Numerical Calculus

2015-2016

= COURSE =

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Course 1
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Chapter 1. Preliminary notions

An element $\tilde{x} \in \mathbb{R}$ which approximates x^* is called the <u>approximation</u> / the <u>approximant</u> of x^*.

Finite and divided differences

$a_i = a + ih$, $i = \overline{0,m}$ - equidistant points

$\boxed{(\Delta_h f)(a_i) = f(a_i) = f(a_{i+1}) - f(a_i)}$ = the finite difference of the first order

$\boxed{(\Delta_h^k f)(a_i) = \Delta_h((\Delta_h^{k-1} f)(a_i))}$ = the k-th order finite difference

$(\Delta^2 f)(a_0) = (\Delta f)(a_1) - (\Delta f)(a_0)$

$(\Delta f)(a_0) = f(a_1) - f(a_0) = 2$

Example

$h = 0{,}25$
$a = 1$
$a_i = a + ih$
$i = \overline{0,4}$
$f_0 = 0$
$f_1 = 2$
$f_2 = 6$
$f_3 = 14$
$f_4 = 17$

a	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
$a_0 = 1$	0	$(\Delta f)(a_0) = 2$	2	2	-11
$a_1 = 1{,}25$	2	4	4	-9	
$a_2 = 1{,}5$	6	8	-5		
$a_3 = 1{,}75$	14	3			
$a_4 = 2$	17				

$\boxed{(\mathcal{D} f)(x_r) = [x_r, x_{r+1}; f] = \frac{f(x_{r+1}) - f(x_r)}{x_{r+1} - x_r}}$ = the first order divided differences

$\boxed{(\mathcal{D}^k f)(x_r) = \frac{(\mathcal{D}^{k-1} f)(x_{r+1}) - (\mathcal{D}^{k-1} f)(x_r)}{x_{r+k} - x_r}}$ = divided difference of k-th order at the point x_r

Example:

$x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$ | $f_0 = 3$, $f_1 = 4$, $f_2 = 7$, $f_3 = 19$

x	f	$\mathcal{D}f$	$\mathcal{D}^2 f$	$\mathcal{D}^3 f$
$x_0 = 0$	3	1	1	0
$x_1 = 1$	4	3	1	
$x_2 = 2$	7	6		
$x_3 = 4$	19			

$$\mathcal{D}f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{1} = 1$$

$$\mathcal{D}^2 f(x_0) = \frac{(\mathcal{D}f)(x_1) - (\mathcal{D}f)(x_0)}{x_2 - x_0} = \frac{2}{2} = 1$$

Taylor's interpolation

$\boxed{P_m(x) = \sum_{k=0}^{m} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0)}$ = Taylor's formula

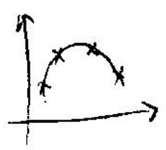
$\boxed{f(x) = P_m(x) + R_m(x)}$ = the approximation formula, where $R_m(x)$ is the remainder (error) for Taylor's interpolation

$\boxed{R_m(x) = \frac{(x-x_0)^{m+1}}{(m+1)!} f^{(m+1)}(\xi)}$ = the remainder

Lagrange interpolation

x_i, $i = \overline{0,m}$, $x_i \neq x_j$, $f : [a,b] \to \mathbb{R}$, $x_i \in [a,b]$, $i = \overline{0,m}$, find the polynomial P of the smallest degree for which $P(x_i) = f(x_i)$, $i = \overline{0,m}$, denoted $\boxed{(L_m f)(x) = \sum_{i=0}^{m} l_i(x) f(x_i)}$

Lagrange interpolation polynomial



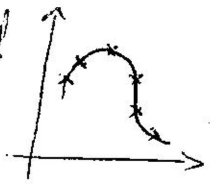
Remark: $L_m f \in \mathbb{P}_m \to$ polynomial of degree m

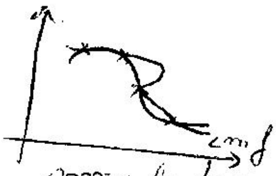
$$u(x) = (x - x_0)(x - x_1) \cdots (x - x_m)$$

$$u_i(x) = \frac{u(x)}{x - x_i}$$

$$l_i(x) = \frac{u_i(x)}{u_i(x_i)} = \frac{u(x)}{(x - x_i)\, u'(x_i)}$$

$(L_m f)(x_i) = f(x_i),\ i=\overline{0,m}$ translated graphically





approx. by Lagrange

Course 2
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Lagrange interpolation

$$\boxed{(L_m f)(x) = \sum_{i=0}^{m} l_i(x) f(x_i)}$$ = the Lagrange interpolation polynomial

Example: Find the Lagrange polynomial that interpolates the data in the following table and find the approx. value of $f(-0,5)$

x	$\overset{x_0}{-1}$	$\overset{x_1}{0}$	$\overset{x_2}{3}$
$f(x)$	8	-2	4

$$(L_2 f)(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2) = \frac{x(x-3)}{4}\cdot 8 + \frac{(x+1)(x-3)}{+3}2 + \frac{(x+1)x}{12}\cdot 4$$

$$l_i = \frac{u_i(x)}{u_i(x_i)},\ i=\overline{0,2},\quad u(x) = \prod_{j=0}^{2}(x-x_j) = (x-x_0)(x-x_1)(x-x_2) =$$

$u_0(x) = (x-x_1)(x-x_2)$
$u_1(x) = (x-x_0)(x-x_2)$
$u_2(x) = (x-x_0)(x-x_1)$

$$l_0(x) = \frac{u_0(x)}{u_0(x_0)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{x(x-3)}{(-1)\cdot(-4)} = \frac{x(x-3)}{4}$$

$$l_1(x) = \frac{u_1(x)}{u_1(x_1)} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-3)}{1\cdot(-3)} = \frac{(x+1)(x-3)}{-3}$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)x}{4\cdot(+3)} = \frac{(x+1)x}{+12}$$

We can reduce the number of operations

$$l_i(x) = \frac{u(x)}{(x-x_i)u_i(x_i)} \Big| \cdot \frac{1}{u(x)} \quad \dots$$

$$\boxed{(L_m f)(x) = \frac{\sum_{i=0}^{m} \frac{A_i f(x_i)}{(x-x_i)}}{\sum_{i=0}^{m} \frac{A_i}{x-x_i}}}$$ = the barycentric form of Lagrange interpolation polynomial

Remark: $f \cong L_m f$, but $\boxed{f = L_m f + R_m f}$ Lagrange interpolation formula

Theorem: Let $\alpha = \min\{x, x_0, \dots x_m\}$ and $\beta = \max\{x, x_0, \dots, x_m\}$. If $f \in C^m[\alpha,\beta]$ and $f^{(m)}$ is derivable on (α,β) then $\forall x \in (\alpha,\beta)$ there exists $\xi \in (\alpha,\beta)$ s.t.

$$\boxed{(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi)}$$

Proof:

$$F(z) = \begin{vmatrix} u(z) & (R_m f)(z) \\ u(x) & (R_m f)(x) \end{vmatrix},\quad u(x_i) = 0,\ i=\overline{0,m},\ R_m f(x_i) = f(x_i) - \underbrace{L_m f(x_i)}_{= f(x_i)} = 0,\ \boxed{i=\overline{0,m}}$$

the polynomial is equal with $L_m f$ in the given points

$$F(x) = 0;\ F(x_i) = \begin{vmatrix} u(x_i) & R_m f(x_i) \\ u(x) & R_m f(x) \end{vmatrix} = 0,\ i=\overline{0,m} \Rightarrow F \text{ has } m+2 \text{ zeros on } (\alpha,\beta) \Rightarrow$$

(because we have m+1 sol. + sol. 0)

$\Rightarrow F'$ has $m+1$ zeros $\Rightarrow \dots \Rightarrow F^{(m+1)}$ has 1 zero
Rolle th.

(1) $\exists\, \xi \in (\alpha,\beta)$ s.t. $F^{(m+1)}(\xi) = 0$

$$F^{(m+1)}(z) = \begin{vmatrix} u^{(m+1)} & (R_m f)^{(m+1)}(z) \\ u(x) & (R_m f)(x) \end{vmatrix} =$$

$$u(z) = (z-x_0)(z-x_1)\dots(z-x_m) \in \mathbb{P}_{m+1}$$

$u^{(m+1)}(z) = (m+1)!$ u derived (m+1) times is $2\cdot3\cdots(m+1)$

$$(R_m f)^{(m+1)}(z) = f^{(m+1)}(z) - (L_m f)^{(m+1)}(z) = 0$$

(2)
$$= \begin{vmatrix} (m+1)! & f^{(m+1)}(z) \\ u(x) & (R_m f)(x) \end{vmatrix}$$

From (1), (2) $\Rightarrow (m+1)!\,(R_m f)(x) = u(x) f^{(m+1)}(\xi) \Rightarrow \boxed{(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi)},\quad \xi \in (\alpha,\beta)$

$$\boxed{|(R_m f)(x)| \le \frac{|u(x)|}{(m+1)!} \|f^{(m+1)}\|_\infty}, \quad x \in [a,b], \quad \|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$

limit of the error $\qquad$ $\|\cdot\|_\infty$ = uniform norm

Example: which is the lim. of error?

x	100	121	144
f	10	11	12

$x = 115$

$$|(R_2 f)(x)| \le \frac{|u(x)|}{3!} \overbrace{\max_{x\in[100,144]} |f'''(x)|}^{M_3 f} = \frac{|u(x)|}{6} \cdot \frac{3}{8} \cdot 10^{-5} = \frac{|(115-100)(115-121)}{16}$$

$$\frac{(115-144)|}{} \cdot 10^{-5}$$

(the error is good, quite small)

$$f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2 \cdot x^{\frac{1}{2}}} = \frac{1}{2} \cdot x^{-\frac{1}{2}}$$

$$f''(x) = +\frac{1}{2} \cdot \frac{-1}{2} \cdot x^{-\frac{1}{2}-1} = -\frac{1}{4} \cdot x^{-\frac{3}{2}}$$

$$f'''(x) = \frac{3}{8} x^{-\frac{5}{2}}$$

$M_3 f = \frac{3}{8} 10^{-5}$ (the smaller x is, the bigger the max is)
we take $x = 100$

$$u(x) = (x-x_0)(x-x_1)(x-x_2) = (115-100)(115-121)(115-144)$$

We have $\lg 2 = 0.301$, $\lg 5 = 0.699$, $\lg 76 = ?$. Study the approx error
$\lg 3 = 0.477$

$x_0 = 2$
$x_1 = 3$
$x_2 = 5$

$76 = 2^2 \cdot 19$
$\lg 76 = 2\lg 2 + \lg 19$ ↳ we have approx. $\lg 19$ → we consider 4 points close to 19

$x_0 = 18 = 2 \cdot 3^2$
$x_1 = 15 = 3 \cdot 5$
$x_2 = 16 = 2^4$
$x_3 = 20 = 2^2 \cdot 3$

76 is really far from 2, 3, 5 (the given values), so the error would be too big.

