Last day, we started Unif $[a, b]$ distn :-

$$f_X(x) = \frac{1}{b-a} \quad \text{if} \quad a \leq x \leq b$$

$$= 0 \quad , \quad \text{o.w.}$$

$$E(X) = \int_a^b x \times \frac{1}{b-a}\, dx = \frac{b+a}{2}$$ (Remark:- you can derive from the expression of MGF as well).

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2 \overset{\text{(check)}}{=} \frac{(b-a)^2}{12} \longrightarrow$$ That also, you can derive from MGF.

Sp. case:- If $a = 0$ & $b = 1$, we have well-known uniform $[0, 1]$ distn :-

Why Unif [0,1] is so special ?? (because it has numerous theoretical applications).

Applications :-

1. In the limiting sense, the discrete uniform distⁿ. (prob. mass $\frac{1}{n}$ on each of the mass points $\left(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n}\right)$) will converge to Unif $[0,1]$. Already, we know that discrete uniform enables us to construct sample version of the moments, which is useful real data analyses.

2. If X is a contⁿ. r.v. with CDF F_X, then F_X will follow uniform $(0,1)$ distⁿ.

This helps us to generate a random from any cont^{n} dist^{n}

Example:– Suppose, we want to generate random number from Cauchy $(0,1)$ dist^{n} $\longrightarrow$ $f_X(x) = \dfrac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$.

f_X p.d.f.

Since F_X (CDF of Cauchy $(0,1)$) follows Unif $(0,1)$, choose one number random from $(0,1)$. (say, 0.89).

If $X \sim$ Cauchy $(0,1)$, then $F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy$

$$= \int_{-\infty}^{x} \frac{1}{\pi(1+y^2)}\, dy$$

Check $= \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x$

Hence, the random number from Cauch $(0,1)$ will be

$\frac{1}{2} + \frac{1}{\pi}\tan^{-1} x = 0.89 \iff x = ??$

3. Monte-Carlo integration

W.L.G, we consider $\int_0^1 f(x)\,dx$ instead of $\int_a^b f(x)\,dx$

$$\int_0^1 f(x)\,dx = \int_0^1 f(x) \times 1\,dx = E_X[f(X)], \text{ where } X \sim \text{Unif}(0,1).$$

Now, based on WLLN, one can approximate $E_X[f(X)]$.

Algorithm:-

Step 1:- Let $x_1, \ldots, x_n$ be random numbers from Unif(0,1).

Step 2:- Compute $f(x_1), f(x_2), \ldots, f(x_n)$.

Step 3:- $$\int_0^1 f(x)\,dx = E_X[f(X)] \underbrace{\approx}_{\text{WLLN}} \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

[Rough idea:- $n^{\delta}\left[\int_0^1 f(x)\,dx - \frac{1}{n}\sum_{i=1}^{n} f(x_i)\right]$, $\delta > 0$]

2. <u>Exponential distⁿ :-</u>

X : Random variable with exp. distⁿ with mean $= \frac{1}{\lambda}$,

The p.d.f. of X will be

$$f_X(x) = \lambda \cdot e^{-\lambda x}, \quad x \geq 0$$
$$= 0, \quad \text{o.w.}$$

The CDF of X will be

$$P[\omega : X(\omega) \leq x] = F_X(x) \overset{\text{check}}{=} \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

$$M_X(t) \overset{\text{check}}{=} \frac{1}{1 - \frac{t}{\lambda}} \quad \text{for } \alpha > t.$$

MGF

& $E(X) \overset{\text{check}}{=} \frac{1}{\lambda}$ & $Var(X) = E(X^2) - \{E(X)\}^2 \overset{\text{check}}{=} \frac{1}{\lambda^2}$

Memory loss property of Exp. distⁿ.

Want to establish that

$$P[X > s + t \mid X > s] = P[X > t].$$

Observe that $P[X > s+t \mid X > s]$

$$= \frac{P[X > s+t \cap X > s]}{P[X > s]} = \frac{P[X > s+t]}{P[X > s]}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} = P[X > t].$$

Remark:- A person aged 10, the survival chance till age 20 will NOT be the same as a person aged 80, the survival chance till age 90. And hence, the lifetime of human cannot be fitted by exp. distⁿ.

3. <u>Gamma dist^n :-</u>

X : Random variable associated with Gamma(α, β) dist^n.

The p.d.f is

$$f_X(x) = \frac{1}{\Gamma\alpha \, \beta^\alpha} e^{-\frac{x}{\beta}} x^{\alpha - 1}, \quad x \geq 0.$$

gamma f^n

Here, α is called the shape parameter & β is the scale parameter.

$\Gamma\alpha$: Gamma f^n

Properties :-

i) $\Gamma x = (x-1)\,\Gamma(x-1)$

ii) If x is an integer, then $\Gamma x = (x-1)!$

iii) $\Gamma\frac{1}{2} = \sqrt{\pi}$

$$M_X(t) = E\left[e^{tX}\right] = \frac{1}{\beta^{\alpha}\Gamma\alpha}\int_0^{\infty} e^{tx}\, e^{-\frac{x}{\beta}}\, x^{\alpha-1}\, dx$$

check carefully

$$= \frac{1}{(1-\beta t)^{\alpha}}$$

And $$E(X^K) = \frac{1}{\beta^{\alpha}\Gamma\alpha}\int_0^{\infty} x^K\, x^{\alpha-1} e^{-\frac{x}{\beta}}\, dx$$

check carefully

$$= \frac{\Gamma(\alpha+K)}{\Gamma\alpha}\,\beta^K$$

Properties of Gam fn

$$E(X) = \frac{\Gamma(\alpha+1)}{\Gamma\alpha}\,\beta = \frac{\alpha\,\Gamma\alpha}{\Gamma\alpha}\,\beta = \alpha\beta,$$

try to convince

$$E(X^2) = \frac{\Gamma(\alpha+2)}{\Gamma\alpha}\,\beta^2 = \alpha(\alpha+1)\,\beta^2,$$ and hence, $$Var(X) = E(X^2) - \{E(X)\}^2$$

$$= \alpha\beta^2.$$

Remark:- Put $\alpha = 1$ & $\beta = \frac{1}{\lambda}$, that will be exp. distⁿ with mean $= \frac{1}{\lambda}$.

(So, everything like $M_x(t)$ or $E(x^K)$ of exp. distⁿ can directly be computed from Gamma distⁿ),

Application:- Earthquake data follow Gamma distⁿ

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