= Consultații Calcul Numeric. =

Intercalare, Aproximare, Quadraturi, Metode de rez. < iterative / directe

Nu se rez.

Cursurile 9, 10, 11 – Met rez. a ec. neliniar < diferentiale
ultimul subiect

– Euler
– Runge Kutta

① Finite and divided differences.
↳ finite differences table – il știu
↳ divided diff. – tabel – il știu

1 rule for each table!

② Interpolation polynomials.
↳ poly. around one point (x_0) (Taylor polynomial).
↳ the error for aprox. Taylor Polynomial (aprox. around x_0)

↳ Lagrange interpolation.
↳ classical form using fundamental polynomial
↳ another form derivated from Bary. poly. (barycentric)
↳ other form, derived from Newton form (with divided diff).
↳ Aiken form

for theoretical problems < classical form / Newton form. }!

→ Bary and Aiken – not required for the error.

↳ the expression of the remainder.

$f = L_m f + R_m f$

$R_m f = f - L_m f$. $\Rightarrow f \approx L_m f$.

+ limit of the error.

Example 6 from course 2.

Find the aprox. value of log 2., knowing log 1 = 0, log 3 = 1.09, log 4 = 1.38, using Lagrange polynomial in classical form and in Newton's form and give the expression of the error.

Classical form:

- the polynomial is not mentioned
- you know is Lagrange because we have a multiple given points

x	x_0: 1	x_1: 3	x_2: 4
$f(x)$	0	1.09	1.38.

$m = 2$.

$$(L_2 f)(x) = \sum_{i=0}^{2} l_i(x)\, f(x_i) = \sum_{i=0}^{2} \frac{u_i(x)}{u_i(x_i)} \cdot f(x_i)$$

$l_i(x)$

$$l_i(x) = \frac{u_i(x)}{u_i(x_i)}$$

0) $l_0(x) = \frac{u_0(x)}{u_0(x_0)} = \frac{(x-x_1)\,(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-3)(x-4)}{(1-3)(1-4)} = \frac{\text{exp}}{6}$

$$u_i(x) = \frac{u(x)}{x-x_i}$$

$$u(x) = \prod_{j=0}^{2} (x-x_j) = (x-x_0)(x-x_1)(x-x_2)$$

1) $l_1(x) = \frac{u_1(x)}{u_1(x_1)} = \frac{(x-x_0)\,(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1)(x-4)}{(3-1)(3-4)} = \frac{\text{exp}}{-2}$.

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{3}$$

avem $l_0, l_1, l_2 \Rightarrow$ acum putem calcula $L_2 f(x)$

$$L_2 f(x) = \frac{(x-x_1)(x-x_2)}{6} \cdot f(x_0) - \frac{(x-1)(x-4)}{2} \cdot f(x_1) + \frac{(x-1)(x-3)}{3} \cdot f(x_3)$$

$$L_2 f(x) = \frac{(x-x_1)(x-x_2)}{6} \cdot 0 - \frac{(x-1)(x-4)}{2} \cdot 1.09 + \frac{(x-1)(x-3)}{3} \cdot 1.38$$

← this is the poly ✓

<u>Newton form</u> (Newton interpolation formula) = Lagrange poly in Newton form

$$L_2 f(x) = N_2 f(x)$$

$$= f(x_0) + \sum_{i=1}^{2} (x-x_0)\dots(x-x_{i-1})(\Delta^i f)(x_0)$$

$$= f(x_0) + \sum_{i=1}^{2} (x-x_0)\dots(x-x_{i-1})\,[x_0, \dots, x_i; f].$$

Making the table (because of $(\Delta^i f)$) → de ordin 1 și 2 because $i = 1:2$.

x	f	$\Delta^1 f$	$\Delta^2 f$
$x_0 = 1$	0	$\frac{1.09-0}{3-1}$	$\frac{0.29-0.545}{4-1} = -0.085$
$x_1 = 3$	1.09	$\frac{1.38-1.09}{4-3}$	
$x_2 = 4$	1.38		

we jump one step. $x_2 - x_0$ nu $x_1 - x_0$

1) $\Delta^1 f = \frac{1.09-0}{3-1} = 0.545$

2) $\Delta^1 f = \frac{1.38-1.09}{4-3} = 0.29$

$L_2 f(x) = N_2 f(x) =$

$= 0 + (x-x_0)(\Delta^1 f)(x_0) + (x-x_0)(x-x_1)(\Delta^2 f)(x_0) =$

$= 0 + (x-1) \cdot 0{,}545 + (x-1)(x-3)(-0{,}085) = \ldots$

The remainder.