A practical method for computing the Lagrange polynomial is the **Aitken's algorithm**
This generates a table

$$\begin{array}{l|l|l|l|l}
x_0 & f_{00} & & & \\
x_1 & f_{10} & f_{11} & & \\
x_2 & f_{20} & f_{21} & f_{22} & \\
\vdots & \vdots & & & f_{33} \\
x_m & f_{m0} & f_{m1} & f_{m2} & f_{m3} \ \dots \ f_{mm}
\end{array}, \quad \text{where} \quad f_{i0} = f(x_i),\ i = \overline{0,m}$$

$$f_{i,j+1} = \frac{1}{x_i - x_j} \begin{vmatrix} f_{jj} & x_j - x \\ f_{ij} & x_i - x \end{vmatrix}, \quad i = \overline{0,m},\ j = \overline{0,i-1}$$

Ex.:
$x = 115$
$x_0 = 100$
$x_1 = 121$
$x_2 = 144$

Because $121 - 115 = 6$ → x_1 will be first
$115 - 100 = 16$ (we sort the nodes respectively to the distance to x)

Course 3
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Approximate $\sqrt{115}$ with precission $\varepsilon = 10^{-3}$ using Aitken's algorithm.

$x_0 = 100$
$x_1 = 121$
$x_2 = 144$

} choose 3 points close to 115 $\Rightarrow$ 121 is the 1st one, 100 second, 144 is 3rd

$$f_{11} = \frac{1}{x_1 - x_0}\begin{vmatrix} f_{00} & x_0 - x \\ f_{10} & x_1 - x \end{vmatrix} = \frac{1}{100-121}\begin{vmatrix} 11 & 6 \\ 10 & -15 \end{vmatrix} = \frac{1}{-21}[11\cdot(-15) - 60]$$

$i = 1, j = 0$

x_0	11
x_1	10
x_2	12

$f_{11} = \frac{1}{\quad}$

$$f_{i,j+1} = \frac{1}{x_i - x_j}\begin{vmatrix} f_{ji} & x_j - x \\ f_{ij} & x_i - x \end{vmatrix}$$

$$f_{21} = \frac{1}{x_2 - x_0}\begin{vmatrix} f_{00} & x_0 - x \\ f_{20} & x_2 - x \end{vmatrix} \qquad f_{22} = \frac{1}{x_2 - x_1}\begin{vmatrix} f_{11} & x_1 - x \\ f_{21} & x_2 - x \end{vmatrix}$$

$i = 2, j = 0 \qquad i = 2, j = 1$

Newton interpolation polynomial

$x_0, x_1, \dots, x_m \to m+1$ nodes
$f(x_0), f(x_1), \dots, f(x_m)$

$$(N_m f)(x) = f(x_0) + \sum_{i=1}^{m} (x - x_0)(x - x_1)\dots(x - x_{i-1})\underbrace{(\mathcal{D}^i f)(x_0)}_{=[x_0, \dots, x_i; f]}$$

$(N_m f)(x) \approx f(x)$

$$\boxed{(N_m f)(x) + (R_m f)(x) = f(x)}$$ Newton interpolation formula

$$\boxed{(R_m f)(x) = (x - x_0)\dots(x - x_m)[x, x_0, \dots, x_m; f]}$$ divided difference of order $m+1$ at the point x

Remark: Divided differences are approximations of derivatives

$$(R_m f)(x) = \frac{(x - x_0)\dots(x - x_m)}{(m+1)!} f^{(m+1)}(\varepsilon) \Rightarrow \boxed{[x, x_0, \dots, x_m; f] = \frac{f^{(m+1)}(\varepsilon)}{(m+1)!}}$$

Example:

$f(x) = \sin \pi x$
$x_0 = 0$
$x_1 = \frac{1}{6}$ Find $L_2 f$ in both forms.
$x_2 = \frac{1}{2}$

$$(L_2 f)(x) = \sum_{i=0}^{2} l_i(x) f(x_i)$$

$$l_i(x) = \frac{u_i(x)}{u_i(x_i)}$$

$u_i(x) = \frac{u(x)}{x - x_i}$

$u(x) = (x - x_0)(x - x_1)(x - x_2)$

$u_0(x) = \frac{u(x)}{x - x_0} = (x - x_1)(x - x_2)$

$u_1(x) = (x - x_0)(x - x_2)$

$u_2(x) = (x - x_0)(x - x_1)$

$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$

$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$

$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$

$(L_m f)(x) = l_0(x)\cdot 0 + l_1(x)\cdot\frac{1}{2} + l_2(x)\cdot 1$

$$(L_2 f)(x) = (N_2 f)(x) = f(x_0) + \sum_{i=1}^{2}(x - x_0)\dots(x - x_{i-1})\underbrace{[x_0, \dots, x_i; f]}_{(\mathcal{D}^i f)(x_0)} =$$

$$= f(x_0) + (x - x_0)\cdot\underbrace{[x_0, x_1; f]}_{\mathcal{D} f(x_0)} + (x - x_0)(x - x_1)\cdot\underbrace{[x_0, x_1, x_2; f]}_{(\mathcal{D}^2 f)(x_0)} = 0 + (x - x_0)\cdot 3 + (x - x_0)(x - x_1)(-3)$$

$$= 3x - 3x\left(x - \frac{1}{6}\right) = 3x\left[1 - \left(x - \frac{1}{6}\right)\right]$$

this polynomial approximates the sin function

	f	$\mathcal{D}f$	$\mathcal{D}^2 f$
$x_0 = 0$	0	3	-3
$x_1 = \frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{2}$	
$x_2 = \frac{1}{2}$	1		

$\frac{3}{2}$: $(1 - \frac{1}{2})/(\frac{1}{2} - \frac{1}{6})$

3: $(\frac{1}{2} - 0)/(\frac{1}{6} - 0)$

$\left(1 - \frac{1}{2}\right)/\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{1}{2}\cdot\frac{6}{2} = \frac{3}{2}$

$\mathcal{D}^2 f(x_0) = \left(\frac{3}{2} - 3\right)/\left(\frac{1}{2}\right) = -\frac{3}{2}\cdot 2 = -3$

Hermite interpolation

$x_k\ ;\ k=\overline{0,m}$ points

$f^{(j)}(x_k)\ ,\ j=\overline{0,r_k}$

$\boxed{P^{(j)}(x_k)=f^{(j)}(x_k)}$ = the Hermite interpolation problem

$\boxed{(H_n f)(x)}$ Hermite interpolation polynomial

$n=m+r_0+r_1+\dots+r_m$

$$\boxed{(H_n f)(x)=\sum_{k=0}^{m}\sum_{j=0}^{r_k} h_{kj}(x)\, f^{(j)}(x_k)}$$

$$\boxed{\begin{cases} h_{kj}^{(p)}(x_\nu)=0, & \nu\neq k \\ h_{kj}^{(p)}(x_k)=\begin{cases}1, & j=p\\ 0, & j\neq p\end{cases} \end{cases}}$$ the Hermite fundamental interpolation polynomials

$$\boxed{u(x)=\prod_{k=0}^{m}(x-x_k)^{r_k+1}}\ ,\ \boxed{u_k(x)=\frac{u(x)}{(x-x_k)^{r_k+1}}}$$

Example: Find the Hermite interpolation polynomial for a function f for which we know $f(0)=1$ $f(1)=-3$ (equivalent with $x_0=0$ multiple node of order 2, $x_1=1$ simple node).
$f'(0)=2$

$x_0=0\ ,\ f(x_0)=1\ ,\ f'(x_0)=2$

$x_1=1\ ,\ f(x_1)=-3$

$m=1$

$r_0=1 \rightarrow$ maximum rank of derivatives at the point x_0

$r_1=0 \rightarrow$ we don't have derivatives $\Rightarrow$ order 0

$n=2 \rightarrow$ degree of polynomials

$$(H_2 f)(x)=\sum_{k=0}^{1}\sum_{j=0}^{r_k} h_{kj}(x)\cdot f^{(j)}(x_k)=h_{00}(x)f(x_0)+h_{01}(x)\cdot f'(x_0)+h_{10}(x)f(x_1)$$

($k=0, j=0$; $k=0, j=1$; $k=1, j=0$)

