The CDF is defined by

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy$$

$\longrightarrow$ It is p.d.f. of the random variable X. C.D.F. of the

Now, it follows from the fundamental theorem of integral calculus that

$$f_X(x) = \frac{d}{dx} F_X(x). \quad \cancel{=} \; f_X(x)$$

$$\left[= \lim_{h \to 0} \frac{F_X(x+h) - F_X(x)}{h} \right].$$

Try to do it by yourself.

Further, note that

$$P[a \le X \le b] = \int_a^b f_X(x)\,dx = \int_{-\infty}^b f_X(x)\,dx - \int_{-\infty}^a f_X(x)\,dx.$$

//dy...

$$= F_X(b) - F_X(a).$$

A few more examples of CDF :-

(Discrete, Ω is finite).

1. $\Omega = \{(i,j)\ ;\ i = 1, 2, 3\ ,\ j = 1, 2\}$.

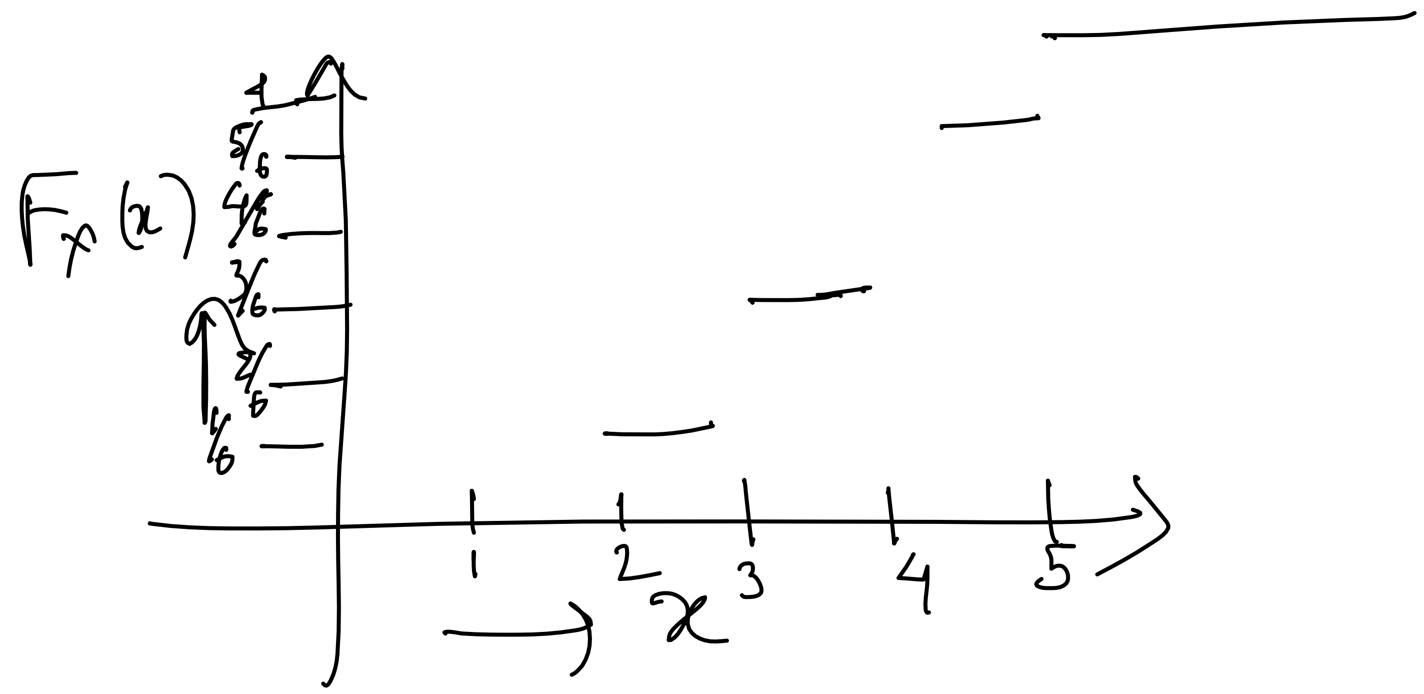
$$P(\{i,j\}) = \frac{1}{6} \quad \forall\ i = 1, 2, 3,\ j = 1, 2$$

