$X$ : Random variable, $F_X$ : CDF of $X$, and $f_X$ : PDF of $X$.

$$F_X(x) = \int_{-\infty}^{x} f_X(z)\,dz \quad \& \quad f_X(x) = \frac{d}{dx} F_X(x).$$

## Deriving CDF from PDF :-

$$f_X(x) = kx(x-1), \quad 0 \le x \le 1$$
$$= 0, \quad \text{o.w.}$$

Note that i) $f_X(x) \ge 0 \quad \forall \; k \ge 0$. Check.

ii) $\int_0^1 kx(x-1)\,dx = 1 \iff k = 6$

Hence, $f_X(x) = 6x(x-1)$ if $0 \le x \le 1$

$= 0$, o.w.

$$F_X(x) = 0 \quad \text{if} \quad x \le 0$$
$$= \int_0^x f_X(y)\,dy = \int_0^x 6y(y-1)\,dy = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right) \quad \text{if} \quad 0 \le x \le 1$$
$$= 1 \quad \text{if} \quad x \ge 1$$

## An example of mixture dist$^n$:

$$F_X(x) = \alpha F_d(x) + (1-\alpha) F_c(x) \quad , \quad 0 \le \alpha \le 1 .$$

$F_d$: Discrete CDF

$F_c$: Continuous CDF

**Note:-** If $\alpha = 1, \Rightarrow X$ is a discrete r.v.

If $\alpha = 0, \Rightarrow X$ is a continuous r.v.

**Recall an earlier example:-** $\Omega = [0,1]$, $P[\{0\}] = \frac{1}{2}$, $I \subset (0,1]$

& $P(I) = \dfrac{\text{Length of } I}{2}$ .

The CDF was

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{2} + \frac{x}{2} & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

In this example,

$$F_d(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

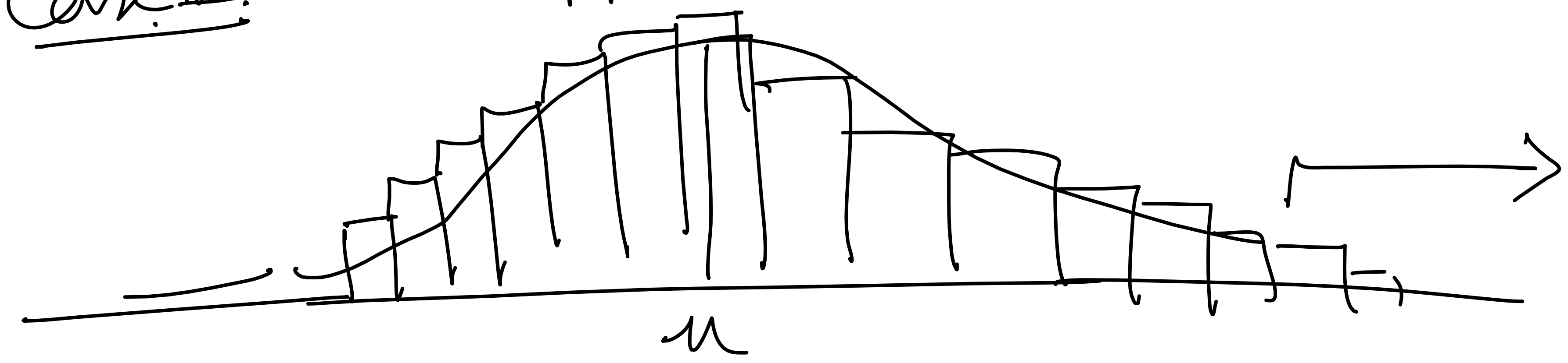
& $$F_c(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

& $\alpha = 1/2$.

Remark:- 1: The mixture of the dist$^n$. (this concept) can be extended for finitely many (or even countably infinite) CDF. For instance, $F_X(x) = \sum_{i=1}^{\infty} \alpha_i F_i(x)$ will be a proper CDF if $\alpha_i \geq 0 \; \forall i$; $\sum_{i=1}^{\infty} \alpha_i = 1$ & $F_i(\cdot)$ are proper CDF $\forall i$.

