Another methodology to find an estimator:-

Method of Moment Estimation (MME):-

$X_1, \dots, X_n \overset{i.i.d.}{\sim} f(x; \theta_1, \dots, \theta_K)$. $\underset{\sim}{\theta} = (\theta_1, \dots, \theta_K)$ is a k-dim of unknown parameter.

Let us denote
$$m_r' = \frac{1}{n}\sum_{i=1}^{n} X_i^r \longrightarrow r\text{-th raw moment.}$$

We know, by WLLN, $\frac{1}{n}\sum_{i=1}^{n} X_i^r \xrightarrow{P} E(X_1^r)$.
$$E(X_1^r) \longrightarrow \mu_r'$$

The MME ~~est~~ is the sol^n of

$$\underbrace{m_r' = \mu_r'(\theta_1, \dots, \theta_K), \quad r = 1, \dots, k.}_{\circledast}$$

Solve $\circledast$, will have $\hat{\theta}_{1,MME}, \dots, \hat{\theta}_{K,MME}$

Examples :-

1. $X_1, \dots, X_n \sim Bin(1, \theta)$, $\theta \in (0,1)$.

$\mu_1' = E(X) = \theta$

$m_1' = \frac{1}{n}\sum_{i=1}^{n} X_i$

Hence, the MME of θ is the solⁿ of

$m_1' = \mu_1' \Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i = \theta$

Hence, here the MME of θ is $\frac{1}{n}\sum_{i=1}^{n} X_i$.

2. $X_1, \dots, X_n \overset{i.i.d}{\sim} \text{Unif}\,(-\theta, \theta)$.

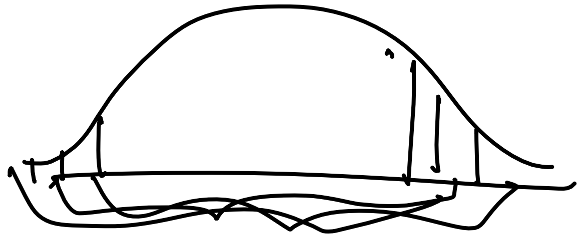
$$E(X) = \mu_1' = \frac{1}{2\theta}\int_{-\theta}^{\theta} x \, dx = 0$$

$$E(X^2) = \mu_2' = \frac{1}{2\theta}\int_{-\theta}^{\theta} x^2 \, dx = \frac{\theta^2}{3}$$

Hence, the sol^n of $m_2' = \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \mu_2' = \frac{\theta^2}{3}$ is the MME of θ. $(*)$

Hence, the MME of θ is

$$\hat{\theta}_{n,MME} = \sqrt{\frac{3}{n}\sum_{i=1}^{n} X_i^2}.$$



Interval estimation:-

$X_1, \dots, X_n \overset{i.i.d}{\sim} f(x;\theta), \quad \theta \in \Theta$

Want to find a random interval

$[L(X_1,\dots,X_n), U(X_1,\dots,X_n)]$ of the unknown parameter θ such that

$$P[L(X_1,\dots,X_n) < \theta < U(X_1,\dots,X_n)] \geq 1-\alpha.$$

$\alpha \in (0,1)$

It is called $(1-\alpha)$ confidence interval of the unknown parameter θ. Here we generally consider a small value of α such that 0.01 or 0.05 or 0.1 (in practice).

Idea:- For a fixed α, we try to minimize the length of the interval $[L(X_1,\dots,X_n), U(X_1,\dots,X_n)]$.

Example 1: $X_1,\dots,X_n \sim N(\mu,\sigma^2)$ $\longrightarrow \sigma^2 \longrightarrow$ known.

Want to find $(1-\alpha)$ CI of μ.

Sol^n:- Since σ^2 is known, we have

$$\frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right)}{\sigma} \sim N(0,1).$$

Denote $Y_n = \dfrac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma}$, where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Let a & b such that

$$P[a < Y_n \leq b] = 1-\alpha$$

$$\Leftrightarrow \quad \Phi(b) - \Phi(a) = 1-\alpha \longrightarrow (1)$$

since $Y_n \sim N(0,1)$, Φ is the CDF of $N(0,1)$.

Now, $$1-\alpha = P\left[a \leq \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \leq b\right]$$

$$\Leftrightarrow \quad 1-\alpha = P\left[a\frac{\sigma}{\sqrt{n}} \leq \bar{X}_n - \mu \leq \frac{b\sigma}{\sqrt{n}}\right]$$

$$\Leftrightarrow \quad 1-\alpha = P\left[\bar{X}_n - b\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X}_n - a\frac{\sigma}{\sqrt{n}}\right].$$

Now, the length of the interval is

$$= (b-a)\frac{\sigma}{\sqrt{n}}.$$

Further, observe that σ & $\sqrt{n}$ are known to us.

Hence, the problem reduces to minimizing $(b-a)$ with the constraint $\Phi(b) - \Phi(a) = 1-\alpha$.

The forms of the soln's of b and a will be

$$b = m \quad \& \quad a = -m \quad \text{———} \quad (2)$$

Using (2) in (1), we have

$$\Phi(m) - \Phi(-m) = 1-\alpha$$

$$\Longleftrightarrow \quad \Phi(m) - \{1 - \Phi(m)\} = 1-\alpha$$

$$\Longleftrightarrow \quad 2\Phi(m) - 1 = 1-\alpha$$

$$\Longleftrightarrow \quad 2\Phi(m) = 2-\alpha \Longleftrightarrow \Phi(m) = 1-\frac{\alpha}{2}$$

$$\Longleftrightarrow \quad m = \Phi^{-1}\left(1-\frac{\alpha}{2}\right)$$

since Φ is a strictly increasing f^{n}.

Hence, $b = \Phi^{-1}\left(1-\frac{\alpha}{2}\right)$

$a = -\Phi^{-1}\left(1-\frac{\alpha}{2}\right)$.

Therefore, ~~the~~ $(1-\alpha)$ CI of μ is (when σ is know)

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i - \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\frac{\sigma}{\sqrt{n}},\ \frac{1}{n}\sum_{i=1}^{n} x_i + \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\frac{\sigma}{\sqrt{n}}\right).$$

Fact :- If σ is unknown, one can construct CI based on a consistent estimator of σ. (denoted by s).

$$\left(s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x}_n)^2} \right).$$

In this case, $(1-\alpha)$ - CI of μ will be

$$\left(\frac{1}{n} \sum_{i=1}^{n} x_i - t_{n-1, 1-\frac{\alpha}{2}} \frac{s}{\sqrt{n}} , \frac{1}{n} \sum_{i=1}^{n} x_i + t_{n-1, 1-\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \right).$$

Testing of hypothesis :- (Neyman - Pearson Approach)

A few facts :-

* Assume that the data is a realization of some random variable

** Assume that the r.v. follows some parametric distⁿ :-

F_θ, where θ is unknown to us.

*** Let $\theta = 5$ or $\theta = 7$.

One hypothesis — Another hypothesis.

Testing of hypothesis will give you an idea, which hypothesis ~~wi~~ will be accepted and which one will be rejected.