$h_{00}(x)=ax^2+bx+c \rightarrow$ in order to find a,b,c, use the formula with the system

$$\begin{cases} h_{00}(x_0)=1 \\ h_{00}'(x_0)=0 \\ h_{00}(x_1)=0 \end{cases}$$ is 0 (given)

$$h_{00}'(x)=2ax+b \Rightarrow \begin{cases} c=1\\ b=0\\ a+b+c=0\end{cases} \Rightarrow \begin{cases} c=1\\ b=0\\ a=-1\end{cases} \Rightarrow h_{00}(x)=\boxed{-x^2+1}$$

$h_{01}(x)=a_1x^2+b_1x+c=\boxed{-x^2+x}$

$$\begin{cases} h_{01}(x_0)=0\\ h_{01}'(x_0)=1\\ h_{01}(x_1)=0\end{cases} \Rightarrow \begin{cases} c_1=0\\ b_1=1\\ a_1+b_1+c_1=0\end{cases} \Rightarrow \begin{cases} c_1=0\\ b_1=1\\ a_1=-1\end{cases}$$

$h_{10}(x)=a_2x^2+b_2x+c=x^2$

$$\begin{cases} h_{10}(x_0)=0\\ h_{10}'(x_0)=0\\ h_{10}(x_1)=1\end{cases} \Rightarrow \begin{cases} c_2=0\\ b_2=0\\ a_2=1\end{cases}$$

$(H_2 f)(x)=(-x^2+1)\cdot 1+(-x^2+x)\cdot 2+x^2(-3)$

Hermite interpolation

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x_k, $k=\overline{0,m}$

$f:[a,b]\to\mathbb{R}$

$f^{(j)}(x_k)$, $k=\overline{0,m}$, $j=\overline{0,r_k}$

$$\boxed{(H_m f)^{(j)}(x_k)=f^{(j)}(x_k)}$$

$$f=H_m f+R_m f \Rightarrow \boxed{f(x)=(H_m f)(x)+(R_m f)(x)}$$

Theorem: If $f\in C^n[\alpha,\beta]$ and $f^{(n)}$ is derivable on (α,β) with $\alpha=\min\{x,x_0,\dots x_m\}$ and $\beta=\max\{x,x_0,\dots,x_m\}$, then there exists $\xi\in(\alpha,\beta)$ such that:

$$(R_m f)(x)=\frac{u(x)}{(n+1)!}f^{(n+1)}(\xi)$$

$$F(z)=\begin{vmatrix}u(z) & (R_m f)(z)\\ u(x) & (R_m f)(x)\end{vmatrix}$$

$F(x)=0$

$F^{(j)}(x_k)=0$, $k=\overline{0,m}$, $j=\overline{0,r_k}$ because $u^{(j)}(x_k)=0$, $k=\overline{0,m}$, $j=\overline{0,r_k}$

$$(R_n^{(j)}f)(x_k)=f^{(j)}(x_k)-\underbrace{(H_m f)^{(j)}(x_k)}_{f^{(j)}(x_k)}=0$$

$k=0\Rightarrow j=\overline{0,r_0}\Rightarrow r_0+1$ values

$k=1\Rightarrow j=\overline{0,r_1}\Rightarrow r_1+1$ values

$k=m\Rightarrow j=\overline{0,r_m}\Rightarrow r_m+1$ values

$$\underbrace{r_0+r_1+\dots+r_m+m+1}_{n}=n+1 \text{ zeros}$$

F has $n+2$ zeros $\underset{\text{Rolle Th.}}{\Rightarrow}$ F' has $n+1$ zeros $\Rightarrow\dots\Rightarrow F^{(n+1)}$ has 1 zero

$$F^{(n+1)}(z)=\begin{vmatrix}u^{(n+1)}(z) & (R_m f)^{(n+1)}(z)\\ u(x) & (R_m f)(x)\end{vmatrix}\quad \exists\,\xi\in(\alpha,\beta) \text{ s.t. } F^{(n+1)}(\xi)=0$$

$$u(x)=\prod_{k=0}^{m}(x-x_k)^{r_k+1}\in P_{n+1}\qquad \left.\begin{matrix}x_0+1\\ r_1+1\\ \dots\\ r_m+1\end{matrix}\right\} r_0+\dots+r_m+m+1=n+1\Rightarrow$$

$$u(x)=\begin{vmatrix}(n+1)! & f^{(n+1)}(z)\\ u(x) & (R_m f)(x)\end{vmatrix}$$

$$(R_m f)^{(n+1)}(x)=f^{(n+1)}(x)-\underbrace{(H_m f)^{(n+1)}(x)}_{}=f^{(n+1)}(x)$$

$=0$, because the deriv. degree ~~of the polynom.~~ is greater than the degree of the polynom. $n+1>n$

so we have $\begin{vmatrix}(n+1)! & f^{(n+1)}(\xi)\\ u(x) & (R_m f)(x)\end{vmatrix}=0$

$$\Rightarrow (n+1)!\cdot(R_m f)(x)=u(x)\cdot f^{(n+1)}(\xi)\Rightarrow \boxed{(R_m f)(x)=\frac{u(x)}{(n+1)!}f^{(n+1)}(\xi)}$$

Corollary: $|(R_m f)(x)|\le\frac{u(x)}{(n+1)!}\|f^{(n+1)}\|_\infty$, $x\in[a,b]$, $\|f^{(n+1)}\|_\infty=\max_{x\in[a,b]}|f(x)|$ maxim of abs. value of derivative of order $n+1$

! If $m=0$ i.e. $n=r_0$ $\Rightarrow$ Hermite Interpolation Problem becomes Taylor Interpolation Problem.

Taylor Interpolation polynomial is:

Example: Find Hermite interpolation formula for $f(x)=xe^x$, which is the limit of error for approximating $f(\frac{1}{2})$?

$f(-1)=-0.3679$ $\quad x_0=-1$

$f(0)=0$ $\quad x_1=0$

$f'(0)=1$ $\quad x_2=1$

$f(1)=2.7183$

Homework: first part of the problem

$r_0=0$ $\quad (f(x_0))$

$r_1=1$ $\quad (\max(f(x_1),f'(x_1)))$

$r_2=0$

$n=r_0+r_1+r_2+m=1+2=3$

$$|(R_m f)(x)|\le\frac{u(x)}{(n+1)!}\|f^{(n+1)}\|_\infty$$

$$\pi(x) = \prod_{k=0} (x-x_k) = (x-x_0)(x-x_1)(x-x_2)$$

$f(x) = xe^x$

$$f'(x) = e^x + xe^x = e^x(x+e^x)$$
$$f''(x) = e^x(1+e^x) = e^x(x+e^x) + e^x = e^x(x+2)$$
$$f'''(x) = e^x(x+2) + e^x = e^x(x+3)$$
$$f^{(4)}(x) = e^x(x+3) + e^x = e^x(x+4)$$

$$M_4 f = \max_{x\in[-1;1]} f^{(4)}(x) = 5e$$

$$|R_3 f(\tfrac{1}{2})| \le \frac{(\frac{1}{2}+1)(\frac{1}{2}-0)^2(\frac{1}{2}-1)}{4!} \cdot M_4 f$$

Hermite interpolation with double nodes

$f:[a,b]\to\mathbb{R}$

$x_0, \dots, x_m$

$f(x), f(x_0)$ $\quad r_0 = 1$ (max. order of derivatives)

$f(x_1), f'(x_1)$ $\quad r_1 = 1$

$f(x_m), f'(x_m)$ $\quad r_m = 1$

$$\Rightarrow n = m + r_0 + \dots + r_m = m + m + 1 = 2m+1$$

$$(H_{2m+1} f)(x_k) = f(x_k), \quad k = \overline{0,m}$$
$$(H_{2+1} f)'(x_k) = f'(x_k)$$

$$\boxed{(H_{2m+1} f)(x) = f(z_0) + \sum_{i=1}^{2m+1} (x-z_0)\dots(x-z_{i-1})(\mathcal{D}^i f)(z_0)}$$

Example: Find the Hermite interpolation polynomial that approximates f, using the classical formula and using divided differences.

$x_0 = -1$
$x_1 = 1$
$f(-1) = -3$
$f'(-1) = 10$
$f(1) = 1$
$f'(1) = 2$

$z_0 = x_0,\ z_1 = x_0$
$z_2 = x_1,\ z_3 = x_1$

	f	$\mathcal{D}f$	$\mathcal{D}^2 f$	$\mathcal{D}^3 f$
$z_0=-1$	-3	10	$(2-10)/(1-(-1))$	2
$z_1=-1$	-3	2	0	
$z_2=1$	1	2		
$z_3=1$	1			

$f'(-1)$ $\quad (2-2)/(z_3-z_1)$ $\quad -4$ $\quad (0+4)/(z_3-z_0)$

$$(\mathcal{D}f)(z_1) = \frac{f(z_2)-f(z_1)}{z_2-z_1} =$$

$$H_3 f(x) = f(z_0) + \sum_{i=1}^{3} (x-z_0)\dots(x-z_{i-1})(\mathcal{D}^i f)(z_0) = -3 + (x-z_0)(\mathcal{D}^1 f)(z_0) + (x-z_0)(x-z_1)(\mathcal{D}^2 f)(z_0) + (x-z_0)(x-z_1)(\mathcal{D}^3 f)(z_0)$$
$$= -3 + (x+1)10 + (x+1)(x+1)4 + (x+1)(x+1)(x-1)2$$
$$= -3 + 10(x+1) + 4(x+1)^2 + 2(x-1)(x+1)^2$$