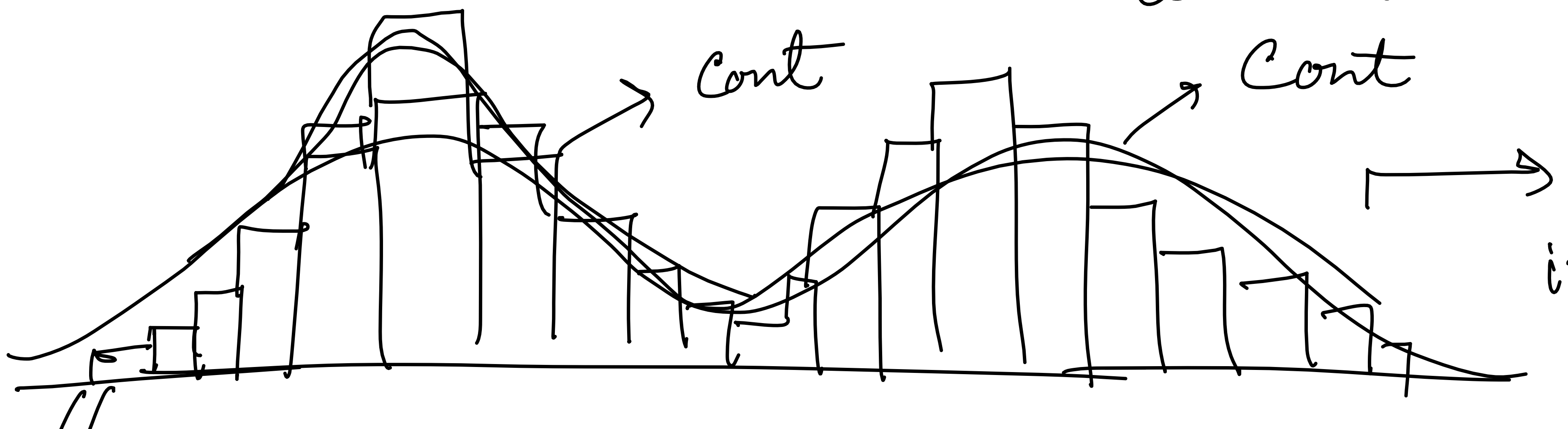
## 2. The significance of this concept:-

**Case 1:-** Suppose the data obtained from $N(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$



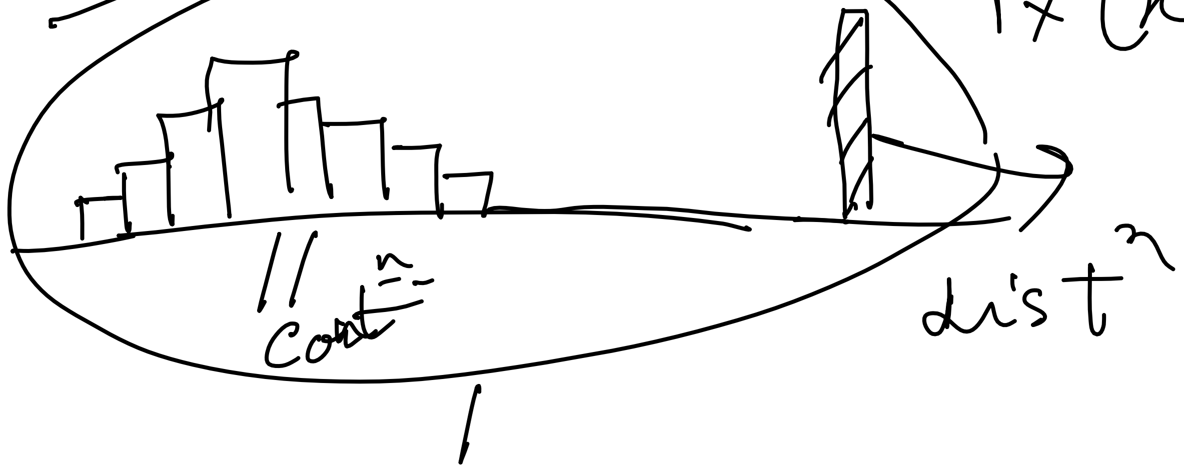

This histogram indicates that the data obtained from unimodal dist$^n$.

**Case 2:-**




This histogram indicates that the data obtained from bimodal dist$^n$.

