To find the most powerful test (MP) :-

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Likelihood ratio test

(To address the question :- What should be the form of the rejection region.)

(Sometimes in the literature, it is called Neyman - Pearson Theorem).

A test will reject H_0 if

$$\Lambda = \frac{\sup_{\theta \in \Theta_0} L(\theta \in \Theta_0 \mid x_1, \ldots, x_n)}{\sup_{\theta \in \Theta_1} L(\theta \in \Theta_1 \mid x_1, \ldots, x_n)}$$

Θ_0 : Parameter space under H_0

Θ_1 : Parameter space under H_1

The MP test at level of significance $= \alpha$ is

$\Lambda < K$, where K is such that $P(\lambda^* < K \mid \theta \in \Theta_0) \leq \alpha$.

A few remarks!–

i) Smaller the ratio, more likely to reject H_0.

ii) LRT is the most Powerful test at a given α.

Recall the first example (Urn containing 10 balls, (Wh & Re) 5 or 7 of them are white.

$H_0: p_0 = 0.5$ ag. $H_1: p_1 = 0.7$. $(p_1 > p_0)$.

Consider $\Lambda = \dfrac{L(0.5 \mid x_1 \ldots, x_n)}{L(0.7 \mid x_1 \ldots x_n)}$

$$= \frac{L(p_0 \mid x_1 \ldots x_n)}{L(p_1 \mid x_1 \ldots x_n)} = \frac{p_0^{\sum_{i=1}^{n} x_i} (1-p_0)^{n-\sum_{i=1}^{n} x_i}}{p_1^{\sum_{i=1}^{n} x_i} (1-p_1)^{n-\sum_{i=1}^{n} x_i}}$$

$$= \left\{ \frac{p_0(1-p_1)}{p_1(1-p_0)} \right\}^{\sum_{i=1}^{n} x_i} \left(\frac{1-p_0}{1-p_1} \right)^n$$

Observations:- $p_0 < p_1 \Rightarrow 1-p_0 > 1-p_1 \Rightarrow \underbrace{\left(\frac{1-p_0}{1-p_1}\right) > 1}_{(*)}$

$\Rightarrow$

$\underbrace{p_0 < p_1}_{(1)}$ $\underbrace{1-p_1 < 1-p_0}_{(2)}$

(1) & (2) $\Rightarrow p_0(1-p_1) < p_1(1-p_0)$

$$\Leftrightarrow \frac{p_0(1-p_1)}{p_1(1-p_0)} < 1 \quad \text{——} \quad (**)$$

Hence, The LRT test will be

$$R = \left\{ \left\{ \frac{p_0(1-p_1)}{p_1(1-p_0)} \right\}^{\sum_{i=1}^{n} x_i} \left(\frac{1-p_0}{1-p_1} \right)^n < k \right\}$$

$$\Leftrightarrow \quad R = \left\{ \left\{ \frac{p_0(1-p_1)}{p_1(1-p_0)} \right\}^{\sum_{i=1}^{n} x_i} < K^* \right\}$$

R ← Rejection regim

$$\Leftrightarrow \quad R = \left\{ \sum_{i=1}^{n} x_i \log \frac{p_0(1-p_1)}{p_1(1-p_0)} < \underbrace{\log K^*}_{K^{**}} \right\}.$$

$$\Leftrightarrow \quad R = \left\{ \sum_{i=1}^{n} x_i > K^{***} \right\}, \text{ when } K^{***} = \frac{K^{**}}{\log \frac{p_0(1-p_1)}{p_1(1-p_0)}}$$

And K^{***} is such that

$$P\left[\sum_{i=1}^{n} x_i > K^{***} \,\middle|\, p = 0.5 \right] \leq \alpha .$$

Another example:-

Suppose $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ $\longrightarrow$ σ^2 is known.

Want to find the MP test $H_0: \mu = \mu_0$ ag. $H_1: \mu = \mu_1$ (suppose $\mu_1 > \mu_0$).

