Probability density functions of the noramlized disk, circle, ball and sphere

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1 PROBLEM FORMULATION

We want to find the probability density function (pdf) of the multivariate random variable $(X', Y', Z') = \frac{(X,Y,Z)}{\sqrt{X^2+Y^2+Z^2}}$. Where (X, Y, Z) is uniformly distributed either in the volume or on the surface of the ball centered at $(0, \rho, 0)$ with radius ρ. We will first consider the $2D$ case with the disk centered at $(0, \rho)$ with radius ρ. We then derive the pdf in the 3-dimensional case.

2 PDF TRANSFORMATION

Let $\vec{X} = (X_1, X_2, ..., X_n)$ be a multivariate random variable distributed according to the pdf $p_{\vec{x}}(x_1, ..., x_n)$, and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be the transformation we apply to our random variable in order to get $\vec{Y} = (Y_1, ..., Y_m) = f(X_1, ..., X_n)$. Let $F_{\vec{y}}(\vec{y})$ be the cumulative distribution function for $\vec{Y} = (Y_1, ..., Y_m)$. Then by definition $F_{\vec{y}}(\vec{y}) = Pr(\vec{Y} \leq \vec{y}) = Pr(\vec{f}(\vec{X}) \leq \vec{y})$, after differentiating both sides:

$$p_{\vec{y}}(y_1, ...y_m) = \frac{\partial}{\partial y_1} ... \frac{\partial}{\partial y_m} \int_{\{\vec{x} \in \mathbb{R}^n | \vec{f}(\vec{x}) \leq \vec{y}\}} p_{\vec{x}}(x_1, ..., x_n) dx_1 ... dx_n$$

3 PDF IN 2D

The problem is easier to solve if we represent the disk S centered at $(0, \rho)$ with radius ρ in polar coordinates. To this end we will construct the ray $(x, y) = (t \sin\theta, t\cos\theta)$ such that $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and intersect it with the boundary of S: $x^2 + (y - \rho)^2 = \rho^2$. Note that we are measuring the angle θ clockwise starting from the vector $(0, 1)$ in order to be consistent with computer graphics conventions (specifically in the context of the rendering equation).

$$\begin{gathered}(t\sin\theta)^2 + (t\cos\theta - \rho)^2 = \rho^2 \\ t^2\sin^2\theta + t^2\cos^2\theta - 2t\rho\cos\theta + \rho^2 - \rho^2 = 0 \\ t^2 - 2t\rho\cos\theta = t(t - 2\rho\cos\theta) = 0 \\ t_1 = 0, t_2 = 2\rho\cos\theta\end{gathered}$$

We find two intersection points. However, t_1 describes only a single point $(0, 0)$ while t_2 describes describes all of the points in ∂S (including $(0, 0)$ for $\theta = \pm\frac{\pi}{2}$). Note that the radius $r(\theta)$ is simply $r(\theta) = |t_2| = 2\rho|\cos\theta|$ with $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and since the cosine is positive in that interval we may drop the absolute value: $r(\theta) = 2\rho\cos\theta$.

3.1 PDF of the normalized disk

Now we can easily compute the normalization of the pdf for uniformly distributed points in S (which as one would expect is $\pi\rho^2$).

$$\begin{aligned}&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\infty H(2\rho\cos\theta - r) C r dr d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\rho\cos\theta} C r dr d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2C\rho^2\cos^2\theta d\theta \\ &= 2C\rho^2 \frac{1}{2}[\theta + \cos\theta\sin\theta]\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = C\pi\rho^2 = 1 \\ &C = \frac{1}{\pi\rho^2}\end{aligned}$$

Then we have the pdf $p(r,\theta) = \frac{\chi_S(r,\theta)}{\pi\rho^2} = \frac{H(2\rho\cos\theta - r)}{\pi\rho^2}$. We want to find the pdf $q(r,\theta) \equiv q(\theta)$ resulting from applying the transformation $r' = 1, \theta' = \theta$.

$$\begin{aligned}q(\theta') &= \frac{d}{d\theta'} \int_{-\frac{\pi}{2}}^{\theta'} \int_0^\infty p(r,\theta) r dr d\theta \\ &= \frac{d}{d\theta'} \int_{-\frac{\pi}{2}}^{\theta'} \int_0^{2\rho\cos\theta} \frac{r}{\pi\rho^2} dr d\theta \\ &= \frac{2\cos^2\theta'}{\pi}\end{aligned}$$

Note that this is equivalent to computing the marginal density with respect to θ. This is not generally the case as we'll see in the normalized circle pdf derivation.

3.2 PDF of the normalized circle

Once again we first compute the normalization (which this time around should not yield 2π since the circle is not centered).

$$\begin{aligned}&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\infty \delta(2\rho\cos\theta - r) C r dr d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2C\rho\cos\theta d\theta \\ &= 4C\rho = 1 \\ &C = \frac{1}{4\rho}\end{aligned}$$

Then we have the pdf $p(r,\theta) = \frac{\chi_{\partial S}(r,\theta)}{4\rho} = \frac{\delta(2\rho\cos\theta - r)}{4\rho}$. We want to find the pdf $q(r,\theta) \equiv q(\theta)$ resulting from applying the transformation $r' = 1, \theta' = \theta$.

$$
\begin{aligned}
q(\theta') &= \frac{d}{d\theta'}\int_{-\frac{\pi}{2}}^{\theta'}\int_0^{\infty} p(r,\theta) r dr d\theta \\
&= \frac{d}{d\theta'}\int_{-\frac{\pi}{2}}^{\theta'}\int_0^{\infty} \frac{\delta(2\rho\cos\theta - r)}{4\rho} r dr d\theta \\
&= \frac{\cos\theta'}{2}
\end{aligned}
$$

Often times Lambert's cosine law is represented through a diagram of a unit circle on top of a shading point. The result above explains why this is the case, since the angular distribution is proportional to the cosine, and equivalent to a cosine distribution over the (upper) unit hemicircle around the normal of a particular shading point.

