Product moments :-

Suppose $g(X_1, \ldots, X_d) = X_1^{k_1} X_2^{k_2} \cdots X_d^{k_d}$.

Now, i) $E[g(X_1, \ldots, X_d)] = E[X_1^{k_1} \cdots X_d^{k_d}]$:- Joint moment of order $(k_1 + k_2 + \cdots + k_d)$.

ii) If $k_i = 1$, $k_1 = \cdots k_{i-1} = k_{i+1} = \cdots = k_d = 0$,

Then $E[g(X_1, \ldots \cancel{X_d})] = E(X_i)$:- First order moment of X_i

iii) If $k_i = 2$, $\underbrace{- \; - \; - \; - \; -}_{} = 0$, then

$E[g(X_1, \ldots, X_d)] = E(X_i^2) \xrightarrow{(ii)}$ Second order moment of X_i.

$Var(X_i) = E(X_i^2) - \{E(X_i)\}^2$.

iv) Suppose $k_i = 1$, $k_j = 1$ $(i \neq j)$, $j > i)$ but $k_1 \cdots = k_{i-1} = k_{i+1} = \cdots = k_{j-1} = k_{j+1} = \cdots k_d = 0$

Then $E[g(X_1, \ldots, X_d)] = E[X_i X_j]$:- Product moment of X_i & X_j.

Def^n:

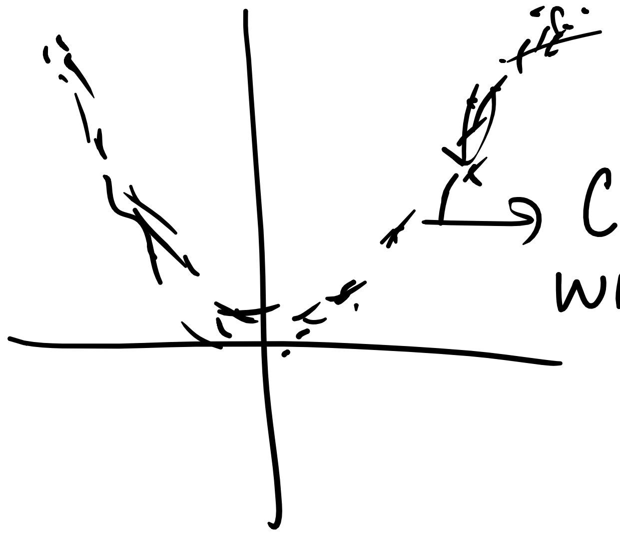
i) $Cov(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j]$.

check it

$= E\{[X_i - E(X_i)][X_j - E(X_j)]\}$

ii) Correlation between X_i and X_j is

$$Corr(X_i, X_j) = \frac{Cov(X_i, X_j)}{\sqrt{Var(X_i)\, Var(X_j)}}$$

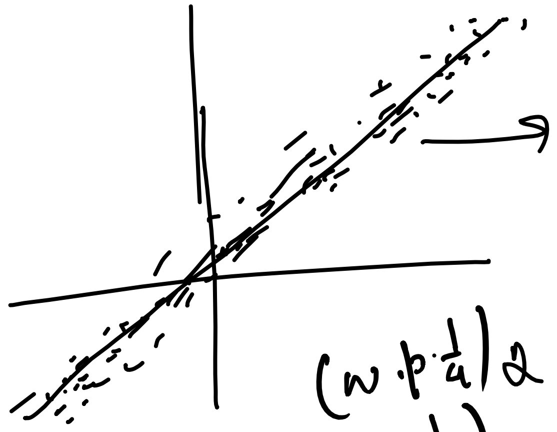


Corr. will NOT be high.

By C.S. inequality, we have

$$-1 \leq Corr(X, Y) \leq 1$$

It measures the $\boxed{\text{linear}}$ relationship bet^n X and Y



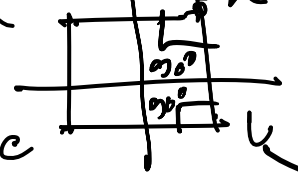
corr will be close to 1.

(w.p $\frac{1}{4}$) a

(w.p $\frac{1}{4}$) c

$(w.p = \frac{1}{4})$ d

(X, Y)



Corr. will be = 0

(w.p $\frac{1}{4}$) b

A few facts:-

i) If $X \perp\!\!\!\perp Y$, then $\rho_{X,Y} = 0$. (Corr. betn. X and Y)

$$\rho_{XY} = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}}$$

, so it is enough to prove that $\operatorname{Cov}(X,Y) = 0$ if $X \perp\!\!\!\perp Y$.

Note that $\operatorname{Cov}(X,Y) = E[XY] - E[X]E[Y]$

$$= \iint xy\, f_{X,Y}(x,y)\, dx\, dy - E[X]E[Y].$$

$X \perp\!\!\!\perp Y$

$$= \iint xy\, f_X(x)\, f_Y(y)\, dx\, dy - E[X]E[Y]$$

$$= \int x f_X(x)\, dx \times \int y f_Y(y)\, dy - E[X]E[Y]$$

$$= E[X]E[Y] - E[X]E[Y] = 0.$$

Remark:- However, $\rho_{X,Y} = 0$ may NOT imply $X \perp\!\!\!\perp Y$.

Connection betn. $X \perp\!\!\!\perp Y$ & $\underset{\mathrm{Cov}(X,Y)=}{\rho_{X,Y}} = 0$ for BVN dist^{n}.