Define $X(i,j) = i + j$

Range of $X = \{2, 3, 4, 5\}$.

$$F_X(x) = P[X \le x] = P[(i,j) : i+j \le x]$$

$$
\begin{aligned}
&= 0 && \text{if } x < 2 \\
&= \frac{1}{6} && \text{if } 2 \le x < 3 \quad \rightarrow (1,1) \\
&= \frac{3}{6} && \text{if } 3 \le x < 4 \quad \rightarrow (1,1),(1,2),(2,1) \\
&= \frac{5}{6} && \text{if } 4 \le x < 5 \quad \rightarrow (1,1),\ (1,2),\ (2,1),\ (2,2),\ (3,1) \\
&= 1 && \text{if } x \ge 5.
\end{aligned}
$$



i) $F_X(x)$ is non-decreasing

ii) $F_X(x)$ is right cont^{n}.

iii) $\lim_{x\to\infty} F_X(x) = 1$ & $\lim_{x\to-\infty} F_X(x) = 0$.

2. Random experiment :- Toss an unbiased until head appears.

$$\Omega = \{H, TH, TTH, TTTH, \ldots\}$$

X : No. of tosses required.

Hence, X can take the values $1, 2, 3, \ldots$

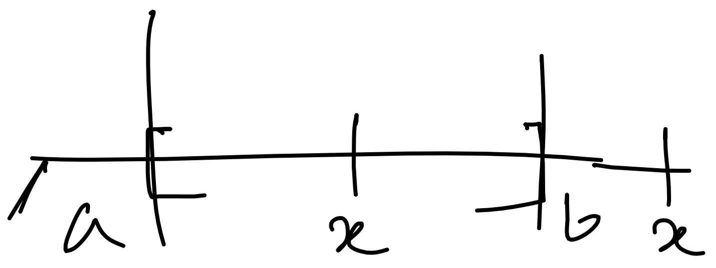
Range spm. $\mathcal{X} = \{1, 2, 3, 4, \ldots\}$

$i = 1, 2, \ldots$ $P[X = i] = \left(\frac{1}{2}\right)^{i-1} \times \left(\frac{1}{2}\right) = \frac{1}{2^i}$.

$$F_X(x) = P[X \leq x] = \begin{cases} 0 & \text{if } x < 1 \\ 1/2 & \text{if } 1 \leq x < 2 \\ \frac{3}{4} = \frac{1}{2} + \left(\frac{1}{2}\right)^2 & \text{if } 2 \leq x < 3 \\ \vdots & \end{cases}$$

In this case also, we observe the same characteristics of $F_X(x)$.

3. $\Omega = [a, b]$. $I \subset \Omega$.



