Probability density functions of the projected offset disk, circle, ball, and sphere

Vassillen Chizhov
University of Saarland, IMPRS
Saarbücken, Germany

1 PROBLEM FORMULATION

We want to find the probability density function (PDF) of the multivariate random variable $(X', Y', Z') = \frac{(X,Y,Z)}{\sqrt{X^2+Y^2+Z^2}}$. Where (X, Y, Z) is uniformly distributed either in the volume or on the surface of the ball centered at $(0, 0, \rho)$ with radius ρ. We will first consider the $2D$ case with the disk centered at $(0, \rho)$ with radius ρ. We then derive the pdf in the 3-dimensional case.

2 NOTATION AND PREREQUISITES

Throughout the text we'll be using the following conventions and notation:

For polar coordinates we'll use the somewhat unusual convention (note that the rotation is measured clockwise from $(0, 1)$, in the context of ray tracing this is usually the normal of the point being shaded in local coordinates):

$$(x, y) = r(\sin\theta, \cos\theta), r \geq 0, \theta \in [-\pi, \pi]$$

This is done in order to be consistent with computer graphics notation, additionally it allows for the form of the probability density functions (PDF) and bidirectional scattering distribution function (BSDF) to be similar to the ones for the $3D$ case. For spherical coordinates we'll be using the standard convention:

$$(x, y, z) = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$
$$r \geq 0, \theta \in [0, \pi], \phi \in [0, 2\pi]$$

With δ we'll denote the scalar Dirac delta, Θ will denote the Heaviside step function. If D is a set, χ_D will be its indicator function, its boundary will be ∂D, and TD will be the set resulting from applying some translation to the set D. The subscripts $_P$, $_S$, and $_C$ will denote polar, spherical, and Cartesian coordinates respectively, specifically in the context of the measure with respect to which a PDF is defined. PDFs will be denoted by p or q, with various subscripts, the corresponding CDFs (cumulative distribution functions) will be denoted with a capitalized version of the literal used to denote the PDF an will have the same subscripts. $U[a, b]$ will denote the uniform distribution on the interval $[a, b]$, $u_1, u_2 \sim U[a, b]$ will denote u_1 and u_2 being (independently) distributed according to $U[a, b]$.

We'll also make use of several mathematical results outlined below.

2.1 Invertible PDF transformation

The following statement can be found it [5]: Let $\vec{X} = (X_1, X_2, ..., X_n)$ be a random vector with the density $p_{\vec{X}}(x_1, ..., x_n)$, and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. For the density $p_{\vec{Y}}(y_1, ..., y_n)$ of $\vec{Y} = \vec{f}(\vec{X})$ we have:

$$\int_G p_{\vec{X}}(\vec{x})\, d\vec{x} = \int_{f(G)} p_{\vec{X}}(f^{-1}(\vec{y})) \left| \frac{\partial(x_1, ..., x_n)}{\partial(y_1, ..., y_n)} \right| d\vec{y}$$

Differentiating both sides yields:

$$p_{\vec{Y}}(\vec{y}) = p_{\vec{X}}(f^{-1}(\vec{y})) \left| \frac{\partial(x_1, ..., x_n)}{\partial(y_1, ..., y_n)} \right| \tag{1}$$

where $|J_{f^{-1}}| = \left| \frac{\partial(x_1,...,x_n)}{\partial(y_1,...,y_n)} \right|$ is the Jacobian determinant of f^{-1} (note also $|J_{f^{-1}}| = |J_f|^{-1}$).

2.2 General PDF transformation

The following statement can be found in [5]: Let $\vec{X} = (X_1, X_2, ..., X_n)$ be a random vector with the PDF $p_{\vec{X}}(x_1, ..., x_n)$, and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be (a Borel-measurable) function. Let the CDF (cumulative distribution function) for $\vec{Y}$ be $P_{\vec{Y}}$ then we can find the density $p_{\vec{Y}}(y_1, ...y_m)$ of $\vec{Y} = \vec{f}(\vec{X})$:

$$P_{\vec{Y}}(\vec{y}) = Pr(\vec{Y} \leq \vec{y}) = Pr(\vec{f}(\vec{X}) \leq \vec{y})$$

Differentiating both sides yields:

$$p_{\vec{Y}}(y_1, ...y_m) = \frac{\partial}{\partial y_1} ... \frac{\partial}{\partial y_m} \int_{\{\vec{x} \in \mathbb{R}^n | \vec{f}(\vec{x}) \leq \vec{y}\}} p_{\vec{X}}(x_1, ..., x_n)\, dx_1 ... dx_n \tag{2}$$

