Aligning 3d orthonormal bases

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1 PROBLEM FORMULATION

Let $\{x, y, z\}$ be an orthonormal basis. Given a unit length vector z' we want to find the orthonormal basis $\{x', y', z'\}$ such that we can get from $\{x, y, z\}$ to $\{x', y', z'\}$ with the shortest rotation around $z \times z'$.

2 SOLUTION

We will first derive a solution for x, and the solution for y would be analogous. Let us decompose x into a sum of two orthogonal vectors, one lying in the plane S spanned by z and z' (as long as z and z' are not parallel) and a vector orthogonal to it. We will denote by $x_{\|}$ the projection of x onto S, and by $x_{\perp}$ the orthogonal vector to S, i.e. $x = x_{\|} + x_{\perp}$. We can further decompose $x_{\|}$ into a sum of a vector parallel to z and a vector parallel to $\hat{z}_{\perp}$, where $\hat{z}_{\perp}$ is unit length, parallel to S, and orthogonal to z. Then we have $x_{\|} = \langle x, z\rangle z + \langle x, \hat{z}_{\perp}\rangle \hat{z}_{\perp}$, where $\langle \cdot, \cdot \rangle$ is the scalar product. Note that $\{x, y, z\}$ is an orthonormal basis, so $\langle x, z\rangle = 0$ and we can write $x_{\|} = \langle x, \hat{z}_{\perp}\rangle \hat{z}_{\perp}$. The vector x' which we are trying to find can also be decomposed in a similar fashion: $x' = x'_{\|} + x'_{\perp}$, such that $x'_{\|} = \langle x', z'\rangle z' + \langle x', \hat{z}'_{\perp}\rangle \hat{z}'_{\perp} = \langle x', \hat{z}'_{\perp}\rangle \hat{z}'_{\perp}$. Since we can get from $\{x, y, z\}$ to $\{x', y', z'\}$ with a rotation around $z \times z'$, then $x'_{\perp} = x_{\perp}$, and also $\langle x', \hat{z}'_{\perp}\rangle = \langle x, \hat{z}_{\perp}\rangle$. Then we have:

$$
\begin{aligned}
x' &= x'_{\|} + x'_{\perp} \\
&= \langle x, \hat{z}_{\perp}\rangle \hat{z}'_{\perp} + x_{\perp} \\
&= \langle x, \hat{z}_{\perp}\rangle \hat{z}'_{\perp} + x - x_{\|} \\
&= x + \langle x, \hat{z}_{\perp}\rangle \hat{z}'_{\perp} - \langle x, \hat{z}_{\perp}\rangle \hat{z}_{\perp} \\
&= x + \langle x, \hat{z}_{\perp}\rangle (\hat{z}'_{\perp} - \hat{z}_{\perp})
\end{aligned}
$$

However, as long as z and z' are not parallel, we can compute $\hat{z}_{\perp}$ and $\hat{z}'_{\perp}$ in the following manner: $\hat{z}_{\perp} = (z' - \langle z, z'\rangle z)/\sqrt{1 - \langle z, z'\rangle^2}$, $\hat{z}'_{\perp} = (\langle z, z'\rangle z' - z)/\sqrt{1 - \langle z, z'\rangle^2}$. Plugging this in, yields:

$$
\begin{aligned}
x' &= x + \langle x, \hat{z}_{\perp}\rangle (\hat{z}'_{\perp} - \hat{z}_{\perp}) \\
&= x + \frac{\langle x, z' - \langle z, z'\rangle z\rangle}{\sqrt{1 - \langle z, z'\rangle^2}} \frac{(\langle z, z'\rangle z' - z - z' + \langle z, z'\rangle z)}{\sqrt{1 - \langle z, z'\rangle^2}} \\
&= x + \frac{\langle x, z' - \langle z, z'\rangle z\rangle}{1 - \langle z, z'\rangle^2} ((\langle z, z'\rangle - 1)(z' + z) \\
&= x - \frac{\langle x, z' - \langle z, z'\rangle z\rangle}{1 + \langle z, z'\rangle} (z' + z) \\
&= x - \frac{\langle x, z'\rangle}{1 + \langle z, z'\rangle} (z' + z)
\end{aligned}
$$

Where the last equality holds due to x and z being orthonormal. Note that this fails when z and z' point in opposite directions (since then there are infinitely many solutions to the problem). The solution for y is similar: $y' = y - \frac{\langle y, z'\rangle}{1 + \langle z, z'\rangle}(z' + z)$.