$m = 1 (x_0, x_1)$

classical method

$$(H_n f)(x) = \sum_{k=0}^{m}\sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k) = h_{00}(x)\cdot f(x_0) + h_{01}(x) f'(x_0) + h_{10}(x)\cdot f(x_1) + h_{11}(x)\cdot f'(x_1) =$$

$m = 1$
$r_0 = 1$
$r_1 = 1$
$n = m + r_0 + r_1 + \dots + r_m = 1 + 2 = 3$

$h_{00}(x) = ax^3 + bx^2 + cx + d$ $\quad \cancel{3ax^2}$ $\qquad h_{00}'(x) = 3ax^2 + 2bx + c$

$h_{00}(x_0) = 1 \qquad -a + b - c + d = 1$
$h_{00}'(x_0) = 0 \qquad 3a - 2b + c = 0$
$h_{00}(x_1) = 0 \qquad a + b + c + d = 0$
$h_{00}'(x_1) = 0 \qquad 3a + 2b + c = 0$

Homework: Find a, b, c, d and also the other equations

Exercise 1

x	x_0 0	x_1 2	x_2 3
$f(x)$	0	10	12
$f'(x)$	5	3	7

$z_0 = x_0 \quad z_1 = x_0$
$z_2 = x_1 \quad z_3 = x_1$
$z_4 = x_2 \quad z_5 = x_2$

$$(\mathcal{D}f)(z_1) = \frac{f(z_2) - f(z_1)}{z_2 - z_1}$$

	f	$\mathcal{D}f$	$\mathcal{D}^2f$	$\mathcal{D}^3f$	$\mathcal{D}^4f$	$\mathcal{D}^5f$
$z_0 = 0$	0	$f'(0) = 5$	$(5-5)/(2-0) = 0$	$(-1-0)/(2-0) = -1/2$	$(0+1/2)/(3-0) = 1/6$	$(2-1/6)/(3-0) = \frac{11}{18}$
$z_1 = 0$	0	5	$(3-5)/(2-0) = -1$	$(-1+1)/(3-0) = 0$	$(6-0)/(3-0) = 2$	
$z_2 = 2$	10	3	$(2-3)/(3-2) = -1$	$(5+1)/(3-2) = 6$		
$z_3 = 2$	10	2	$(7-2)/(3-2) = 5$			
$z_4 = 3$	12	7				
$z_5 = 3$	12					

$(12-10)/(3-2)$

$$\Rightarrow (H_5 f)(x) = f(z_0) + \sum_{i=1}^{5} (x - z_0)\dots(x - z_{i-1})(\mathcal{D}^i f)(z_0)$$
$$= 0 + (x-0)\cdot \mathcal{D}f(0) + (x-0)^2 \mathcal{D}^2 f(z_0) + (x-0)^2\cdot(x-2)\mathcal{D}^3 f(0) +$$
$$+ (x-0)^2(x-2)^2 \mathcal{D}^4 f(0) + (x-0)^2(x-2)^2(x-3)\mathcal{D}^5 f(0)$$
$$= 5x + 0 + \frac{-x+2}{2}x^2 + \frac{1}{6}x^2(x-2)^2 + \frac{11}{18}x^2(x-2)^2(x-3)$$

Birkhoff interpolation

Definition: Let $x_k \in [a,b]$, $k = 0,1,\dots,m$, $x_i \neq x_j$, for $i \neq j$, $r_k \in \mathbb{N}$ and $I_k \subset \{0,1,\dots,r_k\}$, $k = 0,1,\dots,m$, $f : [a,b] \to \mathbb{R}$, s.t. $\exists f^{(j)}(x_k)$, $k = 0,\dots,m$, $j \in I_k$ and denote $n = |I_0| + \dots + |I_m| - 1$, where $|I_k|$ is the cardinal of the set I_k.

The Birkhoff Interpolation Problem (BIP) consists in determining the polynomial P of the smallest degree s.t.
$$P^{(j)}(x_k) = f^{(j)}(x_k),\ k = \overline{0,m},\ j \in I_k$$

In order to check if the BIP has solutions, we consider the polynomial $P(x) = a_n x^n + \dots + a_0$ and the $(n+1)\times(n+1)$ linear system $\boxed{P^{(j)}(x_k) = f^{(j)}(x_k)}$, $k = 0,\dots,m$, $j \in I_k$ (1) having as unknowns the coefficients of the polynomial. If the determinant of (1) is not 0 $\Rightarrow$ BIP has unique solution.

$$\boxed{(B_m f)(x) = \sum_{k=0}^{m} \sum_{j \in I_k} b_{kj}(x) f^{(j)}(x_k)}$$ = Birkhoff interpolation polynomial

$$\boxed{\begin{array}{l} b_{kj}^{(p)}(x_\nu) = 0, \quad k \neq \nu \\ b_{kj}^{(p)}(x_k) = \delta_{j,p} = \begin{cases} 1, & j = p \\ 0, & j \neq p \end{cases} \end{array}}$$ = Birkhoff interpol. poly. fulfil these relations

$$\boxed{\begin{array}{l} f \simeq B_m f \\ f = B_m f + R_m f \end{array}}$$

Exercise 5

$f \in C^2[0,1]$, $\quad x_0 = 0,\ f(0) = 1$; $\quad x_1 = 1,\ f'(1) = \frac{1}{2}$ $\qquad$ Find the corresponding interpolation formula

! If we have only values of the function $\Rightarrow$ Lagrange interpolation
$f(0), f'(0), f(1) \Rightarrow$ Hermite interpolation
if something is missing $\Rightarrow$ Birkhoff interpolation

$I_0 = \{0\}$, $I_1 = \{1\}$

n = degree of poly = no. elems in $I_0, I_1 - 1 = 1 + 1 - 1 = 1$

- check if the problem has solutions

$P(x) = a_1 x + a_0$
$P(x_0) = f(x_0)$
$P'(x_1) = f'(x_1)$

$$\begin{cases} a_1 \cdot 0 + a_0 = 1 \\ a_1 = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} a_0 = 1 \\ a_1 = \frac{1}{2} \end{cases}$$

$$\Delta = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

! If the system is complicated, we don't have to compute the solutions, just write the determ. accord. to the unknowns

$$(B_1 f)(x) = \sum_{k=0}^{1}\sum_{j\in I_k} b_{jk}(x)\, f^{(j)}(x_k) = b_{00}(x) f(x_0) + b_{11}(x) f'(x_1) = 1\cdot 1 + x\cdot\frac{1}{2} = \boxed{\frac{1}{2}x+1}$$

$b_{00}(x) = a_1 x + b_1$

$b_{00}(x_0) = 1$

$b_{00}'(x_1) = 0$

$$\begin{cases} b_{kj}^{(p)}(x_l) = 0, & k \neq l \\ b_{kj}^{(p)}(x_k) = \begin{cases} 1, & j = p \\ 0, & j \neq p \end{cases} \end{cases}$$

$$\begin{cases} a_1 \cdot 0 + b_1 = 1 \\ a_1 = 0 \end{cases} \Leftrightarrow \begin{cases} b_1 = 1 \\ a_1 = 0 \end{cases}$$

$b_{11}(x) = a_2 x + b_2$

$b_{11}(x_0) = 0$

$b_{11}'(x_1) = 1 \Leftrightarrow b_2 = 0,\ a_1 = 1$

Homework: $f'(0) = 1$, $f(1) = 2$, $f'(2) = 1$. Find the approx. value of $f(\frac{1}{3})$

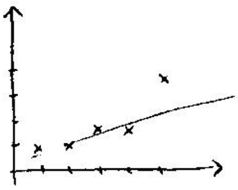
$f'(1), f(2), f(3) \to$ Birkhoff (because we miss $f(1)$)

$f(0), f'(0), f(1) \Rightarrow$ Hermite

Least squares approximation

The least squares approximation φ is determined such that:

- $\left(\sum_{i=0}^{m} [f(x_i) - \varphi(x_i)]^2\right)^{\frac{1}{2}} \to \min$ (discrete case)
- $\left(\int_a^b [f(x) - \varphi(x)]^2\right)^{\frac{1}{2}} \to \min$ (continuous case)



x	1	2	3	4	5
$f(x)$	1	1	2	2	4

$\varphi(x) = ax + b$. Find a, b s.t. φ makes the best function to fit the data

$$E(a,b) = \sum_{i=0}^{4} [f(x_i) - \varphi(x_i)]^2 = \sum_{i=0}^{4} [f(x_i) - (ax_i + b)]^2$$

The min. of the sum is obtained when:

$$\begin{cases} \dfrac{\partial E(a,b)}{\partial a} = 0 \\ \dfrac{\partial E(a,b)}{\partial b} = 0 \end{cases} \Rightarrow \begin{cases} 15a + b = 10 \mid \cdot(-15) \\ 55a + 15b = 37 \end{cases} \Rightarrow \varphi(x) = 0.7x - 0.1$$

$$\begin{cases} 225a - 15b = -150 \\ 55a + 15b = 37 \end{cases}$$

$170a = 113 \Leftrightarrow \boxed{a = \frac{113}{170}} = 0.7$

$b = \left(37 - 55\cdot\frac{113}{170}\right)\cdot\frac{1}{15} = 0.1$

Suppose that $\varphi(x) = \sum_{k=0}^{n} a_k x^k$, $n < m$

Find a_i, $i = \overline{0,n}$ that minimise the sum $E(a_0, \dots, a_n) = \sum_{i=0}^{m} [f(x_i) - \varphi(x_i)]^2$

$$= \sum_{i=0}^{m} \left[f(x_i) - \sum_{k=0}^{n} a_k x_i^k\right]^2$$

The minimum is obtained when

$$\boxed{\frac{\partial E(a_0, \dots, a_n)}{\partial a_j} = 0, \quad j = 0, \dots, n} = \text{normal equations with unique solution}$$

