Problem Set #3

(1) (a)

$\underline{Y = |X|}$

$\forall x \in R \quad Y^{-1}(-\infty, x] = \{\omega : Y(\omega) \le x\} = \{\omega : |X(\omega)| \le x\}$

$= \{\omega : -x \le X(\omega) \le x\}$

$= X^{-1}[-x, \infty) \cap X^{-1}(-\infty, x]$

As X is a r.v. $X^{-1}[-x, \infty) \cap X^{-1}(-\infty, x] \in \mathcal{F}$

$\left[X \text{ is a r.v.} \Rightarrow X^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathcal{B} \right.$

$\Rightarrow X^{-1}[-x, \infty) \in \mathcal{F} \ \& \ X^{-1}(-\infty, x] \in \mathcal{F}$

$\left. \Rightarrow X^{-1}[-x, \infty) \cap X^{-1}(-\infty, x] \in \mathcal{F} \right]$

$\Rightarrow Y^{-1}(-\infty, x] \in \mathcal{F} \quad \forall x \in \mathbb{R}$

$\Rightarrow Y = |X|$ is a r.v.

(b) $\underline{Y = X^2}$; $\quad \forall x \in \mathbb{R} \quad Y^{-1}(-\infty, x] = \{\omega : Y(\omega) \le x\}$

$= \{\omega : X^2(\omega) \le x\}$

$= \{\omega : -\sqrt{x} \le X(\omega) \le \sqrt{x}\}$

$= X^{-1}[-\sqrt{x}, \infty) \cap X^{-1}(-\infty, \sqrt{x}] \in \mathcal{F}$

$\Rightarrow Y = X^2$ is a r.v.

Sly part (c).

(2) $\Omega = [0,1] \quad \mathcal{F}$: Borel σ-field of subsets of Ω

$$X(\omega) = \begin{cases} \omega, & 0 \le \omega \le 1/2 \\ \omega - \frac{1}{2}, & \frac{1}{2} < \omega \le 1 \end{cases}$$

$$X^{-1}(-\infty, x] = \begin{cases} \phi \in \mathcal{F}, & x < 0 \\ [0,x] \cup (\frac{1}{2}, \frac{1}{2}+x] \in \mathcal{F} & 0 \le x < \frac{1}{2} \\ \Omega \in \mathcal{F}, & x \ge \frac{1}{2} \end{cases}$$

$\Rightarrow X^{-1}(-\infty, x] \in \mathcal{F} \quad \forall x \in \mathbb{R} \Rightarrow X$ is a r.v.

(3) $\Omega = \{1, 2, 3, 4\}$

$\mathcal{F} = \{\phi, \Omega, \{1\}, \{2, 3, 4\}\}$

$X(\omega) = \omega + 1 \rightarrow 2, 3, 4, 5$

$$X^{-1}(-\infty, x] = \begin{cases} \phi \ (\in \mathcal{F}) & x < 2 \\ \{1\} \ (\in \mathcal{F}) & 2 \le x < 3 \\ \{1, 2\} \notin \mathcal{F} & 3 \le x < 4 \end{cases}$$

$\Rightarrow X$ is not a r.v.

(4) $\Omega = \{\underbrace{1H, \ldots, KH}_{\text{all hearts}}, \underbrace{1S, \ldots KS}_{\text{Spades}}, \underbrace{1D, \ldots KD}_{\text{Diamond}}, \underbrace{1C, \ldots KC}_{\text{club}}\}$