2.3 Coarea formula applied to the Dirac delta

A variant of the following statement can be found in [2]: Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, such that $|\nabla f| \neq 0$, and $f^{-1}(0)$ be a smooth hypersurface in $\mathbb{R}^n$, and $g(x)$ be a smooth function, then using the coarea formula from geometric measure theory:

$$\int_{R^n} \delta(f(x)) g(x)\, dx = \int_{f^{-1}(0)} \frac{g(x)}{|\nabla f(x)|}\, d\sigma(x)$$

where $d\sigma$ is the measure on the hypersurface $f^{-1}(0)$. Equivalently, if we set $S = f^{-1}(0)$

$$\int_{R^n} \delta(f(x)) |\nabla f(x)| g(x)\, dx = \int_{R^n} \delta_S(x) g(x)\, dx$$
$$= \int_S g(x)\, d\sigma(x) \tag{3}$$

2.4 Inverse transform sampling

A variant of the following statement can be found in [1]: Let P be a continuous CDF, and let P^{-1} be its inverse function (using the infiumu because CDFs are weakly monotonic and right-continuous): $P^{-1}(u) = \inf x | F(x) \geq u, 0 < u < 1$. If U is a uniform random variable on $(0, 1)$ then $P^{-1}(U)$ has P as its CDF:

$$Pr(P^{-1}(U) \leq x) = Pr(U \leq P(x)) = P(x)$$

3 PDF IN 2D

In polar coordinates the uniform probability density function on the disk D with center $(0,0)$ and radius ρ, and the uniform PDF on its boundary ∂D (the circle) are given by:

$$p_{D_P}(r) = \frac{\chi_{D_P}(r)}{\pi\rho^2} r = \frac{\Theta(\rho - r)r}{\pi\rho^2}$$
$$p_{\partial D_P}(r) = \frac{\chi_{\partial D}(r)}{2\pi\rho} r = \frac{\delta(\rho - r)r}{2\pi\rho} = \frac{\delta_{\partial D_P}(r)r}{2\pi\rho}$$

Where we have used the relationship (3) to show $\delta(\rho - r) = \delta_{\partial D}(r)$. We will translate the disk so that its center is at $(0,\rho)$. To that end we will transform it into Cartesian coordinates:

$$p_{D_C}(x,y) = \frac{p_{D_P}(r)}{r} = \frac{\Theta(\rho - \sqrt{x^2+y^2})}{\pi\rho^2}$$
$$p_{\partial D_P}(x,y) = \frac{p_{\partial D_P}(r)}{r}$$

We have used (2), where the division by r is due to the Jacobian determinant. Translating the disk yields: $p_{TD_C}(x,y) = p_{D_C}(x, y-\rho)$ (the Jacobian determinant for the translation is 1). Transforming the translated PDF back to polar coordinates gives:

$$p_{TD_P}(r,\theta) = p_{TD_C}(x,y)r = \frac{\Theta(\rho - \sqrt{x^2+(y-\rho)^2})}{\pi\rho^2} r$$
$$= \frac{\Theta(\rho - \sqrt{\rho^2+r^2-2r\rho\cos\theta})r}{\pi\rho^2}$$
$$p_{\partial TD_P}(r,\theta) = p_{\partial TD_C}(x,y)r = \frac{\delta(\rho - \sqrt{\rho^2+r^2-2r\rho\cos\theta})r}{2\pi\rho}$$

Note that $\rho - \sqrt{\rho^2+r^2-2r\rho\cos\theta} = 0$ has roots at $r = 0$ and $r = 2\rho\cos\theta$, and we can ignore the 0 since it is achieved for $\theta = \pm\frac{\pi}{2}$. Thus one possible parametrisation of the boundary is $\partial TD_P = \{(2\rho\cos\theta, \theta), \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$. We can directly rewrite the PDF for the disk since:

$$\Theta(f(x)) = \chi_{\{x|f(x)\geq 0\}}(x)$$
$$\Theta(\rho - \sqrt{\rho^2+r^2-2r\rho\cos\theta}) = \chi_{TD_P}(r) = \Theta(2\rho\cos\theta - r)$$
$$p_{TD_P}(r,\theta) = \frac{\Theta(2\rho\cos\theta - r)r}{\pi\rho^2}$$

The same does not hold for $p_{\partial TD_P}(r,\theta)$ due to the properties of the Dirac delta. However we can note that the magnitude of the gradient of $\rho - \sqrt{\rho^2+r^2-2r\rho\cos\theta}$ is 1 and thus:

$$\delta(\rho - \sqrt{\rho^2+r^2-2r\rho\cos\theta}) = \delta_{\partial TD_P}(r,\theta)$$

This is not a coincidence since translating δ_G yields δ_{TG} [4]. Also note that we have swapped the places of ∂ and T since $T\partial D = \partial TD$.