$$(R_2 f)(x) = \frac{u(x)}{(2+1)!} f^{(2+1)}(\varepsilon), \quad \varepsilon \in (1, 4)$$

$$= \frac{\overbrace{(x-1)(x-3)(x-4)}^{u(x)}}{6} \cdot \ln''' \varepsilon$$

Limit of the error

$$|(R_2 f)(x)| \le \frac{u(x)}{6} \cdot \|\max_{x \in [1,4]} |f'''(x)| \quad |f^{(3)}(x)| = \frac{2}{x^3}$$

$$|(R_2 f)(2)| = \frac{|(2-1)(2-3)(2-4)|}{6} = \frac{2}{1} = 2$$

Next: Hermite interpolation polynomial.
↳ when we know derivatives of the functions, apply this interpolaction..

If we know $f(0)$, $f'(0)$
$f(1)$, $f'(1)$
$\cancel{f(2)}$, $f'(2)$
$(f(2))$

we miss value of the function
⇓
Birkoff interpolation

if we would have this ⇒ Hermite interpol.

sau dacă am avea $f''(2)$ în loc de $f'(2)$

$\Downarrow$

tot Birkhoff interpolation.

E Birkhoff dacă
- funcții cu valori lipsesc.
- derivatele nu sunt toate de ordin ...

Dacă toate sunt prezente $\Rightarrow$ Hermite interpolation.

$\boxed{\text{Ex1}}$ Find the interp. poly. corresp. to the data

$f(-1)=0,\ f(0)=1,\ f'(0)=-2.$

we don't have $f'(-1)$ $\Rightarrow$ Birkhoff?

No, we can compute $f'(-1)$!

And we don't miss the value of the func, we have $f(-1)$!. ($f'(-1)$ is missing).

Hermite interpol.:

$$(H_m f)(x) = \sum_{k=0}^{m} \sum_{j=0}^{r_k} h_{kj}(x) \cdot f^{(j)}(x_k) =$$

$x_0 = -1$

$x_1 = 0$

$m = 1$

$r_0 = 0$

$r_1 = 1$

degree of the poly $= m + r_0 + r_1 = 1+0+1 = \underline{\underline{2}}$

$$= \underbrace{h_{00}(x) \cdot f^{(0)}(x_0)}_{k=0} + \underbrace{h_{10}(x) \cdot f^{(0)}(x_1) + h_{11}(x) \cdot f^{(1)}(x_1)}_{k=1}$$

(Exp 3)

→ ydep from (H2f)(x).

1) $h_{00}(x) = ax^2 + bx + c$ → write the system

from énoncé of problem

$$\begin{cases} h_{00}(x_0) = ax_0^2 + bx_0 + c = 1 \\ h_{00}(x_1) = ax_1^2 + bx_1 + c = 0 \\ h_{00}^{(1)}(x_1) = \qquad = 0 \end{cases}$$

→ 0, when the first index from h_{00} differs from the index of x (x_1) $0 \neq 1$

→ 1, when they don't differ (0 = 0)

$h_{00}(x_0)$

$h_{00}^{(0)}(x_0) =$

$\boxed{2 \text{ conditions}}$!

$$\begin{cases} a + b + c = 1 \\ c = 0 \\ b = 0 \end{cases} \qquad \begin{cases} a = 1 \\ b = 0 \\ c = 0. \end{cases} \checkmark$$

2) $h_{10}(x) = a_1x^2 + b_1x + c_1$

$$\begin{cases} h_{10}(x_0) = 0 \\ h_{10}(x_1) = 1 \\ h'_{10}(x_1) = 0 \end{cases} \qquad \begin{cases} a_1 - b_1 + c_1 = 0 \\ c_1 = 1 \\ b_1 = 0 \end{cases} \qquad \begin{cases} a_1 = 0 \\ b_1 = 0 \\ c_1 = 1 \end{cases} \checkmark$$

3) $h_{11}(x) = a_2x^2 + b_2x + c_2$

$$\begin{cases} h_{11}(x_0) = 0 \\ h_{11}(x_1) = 0 \\ h'_{11}(x_1) = 1 \end{cases} \qquad \begin{cases} a_2 - b_2 + c_2 = 0 \\ c_2 = 0 \\ b_2 = 1 \end{cases} \qquad \begin{cases} a_2 = +1 \\ b_2 = 1 \\ c_2 = 0 \end{cases} \checkmark$$

Replace in $(H_2 f)(x)$ (Exp 3).

$$= \underbrace{x^2 \cdot 0}_{\uparrow} + (x^2+1)\cdot 1 + (x^2+x)(-2)$$

replace a, b, c in
$h_{00}(x)$

$\boxed{\text{Obs}}$

→ Lagrange - când știm numai valoarea funcției

→ Hermite - când știm atât val. funcției cât și niște derivate. (de acelaș ordin)

→ Birkoff. - când lipsesc val. func dar avem derivate (nu neapărat de acelaș ordin).