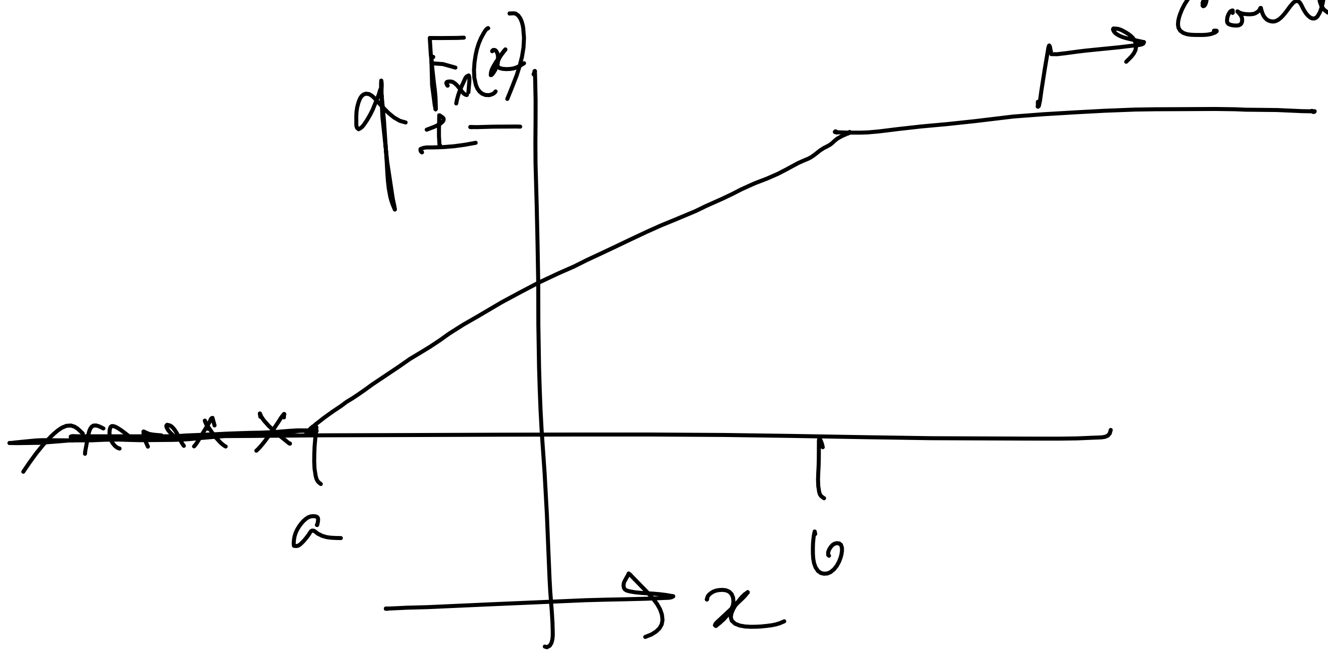
$$P[I] = \frac{\text{length of the interval of } I}{(b-a)}$$

$X(\omega) = \omega$, $\mathcal{X} = [a, b]$.

Range space.

$$F_X(x) = P[X \le x] = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \le x < b. \\ 1 & \text{if } x \ge b. \end{cases}$$