Exercise:

x	0	1	2	3
$f(x)$	-4	0	4	-2

Find the corresponding least squares poly. of first degree

$$\boxed{E(a,b) = \sum_{i=0}^{m} [f(x_i) - \varphi(x_i)]^2 = \sum_{i=0}^{m} [f(x_i) - (ax_i + b)]^2}$$

do not compute this, just derivate it

$\varphi(x) = ax + b$

$$E(a,b) = \sum_{i=0}^{3} [f(x_i) - \varphi(x_i)]^2 = [-4 - (a\cdot 0 + b)]^2 + [0 - (a+b)]^2 + [4 - (2a+b)]^2 + [-2 - (3a+b)]^2$$

$$\frac{\partial E}{\partial a} = 2[0 - (a+b)](-1) + 2[4 - (2a+b)](-2) + 2[-2 - (3a+b)]\cdot(-3) = 0$$

$$= +2a + b - 16 + 8a + 4b + 12 + 18a + 6b = 28a + 11b - 4 = 0$$

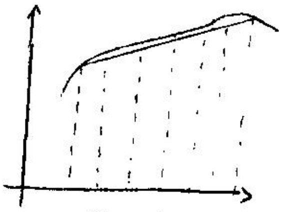
$$\frac{\partial E}{\partial b} = 2(-4-b)(-1) + 2(-a-b)(-1) + 2(4-2a-b)(-1) + 2(-2-3a-b)(-1) = 0$$

$$= 8 + 2b + 2a + 2b - 8 + 4a + 2b + 4 + 6a + 2b = 0$$

$$= 12a + 8b + 4 = 0$$

$$\begin{cases} 28a + 11b - 4 = 0 \;|\cdot 8 \\ 12a + 8b + 4 = 0 \;|\cdot(-11) \end{cases} \Leftrightarrow \begin{cases} 224a + 88b - 32 = 0 \\ -142a - 88b - 44 = 0 \end{cases}$$

$$82a \;/\; -76 \Leftrightarrow \boxed{a = \frac{76}{82}}$$



Romberg's iterative generation method

$QT_0(f) = \frac{h}{2}[f(a)+f(b)]$, $h = b-a$

$QT_1(f) = \frac{h}{4}\left[f(a) + 2f\left(a+\frac{h}{2}\right) + f(b)\right]$ OR $QT_1(f) = \frac{1}{2}QT_0(f) + hf\left(a+\frac{h}{2}\right)$

Romberg quadrature →

The general formula: $\boxed{QT_k(f) = \frac{1}{2}QT_{k-1}(f) + \frac{h}{2^k}\sum_{j=1}^{2^{k-1}} f\left(a + \frac{2j-1}{2^k}h\right)}$, $k = 1, 2,$

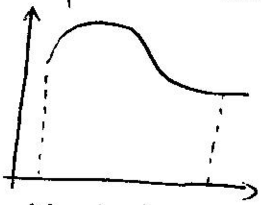
we compute $QT_0(f)$, $QT_1(f)$, ... until we meet $|QT_n(f) - QT_{n-1}(f)| \le \varepsilon$

- The k-th element generated by the Simpson formula

$\boxed{QS_k(f) = \frac{1}{3}[4QT_{k+1}(f) - QT_k(f)]}$. $k = 0, 1, \ldots$ = **repeated** Simpson's formula

$QS_0 = \frac{h}{6}\left[f(a) + 4f\left(a+\frac{h}{2}\right) + f(b)\right]$

Adaptive Quadrature Methods



The methods that adapt the steps according to the need of the function are called adaptive quadrature methods.

Approximate: $I = \int_a^b f(x)\,dx$ with precision ε.

First step: Apply Simpson's formula.

The algorithm:

```
function I = adquad(a, b, er)
    I1 = Simpson(a, b)
    I2 = Simpson(a, (a+b)/2) + Simpson((a+b)/2, b)
    if |I1 - I2| < 15 * er
        I = I2
    else                 return
        I = adquad(a, (a+b)/2, er/2) + adquad((a+b)/2, b, er/2)
    end-if
end-function
```

Generated quadrature formulas

Ex.: We know $f'(a)$, $f(b)$ and $f(x) = (x-b)f'(a) + f(b) + (R_1f)(x)$

Obs: It's not Birkhoff because $f(a)$ is missing

Find $f(a)$.

$$(B_mf)(x) = \sum_{k=0}^{m}\sum_{j\in I_k} b_{kj}\, f^{(j)}(x_k)$$

$x_0 = a$
$x_1 = b$

$I_0 = \{1\}$ $\quad I_1 = \{0\}$ $\quad m = 0$ $\quad n = |I_0| + |I_1| - 1 = 1$

$(B_1f)(x) = \underbrace{b_{01}(x)f'(x_0)}_{k=0} + \underbrace{b_{10}(x)\cdot f(x_1)}_{k=1}$

$(B_1f)(x) = (x-b)f'(a) + f(b) \simeq f \Rightarrow \int_a^b f(x)\,dx \simeq$

$\simeq \int_a^b (B_1f)(x)\,dx = A_0f'(a) + A_1f(b) \overset{?}{=} -\frac{(a-b)^2}{2}f'(a) + (b-a)f(b)$

$A_0 = \int_a^b (x-b)\,dx = \frac{(x-b)^2}{2}\Big|_a^b$

$A_1 = \int_a^b 1\,dx = x\Big|_a^b$

$A_0 = -\frac{(a-b)^2}{2}$ $\quad A_1 = b-a$

$b_{01}(x) = m\cdot x + n$

$b_{01}(x) = a_1x + b_1 = x - b$

$b_{01}'(x_0) = 1$

$b_{01}(x_1) = 0$

$\Rightarrow \begin{cases} a_1 = 1 \\ a_1 b + b_1 = 0 \end{cases} \Leftrightarrow \begin{cases} a_1 = 1 \\ b_1 = -b \end{cases}$

$b_{10}(x) = a_2x + b_2$

$b_{10}'(x_0) = 0$

$b_{10}(x_1) = 1$

$\Rightarrow \begin{cases} a_2 = 0 \\ a_2 b + b_2 = 1 \end{cases} \Leftrightarrow \begin{cases} a_2 = 0 \\ b_2 = 1 \end{cases}$

$b_{10}(x) = a_2x + b_2 = 1$

Interpolary Quadrature Formulas

Quadrature formulas of Gauss type

$\int_a^b w(x)\cdot f(x)dx = \sum_{k=1}^{m} A_k f(x_k) + R_m(f)$ - max. degree of exactness

- A_k and x_k, $k=1,\dots,m$ form] are $2m$ unknown parameters $\Rightarrow$ $2m$ equations obtained s.t. the formulas is exact for any polynomial degree at most $2m-1$

For general case denote:

$l_k(x) = x^k$; $k = 0,\dots,2m-1$ and obtain the system

$$\begin{cases} \sum_{k=1}^{m} A_k l_0(x_k) = \int_a^b w(x) l_0(x)\,dx \\ \sum_{k=1}^{m} A_k l_1(x_k) = \int_a^b w(x) l_1(x)\,dx \\ \dots \\ \sum_{k=1}^{m} A_k l_{2m-1}(x_k) = \int_a^b w(x) l_{2m-1}(x)\,dx \end{cases} \iff \begin{cases} A_1 + A_2 + \dots + A_m = \mu_0 \\ A_1 x + A_2 x_2 + \dots + A_m x_m = \mu_1 \\ \dots \\ A_1 x_1^{2m-1} + A_2 x_2^{2m-1} + \dots + A_m x_m^{2m-1} = \mu_{2m-1} \end{cases}$$

For $w(x)=1$, the nodes are roots of Legendre orthogonal polynomial:

$$u(x) = \frac{m!}{2(m)!}\left((x-a)^m (x-b)^m\right)^{m}$$

and for finding the coefficients we use the first m equations from the system.

Ex.: $m=1 \Rightarrow \int_a^b f(x)\,dx = A_1 f(x_1) + R_1(f)$

$$\begin{cases} A_1 = \mu_0 = b-a \\ A_1 x_1 = \mu_1 = \frac{b^2-a^2}{2} \\ \mu_0 = \int_a^b 1\,dx = b-a \\ \mu_1 = \int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2-a^2}{2} \end{cases}$$

$A_1 = b-a$

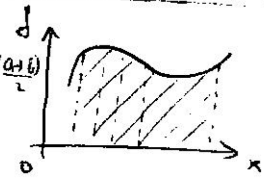
$x_1 = \frac{b+a}{2}$

$u(x) = \frac{1}{2}\left[(x-a)(x-b)\right]'$

$(x^2-(a+b)x+ab)' = \frac{1}{2}(2x-a-b) = x - \frac{a+b}{2} \Rightarrow$

$u(x) = \alpha \Rightarrow 1 = \frac{a+b}{2}$

$$\boxed{\int_a^b f(x)\,dx = (b-a)\, f\left(\frac{b+a}{2}\right) + R_1(f)}$$ = rectangle quadrature formula



Repeated rectangle quadrature formula:

$$\int_a^b f(x)\,dx = \frac{b-a}{m}\sum_{i=1}^{m} f(x_k) + R_m(f), \quad R_m(f) = \frac{(b-a)^3}{24m^2} f''(\varepsilon), \ \varepsilon \in [a,b]$$

with $x_1 = a + \frac{b-a}{2m}$, $x_i = x_1 + (i-1)\frac{b-a}{m}$, $i=2,\dots,m$

we have:

$$|R_m(f)| \le \frac{(b-a)^3}{24 m^2} M_2 f \qquad M_2 f = \max_{x\in[a,b]} |f''(x)|$$

Course 8
18.04.2016

Numerical methods for solving linear systems

Clasification
- direct methods → low number of unknowns ($\le$ several 10.000)
- iterative methods → medium no. of unknowns
- semiiterative methods → large no. of unknowns (aprox. of the sol.)

