## Hypergeometric dist$^n$ :-

$n > N-M$
$n-(N-M)$

Suppose, an urn contains $N$ balls

$M$ of them are white

$N-M$ of them are red.

$(M \leq N)$.

**Experiment:-** $n$ many balls drawn randomly at a time.

$X$: No. of white balls among those $n$ balls.

$$P[X = x] = \begin{cases} \dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} & \text{if } x \in [\min(0, n-(N-M)), \max(n, M)] \\ 0 & \text{, otherwise} \end{cases}$$

**Needs to check:-** i) $P[X = x]$ is a p.m.f.

ii) Derivation of MGF, $E(X)$, $Var(X)$.

## I. Poisson distⁿ :-

Poisson distⁿ is nothing but a binomial distⁿ when $n \to \infty$, $p \to 0$ and $np \longrightarrow \lambda (>0)$

Probability mass fⁿ

$$P[X = x] = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$$\longrightarrow \frac{n!}{x!\,(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \quad (\text{since } np \to \lambda)$$

$$= \frac{\lambda^x}{x!} \times \frac{n!}{(n-x)!\,n^x} \times \left(1 - \frac{\lambda}{n}\right)^n \times \left(1 - \frac{\lambda}{n}\right)^{-x}$$

Observe that $\frac{\lambda}{n} \to 0$ as $n \to \infty$.

First term :- $\dfrac{\lambda^x}{x!}$ will remain same as $n \to \infty$.

Second term :- $\dfrac{n!}{(n-x)!\, n^x} = \dfrac{n \times (n-1) \times \cdots \times (n-x+1)}{n^x} \longrightarrow 1$ as $n \to \infty$.

Third term :- $\left(1 - \dfrac{\lambda}{n}\right)^n \longrightarrow e^{-\lambda}$ as $n \to \infty$

Fourth term :- $\left(1 - \dfrac{\lambda}{n}\right)^{-x} \longrightarrow 1$ as $n \to \infty$.

Hence $\left(\text{First} \times \text{Second} \times \text{Third} \times \text{Fourth}\right) \xrightarrow{n \to \infty} \dfrac{e^{-\lambda} \lambda^x}{x!}$

<u>P.M.F. of Poisson dist^n :-</u>

$$P[X = x] = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, \ldots$$

## Comparison btwn Binomial & Poisson distn.

**Exp:-** Two dice are rolled 100 times.

X : The number of double 6s.

$X \sim Bin\left(n=100,\ p=\frac{1}{36}\right)$.

Want to approximate by $X \sim Poisson\left(100 \times \frac{1}{36} \approx 2.78\right)$.

| $P[X=x]$ $x$ | Binomial $Bin(100, \frac{1}{36})$ | Poisson $P(100 \times \frac{1}{36} \approx 2.78)$ |
|---|---|---|
| 0 | 0.0596 | 0.0620 |
| 1 | 0.1705 | 0.1725 |
| 2 | 0.2414 | 0.2397 |
| 3 | 0.2255 | 0.2221 |
| 4 | 0.1564 | 0.1544 |
| 5 | 0.0858 | 0.0858 |
| 6 | 0.0889 | 0.0889 |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |
| 11 | | |

A few properties :-

i) $P[X=x]$ is a p.m.f.

$\downarrow$ Poisson dist$^n$ with parameter $\lambda\ (>0)$.

$$P[X=x] = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x=0,1,\ldots$$
$$= 0, \quad \text{o.w.}$$

Observe that $P[X=x] \geq 0 \quad \forall x = 0, \ldots$
$\lambda > 0$

Next,
$$\sum_{x=0}^{\infty} P[X=x] = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!}$$
$$= e^{-\lambda} \times e^{\lambda} = 1.$$

Hence, $P[X=x]$ is a p.m.f.

ii) $M_X(t) = E[e^{tX}]$

MGF

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda \cdot e^t)^x}{x!} = e^{-\lambda} \times e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

(can prove from MGF of Bin(n,p) when $n \to \infty$, $p \to 0$ & $np \to \lambda$).

iii) $$E(X) = \sum_{x=0}^{\infty} x \times \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \times \lambda \times \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = e^{-\lambda} \times \lambda \times e^{\lambda}$$

$$= \lambda.$$

ii) $E(x^2) = \sum_{x=0}^{\infty} x^2 x \frac{e^{-\lambda}\lambda^x}{x!}$ check it $= \lambda + \lambda^2$ (To compute, $E(x^2) = E[x(x-1)] + E(x)]$

So, $Var(x) = E(x^2) - \{E(x)\}^2$

$= \lambda + \lambda^2 - \lambda^2 = \lambda.$

## Applications of Poisson distⁿ

i) Poisson distⁿ has been used in the analysis of telephone system.

ii) Number of Alpha particles emitted from a radioactive element.

## Continuous distⁿ :-

### i) Continuous Uniform distⁿ :-

X : Uniform distⁿ on the interval $[a, b]$

Then p.d.f. will be

$$f_X(x) = \frac{1}{b-a} \quad \text{if} \quad a \le x \le b$$

$$= 0 \quad , \quad \text{o.w.}$$

Note that $f_X(x)$ is a p.d.f. because

$f_X(x) \ge 0 \ \forall x$ & $\int_a^b f_X(x)\,dx = 1$.

Now, want to derive CDF

$$F_X(x) = \int_a^x f_X(y)\,dy = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b. \end{cases}$$

$$M_X(t) = E\left[e^{tX}\right] = \int_a^b e^{tx} \times \frac{1}{b-a}\,dx$$

$$= \frac{e^{tb} - e^{ta}}{t(b-a)}, \quad t \neq 0.$$