3.1 PDF of the projection of the offset disk

We have the PDF $p_{TD_P}(r,\theta) = \frac{\chi_{TD_P}(r,\theta)r}{\pi\rho^2} = \frac{\Theta(2\rho\cos\theta - r)r}{\pi\rho^2}$. We want to find the PDF $q_{TD_P}(r,\theta) \equiv q_{TD_P}(\theta)$ resulting from applying the transformation $r' = 1, \theta' = \theta$ (a projection onto the unit hemicircle in the same half space as $(0,1)$ - usually the normal in local space):

$$q_{TD_P}(\theta') = \frac{\partial}{\partial\theta'}\int_{-\frac{\pi}{2}}^{\theta'}\int_0^{\infty} p_{TD_P}(r,\theta)\,dr\,d\theta$$
$$= \int_0^{\infty}\frac{\Theta(2\rho\cos\theta' - r)r}{\pi\rho^2}\,dr$$
$$= \int_0^{2\rho\cos\theta'}\frac{r}{\pi\rho^2}\,dr$$
$$= \frac{2\cos^2\theta'}{\pi}$$

We have applied (2) to get the first equality (since the projection is not invertible), note however that this is equivalent to computing the marginal density for θ. This is expected considering the definition of marginalization, the argument of Θ and what the projection actually does.

3.2 Sampling from the projection of the offset disk

There are two obvious ways to sample from this PDF. The first is to generate samples uniformly in the unit disk with center $(0,0)$, translate them with the normal, and then normalize the resulting points. Generating points uniformly in the disk with center $(0,0)$ and radius ρ can be done (it's also possible through rejection sampling) by inverting the cumulative marginal distributions (we don't need conditional distributions since the joint PDF is separable):

$$u_1, u_2 \sim U[0,1]$$
$$P_{D_P,\theta}(\theta') = \int_{-\pi}^{\theta'}\int_0^{\infty} p_{D_P}(r)\,dr\,d\theta$$
$$= \int_{-\pi}^{\theta'}\int_0^{\rho}\frac{r}{\pi\rho^2}\,dr\,d\theta = \frac{\theta'}{2\pi} + 1 = u_1 \rightarrow$$
$$\theta' = P_{D_P,\theta}^{-1}(u_1) = 2\pi(u_1 - 1) \simeq 2\pi u_1$$
$$P_{D_P,r}(r') = \int_{-\pi}^{\pi}\int_0^{r'} p_{D_P}(r)\,dr\,d\theta = r'^2 = u_2 \rightarrow$$
$$r' = P_{D_P,r}^{-1}(u_2) = \rho\sqrt{u_2}$$

Where we have used the inverse transform sampling method. Note that $1-u_1$ and u_1 have the same distribution so using $2\pi(u_1-1)$ and $2\pi u_1$ is the same. Having a normal n and setting $\rho = 1$ we can get uniformly distributed points in the unit disk with center n by using $n + \sqrt{u_2}(\sin 2\pi u_1, \cos 2\pi u_1)$. The projection onto the unit hemicircle in the same half space as n can be computed by normalizing the resulting points. The method generates $\cos^2\theta$ distributed points on the hemisphere, however it has a singularity near 0 which yields poor numerical precision for points sampled close to $(0,0)$.Additionally the method does not require a second vector orthogonal to the normal (even if it is trivial to construct such a vector $(-n_y, n_x)$ unlike in the 3-dimensional case).

The second method consists of directly using the derived pdf to sample the points:

$$u_1 \sim U[0,1]$$
$$Q_{TD_P,\theta}(\theta') = \int_{-\pi}^{\theta'} \frac{2\cos^2\theta'}{\pi}\,d\theta$$
$$= \frac{\theta' + \sin\theta'\cos\theta'}{\pi} + 1 = u_1$$

Note that the expression doesn't have a closed form inverse, so the remaining option is a numerical inversion technique. Then the sample points are given by $\sin(\theta'(u_1))n_\perp + \cos(\theta'(u_1))n$.