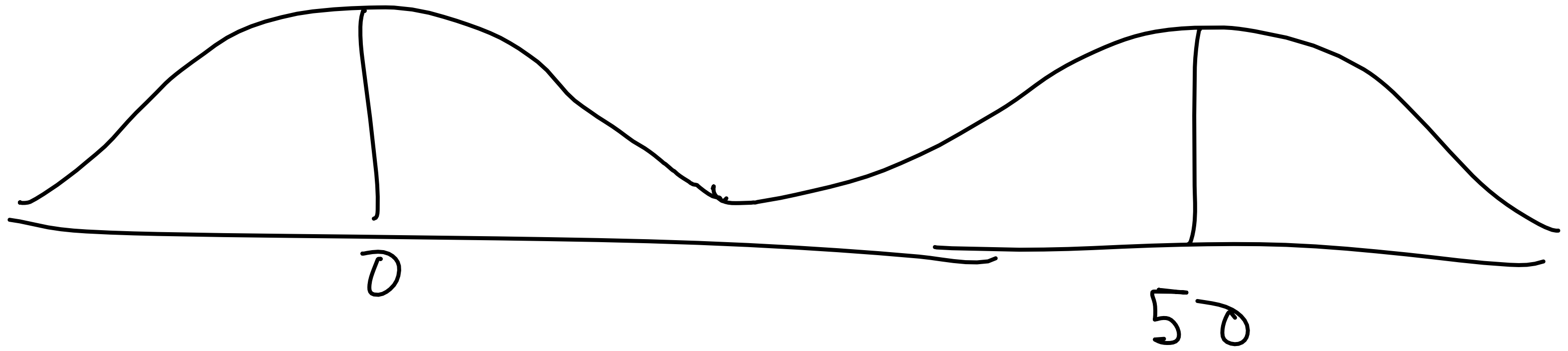
**Case 3:-**




$$F_X(x) = \alpha F_1(x) + (1-\alpha) F_2(x)$$

$\alpha = 0.7$, $F_1 = N(0,1)$, $F_2 = N(50,1)$

$(1-\alpha) = 0.3$



(New concept):- **Expectation:-** Suppose, $X$ is a discrete r.v.

$X$ : $x_1, x_2, \ldots, \ldots$ (mass points)

$P[X = x_i]$ : $p_1, p_2, \ldots$ (mass probabilities).

It should be a "measurable" function. (Discrete case).

Then $E[g(X)] = \sum_{k=1}^{\infty} g(x_k)\, p_k$ provided

$$\sum_{k=1}^{\infty} |g(x_k)|\, p_k = E[|g(X)|] < \infty.$$

If $X$ is a continuous r.v, then

$E[g(X)] = \int g(x)\, f_X(x)\, dx$ provided $E[|g(X)|]$

CDF: $F_X$ PDF: $f_X$

$$= \int |g(x)|\, f_X(x)\, dx < \infty.$$

Explanation:— Suppose $g(x) = X$, then our claim is that

$EX$ exists $(< \infty)$ if $E|X| < \infty$.

One way is trivial:— $X \le |X|$

$\Rightarrow E(X) \le E|X|$

So, $E(X)$ exists if $E|X| < \infty$.

Other way:— (Why it is equivalent):—

$$\underset{\text{any } f}{f} = \underset{\ge 0}{f^+} - \underset{\ge 0}{f^-}$$

$$\underset{\text{any r.v.}}{X} = \underset{\ge 0}{X^+} - \underset{\ge 0}{X^-}$$

And moreover, $|X| = X^+ + X^-$

Now, $E|X| < \infty \Rightarrow EX^+$ & $EX^-$ both are finite

Next, If $EX^+$ & $EX^-$ are finite, then $EX = EX^+ - EX^-$ will also be finite

And hence, $EX < \infty \Longleftrightarrow E|X| < \infty$.

Some special case:-

i) $g(X) = X$, then $E(X)$: Mean of the dist$^n$.

ii) $g(X) = X^n$, then $EX^n$: $n$-th order moment about origin ($=0$).

iii) $g(X) = (X-a)^n$, then $E(X-a)^n$: $n$th order moment about $a$.

iv) If $a = E(X)$, then

$$\mu_n := E[X - E(X)]^n : n\text{-th order central moment.}$$

A "vague" discussion (why $E(X)$ is called mean):-

Suppose, $\mathcal{X} = \{x_1, \dots, x_n\}$ obtained from $F$ (CDF, and it is unknown).

data

$$E_{\mathcal{X}} = \int x \, d\hat{F}_n(x)$$

$\hat{F}_n$ ⟶ estimated version (One of the simplest estimate of $F$ is that you are putting mass $\frac{1}{n}$ on each obs.)

$$= \frac{1}{n} \sum_{i=1}^{n} x_i \text{ (average)}.$$

Empirical CDF ⟷ $\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1\{x_i \le x\}$

$E(X)$
$= \int x f_X(x) dx$
$= \int x \, dF(x)$
$\Downarrow$
estimate

A paradox related to Expectation:—