4) **Beta distribution:-**

X : Random variable

$$f_X(x) = \frac{1}{Beta(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \le x \le 1,$$ where

$$Beta(\alpha,\beta) = \frac{\Gamma\alpha\,\Gamma\beta}{\Gamma\alpha+\beta}.$$

Remark:- If $\alpha = 1$ & $\beta = 1$, then $f_X(x)$ will coincide with the uniform density over $(0,1)$.

$M_X(t)$ & $E(X^k)$: Try to do it by yourself.

5) **Weibull distribution:-** X : Random variable

$$\text{p.d.f.} \leftarrow f_X(x) = \underset{\substack{\downarrow \\ \text{norming constant}}}{c} \times e^{-\left(\frac{x}{\beta}\right)^k} x^{\alpha-1}, \quad x \ge 0.$$

norming constant, ie, c is such that $\int_0^\infty c\, e^{-\left(\frac{x}{\beta}\right)^k} x^{\alpha-1}\,dx = 1$

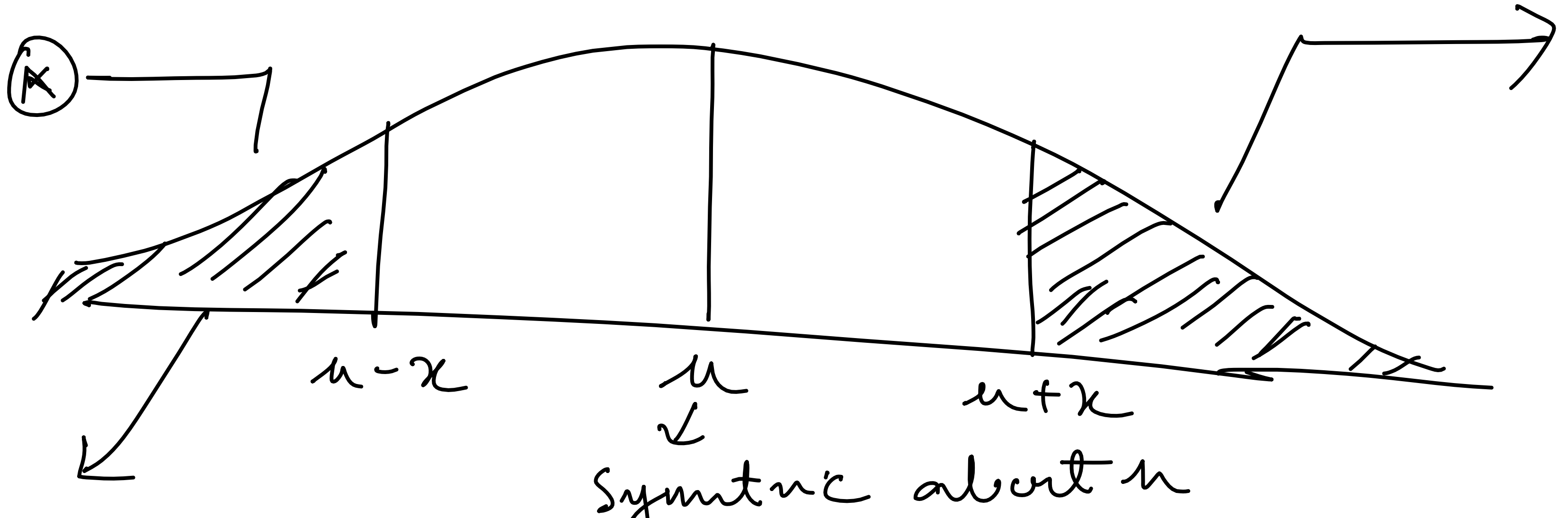
6. Normal distribution (Gaussian distⁿ) :-

Gauss proposed this distⁿ, and that's why it is called Gaussian distⁿ.

X : Random variable.

f_X : p.d.f. of X.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma \in \mathbb{R}^+.$$



Observe that $f_X(x) \geq 0 \ \forall x \in \mathbb{R}$ &

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \text{ (check!)}.$$

Hence, $f_X(x)$ is a proper p.d.f.

Note that $f(\mu + x) = f(\mu - x) \ \forall x$

Hence, f is symmetric about μ.

Moreover, (*) indicates that $\longrightarrow P[X > \mu + x] = P[X < \mu - x] \ \forall x$.

$$M_X(t) = E\left[e^{tX}\right] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

$\downarrow$ MGF

put $\frac{x-\mu}{\sigma} = z \Rightarrow dx = \sigma\, dz.$

$$\Rightarrow M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-\frac{z^2}{2}} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left\{z^2 - 2t\sigma z + t^2\sigma^2 - t^2\sigma^2\right\}} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \times e^{\frac{1}{2}t^2\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} dz$$

$$= e^{t\mu + \frac{1}{2}t^2\sigma^2} \times \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} dz}_{=1}$$

$$= e^{t\mu + \frac{1}{2}t^2\sigma^2}$$

iii) $E(X) = \left.\frac{d}{dt} M_X(t)\right|_{t=0}$.

