Marginal Probability mass f^{n} from Joint probability mass f^{n}

Suppose $P_{X,Y}(x_i, y_j)$: Joint p.m.f. $i=1,\dots,n$ & $j=1,\dots,m$.

$$P_X(x) = \sum_{j=1}^{m} P_{X,Y}(x, y_j) \longrightarrow \text{Marginal p.m.f. of } X.$$

Marginal p.m.f. of X

Similarly, $P_Y(y) = \sum_{i=1}^{n} P_{X,Y}(x_i, y) \longrightarrow$ Marginal p.m.f. of Y.

This def^{n} can be extended for d-dimensional case also.

$$(X_1, \dots X_d) \xrightarrow{\text{p.m.f.}} P_{X_1,\dots,X_d}(x_1,\dots,x_d) = P[X_1 = x_1, \dots, X_d = x_d].$$

Then $P_{X_1}(x) = \sum_{x_2,\dots,x_d} P[X_1 = x, X_2 = x_2, \dots, X_d = x_d]$.

Marginal p.m.f. of X_1

Continuous Random Vector (Variables) :-

(X, Y) : Bivariate Random vectors.

F_{XY} : Joint CDF of (X, Y).

$f_{X,Y}$: Joint PDF of (X, Y).

$$F_{X,Y} = P\left[\omega : X(\omega) \leq x,\ Y(\omega) \leq y\right] = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u, v)\, du\, dv$$

For a general set A, we can write

$$P\left[(X, Y) \in A\right] = \iint_{A} f_{X,Y}(u, v)\, du\, dv.$$

If $F_{X,Y}$ is given, then

$$f_{X,Y}(u,v) = \frac{\partial^2 F_{X,Y}(u,v)}{\partial u\, \partial v}.$$

Next, Marginal CDF & PDF :-

$$\underbrace{F_X(x)}_{\text{Marginal CDF of } X} = P[X \le x] = \underbrace{\lim_{y\to\infty} P[X \le x, Y \le y]}_{\lim_{y\to\infty} F_{X,Y}(x,y)}.$$

$$= \int_{-\infty}^{x} \left[\int_{-\infty}^{\infty} f_{X,Y}(u,v)\, dv \right] du. \quad (\star)$$

Remark :- Integral sign can be interchange & become Fubini's theorem is applicable here $(f_{X,Y}(\cdot,\cdot) \ge 0)$. $\forall (\cdot,\cdot)$,

$$\underbrace{f_X(u)}_{\text{Marginal PDF}} = \left.\frac{d}{dx} F_X(x)\right|_{x=u} = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\, dv$$

Remark:– These concepts can be extended for any finite dimensional case.

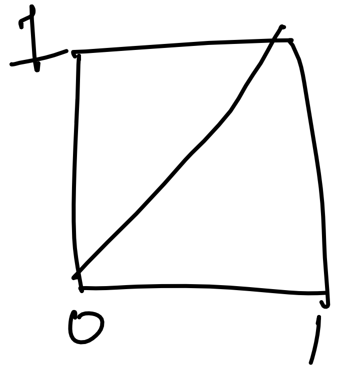
Example:– $f_{X,Y}(x,y) = \frac{12}{7}(x^2 + xy)$, $0 \le x \le 1$, $0 \le y \le 1$.

[Check that $f_{X,Y}(\cdot,\cdot)$ is a proper joint p.d.f.

i.e., $f_{X,Y}(\cdot,\cdot) \ge 0 \ \forall (\cdot,\cdot)$ & $\iint f_{X,Y}(u,v)\,du\,dv = 1$].