Ace → 1H, King → KH

$$X(\omega) = \begin{cases} 4 & \text{ace} \\ 3 & \text{King} \\ 2 & \text{Queen} \\ 1 & \text{Jack} \\ 0 & \text{o/w} \end{cases} \qquad \mathcal{F} = \text{power set}$$

$$X^{-1}(-\infty, x] = \begin{cases} \phi & \text{If } x < 0 \\ \{2H, \ldots 10H, 2S, \ldots, 10S, 2D, \ldots 10D, 2C, \ldots 10C\} & \text{If } 0 \le x < 1 \\ \{2H, \ldots JH, 2S, \ldots JS, 2D, \ldots JD, 2C, \ldots JC\} & \text{If } 1 \le x < 2 \\ \{2H, \ldots QH, 2S, \ldots QS, 2D, \ldots QD, 2C, \ldots QC\} & 2 \le x < 3 \\ \{2H, \ldots KH, 2S, \ldots KS, 2D, \ldots KD, 2C, \ldots KC\} & 3 \le x < 4 \\ \Omega & x \geq 4 \end{cases}$$

$\Rightarrow X^{-1}(-\infty, x] \in \mathcal{F} \ \forall x \in \mathbb{R}$

$\Rightarrow X$ is a r.v.

d.f

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{36}{52} & 0 \le x < 1 \\ \frac{40}{52} & 1 \le x < 2 \\ \frac{44}{52} & 2 \le x < 3 \\ \frac{48}{52} & 3 \le x < 4 \\ 1 & x \geq 4 \end{cases}$$

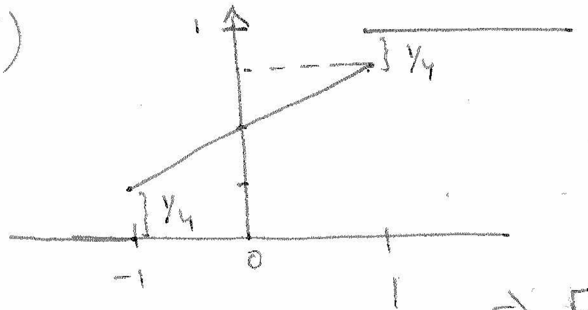
jumps at $x = 0 \to \frac{9}{13}$

$x = 1 \to \frac{1}{13}$

$x = 2 \to \frac{1}{13}$

$x = 3$ & $x = 4 \to \frac{1}{13}$

(5)



(i) $F(.)$ is non decreasing ✓

(ii) $F(.)$ is right contin everywhere

(iii) $F(-\infty) = 0$ & $F(\infty) = 1$

$\Rightarrow F(.)$ is a d.f.

$$P\left(-\tfrac{1}{2} < X \le \tfrac{1}{2}\right) = F\left(\tfrac{1}{2}\right) - F\left(-\tfrac{1}{2}\right)$$

$$= \frac{5}{8} - \frac{3}{8} = \frac{2}{8}$$

$$P(X=0) = F(0) - F(0-) = 0$$

$$P(X=1) = F(1) - F(1-) = 1 - \frac{3}{4} = \frac{1}{4}$$

$$P(-1 \le X < 1) = F(1-) - F(-1-)$$

$$= \frac{3}{4} - 0 = \frac{3}{4}$$

Discrete part

$$\alpha F_d(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{4} & -1 \le x < 1 \\ \frac{1}{2} & x \geq 1 \end{cases}$$

Explain how you get this.

Continuous part

$$(1-\alpha) F_c(x) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{4} & -1 \le x < 1 \\ \frac{1}{2} & x \geq 1 \end{cases}$$

$$\Rightarrow \quad \alpha = \frac{1}{2} \ \& \quad F_d(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} & -1 \le x < 1 \\ 1 & x \geq 1 \end{cases}$$

$\downarrow$ discrete d.f.

$$F_c(x) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{2} & -1 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

continuous d.f.

(5) (a) $F(x)$ is not right continuous at $x = \frac{1}{2}$

$\Rightarrow F(.)$ is NOT a d.f.

(b) & (c) the n.s.c. holds for these 2 and hence they are d.f.s.