Perturbation of linear systems

Consider the general form of a linear system: $Ax = b$, $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$ matrix of coef.

The number $\boxed{cond(A) = \|A\| \cdot \|A\|^{-1}}$ is called conditioning number

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ vector of unknowns $\quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ → vector of free terms

- It measures the sensibility of the solution x of the system $Ax = b$ to the perturbation of A and b.
- The system is $\begin{cases} \text{good conditioned } (cond(A) \le 1000) \\ \text{ill conditioned } (cond(A) > 1000) \end{cases}$

If b is perturbed $\Rightarrow cond(A) \frac{\|\delta b\|}{\|b\|}$

A is perturbed $\Rightarrow cond(A) \frac{\|\delta A\|}{\|A\|}$

Direct methods for solving linear systems

- Cramer's method → if $n = 100$, $n \times n$ system $\Rightarrow \begin{cases} (n+1)! \text{ aditions} \\ (n+1)! \text{ multiplications} \\ n \text{ divisions} \end{cases}$ ⇒ aprox. 10^{94} years for solving the system!

- Gauss's method

Consider the linear system $Ax = b$ i.e.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

<u>Steps</u> - reduce the system from above to an equivalent one, $Ux = d$, U = upper triangular matrix
- solve the upper triangular linear system $Ux = d$, by backward substitution

! At least one of the elements on the first column is nonzero (a chosen non-zero elem.), otherwise the system is singular

If a_{kk}, the pivot is $\neq 0$, denote $m_{ik} = \frac{a_{ik}^k}{a_{kk}^k}$ and we get:

$$a_{ij}^{k+1} = a_{ij}^k - m_{ik} a_{kj}^k, \quad j = k, \dots, n$$

$$b_i^{k+1} = b_i^k - m_{ik} b_k^k, \quad i = k+1, \dots, n$$

After $n-1$ steps, we obtain the following system:

$$\begin{pmatrix} a_{11}^1 & a_{12}^1 & \dots & a_{1n}^1 \\ 0 & a_{22}^2 & \dots & a_{2n}^2 \\ 0 & 0 & \dots & a_{3n}^3 \\ \vdots & & & \\ 0 & 0 & \dots & a_{nn}^{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{pmatrix}$$

→ $n-1$ an is the same because we don't do anything on the last step

Gauss algorithm yields to $m_{21} = \frac{a_{21}}{a_{11}} = \frac{1}{0.0001} \Rightarrow \dots \Rightarrow y = 0.(9998) \approx 1$ → replacing the first ec. we get $x = 1.00(1000) \approx 1$. By division with a pivot of small values ⇒ possible errors

How to avoid this?

- Partial pivoting - find an index p from the set $\{k, \dots, n\}$ s.t.

$$|a_{p,k}^k| = \max_{i=k,n} |a_{i,k}^k|$$

- **Total pivoting** - finding $p, q \in \{k, \dots, n\}$ s.t.

$$|a_{p,q}^{k}| = \max_{i,j=\overline{k,n}} |a_{ij}^{k}|$$

The pivot should be the maximum on each column, in absolute value.

<u>Example</u>: Solve the system

(1) $\begin{cases} 2x + y = 3 \\ 3x - 2y = 1 \end{cases}$ $\left(\begin{array}{cc|c} 2 & 1 & 3 \\ 3 & -2 & 1 \end{array}\right) \simeq \left(\begin{array}{cc|c} 3 & -2 & 1 \\ 2 & 1 & 3 \end{array}\right) \simeq \left(\begin{array}{cc|c} 3 & -2 & 1 \\ 0 & \frac{7}{3} & \frac{7}{3} \end{array}\right) \Rightarrow \begin{cases} 3x - 2y = 1 \\ 0 + \frac{7}{3}y = \frac{7}{3} \end{cases} \Rightarrow$

$\begin{cases} y = 1 \\ 3x - 2 = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$

(2) $\begin{cases} x_1 + x_2 + x_3 = 4 \\ 2x_1 - 2x_2 + 3x_3 = 5 \\ x_1 - x_2 + 4x_3 = 5 \end{cases}$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -2 & 3 & 5 \\ 1 & -1 & 4 & 5 \end{array}\right) \simeq \left(\begin{array}{ccc|c} 2 & -2 & 3 & 5 \\ 1 & 1 & 1 & 4 \\ 1 & -1 & 4 & 5 \end{array}\right) \begin{array}{c} L_2 - \frac{1}{2}L_1 \\ L_3 - \frac{1}{2}L_1 \end{array} \left(\begin{array}{ccc|c} 2 & -2 & 3 & 5 \\ 0 & 2 & -\frac{1}{2} & \frac{3}{2} \\ 0 & +0 & \frac{5}{2} & \frac{5}{2} \end{array}\right)$

$\begin{cases} \frac{5}{2}x_3 = \frac{5}{2} \\ 2x_2 - \frac{1}{2}x_3 = \frac{3}{2} \\ 2x_1 - 2x_2 + 3x_3 = 5 \end{cases} \Rightarrow \begin{cases} \boxed{x_3 = 1} \\ 2x_2 - \frac{1}{2} = \frac{3}{2} \\ 2x_1 - 2x_2 + 3 = 5 \end{cases} \Leftrightarrow \begin{cases} x_3 = 1 \\ \boxed{x_2 = 1} \\ 2x_1 - 2 = 2 \end{cases} \Leftrightarrow \begin{array}{l} x_3 = 1 \\ x_2 = 1 \\ \boxed{x_1 = 2} \end{array}$

- **Total elimination method (Gauss-Jordan)**

- make 0's bellow and above the ~~first~~ diagonal

- **Factorization method (LU method)**

- applied if the matrix is strictly diagonally dominant ($|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}|$, for $i = \overline{1,n}$)

ex.: $|a_{11}| > |a_{12}| + |a_{13}| + \dots + |a_{1n}|$

$\dots$

$|a_{nn}| > |a_{n1}| + \dots + |a_{nn-1}|$

Remark: In this case, Gauss elimination can be done without row or column interchanges.

Theorem: If A is strictly diagonally dominant $\Rightarrow$ it can be factored into a power of the lower triangular matrix L and the upper triangular matrix U, namely $A = LU$.

$L = \begin{pmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & \dots & 0 \\ \vdots & & & \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{pmatrix}$ $U = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & & & \\ 0 & 0 & \dots & u_{nn} \end{pmatrix}$

First stage: solve $Lz = b$
Second stage: solve $Ux = z$

- Doolittle method = all diagonal elems. of L are 1
- Crout method = all diagonal elems. of U are 1
- Choleski method = $l_{ii} = u_{ii}$, $i = \overline{1,n}$

Doolittle method:

$l_{i,k} := \frac{a_{ik}^{(k-1)}}{a_{k,k}^{(k-1)}}, \ i = \overline{k+1, n}$, $t^{(k)} = \begin{bmatrix} 0 \\ 0 \\ l_{k+1,k} \\ \vdots \\ l_{n,k} \end{bmatrix}$ 0's on the first k lines , $M_k = I_n - t^{(k)} e_k \in M_{n\times n}(\mathbb{R})$

M_k = Gauss matrix

$l_{i,k}$ = Gauss multiplier

$t^{(k)}$ = Gauss vector

$e_k = (0 \dots 1 \dots 0)$ = k unit vector of dimension n

I_n = identity matrix

• If $A \in M_{n\times n}(\mathbb{R}) \Rightarrow U = M_{n-1} \cdot M_{n-2} \cdot \dots M_2 \cdot M_1 \cdot A$; $L = M_1^{-1} \cdot M_2^{-1} \dots M_{n-1}^{-1}$

Example:

$A = \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix}$

$n = 2$

$k = \overline{1,1} = 1 \ (k = \overline{1, n-1})$

$i = 2 \ (i = \overline{k+1, n})$

$\Rightarrow l_{2,1} = \frac{a_{21}}{a_{11}} = \frac{6}{2} = 3$

$t^{(1)} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} (1 \ \ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$

$U = M_1 \cdot A = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a-3b & b \\ c-3d & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{matrix} b = 0 \\ d = 1 \\ a = 1 \\ c = 3 \end{matrix}$

$L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

Iterative methods for solving linear systems

Course 9
25.04.2016

$Ax = b$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

An iterative method for solving linear systems consists of converting the system $Ax=b$ to the form $x = b' - Bx$.

