Random variable:-

Motivating Example:- A coin is tossed thrice.

$\Omega = \{HHH, HHT, HTT, HTH, TTT, TTH, THH, THT\}$.

Defn: : Random variable is a real valued function, which is mapping from Ω to $\mathbb{R}$. (Range space can be $\mathbb{R}^d$ or $\mathcal{H}$; however, in the literature, we call them random vector).

Suppose,

X: No. of heads in an outcome.

Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}$.

Then $X(\omega_3) = 1$. [Since the value of the random variable depends on the randomness of Ω, that's why it is called Random variable], Random variable

Discrete Random variable. | Continuous Random variable | Mixture of Discrete & Continuous random variable.

Discrete random variable:- A random variable is discrete if it can take only finitely many or contable infinitely many values.

Example related to the contable infinitely may values:-

Random experiment:- Tossing a coin.

X : No. of tosses required until a head appear.

Here X can take any values of 1, 2, 3, 4, - - , , .

Recall the first example:-

$\Omega = \{HHH, HHT, HTT, HTH, TTT, TTH, THH, THT\}$.

X: No. of heads in an outcome.

$$\left.\begin{aligned} P[X=0] &= \frac{1}{8}. \\ P[X=1] &= \frac{3}{8} \\ P[X=2] &= \frac{3}{8} \\ P[X=3] &= \frac{1}{8}. \end{aligned}\right\} \longrightarrow$$

These are probability mass functions of the "distribution" associated with the random variable X.

Probability mass $f^{\underline{n}}$ (p.m.f.):- Let X be a random variable, which can take the values $x_1, x_2, \ldots$ so on. Then there is a $f^{\underline{n}}$ 'p' such that $p(x_i) = P[X = x_i]$, and $\sum_{i} p(x_i) = 1$.

(i is indexed over all mass points)

Overall, $p(x_i)$, $i=1,\dots$ has the following properties

i) $p(x_i) \geq 0 \quad \forall\, i=1,\dots$

ii) $\sum_{i=1}^{n} p(x_i) = 1$.

Then 'p' is called the probability mass $f^{\underline{n}}$ (p.m.f.).

Next concept :- Cumulative distribution $f^{\underline{n}}$ (CDF)

Let X be a random variable, and it CDF F is defined as

$$F_X(x) = P[X \leq x]$$

[Precisely speaking,

$$F_X(x) = P[\omega \in \Omega : X(\omega) \leq x]$$
]

Earlier Example

$P[X=0] = \frac{1}{8}$

$P[X=1] = \frac{3}{8}$

$P[X=2] = \frac{3}{8}$

$P[X=3] = \frac{1}{8}$

For the example we considered,

$$F_X(x) = P[X \le x]$$

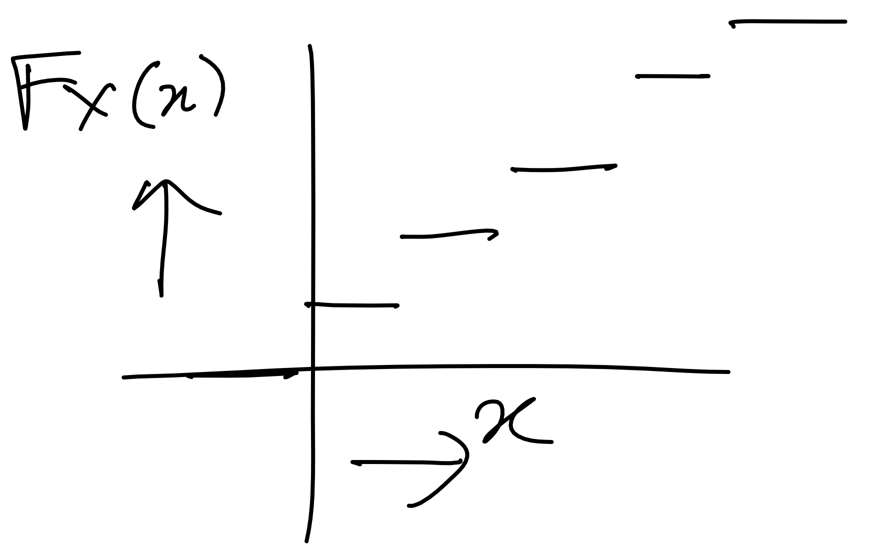
$$= 0 \quad \text{if} \quad x < 0$$

$$= 0 + \frac{1}{8} = \frac{1}{8} \quad \text{if} \quad 0 \le x < 1.$$

$$= 0 + \frac{1}{8} + \frac{3}{8} = \frac{4}{8} \quad \text{if} \quad 1 \le x < 2$$

$$= 0 + \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8} \quad \text{if} \quad 2 \le x < 3$$

$$= 0 + \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1 \quad \text{if} \quad x \ge 3$$



Theoretical explanation

$A_1 = \{\omega : X(\omega) \le x_1\}$

$A_2 = \{\omega : X(\omega) \le x_2\}$

Given that $x_1 \le x_2$

$\Rightarrow A_1 \subseteq A_2 \quad \forall x_1 \le x_2$

$\Rightarrow P(A_1) \le P(A_2) \quad \forall x_1 \le x_2$

$\Rightarrow F_X(x_1) \le F_X(x_2) \quad \forall x_1 \le x_2$

Remark:- The earlier example indicates that

i) $F_X(x)$ is a non-decreasing f^n.

ii) $F_X(x)$ is a righ continuous f^n w.r.t. x.

iii) $\lim_{x \to \infty} F_X(x) = 1$ & $\lim_{x \to -\infty} F_X(x) = 0$.

Continuous Random variable:-

Real Examples:- 1. Life time of electric bulb.

2. Height of people in a city.

Motivation:- Indicates from the real life examples that the sample space Ω will be an uncontable set (for instance, Ω is an interval).

Probability density function (p.d.f.):-

[Essentially continuous counterpart of p.m.f.]

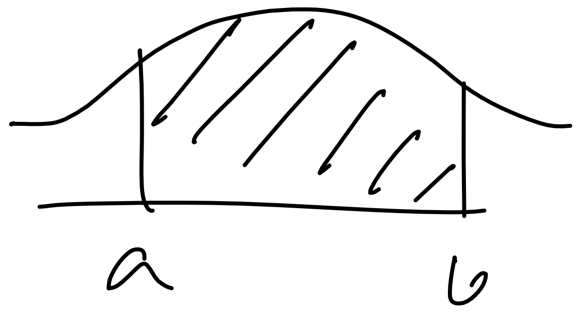
Suppose $f_X(x)$ is the p.d.f. associated with the random variable X, then

i) $f_X(x) \geq 0 \quad \forall x$.

ii) $\int f_X(x)\,dx = 1$.

If any function 'f' satisfies (i) & (ii), we can call it a probability density f^n^ (p.d.f.).

<u>Remark:-</u> X will be a continuous random variable if $P[a < X < b] = \int_a^b f_X(x)\,dx$.



Hence, the CDF of X (continuous) will be

$$F_X(x) = P[\omega : X(\omega) \leq x] = \int_{-\infty}^{x} f_X(x)\, dx$$ [follows from the previous equality].