$$P[X \le x] = P[\omega : \omega \le x]$$



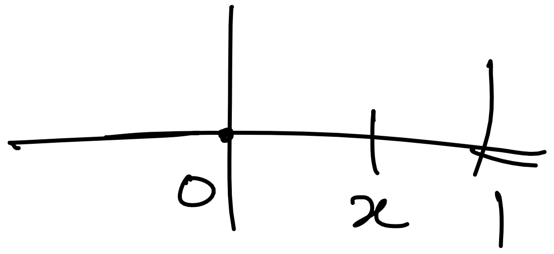
Continuous everywhere

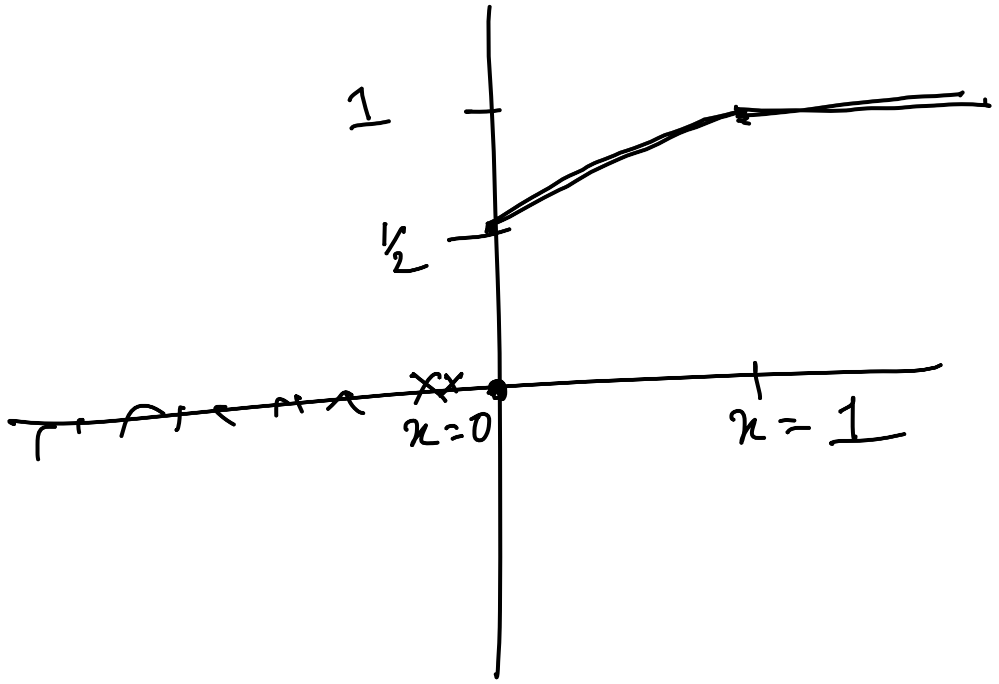
4. $\Omega = [0, 1]$

$P[\{0\}] = \frac{1}{2}$, $\quad I \subset [0, 1]$, and $P[I] = \dfrac{\text{length of } I}{2}$.

$X(\omega) = \omega$

$$F_X(x) = P[X \le x] = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ \frac{1}{2} + \frac{x}{2} & \text{if } 0 < x < 1. \\ 1. & \text{if } x \ge 1 \end{cases}$$





<u>Fact :-</u> Suppose, a function F_X satisfying

i) F_X is non-decreasing.

ii) F_X is right continuous.

iii) $\lim_{x \to \infty} F_X(x) = 1$ & $\lim_{x \to -\infty} F_X(x) = 0$.

Then $F_X(x)$ will be the CDF of a some random variable X.

<u>The significance :-</u> If you have one CDF, there must be one random variable associated with that CDF.

A few more remarks :-

i) If X is an r.v. with CDF F. Then

$$P[a < X \leq b] = F_X(b) - F_X(a).$$

(Reason: $\{-\infty < X \leq b\} = \underbrace{\{-\infty < X \leq a\} \cup \{a < X \leq b\}}_{\text{disjoint}}$

$$\Rightarrow \underbrace{P[-\infty < X \leq b]}_{\text{def}^n} = \underbrace{P[-\infty < X \leq a]}_{\text{def}^n} + P[a < X \leq b]$$

$$\Rightarrow F_X(b) = F_X(a) + P[a < X \leq b]$$

$$\Leftrightarrow P[a < X \leq b] = F_X(b) - F_X(a).$$

2. Suppose, $\underset{\text{r.v.}}{\underset{\downarrow}{X}} \sim \underset{\text{C.D.F}}{\underset{\downarrow}{F}}$

Then $P[X = x] = F_X(x) - F_X(x-)$, where

$$F_X(x-) = \lim_{h \to 0+} F_X(x-h).$$

If X is cont$^{\text{n}}$ r.v., then $\lim_{h \to 0+} F_X(x-h) = F_X(x)$,

and hence, $P[X = x] = F_X(x) - \lim_{h \to 0+} F_X(x-h)$

$$= F_X(x) - F_X(x) = 0.$$

3. Number of points of discontinuity of CDF F_X is atmost countable. (Try to prove yourself).

One example :- Finding density f^{n} from CDF.

Recall $\Omega = [a, b]$, $I \subset \Omega$, $P(I) = \dfrac{\text{Length of } I}{(b-a)}$

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \le x < b \\ 1 & \text{if } x \ge b \end{cases}$$

p.d.f.

$\uparrow$

$$f_X(x) = \frac{1}{b-a} \quad \text{if} \quad a \le x \le b$$

$$= 0 \quad \text{otherwise.}$$