$Hf^{(j)}(x_k)$, $j = \overline{0, r_k}$, $k = \overline{0, m}$

$Bf^{(j)}(x_k)$, $j \in I_k \subset \{0, 1, \dots, r_k\}$.

$f(0)$, $r_0 = 0$
$f(1)$, $f'(1)$, $r_1 = 1$

$f(0) = I_0 = \{0\}$
$f'(1) = I_1 = \{1\}$
$f(2), f''(2) \quad I_2 = \{0, 2\}$.

Example: Find the coefficients of the following quadrature formula:

$$\int_0^1 f(x)\,dx = A f^{(1)}(0) - B f^{(1)}(0) + C f(1)$$

(find A, B, C)

→ approximate the func → use interpolation procedure

→ approx $f(x)$ → approx using Birkoff interpol.
(pt. că nu avem $f(0)$, doar $f'(0)$ și $f''(0)$.

$f(x) \simeq (Bf)(x)$.

valorile ce iau după ordin pt. $f(0) < f'(0) < f''(0)$.

$f^{(1)}(0)$, $f^{(1)}(0)$ → $x_0 = 0$ $\quad I_0 = \{1, 2\}$

$f(1)$ → $x_1 = 1$ $\quad I_1 = \{0\}$.

$m = 1$

$n = |I_0| + |I_1| - 1$

$n = 2 + 1 - 1 = 2$

⇓

second degree polynomial.

$$(B_2 f)(x) = \sum_{k=0}^{1} \sum_{j \in I_k} b_{kj}(x) f^{(j)}(x_k) =$$

$$= \underbrace{b_{01}(x) \cdot f^{(1)}(x_0)} + \underbrace{b_{02}(x) f^{(2)}(x_0)} + \underbrace{b_{10}(x) f^{(0)}(x_1)}.$$

1) $b_{01}(x) = ax^2 + bx + c$

$$\begin{cases} b_{01}'(x_0) = 1 \\ b_{01}''(x_0) = 0 \\ b_{01}(x_1) = 0 \end{cases}$$

$c = 1$

... Hermite like ...

aplicând Birkoff.

După ce am găsit polynomul, facem integrala

$\int_0 \text{poly Bir.} \Rightarrow$ rezultă A, B, C.

! For now only < divided / finite differences interpolations.

Dacă nu știm să derivăm, putem întreba.

$\boxed{3}$ Numerical integration of function ← trapezium not but < repeated trap. form / Simpson formula } must know.

~~Repeated~~ trapezium formula

Approximate $\int_0^2 \frac{1}{x+4}$ with $\varepsilon = 10^{-3}$ precission.

$$|(R_n f)(x)| \leq \frac{(b-a)^3}{12n^2} M_2 f = \frac{(2-0)^3}{12n^2} M_2 f < \underbrace{\frac{1}{1000}}_{\varepsilon}$$

$\boxed{\begin{aligned} a&=0 \\ b&=2 \end{aligned}}$

$$\Downarrow$$

$$\frac{2}{3n^2} \cdot M_2 f < \frac{1}{1000}$$

$$M_2 f = \max_{a \leq x \leq b} |f''(x)| = \frac{2}{4^3}$$

$$f(x) = \frac{1}{x+4}$$

$$f'(x) = \frac{-1}{(x+4)^2}$$

$$f''(x) = \frac{2}{(x+4)^3}$$ → maximul este în 0

$$\left|(R_n f)(x)\right| \le \frac{2}{4^3} \cdot \frac{2}{3n^2} < \frac{1}{1000}$$

$$1000 < 16 \cdot 3 \cdot n^2$$

$\Downarrow$

scoatem n (cel mai mic n care îndeplinește condiția)

După ce aflăm n

$\Downarrow$

calculăm formula repeated trapezium formula. Ls genurăm puncte în intervalul a, b și le înlocuim.

Din C7 – e nu se cer.

Se cere – Gauss type formula.

C8 – linear systems (nu se dă)

C9 Jacobi – we should know to obtain several iterations.