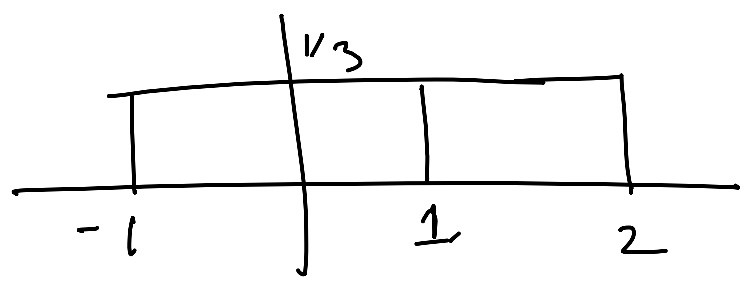
Last day's example !—

$$f_X(x) = \begin{cases} 1/3 & \text{if } -1 \leq x \leq 2 \\ 0 & , \text{ o.w.} \end{cases}$$



Want to derive the distn (p.d.f.) of $Y = X^2$

The range of X is -1 to 2.

Hence, the range of Y is 0 to 4.

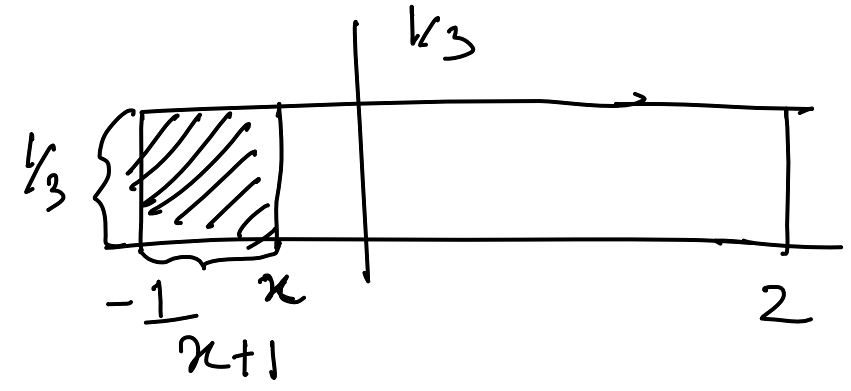
The CDF of Y is

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ P[\omega : Y(\omega) \leq y] & \text{if } 0 \leq y \leq 4 \\ 1 & \text{if } y > 4. \end{cases}$$

If $0 \le y \le 1$.

so, $P[Y \le y] = P[X^2 \le y] = P[-\sqrt{y} \le X \le \sqrt{y}]$

$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$. $\quad$ (*)



Notice that

$$F_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{1}{3}(x+1) & \text{if } -1 \le x \le 2, \\ 1 & \text{if } x > 2 \end{cases} \quad (**)$$

When $0 \le y \le 1$, we have $P[Y \le y] \overset{\text{Using (*)}}{=} F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$\Leftrightarrow 0 \le \sqrt{y} \le 1$

$\Rightarrow -\sqrt{y} \ge -1$

$= \frac{1}{3}(\sqrt{y}+1) - \frac{1}{3}(-\sqrt{y}+1)$

$= \frac{2}{3}\sqrt{y}$ when $0 \le y \le 1$.

The p.d.f. will be $f_Y(y) = \frac{d}{dy}\left(\frac{2}{3} y^{\frac{1}{2}}\right) = \frac{1}{3\sqrt{y}}$ if $0 \le y \le 1$.

If $1 \le y \le 4$, we have.

$$P[X^2 \le y] = F_X(\sqrt{y}) - \underset{0}{\underset{\shortparallel}{F_X(-\sqrt{y})}}$$

$\cancel{\phi}$

$y \ge 1$

$\Rightarrow \sqrt{y} \ge 1.$

$\Rightarrow -\sqrt{y} \le -1.$

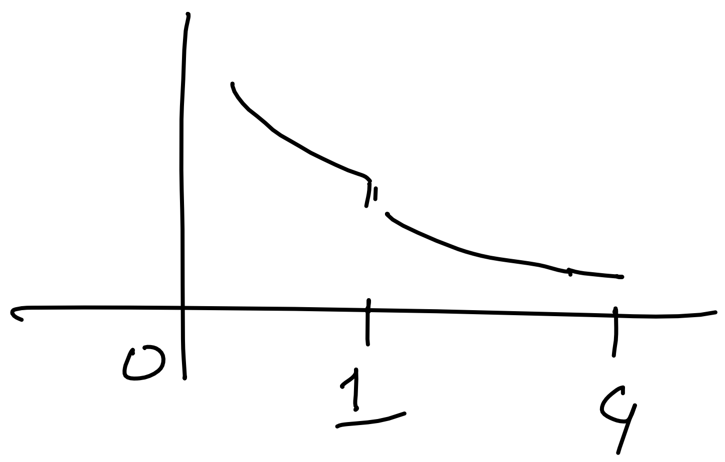
$$= F_X(\sqrt{y}) = \frac{1}{3}(\sqrt{y}+1)$$

$\longmapsto$ Using (**)

The p.d.f. will be $f_Y(y) = \frac{d}{dy}\left(\frac{1}{3}\sqrt{y}+1\right) = \frac{1}{6\sqrt{y}}$ if $1 \le y \le 4.$

$$f_Y(y) = \frac{1}{3\sqrt{y}} \quad \text{if } 0 \le y \le 1$$

$$= \frac{1}{6\sqrt{y}} \quad \text{if } 1 \le y \le 4$$



Method 2:- (Jacobian approach) :-

X : Random variable with p.d.f. $f_X(x)$.