3.3 PDF of the projection of the offset circle

We have the PDF:

$$p_{\partial TD_P}(r,\theta) = \frac{\chi_{\partial TD_P}(r,\theta)r}{2\pi\rho} = \frac{\delta(\rho - \sqrt{\rho^2 + r^2 - 2r\rho\cos\theta})r}{2\pi\rho}$$

We want to find the PDF $q_{\partial TD_P}(r,\theta) \equiv q_{\partial TD_P}(\theta)$ resulting from applying the transformation $r' = 1, \theta' = \theta$:

$$q_{\partial TD_P}(\theta') = \frac{\partial}{\partial\theta'}\int_{-\frac{\pi}{2}}^{\theta'}\int_0^\infty p_{\partial TD_P}(r,\theta)\,dr\,d\theta$$
$$= \frac{\partial}{\partial\theta'}\int_{-\frac{\pi}{2}}^{\theta'}\int_0^\infty \frac{\delta(\rho - \sqrt{\rho^2 + r^2 - 2r\rho\cos\theta})r}{2\pi\rho}\,dr\,d\theta$$
$$= \frac{\partial}{\partial\theta'}\int_{-\frac{\pi}{2}}^{\theta'}\int_0^\infty \frac{r}{2\pi\rho}\frac{\delta_{\partial TD_P}(r,\theta)}{|1|}\,dr\,d\theta$$
$$= \frac{\partial}{\partial\theta'}\int_{\frac{\pi}{2}}^{\theta'} \frac{2\rho}{2\pi\rho}\,d\theta = \frac{1}{\pi}$$

We get a uniform distribution on the unit hemicircle in the same half-space as $(0,1)$. The first equality holds due to (2), the third equality holds due to the fact that the magnitude of the gradient of the argument of the delta is equal to $|1|$, where we have used the relationship from (3). We get the fourth equality by rewriting the integral due to (3) as an integral over the curve $(2\rho\cos\theta,\theta)\theta \in [-\frac{\pi}{2},\frac{\pi}{2}]$ which has differential length element $d\sigma = 2\rho\,d\theta$. We could have used $p_{\partial TD_P}(r,\theta) = \frac{\delta_{\partial TD_P}(r,\theta)r}{2\pi\rho}$ directly, which we will do in the 3-dimensional case.

3.4 Sampling from the projection of the offset circle

There are three obvious ways to sample from this PDF. The first is to generate samples uniformly in the unit disk with center $(0,0)$ (can be done through rejection sampling also), normalize them, translate them with the normal n, and then normalize the resulting points. However for points that fall close to $(0,0)$ before or after the translation one may encounter numerical issues due to the normalization.

The second is to generate uniformly distributed points on the unit circle with center $(0,0)$, translate them with the normal n, and then normalize them. Once again normalization may produce inaccurate results for points near $(0,0)$. Generating uniformly distributed points on the unit circle with center $(0,0,0)$ can be done in the following way:

$$u_1 \sim U[0,1]$$
$$P_{\partial D_P,\theta}(\theta') = \int_{-\pi}^{\theta'}\int_0^\infty p_{\partial D_P}(r)\,dr\,d\theta$$
$$= \int_{-\pi}^{\theta'} \frac{\rho}{2\pi\rho}\,d\theta = \frac{\theta'}{2\pi} + \frac{1}{2} = u_1 \rightarrow$$
$$\theta' = P^{-1}_{\partial D_P,\theta}(u_1) = (2u_1 - 1)\pi \simeq 2\pi u_1$$

Note that the last equivalence will produce points distributed in the same way because of the periodicity of sin and cos. The points on the unit circle are then given by $(\sin 2\pi u_1, \cos 2\pi u_2)$.