After an approximation of $x^{(0)}$, the seq. of approximations of the solution $x^{(0)}, x^{(1)} \dots x^{(k)}$ is given by

Yacobi iterative method

Consider $\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$ $\quad a_{11}, a_{22}, \dots, a_{nn} \neq 0$

We solve each equation for one of the variables:

$$\begin{cases} x_1 = u_{12}x_2 + \dots + u_{1n}x_n + c_1 \\ x_2 = u_{21}x_1 + \dots + u_{2}x_n + c_2 \\ \dots \\ x_n = u_{n1}x_1 + \dots + u_{nn-1}x_n + c_2 \end{cases}, \text{ where } u_{ij} = -\frac{a_{ij}}{a_{ii}},\ c_i = \frac{b_i}{a_{ii}},\ i = \overline{1,n}$$

$$\boxed{x_i^{(k)} = \frac{b_i - \sum_{j=1, j\neq i} a_{ij} x_j^{(k-1)}}{a_{ii}}}, \quad i = 1, 2, \dots, n,\ k \geq 1$$

- The iterative process terminates when a convergence criterion is satisfied.
- Stopping criterions: $|x^{(k)} - x^{(k-1)}| < \varepsilon$ or $\frac{|x^{(k)} - x^{(k-1)}|}{|x^{(k)}|} < \varepsilon$ with $\varepsilon > 0$ - a prescribed tolerance

Gauss-Seidel iterative method

Each x is obtained using the approximations of the other variables

For a $n \times n$ system, the $k+1$ th approx. is:

$$\begin{cases} x_1^{(k+1)} = u_{12}x_2^{(k)} + \dots + u_{1n}x_n^{(k)} + c_1 \\ x_2^{(k+1)} = u_{21}x_1^{(k+1)} + u_{23}x_3^{(k)} + \dots + u_{2n}x_n^{(k)} + c_2 \\ \vdots \\ x_n^{(k+1)} = u_{n1}x_1^{(k+1)} + \dots + u_{nn-1}x_{n-1}^{(k+1)} + c_n \end{cases}$$

$$\boxed{x_i^{(k)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}}{a_{ii}}}$$

- stopping criterions: $|x^{(k)} - x^{(k-1)}| < \varepsilon$ or $\frac{|x^{(k)} - x^{(k-1)}|}{|x^{(k)}|} < \varepsilon$, with ε - a prescribed to tolerance, $\varepsilon > 0$

Relaxation method (SOR method)

$$\boxed{x_i^{(k)} = \underbrace{\frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right)}_{\text{Gauss-Siedel it. meth.} \cdot \omega} + (1-\omega)x_i^{(k-1)}}, \text{ for each } i = \overline{1,n},\ k \geq 1$$

- if $0 < \omega < 1 \Rightarrow$ under relaxation meth. — used for systems that are not convergent
- if $\omega > 1 \Rightarrow$ over relaxation method — accelerate convergence

- The method converges for $0 < \omega < 2$.

The matriceal formulations of the iterative methods

Split the matrix A into the sum $A = D + L + U$, where D is the diagonal of A, L is the lower triangular part of A and U is the upper triangular of A.

$$D = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ 0 & \dots & 0 & a_{nn} \end{pmatrix} \quad U = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ \dots & & & & 0 \end{pmatrix} \quad L = \begin{pmatrix} 0 & 0 & \dots & 0 \\ a_{21} & 0 & \dots & 0 \\ \vdots \\ a_{n1} & a_{n2} & \dots & 0 \end{pmatrix}$$

$\Rightarrow Ax = b \Leftrightarrow (D + L + U)x = b$

Matriceal forms:

$$\boxed{\begin{aligned} &Dx^{(k)} = -(L+U)x^{(k-1)} + b \\ &(D+L)x^{(k)} = -Ux^{(k-1)} + b \\ &(D+\omega L)x^{(k)} = ((1-\omega)D - \omega U)x^{(k-1)} + \omega b \end{aligned}}$$

Jacobi method

Gauss-Seidel method

Relaxation method

Theorem (Convergence): If A is strictly diagonally dominant, then all 3 methods converge for any choice of the starting vector $x^{(0)}$.

Example: Consider the linear system

$$\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Perform 2 iterations of the Jacobi and Gauss methods $x = [3, 11]$

Jacobi: $\begin{cases} 4x + y = 3 \\ 2x + 5y = 1 \end{cases}$

$$\begin{cases} x^{(1)} = \frac{3-y^{(0)}}{4} = \frac{3-11}{4} = \frac{-8}{4} = -2 \Rightarrow x^{(2)} = 1 \\ y^{(1)} = \frac{1-2x^{(0)}}{5} = \frac{1-6}{5} = \frac{-5}{5} = -1 \Rightarrow y^{(2)} = 1 \end{cases}$$

Gauss-Siedel:

$$x^{(1)} = \frac{3-y^{(0)}}{4} = -2$$

$$y^{(1)} = \frac{1-2x^{(1)}}{5} = 1$$

$$x^{(2)} = \frac{3-y^{(1)}}{4} = \frac{1}{2}$$

$$y^{(2)} = \frac{1-2x^{(2)}}{5} = 0$$

Relaxation

$$x^{(1)} = \omega\frac{3-y^{(0)}}{4} + (1-\omega)x^{(0)}, \quad \omega = 1.2$$

$$= 1.2 \cdot (-2) + 0.2 \cdot 3 = -2.4 - 0.6 = -3$$

$$y^{(1)} = \omega + (0.2) \cdot y^{(0)} = 1,2 + (-2,2) = -1$$

! Fără metoda lui Gauss (nu Gauss-Siedel) nu va fi la examen.

Numerical methods for solving nonlinear equations

Consider the equation $f(x) = 0$, $x \in \mathbb{R}$. We attach a mapping $F: D \to D$, $D \subset \mathbb{R}^m$ to this equation. Let $(x_0, \dots, x_{m-1}) \in D$. Using F and the no. $x_0, \dots, x_{m-1}$ we construct the seq. $x_0, \dots, x_m$, with

$$x_i = F(x_{i-m}, \dots, x_{i-1}), \quad i = m, \dots$$

We need to choose F and $x_0, \dots, x_{m-1} \in D$ s.t. the seq. would converge to the solution $f(x)=0$, $x \in \mathbb{R}$

- $x_0, x_1, \dots, x_n$ = starting points
- if we have 1 starting point $\Rightarrow$ F is a one step method
- — " — more than 1 — " — $\Rightarrow$ F is a multistep method

Definition: If the seq. $x_0, \dots, x_n$ converges to the sol. of the eq. $f(x) = 0 \Rightarrow F$-method is convergent, otherwise is divergent.

Definition: Let $\alpha \in \mathbb{R}$ be a solution of the eq. $f(x) = 0$ and $x_0, \dots, x_n \dots$ be the seq. generated by a given F-method. The number p having the property

$$\boxed{\lim_{x_i \to \alpha} \frac{\alpha - F(x_{i-m+1}, \dots, x_i)}{(\alpha - x_i)^p} = C \neq 0}, \quad C = \text{constant}$$

" the order of the F-method

One-step methods

Let F be a one-step method, for a given x_i we have $x_{i+1} = F(x_i)$

- If $p = 1 \Rightarrow$ the convergence condition is $|F'(x)| < 1$
- If $p > 1 \Rightarrow$ there always exists a neighborhood of α where the F-method converges.

Information of f at a single point $\Rightarrow$ Taylor interpolation.

$$\boxed{F_m^T(x_i) = x_i + \sum_{k=1}^{m-1} \frac{(-1)^k}{k!}[f(x_i)]^k g^{(k)}(f(x_i)), \text{ where } g = f^{-1}} = \text{method for approximating } \alpha$$

[2] $\boxed{|\alpha - F_m^T(x_i)| \leq \frac{1}{m!}[f(x_i)]^m M_m g}$, with $M_m g = \sup_{y \in V(0)} |g^{(m)}(y)|$ $\to$ upper bound for the abs. error in approximating x_{i+1}

Numerical methods for solving non-linear equations in $\mathbb{R}$

09.05.2016

One-step methods

$m = 2$

$$F_2^T(x_i) =$$

The higher the order of a method is, the faster the method converges. Still, this does not mean that higher order means more efficient. The order 2 and 3 are the most efficient.

$$\boxed{x_{i+1} = F_2^T(x_i) = x_i - \frac{f(x_i)}{f'(x_i)}, \quad i = 0, 1, \ldots} = \text{Newton's method}$$

$$\boxed{|x_n - \alpha| \le |x_n - x_{n-1}|, \quad n > n_0}$$

Remark: The starting value is chosen randomly. If, after a fixed number of iteration, the required precision is not achieved, i.e. condition $|x_n - x_{n-1}| \le \varepsilon$, does not hold for a prescribed positive ε, the computation has to be started over with a new starting value.

$$x_0, \quad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$|x_{i+1} - x_i| < \varepsilon$$

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_0)}, \quad k = 0, 1, \ldots} = \text{modified form of Newton's method.}$$

$$\boxed{y - y_0 = f'(x_0)(x - x_0)} \to \text{general eq. of the tangent line in the point } (x_0, y_0)$$

$$-y_0 = f'(x_0)(x - x_0) \Rightarrow x_0 f'(x_0) - f(x_0) = x f'(x_0) \Rightarrow \boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

The algorithm:

Let x_0 be the initial approximation

for $n = 0, 1, \ldots IT\ max$

$$x_{n+1} \leftarrow x_n - \frac{f(x_n)}{f'(x_n)}$$

A stopping criterion is:

$$|f(x_n)| \le \varepsilon \text{ or } |x_{n+1} - x_n| \le \varepsilon \text{ or } \frac{|x_{n+1} - x_n|}{|x_{n+1}|} \le \varepsilon, \text{ where } \varepsilon \text{ is a}$$

specified tollrance value.

Ex.: $x^3 - x^2 - 1 = 0$, accuracy of 10^{-4}. use $x_0 = 1$

$\cancel{x_4 = x_3 - \frac{f(x_4)}{f'(x_4)}}$

$f'(x) = 3x^2 - 2x$

$\cancel{f''(x) = 6x - 2}$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1}{1} = 1+1 = 2$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{8}{2} - \frac{3}{8} = \frac{13}{8}$

$\cancel{x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{13}{8} - }$

If we continue we get that the seq. converges to 1. ...