Recall that $E(X^r)$ is the coeff. of $\frac{t^r}{r!}$ in $M_X(t)$.

In this case, $M_X(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}$

$$= 1 + \left(t\mu + \tfrac{1}{2}t^2\sigma^2\right)$$

coeff. of $\frac{t^1}{1!}$

$$+ \frac{1}{2!}\left(t\mu + \tfrac{1}{2}t^2\sigma^2\right)^2 + \cdots$$

Then, $E(X) = \mu$.

and $E(X^2)$ $\left(\text{coeff. of } \frac{t^2}{2!}\right) = \sigma^2 + \mu^2$.

So, $Var(X) = E(X^2) - \{E(X)\}^2 = \sigma^2 + \mu^2 - \mu^2$

$$= \sigma^2.$$

Standard Normal $\text{dist}^{\underline{n}}$:

If $\mu = 0$ & $\sigma = 1$, then in literature, it is called standard normal $\text{dist}^{\underline{n}}$ (And, it is denoted by $N(0,1)$).

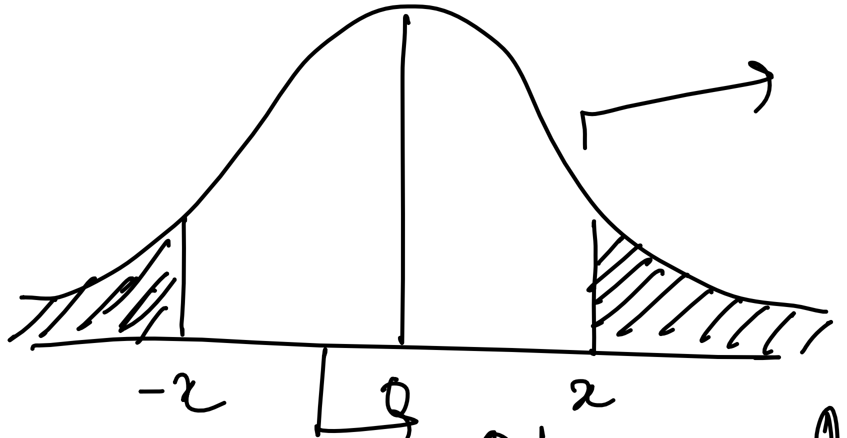
X: Random variable.

p.d.f. of X $\leftarrow$ $\phi_X(x) \doteq \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$

CDF of $N(0,1)$-random variable is denoted by $\Phi(x)$; defined as

$$\Phi(x) = P[\omega : \underset{N(0,1)}{\underset{\downarrow}{X}}(\omega) \le x]$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \int_{-\infty}^{x} \phi_X(y)\, dy.$$



(**)

Observe that $\Phi_X(-x) = P[X \le -x] \underbrace{= P[X \ge x]}_{\text{follows from (**)}} = 1 - P[X \le x]$

// defn.

$= 1 - \Phi_X(x)$

Hence, we have $\forall\, x \in \mathbb{R}$,

$$\Phi_X(x) + \Phi_X(-x) = 1 \quad \text{———} \quad (***)$$

Since (***) is true for all x, we have

$$\Phi_X(0) + \Phi_X(-0) = 1.$$

$$\iff \Phi_X(0) = 1/2.$$

Therefore, Med of $N(0,1)$ is $\frac{1}{2}$. Mean

Already, we have proved that $E(X) = \frac{1}{2}$.

Furthermore, Mode of $N(0,1) = 0$.

Result :- If $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0,1)$. Footnote

Have to show that CDF of $\frac{X-\mu}{\sigma} = Y$ (say) is same as that of $N(0,1)$ random variable i.e. $\Phi(\cdot)$

Proof :- Let $Y = \frac{X-\mu}{\sigma}$.

CDF of Y at the point m:

CDF of Y

$$F_Y(m) = P[Y \le m]$$

$$= P\left[\frac{X-\mu}{\sigma} \le m\right]$$

$$= P[X \le \mu + \sigma m]$$

$$= \int_{-\infty}^{\mu+\sigma m} \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Transform $l = \frac{x-\mu}{\sigma}$

$$= \int_{-\infty}^{m} \frac{1}{\sqrt{2\pi}} e^{-\frac{l^2}{2}} dl$$

def^n: $= \Phi(m)$

Hence $Y \sim N(0,1)$.

Example:— Suppose $X \sim N(6, 2)$.

Want to compute $P[X \leq 5]$.

$$P[X \leq 5] = P\left[\frac{X-6}{\sqrt{2}} \leq \frac{5-6}{\sqrt{2}}\right] = P\left[Y \leq -\frac{1}{\sqrt{2}}\right]$$

Standardization by μ & σ

$Y \to N(0,1)$.

$$= \Phi\left(-\frac{1}{\sqrt{2}}\right) = 1 - \Phi\left(\frac{1}{\sqrt{2}}\right).$$