Consider the LRT

$$\Lambda = \frac{L(\mu_0 \mid x_1, \dots, x_n)}{L(\mu_1 \mid x_1, \dots, x_n)} = \frac{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2}}{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_1)^2}}$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left\{\sum_{i=1}^{n}(x_i-\mu_0)^2 - \sum_{i=1}^{n}(x_i-\mu_1)^2\right\}\right)$$

Now, the rejection region of the MP test ~~xx~~ will be

$$R = \{ \lambda < K \}$$

$$\Rightarrow R = \left\{ \exp\left(-\frac{1}{2\sigma^2} \left(\sum_{i=1}^{n} (x_i - \mu_0)^2 - \sum_{i=1}^{n} (x_i - \mu_1)^2 \right)\right) < K \right\}.$$

$$\Rightarrow R = \left\{ \sum_{i=1}^{n} (x_i - \mu_0)^2 - \sum_{i=1}^{n} (x_i - \mu_1)^2 > K^* \right\}$$

$$\Rightarrow R = \left\{ 2n \bar{x}_n (\mu_1 - \mu_0) + n (\mu_0^2 - \mu_1^2)^2 > K^* \right\}, \quad \bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\Rightarrow R = \left\{ 2n \bar{x}_n (\mu_1 - \mu_0) > K^{**} \right\}$$

$$\Rightarrow R = \left\{ \bar{X}_n > K^{***} \right\},$$ where K^{***} is such that $P[\bar{X}_n > K^{***} \mid \mu = \mu_0] = \alpha$.

Explicit expression of $K^{***} = L$

$$\alpha = P\left[\overline{X_n} > L \mid \mu = \mu_0\right]$$

$$\Leftrightarrow P\left[\frac{\overline{X_n} - \mu_0}{\sigma/\sqrt{n}} > \frac{L - \mu_0}{\sigma/\sqrt{n}}\right] = \alpha$$

$$\Leftrightarrow P\left[Z > \frac{L - \mu_0}{\sigma/\sqrt{n}}\right] = \alpha \quad \text{, where } Z \sim N(0,1).$$

$$\Leftrightarrow \quad 1 - \underbrace{\Phi}_{\text{CDF of } N(0,1).}\left(\frac{L - \mu_0}{\sigma/\sqrt{n}}\right) = \alpha.$$

check

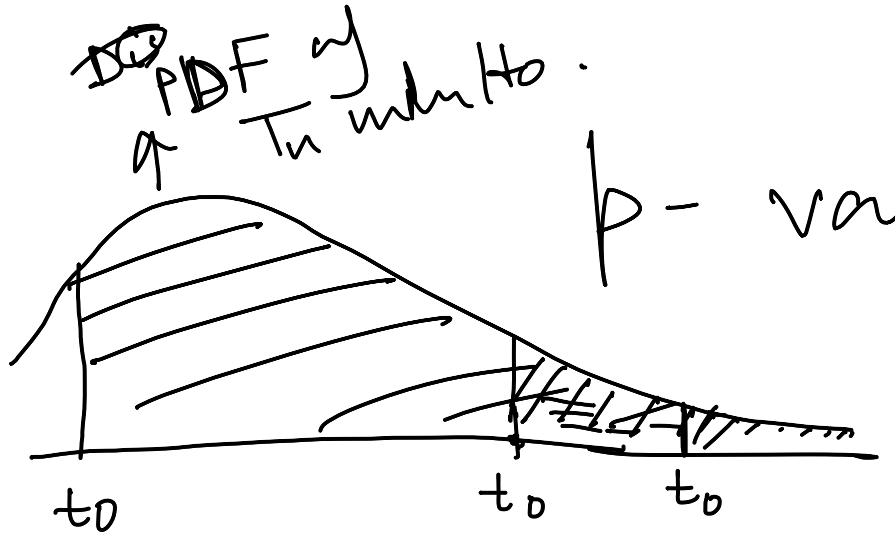
$$\Leftrightarrow L = \mu_0 + \frac{\sigma}{\sqrt{n}}\,\Phi^{-1}(1-\alpha).$$

p-value (Another way to summarize the performance of the test):-

p-value:- The smallest significance level at which H_0 would be rejected.

Let $X_1, \ldots, X_n$ be an r.s.

$t_0 = T_n(X_1, \ldots, X_n)$, where T_n is the test statistic.



$$p\text{-value} = P_{H_0}\left[T_n > t_0\right]$$

Recall the first example:-

$p = 0.5$ ag. $p = 0.7$

Let $X = 2$ (No. of white ball in 10 draws).

$$\text{p-value} = P_{H_0}[X > 2]$$

$$= P[X > 2 \mid p = 0.5]$$

$$= 1 - P[X \leq 1 \mid p = 0.5]$$

$\underset{\text{check}}{=}$ 0.9892

Under H_0, it is NOT rare event
(Cannot disagree with the null).