Hermite inverse interpolation

Courx 12
16.05.2016

$f(x)=0$

$g=f^{-1}$ $f(\alpha)=0$

$\alpha=g(0)$

Consider the equation $f(x)=0$, $x\in\Omega$, $f:\Omega\to\mathbb{R}$, $\Omega\in\mathbb{R}$.
Assume α is a solution of the equation $f(x)=0$, $V(\alpha)$ is a neighborhood of α.

If $y_k=f(x_k)$, where $x_k\in V(\alpha)$, $k=0,\dots,m$ are approximations of α, $r_k\in\mathbb{N}$ $\Rightarrow$ there exists $g^{(j)}(y_k)=(f^{-1})^{(j)}(y_k)$, $j=0,\dots,r_k$; one considers the Hermite

We approximate g using the Hermite polynomial

$$\boxed{(H_m g)(y)=\sum_{k=0}^{m}\sum_{j=0}^{r_k} h_{kj}(y)\,g^{(j)}(y_k)}$$

- Taking into account that:

$$\alpha=g(0)\approx(H_m g)(0)\ \longrightarrow\ \boxed{F_m^H(x_0,\dots,x_m)=(H_m g)(0)}$$

approximation method for α

Birkhoff inverse interpolation

$\boxed{(B_m g)(y)=\sum_{k=0}^{m}\sum_{j\in I_k} b_{kj}(y)\,g^{(j)}(y_k)}$ = the Birkhoff polynomial

satisfies the conditions $(B_m g)^{(j)}(y_k)=g^{(j)}(y_k)$, $j\in I_k$, $k=0,\dots,m$

- Taking into account that $\alpha=g(0)\approx(B_m g)(0)$

$$\Rightarrow\ \boxed{F_m^B(x_0,\dots,x_m)=(B_m g)(0)}$$

Numerical methods for solving differential equations

We consider a Cauchy problem: $y'=f(x,y)$, $y(x_0)=y_0$ (5), $f:D$, $D=\{(x,y)\in\mathbb{R}^2 \mid |x-x_0|\le a,\ |y-y_0|\le b\}$, $a,b\in\mathbb{R}_+$, continuous and derivable.

Taylor interpolation method

Let $f\in C^p(D)$ and y be a solution of the problem (5). We attach Taylor interp. formula to y, with respect to x_0.

$$y=T_p y+R_p y,$$

$$(T_p y)(x)=y(x_0)+\frac{x-x_0}{1!}y'(x_0)+\dots+\frac{(x-x_0)^p}{p!}y^{(p)}(x_0).$$

$$(R_p y)(x)=\frac{(x-x_0)^{p+1}}{(p+1)!}y^{(p+1)}(\xi),\quad \xi \text{ between } x_0 \text{ and } x$$

Denoting $y^{(k)} = f^{(k-1)}$ $\Rightarrow$

$$\boxed{(T_p y)(x) = y(x_0) + \frac{x-x_0}{1!} f(x_0, y(x_0)) + \frac{(x-x_0)^2}{2!} f'(x_0, y(x_0)) + \dots + \frac{(x-x_0)^p}{p!} f^{(p-1)}(x_0, y(x_0))}$$ = Taylor polynomial

For equidistant points $x_i = x_0 + ih$, with $y_i = y(x_i)$, $i = 0, \dots, N$, $h = \frac{b-a}{N}$ we have

$$\boxed{y_{i+1} = y_i + h T_n(x_i, y_i)}$$ = Taylor interpolation method of order n

$$T_n(x_i, y_i) = f(x_i, y_i) + \frac{h}{2!} f'(x_i, y_i) + \dots + \frac{h^{n-1}}{n!} f^{(n-1)}(x_i, y_i)$$

Euler's method

- obtained from the general form 1, where $n = 1$ $\Rightarrow$

$$\boxed{y_{i+1} = y_i + h f(x_i, y_i)}$$ Euler's method

Algorithm for Euler's method

$h \leftarrow \frac{b-a}{N}$

$y \leftarrow y_0$

for $i = 0, 1, \dots, N-1$

$\quad y_{i+1} \leftarrow y_i + h f(x_i, y_i)$

end

Example: $f(x, y) = 2x - y$

$y_1 = y_0 + h f(x_0, y_0) = -1 + \frac{1}{10} \cdot f(0, -1) = -1 + \frac{1}{10} \cdot 1 = \frac{-9}{10} = -0.9$

$y_0 = -1$

$x_0 = 0$

$h = \frac{b-a}{N} = \frac{1+0}{10} = \frac{1}{10}$

Runge-Kutta methods

$$\boxed{T_2(x, y) = f(x, y) + \frac{h}{2} f'(x, y)}$$ $h = \frac{b-a}{N}$, N - given

We have $f'(x, y) = \frac{\partial f}{\partial x}(x, y) + \frac{\partial f}{\partial y}(x, y) \cdot y'(x)$

$$\Rightarrow T_2(x, y) = f(x, y) + \frac{h}{2}\frac{\partial f}{\partial x}(x, y) + \frac{h}{2}\frac{\partial f}{\partial y}(x, y)\, y'(x)$$

$$\boxed{y_{i+1} = y_i + h f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} f(x_i, y_i)\right), \quad y_0 = \alpha, \ i = 0, \dots, N-1}$$

the midpoint method

(Runge-Kutta method of second order)

$y_0 = \alpha$

$k_1 = h f(x_i, y_i)$

$k_2 = h f\left(x_i + \frac{h}{2}, y_i + \frac{1}{2} k_1\right)$

$k_3 = h f\left(x_i + \frac{h}{2}, y_i + \frac{1}{2} k_2\right)$

$k_4 = h f(x_{i+1}, y_i + k_3)$

$$\Rightarrow \boxed{y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}, \quad i = 0, \dots, N-1$$

Runge-Kutta method of fourth order

For the exam

! Doar exerciții (Orice document pe hârtie)

$\boxed{C1}$ - preliminary notions (remember)
- how to compute finite and divided differences
  ↳ Newton, Hermite poly with double nodes
  ↳ how to comp. the table

$x_0, x_1, \dots, x_n$
$f(x_0)$ $f'(x_0)$ } double
$f(x_1)$ $f'(x_1)$ } info.

- Taylor interpolation → formula
  ↳ remainder
  ↳ good approx. prop. around x_0
- Lagrange interpolation (if we have points and value at these points)
  ↳ tables → find poly. that approx. the data
  ↳ $f(x_i) = L_m f(x_i)$ (Lagrange interp. cond.)

$\boxed{C2}$ - Lagrange → how to obtain the error
  ↳ a limit of the error
  ↳ int. formula (which includes also the error)
- no Aitken's algorithm

$\boxed{C3}$ - Newton's form → the one with divided differences
  → compute the table + use the formula
- Hermite interpolation → when we know values of function + some derivatives (until an interval)
  ex.: x_0, $f(x_0)$, $f'(x_0)$ $r_0 = 1$
  x_1, $f(x_1)$ $r_1 = 0$
  x_2, $f(x_2)$, $f'(x_2)$, $f''(x_2)$ $r_2 = 2$
  ↳ if this is missing → Birkhoff

$\boxed{C4}$ - form of the remainder
- Hermite interpolation with double nodes

$\boxed{C5}$ - Birkhoff interpolation (used when something is missing)
  ↳ there are cases when the problem cannot be solved
  ↳ how to check that the problem has solution
- least squared interpolation
  ↳ how we form the system
  ↳ how to find the coeff.

$\boxed{C6}$ - numerical integr. of functions
  ↳ how to apply simple and repeated formulas
  ↳ find the value of n, approx. the interval with a given precision
  ↳ the smallest n s.t. we get the given precision
- no direct methods

$\boxed{C7}$ - no Romberg iterative generation / Adaptive quadrature form.
- general quadrature formulas (starting with any interp. formula → quadrat
- quadrature formulas of Gauss type
  ↳ particular case of rectangle
- perturbations linear system (not important)
- direct methods
- no Gauss method
- no LU method/Gauss-J/Doolitle

C8
- Jacobi iterative methods
- numerical methods for solving linear ec.
- ~~how~~ inverse Lagrange
- bisection
- false position

Examples:

(1) Lagrange

x	0	4	8
$f(x)$	2	6	10

$m = 2$ (the nodes are denoted from 0 to m) — degree of the polynomial

$(L_2 f)(x) = \sum_{i=0}^{2} l_i(x) f(x_i) = l_0(x)\cdot 2 + l_1(x)\cdot 6 + l_2(x)\cdot 10$

$l_0(x) = \frac{u_0(x)}{u_0(x_0)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-4)(x-8)}{32}$

$u(x) = \prod_{j=0}^{m} (x-x_j) = (x-x_0)(x-x_1)(x-x_2)$

$u_0(x) = \frac{u(x)}{x-x_0} = \frac{\cancel{(x-x_0)}(x-x_1)(x-x_2)}{\cancel{x-x_0}} = (x-x_1)(x-x_2)$

---

(2)

x	2	4	6
$f(x)$	-1	2	5
$f'(x)$	4		8

Hermite (because we have derivative)

$k = 0, 2$

$(H_4 f)(x) = h_{00}(x)\cdot f^{0}(x_0) + h_{01}(x) f'(x_0) +$
$+ h_{10}(x) f^{0}(x_1)$
$+ h_{20}(x) f^{0}(x_2) + h_{21}(x) f'(x_2)$

$r_0 = 1$
$r_1 = 0$
$r_2 = 1$

$n = m + r_0 + r_1 + r_2 = 2 + 1 + 0 + 1 = 4$

degree poly.

m = index of the nodes $(k = 0, \dots, m)$