$Y = g(X) \longrightarrow$ What will be the p.d.f.

Here we assume g is monotone.

It states that if g is monotone, the p.d.f. of $Y = g(X)$ will be

$$f_Y(y) = f_X(g^{-1}(y))\,|J|, \quad J \to \text{Jacobian}$$

p.d.f. of Y

where $J = \dfrac{dx}{dy}$.

[Rough idea:- $g(X) \xrightarrow{CDF} G$

$$G(y) = P[w : g(X)(w) \le y] = P[w : X(w) \le g^{-1}(y)] = F_X(g^{-1}(y))$$

CDF of X]

$$g(y) = \frac{d}{dy} F_X(g^{-1}(y)) = \frac{d}{dx} F_X(g^{-1}(y)) \frac{dx}{dy}$$

$$= f_X(g^{-1}(y)) \frac{dx}{dy}$$

Example :- $X \sim \text{Unif}(0,1)$.

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0, & \text{o.w.} \end{cases}$$

$e^{-y/2}$

Want to derive the CDF or the PDF of Y.

$Y = g(X) = -2\ln X$. ⊗

Observe that $0 < X < 1 \Rightarrow 0 \leq Y \leq \infty$.

Here, $g^{-1}(y) = e^{-\frac{y}{2}}$ & $\frac{dy}{dx} = -\frac{2}{x}$ (from ⊗) using ⊗

$$\Rightarrow \left|\frac{dx}{dy}\right| = \frac{x}{2} = \frac{e^{-y/2}}{2}$$

Hence, $f_Y(y) = f_X(g^{-1}(y))\,|J| = 1 \times \frac{1}{2}e^{-y/2}$ if $0 \leq y \leq \infty$

$= 0$, o.w.

Example :- $X \sim N(0,1)$.

Want to derive the CDF or PDF of $Y = e^X$.

Since $X \sim N(0,1)$, we have $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$

$= 0 \quad$, o.w.

$Y = e^X \Rightarrow \quad X \in (-\infty, \infty)$.

$Y \in (0, \infty)$.

Observe that $g(x) = e^x$ is a monotone $f^{\underline{n}}$.

$y = g(x) = e^x \Rightarrow g^{-1}(y) = \ln y \quad$ if $0 \leq y \leq \infty$.

$\frac{dy}{dx} = e^x$. Now, $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$.

$\Rightarrow \frac{dx}{dy} = \frac{1}{e^x} = \frac{1}{y}$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\ln y)^2} \times \frac{1}{y}, \quad 0 \leq y \leq \infty$.

$\hookrightarrow$ log-normal density.

Example:- Suppose $X \sim N(0,1)$.

Want to derive the CDF or PDF of $Y = X^2$.

Observe that $\Leftarrow$ $Y = X^2$ is NOT monotone.

$x = \sqrt{Y}$

$Y = X^2 : (-\infty, 0) \longrightarrow (0, \infty)$. $\longrightarrow$ Here $f(x) = x^2$ is monotone (decreasing)

$Y = X^2 : (0, \infty) \longrightarrow (0, \infty)$. $\longrightarrow$ Here $f(x) = x^2$ is monotone (increasing).

Case 1:- If $x \in (-\infty, 0)$, $\dfrac{dx}{dy} = -\dfrac{1}{2\sqrt{y}}$

Case 2:- if $x \in (0, \infty)$, $\dfrac{dx}{dy} = \dfrac{1}{2\sqrt{y}}$.

Hence, $$f_Y(y) = f_X(-\sqrt{y}) \left| \frac{dx}{dy} \right| + f_X(\sqrt{y}) \left| \frac{dx}{dy} \right|, \quad y > 0$$

$$= \frac{1}{2\sqrt{y}} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} + \frac{1}{2\sqrt{y}} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}}, \quad y > 0$$

$\left[f_X(g^{-1}(y)) = f_X(\sqrt{y} \text{ or } -\sqrt{y}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \right]$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} y^{-\frac{1}{2}}, \quad y \geq 0.$$

This density function is well-known χ^2 distribution with 1 degree of freedom.

Alternative Method:-

$$F_Y(y) = P[Y \leq y] = P[X^2 \leq y] = P[-\sqrt{y} \leq X \leq \sqrt{y}]$$

$$= \Phi_X(\sqrt{y}) - \Phi_X(-\sqrt{y}).$$

Using symmetricity of $N(0,1)$

$$= \Phi(\sqrt{y}) - [1 - \Phi_X(\sqrt{y})] = 2\Phi_X(\sqrt{y}) - 1.$$

Hence, $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}[2\Phi_X(\sqrt{y}) - 1] = \frac{d}{dx}[2\Phi_X(\sqrt{y}) - 1]\frac{dx}{dy}$

$$= 2\phi(\sqrt{y}) \times \frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi}} e^{-y/2} y^{-\frac{1}{2}}, \; y > 0.$$

<u>Remark:-</u> i) If $X \sim N(0,1)$, then $Y = X^2 \sim \chi^2_1$

↳ Chi-square distⁿ with 1 degree of freedom.

ii) If we want to generate the data from χ^2_1 distⁿ, first generate the data $x = \{x_1, \ldots, x_n\}$ from $N(0,1)$ distⁿ, and afterwards, $x^* = \{x_1^2, x_2^2, \ldots, x_n^2\}$ will be the data from χ^2_1 distⁿ.