(7) $F(.)$ is ~~not~~ non-decreasing

$F(\infty)=1$, $F(-\infty)=0$

$F(x)$ is right cont $\forall x \in \mathbb{R}$

$\Rightarrow F(.)$ is a d.f.

$$P(X>6) = 1-P(X\le 6) = 1-F(6) = 1-\left(1-\frac{2}{3}e^{-6/3}-\frac{1}{3}e^{-[6/3]}\right)$$

$$= 1-\left(1-\frac{2}{3}e^{-2}-\frac{1}{3}e^{-2}\right) = e^{-2}.$$

$$P(X=5) = F(5)-F(5-)$$

$$=\left(1-\frac{2}{3}e^{-5/3}-\frac{1}{3}e^{-[5/3]}\right)-\left(1-\frac{2}{3}e^{-5/3}-\frac{1}{3}e^{-\left[\frac{5-}{3}\right]}\right)$$

$$=\left(1-\frac{2}{3}e^{-5/3}-\frac{1}{3}e^{-1}\right)-\left(1-\frac{2}{3}e^{-5/3}-\frac{1}{3}e^{-1}\right)=0$$

$$P(5\le X\le 8) = F(8)-F(5-)$$

$$= \text{— — —}$$

(8) $$P(-2\le X<5) = F(5-)-F(-2-)$$

$$= \frac{1}{2}-0 = \frac{1}{2}$$

$$P(0<X<5.5) = F(5.5-)-F(0)$$

$$=\left(\frac{1}{2}+\frac{1}{8}\right)-\frac{1}{2} = \frac{1}{8}$$

$$P(1.5<X\le 5.5 \mid X>2) = \frac{P(2<X\le 5.5)}{P(X>2)}$$

$$= \frac{F(5.5)-F(2)}{1-F(2)} = \frac{\left(\frac{1}{2}+\frac{1}{8}\right)-\frac{1}{2}}{1-\frac{1}{2}} = \frac{1}{4}$$

(9) For $x_1 < x_2$

$$F(x_2) - F(x_1) = \sum_1^n \alpha_i \big(\underbrace{F_i(x_2) - F_i(x_1)}_{\geq 0 \ \forall i} \big) \geq 0$$

$\Rightarrow F(x)$ is non-decreasing.

$$F(-\infty) = \sum \alpha_i F_i(-\infty) = 0 \quad \& \quad F(\infty) = 1$$

$$F(x+) = \lim_{z \downarrow x} F(z) = \lim_{z \downarrow x} \sum \alpha_i F_i(z)$$

$$= \sum \alpha_i F_i(x+) = \sum \alpha_i F_i(x) = F(x)$$

$\Rightarrow F(\cdot)$ is a d.f.

(10) $G(\infty) = F_1(\infty) + F_2(\infty) = 2 \neq 1$

$\Rightarrow G(\cdot)$ is NOT a d.f.

(11) Right cont at $x = 0 \Rightarrow F(0) = F(0+)$

$$\Rightarrow 0 = \alpha + k \quad -(i)$$

$$F(\infty) = 1 \Rightarrow \alpha = 1 \Rightarrow k = -1$$

(12) Right cont at $x = 3 \Rightarrow F(3) = F(3+)$

$$\Rightarrow \frac{6+c}{8} = 1 \Rightarrow c = 2$$

f^n $F(\cdot)$ is having jump at $\underline{x = 0}$ only (magnitude $2/8$)

discrete part of d.f

$$\alpha F_d(x) = \begin{cases} 0 & x < 0 \\ \frac{2}{8} & x \geq 0 \end{cases}$$

Cont part

$$(1-\alpha)\,F_c(x) = \begin{cases} 0 & \text{If } x < 0 \\ x/8 & 0 \le x < 1 \\ x^2/8 & 1 \le x < 2 \\ \frac{2x}{8} & 2 \le x \le 3 \\ \frac{6}{8} & x > 3 \end{cases}$$

$$\Rightarrow \alpha = \frac{2}{8} = \frac{1}{4} \quad \& \quad (1-\alpha) = \frac{3}{4}$$

