^{r} g_i(X)\right) = \sum_{i=1}^{r} E\left(g_i(X)\right).$$

Proof: Existence (discrete case):-

Enough to show that $\sum_{K=1}^{\infty} \left|\sum_{i=1}^{r} g_i(x_K)\right| p_K < \infty$.

Observe that $\sum_{K=1}^{\infty} \left|\sum_{i=1}^{r} g_i(x_K)\right| p_K \le \sum_{K=1}^{\infty} \sum_{i=1}^{r} \left|g_i(x_K)\right| p_K$

(using triangular inequality)

$$= \sum_{i=1}^{r} \left(\sum_{K=1}^{\infty} |g_i(x_K)| \, p_K \right)$$ (interchanging $\sum\sum$ because

$< \infty$ since $E(g_i(x))$ exists $|g| \geq 0$ & $p_K \geq 0$)

$$\sum_{K=1}^{\infty} |g_i(x_K)| \, p_K < \infty$$

$$\Rightarrow E\left(\sum_{i=1}^{r} g_i(x)\right) \text{ exists.}$$

Further,
$$E\left(\sum_{i=1}^{r} g_i(x)\right) = \sum_{K=1}^{\infty} \left(\sum_{i=1}^{r} g_i(x_K)\right) p_K$$

needs to verify.

$$= \sum_{i=1}^{r} \sum_{K=1}^{\infty} g_i(x_K) \, p_K$$

$$= \sum_{i=1}^{r} E(g_i(x))$$

For cont^{n}., the proof will remain same, only the sum will be replaced by integration.

Result:- Existence of higher order moments

$\Rightarrow$ " " lower " "

Mathematically speaking, If $E|X|^m < \infty \Rightarrow E|X|^k < \infty$ $\forall\ k \le m$.

Proof:- $E|X|^k = \int |x|^k f_X(x)\, dx$.

$$= \int_{|x|\le 1} |x|^k f(x)\, dx + \int_{|x|>1} |x|^k f(x)\, dx.$$

$$\le \int_{-\infty}^{\infty} f(x)\, dx + \int_{-\infty}^{\infty} |x|^m f(x)\, dx.$$

$|x|$ is replace by 1 — since $|x| > 1$ and $m \ge k$

$$= 1 + E|X|^m < \infty.$$ It proves the result.