Claim:- $(X,Y) \sim \text{BVN}(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$.

$X \sim N(\mu_x, \sigma_x^2)$

$Y \sim N(\mu_y, \sigma_y^2)$.

Then $\rho_{X,Y} = \rho$.

Outline:-

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

$$= \frac{\mathrm{Cov}(X,Y)}{\sigma_x \sigma_y}$$

$$= \frac{E\left[X - \overset{\mu_x}{\cancel{E[X]}}\right]\left[Y - \overset{\mu_y}{\cancel{E[Y]}}\right]}{\sigma_x \sigma_y}$$

$$= \frac{\iint (x-\mu_x)(y-\mu_y) \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]} dx\,dy}{\sigma_x \sigma_y}$$

Tranfom $\frac{x-\mu_x}{\sigma_x} = u$ & $\frac{y-\mu_y}{\sigma_y} = v$.

and follow the same arguments as we did for deriving the marginal density, we have

$$\rho_{X,Y} = \frac{\rho\,\sigma_x\sigma_y}{\sigma_x\sigma_y} = \rho,$$ and hence $Corr(X,Y) = \rho$ if $(X,Y) \sim BVN(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$.

Note that if $(X, Y) \sim BVN(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2; \rho)$,

then $f_{X,Y}(x,y) = \dfrac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]}$.

If $\rho = 0$, then

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}.$$

Joint density

$$= \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)\right]^2}}_{\text{Marginal density of } X} \times \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-\frac{1}{2}\left[\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}}_{\text{Marginal density of } Y.}$$

$\Rightarrow X \perp\!\!\!\perp Y$. Overall, if $(X, Y) \sim BVN(\cdot, \cdot, \cdot, \cdot, \rho)$, then $\rho_{X,Y} = 0 \iff X \perp\!\!\!\perp Y$.

$M_X(t) = E[e^{tX}]$
$tX = \langle t, X \rangle$

Moment generating function for random vector:-

$\underset{\sim}{X} = (X_1, \dots, X_d)$ & $\underset{\sim}{t} = (t_1, \dots, t_d) \mapsto$ d-dim fixed point.

d-dim Random vector

$$M_{\underset{\sim}{X}}(\underset{\sim}{t}) = E\left[e^{\langle \underset{\sim}{t}, \underset{\sim}{X} \rangle}\right] = E\left[e^{\underset{\sim}{t}^T \underset{\sim}{X}}\right] = E\left[e^{t_1 X_1 + t_2 X_2 + \dots + t_d X_d}\right]$$

provided the Expectation exists in some nbd of $(t_1, \dots, t_d) = (0, \dots, 0)$.

Similar fact:-

$$E\left[X_1^{k_1} \cdots X_d^{k_d}\right] = \left.\frac{\partial^{k_1 + \dots + k_d} M_{\underset{\sim}{X}}(\underset{\sim}{t})}{\partial t_1^{k} \, \partial t_2^{k_2} \cdots \partial t_d^{k_d}}\right|_{\underset{\sim}{t} = (0, \dots, 0)} = E(X_1^{k_1} \cdots X_d^{k_d}).$$

Remark:- Like univariate case, here also $M_{\underset{\sim}{X}}(t)$ characterizes the dist^n of X as long as $M_X(\underset{\sim}{t})$ exists.

i.e, $M_{\underset{\sim}{X}}(\underset{\sim}{t}) = M_{\underset{\sim}{Y}}(\underset{\sim}{t}) \quad \forall \underset{\sim}{t}$

$$\Leftrightarrow \underset{\sim}{X} \overset{d}{=} \underset{\sim}{Y}. \quad \text{———} \quad (*)$$

Application:- If $\underset{\sim}{X} \sim F$ & $\underset{\sim}{Y} \sim G$.

Want to test $H_0 : F = G$ ag. $H_1 : F \neq G$,

(*) will help us to carry out the test H_0 against H_1 as long as $M_{\underset{\sim}{X}}(\underset{\sim}{t})$ & $M_{\underset{\sim}{Y}}(\underset{\sim}{t})$ exist.

Result :- If $(X, Y) \sim BVN(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$, then

$$M_{(X_1, Y)}(\underset{\sim}{t}) = E\left[e^{t_1 X_1 + t_2 \cancel{Y_2}}\right]$$

(t_1, t_2)

$$= e^{(t_1 \mu_x + t_2 \mu_y) + \frac{1}{2}\underbrace{(t_1^2 \sigma_1^2 + t_2^2 \sigma_2^2 + 2 t_1 t_2 \rho \sigma_1 \sigma_2)}_{**}}$$

Try to do it by yourself.

Corollary :- $(X, Y) \sim BVN(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$.

Then $X + Y \sim UVN(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2 + 2\rho\sigma_x\sigma_y)$.

Proof :- In $(**)$, consider $t_1 = t_2 = t$, we then have.

$X \sim N(\mu, \sigma^2)$
$M_X(t) = e^{t\mu + \frac{t^2}{2}\sigma^2}$

$$E\left[e^{t(X+Y)}\right] = e^{t(\mu_x + \mu_y) + \frac{t^2}{2}(\sigma_x^2 + \sigma_y^2 + 2\rho\sigma_1\sigma_2)}$$

$\longmapsto$ MGF of $N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2 + 2\rho\sigma_x\sigma_y)$

$\Rightarrow (X+Y) \sim UVN(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2 + 2\rho\sigma_x\sigma_y)$.