&

$$F_d(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \quad \& \quad F_c(x) = \begin{cases} 0 & x < 0 \\ x/6 & 0 \le x < 1 \\ x^2/6 & 1 \le x < 2 \\ x/3 & 2 \le x \le 3 \\ 1 & x > 3 \end{cases}$$

(13) $Y = X^+$

$P(Y \le y) = 0$ If $y < 0$

If $y = 0$ $P(Y \le 0) = P(X^+ \le 0) = P(X^+ = 0)$

$= P(X \le 0) = F(0)$

If $y > 0$ $P(Y \le y) = P(X^+ \le y)$

$= P(X \le y) = F(y)$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ F(y) & y \geq 0 \end{cases}$$

$$Z = |X|$$

$$F_Z(z) = P(|X| \le z)$$

$$= P(-z \le X \le z)$$

$$= \begin{cases} F(z) - F(-z-) & \text{If } z \ge 0 \\ 0 & \text{If } z < 0 \end{cases}$$

(14) For $x_1 < x_2$

$$F(x_2) - F(x_1) = \int_{-\infty}^{\infty} F_1(x_2 - y)\, dF_2(y) - \int_{-\infty}^{\infty} F_1(x_1 - y)\, dF_2(y)$$

$$= \int_{-\infty}^{\infty} \Big(\underbrace{F_1(x_2 - y) - F_1(x_1 - y)}_{\ge 0} \Big)\, dF_2(y)$$

$$\ge 0 \quad \forall\, x_1 < x_2$$

$$F(\infty) = \int_{-\infty}^{\infty} F_1(\infty)\, dF_2(y) = F_2(\infty) - F_2(-\infty) = 1$$

$$F(-\infty) = 0$$

$$F(x+) = \int_{-\infty}^{\infty} F_1((x+) - y)\, dF_2(y)$$

$$= \int_{-\infty}^{\infty} F_1(x - y)\, dF_2(y) = F(x)$$

$\Rightarrow$ $F(.)$ as defined is a d.f.

(15)

(a) $f(1) < 0 \Rightarrow f(.)$ is not a p.m.f.

(b) $f(x) \geq 0 \;\; \forall x$

$$\sum_x f(x) = e^{-\lambda} \sum_0^{\infty} \frac{\lambda^x}{x!} = 1$$

$\Rightarrow f(.)$ is a p.m.f.

(c) $\sum_x f(x) \neq 1 \Rightarrow f(.)$ is not a p.m.f.

(16) $f(x) \geq 0 \quad \forall\, c \in [0, 1]$.

$\sum_x f(x) = 1$ also $\forall\, c \in [0,1]$

$\Rightarrow \forall c \ni 0 \leq c \leq 1$, $f(.)$ is p.m.f

(17). Suppose

$P(X=-3) = P(X=-2) = P(X=-1) = P(X=1) = P(X=2) = P(X=3) = p$

$\Rightarrow P(X<0) = 3p = P(X>0) = P(X=0)$

$P(X<0) + P(X=0) + P(X>0) = 1$

$\Rightarrow p = \frac{1}{9}$

p.m.f.

$X=x$	-3	-2	-1	0	1	2	3
$P(X=x)$	$1/9$	$1/9$	$1/9$	$3/9$	$1/9$	$1/9$	$1/9$

$$F_X(x) = \begin{cases} 0 & x < -3 \\ 1/9 & -3 \le x < -2 \\ 2/9 & -2 \le x < -1 \\ 3/9 & -1 \le x < 0 \\ 6/9 & 0 \le x < 1 \\ 7/9 & 1 \le x < 2 \\ 8/9 & 2 \le x < 3 \\ 1 & x \geq 3 \end{cases}$$

(18) X : r.v. denoting total # of clocks working after 300 days.