The third method generates points directly on the unit hemicircle in the same half-space as the normal n. It requires computing $n_\perp = (-n_x, n_y)$ which is trivial unlike in the 3-dimensional case. The generation is done in a similar way:

$$u_1 \sim U[0,1]$$
$$P'_{\partial D_P,\theta}(\theta') = \int_{-\frac{\pi}{2}}^{\theta'}\int_0^\infty 2p_{\partial D_P}(r)\,dr\,d\theta$$
$$= \int_{-\frac{\pi}{2}}^{\theta'} \frac{\rho}{\pi\rho}\,d\theta = \frac{\theta'}{\pi} + \frac{1}{2} = u_1 \rightarrow$$
$$\theta' = P'^{-1}_{\partial D_P,\theta}(u_1) = (2u_1 - 1)\frac{\pi}{2}$$

Then the sampled points are given by $\sin((2u_1-1)\frac{\pi}{2})n_\perp + \cos((2u_1-1)\frac{\pi}{2})n$. This method does not suffer from singularities near $(0,0)$ and requires less computation so it is to be preferred.

Often times Lambert's cosine law is represented through a diagram of a unit circle on top of a shading point. The result above explains why this is the case, since the angular distribution is proportional to the cosine, and equivalent to a cosine distribution over the (upper) unit hemicircle around the normal of a particular shading point.

4 PDF IN 3D

The derivation is similar to the 2D case:

In spherical coordinates the uniform probability density function on the ball B with center $(0,0,0)$ and radius ρ, and the uniform PDF on its boundary ∂B (the sphere) is given by:

$$p_{B_S}(r) = \frac{3\chi_{B_S}(r)}{4\pi\rho^3}r^2\sin\theta = \frac{3\Theta(\rho - r)r^2\sin\theta}{4\pi\rho^3}$$
$$p_{\partial B_S}(r) = \frac{\chi_{\partial B_S}(r)}{4\pi\rho^2}r^2\sin\theta = \frac{\delta(\rho - r)r^2\sin\theta}{4\pi\rho^2} = \frac{\delta_{\partial B_S}(r)r^2\sin\theta}{4\pi\rho^2}$$

We will translate the ball so that its center is at $(0,0,\rho)$. To that end we will transform it into Cartesian coordinates:

$$p_{B_C}(x,y,z) = \frac{p_{B_S}(r)}{r^2\sin\theta} = \frac{3\Theta(\rho - \sqrt{x^2+y^2+z^2})}{4\pi\rho^3}$$
$$p_{\partial B_S}(x,y,z) = \frac{p_{\partial B_S}(r)}{r^2\sin\theta}$$

We have used (1) , where the division by $r^2\sin\theta$ is due to the Jacobian determinant. Translating the ball yields: $p_{TB_C}(x,y) = p_{B_C}(x,y,z-\rho)$ (the Jacobian determinant for the translation is

1). Transforming the translated PDF back to spherical coordinates gives:

$$p_{TB_S}(r,\theta) = p_{TB_C}(x,y,z)r^2\sin\theta$$
$$= \frac{3\Theta(\rho - \sqrt{\rho^2 + r^2 - 2r\rho\cos\theta})r^2\sin\theta}{4\pi\rho^3}$$
$$p_{\partial TB_S}(r,\theta) = p_{\partial TB_C}(x,y,z)r^2\sin\theta$$

The equality for the sphere's PDF holds due the fact that the translated δ_G is simply δ_{TG}. The parametrisation of the boundary is the same as in the $2D$ case: $\partial TB_S = \{(2\rho\cos\theta,\theta,\phi), \theta\in[0,\frac{\pi}{2}], \phi\in[0,2\pi]\}$. Then we can rewrite the PDFs as:

$$p_{TB_S}(r,\theta) = \frac{3\Theta(2\rho\cos\theta - r)r^2\sin\theta}{4\pi\rho^3}$$
$$p_{\partial TB_S}(r,\theta) = \frac{\delta_{\partial TB_S}(r,\theta)r^2\sin\theta}{4\pi\rho^2}$$

4.1 PDF of the projection of the offset ball

We have the PDF $p_{TB_S}(r,\theta) = \frac{3\chi_{TB_S}(r,\theta)r^2\sin\theta}{4\pi\rho^3} = \frac{3\Theta(2\rho\cos\theta - r)r^2\sin\theta}{4\pi\rho^3}$. We want to find the PDF $q_{TB_S}(r,\theta,\phi) \equiv q_{TB_S}(\theta)$ resulting from applying the transformation $r' = 1, \theta' = \theta, \phi' = \phi$ (a projection onto the unit hemisphere in the same half space as $(0,0,1)$ - usually the normal in local space):

$$q_{TB_S}(\theta') = \frac{\partial}{\partial\theta'}\frac{\partial}{\partial\phi'}\int_0^{\phi'}\int_0^{\theta'}\int_0^{\infty} p_{TB_S}(r,\theta)\,dr\,d\theta\,d\phi$$
$$= \int_0^{2\rho\cos\theta}\frac{3r^2\sin\theta}{4\pi\rho^3}\,dr$$
$$= \frac{2\cos^3\theta'\sin\theta'}{\pi}$$

The first equality holds due to (2). Note that as in the $2D$ case this was equivalent to computing the marginal distribution for ϕ,θ once again due to the same reasons.