Suppose, we want to compute $P[\omega : X(\omega) > Y(\omega)]$

$$A = \{(x,y) : 0 \le y \le x \le 1\}.$$



$$P[\omega : X(\omega) \geq Y(\omega)] = P[X \geq Y]$$

$$= \iint_A f_{X,Y}(u, v)\, du\, dv, \text{ where } A = \{(x, y) : 0 \leq y \leq x \leq 1\}.$$

$$= \int_0^1 \left[\int_0^u f_{X,Y}(u, v)\, dv \right] du.$$

$$= \frac{12}{7} \int_0^1 \left[\int_0^u (u^2 + uv)\, dv \right] du$$

Check this result

$$= \frac{9}{14}$$

For this example, the marginal density of X will be

$$f_X(x) = \int_0^1 f_{X,Y}(x,v)\,dv = \frac{12}{7}\left(x^2 + \frac{x}{2}\right), \quad 0 \le x \le 1.$$

$$= \frac{12}{7}\int_0^1 (x^2 + xv)\,dv$$

Multivariate Joint distn (Necessary properties) :-

1) $F_{X_1,\dots,X_d}(x_1,\dots,x_d) \to 1$ if $x_i \to \infty$ for all $i = 1,\dots,d$.

2) $F_{X_1,\dots,X_d}(x_1,\dots,x_d) \to 0$ if at least $x_i \to -\infty$

$$\left[\, P[X_1 \le x_1 \cap \dots \cap X_d \le x_d] \le P[X_1 \le x_1] \Rightarrow 0 \text{ if } x_1 \to -\infty \,\right]$$

suppose $x_1 \to -\infty$

3) $F_{X_1,\dots,X_d}(x_1,\dots,x_d)$ is non-decreasing with respect to each component.

4) $F_{X_1,\dots,X_d}(x_1,\dots,x_d)$ is right contn with respect to each component.

Remark:- However, if a function $G_{X_1,\ldots,X_d}(x_1,\ldots,x_d)$ satisfies 1), 2), 3) and 4), it does not give any guarantee that $G_{X_1,\ldots,X_d}(x_1,\ldots,x_d)$ will be a CDF. In other words, these are all necessary properties but NOT sufficient properties for $d \geq 2$.

Rough idea:- Notion of monotonicity in multidimension / abstract space

i) $f: \mathbb{R} \to \mathbb{R}$, when $f(x) \uparrow x$, i.e, $f(x) \geq f(y)$
$\Leftrightarrow x \geq y$.

In other words, you can write

$f(x) \uparrow x$, i.e, when $\{f(x) - f(y)\} \times (x-y) \geq 0$. $\forall x, y$. (*)

Extension:- $\quad f: \mathbb{R}^d \to \mathbb{R}^d$

$\underset{\sim}{f}(\underset{\sim}{x}) \uparrow \underset{\sim}{x}$, when $\langle \underset{\sim}{f}(\underset{\sim}{x}) - \underset{\sim}{f}(\underset{\sim}{y}), \underset{\sim}{x} - \underset{\sim}{y} \rangle \geq 0$ $\quad \forall\ \underset{\sim}{x} \in \mathbb{R}^d, \underset{\sim}{y} \in \mathbb{R}^d.$

$f: \mathcal{H} \to \mathcal{H}$

$f(\underset{\sim}{x}) \uparrow x$, when $\langle f(\underset{\sim}{x}) - f(\underset{\sim}{y}), \underset{\sim}{x} - \underset{\sim}{y} \rangle \geq 0$ $\quad \forall\ \underset{\sim}{x} \in \mathcal{H}, \underset{\sim}{y} \in \mathcal{H}.$

One counter example (about those four properties only necessary properties for $d \geq 2$).

Suppose $F_{X_1, X_2}(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 + x_2 < 0 \\ 1 & \text{if } x_1 + x_2 \geq 0. \end{cases}$

Check that $F_{X_1, X_2}(x_1, x_2)$ satisfies the properties 1), 2), 3) & 4).

Suppose $(a_1, b_1) = (0, 2)$ & $(a_2, b_2) = (-1, 1)$.

$P[a_1 \leq X_1 \leq b_1, \ a_2 \leq X_2 \leq b_2] = F_{X_1 X_2}(2, 1) - F_{X_1, X_2}(0, 1) - F_{X_1, X_2}(2, -1) + F_{X_1, X_2}(0, -1)$

$= 1 - 1 - 1 - 0 = -1 < 0.$