X can takes values in $\{0, 1, 2, 3\}$

$$P(X=0) = P(A^c B^c C^c) = P(A^c) P(B^c) P(C^c)$$
$$= (1-0.95)(1-0.9)(1-0.8) = p_0, \text{ say}.$$

$$P(X=1) = P(A B^c C^c \cup A^c B C^c \cup A^c B^c C)$$
$$= P(A) P(B^c) P(C^c) + P(A^c) P(B) P(C^c) + P(A^c) P(B^c) P(C)$$
$$= - -$$
$$= p_1 \text{ say}$$

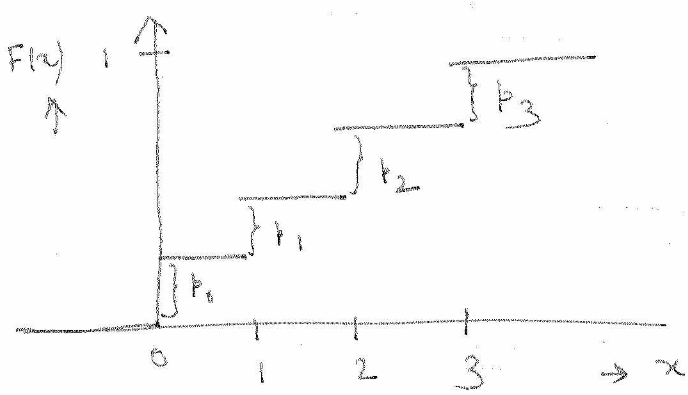
$$P(X=2) = P(A B C^c \cup A B^c C \cup A^c B C)$$
$$= P(A) P(B) P(C^c) + P(A) P(B^c) P(C) + P(A^c) P(B) P(C)$$
$$= - - = p_2 \text{ say}$$

$$P(X=3) = P(ABC) = P(A) P(B) P(C) = .95 \times .9 \times .8$$
$$= p_3 \text{ say}$$

d.f. p.m.f

$X=x$	0	1	2	3
$P(X=x)$	p_0	p_1	p_2	p_3

d.f.



(19) $f(x) \geq 0 \quad \forall x$

$$\int_{-\infty}^{\infty} f(x)\,dx = \theta^2 \int_0^{\infty} x e^{-\theta x}\,dx = \theta^2 \cdot \frac{\Gamma 2}{\theta^2} = 1$$

$\Rightarrow f(\cdot)$ is a p.d.f.

$$P(2 < X < 3) = F(3) - F(2)$$

$$\& \; P(X > 5) = 1 - F(5)$$

where, $F(x) = \int_{-\infty}^{x} f(x)\,dx = \begin{cases} \theta^2 \int_0^{x} x e^{-\theta x}\,dx & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$

integration by parts

(20) If $c \geq 0$ then $f(x) \geq 0 \quad \forall x$

$$\int_0^{\infty} f(x)\,dx = 1$$

$$\Rightarrow c\int_0^{\infty} (x+1)e^{-\lambda x}\,dx = 1$$

$$\text{i.e. } c\left[\frac{\Gamma 2}{\lambda^2} + \frac{\Gamma 1}{\lambda}\right] = 1$$

$$\Rightarrow c = \frac{\lambda^2}{1+\lambda}$$

d.f. $F(x) = \int_0^x f(x)\,dx = \frac{\lambda^2}{1+\lambda}\left[\int_0^x (1+x)e^{-\lambda x}\,dx\right]$ if $x \geq 0$

$= 0$ if $x < 0$

by parts.

(21)

$f(x) \geq 0 \quad \forall x$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-3}^{3} \frac{x^2}{18}\,dx = \frac{1}{18}\left.\frac{x^3}{3}\right|_{-3}^{3} = \frac{1}{18}\cdot 18 = 1$$

$\Rightarrow f(.)$ is a p.d.f.

$$F_X(x) = \begin{cases} 0 & \text{if } x < -3 \\ \frac{x^3+27}{54} & -3 \leq x \leq 3 \\ 1 & x > 3 \end{cases}$$

$P(|X| < 1) = F(1) - F(-1)$

$= \frac{28}{54} - \frac{26}{54}$

$P(X^2 < 9) = 1$