<u>Remark:-</u> How to choose "one" u.e. among those infinitely many u.e.??

Available in the literature in the last 70 years

i) Among those u.e, choose that one, which has the minimum variance. (This concept is related to MVUE/UMVUE).

New

ii) Among those u.e, choose that one, which has the minimum other criteria of spreadness such as Median absolute deviation about median/ Mean absolute deviation about mean.

Example:- Let $X_1, \dots, X_n \sim \text{Unif}(0, \theta)$, $\theta > 0$.

Consider $T_n^{(1)} = 2X_1$, $T_n^{(2)} = X_1 + X_2$, $T_n^{(3)} = 2 \times \frac{1}{n} \sum_{i=1}^{n} X_i$,

$$T_n^{(4)} = \frac{n+1}{n} X_{(n)}.$$

Study:-

i) $E\left(T_n^{(1)}\right) = 2E(X_1) = 2 \times \frac{\theta}{2} = \theta.$

ii) $E\left(T_n^{(2)}\right) = E(X_1) + E(X_2) = \frac{\theta}{2} + \frac{\theta}{2} = \theta.$

iii) $E\left(T_n^{(3)}\right) = 2 \times \frac{1}{n} \sum_{i=1}^{n} E(X_i) = 2 \times \frac{1}{n} \sum_{i=1}^{n} \frac{\theta}{2} = \theta.$

iv) $E\left(T_n^{(4)}\right) = \frac{n+1}{n} \int_0^{\theta} y \times f_{X_{(n)}}(y)\, dy = \frac{n+1}{n} \int_0^{\theta} y \times \frac{n}{\theta^n} y^{n-1} dy$ //check!!

However, $T_n^{(4*)} = X_{(n)}$, then $E\left(T_n^{(4*)}\right) = \frac{n}{n+1}\theta \xrightarrow[\text{as } n \to \infty]{} \theta$ $= \frac{n+1}{n} \times \frac{n}{n+1} \theta = \theta.$

$T_n^{(4)} = \frac{n+1}{n} X_{(n)}$

Remark:- $T_n = a_{p,n} X_{(n-p)}$, where p is a fixed number will be an u.e. of θ.

Consistency:- **Setup:-** $X_1, \ldots, X_n \sim f(x|\theta)$ $\longrightarrow$ unknown parameter.

$T_n = T_n(X_1, \ldots, X_n)$ is said to be a consistent estimator of θ if $T_n \xrightarrow{P} \theta$ as $n \to \infty$.
$\longrightarrow$ converges in prob.

In other words, $\forall\ \varepsilon > 0$, $P\left[|T_n - \theta| > \varepsilon\right] \longrightarrow 0$ as $n \to \infty$.

<u>Example:-</u> $X_1, \ldots, X_n \sim N(\mu, 1)$.

$$T_n^{(1)} = X_1 \quad \text{and} \quad T_n^{(2)} = \frac{1}{n}\sum_{i=1}^{n} X_i .$$

<u>~~Study~~ :-</u> <u>Remark:-</u> Both $T_n^{(1)}$ & $T_n^{(2)}$ are u.e. of μ.

<u>Checking of consistency:-</u>

<u>$T_n^{(1)}$:-</u> Take any $\varepsilon > 0$,

$$P\left[|T_n^{(1)} - \mu| > \varepsilon\right] = P\left[|X_1 - \mu| > \varepsilon\right]$$

$$= P\left[X_1 > \mu + \varepsilon \cup X_1 < \mu - \varepsilon\right] = P\left[X_1 > \mu + \varepsilon\right] + P\left[X_1 < \mu - \varepsilon\right]$$

$$= P\left[\frac{X_1 - \mu}{1} > \varepsilon\right] + P\left[\frac{X_1 - \mu}{1} < -\varepsilon\right] = \{1 - \Phi(\varepsilon)\} + \Phi(-\varepsilon) = \{1 - \Phi(\varepsilon)\} + \{1 - \Phi(\varepsilon)\}$$

$$= 2(1 - \Phi(\varepsilon)).$$

Hence, $T_n^{(1)} = X_1$ is NOT a consistent estimator of μ.