4 PDF IN 3D

The derivation is similar to the $2D$ case. We intersect a ray $(x, y, z) = (t\sin\theta\cos\phi, t\cos\theta, t\sin\theta\sin\phi)$, where $\theta \in [0, \frac{\pi}{2}], \phi \in [0, 2\pi]$, with $\partial S : x^2 + (y-\rho)^2 + z^2 = \rho^2$. Once again we measure θ clockwise (if the coordinate system is right-handed) starting from $(0, 1, 0)$ in order to be consistent with computer graphics conventions.

$$
\begin{aligned}
(t\sin\theta\cos\phi)^2 + (t\cos\theta - \rho)^2 + (t\sin\theta\sin\phi)^2 &= \rho^2 \\
t^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + t^2\cos^2\theta - 2t\rho\cos\theta + \rho^2 - \rho^2 &= 0 \\
t^2 - 2t\rho\cos\theta = t(t - 2\rho\cos\theta) &= 0 \\
t_1 = 0, t_2 &= 2\rho\cos\theta
\end{aligned}
$$

We find two intersection points. However, t_1 describes only a single point $(0, 0)$ while t_2 describes describes all of the points in ∂S (including $(0, 0)$ for $\theta = \frac{\pi}{2}$). Note that the radius $r(\theta)$ is simply $r(\theta) = |t_2| = 2\rho|\cos\theta|$ with $\theta \in [0, \frac{\pi}{2}], \phi \in [0, 2\pi]$, and since the cosine is positive in that interval we may drop the absolute value: $r(\theta) = 2\rho\cos\theta$.

4.1 PDF of the normalized ball

Now we can easily compute the normalization of the pdf for uniformly distributed points in S (which as one would expect is $\frac{4}{3}\pi\rho^3$).

$$
\begin{aligned}
&\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}}\int_0^{\infty} H(2\rho\cos\theta - r) C r^2 \sin\theta dr d\theta \\
&= \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}}\int_0^{2\rho\cos\theta} C r^2 \sin\theta dr d\theta \\
&= 2\pi\int_0^{\frac{\pi}{2}} \frac{8}{3} C\rho^3 \cos^3\theta\sin\theta d\theta \\
&= \frac{4}{3} C\pi\rho^3 = 1 \\
C &= \frac{3}{4\pi\rho^3}
\end{aligned}
$$

Then we have the pdf $p(r, \phi, \theta) = \frac{3}{4\pi\rho^3}\chi_S(r, \phi, \theta)$, $p(r, \phi, \theta) = \frac{3}{4\pi\rho^3} H(2\rho\cos\theta - r)$. We want to find the pdf $q(r, \phi, \theta) \equiv q(\theta)$ resulting from applying the transformation $r' = 1, \phi' = \phi, \theta' = \theta$.

$$
\begin{aligned}
q(\theta') &= \frac{\partial}{\partial\phi'}\int_0^{\phi'} d\phi \frac{\partial}{\partial\theta'}\int_0^{\theta'}\int_0^{\infty} p(r, \phi, \theta) r^2 \sin\theta dr d\theta \\
&= \frac{\partial}{\partial\theta'}\int_0^{\theta'}\int_0^{2\rho\cos\theta} \frac{3r^2\sin\theta}{4\pi\rho^3} dr d\theta \\
&= \frac{\partial}{\partial\theta'}\int_0^{\theta'} \frac{8\rho^3\cos^3\theta\sin\theta}{4\pi\rho^3} d\theta \\
&= \frac{2\cos^3\theta'\sin\theta'}{\pi}
\end{aligned}
$$

Note that this was equivalent to computing the marginal distribution with respect to ϕ, θ. This is not generally the case as we'll see in the normalized sphere pdf derivation.

4.2 PDF of the normalized sphere

We can compute the normalization of the pdf for uniformly distributed points in ∂S:

$$
\begin{aligned}
&\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}}\int_0^{\infty} \delta(r - 2\rho\cos\theta) C r^2 \sin\theta dr d\theta \\
&= 2\pi\int_0^{\frac{\pi}{2}} 4C\rho^2\cos^2\theta\sin\theta d\theta \\
&= \frac{8}{3} C\pi\rho^2 = 1 \\
C &= \frac{3}{8\pi\rho^2}
\end{aligned}
$$

Then we have the pdf $p(r, \phi, \theta) = \frac{3}{8\pi\rho^2}\chi_{\partial S}(r, \phi, \theta) = \frac{3}{8\pi\rho^2}\delta(r - 2\rho\cos\theta)$. We want to find the pdf $q(r, \phi, \theta) \equiv q(\theta)$ resulting from applying the transformation $r' = 1, \phi' = \phi, \theta' = \theta$.

$$
\begin{aligned}
q(\theta') &= \frac{\partial}{\partial\phi'}\int_0^{\phi'} d\phi \frac{\partial}{\partial\theta'}\int_0^{\theta'}\int_0^{\infty} p(r, \phi, \theta) r^2 \sin\theta dr d\theta \\
&= \frac{\partial}{\partial\theta'}\int_0^{\theta'}\int_0^{2\rho\cos\theta} \frac{3}{8\pi\rho^2}\delta(r - 2\rho\cos\theta) r^2 \sin\theta dr d\theta \\
&= \frac{\partial}{\partial\theta'}\int_0^{\theta'} \frac{3}{2\pi}\cos^2\theta\sin\theta d\theta \\
&= \frac{3}{2\pi}\cos^2\theta'\sin\theta'
\end{aligned}
$$