$\rightarrow$ Convergence in prob.

Examples :- Suppose $X_1, \ldots, X_n \sim \text{Unif}(0, \theta)$.

Want to show that $\text{Max}(X_1, \ldots, X_n) \xrightarrow{P} \theta$.

$$\text{Max}(X_1, \ldots, X_n) = X_{(n)}$$

Proofs :-

Using sufficient condition :-

$E(X_{(n)}) \longrightarrow \theta$ as $n \to \infty$ & $\text{Var}(X_{(n)}) - \underbrace{\text{Var}(\theta)}_{=0} \longrightarrow 0$ as $n \to \infty$

$\Longleftrightarrow \text{Var}(X_{(n)}) \longrightarrow 0$ as $n \to \infty$.

Using the distn fn & density fn of $X_{(n)}$, we have

$$f_{X_{(n)}}(y) = \frac{n y^{n-1}}{\theta^n}, \quad 0 < y < \theta.$$

p.d.f. of $X_{(n)}$

$$\text{So, } E[X_{(n)}] = \int_0^\theta y \times \frac{n y^{n-1}}{\theta^n}\, dy = \frac{n}{n+1}\theta \longrightarrow \theta \text{ as } n \to \infty.$$

Note that

$$\operatorname{Var}[X_{(n)}] = E[X_{(n)}^2] - \{E[X_{(n)}]\}^2$$

$$E[X_{(n)}^2] = \int_0^\theta y^2 \times \frac{n \times y^{n-1}}{\theta^n}\, dy \overset{\text{Check}}{=} \frac{n}{n+2}\theta^2$$

So, $\operatorname{Var}[X_{(n)}] = \frac{n}{n+2}\theta^2 - \left\{\frac{n}{n+1}\theta\right\}^2$

$\longrightarrow 0$ as $n \to \infty$.

Hence, $X_{(n)} \xrightarrow{P} \theta$.

From def^n :-

For any $\epsilon > 0$,

$P[|X_{(n)} - \theta| < \varepsilon] \longrightarrow 1$ as $n \to \infty$

$\Leftrightarrow P[|X_{(n)} - \theta| > \varepsilon] \longrightarrow 0$ as $n \to \infty$

Consider,

$$P\left[\,|X_{(n)} - \theta| \le \varepsilon\,\right]$$

$$= P\left[\,\theta - \varepsilon \le X_{(n)} \le \theta + \varepsilon\,\right]$$

$$= P\left[X_{(n)} \le \theta + \varepsilon\right] - P\left[X_{(n)} \le \theta - \varepsilon\right]$$

$$= 1 - P\left[X_{(n)} \le \theta - \varepsilon\right]$$

$$= 1 - P\left[X_1 \le \theta - \varepsilon,\ X_2 \le \theta - \varepsilon, \ldots, X_n \le \theta - \varepsilon\right]$$

In this case $P[|X_{(n)} - \theta| \le \varepsilon] = 1$.

$\longrightarrow$ [Note that if $\varepsilon > \theta$, $P[X_i \le \theta - \varepsilon] = 0$.

$$= 1 - \prod_{i=1}^{n} P\left[X_i \le \theta - \varepsilon\right] \quad \text{since } X_i\text{'s are indep.}$$

$$= 1 - \prod_{i=1}^{n} \int_0^{\theta-\varepsilon} \frac{1}{\theta}\, dy = 1 - \prod_{i=1}^{n} \left(\frac{\theta - \varepsilon}{\theta}\right) = 1 - \left(1 - \frac{\varepsilon}{\theta}\right)^n$$

$\longrightarrow 1$ as $n \to \infty$ since $\varepsilon < \theta$ is considered here!

Hence, $X_{(n)} \xrightarrow{P} \theta$.

Try to prove by yourself.

A few results :-

i) $X_{(n)} \xrightarrow{P} X \implies aX_{(n)} \xrightarrow{P} aX$ for any $a \in \mathbb{R}$.

ii) $X_{(n)} \xrightarrow{P} X \implies g(X_{(n)}) \xrightarrow{P} g(X)$ for any continous f^n g.

$\mapsto$ Continous Mapping theorem.

iii) **Slutskey's result :-**

$X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$

Then (a) $X_n + Y_n \xrightarrow{P} X + Y$.

(b) $X_n Y_n \xrightarrow{P} XY$.

(c) $\dfrac{X_n}{Y_n} \xrightarrow{P} \dfrac{X}{Y}$ if $P[Y=0] = 0$.

A few more remarks :-

$X_1, \dots, X_n \sim U(0, \theta)$

Already, we have established $X_{(n)} \xrightarrow{P} \theta$.

This implies that $X_{(n)}^2 + 2 \xrightarrow{P} \theta^2 + 2$.

or $X_{(n)}^{101} \xrightarrow{P} \theta^{101}$.

Weak law of large numbers (WLLN) :-

General statement :- $\{X_n\}$ is a sequence of random variables

$\{X_n\}$ will satisfy WLLN if $\exists\, a_n \geq 0$ & $b_n \geq 0$

& $\dfrac{X_n - a_n}{b_n} \xrightarrow{P} c$.

Well known versions:-

i) Let $\{X_n\}$ be a sequence of i.i.d. random variables with $Var(X_1) < \infty$. Then $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{P} E(X_1)$.

Proof:- For any $\varepsilon > 0$, we have

$$P\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - E(X_1)\right| > \varepsilon\right]$$

$$\leq \frac{E\left[X_1 - E(X_1)\right]^2}{n\varepsilon^2}$$

$$= \frac{Var(X_1)}{n\varepsilon^2} \longrightarrow 0 \text{ as } n \to \infty$$

since $Var(X_1) < \infty$.

Note that

$$\left[E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = E(X_1)\right]$$

$$Var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n} E\left[X_1 - E(X_1)\right]^2$$

2. <u>Kinchin's WLLN :-</u>

Let $X_1, \dots, X_n$ be i.i.d. sequence of random variables and $E|X_1| < \infty$.

Then $\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{P} E(X_1)$ as $n \to \infty$.

<u>Examples :- 1 :-</u> $X_1, \dots, X_n \sim \text{Ber}(1, \theta)$, $0 < \theta < 1$.

$E(X_1) = \theta$.

$\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{P} \theta$.

<u>Example 2 :-</u> $X_1, \dots, X_n \sim N(\mu, 1)$.

$E(X_1) = \mu$ & $Var(X_1) = 1$.

In this case, $\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{P} \mu$.

Example 3 :- $X_1, \dots, X_n \overset{i.i.d}{\sim} t_1$. (Cauchy)

$$f_X(x) = \frac{1}{\pi\{1 + (x-\mu)^2\}}, \quad x \in \mathbb{R}, \quad \mu \in \mathbb{R}.$$

However, $E(X)$ & $Var(X)$ do NOT exist.

Hence, WLLN is NOT applicable on t_1.

In fact, $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \sim t_1$.

Example 4 :- $X_1, \dots, X_n \overset{i.i.d}{\sim} t_2$

$$f_X(x) = \frac{1}{c \cdot \{1 + (x-\mu)^2\}^{3/2}} \quad x \in \mathbb{R}, \ \mu \in \mathbb{R},$$

Here $E(X)$ exists but $Var(X)$ does NOT exist.

Hence, Kinchin's WLLN is applicable but earlier one cannot use