Checking consistency of $T_n^{(2)}$:-

(*) $X_1, \ldots, X_n \sim N(\mu, 1)$
$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i \sim N(\mu, \frac{1}{n})$.

Take any $\varepsilon > 0$,

$$P\left[\left|T_n^{(2)} - \mu\right| > \varepsilon\right] = P\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| > \varepsilon\right]$$

$$= P\left[\frac{1}{n}\sum_{i=1}^{n} X_i - \mu > \varepsilon\right] + P\left[\frac{1}{n}\sum_{i=1}^{n} X_i - \mu < -\varepsilon\right]$$

$$= P\left[\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\frac{1}{\sqrt{n}}} > \sqrt{n}\,\varepsilon\right] + P\left[\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\frac{1}{\sqrt{n}}} < -\sqrt{n}\,\varepsilon\right]$$

Using (*)

$$= P\left[\underbrace{Z}_{N(0,1)} > \sqrt{n}\,\varepsilon\right] + P\left[Z < -\sqrt{n}\,\varepsilon\right]$$

$$= 1 - \Phi(\sqrt{n}\,\varepsilon) + \Phi(-\sqrt{n}\,\varepsilon) \longrightarrow 0 \text{ as } n \to \infty.$$

Hence, $T_n^{(2)} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is a consistent estimator of μ.

<u>Remark:-</u> Consistency does NOT guarantee the unbiasedness

<u>For example:-</u> $X_1, \ldots, X_n \sim U(0, \theta)$.

$T_n = X_{(n)}$ is a consistent estimator of θ but not unbiased estimator.

<u>Result (sufficient condition):-</u> Let $\{T_n\}_{n \geq 1}$ be a sequence of consistent estimator of an unknown parameter θ.

Then $E(T_n) \longrightarrow \theta$ as $n \to \infty$ & $Var(T_n) \to 0$ as $n \to \infty$.

Proof :- $T_n \xrightarrow{P} \theta$

$\Rightarrow \forall \varepsilon > 0, \quad P[|T_n - \theta| > \varepsilon] \to 0$ as $n \to \infty$.

Consider $P[|T_n - \theta| > \varepsilon] = P[(T_n - \theta)^2 > \varepsilon^2]$

$$\leq \frac{E(T_n - \theta)^2}{\varepsilon^2} \quad \text{by Markov's inequality}$$

$$= \frac{E[T_n - E(T_n) + E(T_n) - \theta]^2}{\varepsilon^2}$$

$$= \frac{E[T_n - E(T_n)]^2 + \{E(T_n) - \theta\}^2 + 2\{E(T_n) - \theta\} \underbrace{E(T_n - E(T_n))}_{=0}}{\varepsilon^2}$$

$$= \frac{Var(T_n) + \{E(T_n) - \theta\}^2 + 0}{\varepsilon^2} \to 0 \text{ if } Var(T_n) \to 0 \text{ as } n \to \infty$$

$\& \ E(T_n) \to \theta$ as $n \to \infty$.

Result :- (Continuous Mapping theorem) :-

If T_n is a consistent estimator of θ, then $g(T_n)$ will be a consistent estimator of $g(T_n)$ if g is continuous $f^{\underline{n}}$.

Ex :- $X_1, \dots, X_n \sim U(0,\theta)$.

$$X_{(n)} \xrightarrow{P} \theta$$

So, $e^{X_{(n)}} \xrightarrow{P} e^{\theta}$ since e^x is a cont$^{\underline{n}}$. $f^{\underline{n}}$ of X.

$$\overline{X}_n \xrightarrow{P} \theta.$$

$$e^{\overline{X}_n} \xrightarrow{P} e^{\theta}.$$

Remark :- Consistent estimator is NOT unique.

Sufficient Statistic