4.2 Sampling from the projection of the offset ball

One can sample from the PDF above without building tangents. This can be achieved by sampling from the unit ball (either through rejection sampling or by inverse transform sampling), translating the resulting points with the normal n, and normalizing. Note that there is a singularity at $(0,0,0)$ for the normalization, so points that fall close to it after the translation may be inaccurate.

If tangents are used, then one may sample directly from the PDF through inverse transform sampling.

4.3 PDF of the projection of the offset sphere

We have the PDF:

$$p_{\partial TB_S}(r,\theta) = \frac{\chi_{\partial TB_S}(r,\theta)r^2\sin\theta}{4\pi\rho^2} = \frac{\delta_{\partial TB_S}(r,\theta)r^2\sin\theta}{4\pi\rho^2}$$

We want to find the PDF $q_{\partial TB_S}(r,\theta,\phi) \equiv q_{\partial TB_S}(\theta,\phi)$ resulting from applying the transformation $r' = 1, \theta' = \theta, \phi' = \phi$:

$$q_{\partial TB_S}(\theta') = \frac{\partial}{\partial\theta'}\frac{\partial}{\partial\phi'}\int_0^{\phi'}\int_0^{\theta'}\int_0^{\infty} p_{TB_P}(r,\theta)\,dr\,d\theta\,d\phi$$
$$= \frac{\partial}{\partial\theta'}\frac{\partial}{\partial\phi'}\int_0^{\phi'}\int_0^{\theta'}\int_0^{\infty}\frac{\delta_{\partial TB_S}(r,\theta)r^2\sin\theta}{4\pi\rho^2}\,dr\,d\theta\,d\phi$$
$$= \frac{\partial}{\partial\theta'}\frac{\partial}{\partial\phi'}\int_0^{\phi'}\int_0^{\theta'}\frac{2\rho^2\sin 2\theta}{4\pi\rho^2}\,d\theta\,d\phi$$
$$= \frac{\sin 2\theta}{2\pi} = \frac{\cos\theta\sin\theta}{\pi}$$

The first equality holds due to (2) and the third equality holds due to (3). We get the fourth equality by rewriting the integral due to (3) as an integral over the surface ∂TB_S which has differential area element $d\sigma = 4\rho^2\sin 2\theta\,d\theta\,d\phi$. Since the PDF is cosine weighted it can be used to sample from the diffuse BSDF without building tangents [3].

4.4 Sampling from the projection of the offset sphere

One can sample from the PDF above without building tangents. This can be achieved by sampling the unit sphere (either through inverse transform sampling or rejection sampling inside the unit ball and a subsequent normalization), translating the resulting points with the normal n, and normalizing. Note that there is a singularity at $(0,0,0)$ for the normalization, so points that fall close to it after the translation may be inaccurate. Also if we use rejection sampling, points that also fall close to $(0,0,0)$ before the translation may also be inaccurate due to the normalization.

If tangents are used, then one may sample directly from the PDF through inverse transform sampling.

REFERENCES

[1] [n. d.]. Inverse transform sampling. Retrieved January 27, 2019 from https://en.wikipedia.org/wiki/Inverse_transform_sampling

[2] [n. d.]. Properties of the Dirac delta in n dimensions. Retrieved January 27, 2019 from https://en.wikipedia.org/wiki/Dirac_delta_function#Properties_in_n_dimensions

[3] Edd Biddulph. 2017. Lambertian Reflection Without Tangents. Retrieved January 27, 2019 from http://amietia.com/lambertnotangent.html

[4] Levent Onural. 2006. Impulse functions over curves and surfaces and their applications to diffraction. *J. Math. Anal. Appl.* 322, 1 (2006), 18 – 27. https://doi.org/10.1016/j.jmaa.2005.07.012

[5] Lars Winterfeld. [n. d.]. Transformation of Probability Densities. Retrieved January 27, 2019 from https://en.wikibooks.org/wiki/Probability/Transformation_of_Probability_Densities