Bonferroni's inequality:-

Statement:- If $A_1, A_2, \ldots, A_n$ are events, then

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1).$$

Proof:-

$$P\left(\bigcap_{i=1}^{n} A_i\right) = 1 - P\left(\left(\bigcap_{i=1}^{n} A_i\right)^c\right)$$

$$= 1 - P\left(\bigcup_{i=1}^{n} A_i^c\right) \quad \longrightarrow \text{De morgan's law}$$

$$\geq 1 - \sum_{i=1}^{n} P(A_i^c) \quad \text{(Using subadditivity of P function)}$$

or Bool's inequality

$$= 1 - \sum_{i=1}^{n}\left\{1 - P(A_i)\right\}$$

$$= 1 - n + \sum_{i=1}^{n} P(A_i) = \sum_{i=1}^{n} P(A_i) - (n-1)$$

(Proved).

Significance of these inequalities:- These type of inequalities can be applied to compute type-I error for multiple testing of hypotheses problem.

A few elementary counting rules:-

1. If the first element of a pair can be selected in n_1 ways, and for each of these n_1 ways, the second element can be selected in n_2 ways, then the total no. of pairs will be $n_1 n_2$.

(This result can be extended for finitely many case).

ii) Suppose, there are n many distinct objects $(a_1, \ldots, a_n)$.
Want to draw $k \ (\leq n)$ many objects. The total number of selection (without replacement)

1. If order does matter $= {}^{n}P_{k} = \dfrac{n!}{(n-k)!}$

2. If order does NOT matter $= {}^{n}C_{k} = \dfrac{n!}{k!\,(n-k)!}$

(you can check it by $n=4$, $k=2$).

Conditional probability:-

Motivating example:-

T_+ = High blood concentration (positive test)

T_- = Low blood concentration (negative test)

D_+ = Toxicity (disease present).

D_- = No toxicity (No disease).

	D_+	D_-	Total
T_+	25	14	39
T_-	18	78	96
	43	92	135

$\Longleftrightarrow$

Proportion-wise (i.e, "probality")

	D_+	D_-	
T_+	0.185	0.104	0.289
T_-	0.133	0.578	0.711
	0.318	0.682	1.00

So, we have $P(D^+) = 0.318 \; (\approx 32\%)$

$$P(D^+ \mid T^+) = \frac{25}{39} \approx 0.640 \; (\approx 64\%).$$

Mathematical def^n of conditional probability :-

Suppose, A and B are two events.

The prob. A given B is denoted by $P(A|B)$.

It is defined as $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

Remark :- i) Here Ω is not the sample space; rather B is the sample space.

ii) Check that $P(A|B)$ is also a proper probability function.

Multiplication rule :-

Since $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$$\Leftrightarrow P(A \cap B) = P(A|B)\, P(B).$$

Remark :- This result can be extended for finitely many events as well.

$$P\left(\bigcap_{i=1}^{n} C_i\right) = P(C_1)\, P(C_2|C_1)\, P(C_3|C_1 \cap C_2) \cdots P(C_n | C_1 \cap C_2 \cdots \cap C_{n-1}).$$

(Check it!).

Example:- An urn contains 3 red balls and 1 blue ball. Two balls are selected without replacement. Want to know what is the prob. that both are red.

Sol^n:- R_1 : A red ball is drawn on the first trial.

R_2 : A " " " " " " second trial.

$$P(R_1 \cap R_2) = P(R_1)\, P(R_2 | R_1)$$

Using Multiplication rule

$$= \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$$

Mathematical Application of Conditional prob

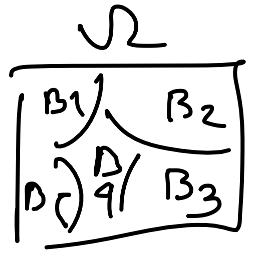
Vaguely

Theorem of Total prob. Bayes' theorem.

Theorem of Total probality :-

Statement :- Let $B_1, \dots, B_n$ be such that

$\bigcup_{i=1}^{n} B_i = \Omega$ and $B_i \cap B_j = \phi \ \forall \ i \neq j$.



with $P(B_i) > 0 \ \forall \ i$.

Then for any event A, we have

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i)\, P(B_i).$$

Proof :- First observe that

$$P(A) = P(A \cap \Omega)$$

$$= P\left(A \cap \left(\bigcup_{i=1}^{n} B_i\right)\right) \longrightarrow ①$$

$$= P\left(\bigcup_{i=1}^{n} (A \cap B_i)\right)$$ (Using Distributive law).

Since B_i's are disjoint $\implies$ $A \cap B_i$'s will also be disjoint.

From (1) $\implies$

Hence, $P\left(\bigcup_{i=1}^{n} (A \cap B_i)\right) = \sum_{i=1}^{n} P(A \cap B_i)$. — (2)

multiplicative rule

$$= \sum_{i=1}^{n} P(A \mid B_i)\, P(B_i)$$

Using (2) in (1), we have

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i)\, P(B_i) \quad \text{(Proved)}.$$